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Mini-Workshop: A Geometric Fairytale full of Spectral Gaps and Random Fruit

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Abstract. In many situations, most prominently in quantum mechanics, it is important to understand well the eigenvalues and associated eigenfunctions of certain self-adjoint differential operators. The goal of this workshop was to study the strong link between spectral properties of such operators and the underlying geometry which might be randomly generated. By combining ideas and methods from spectral geometry and probability theory, we hope to stimulate new research including important topics such as Bose–Einstein condensation in random environments.

Mathematics Subject Classification (2020): 47A75, 58J50, 60K37, 60K40, 81Q10, 81Q35, 82D03.

Introduction by the Organizers

The workshop *A Geometric Fairytale full of Spectral Gaps and Random Fruit* was organised by Joachim Kerner (Hagen), Matthias Täufer (Hagen), and Pavlo Yatsyna (Espoo). In addition to the three organizers, fourteen researchers from all over the world attended the workshop. Based on recent advances in spectral geometry (e.g. related to Pólya's conjecture) and in the field of Bose–Einstein condensation in random environments (related to the Kac–Luttinger conjecture), the aim of the workshop was to bring together researchers from those and related fields in order to stimulate further research along those lines. More explicitly, it appears that the fields involved would benefit a great deal from a stronger connection between them.

A classical problem in spectral geometry is to study the eigenvalues of the two-dimensional Laplacian

$$-\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

on a bounded domain $\Omega \subset \mathbb{R}^2$ and subject to Dirichlet boundary conditions. The eigenvalues $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots$ form a sequence of real numbers tending to infinity and constitute the spectrum of the self-adjoint operator $-\Delta$. One goal is then to derive (good) upper and lower bounds on the lowest eigenvalue $\lambda_1(\Omega)$ in terms of *geometrical* quantities of Ω such as volume, inradius, etc.; a famous inequality in this context is named after Rayleigh, Faber, and Krahn. Alternatively, one might ask to derive such bounds for the so-called *spectral gap*

$$\lambda_2(\Omega) - \lambda_1(\Omega) \ .$$

On a quantum-mechanical level, the spectral gap measures the smallest amount of energy that is necessary to excite a particle occupying the ground state associated with $\lambda_1(\Omega)$. The spectral gap is also an important quantity from a mathematical point of view and we shall refer to the proof of the so-called fundamental gap conjecture by B. Andrews and J. Clutterbuck as an example of this fact.

Quite interestingly, it turns out that understanding the asymptotic behaviour of the eigenvalues of the (Dirichlet-)Laplacian is crucial in order to conclude so-called Bose–Einstein condensation (BEC) in a gas of non-interacting bosons, or to say something about its type. BEC is an important quantum-mechanical phenomenon that might occur in bosonic many-particle systems (typically at low temperatures if at all); indeed, a current goal in mathematical physics is to prove existence of BEC in a three-dimensional Bose gas with (suitable, strong) particle-particle interactions in the thermodynamic limit. It should be noted that, in various scaling-limits, existence of BEC could already be proved and a lot of progress has been achieved in the last twenty-some years. Now, going back to non-interacting Bose gases, one might ask whether BEC exists also in a *random environment* and of what type it is. Of course, the same question is sensible for a Bose gas with interactions and it is one hope that, starting with this workshop, there will be progress regarding this question in the near future. In any case, random here can mean that the domain $\Omega \subset \mathbb{R}^d$ on which one considers the d-dimensional (Dirichlet-)Laplacian is a domain that is generated via a random process. For example, based on a Poisson point process, one may randomly distribute balls of radius $R > 0$ in space and then consider the (Dirichlet-)Laplacian on the random domain

$$\Omega_{L,\omega} := (-L/2, +L/2)^d \setminus \bigcup_j B_R(x_j(\omega)) \ .$$

Here, $x_j(\omega) \in \mathbb{R}^d$ is the random centre of the j-th ball $B_R(x_j(\omega))$. Now, the spectral gap $\lambda_2(\Omega_{L,\omega}) - \lambda_1(\Omega_{L,\omega})$ is also random and depends, in addition, on the side-length of the cube $(-L/2, +L/2)^d$. The Kac–Luttinger conjecture then makes a statement about the type of BEC (so-called type-I BEC with a sole macroscopic occupation of the ground state) on $\Omega_{L,\omega}$ and in $d = 3$. At this point, we just mention that, in order to prove this conjecture, one has to show that the spectral gap does not converge to zero too fast in the limit $L \to \infty$. Also, it is interesting to mention that the recent proof of the Kac–Luttinger conjecture by A. S. Sznitman employs a quantitative version of the Rayleigh–Faber–Krahn inequality which is one indication of the strong link between the fields represented at the workshop.

Of course, spectral geometric considerations are not limited to domains in Euclidean space but also play an important role in the context of other mathematical structures such as graphs (or lattices), quantum graphs and manifolds. Due to advances in nano-technology, it has become increasingly important to understand spectral properties of such structures. Therefore, investigations of spectral gaps beyond classical topics might prove fruitful in the future.

Mini-Workshop: A Geometric Fairytale full of Spectral Gaps and Random Fruit

Table of Contents

Abstracts

A variational formulation for Dirac Operators in bounded domains and applications to spectral geometric inequalities

Rafael D. Benguria

(joint work with Pedro S. Antunes, Vladimir Lotoreichik, Thomas Ourmières-Bonafos)

Consider $\Omega \subset \mathbb{R}^2$, a C^∞ simply connected domain, and let $n = (n_1, n_2)^T$ be the outward pointing normal on $\partial\Omega$.

The Dirac operator with *infinite mass boundary conditions* in $L^2(\Omega, \mathbb{C}^2)$ is defined as,

$$D^\Omega \equiv \begin{pmatrix} 0 & -2\,i\,\partial_z \\ -2\,i\,\partial_{\bar z} & 0 \end{pmatrix}.$$

Here,

$$\mathrm{dom}(\mathrm{D}^\Omega) = \{\mathrm{u} = (\mathrm{u}_1, \mathrm{u}_2)^{\mathrm{T}} \in \mathrm{H}^1(\Omega, \mathbb{C}^2) | \mathrm{u}_2 = i\mathbf{n}\,\mathrm{u}_1 \text{ on } \partial\Omega\}.$$

We have set $\mathbf{n} = n_1 + i\,n_2$ and

$$\partial_z = \frac{1}{2}\left(\partial_1 - i\partial_2\right), \qquad \partial_{\bar z} = \frac{1}{2}\left(\partial_1 + i\partial_2\right).$$

The Dirac operator with *infinite mass boundary conditions* is self-adjoint [6, 2]. Its spectrum is symmetric with respect to the origin, consisting of eigenvalues of finite multiplicity, with,

$$... \le E_k(\Omega) \le \cdots \le -E_1(\Omega) < 0 < E_1(\Omega) < \ldots E_k(\Omega) \le \ldots$$

The following geometrical bound was proved in [3],

$$E_1(\Omega) \ge \sqrt{\frac{2\pi}{|\Omega|}},$$

where $|\Omega|$ is the area of the domain Ω. By analogy with the Rayleigh-Faber–Krahn inequality it is natural to conjecture (see, [1], Conjecture 1):

Dirac–Rayleigh–Faber–Krahn–Conjecture

$$E_1(\Omega) \ge \sqrt{\frac{\pi}{|\Omega|}}\, E_1(\mathbb{D}), \tag{1}$$

where $\mathbb{D}$ is the unit disk, and one expects equality if and only if Ω is a disk.

In Sections 7 and 8 of our recent article [1] we present strong numerical evidence supporting this conjecture. Nevertheless this conjecture is still open.

In [1] we prove the following isoperimetric upper bound on $E_1(\Omega)$.

Theorem 1. Let $\Omega \subset \mathbb{R}^2$ be a C^∞ simply connected domain. Then we have

$$E_1(\Omega) \le \frac{|\partial\Omega|}{\pi r_i^2 + |\Omega|} E_1(\mathbb{D}), \tag{2}$$

with equality if and only if Ω is a disk.

Here $|\Omega|$ is the area, $|\partial\Omega|$ the perimeter and r_i the inradius of the domain Ω.

What we actually prove is

Theorem 2. Let $\Omega \subset \mathbb{R}^2$ be a C^∞ simply connected domain. Then we have

$$E_1(\Omega) \le \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 + 8\pi E_1(\mathbb{D})(E_1(\mathbb{D})-1)(\pi r_i^2 + |\Omega|)}}{2(\pi r_i^2 + |\Omega|)}$$

with equality if and only if Ω is a disk.

Theorem 1 follows from Theorem 2 using $\pi r_i^2 \le |\Omega| \le |\partial\Omega|^2/(4\pi)$.
The proof of Theorem 2 is obtained by combining a new variational characterization of $E_1(\Omega)$, inspired by min-max techniques for operators with gaps introduced in [5], and the classical proof of Szegő about the first nontrivial Neumann eigenvalue of the Laplacian in $\mathbb{R}^2$ [7].

Consider the quadratic form

$$q^\Omega_{E,0}(u) \equiv 4\int_\Omega |\partial_{\bar z} u|^2\,dx - E^2\int_\Omega |u|^2\,dx + E\int_{\partial\Omega} |u|^2\,ds,$$

with $\operatorname{dom}\left(q^\Omega_{E,0}\right) = C^\infty(\bar\Omega, \mathbb{C})$.

For $E > 0$, $q^\Omega_{E,0}$ is bounded below with dense domain and we consider q^Ω_E the closure in $L^2(\Omega)$ of $q^\Omega_{E,0}$. Then, we define the first min-max level,

$$\mu^\Omega(E) = \inf_u \frac{q^\Omega_E(u)}{\int_\Omega |u|^2\,dx}.$$

where the infimum is taken over $\operatorname{dom}(q^\Omega_E) \setminus \{0\}$. Then we have the following variational characterization ofthe first non–negative eigenvalue.

Theorem 3. [1] $E > 0$ is the first non–negative eigenvalue of D^Ω if and only if $\mu^\Omega(E) = 0$.

Heuristics: Let $(u, v)^T \in \operatorname{dom}(D^\Omega)$ be an eigenfunction with eigenvalue E. In Ω the eigenvalue equation reads,

$$-2i\partial_z v = E\,u, \qquad -2i\partial_{\bar z} u = E\,v.$$

Assuming the equations are valid up to the boundary, using the infinite mass boundary conditions, we get the following boundary condition on u,

$$\bar n \partial_{\bar z} u + \frac{E}{2} u = 0, \qquad \text{on } \partial\Omega.$$

Now from the equations for u and v we get,

$$-4\partial_z\partial_{\bar z} u = E^2 u, \qquad \text{in } \Omega.$$

Taking the scalar product with u, integrating by parts, and using the boundary condition formally gives $q^\Omega_E(u)$. This is the reason for introducing q^Ω_E.

In order to use the function $\mu^\Omega(E)$ to estimate $E_1(\Omega)$ we need the following:

Lemma 4. [1]. The map $\mu^\Omega : E \ge 0 \to \mu^\Omega(E)$ satisfies:
i) $\mu^\Omega(E)$ is a continuous and concave function on $\mathbb{R}_+$.

ii) We have $\mu^{\Omega}(0) = 0$, and there exists $E_*^{\Omega} > 0$ such that for all $(0, E_*^{\Omega})$, $\mu^{\Omega}(E) > 0$.

