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Geometrie

Organized by
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Abstract. The workshop *Geometrie*, organized by Aaron Naber (Evanston), André Neves (Chicago) and Burkhard Wilking (Münster) was well attended with over 42 participants (35 in person and 7 online) with broad geographic representation from all continents, and held in a very active atmosphere. During the meeting, various interesting topics in geometry were discussed, such as geometric flows, Einstein manifolds and spaces with sectional curvature bounds.

Mathematics Subject Classification (2020): 53-XX.

Introduction by the Organizers

The workshop consisted of 18 one hour talks and 3 half hour after dinner talks (Monday, Tuesday, Thursday). The after dinner talks were given by PhD students and very recent PhD's. All the speakers did an excellent job, which were the main contributions to the good atmosphere at the workshop.

Among all the talks, four were focused on geometric flows. Two of them were related to Ricci flow and the other two related to mean curvature flow. Hamilton's pinching conjecture states that three dimensional Ricci pinched manifolds are either flat or compact. Miles Simon used stability estimates for the initial data of Ricci flows to present a proof of Hamilton's pinching conjecture with the additional assumption of bounded sectional curvature. Peter Topping proved the long time existence of Ricci flow on a non-compact Ricci pinched manifold. As a corollary he obtained Hamilton's pinching conjecture without any additional assumptions on the sectional curvature. The other two talks were related to mean curvature flow. Gerhard Huisken studied inverse mean curvature flow with entire graphs as initial data. He proved long time existence for graphs with superlinear growth that are δ-starshaped. For linear growth a diffusion effect occurs and the solution converges in finite time. Robert Haslhofer classified noncollapsed translators in $\mathbb{R}^4$. As an application he obtained a classification of certain blowup limits of mean curvature flow in $\mathbb{R}^4$.

Michael Wiemeler, Jan Nienhaus and Anton Petrunin talked about manifolds with non-negative and positive sectional curvature. A famous conjecture by Hopf states that an even dimensional positively curved manifold has positive Euler characteristic. Under the additional assumption of an isometric five-torus action on the manifold Michael Wiemeler proved the Hopf conjecture. For this he gave a classification of fixed point sets of a five-torus action on a positively curved manifold. He also presented a classification up to rational cohomology of even dimensional manifolds with vanishing odd-degree cohomology that admit an action by a seven-torus. Jan Nienhaus extended Wiemeler's results to four- and six-torus actions. In particular he was able to improve Lee Kennard's four-periodicity theorem, using that the normal bundle of a fixed point set of a torus action always admits a complex structure. Anton Petrunin showed, that five point metric spaces can be embedded into a non-negatively curved Riemannian manifold, if and only if the Lang-Schroeder-Sturm inequalities are satisfied.

A mordern approach to the Bochner technique was presented by Matthias Wink. In particular he showed, that n-manifolds with $\lceil \frac{n}{2} \rceil$-positive curvature operator are rational homology spheres. Furthermore he presented applications to the Kähler curvature operator on a Kähler manifold. Eleonora di Nezzas talk was focused on Kähler geometry. In fact she studied geodesics in a certain p-distance on the spaces of Kählerpotentials and plurisubharmonic functions on a compact Kähler manifold. Olivier Biquard gave a classification of toric hermitian gravitational instantons.

Two talks related to positive scalar curvature were given by Chao Li and Christos Mantoulidis. Chao Li studied manifolds with $\lambda_1(-\Delta + \frac{1}{2}R) > 0$, where λ_1 is the eigenvalue of the self-adjoint elliptic operator with Dirichlet boundary conditions. The study of these manifolds is motivated by the study of hypersurfaces in manifolds with positive scalar curvature. An application of his main result is a proof of the three dimensional Bernstein conjecture, which states that an immersed, complete, two-sided, stable minimal hypersurface of the four-dimensional Euclidean space is flat. Christos Mantoulidis proved that every closed and oriented four-manifold that admits a metric of positive scalar curvature can be obtained from a four-orbifold with vanishing first Betti number that admits a metric with positive scalar curvature by 0- and 1-surgeries. Additionally the second Betti number of the orbifold is bounded by the second Betti number of the original manifold. This partially answers the question if any four-manifold admitting positive scalar curvature can be obtained by performing surgeries of codimension at least three on “simple” manifolds.

Three of the talks were related to Einstein metrics. The famous Alekseevsky conjecture states that any homogeneous Einstein manifold with negative Einstein constant must be diffeomorphic to $\mathbb{R}^n$. Christoph Böhm presented a proof of this conjecture by using the more general approach of studying invariant Einstein metrics on manifolds with a group action of one orbit type. Ursula Hamenstädt and Frieder Jäckel talked about a joint work on metrics that are almost Einstein and close to a hyperbolic metric. As an application of their main result they showed that hyperbolic manifolds still carry hyperbolic metrics even after certain "drilling and filling" constructions are performed.

The remaining five talks were given by Robin Neumayer, Daniel Stern, Karl-Theodor Sturm, Guofang Wei and Liam Mazurowsky. Robin Neumayer studied the existence of minimizers and the rigidity of optimal Sobolev inequalities. In particular she showed that optimal Sobolev inequalities for a bounded domain with boundary of class C^2 are sharp. As a corollary a rigidity statement was obtained. Daniel Stern studied the existence of harmonic maps on closed manifolds, such that the codomain has non-trivial l-th homotopy group. In particular he could find harmonic maps with Morse-index bounded by $l + 1$ and small singular sets. He applied this result to study Schrödinger operators on manifolds with dimensions between three and five. Karl-Theodor Sturm constructed and analysed conformally invariant fields on manifolds of even dimension. In particular these fields share a quasi invariance property under conformal transformation and define so called *Liouville Quantum Gravity measures*, which also fulfill a quasi invariance property. Guofang Wei proved that there is no lower bound on the fundamental gap in terms of the diameter of the Laplacian and the Schrödinger operator with Dirichlet boundary conditions on a convex domain in a hyperbolic space. This is a sharp contrast to the flat or spherical case. Liam Mazurowski applied Zhou's and Zhu's Min-Max theory to non-compact manifolds to construct submanifolds with constant mean curvature in asymptotically flat manifolds.

Workshop: Geometrie

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Abstracts

Comparing Riemannian manifolds using spinors

John Lott

In 1998, Llarull proved that if g is a Riemannian metric on S^n with $g \geq g_{stan}$ and $R \geq n(n-1)$ then $g = g_{stan}$ [3]. Here g_{stan} is a Riemannian metric on S^n with constant sectional curvature 1 and R denotes the scalar curvature. In particular, one cannot simultaneously increase the Riemannian metric and the scalar curvature of the sphere. Although the statement of Llarull's theorem sounds elementary, the only known proof uses spinors.

In 2002, Goette-Semmelmann extended Llarull's result, replacing the standard sphere by a compact Riemannian manifold M with nonnegative curvature operator. **Theorem** [2] Let N and M be compact connected Riemannian manifolds of dimension n. Let $f : N \to M$ be a smooth spin map. Suppose that
1. f is Λ^2-nonincreasing.
2. M has nonnegative curvature operator.
3. $R_N \geq f^* R_M$.
4. $\chi(M) \neq 0$ and f has nonzero degree.