To prove the Theorem 2 we construct an adequate test function for q_E^{Ω} [1]. We do so following the strategy of [7], transplanting the eigenfunction of the unit disk in Ω using a conformal map. We then use our Lemma 3.

Remark. It is interesting to note that our upper bound on $E_1(\Omega)$ given by (2) and the conjectured Rayleigh–Faber–Krahn inequality for the Dirac operator with infinite mass boundary condition given by (1) together with the onvious inequality $|\Omega| \geq \pi\, r_i^2$ would imply one of the Bonnesen inequalities [4], namely,

$$|\Omega| \leq r_i \left(|\partial\Omega| - \pi\, r_i\right).$$

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Principal frequencies and inradius

Lorenzo Brasco

(joint work with Francesca Bianchi)

Let $0 < s < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. We consider its *first eigenvalue of the fractional Dirichlet-Laplacian of order s*, defined by

$$\lambda_1^s(\Omega) = \inf_{\varphi \in C_0^\infty(\Omega)} \Big\{ [\varphi]^2_{W^{s,2}(\mathbb{R}^N)} \,:\, \|\varphi\|_{L^2(\Omega)} = 1 \Big\}.$$

Here functions $C_0^\infty(\Omega)$ are considered to be extended by 0 outside Ω and the quantity $[\,\cdot\,]_{W^{s,2}(\mathbb{R}^N)}$ is the *Sobolev-Slobodeckiĭ seminorm*, given by

$$[\varphi]^2_{W^{s,2}(\mathbb{R}^N)} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{N+2\,s}}\, dx\, dy, \qquad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N).$$

The case of the classical Dirichlet-Laplacian is formally recovered in the limit as s goes to 1, by recalling the celebrated *Bourgain-Brezis-Mironescu formula*, i.e.

$$\lim_{s \nearrow 1} (1-s)\, [\varphi]^2_{W^{s,2}(\mathbb{R}^N)} = C_N \int_{\mathbb{R}^N} |\nabla \varphi|^2\, dx, \qquad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N).$$

see [8, Corollary 3.20]. This can be made more precise, by using $\Gamma-$convergence arguments, see [6, Theorem 1.2].

We aim at proving estimates on $\lambda_1^s(\Omega)$, in terms of simple geometric features of the open set Ω. In particular, we focus on the *planar case* (i.e. we take $N=2$) and we want to compare $\lambda_1^s(\Omega)$ with the *inradius* of Ω. The latter is defined by

$$r_\Omega = \sup\Big\{r>0\,:\ \exists \text{ a disk of radius } r\subseteq \Omega\Big\}.$$

By using the scaling properties of λ_1^s and its monotonicity with respect to set inclusion, it is easily seen that we have

$$\lambda_1^s(\Omega)\le \lambda_1^s(B_1)\,\left(\frac{1}{r_\Omega}\right)^{2\,s},\qquad \text{where } B_1=\{x\in\mathbb{R}^2\,:\,|x|<1\}.$$

One may wonder whether it is possible to "revert" this estimate. In other words, we inquire whether it is possible to bound λ_1^s from below in terms of the inradius only, up to some universal constant multiplicative factor.

Already in the case of the classical Dirichlet-Laplacian, this is known to be false in general, but it becomes feasible under specific geometric and/or topological assumptions. For example, we can prove such an estimate in the following cases: for open convex sets (this is a result by Hersch, see [11]); for simply connected open sets (this is due to Makai [12] and, independently, to Hayman [10], see also Ancona [1] and Bañuelos & Carroll [2] for different proofs); or for open sets with given topology. By the latter, we mean *multiply connected sets of order k*, with $k\in\mathbb{N}\setminus\{0\}$, i.e. open planar sets whose complement has k connected components (in the one-point compactification of $\mathbb{R}^2$). In this case, the relevant lower bound on the first eigenvalue is due to Osserman [13] and Taylor [14], with different proofs. We also mention Croke [7] for a refinement of Osserman's method and Graversen & Rao [9] for a slightly suboptimal result, using probabilistic techniques.

Coming back to the fractional case, Hersch's result has been extended to the case of λ_1^s by Bañuelos, Latała and Méndez-Hernández, see [3]. In [4, 5], we show that it is possible to extend to the fractional case the results for simply and multiply connected sets, as well. Precisely, we have the following result:

Theorem. *Let $1/2<s<1$, there exists a constant $\vartheta_s>0$ such that for every $\Omega\subseteq\mathbb{R}^2$ open multiply connected set of order $k\in\mathbb{N}\setminus\{0\}$, we have*

$$\lambda_1^s(\Omega)\ge \frac{\vartheta_s}{k^s}\,\left(\frac{1}{r_\Omega}\right)^{2\,s}. \tag{1}$$

Moreover, the constant ϑ_s has the following asymptotic behaviours

$$\vartheta_s\sim (2\,s-1)^2,\quad \text{for } s\searrow\frac{1}{2}\qquad \text{and}\qquad \vartheta_s\sim\frac{1}{1-s},\quad \text{for } s\nearrow 1.$$

We also show, by constructing suitable examples, that the estimate (1) is optimal:

- in its dependence on s, as $s\nearrow 1$;
- in its dependence on k, as $k\to\infty$.

In particular, from (1) it is possible to recover the Osserman-Taylor result for the classical Dirichlet-Laplacian, by taking the limit as $s \nearrow 1$ and using the Bourgain-Brezis-Mironescu formula.

The restriction $s > 1/2$ is optimal, since in [4] we show that such an estimate is not possible for $0 < s \leq 1/2$. As for the asymptotic behaviour of ϑ_s at the critical threshold $s = 1/2$, in the simply connected case (i.e. for $k = 1$) we can improve the asymptotics above to

$$\vartheta_s \sim 2\,s - 1, \quad \text{for } s \searrow \frac{1}{2},$$

which is sharp, as showed in [5]. We conjecture that this should be the optimal behaviour of ϑ_s for $k \geq 2$, as well.

The proof of the result above is constructive, thus the constant obtained in the lower bound (1) can be made explicit. However, this is not likely to be optimal. Thus, the previous result naturally leads to the following

Open problem. *Let $1/2 < s < 1$ and $k \in \mathbb{N} \setminus \{0\}$. Find the sharp constant $T_{s,k}$ in* (1)*, i.e. determine the value of the following shape optimization problem*

$$T_{s,k} = \inf \left\{ \lambda_1^s(\Omega)\,(r_\Omega)^{2\,s} \ : \ \begin{array}{c} \Omega \subset \mathbb{R}^2 \textit{ open multiply connected set} \\ \textit{of order } k, \textit{ with } r_\Omega < +\infty \end{array} \right\}.$$

We point out that, already for the case of the classical Dirichlet-Laplacian and for simply connected sets (i.e. for $k = 1$), this is a major open problem.

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Elements of random convex geometry

Pierre Calka
(joint work with Y. Demichel, N. Enriquez, A. Goldman and J. E. Yukich)

The talk surveys some of the models, recent questions and results related to random convex geometry in the Euclidean setting. We concentrate on particular problems which satisfy the following pattern: first, we consider a random point set of $\mathbb{R}^d$, $d \geq 2$, which is almost surely locally finite, second we use this point process to construct graphs, partitions or random (convex) sets by a deterministic procedure and finally we analyse some asymptotic configuration in a broad sense.

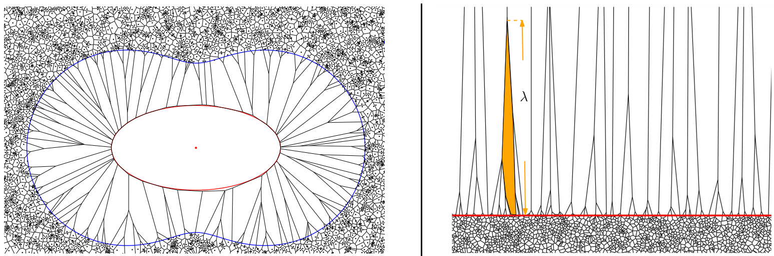
The choice of the distribution of the point process is crucial as the interaction between the random points should be reflected by the properties of the constructed geometric shape. We opt classically for the Poisson point process, i.e. an interaction-free point process in $\mathbb{R}^d$ which has the fundamental property that the numbers of points falling inside disjoint Borel subsets of $\mathbb{R}^d$ are independent and Poisson distributed. In particular, when the property of translation-invariance is added, the Poisson point process is characterized by only one real parameter called the intensity which is the mean number of points per unit volume.

We describe below two classical deterministic constructions based on a homogeneous Poisson point process in the Euclidean space: the Poisson-Voronoi tessellation and the convex hull of a Poisson input. They lead to random geometric models which we comprehend through their asymptotic behavior: in the first case, we investigate rare configurations in the partition while in the second case, we describe the obtained shape when the size of the input goes to infinity.

Large Poisson-Voronoi cells

A well-known geometric construction consists in partitioning $\mathbb{R}^d$ into convex polytopes through the Voronoi procedure associated with the homogeneous Poisson point process of intensity 1. More precisely, we associate to any Poisson point its cell, i.e. the set of points of $\mathbb{R}^d$ which are closer to that particular Poisson point, called its nucleus, than to any other Poisson point. A conjecture stated by D. G. Kendall in the forties in the particular case of the plane and proved by D. Hug and R. Schneider in a very general setting [6] claims that the large cells of the tessellation should be almost circular. In a common work with A. Goldman [5], we calculate the mean of the spectral function of the Dirichlet-Laplacian of the typical Poisson-Voronoi cell and deduce in particular that the first eigenvalue $\lambda_1(\mathcal{C})$ of the typical cell $\mathcal{C}$ has the same asymptotic distribution function as the first eigenvalue of its inball centered at its nucleus, i.e. when $t \to 0$,

$$\log \mathbb{P}(\lambda_1(\mathcal{C}) \leq t) \sim -4\pi j_0^2 t^{-1}$$



(a) The cell K_λ when K is an ellipse (b) The elongated cell $\mathcal{C}^{(\lambda)}$

where j_0 is the first positive zero of the Bessel function J_0. This result is another confirmation of the spherical shape of the large cells.