Then $R_N = f^* R_M$.

The proof of the Goette-Semmelmann theorem uses a Clifford module which, when N and M are both spin, is $S_N \otimes f^* S_M$. Motivated by questions of Gromov, I extended the result to manifolds with boundary.

Theorem [4] Let N and M be compact connected Riemannian manifolds-with-boundary of even dimension n. Let $f : (N, \partial N) \to (M, \partial M)$ be a smooth spin map. In additions to the assumptions of the previous theorem, suppose that
1. ∂f is distance-nonincreasing.
2. ∂M has nonnegative second fundamental form.
3. $H_{\partial N} \geq (\partial f)^* H_{\partial M}$.

Then $R_N = f^* R_M$ and $H_{\partial N} = (\partial f)^* H_{\partial M}$. If M is flat then N is Ricci-flat.

Here H denotes the mean curvature. In later improvements, Wang-Xie-Yu showed that the result is also true when n is odd [6] and Wang-Xie showed that flatness of M implies flatness of N [5]. In fact, Wang-Xie-Yu extending the theorem to manifolds-with-corners and Wang-Xie showed that the polyhedral analog implies the validity of the Stoker conjecture concerning convex polyhedra.

As a further application of the technique of comparing spinors, I described how to define a spinorial quasilocal mass for a compact manifold-with-boundary, in analogy to Witten's proof of the positive mass theorem for a noncompact asymptotically Euclidean spin manifold. Earlier work in this direction was done by Dougan-Mason [1] and Zhang [7]. There are two basic issues in defining a spinorial quasilocal mass. First, it is not clear how to put in the comparison manifold.

Second, it is not clear how to impose boundary conditions. Both of these issues are resolved by comparing spinors.

Let N and M be compact connected Riemannian manifolds-with-boundary of dimension n. Let $f : (N, \partial N) \to (M, \partial M)$ be a smooth spin map so that for any component C of ∂N, the restriction $\partial f\Big|_C : C \to (\partial f)(C)$ is an isometric diffeomorphism. For simplicity, suppose that n is even, and that N and M are spin. Then

$$S_N \otimes f^* S_M\Big|_{\partial N} \cong \Lambda^* T^* N\Big|_{\partial N} \cong \Lambda^* T^* \partial N \oplus (\tau^n \wedge \Lambda^* T^* \partial N),$$

where τ^n is a unit conormal to ∂N. There is a canonical section of $\Lambda^* T^* \partial N$, namely the constant function 1. It turns out that one can always solve the Dirac equation on $C^\infty(N, S_N \otimes f^* S_M)$ with boundary value $1 + \tau^n \wedge \phi$ for some $\phi \in \Omega^*(\partial N)$, although perhaps not uniquely. Let ψ be the solution with minimal L^2-norm. Then we define the quasilocal mass of N, relative to M, by

$$\mathcal{M} = - \int_{\partial N} \langle \psi, \nabla_{e_n} \psi \rangle dA,$$

where e_n is the inward-pointing unit normal vector. Some basic properties are

(1) If $N = M$ and f is the identity map then $\mathcal{M} = 0$.
(2) If M has nonnegative curvature operator and $R_N \geq |\Lambda^2 df| \, f^* R_M$ then $\mathcal{M} \geq 0$.

In the weak field limit, i.e. when M is close to N, the quasilocal mass $\mathcal{M}$ is approximated by the Brown-York mass. There is also a spacetime extension of the definition of $\mathcal{M}$.

References

[1] A. Dougan and L. Mason, *Quasilocal mass constructions with positive energy*, Phys. Rev. Lett. **67** (1991), 2119–2122.
[2] S. Goette and U. Semmelmann, *Scalar curvature estimates for compact symmetric spaces*, Differential Geom. Appl. **16** (2002), 65–78.
[3] M. Llarull, *Sharp estimates and the Dirac operator*, Math. Ann. **310** (1998), 55–71.
[4] J. Lott, *Index theory for scalar curvature on manifolds with boundary*, Proc. Amer. Math. Soc. **149** (2021), 4451–4459.
[5] J. Wang and Z. Xie, *On Gromov's dihedral ridigidity conjecture and Stoker's conjecture*, preprint, https://arxiv.org/abs/2203.09511 (2022)
[6] J. Wang, Z. Xie and G. Yu, *On Gromov's dihedral extremality and rigidity conjectures*, preprint, https://arxiv.org/abs/2112.01510 (2021)
[7] X. Zhang, *A new quasi-local mass and positivity*, Acta Math. Sin. (Engl. Ser.) **24** (2008), 881–890.

Geodesic distance

Eleonora Di Nezza
(joint work with Chinh Lu)

Let X be a compact Kähler manifold of complex dimension n and fix a Kähler metric ω normalized such that $\int_X \omega^n = 1$. By the $\partial\bar{\partial}$-lemma there is a correspondence between the space of Kähler metrics in the cohomolgy class $\{\omega\} \in H^{1,1}(X, \mathbb{R})$ and the space of Kähler potentials

$$\mathcal{H} := \{u \in C^\infty(X, \mathbb{R}) \ : \ \omega_u := \omega + i\partial\bar{\partial}u > 0\}.$$

This space of smooth potential is a subset of PSH(X, ω), the set of ω-*plurisubharmonic* functions. We say that a function $u : X \to \mathbb{R} \cup \{-\infty\}$ is quasi-plurisubharmonic (qpsh) if locally $u = \rho + \varphi$ where φ is plurisubharmonic (psh) and ρ is smooth. A qpsh function u is ω-psh if $\omega + i\partial\bar{\partial}u \geq 0$ in the weak sense of currents.

Motivated by the study of canonical metrics on X, in [1] Mabuchi introduced a Riemannian structure on the space of Kähler potentials, giving rise to the notion of length of a smooth path $\gamma : [0, 1] \to \mathcal{H}$

$$\ell_2(\gamma) := \int_0^1 \sqrt{\int_X |\dot{\gamma}_t|^2 \omega_{\gamma_t}^n}\, dt,$$

and consequently the notion of distance between φ_0, φ_1, two elements in $\mathcal{H}$:

$$d_2(\varphi_0, \varphi_1) := \inf\{\ell_2(\gamma) \quad \gamma : [0, 1] \to \mathcal{H},\ \gamma(0) = \varphi_0, \gamma(1) = \varphi_1\}.$$

Mimicking the finite dimensional setting in Riemannian geometry, one can also define geodesics by the Euler-Lagrange equation of the energy functional associated to the L^2-metric. It turns out that such geodesics can be explicit expressed as an *envelope*.

For a curve $[0, 1] \ni t \mapsto u_t \in$ PSH(X, ω) we define

$$X \times D \ni (x, z) \mapsto U(x, z) := u_{\log|z|}(x), \tag{1}$$

where $D = \{z \in \mathbb{C},\ 1 < |z| < e\}$ and $\pi : X \times D \to X$ is the projection on the first factor. We say that $t \mapsto u_t$ is a subgeodesic if $(x, z) \mapsto U(x, z)$ is a $\pi^*\omega$-psh function on $X \times D$.