A derived problem concerns the behavior of a planar Poisson-Voronoi cell K_λ conditioned on containing a large multiple λK of a fixed shape K, see Figure (a). With Y. Demichel and N. Enriquez [1], we provide precise asymptotic estimates for the mean number of vertices, mean defect area and mean defect perimeter in the spirit of A. Rényi & R. Sulanke's work for random convex hulls [7]. For instance, there exist explicit constants c_1 and c_2 depending only on K such that when $\lambda \to \infty$

$$\mathbb{E}(\mathrm{Vol}_2(K_\lambda \setminus (\lambda K))) \sim \begin{cases} c_1\lambda^{\frac{2}{3}} & \text{if } K \text{ has a } \mathcal{C}^2\text{-boundary} \\ c_2\lambda & \text{if } K \text{ is a polygon} \end{cases} .$$

In a companion work, we focus on the neighboring elongated cells around the large cell K_λ. More precisely, we consider a simplified model where the Poisson points are located in the lower half-plane and study the typical Poisson-Voronoi cell $\mathcal{C}^{(\lambda)}$ whose maximal height is $\lambda > 0$, see Figure (b). We prove that up to rescaling, $\mathcal{C}^{(\lambda)}$ has a limiting shape, i.e. if $F^{(\lambda)}(x,y) = (\lambda^{-\frac{1}{3}}x, \lambda^{-1}y)$, $x \in \mathbb{R}$, $y > 0$ and $Z^{(\lambda)}$ is the Voronoi nucleus associated with $\mathcal{C}^{\lambda}$,

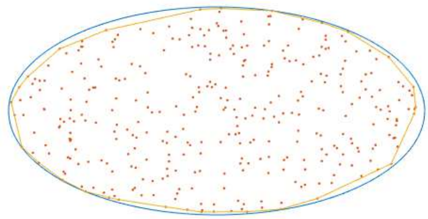
$$F^{(\lambda)}(\mathcal{C}^{(\lambda)} - Z^{(\lambda)}) \xrightarrow{D} \mathcal{C}^{(\infty)}$$

where $\mathcal{C}^{(\infty)}$ is a random apeirogon whose distribution is encoded by an explicit Markov chain in $\mathbb{R}^2$, see [2].

Typical and maximal fluctuations of random convex hulls

We consider the convex hull K_λ of the intersection of a homogeneous Poisson point process of intensity $\lambda > 0$ in $\mathbb{R}^d$ with a convex body K of $\mathbb{R}^d$ with a $\mathcal{C}^2$ boundary and positive Gaussian curvature, see e.g. Figure (c).

The asymptotics of its global characteristics, such as its number of faces or its defect volume have been largely investigated since A. Rényi & R. Sulanke's seminal paper [7]. In a common work with J. E. Yukich [3], we adopt a slightly different approach by considering the set of facets of K_λ and studying both the typical and maximal distributions of two functionals of those facets: the distance to the boundary ∂K and the $(d-1)$-dimensional volume. In particular, we derive limit distributions for both the maximal local roughness $MLR(K_\lambda)$, which is the maximal distance from a facet of K_λ to ∂K, and the maximal facet volume



(c) The random polytope K_λ when K is an ellipse

$MFV(K_\lambda)$. Indeed, under some general assumptions on K, there exist explicit constants a_i and b_i depending only on dimension d and on K such that

$$MLR(K_\lambda) = \lambda^{-\frac{2}{d+1}} (a_0(a_1 \log \lambda + a_2 \log(\log \lambda) + a_3 + \xi_\lambda))^{\frac{2}{d+1}}$$

and

$$MFV(K_\lambda) = \lambda^{-\frac{d-1}{d+1}} (b_0(b_1 \log \lambda + b_2 \log(\log \lambda) + b_3 + \psi_\lambda))^{\frac{d-1}{d+1}}$$

where both variables ξ_λ and ψ_λ converge to the Gumbel distribution, i.e. for $t \in \mathbb{R}$,

$$\lim_{\lambda\to\infty} \mathbb{P}(\xi_\lambda \le t) = \lim_{\lambda\to\infty} \mathbb{P}(\psi_\lambda \le t) = e^{-e^{-t}}.$$

Our calculations also show that the model belongs to a so-called *KPZ universality class* where the denomination comes from the celebrated Kardar-Parisi-Zhang equation [4]. Additional results concern the location and shape of the facet which reaches the maximal volume: we prove that the Gauss curvature at the point on ∂K which is the closest to the facet with maximal volume converges to the minimum of the Gauss curvature along the boundary of K. Moreover, we obtain an explicit limit distribution for the location of that closest boundary point. Similar results occur for the location of the facet which maximizes the distance to ∂K. Finally, in the case of the facet with maximal volume, we prove that up to affine transformation, its shape converges to the shape of a regular simplex.

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Spectral gap condition for type-I Bose–Einstein condensation in random potentials

Wolfgang Spitzer
(joint work with Joachim Kerner, Maximilian Pechmann)

We consider non-interacting bosons in a non-negative random potential V_ω in $\mathbb{R}^d$, $\omega \in \Omega$, $d \in \mathbb{N}$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is some probability space. The single particle energy is described by the random Schrödinger operator

$$h_\omega := -\Delta + V_\omega \geq 0\,, \quad \omega \in \Omega\,,$$

defined as self-adjoint operators on (a suitable domain of) $\mathsf{L}^2(\mathbb{R}^d)$. Almost surely (with respect to $\mathbb{P}$), the infimum of the spectrum of h_ω will be 0 in our models.

Equilibrium states of the Bose gas are parametrized by the temperature $T > 0$ and the constant (particle) density $\rho > 0$ or by T and the so-called chemical potential $\mu < \inf \operatorname{spec}(h_\omega) = 0$. Instead of T we use $\beta := 1/T > 0$.

We need to start with a microscopic description and consider N bosons inside the box $\Lambda = (-L/2, L/2)^d \subset \mathbb{R}^d$ for some $L > 0$. Since we keep the particle density ρ fixed we choose L such that $\rho = N/L^d$, and thus $L = L_N = (N/\rho)^{1/d}$.

The operator $h_N = h_{N,\omega} := h_\omega \restriction \Lambda$ reduced to the box Λ (with some boundary conditions) has discrete, random eigenvalues $E_N^j = E_{N,\omega}^j$ such that $0 \leq E_N^1 \leq E_N^2 \leq E_N^3 \leq \cdots$ with normalized eigenfunctions $\varphi_N^j = \varphi_{N,\omega}^j \in \mathsf{L}^2(\Lambda)$, $j \in \mathbb{N}$. The (effective) one-particle operator on $\mathsf{L}^2(\Lambda)$ (in the grand-canonical ensemble),

$$0 \leq D_{N,\beta,\mu} := \frac{1}{\exp(\beta(h_N - \mu)) - 1} = \sum_{j \geq 1} \underbrace{\frac{1}{\exp(\beta(E_N^j - \mu)) - 1}}_{=: n_N^j} |\varphi_N^j\rangle\langle\varphi_N^j|$$

describes the physics of the Bose gas in equilibrium. We call n_N^j the expected occupation number of bosons in the state φ_N^j at inverse temperature β and chemical potential μ. The value of $\mu = \mu_{N,\omega} < E_N^1$ is uniquely chosen such that the total number of bosons in Λ, $N = \operatorname{trace} D_{N,\beta,\mu} = \sum_{j \geq 1} n_N^j$, or that the particle density,

$$\rho = \frac{N}{|\Lambda|} = \frac{1}{L^d} \sum_{j \geq 1} \frac{1}{\exp(\beta(E_N^j - \mu)) - 1} = \frac{1}{L^d} \sum_{j \geq 1} n_N^j\,. \tag{1}$$

Our main concern is the fate of the fraction of random occupation numbers, n_N^j/N (or equivalently of n_N^j/L^d), in the *thermodynamic* limit $N \to \infty$. If there is a $\hat{c} \in \mathbb{N}$ such that in probability

$$\limsup_{N \to \infty} \frac{n_N^j}{N} \begin{cases} > 0 & \text{if } j = 1, \ldots, \hat{c} \\ = 0 & \text{if } j \geq \hat{c} + 1 \end{cases} \tag{2}$$

then we call this type-I *Bose–Einstein condensation* (BEC) in probability.

We introduce the random integrated density of states $\mathcal{N}_N^{\mathrm{I}}$ and the random density of states measure $\mathcal{N}_N$: For $E \in \mathbb{R}$, let

$$\frac{\#\{j \in \mathbb{N} : E_N^j < E\}}{|\Lambda|} =: \mathcal{N}_N^{\mathrm{I}}(E) =: \int_{(-\infty,E)} \mathrm{d}\mathcal{N}_N(\tilde{E}) \,.$$

This allows us to rewrite (1) in the form

$$\rho = \int_{\mathbb{R}} \frac{\mathrm{d}\mathcal{N}_N(E)}{\exp(\beta(E-\mu)) - 1} \,.$$

Assuming that the random measure $\mathcal{N}_N$ converges vaguely to some *non-random* measure $\mathcal{N}_\infty$ as $N \to \infty$ then we define the critical density (as a function of β)

$$\rho_{\mathrm{crit}}(\beta) := \sup_{\mu<0} \int_0^\infty \frac{\mathrm{d}\mathcal{N}_\infty(E)}{\exp(\beta(E-\mu) - 1} = \int_0^\infty \frac{\mathrm{d}\mathcal{N}_\infty(E)}{\exp(\beta E) - 1} \,.$$

It turns out that in many random models as the one we present below, the corresponding (non-random) integrated density of states, $\mathcal{N}_\infty^{\mathrm{I}}$, satisfies a *Lifshitz-tail* behavior (see e.g. [5]) at the bottom of the spectrum of h_ω, that is, as $E \downarrow 0$

$$\mathcal{N}_\infty^{\mathrm{I}}(E) \approx \exp[-CE^{-d/2}] \ll E \,, \tag{3}$$

for some constant $C > 0$. This implies that the critical density $\rho_{\mathrm{crit}}(\beta)$ is finite in any spatial dimension $d \geq 1$, contrary to the free case ($V_\omega = 0$), where this critical density is finite only if $d \geq 3$. If the density $\rho > \rho_{\mathrm{crit}}(\beta)$, then the remaining $L^d(\rho - \rho_{\mathrm{crit}}(\beta))$ particles in Λ have to occupy the low energy states. Indeed (see [3, Theorem 4.1] and [2, Theorem 2.5]), using (3), it is not difficult to prove a weaker form of BEC called g(eneralized) BEC introduced by Girardeau, that is,

$$\mathbb{P}\Big(\lim_{E\downarrow 0} \limsup_{N\to\infty} \sum_{j\in\mathbb{N}: E_N^j \leq E} \frac{n_N^j}{N} > 0\Big) = 1 \,. \tag{4}$$

This by itself does not imply type-I BEC and other possibilities may occur (type-II and type-III, see [2] and references to M. van den Berg, J.T. Lewis, and J.V. Pulé).

Let us now describe the random model that was studied by Kac and Luttinger [1]. To this end, let $X = \{\xi_k\}$ denote a Poisson point process on $\mathbb{R}^d$ of intensity $\nu > 0$ with random points $\xi_k = \xi_k(\omega) \in \mathbb{R}^d$ for $\omega \in \Omega$ and some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ so that for $A \subset \mathbb{R}^d$ with Lebesgue measure $|A| < \infty$,

$$\mathbb{P}\big(\omega \in \Omega : \#\{X(\omega) \cap A\} = m\big) = \frac{(\nu|A|)^m}{m!} \exp(-\nu|A|) \,, \quad m \in \mathbb{N}_0 \,.$$

Let $u : \mathbb{R}^d \to [0,\infty)$ be a fixed single-site (usually, compactly supported) potential and $\gamma > 0$. Then, we define the random potential

$$V_\omega(x) := \gamma \sum_k u(x - \xi_k(\omega)) \,, \quad \omega \in \Omega \,, x \in \mathbb{R}^d \,. \tag{5}$$

A special case of interest is when $u = 1_{B(0,a)}$, the indicator function of the closed ball $B(0,a)$ with center 0 and radius $a \geq 0$, particularly the limit $\gamma \to \infty$ so that

$$h_\omega = -\Delta \restriction \Big(\mathbb{R}^d \setminus \bigcup_k B(\xi_k(\omega), a)\Big) \,, \tag{6}$$

supplied with Dirichlet boundary conditions. This is the hard obstacle model for which Kac and Luttinger proved gBEC in the sense of (4) and conjectured that only the ground state is macroscopically occupied, that is, type-I BEC with $\hat{c} = 1$.