Given $\varphi_0, \varphi_1 \in \mathcal{H}$, for $(x, z) \in X \times D$ we define

$$\Phi(x, z) := \sup\{U(x, z) \ : \ t \mapsto u_t \text{ is a subgeodesic and } \limsup_{t \to 0,1} u_t \leq \varphi_0, \varphi_1\}.$$

The curve $t \mapsto \varphi_t$ constructed from Φ via (1) is the plurisubharmonic (psh) geodesic segment connecting φ_0 and φ_1.

Such a description turns out to be very useful and handy since geodesics joining two elements in $\mathcal{H}$ are not smooth in general. The optimal regularity is indeed only $C^{1,1}$. Nevertheless, it was proved by Chen [2] that such geodesics are distance minimizing, i.e.

$$d_2(\varphi_0,\varphi_1)=\sqrt{\int_X |\dot\varphi_t|^2\omega^n_{\varphi_t}},\quad \forall t\in[0,1].$$

Later on, Darvas [3] defined a family of L^p distances d_p, $p\geq 1$, (coming from a Finsler structure on the tangent spaces) between two potentials $\varphi_0,\varphi_1\in\mathcal{H}$ proving that, once again

$$d_p(\varphi,\varphi_1)^p=\left(\int_X |\dot\varphi_t|^p\omega^n_{\varphi_t}\right)^{1/p},\quad \forall t\in[0,1], \tag{2}$$

where φ_t is the psh geodesic segment defined above.

It follows from the equality in (2) that $(\mathcal{H},d_p)$ is a metric space. On the other hand, the latter is not complete since we know already that geodesics do not stay in $\mathcal{H}$.

Darvas also proved that the metric completion $(\mathcal{H},d_p)$ can be identified with the energy class $\mathcal{E}^p(X,\omega)$. This class is a subset of (possibly singular) ω-psh functions with an extra integrability assumption: we say that a ω-psh function u belongs to $\mathcal{E}^p(X,\omega)$ if and only if $\int_X |u|^p\omega_u^n<+\infty$. Examples of functions in $\mathcal{E}^p$ are functions that locally (in a small coordinate ball) write as $-(-\log|z|)^\alpha$ with $\alpha\in(0,1)$ such that $p<\frac{n(1-\alpha)}{\alpha}$. In particular $\mathcal{E}^p$ contains a lot of unbounded potentials.

The main goal of the joint paper with Chinh Lu [4] is to investigate under which condition on $\varphi_0,\varphi_1\in\mathcal{E}^p(X,\omega)$, the identity in (2) holds. It is easy to construct examples of bounded (but not smooth!) endpoints such that (2) does not hold. Our result give a sufficient condition under which we have a positive answer:

Theorem 1. *Assume $\varphi_0,\varphi_1\in \mathrm{Ent}(X,\omega)$ and let φ_t be the psh geodesic connecting varphi$_0$ to φ_1. If $\varphi_0-\varphi_1$ is bounded then*

$$\int_X |\dot\varphi_t|^p\omega^n_{\varphi_t} \text{ is constant in } t\in[0,1].$$

If in addition $\varphi_0,\varphi_1\in\mathcal{E}^p(X,\omega)$, then

$$d_p^p(\varphi_0,\varphi_1)=\left(\int_X |\dot\varphi_t|^p\omega^n_{\varphi_t}\right)^{1/p},\quad \forall t\in[0,1].$$

Here $\mathrm{Ent}(X,\omega)$ consists of functions $u\in\mathcal{E}(X,\omega)$ whose Monge-Ampère measure has finite entropy:

$$\mathrm{Ent}(\omega^n,\omega_u^n):=\int_X \log\left(\frac{\omega_u^n}{\omega^n}\right)\omega_u^n<+\infty.$$

References

[1] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds. I*, Osaka J. Math. **24** (1987), 227–252.

[2] X.-X. Chen, *The space of Kähler metrics*, J. Differential Geom. **56** (2000), 189–234.

[3] T. Darvas, *The Mabuchi geometry of finite energy classes*, Adv. Math. **285** (2015), 182–219.

[4] E. Di Nezza and C. H. Lu, *Geodesic distance and Monge-Ampère measures on contact sets*, Analysis Math. (2022).

Initial stability estimates for Ricci flow and three dimensional Ricci-pinched manifolds

Miles Simon

(joint work with Alix Deruelle, Felix Schulze)

We consider the following setup:
$(N, g(t))_{t\in(0,T)}$ is a smooth, complete, connected solution to Ricci flow $\frac{\partial g}{dt}(t) = -2\mathrm{Ric}(g(t))$ (introduced by Hamilton '82) with

$$\mathrm{Ric} \geq 0 \quad (a)$$

$$|\mathrm{Rm}| \leq \frac{c_0^2}{t} \quad (b)$$

The evolution of distance under Ricci flow satisfies the following estimates (Hamilton/Perelman)

$$d_s e^{t-s} \geq d_t \geq d_s - c_0\gamma(n)\sqrt{t-s} \quad (c)$$

for $0 < s < t$ where $d_t = d(g(t))$.

A prominent example is:
Expanding Solitons with bounded non-negative curvature operator:
$g(t) := t(\Phi_t)^* g(1), t \in (0, \infty)$, where $\Phi_t : N \to N$ are diffeomorphisms, and $\mathcal{R}(g) \geq 0$.
The solution is in fact a Gradient soliton: there exists $f : M \to \mathbb{R}$ such that $\partial_t \Phi_t = -t^{-1}\nabla^g f \circ \Phi_t$.

(c) implies there is a uniform non-degenerate limit

$$d_0 := \lim_{t\searrow 0} d(g(t)).$$

We wish to compare regions of solutions.
We consider $M_1 \subseteq N_1$, $M_2 \subseteq N_2$, which are open, connected regions, $(N_1, g_1(t))$ $(N_2, g_2(t))$ $t \in (0,T)$ are smooth, complete , bounded curvature solutions to Ricci flow satisfying $|\mathrm{Rm}(g_i(t))| \leq \frac{c_0^2}{t}(a)$,$\mathrm{Ric} \geq 0, (b)$ for $i = 1, 2$ and hence (c)

$$d_{i,s} \geq d_{i,t} \geq d_{i,s} - c_0\gamma(n)\sqrt{t-s} \quad (c)$$

for $0 < s < t$ where $d_{i,t} = d(g_i(t))$, and $d_{i,0} := \lim_{t\to 0} d_{i,t}$ and there exists an isometry $\psi_0 : (M_1, d_{1,0}) \to (M_2, d_{2,0})$. We assume two regularity conditions on the initial data of the regions $(M_i, d_{i,0})$. the first one is : "all tangent cones are $(\mathbb{R}^n, d_{euc})$". In the setting we are considering Cheeger-Colding/ Colding theory implies this is equivalent to a *uniform Reifenberg condition*:

$(R1)$ for all $p \in M$ and for all $\varepsilon > 0$, there exist $r > 0$ and a neighborhood $U_p \subset\subset M$ such that $d_{GH}(B_{s^{-1}d_0}(x,1), \mathbb{B}(0,1)) < \varepsilon,$ for all $s < r,$ and for all $x \in U_p$ such that $B_{d_0}(x,s) \subset\subset U_p$.