In order to prove type-I BEC we assumed properties of the integrated density of states $\mathcal{N}_N^{\mathrm{I}}, \mathcal{N}_\infty^{\mathrm{I}}$ (see [2, Assumptions 2.2]) to hold and of the spectral gap for general random Schrödinger operators h_ω (see [2, (2.10)]. For the first set of conditions we most crucially assumed (cf. (3)), amongst other conditions, that there is some $\eta \in (0,1)$ and some $c_1 > 0$ such that $\lim_{N\to\infty} N^{1-\eta}\mathcal{N}_\infty^{\mathrm{I}}\big(c_1/(\ln(N))^{2/d}\big) = 0$. Then we defined the events $\Omega_N^{\hat{c}} \subset \Omega$ so that (i) $E_N^{\hat{c}+1} - E_N^1 \geq c_2 N^{-1+\eta}$ and (ii) $E_N^1 \leq c_3/(\ln(N))^{2/d}$. The (spectral) gap condition is now the following: there exist constants $\hat{c} \in \mathbb{N}, 0 < c_2, c_3 < c_1$ such that $\lim_{N\to\infty} \mathbb{P}(\Omega_N^{\hat{c}}) = 1$. If these conditions are all satisfied and if $\rho > \rho_{\mathrm{crit}}(\beta)$ then our main result in [2] says that for $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\Big|\frac{1}{N}\sum_{j=1}^{\hat{c}} n_N^j - \frac{\rho - \rho_{\mathrm{crit}}(\beta)}{\rho}\Big| < \varepsilon\right) = 1$$

and $\lim_{N\to\infty} \mathbb{P}\big(n_N^j/N \geq \varepsilon\big) = 0$ for all $j \geq \hat{c}+1$, that is, type-I BEC in probability. We also verified that the gap condition is satisfied in the one-dimensional hard obstacle model (6) with $a = 0$ for any $\nu > 0$ with $\hat{c} = 1$; this should extend to any $a > 0$. Recently, Alain-Sol Sznitman [6] proved that the gap condition also holds in the higher dimensional hard obstacle model (6) for any $a > 0$ and any $\nu > 0$, also with $\hat{c} = 1$. In fact, he proved a much stronger result with a presumably optimal logarithmic scaling of the energy gap. The conditions on $\mathcal{N}_\infty^{\mathrm{I}}$ were known to be satisfied [5].

Altogether this implies for $\rho > \rho_{\mathrm{crit}}(\beta)$ type-I BEC in probability with $\hat{c} = 1$ in the hard obstacle model (6) in all dimensions $d \geq 1$. This solves the nearly 50 year old conjecture of Kac and Luttinger. An open problem concerns type-I BEC in the soft obstacle model (5) for $0 < \gamma < \infty$, see [4] for a result in dimension one.

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On the spectral gap in the Kac–Luttinger model and Bose–Einstein condensation

Alain-Sol Sznitman

In this talk we report on the results of the preprint [1].We consider the Dirichlet eigenvalues of the Laplacian among a Poissonian cloud of hard spherical obstacles of fixed radius in large boxes of R^d, $d \geq 2$. In a large box of side-length 2ℓ centered at the origin, the lowest eigenvalue is known to be typically of order $(\log \ell)^{-2/d}$. We show in [1] that with probability arbitrarily close to 1 as ℓ goes to infinity, the spectral gap stays bigger than $\sigma(\log \ell)^{-(1+2/d)}$, where the small positive number σ depends on how close to 1 one wishes the probability. Incidentally, the scale $(\log \ell)^{-(1+2/d)}$ is expected to capture the correct size of the gap. Our result involves the proof of new deconcentration estimates. Combining this lower bound on the spectral gap with the results of Kerner-Pechmann-Spitzer [2], we infer a type-I generalized Bose–Einstein condensation in probability for a Kac–Luttinger system of non-interacting bosons among Poissonian spherical impurities, with the sole macroscopic occupation of the one-particle ground state when the density exceeds the critical value.

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Nodal count via topological data analysis

Iosif Polterovich

(joint work with Lev Buhovsky, Jordan Payette, Leonid Polterovich, Egor Shelukhin and Vukašin Stojisavljević)

Consider the Laplace–Beltrami operator Δ on a compact connected n-dimensional Riemannian manifold M. If $\partial M \neq 0$, we assume for simplicity the Dirichlet boundary conditions. The spectrum of the Laplacian is discrete, and the eigenvalues form a sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \nearrow +\infty$. The corresponding eigenfunctions f_j, $\Delta f_j = \lambda_j f_j$, form an orthonormal basis in $L^2(M)$.

Let $\mathcal{Z}_f$ denote the *nodal* (i.e. zero) set of a function f. A *nodal domain* of f is a connected component of the set $M \setminus \mathcal{Z}_f$. Nodal patterns of Laplace eigenfunctions have fascinated researchers since Chladni's experiments with vibrating plates at the turn of the XIXth century. In particular, it has been observed that these patterns tend to exhibit increasingly complex behaviour as $\lambda_j \to \infty$. A fundamental result providing the control of the number of the nodal domains of an eigenfunction is the following theorem due to R. Courant.

Theorem 1 (R. Courant, 1923) *A Laplace eigenfunction f_j has at most j nodal domains.*

Denote by $m_0(f)$ the number of nodal domains of f. Together with Weyl's law, Courant's theorem implies

$$m_0(f_j) = O\left(\lambda_j^{n/2}\right). \tag{1}$$

There have been many attempts to find an appropriate generalization of this statement in various directions: to linear combinations of eigenfunctions (the so-called *Courant–Herrmann conjecture*, see [2] and references therein), to their products [1], to other operators, to higher topological invariants of nodal domains. However, using the construction obtained in [3] one can show that none of these generalizations hold in a straighforward sense [4]. Still, it turns out that these and other extensions of Courant's theorem can be obtained if one counts the nodal domains in a *coarse* way, i.e. ignoring small oscillations.

Following [7] we say that a nodal domain Ω of a function f is δ-*deep* for some $\delta > 0$ if $\max_\Omega |f| > \delta$. Let $m_0(f, \delta)$ be the number of δ-deep nodal domains of a function f, and $W^{k,p}(M)$ be the Sobolev space of integer order k based on $L^p(M)$.

One of our main results shows that $m_0(f, \delta)$ is controlled by the appropriate Sobolev norms of f.

Theorem 2 ([4]) *Let $f \in W^{k,p}(M)$ for some $k > \frac{n}{p}$, where $n = \dim M$. Then, for any $\delta > 0$,*

$$m_0(f, \delta) \le C\delta^{-\frac{n}{k}} \|f\|_{W^{k,p}}^{\frac{n}{k}},$$

where the constant C depends only on M, k, p.

The proof of this result uses multiscale polynomial approximation in Sobolev spaces and the theory of persistence barcodes originating in topological data analysis. In particular, our methods develop some ideas of [8, 5, 6]. Note that Theorem 2 can be extended to persistent Betti numbers of arbitrary degree

$$m_r(f, \delta) = \dim \mathrm{Im}\left(H_r(\{|f| > \delta\}) \to H_r(M \setminus \mathcal{Z}_f)\right),$$

where H_r stands for the r-th homology group with coefficients in a field.

Theorem 2 admits various applications for eigenfunctions of elliptic differential and pseudo-differential operators. We present one of them below, which gives a coarse version of the Courant–Herrmann conjecture mentioned above.

Let $\mathcal{F}_\lambda$ denote the subspace spanned by all eigenfunctions with eigenvalues $\le \lambda$. Given an L^2-normalized $f \in \mathcal{F}_\lambda$, one can use elliptic regularity to control $\|f\|_{W^{k,2}}$ in terms of λ. This implies

Theorem 3 ([4]) *Let M be as above and $k > \frac{n}{2}$ be an integer. Then for any $\delta > 0$ and any $f \in \mathcal{F}_\lambda$ with $\|f\|_{L^2} = 1$,*

$$m_0(f, \delta) \le C\delta^{-\frac{n}{k}} (\lambda + 1)^{\frac{n}{2}},$$

where the constant C depends only on M and k.

Note that the exponent of λ is sharp and is consistent with estimate (1). In two dimensions, versions of this result have been proved earlier in [7, 6].

One can also obtain a counterpart of Theorem 2 for the *coarse zero count*, i.e. for the number of connected components of the set $\{|f| < \delta\}$ which contain zeros of f. This result implies, in particular, a coarse version of Bézout's theorem for linear combinations of Laplace eigenfunctions [4]. Here, following the idea of Donnelly and Fefferman, the analogue of the degree of a polynomial is given by the square root of the corresponding eigenvalue.

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Bose–Einstein Condensation beyond the Gross–Pitaevskii Regime

Christian Brennecke

(joint work with Arka Adhikari and Benjamin Schlein)

Understanding the low-energy properties of the weakly interacting Bose gas is a challenging problem in mathematical physics. In this talk, we consider N interacting bosons that move in $\Lambda = \mathbb{T}^3$ and whose energies are described by

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{1\leq i<j\leq N} N^{2-2\kappa} v(N^{1-\kappa}(x_i - x_j)), \tag{1}$$

acting in $L^2_s(\Lambda^N) = \bigotimes_{\text{sym}}^N L^2(\Lambda)$, the space of wave functions that are invariant under permutations of the particle coordinates (as appropriate for bosons). Here, $\kappa \in (0, 1/43)$ is assumed to be small, but positive and the unscaled potential $v \in L^3(\mathbb{R}^3)$ is assumed to be radially symmetric, pointwise non-negative and of compact support. Observe that, by a simple change of variables, the system is equivalent to a system of particles moving in a box of side length $L = N^{1-\kappa}$ and interacting with the unscaled potential $v(\cdot)$. In other words, the limit $N \to \infty$ corresponds to taking a joint infinite number of particles and low density limit: the particle density equals in the rescaled system $\rho = N^{2-3\kappa} \to 0$ as $N \to \infty$.

Heuristically, one may think of the ground state energy E_N to be equal to $N(N-1)/2$, the number of pairs of N particles, times the ground state energy of two bosons. After changing to relative and center of mass coordinates, and a change of variables to rescale the potential $N^{2-2\kappa}v(N^{1-\kappa}.)$, we thus expect that

(2)
$$E_N = N^{1+\kappa}\inf\Big\{\int_{\mathbb{R}^3}|\nabla f|^2+\frac{1}{2}v|f|^2 : \lim_{|x|\to\infty}|f(x)|=1\Big\}(1+o(1)) = 4\pi\mathfrak{a}N^{1+\kappa}(1+o(1)),$$

where $\mathfrak{a}$ denotes the scattering length of v. Eq. (2) is well-known and has been proved (for more general systems in the thermodynamic limit) rigorously in [11].

Moreover, extending the arguments from [9, 10] to small values of $\kappa>0$, one can also show that the unique, positive ground state $\psi_N \in L^2_s(\Lambda^N)$ of the system exhibits complete Bose–Einstein condensation (BEC) into the zero momentum mode $\varphi_0 \equiv 1_{|\Lambda} \in L^2(\Lambda)$. Mathematically, this means that

$$\lim_{N\to\infty}\langle\varphi_0,\gamma_N^{(1)}\varphi_0\rangle = 1 \tag{3}$$

where $\gamma_N^{(1)} = \operatorname{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$ denotes the one-particle reduced density of ψ_N. A proof of (3) in the usual thermodynamic limit is a major open problem in mathematical physics. The goal of [1] is to provide instead a novel proof of (3) in the simpler scaling regimes (1), extending the methods of [7, 2, 3, 4, 5] (see [8] for an alternative approach) and providing strong quantitative control on the expected number of excitations $\mathcal{N}_+$, defined by

$$\mathcal{N}_+ = \sum_{i=1}^N (1-|\varphi_0\rangle\langle\varphi_0|)_{x_i},$$

and on its higher moments $\mathcal{N}_+^k$, for $k\in\mathbb{N}$. Such strong quantitative control can be used to determine the excitation spectrum above E_N, for sufficiently small values of $\kappa>0$ (see [1, 6] for the details).

Proving BEC is intimately related to studying the fluctuations of H_N around the leading order contribution $4\pi\mathfrak{a}N^{1+\kappa}$ to E_N, and thus also to the construction of suitable trial states that model the ground state. In [1], we approximate

(4)
$$\begin{aligned}\psi_N &\approx C\prod_{1\le i<j\le N}\Big(1+\frac{1}{N}\eta(x_i-x_j)\Big)\exp\Big(\frac{1}{2}\sum_{p\in P_L}\tau_p b_p^*b_{-p}^* - \text{h.c.}\Big)\varphi_0^{\otimes N}\\ &\approx \exp\Big(\frac{1}{2N}\sum_{r\in P_H, p,q\in P_L}\eta_r\, a^*_{p+r}a^*_{q-r}a_pa_q - \text{h.c.}\Big)\exp\Big(\frac{1}{2}\sum_{p\in P_L}\tau_p b_p^*b_{-p}^* - \text{h.c.}\Big)\varphi_0^{\otimes N}\\ &=: U\exp\Big(\frac{1}{2}\sum_{p\in P_L}\tau_p b_p^*b_{-p}^* - \text{h.c.}\Big)\varphi_0^{\otimes N}.\end{aligned}$$

for suitable coefficients $(\eta_p)_{p\in 2\pi\mathbb{Z}^3}$ (related to the two-body scattering problem) and $(\tau_p)_{p\in2\pi\mathbb{Z}^3}$, (determining a quasi-free ground state after correlations have been removed) so that our key task is to analyze the renormalized Hamiltonian UH_NU^*, with the unitary map U defined in (4) (using the standard formalism of second quantization). A careful analysis shows that extracting the two-body correlations via U leads, on the level of the Hamiltonian, to the renormalization of the singular potential $N^{3-3\kappa}N^{\kappa}v(N^{1-\kappa}(\cdot))$ with Fourier coefficients $N^{\kappa}\widehat{v}(\cdot/N^{1-\kappa})$ to a regularized potential with Fourier coefficients

$$N^{\kappa}\widehat{v}(\cdot/N^{1-\kappa}) \to 8\pi\mathfrak{a}\, N^{\kappa}\mathbf{1}_{P_H^c}(\cdot).$$

Here, P_H denotes a suitable set of large momenta; in particular, its complement P_H^c is of finite support. Once the renormalization is established with sufficiently strong control on the errors, standard arguments imply the lower bound

$$H_N \geq 4\pi\mathfrak{a}N^{1+\kappa} + c\sum_{i=1}^{N}(1-|\varphi_0\rangle\langle\varphi_0|)_{x_i} - O(N^{43\kappa})$$

for some $c > 0$, which implies the leading order order contribution to E_N as in (2) as well as complete BEC as in (3) (recalling that $\kappa \in (0, 1/43)$).

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Upper bounds for the ground state energy of dilute Bose gases

Serena Cenatiempo

(joint work with G. Basti, A. Giuliani, A. Olgiati, G. Pasqualetti, B. Schlein)

We consider N bosons in the three dimensional torus $\Lambda_T = [-T/2, T/2]^3$, interacting via a two-body non negative, radial and compactly supported potential V with scattering length $\mathfrak{a}$. In the units where the particle mass is set to $m = 1/2$ and $\hbar = 1$, the Hamilton operator has the form

$$H_T = -\sum_{i=1}^{N} \Delta_{x_i} + \sum_{1\leq i<j\leq N} V(x_i - x_j) \tag{1}$$

and acts on the Hilbert space $L^2_s(\Lambda_T^N)$, the subspace of $L^2(\Lambda_T^N)$ consisting of functions which are symmetric with respect to permutations of the N particles. We are interested in the limit $N, T \to \infty$ with $\rho = N/|\Lambda_T|$ fixed, known as *thermodynamic limit*. Let us denote by $E(N,T)$ the ground state energy of the system described by (1). Then the specific ground state energy, defined by

$$e(\rho) = \lim_{N,T\to\infty,\, \rho=N/T^d} \frac{E(N,T)}{T^d} \tag{2}$$

admits the following expansions in the dilute limit $\rho\mathfrak{a}^3 \ll 1$:

$$e(\rho) = 4\pi\rho^2\mathfrak{a}\Big[1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho\mathfrak{a}^3} + o\big(\sqrt{\rho\mathfrak{a}^3}\big)\Big]\,. \tag{3}$$

Remarkably (3) depends on the interaction only through the scattering length $\mathfrak{a}$. Indeed the same expansion is expected to hold for a gas of N hard spheres in the three dimensional torus Λ_T, whose ground state energy is defined as

$$E^{\mathrm{hc}}(N,T) = \inf \frac{\langle \Psi, \sum_{j=1}^{N} -\Delta_{x_j}\Psi\rangle}{\|\Psi\|^2} \tag{4}$$

with the infimum taken over all $\Psi \in L^2(\Lambda_T^N)$, symmetric with respect to permutations of the N particles and satisfying the hard-sphere condition $\Psi(x_1, \dots, x_N) = 0$, if there exist $i, j \in \{1, \dots, N\}$, $i \neq j$, with $|x_i - x_j| < \mathfrak{a}$.

The expansion (3) has been first predicted in [6, 13] under the assumption that systems described by (1) exhibit a macroscopic occupation of the zero momentum mode, a phenomenon known as Bose–Einstein condensation. Even though a proof of condensation in the thermodynamic limit is still beyond reach of the current available methods, upper and lower bounds compatible with (3) have been recently shown for non negative interactions [16, 10, 11, 2] (interestingly the conditions under which the lower bound can be shown are more general, and include the hard spheres gas [11], while the upper bound requires $V \in L^3(\mathbb{R}^3)$). In fact, to make the heuristics from [6, 13] rigorous, it is sufficient to show the occurrence of condensation on sufficiently large - but finite - length scales $L \ll T$. More precisely it is possible to derive lower (resp. upper) bounds compatible with (3) starting from the analysis of the problem on finite size boxes Λ_L with side length $L = (\rho\mathfrak{a})^{-1/2}(\rho\mathfrak{a}^3)^{-\alpha}$ with $\alpha > 0$ (resp. $\alpha > 1/2$), see e.g. [1].

Aim of this talk is to describe trial states resolving the asymptotic expansion (3) at different orders. A pioneering observation due to [5, 7, 12] is that to get an upper bound for the Hamiltonian (1) (a similar observation holds for (4)), it is enough to modify the non interacting ground state $\Psi_N^{\mathfrak{a}=0}(x_1,\ldots,x_N) \equiv 1$, by adding correlations produced by two-body scattering events. We consider

$$\Psi_N(x_1,\ldots,x_N) = \prod_{i<j}^{N} f_\ell(x_i - x_j) \tag{5}$$

where $\mathfrak{a} \ll \ell \ll T$ is a parameter that will be fixed later and f_ℓ is the ground state solution of the Neumann problem

$$\left(-\Delta + \frac{1}{2}V\right) f_\ell = \lambda_\ell f_\ell \tag{6}$$

on the ball $|x| \leq \ell$, and with the normalization $f_\ell(x) = 1$ for $|x| = \ell$. States as in (5) are highly correlated. However for $\ell = \rho^{-1/3}$ the computation of their energy is easy (see e.g. [4]) and one obtains the upper bound $e(\rho) \leq 4\pi\rho^2\mathfrak{a}\big(1+C(\rho\mathfrak{a}^3)^{1/3}\big)$ first derived in [8]. A careful analysis of the cancellations between the expectation of the energy of Ψ_N and its norm allows to take into account of correlations among particles at distances up to $\ell = \kappa(\rho\mathfrak{a})^{-1/2}$ with $\kappa \ll 1$, this leading to an upper bound of the form $e(\rho) \leq 4\pi\rho^2\mathfrak{a}\big(1 + C(\rho\mathfrak{a}^3)^{1/2}\big)$ which also holds for hard sphere bosons [3].

An alternative route to build a trial state matching (3) is to model correlations among particles via unitary operators acting on the bosonic Fock space

$$\mathcal{F}(\Lambda_T) = \bigoplus_{n\geq 0} L_s^2(\Lambda_T^n) = \bigoplus_{n\geq 0} L^2(\Lambda_T)^{\otimes_s n}\,.$$

On this space, we model a condensate with an expected number of particles N_0 via a coherent state, namely by acting with the operator $W_{N_0} = \exp[\sqrt{N_0}a_0^* - \sqrt{N_0}a_0]$ on the vacuum state Ω in $\mathcal{F}(\Lambda_T)$. Then we act on the coherent state with the unitary operator $T_\nu = \exp[B(\nu)]$ with

$$B(\nu) = \sum_{\substack{p\in(2\pi/T)\mathbb{Z}^d\\ p\neq 0}} \left(\nu_p a_p^* a_{-p}^* - \bar{\nu}_p a_p a_{-p}\right), \tag{7}$$

where for $p \in (2\pi/T)\mathbb{Z}^d$ the operator a_p^* is defined by

$$\left(a_p^*\Psi\right)^{(n)}(x_1,\ldots,x_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \frac{e^{ip\cdot x_j}}{\sqrt{|\Lambda_T|}}\,\psi^{(n-1)}(x_1,\ldots,x_{j-1},x_{j+1},\ldots x_n)\,,$$

and a_p is its adjoint. Operators of the form T_ν as above model the scattering from the condensate of pairs of particles with opposite momenta $p, -p$. In fact one could interpret $a_p^*a_{-p}^*$ as obtained by a quartic operator $a_p^*a_{-p}^*a_0a_0$ by replacing both operators a_0 by the constant factor $\sqrt{N_0}$ (this substitution, first introduced in [6] and known as *c-number substitution*, can in fact be rigorously justified). Trial states of the form $\Psi_{\nu,N_0} = T_\nu W_{N_0}\Omega$ are easy to deal with, being the action of T_ν and W_{N_0} on the operators a_p^*, a_p explicit. Choosing the correct ν_p one can show that they have to the leading order the correct ground state energy. However, as shown in [9, 15], they cannot capture the sub-leading correction. To reach the precision of (3) one needs to further apply the exponential of a cubic expression in the creation and annihilation operators a_p^* and a_p, which can be thought as obtained by applying the c-number substitution to an operator of the form $a^*_{r+v}a^*_{-r}a_v a_0$. The main challenge to be overcome is that the action of this operator is not explicit; moreover it is not possible to expand the exponential of the cubic operator who is expected to give the correct correction unless we consider small boxes with side length of the order $(\rho\mathfrak{a})^{-1/2}(\rho\mathfrak{a}^3)^{-\alpha}$ with $\alpha < 1/3$. In [2] we developed new ideas allowing to perturb the state Ψ_{ν,N_0} by the exponential of a cubic operator on the large boxes needed to derive a result valid in the thermodynamic limit. Compared to [16] our trial state applies to a larger class of potential and is substantially simpler.