The second regularity condition is:

$(R2)$ For any $x_0 \in M$, there is a radius $R = R(x_0) > 0$ such that $B_{d_0}(x_0, 4R) \subset\subset M$ and points $a_1, \dots, a_n \in B_{d_0}(x_0, 3R)$ such that the map

$$D_0 : \begin{cases} B_{d_0}(x_0, 4R) & \to & \mathbb{R}^n \\ x & \to & (d_0(a_1, x) - d_0(a_1, x_0), \dots, d_0(a_n, x) - d_0(a_n, x_0)), \end{cases}$$

is a $(1 + \varepsilon_0)$ bi-Lipschitz homeomorphism on $B_{d_0}(x_0, 2R)$.

In order to compare the solutions $(M_1, g_1(t))$ and $(M_2, g_2(t))$ we need a 'good gauge'. Our Gauge is the Ricci DeTurck Flow/Ricci Hamronic map heat flow: We consider Dirichlet solutions $\frac{\partial}{\partial t} F_i(\cdot, t) = \Delta_{g_i(t)} F_i(\cdot, t)$ to Ricci Harmonic map heat flow, $\tilde{g}_i(t) = (F_i(t))_*(g_i(t))$ and we compare $\tilde{g}_1(t)$ and $\tilde{g}_2(t)$.

Theorem 1. *Let $(N_i^n, g_i(t))_{t \in (0,T)}$, $i = 1, 2$, be smooth solutions to Ricci flow with* $\mathrm{Ric} \geq 0$ *and* $|\mathrm{Rm}(\cdot, t)| \leq \frac{c_0^2}{t}$ *and hence (c). Assume that on $M_1 \subseteq N_1$ and $M_2 \subseteq N_2$ there exists an isometry $\psi_0 : (M_1, d_{1,0}) \to (M_2, d_{2,0})$. We assume (R1), (R2) hold on M_1 and M_2 and that $x_0 \in M_1$. Then there exists an $R_0 \in (0, 1)$ and $T_0 > 0$ depending on n, ε_0 and x_0 and Dirichlet solutions to Ricci Harmonic map heat flow $F_1 : B_{d_{1,0}}(x_0, \frac{3}{2}R_0) \times [0, T_0] \to \mathbb{R}^n$ with initial and boundary values given by D_0, and $F_2 : B_{d_{2,0}}(\psi_0(x_0), \frac{3}{2}R_0) \times [0, T_0] \to \mathbb{R}^n$, with initial and boundary values given by $D_0 \circ \psi_0^{-1}$, such that $F_1(t)$ and $F_2(t)$ are smooth, uniformly Bi-Lipschitz diffeos for $t > 0$.*

The solutions $(\tilde{g}_1(t))_{t \in (0,T_0)}$ and $(\tilde{g}_2(t))_{t \in (0,T_0)}$ to δ-Ricci-DeTurck flow defined on $\mathbb{B}(0, R_0) \times (0, T_0)$ satisfy the following: There exist $C_0 > 0$ depending on n, ε_0 and x_0 such that if $t \in (0, T_0]$:

$$|\tilde{g}_1(t) - \tilde{g}_2(t)|_\delta \leq \exp\left(-\frac{C_0}{t}\right), \quad \textit{on } \mathbb{B}(0, R_0).$$

An application: The Hamilton conjecture

Let (M, g) be a smooth, complete manifold with bounded and uniformly pinched non-negative Ricci curvature, that is

$\mathrm{R} \geq 0$ and the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ of the Ricci curvature satisfy $a\mathrm{R} \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \mathrm{R}$ where $a > 0$.

Hamilton Conjecture: Either (M, g) is compact or (M, g) is flat.
Assume by contradiction, (M, g_0) is as above but not compact and not flat (*)

Previous works: John Lott (2019):

Let (M^3, g_0) be as in (*). Then there exists a $(M, g(t))_{t \in [0,\infty)}$ solution to Ricci flow with $0 < b\mathrm{R} \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \mathrm{R}$ for all $t > 0$ and

$AVR(M, g(t)) = \lim_{r \to \infty} \frac{vol(B_{g(t)})(r)}{r^3} = v_0 > 0$ (is a non-zero constant independent of time) and $|\mathrm{Rm}(\cdot, t)| \leq \frac{c_0^2}{t}$ for all $t \in [0, \infty)$. Without loss of generality $v_0 < \omega_3$: otherwise the equality case of the Bishop-Gromov-Inequality implies the space is flat, which leads to a contradiction. We blow down the solution, to obtain a solution $(N_1, g_1(t))_{t \in (0,\infty)}$ satisfying $\mathrm{Ric}(t) \geq 0$ and $|\mathrm{Rm}(\cdot, t)| \leq$

$\frac{c_0^2}{t}$ for all $t \in [0, \infty)$. Hence (c) is satisfied. The initial metric $(N_1, d_{1,0})$ is a cone and has $AVR = v_0 < \omega_3$. Using results from RCD/Alexandorv theory (Sturm, Lott/Villani, (2005 $\rightarrow$), Ketterer (2011), Ambrosio-Erbar-Gigli-Savaré-Sturm (2011-2013), Cavelletti/Millman (2016), Alexandrov, Reshetnyak, (1900's) , Lytchack/Stadler (2018) ...) we see that the cone is in fact a cone over an Alexandrov space with sec ≥ 1 and may be approximated in the Gromov-Hausdorff sense by smooth spaces with curvature operator $\mathcal{R} \geq 0$. A result of Hochard shows that both $(N_1, d_{1,0})$ and $(N_2, d_{2,0})$ satisfy (R1) away from the tip. Hence, using the theory of Alexandrov Spaces (Gromov-Burgao-Perleman (1992)) we see that also (R2) is satisfied away from the tip.

Using previous results of the authors, we see that there is an expanding gradient soliton $(N_2, g_2(t))$ coming out of $(N_1, d_{1,0})$. A result of Hochard shows that both solutions $(N_1, g_1(t))$ and $(N_2, g_2(t))$ satisfy (R1) away from the tip, and so we may compare the solutions away from the tip of the cone, using the first part of the talk. This shows us that $(N_2, g_2(t))$ satisfies

$$\mathrm{Ric}(g_2(1)) \geq \lambda_0 \mathrm{R}_{g_2(1)} g_2(1) - \exp\left(-C_0 f\right) g_2(1), \quad \text{on } N_2.$$

where f is the potential of the expanding gradient soliton. Results of A. Deruelle (2017) then show that $(N_2, g_2(t))$ is isometric to $(\mathbb{R}^3, d_{euc})$. Hence $(N_2, d_{2,0})$ is isometric to $(\mathbb{R}^3, d_{euc})$ and hence $(N_1, d_{1,0})$ is isometric to $(\mathbb{R}^3, d_{euc})$ and hence $(N_1, g_1(t))$ is flat, and has asymptotic volume ratio equal to ω_3, which is a contradiction to the fact that asymptotic volume ratio of $(N_1, g_1(t)))$ is $v_0 < \omega_3$.

The case of non-bounded curvature (Man-Chun Lee/Peter Topping): In a paper which appeared shortly after ours, Man-Chun Lee/ Peter Topping (2022) show that any smooth, complete non-compact Ricci pinched manifold (M, g_0) has a solution which remains Ricci pinched and satisfies (a),(b) and (c). In particular $|\mathrm{Rm}(\cdot, t)| \leq \frac{c}{t} < \infty$ for all $t > 0$ and hence, the results of this talk imply that $(M, g(t))$ is flat for all $t \in (0, \infty)$ and hence (M, g_0) is flat. That is, the Conjecture of Hamilton is verified also in the case where one doesn't assume, that curvature is bounded.

Existence of Extremals and Rigidity for Optimal Sobolev Inequalities

Robin Neumayer

(joint work with Francesco Maggi and Ignacio Tomasetti)

Introduction. Sobolev inequalities, which relate the integrability or regularity of a function to the integrability of its derivatives, are a fundamental tool across analysis and geometry. A classical example is the optimal Sobolev inequality on Euclidean space: for $n \geq 2$ and $p \in (1, n)$ fixed, any function $u \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \geq S_{n,p} \|u\|_{L^{p^\star}(\mathbb{R}^n)} \,. \tag{1}$$

The critical exponent $p^\star = np/(n-p)$ is the unique value making the left- and right-hand sides of (1) scale the same way under dilations $u(x) \mapsto u(x/\alpha)$ for $\alpha > 0$. Optimal Sobolev constants often encode geometric information about their underlying domain or manifold. For example, Ledoux showed in [6] that if a complete Riemannian manifold (M^n, g) with non-negative Ricci curvature admits a Sobolev inequality of the form (1) with optimal constant S_g, then $S_g \leq S_{n,p}$, with $S_g = S_{n,p}$ if and only if (M^n, g) is isometric to Euclidean space. A stability result corresponding to this geometric comparison theorem was shown in [13]; for a comparison theorem for optimal log-Sobolev constants, a different type of stability was shown in [7] for manifolds with (almost) non-negative scalar curvature.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^1 boundary. The Sobolev inequality (1) holds for any $u \in C_c^\infty(\Omega)$, with the same optimal constant $S_{n,p}$. This is just one slice of the global picture, though; functions that do not vanish on the boundary also enjoy a Sobolev inequality once a term with a trace norm on $\partial\Omega$ is included. The optimal form of such a Sobolev inequality is defined through a family of variational problems with two critical constraints: for $T \geq 0$, let

$$(2) \qquad \Phi_\Omega(T) = \inf\left\{ \left(\int_\Omega |\nabla u|^p\right)^{1/p} \ : \ \int_\Omega |u|^{p^\star} = 1, \ \int_{\partial\Omega} |u|^{p^\sharp} = T^{p^\sharp} \right\}.$$

Here $p^\sharp = (n-1)p/(n-p)$ is the critical exponent making this norm scale the same way as the other two; the critical scaling of the norms makes $\Phi_\Omega(T)$ invariant under dilations as well as translations of the domain Ω. By definition, $\Phi_\Omega(T)$ is the optimal constant in a family of Sobolev inequalities:

$$(3) \quad \|\nabla u\|_{L^p(\Omega)} \geq \Phi_\Omega(T)\|u\|_{L^{p^\star}(\Omega)} \qquad \text{whenever} \qquad \|u\|_{L^{p^\sharp}(\partial\Omega)}/\|u\|_{L^{p^\star}(\Omega)} = T.$$

The particular slice $T = 0$ is the inequality (1), i.e. $\Phi_\Omega(0) = S_{n,p}$. A constant test function shows that $\Phi_\Omega(\mathrm{I}(\Omega)^{1/p^\sharp}) = 0$, where we set $\mathrm{I}(\Omega) := \mathrm{Per}(\Omega)/|\Omega|^{\frac{n-1}{n}}$.

In [10], Maggi and Villani proved a geometric comparison theorem for the optimal Sobolev constants $\Phi_\Omega(T)$, showing that balls have the *worst* optimal Sobolev constant. More precisely, letting $B = \{|x| < 1\} \subset \mathbb{R}^n$, they showed that

$$(4) \qquad \Phi_\Omega(T) \geq \Phi_B(T) \qquad \text{for all } T \in [0, \mathrm{I}(B)^{1/p^\sharp}].$$

They also proved existence and characterization of minimizers for the variational problem $\Phi_B(T)$ for all T in this parameter range. Scaling shows that half spaces have the *best* optimal Sobolev constant, i.e. $\Phi_\Omega(T) \leq \Phi_H(T)$ for all $T > 0$, where $H = \{x \cdot e_n > 0\} \subset \mathbb{R}^n$. In [8], Maggi and the author established the existence and characterization of minimizers of $\Phi_H(T)$ for all $T > 0$ (see also [3] when $p = 2$).

Main Results. Two main open problems about $\Phi_\Omega(T)$ motivate the paper [9].

(A) When do minimizers of the variational problem (2) exist? Equivalently, when do extremal functions exist in the sharp Sobolev inequality (3)?

(B) Does rigidity hold in the geometric comparison theorem (4)?

From scaling and the characterization of extremal functions of (1) due to Aubin [2] and Talenti [12], it is easily shown that for $T = 0$, minimizers of (2) cannot exist unless $\Omega = \mathbb{R}^n$. For $T > 0$, in [9] we use the characterization of $\Phi_H(T)$ from [8] to prove that if existence fails on an open bounded domain Ω with C^1 boundary, it can only occur because a minimizing sequence concentrates at exactly one point located on $\partial\Omega$. Assuming further regularity of $\partial\Omega$ and restricting the dimension, we rule out this possibility and prove the following existence theorem.

Theorem 1. Fix $p > 1$ and $n > 2p$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary of class C^2. For every $T \in (0, \infty)$, a minimizer of (2) exists.

Question (B) was posed as an open problem in [10], and the proof of (4) implies the following *rigidity criterion*: If Ω is connected and $\Phi_\Omega(T) = \Phi_B(T)$ for some $T \in (0, \mathrm{I}(B)^{1/p^\sharp})$, *and additionally a minimizer of* (2) *exists for this* T, then Ω is a ball. The connectedness assumption is necessary for rigidity to hold; consider the union of a ball and any other domain. Thanks to this rigidity criterion, we obtain an affirmative answer to Question (B) under the assumptions of Theorem 1.

Corollary 2. Fix $p > 1$ and $n > 2p$. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected domain with boundary of class C^2. If $\Phi_\Omega(T) = \Phi_B(T)$ for some $T \in (0, \mathrm{I}(B)^{1/p^\sharp}]$, then Ω is a ball.