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The Bose gas in a box with Neumann boundary conditions

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(joint work with Robert Seiringer)

We consider a gas of N interacting bosons trapped in a box $\Lambda = [-L/2, L/2]^3$ described through the Hamiltonian

$$H_N = -\sum_{i=1}^{N} \Delta_i + \kappa \sum_{i<j}^{N} V(x_i - x_j) \tag{1}$$

acting on $L^2_s(\Lambda^N)$, the space of permutation symmetric wave functions in $L^2(\Lambda^N)$. We impose Neumann boundary conditions on Λ. The interaction potential V is nonnegative, spherically symmetric, compactly supported and bounded, and κ is a positive coupling constant. We call $\mathfrak{a}$ the scattering length of V, defined through the two-body problem (written in the relative coordinates)

$$\left[-\Delta + \frac{\kappa}{2} V(x)\right] f_0(x) = 0 \tag{2}$$

with the boundary condition that $f_0(x) \to 1$, as $|x| \to \infty$ (in (2), Δ indicates the Laplacian on $\mathbb{R}^3$). The solution f_0, outside the support of V, takes the form

$$f_0(x) = 1 - \frac{\mathfrak{a}}{|x|}, \tag{3}$$

where $\mathfrak{a}$ is the scattering length of κV.

We focus on the dilute Bose gas, where the parameter $\rho\mathfrak{a}^3$ is small ($\rho = N/|\Lambda|$ is the density of the gas). We define the ground state energy of (1) as

$$E(N, L) = \inf_{\psi \in L^2_s(\Lambda^N),\, \|\psi\|=1} \langle \psi, H_N \psi \rangle,$$

and the ground state energy per particle in the thermodynamic limit as

$$e(\rho) = \lim_{\substack{N,L\to\infty \\ \rho = N/|\Lambda|}} \frac{E(N, L)}{N}. \tag{4}$$

In the dilute regime the first two terms of the expansion of $e(\rho)$, for small ρa^3, are known:

$$e(\rho) = 4\pi\rho\mathfrak{a} \left[1 + \frac{128}{15\sqrt{\pi}} (\rho\mathfrak{a}^3)^{1/2} + o((\rho\mathfrak{a}^3)^{1/2})\right]. \tag{5}$$

This is the Lee-Huang-Yang formula, proved in [12, 9, 10, 2]. This expression shows the universality of the model, because the ground state energy only depends on the scattering length and no other detail of the interaction is important. Another relevant regime for the Bose gas is the Gross–Pitaevskii regime, where the box length L is proportional to the number of particle N, which is a large parameter (this is also a dilute regime). In this limit, an analogue of the Lee-Huang-Yang formula and the excitation spectrum of the Hamiltonian have been obtained in [4].

In the thermodynamic limit, one of the crucial difficulties one has to face is the absence of an energy gap. A way to address this issue consists in partitioning the thermodynamic limit volume in cells of side length ℓ and studying a localized problem in each cell. The length ℓ needs to be chosen as a suitable function of ρ (which remains fixed in the thermodynamic limit) and large enough to provide the desired precision in the ground state energy. To obtain a lower bound for $e(\rho)$, since V is positive, we can neglect interactions between particles in different cells. Neumann boundary conditions are imposed in each cell, which guarantees that the Laplacian on Λ is estimated from below by the sum of the Laplacians restricted to each cell. Adding the lower bounds in the different cells and minimizing over all the possible ways of distributing the particles in the cells we obtain the lower bound

$$E(N,L) \geq \inf_{\{n_k\}:\sum_k n_k = N} \ell^{-2} \sum_{k=1}^{M^3} e_{n,\ell}, \tag{6}$$

where $e_{n,\ell}$ is the ground state energy of

$$H_{n,\ell} = -\sum_{i=1}^{n} \Delta_i + \kappa \sum_{i<j}^{n} \ell^2 V\big(\ell(x_i - x_j)\big) \tag{7}$$

acting on $L^2_s(\Lambda_1^n)$, where $\Lambda_1 = [-1/2, 1/2]^3$, with Neumann boundary conditions (notice the rescaling of lengths).

We present now results from [6]. The first result deals with the localized problem described by (7); it is an estimate on the ground state energy of $H_{n,\ell}$ and a bound on the rate of Bose–Einstein condensation in the constant wave function $\varphi_0 = 1$.

Theorem 1. *Let V be positive, compactly supported, spherically symmetric and bounded. Assume that $\kappa > 0$ is a fixed, small enough constant independent of all parameters and $n\ell^{-1} \leq 1$. Then, the ground state energy $e_{n,\ell}$ of $H_{n,\ell}$ is such that*

$$\Big| e_{n,\ell} - 4\pi\mathfrak{a}\frac{n^2}{\ell} \Big| \leq C\Big(\frac{n}{\ell} + \frac{n^2}{\ell^2}\ln(\ell)\Big) \tag{8}$$

for a constant $C > 0$ depending only on V and κ.

Furthermore, let $\psi_n \in L^2_s(\Lambda_1^n)$ be a normalized wave function, with

$$\langle \psi_n, H_{n,\ell}\psi_n \rangle \leq e_{n,\ell} + \zeta$$

for some $\zeta > 0$. Let $\gamma_n^{(1)} = Tr_{2,\dots,n}|\psi_n\rangle\langle\psi_n|$ be the one-particle reduced density matrix associated with ψ_n. Then there exists a constant $C > 0$ depending only on V and κ such that

$$1 - \langle \varphi_0, \gamma_n^{(1)} \varphi_0 \rangle \leq C\Big(\frac{\zeta}{n} + \frac{1}{\ell}\Big) \tag{9}$$

where $\varphi_0(x) = 1$ for all $x \in \Lambda_1$.

From Theorem 1, we can easily deduce an estimate for the ground state energy of H_N in (1) in the thermodynamic limit, as stated in the following corollary.

Corollary 2. *Let V satisfy the same assumptions as in Theorem 1 and $\kappa > 0$ be small enough. Then there exists a constant $C > 0$ such that $e(\rho)$ satisfies*

$$e(\rho) \geq 4\pi\mathfrak{a}\rho\Big(1 - C(\rho\mathfrak{a}^3)^{1/2}\ln(1/\rho)\Big) \tag{10}$$

for ρ small enough.

If we take $n = \ell$, we recover the Gross–Pitaevskii regime. The ground state energy (8) presents a logarithmic correction in the error term that was not present in the periodic case, studied in [3, 5]. This is optimal and is intrinsic to the Neumann boundary conditions. The rate in n for Bose–Einstein condensation in (9) instead coincides with the rate obtained in the periodic case (and it is optimal). The lower bound (10) is not optimal (as showed in [9] and as we can see from (5)), but the method to achieve it is new and offers several advantages. Here the lower bound for $e(\rho)$ simply follows by Neumann bracketing from the study of the localized problem in the cell, and no further localization procedure is necessary (while [9] deals with a modified kinetic energy, simulating the Neumann Laplacian in a periodic boundary conditions setting). The strategy to prove Theorem 1 is similar to [3, 5]; the lack of translation invariance however makes the description of correlations much more complicated. We describe correlations through the solution of an analogue of the two-body problem (2). In the Neumann setting, the two-body problem naturally lives in a six-dimensional space, and there is no decoupling in center of mass and relative coordinates. The solution cannot be explicitly computed, and precise estimates need to be obtained to control the many-body analysis. Corollary 2 follows from (8) upon the choice $\ell \sim \rho^{-1/2}$. Taking ℓ larger would lead to a better precision in (10) (the choice $\ell \simeq \rho^{-1/2-\varepsilon}$ leads to (5), as showed in [9]). This however requires a more precise study of $H_{n,\ell}$ with larger n/ℓ. Such study has been done so far only in the translation invariant setting [1, 8, 7]. The extension to the Neumann case would provide an alternative proof of a lower bound for the Lee-Huang-Yang formula (5); moreover, obtaining the excitation spectrum in the cells of side length $\rho^{-1/2-\varepsilon}$ with Neumann boundary conditions would allow to extend the Lee-Huang-Yang formula to positive temperature, improving the leading order results [11, 13].

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Bose–Einstein condensation in 1D noninteracting Bose gases with soft Poissonian obstacles

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(Conventional) *Bose–Einstein condensation* (BEC) is a macroscopic occupation of a one-particle state and occurs, under certain circumstances, in bosonic particle systems. In the case of noninteracting Bose gases (bosonic particle systems without interparticle interaction), a necessary but not sufficient requirement for the occurrence of BEC is the presence of *generalized Bose–Einstein condensation* (g-BEC). This broader definition only requires a macroscopic occupation of an arbitrarily small energy band of one-particle states [1, 2, 3, 4, 11]. Depending on the quantity of macroscopically occupied one-particle states in the condensate one then distinguishes three types: Type-I g-BEC is said to occur if the number of macroscopically occupied one-particle states is finite but at least one. If there are infinitely many macroscopically occupied one-particle states, the condensation is said to be of type II. Lastly, a generalized condensate in which none of the one-particle states are macroscopically occupied is called a type-III g-BEC. Showing the occurrence of g-BEC is easier than the occurrence of BEC and involves verifying that a certain critical density is finite as a main step. Proving BEC or, similarly, determining the type of g-BEC, however, seems to require fairly accurate knowledge about the gaps between the eigenvalues of the corresponding one-particle (random) Schrödinger operator at the bottom of the spectrum [2, 9], which is often difficult to obtain. Note that the definition of type-I g-BEC is more restrictive than our definition of BEC.

Random potentials are known to be able to trigger and enhance the occurrence of g-BEC in noninteracting Bose gases, see, for example, [11] and [8, Appendix A]. Thus, the study of Bose gases in such potentials is of great interest. This holds especially true for Poisson random potentials as they are commonly used to model systems with structural disorder. Although it is believed that repulsive interactions between the particles eventually need to be taken into account, exploring noninteracting Bose gases with respect to BEC is nevertheless an important first step and of independent interest [11].