Finally, we obtain the following weak rigidity theorem without additional restrictions on n or $\partial\Omega$.

Theorem 2. Fix $n \geq 2$ and $p \in (1, n)$. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected domain with boundary of class C^1. If there exists $T_* > 0$ such that $\Phi_\Omega(T) = \Phi_B(T)$ for all $T \in (0, T_*)$, then Ω is a ball.

The proof of Theorem 2 again uses the rigidity criterion above; based on the characterization of minimizers of $\Phi_H(T)$ from [8] and an analysis of the Euler-Lagrange equation asymptotically satisfied by a concentrating minimizing sequence, we prove that minimizers of (2) exist for $T > 0$ sufficiently small under the assumptions of Theorem 2.

Open problems. There are quite a few open problems related to this program. First, can one show Theorem 1 in all dimensions? In [9], we build an explicit ("Aubin-type" [1]) test function and expand its energy to rule out concentration. The dimension restriction comes from the tail decay rate of extremals of $\Phi_H(T)$; this issue is familiar from the Yamabe problem and one may hope to construct a "Schoen-type" [11] global test function to show existence in low dimensions.

Second, under the assumptions of Corollary 2, is the comparison theorem (4) *stable*, i.e. if $\Phi_\Omega(T) \approx \Phi_B(T)$ for some $T \in (0, \mathrm{I}(B)^{1/p^\sharp}]$, then is Ω close to a ball in a suitable sense? A starting point here is to analyze the mass transportation proof of (4) from [10] and to show that the optimal transport map taking a minimizer of $\Phi_\Omega(T)$ to a minimizer of $\Phi_B(T)$ is close to the identity.

Finally, when $p = 2$, the variational problem (2) is related to the Yamabe problem for manifolds with boundary [4, 5], where one seeks a conformal metric of constant scalar curvature and constant mean curvature boundary on a given Riemannian manifold with boundary. In the conformally flat case $(M, g) = (\Omega, g_{\mathrm{euc}})$, this is equivalent to showing the existence of critical points of the energy $\int_\Omega |\nabla u|^2 + c_n \int_{\partial\Omega} h\, u^2$ in the same constraint space as in (2). Here h is the mean curvature of

$\partial\Omega$. Can our analysis in [9] be refined to produce a one-parameter family $\{g_T\}_{T>0}$ of Yamabe metrics with the prescribed ratio $\mathrm{vol}(\partial M, g_u)/\mathrm{vol}(M, g_u) = T$?

References

[1] T. Aubin. *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*. J. Math. Pures Appl., (9) **55** (1976), no. 3 , 269–296.

[2] T. Aubin. *Problèmes isopérimétriques et espaces de Sobolev*. J. Differential Geom., **11** (1976), no. 4, 573–598.

[3] E. Carlen and M. Loss. *On the minimization of symmetric functionals*. Rev. Math. Phys., **6** (1994), 5A, 1011–1032. Special issue dedicated to Elliott H. Lieb.

[4] J. Escobar. *The Yamabe problem on manifolds with boundary*. J. Differential Geom. **35** (1992), no. 1, 21-84.

[5] J. Escobar. *Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary*. Indiana University Math. J., **45** (1996), 917–943.

[6] M. Ledoux, *On manifolds with non-negative Ricci curvature and Sobolev inequalities*, Comm. Anal. Geom. **7** (1999), 347–353.

[7] M.C. Lee, A. Naber, and R. Neumayer. *d_p Convergence and ϵ-regularity theorems for entropy and scalar curvature lower bounds*. Accepted to Geom. Topol.

[8] F. Maggi and R. Neumayer. *A bridge between Sobolev and Escobar inequalities and beyond*. J. Funct. Anal., **273** (2017), no. 6, 2070–2106.

[9] F. Maggi, R. Neumayer, and I. Tomasetti, *Rigidity theorems for best Sobolev inequalities*. Submitted for publication, preprint available at arXiv:2206.12386.

[10] F. Maggi and C. Villani, *Balls have the worst best Sobolev inequalities*, J. Geom. Anal. **15** (2005), no. 1, 83–121.

[11] R Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), 479–495.

[12] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 573–598.

[13] C. Xia, *Complete manifolds with nonnegative Ricci curvature and almost best Sobolev constant*, Illinois J. Math. **45** (2001), 1253–1259.

An improved 4-periodicity theorem and its applications to positively curved manifolds with symmetry

Jan Nienhaus

When studying positively curved manifolds with torus actions one important tool is the 4-periodicity theorem of Lee Kennard [1]. It states that whenever $N^{n-k} \hookrightarrow M^n$ is a $\dim(N)$-connected inclusion of submanifolds with $k \leq \frac{n}{3}$, then $H^*(M, \mathbb{Q})$ is 4-periodic, i.e. is a rational sphere or admits an element $x \in H^4(M, \mathbb{Q})$ such that multiplication with x is an isomorphism in all degrees. This is a main ingredient of Kennard-Wiemeler-Wikling's rational cohomology classification of fixed point components of T^5-actions on positively curved manifolds [2]. They are there shown to have the cohomology rings of spheres or complex or quarternionic projective spaces. In particular, positively curved manifolds with symmetry rank at least 5 have $\chi(M) > 0$ in even dimensions.

When trying to improve these results, one finds the $k \leq \frac{n}{3}$-condition to be the main obstacle for going to T^4-symmetry. However, $S^8 \subset Ca\mathbb{P}^2$ gives an example with $k = \frac{n}{2}$ that is not 4-periodic. To circumvent this counterexample, one has to incorporate more geometric information: If $N \subset M$ is a fixed point component of some S^1-action, as is always the case in the applications, the isotropy action induces a complex structure on the normal bundle $\mathcal{V}N$. It turns out that this is already enough additional data. We prove that for $N \subset M$ $\dim(N)$-connected with $k \leq \frac{n}{2}$ and complex normal bundle, $H^*(M, \mathbb{Q})$ is 4-periodic. The main new idea is to move relations arising from the action of the Steenrod algebras on $H^*(BU, \mathbb{Z}_p)$ to the cohomology of M. Using this new 4-periodicity theorem we prove that fixed point components of effective isometric T^4-actions on positively curved closed manifolds have the rational cohomology ring of spheres or complex/quarternionic projective spaces. In particular, $\chi(M) > 0$ if n is even.

References

[1] L. Kennard, *On the Hopf conjecture with symmetry*, Geometry & Topology 17 (2013) 563–593.

[2] L. Kennard, M. Wiemeler, B. Wilking, *Splitting of torus representations and applications in the Grove symmetry program*, Arxiv 2021 (https://doi.org/10.48550/arXiv.2106.14723).

Inverse mean curvature flow and the relation to fast diffusion

Gerhard Huisken

(joint work with Panagiota Daskalopoulos)

The lecture explores inverse mean curvature flow

$$\frac{\partial}{\partial t} F = \frac{1}{H}\nu,\ F(\cdot, t)\colon \mathbb{R}^n \to \mathbb{R}^{n+1},$$

ν exterior unit normal, $H > 0$ mean curvature, for initial data that are entire graphs $M_0^n = \operatorname{graph} u_0$ with $u_0\colon \mathbb{R}^n \to \mathbb{R}$. We prove long time existence if the initial data have super-linear growth

$$u_0(x) \to \infty,\ |\nabla u(x)| \to \infty \text{ as } |x| \to \infty$$

such as $u(x) = |x|^q$, $q > 1$ and are δ-starshaped around some point $x_0 \in \mathbb{R}^{n+1}$:

$$0 < \delta \leq H\langle F - x_0, \nu\rangle$$

In case of linear growth, assuming u_0 is convex and

$$\alpha_0|x| \leq u_0(x) \leq \alpha_0|x| + k, 0 < k \in \mathbb{R}$$
$$0 < c_0 \leq H \cdot u \leq c_1 < \infty$$

We show that a fast diffusion effect occurs and $u(\cdot, t)$ converges in finite time to a horizontal plane in $C^{1,\alpha}_{loc}$ with height $h \in (0, k)$.

Stability of Einstein metrics and effective hyperbolization in large Hempel distance

Ursula Hamenstädt
(joint work with Frieder Jäckel)

The talk is devoted to the explaining the following result and its applications [2]. In its formulation, the *thick part* of a Riemannian manifold M is the subset of all points of injectivity radius bigger than a fixed *Margulis constant* for the dimension and the given curvature bounds. Its complement is the *thin part* of M.

Theorem 1. For $\epsilon > 0$, $b > 1$, $\delta < 2 - b$ consider a finite volume Riemannian manifold $(M, \overline{g})$ of dimension 3 with the following curvature properties.

(1) The sectional curvature is contained in the interval $[-1-\epsilon, -1+\epsilon]$.
(2) If d is the distance function, then for any $x \in M$, the Ricci curvature satisfies
$$e^{bd(x,\mathrm{M}_{\mathrm{thick}})}|\int e^{-(2-\delta)d(x,y)}|\mathrm{Ric} + 2\overline{g}|^2 dy| \le \epsilon^2.$$
(3) In the thin part of M, the curvature is constant -1.

There exists ϵ_0 such that if $\epsilon \le \epsilon_0$, then there exists a metric of constant curvature -1 which is close to $\overline{g}$ in the C^2-topology.

The result extends an unpublished result of Tian [3] and also relies on earlier work of Bamler [1].

A component of the thin part of a hyperbolic 3-manifold either is a solid torus whose core curve is a short closed geodesic, or a cusp, which is a submanifold diffeomorphic to $T^2 \times [0, \infty)$ where T^2 is a two-torus. In particular, the boundary of a Margulis tube is diffeomorphic to the boundary of a cusp. As a consequence, Margulis tubes can be surgered from the manifold and replaced by cusps, and cusps can be surgered from the manifold and replaced by Margulis tubes.

We explain how Theorem 1 can be used to proof the following drilling and filling theorem, improving earlier results of Brock and Bromberg, and Hodgson and Kerckhoff with a different proof.

Theorem 2.

(1) The manifold obtained from drilling sufficiently sparsely distributed Margulis tubes with sufficiently short core geodesics from a hyperbolic 3-manifold M admits a finite volume hyperbolic metric C^2-close to the original metric on the complement of the surgered tubes.
(2) The manifold obtained from removing sufficienlty sparsely distributed cusps from a hyperbolic 3-manifold M and gluing in solid tori whose meridians correspond to sufficiently long curves on the boundary tori admits a finite volume hyperbolic metric which is C^2-close to the original metric on the complement of the filled cusps.

We discuss how this drilling and filling result enters into the proof of the following second application of Theorem 1.

The *curve graph* of a closed surface is the graph whose vertices are simple closed curves and where two such curves are connected by an edge of length one if they can be realized disjointly. The *Hempel distance* of a closed 3-manifold M, glued from two handlebodies H_1, H_2 of genus $g \geq 2$ with an orientation reversing diffeomorphism $f : \partial H_1 \to \partial H_2$, is the minimal distance in $\partial H_1 = \partial H_2$ between a diskbounding simple closed curve in H_1 and a diskbounding simple closed curve in H_2.

Theorem 3. For all $g \geq 2$ there exists numbers $R = R(g) > 0, C = C(g) > 0$ with the following property. If the Hempel distance δ of a closed 3-manifold M, glued from two handlebodies of genus g, is at least R, then M admits a hyperbolic metric, and the volume of this metric is at least $C\delta$.

A much more general result in this direction is due to Perelman, but the proof of hyperbolization we give and the volume estimate are new.

References

[1] R. Bamler, *Construction of Einstein metrics by generalized Dehn filling*, Journal of the European Math. Soc. **14** (2012), 878–909.

[2] U. Hamenstädt, F. Jäckel, *Stability of Einstein metrics and effective hyperbolization in large Hempel distance*, arXiv:2206.10438.

[3] G. Tian, *A pinching theorem on manifolds with negative curvature*, unpublished manuscript.

Non-compact Einstein-manifolds with symmetry

Christoph Böhm

(joint work with Ramiro Lafuente)

One of the applications of Theorem 3 yields the proof of the long standing

Theorem 1 (Alekseevskii Conjecture)**.** A homogeneous Einstein manifold (M^n, g) with negative Einstein constant is diffeomorphic to $\mathbb{R}^n$.

The conjecture was known for $n \leq 5$ and $n = 7$ (Arroyo and Lafuente, 2017) and in certain cases (Jablonski, 2015). Is was not known whether $\mathrm{Sl}_{\mathbb{C}}(2)$ does admit a homogeneous Einstein metric. It is still open, whether $\mathrm{Sl}_{\mathbb{C}}(2)$ does admit a homogeneous metric with $\mathrm{ric}(g) < 0$. The case $\mathrm{Sl}_{\mathbb{R}}(3)$ was also open. It was known that $\mathrm{Sl}_{\mathbb{R}}(3)$ admits homogeneous metrics with $\mathrm{ric}(g) < 0$. Homogeneous Einstein metrics on $\mathbb{R}^n$ are isometric to Einstein solvmanifolds [BL22].

Conjecture (Dynamical Alekseevskii Conjecture)**.** Let M^n be a homogeneous, simply-connected manifold which is not contractible. Then all homogeneous Ricci flow solutions on M^n have finite extinction time.

It is known that such homogeneous spaces M^n admit a homogeneous metric g_+ with positive scalar curvature (for each representation $M^n = G/H$ indeed). The above conjecture then claims, that for any other homogeneous initial metric g_0 on M^n, possibly having negative scalar curvature or even negative Ricci curvature, the corresponding Ricci flow solution has finite extinction time. Clearly, a homogeneous Einstein metric g_- with negative Einstein constant, which does not exist by Theorem 1, would provide a counterexample to the dynamical Alekseevskii Conjecture. The dynamical Alekseevskii Conjecture is known to be true for compact homogeneous spaces [B15].