The *Kac–Luttinger conjecture* presumes that g-BEC in noninteracting Bose gases in Poisson random potentials that have compactly supported, nonnegative measurable functions as their single-site potentials is generally of type I or II, that is, BEC occurs [6, 7, 12]. To the best of our knowledge, however, the type of g-BEC in random potentials at positive temperatures has previously been rigorously determined only for one-dimensional Poisson random potentials whose single-site potentials consist of the Dirac delta function δ. The Luttinger–Sy model [14, 15] has a Poisson random potential on $\mathbb{R}$ with, informally, a single-site potential of the form $\gamma\delta$ where $\gamma = \infty$, that is, one has Dirichlet boundary conditions at all atoms of each realization of the Poisson random measure. This model is easier to explore, because the singularity of this random potential eliminates quantum tunneling effects [5, p. 3]. It has been proved that in this Luttinger–Sy model a type-I g-BEC, where only the ground state of the corresponding one-particle random Schrödinger operator is macroscopically occupied, occurs in probability and in the rth mean, $r \geq 1$, in [9], and in a slightly different setting $\mathbb{P}$-almost surely [12], if and only if the particle density is larger than a critical density. In addition, it has been shown that in the Luttinger–Sy model with finite interaction strength, that is, in the case of a Poisson random potential on $\mathbb{R}$ with, informally, a single-site potential of the form $\gamma\delta$ with $\gamma > 0$, a type-I g-BEC occurs with probability arbitrarily close to one for particle densities larger than a critical density as long as one allows sufficiently many one-particle states to be macroscopically occupied [8]. Despite their singularities, Poisson random potentials on $\mathbb{R}$ with such single-site potentials, and in particular the infinite potential strength of the Luttinger–Sy model, are believed to be good approximations with respect to the occurrence of BEC for noninteracting Bose gases in more realistic Poisson random potentials on $\mathbb{R}$, such as ones that have nonnegative, bounded functions as their single-site potentials [10, p. 14], [12, p. 8].

In the work [16], we explore this last statement, that is, we study one-dimensional noninteracting Bose gases in Poisson random potentials on $\mathbb{R}$ with *soft obstacles*, that is, with single-site potentials that are nonnegative, compactly supported, and bounded measurable functions with respect to the occurrence of BEC in the thermodynamic limit and in the grand-canonical ensemble at positive temperatures. For this model, we confirm the Kac–Luttinger conjecture in the following sense. Under the assumption that the particle density is larger than a finite critical density, we prove: A type-I g-BEC in which only the ground state is macroscopically occupied occurs with a probability arbitrarily close to one if the random potential has a, in a certain sense, sufficiently large strength. The probability for this kind of condensation converges to one and, consequently, such a type-I g-BEC occurs in probability and in the rth mean, $r \geq 1$, if the strength of the Poisson random potential converges in a certain sense but arbitrarily slowly to infinity in the thermodynamic limit. One also obtains a probability arbitrarily close to one for the occurrence of type-I g-BEC in the case of a Poisson random potential of any fixed strength. However, our upper bound for the number of macroscopically occupied one-particle states depends on the strength of the random potential and thus may be a large (yet finite) number in this case. As a side note we mention that the same results hold true for the Luttinger–Sy model with finite interaction strength, and we thus confirm and extend the results in [8] while using a different, more direct method.

Lastly, the Kac–Luttinger conjecture in the sense of a type-I g-BEC occurrence in probability has now also been confirmed in the case of hard Poissonian obstacles in dimensions $d \geq 2$, in the very recent work [17]. We would also like to mention that in any dimensions the physical intuition favors the occurrence of type-I g-BEC in probability in the case of soft Poissonian obstacles as well [11, 12]. It is because particles in the condensate should have an energy of almost zero. Consequently, a finitely tall hill (soft obstacles) should appear as infinitely tall (as hard obstacles) for these particle. This open problem remains to be proven.

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Spectral Estimates for Infinite Quantum Graphs

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An *infinite metric graph* $\mathcal{G}$ is obtained from a locally finite combinatorial graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ by identifying its edges $e \in \mathcal{E}$ with intervals of certain lengths $\ell(e) > 0$, $e \in \mathcal{E}$. Schrödinger operators on metric graphs are often called *quantum graphs* or *quantum graph operators*. The *Kirchhoff Laplacian* $\mathbf{H}$ provides the analog of the Laplace–Beltrami operator in this setting. It is an unbounded, self-adjoint and non-negative operator in the L^2-space $L^2(\mathcal{G})$ of the infinite metric graph $\mathcal{G}$. A natural question is to find *spectral estimates*, that is, estimates for the infimum of the spectrum

$$\lambda_0(\mathbf{H}) = \inf\{\lambda |\, \lambda \in \sigma(\mathbf{H})\}$$

in terms of *geometric data* of the metric graph $\mathcal{G}$. In the following, we discuss estimates in terms of *isoperimetric constants*, which were recently proved in [4].

The use of isoperimetric constants for spectral estimates has a long tradition and applies to several geometric settings. Such estimates were first obtained for manifolds by Cheeger [2] and Buser [1]. A Cheeger estimate for finite metric graphs (i.e. finitely many vertices and edges) was proved in [5]. For infinite combinatorial graphs $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$, a definition of isoperimetric constant $\alpha_{\mathrm{comb}}(\mathcal{G}_d)$ and Cheeger estimate for the discrete graph Laplacian were established in [3].

In [4], the isoperimetric constant of an infinite metric graph $\mathcal{G}$ was defined as

$$\alpha(\mathcal{G}) := \inf_{\widetilde{\mathcal{G}}} \frac{\mathrm{area}(\partial\widetilde{\mathcal{G}})}{\mathrm{Vol}(\widetilde{\mathcal{G}})},$$

where the infimum is taken over all *finite, connected subgraphs* $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}})$ of $\mathcal{G}$. Moreover, $\mathrm{vol}(\widetilde{\mathcal{G}})$ is the Lebesgue volume of $\widetilde{\mathcal{G}} \subset \mathcal{G}$, $\partial\widetilde{\mathcal{G}} \subset \mathcal{V}$ is the topological boundary, and $\mathrm{area}(\partial\widetilde{\mathcal{G}}) := \sum_{v \in \partial\widetilde{\mathcal{G}}} \#\{e \in \widetilde{\mathcal{E}} | e \text{ is adjacent to } v\}$. The following estimates can be viewed as the analogs of the results in [1, 2].

Theorem 1 ([4])**.** *Let $\mathcal{G}$ be an infinite metric graph. Then*

$$\frac{1}{4}\alpha(\mathcal{G})^2 \leq \lambda_0(\mathbf{H}) \leq \frac{\pi^2}{2} \frac{\alpha(\mathcal{G})}{\inf_{e \in \mathcal{E}} \ell(e)}. \tag{1}$$

The main discovery is the *combinatorial structure* of the isoperimetric constant $\alpha(\mathcal{G})$, which is defined by taking the infimum over subgraphs. It allows to investigate $\alpha(\mathcal{G})$ by methods from *discrete geometry and combinatorics*, see [4]. Moreover, this combinatorial structure does not appear in the isoperimetric constant of finite metric graphs [5].

In particular, one may establish connections between $\alpha(\mathcal{G})$ and the isoperimetric constant $\alpha_{\mathrm{comb}}(\mathcal{G}_d)$ of the underlying combinatorial graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ of $\mathcal{G}$. Among others, this leads to the following result.

Corollary 1 ([4])**.** *Let $\mathcal{G}$ be an infinite metric graph. If $\alpha_{\mathrm{comb}}(\mathcal{G}_d) > 0$, then the following equivalences hold.*

- $\lambda_0(\mathbf{H}) > 0$ *if and only if* $\sup_{e\in\mathcal{E}} \ell(e) < \infty$.
- *The spectrum of* $\mathbf{H}$ *is purely discrete if and only if for every* $\varepsilon > 0$*, there are only finitely many edges with* $\ell(e) \geq \varepsilon$.

On the other hand, if $\alpha_{\mathrm{comb}}(\mathcal{G}_d) = 0$ *and* $\inf_{e\in\mathcal{E}} \ell(e) > 0$*, then* $\lambda_0(\mathbf{H}) = 0$.

The isoperimetric constant $\alpha_{\mathrm{comb}}(\mathcal{G}_d)$ is a central object in spectral graph theory and the study of random walks. In particular, the positivity of $\alpha_{\mathrm{comb}}(\mathcal{G}_d)$ has been studied for several classes of graphs. E.g., for a Cayley graph $\mathcal{G}_d$ of a finitely generated group Γ, $\alpha_{\mathrm{comb}}(\mathcal{G}_d) = 0$ exactly when Γ is amenable. From this point of view, the above result connects the investigation of $\lambda_0(\mathbf{H})$ to a well-studied problem in discrete geometry.

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A Discovery Tour in Random Riemannian Geometry

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(joint work with Eva Kopfer and Karl-Theodor Sturm)

Let (M, g) be a (connected, smooth) n-dimensional Riemannian manifold of bounded geometry, and denote by Δ_g its Laplace–Beltrami operator. The *Fractional Gaussian Field* $h := h_{s,m}$ on M with regularity parameter $s \in \mathbb{R}$ and mass parameter $m > 0$ is the centered Gaussian field of distributions on M with correlation

$$\mathrm{Cov}\big[\langle h \mid \phi\rangle, \langle h \mid \psi\rangle\big] = \big\langle (m^2 - \tfrac{1}{2}\Delta_g)^{-1}\phi \big| \psi \big\rangle$$

for every pair of test functions ϕ, ψ on M.

When M is additionally compact, the same construction makes sense also in the massless case $m = 0$, in which case h is only defined up to additive constants. The same construction of massless Fractional Gaussian Fields is in fact possible also when $M = \mathbb{R}^d$ is a standard Euclidean space, in which case the whole family of such fields (parametrized by s and n) is well studied, see the survey [3]. It includes, e.g.: white noise, fractional Brownian motions, Gaussian Free Fields, log-correlated Gaussian fields, the membrane model, the odometer for the sandpile model, and a variety of other interesting objects.

For $s > 0$, the above covariance operator is an integral operator, represented by the fractional massive Green kernel

$$G_{s,m}(x,y) \coloneqq \frac{1}{\Gamma(s)} \int_0^\infty e^{-m^2 t} t^{s-1} \, p_t(x,y) \, \mathrm{d}t$$

defined via the standard heat kernel p_t of M. The regularity properties of the kernel $G_{s,m}$ play a key role in establishing regularity properties of the fields $h^\bullet$ via Kolmogorov–Chentsov estimates and the *noise distance* of the fields

$$\begin{aligned} \rho_{s,m}(x,y) &\coloneqq \mathbb{E}\big[|h^\bullet(x) - h^\bullet(y)|^2\big]^{1/2} \\ &= \big[G_{s,m}(x,x) + G_{s,m}(y,y) - 2\, G_{s,m}(x,y)\big]^{1/2} . \end{aligned}$$

As it turns out, for $s > n/2$ the random field $h^\bullet \colon \omega \mapsto h^\omega$ is in fact a random continuous function on M, and we study the random perturbation $g^\bullet \colon \omega \mapsto g^\omega \coloneqq e^{2h^\omega} g$ of the reference metric g by a conformal factor $h^\bullet$. When $n = 2$, the critical case $s = n/2$ —here beyond our scope— corresponds to the celebrated *Liouville Quantum Gravity*, addressed in [2] for even $n \geq 2$.

In [1] we rather study how basic objects related to the *random Riemannian manifold* $(M, g^\bullet)$ change under the influence of the noise. These include:

- geometric quantities, e.g.: *intrinsic distance*, *diameter*, *volume*;
- functional-analytic quantities, e.g.: *spectral bound* (in the case of non-compact M), or *spectral gap* (in the case of closed M);
- probabilistic objects, e.g.: *heat kernels*, *Brownian motions* and their Dirichlet forms.

We show how to quantify these dependencies in terms of key parameters of the noise and discuss explicit examples on spheres, tori, and hyperbolic spaces.

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Mathematics of magic angles

Simon Becker

(joint work with Mark Embree, Tristan Humbert, Jens Wittsten, and Maciej Zworski)

Magic angles are a hot topic in condensed matter physics: when two sheets of graphene are twisted by those angles the resulting material is superconducting, see Fig. 1.

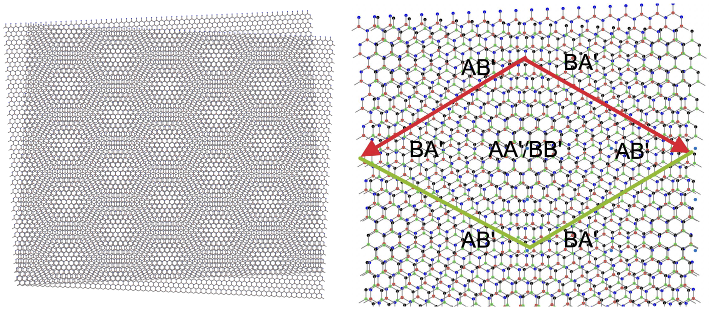

Figure 1. Left: Moiré pattern for twisting angle $\theta = 5°$. Right: One moiré fundamental cell, with (A=red, B=blue) and (A'=green, B'=black) indicating vertices of single honeycomb lattices of graphene, respectively.

The mathematics is described by a matrix-valued Dirac-type operator

$$H = \begin{pmatrix} 0 & D(\alpha)^* \\ D(\alpha) & 0 \end{pmatrix} \text{ with } D(\alpha) = \begin{pmatrix} 2D_{\bar z} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar z} \end{pmatrix}$$

whose spectral properties are thought to determine which angles are magical. It comes from a 2019 PR Letter by Tarnopolsky–Kruchkov–Vishwanath [4]. The parameter α is inversely proportional to the twisting angle of the lattices. The potential U is the tunnelling potential of the electron hopping from the upper to the lower layer satisfying basic honeycomb symmetries

$$U(z+\mathbf{a}) = \bar\omega^{a_1+a_2}U(z) \text{ and } U(\omega z) = \omega U(z) \text{ where}$$

$$\omega = e^{2\pi i/3} \text{ and } \mathbf{a} = a_1\zeta_1 + a_2\zeta_2 \text{ with } \zeta_j = \frac{4\pi i \omega^j}{3}.$$

The mathematics behind this is an elementary blend of representation theory (of the Heisenberg group in characteristic three), Jacobi theta functions and spectral instability of non-self-adjoint operators (involving Hörmander's bracket condition in a very simple setting), see [1].

The Hamiltonian H is periodic with respect to the lattice $\Lambda = 3\zeta_1\mathbb{Z} + 3\zeta_2\mathbb{Z}$. Thus, by applying the Bloch-Floquet transform, we study

$$H_{\mathbf{k}} = \begin{pmatrix} 0 & D(\alpha)^* + \overline{\mathbf{k}} \\ D(\alpha) + \mathbf{k} & 0 \end{pmatrix} : H^1(\mathbb{C}/\Lambda) \to L^2(\mathbb{C}/\Lambda).$$

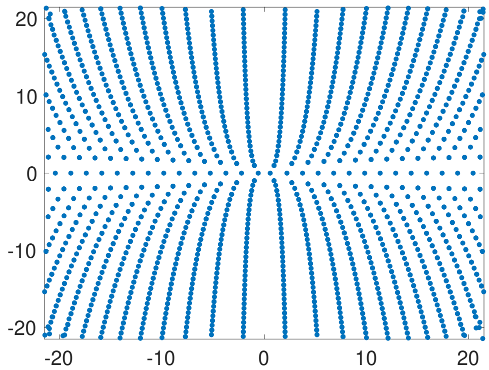

FIGURE 2. Magic parameters α

A magic angle α is then characterized by the *flat-band condition*

$$\alpha \text{ is magic } \Leftrightarrow 0 \in \bigcap_{\mathbf{k}\in\mathbb{C}} \operatorname{Spec}(H_{\mathbf{k}}) \Leftarrow 1/\alpha \in \operatorname{Spec}(T_{\mathbf{k}})$$

where $T_{\mathbf{k}} = (2D_{\bar{z}} - \mathbf{k})^{-1} \begin{pmatrix} 0 & U(z) \\ U(-z) & 0 \end{pmatrix}$ for any/some $\mathbf{k} \in \mathbb{C} \setminus \Lambda^*$.

Recent mathematical progress includes the proof of existence of generalized magic angles and computer assisted proofs of existence of real ones [5] and analytical progress on the existence and distribution of magic angles [2, 3], see Fig. 2.

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Spectral gaps on domains in hyperbolic space, and on metric trees

Julie Clutterbuck

(joint work with Theodora Bourni, Xuan Hien Nguyen, Alina Stancu, Guofang Wei, Valentina-Mira Wheeler, Mitchell Wolswinkel, Huateng Zhu)

We consider the Dirichlet eigenvalue problem: On a bounded domain $\Omega \subseteq M^n$,

$$-\Delta u + Vu = \lambda u \text{ in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Here $V \geq 0$ is a convex potential. The eigenvalues are

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

The *spectral gap* is $\Gamma(\Omega, V) = \lambda_2 - \lambda_1$. Here we are concerned with finding lower bounds on the gap, and the corresponding optimal domains and potentials.

Theorem 1. *For bounded convex domains in $\mathbb{R}^n$ of diameter D,*

$$\Gamma(\Omega, V) \geq \Gamma(I_D, 0)$$

where I_D is the interval of length D.

This was independently conjectured by several people [3, 15, 16]. It was proved by Lavine for $n = 1$, and by Andrews–Clutterbuck for $n > 1$ [1, 10].

Naturally one is curious about domains in manifolds. In the case of the sphere, this was resolved by Dai, He, Seto, Wang, Wei, and Zhang (in various subsets) [8, 9, 13]:

Theorem 2. *For a convex $\Omega \subseteq \mathbb{S}^n$ with $D = diameter(\Omega)$,*

$$\Gamma(\Omega, 0) \geq 3\pi^2/D^2.$$

However in the corresponding case in hyperbolic space (ie constant negative curvature) we find that the spectral gap can be arbitrarily small:

Theorem 3 (Bourni–Clutterbuck–Nguyen–Stancu–Wei–Wheeler [5, 6])**.** *For all $D > 0$, and $\epsilon > 0$, there exists a convex domain $\Omega \subset \mathbb{H}^n$ with diameter D and*

$$\lambda_2 - \lambda_1 \leq \frac{\epsilon}{D^2}.$$

Some comments on this result follow. In both $\mathbb{R}^n$ and $\mathbb{S}^n$, the first Dirichlet eigenfunction on convex domains is *log-concave.* A function f is log-concave if $\log f$ is concave. Log-concavity implies that the curvature of the level sets of f is positive, and so implies the convexity of the super-level sets of f. It is a result of Brascamp-Lieb [7] that for convex domains in $\mathbb{R}^n$, the first eigenfunction is log-concave; the analogous result for domains in $\mathbb{S}^n$ was used by Dai et al in their work on the spectral gap. Log-concavity is crucial to the spectral gap results in $\mathbb{R}^n$ and $\mathbb{S}^n$.

However Shih [14] has an example of a convex domain in $\mathbb{H}^2$ where u_1 is *not* log-concave. In the upper half space model for $\mathbb{H}^2$, we can use separation of variables

$$u(r,\theta) = f(r)h(\theta).$$

If $-\Delta u = \lambda u$, then

$$\begin{aligned} r^2 f_{rr} + r f_r &= -\mu f \quad r \in (1, R) \\ h_{\theta\theta} + \lambda(\csc\theta)^2 h &= \mu h \quad \theta \in (\theta_0, \theta_1) \end{aligned}$$

with Dirichlet boundary conditions. Here μ couples the two equations together and has a discrete set of possible values.

In the upper half-space model, geodesics are vertical lines and half-circles centred on $y = 0$, so this is convex. Shih very carefully estimated the first eigenfunction, and was able to show that, if one chooses the proportions of the domain carefully, then there is a point where $D^2 \log u_1 > 0$.

We used a similar construction to show that for this kind of domain, the ground state can have two distinct maxima. Not only are the superlevel sets of u not convex— they are not even connected. The two sides of the domain can be separated by a region of very small area which allows the first eigenfunction u_1 and the second eigenfunction u_2 to be very close in absolute value, although u_2 changes sign. This implies that λ_1 and λ_2 are very close, and so the spectral gap is small.

In subsequent work, members of our team showed that even using stronger notions of convexity is not enough to permit a lower bound of the kind found in $\mathbb{R}^n$ or $\mathbb{S}^n$:

Theorem 4 (Nguyen–Stancu–Wei [11]). *For any geodesic ball $B \subset \mathbb{H}^n$*

$$\Gamma(B, 0) \leq \frac{C(n)}{D^2}.$$

Moreover, for any horoconvex domain $\Omega \subset \mathbb{H}^n$ with diameter D sufficiently large,

$$\Gamma(\Omega, 0) \leq \frac{C(n)}{D^2}.$$

Spectral gap for the Schrödinger operator on metric trees. Motivated by the previous results, we consider tree graphs as having an (appropriately defined) negative curvature, and study the spectral gap (in a forthcoming paper). There has been extensive work on this problem in recent years: see the survey by Berkolaiko and Kuchment [4], and for trees specifically, Rohleder [12] among others.

For a tree graph $\mathcal{G}$ we consider the eigenvalue problem

$$\begin{aligned} -f_i'' + V f_i &= \lambda_i f_i \text{ on edges} \\ f(v_j) &= 0 \text{ at each vertex } v_j \text{ of degree 1} \\ \sum_k f_{\nu_k}(v_j) &= 0 \text{ at each vertex } v_j \text{ of degree 2 or more,} \end{aligned}$$

where we take the sum over edges e_k incident to v_j, and subscript ν_k denotes the derivative along a path along edge e_k concluding at the vertex v_j.

We consider potentials V that are *edgewise* convex, in the sense that along any path in $\mathcal{G}$, V is convex. This is a quite restrictive condition. For such graphs we are able to show that the minimiser of the spectral gap must be edgewise affine (that is, affine along each edge, and continuous across vertices).

Theorem 5 (Clutterbuck–Wolswinkel–Zhu)**.** *Let $\mathcal{G}$ be a tree graph. For every edgewise convex potential V, there is a convex,* edgewise affine *function L_V such that*

$$\Gamma(\mathcal{G}, V) \geq \Gamma(\mathcal{G}, L_V).$$

The methods here are those of [2, 10] for intervals, extended to graphs.

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