Surprisingly, to prove the Alekseevskii Conjecture, we have to consider the following much more general situation. We will assume that a (connected) Lie group G acts on (M^n, g) isometrically (almost effectively and properly) and that all orbits are principal. In this case the orbit space

$$B = M^n/G$$

is a smooth manifold (no boundary, $\dim B = d = n - \dim G$). We call an isometric action of G on (M^n, g) *principal with compact base* if all orbits are principal and the orbit space B is compact.

Example: Let B be a compact (smooth) manifold, G be a non-compact Lie group and set

$$M^n := G \times B\,.$$

Theorem 2. If (M^n, g) admits an isometric G-principal action with compact base and if $\mathrm{ric}(g) < 0$, then G is non-unimodular or semisimple.

A Lie group G is unimodular if the Haar measure of G is right-invariant. The subgroup $\Delta(m)$ of upper triangular matrices in $\mathrm{Sl}_{\mathbb{R}}(m)$ is non-unimodular. Nilpotent Lie groups are unimodular. The subgroup $N(m)$ of $\Delta(m)$ with diagonal elements all equal to 1 is nilpotent. A Lie group G is semisimple if and only if the nilradical N, (the largest normal nilpotent subgroup of G) vanishes. Semisimple Lie groups are classified, and they are unimodular. $\mathrm{Sl}_{\mathbb{R}}(m)$ is semisimple. The case $\dim B = 0$ is due to Dotti-Miatello (1984).

Theorem 3. If an Einstein manifold (M^n, g) with $\lambda = -1$ admits an isometric G-principal action with compact base, G non-unimodular, then the nilradical N of G acts polarly on M^n and the N-orbits are nilsolitons.

An action is polar if and only if the horizontal distribution $\mathcal{H} = \mathcal{V}^\perp$ is integrable, $\mathcal{V}_p = T_p(N.p)$ for all $p \in M^n$. A nilsoliton is a left-invariant metric on N, which is a (non-gradient) Ricci soliton. Theorem 3 generalizes the famous result of J. Lauret [L10] that Einstein solvmanifolds are standard. There exist examples of product Einstein manifolds $(M^n = G \times B, g)$ as in Theorem 3.

Under the above assumptions, considering the Riemannian submersion given by the N-action on (M^n, g), the Ricci tensor $\mathrm{ric}(g)$ of (M, g) is given by

$$\begin{aligned}
\mathrm{ric}(U,U) &= \mathrm{ric}^F(U,U) + \langle L_Z U, U\rangle \\
&\quad + \langle A\,U, A\,U\rangle - \sum_{j=1}^{d} \langle (\nabla_{X_j} L)_{X_j} U, U\rangle\,, \\
\mathrm{ric}(U,X) &= -\sum_{i=1}^{n-d} \langle (\nabla_{U_i} T)_{U_i} U, X\rangle + \langle \nabla_U Z, X\rangle \\
&\quad + \sum_{j=1}^{d} \langle (\nabla_{X_j} A)_{X_j} X, U\rangle - 2\,\langle A_X, T_U\rangle, \\
\mathrm{ric}(X,X) &= \overline{\mathrm{ric}}(\bar X, \bar X) - 2\,\|A_X\|^2 - \|L_X\|^2 + \langle \nabla_X Z, X\rangle\,.
\end{aligned}$$

Here L_X denotes the shape operator of the fibres F in the normal direction X; X horizontal vector field, often even basic. Note: Basic vector fields are N-invariant. Z denotes the mean curvature vector of the fibres F. ric^F denotes the Ricci tensor of the fibres, $(F, g_F = g|_{TF})$. $\overline{\mathrm{ric}}$ denotes the Ricci tensor of the base $(B, \bar g)$. Note that apriori we know nothing about the geometriy of the base. A denotes the A-tensor, measuring the integrability of the horizontal distribution.

We are able to obtain sharp estimates for the above equations, using sharp estimates for ric^F, coming from real GIT (geometric invariant theory), a generalized Helmholtz-decomposition for smooth vector fields on compact Riemannian manifolds and the Bochner formula for $\overline{\mathrm{ric}}$.

If G is semisimple, we use the Iwasawa decomposition of G, which enables us to apply Theorem 3 also in this case.

References

[B15] C. Böhm, *On the long time behavior of homogeneous Ricci flows*, Comment. Math. Helv. 90 (2015), no. 3, 543–571.

[BL22] C. Böhm, R. Lafuente, *Homogeneous Einstein metrics on Euclidean spaces are Einstein solvmanifolds*, Geom. Topol. 26 (2022), no. 2, 899–936.

[BL21] C. Böhm, R. Lafuente, *Non-compact Einstein manifolds with symmetry*, arXiv:2107.04210.

[L10] J. Lauret, *Einstein solvmanifolds are standard*, Ann. of Math. (2) 172 (2010), no. 3, 1859–1877.

Harmonic maps and extremal Schrödinger operators

Daniel Stern

(joint work with Mikhail Karpukhin)

Questions about the existence and regularity of harmonic maps have inspired many important developments in geometric analysis since the 1960s, beginning with the foundational work of Eells and Sampson [EeSa64] proving existence of harmonic maps minimizing the energy in any homotopy class of maps from a given closed manifold M to a closed target N of nonpositive sectional curvature. In the 1970s, major breakthroughs in the existence theory for harmonic maps from surfaces M^2 were made by several authors, most notably Sacks and Uhlenbeck [SaUhl77].

In the special case of sphere-valued maps from surfaces $u : M^2 \to S^k$, it was observed in [ESI08, Na96] that harmonic maps correspond to critical points for the eigenvalue functionals

$$\bar{\lambda}_j(M, g) := \lambda_j(\Delta_g) \cdot Area(M, g)$$

in a fixed conformal class $[g]$, via the assignment $u \mapsto g_u = \frac{1}{2}|du|_g^2 g$. In particular, maximization of $\bar{\lambda}_j$ in a given conformal class gives rise to a harmonic map $u : (M^2, [g]) \to S^k$ into some sphere S^k . In an earlier paper [KS20], we studied a natural min-max construction for harmonic maps $M^2 \to S^k$ from surfaces to spheres, and showed that the construction stabilized for k sufficiently large, yielding a metric maximizing the normalized first Laplace eigenvalue $\bar{\lambda}_1(M, g)$ in a given conformal class. In addition to giving a new existence proof for $\bar{\lambda}_1$-maximizing metrics (previously established in [Pet14] and [KNPP20]), our approach yielded regularity and stability results for a generalized version of the $\bar{\lambda}_1$-maximization problem [KNPS], with applications to variational problems for Steklov eigenvalues and free boundary minimal surfaces [KS21].

The initial motivation for the work described in this talk was an attempt to generalize the results of [KS20] to higher dimensions–namely, to produce a canonical family of sphere-valued harmonic maps on any Riemannian manifold, and relate these maps to a spectral optimization problem generalizing the conformal maximization of $\bar{\lambda}_1(M, g)$ from the two-dimensional setting. In the course of solving this problem, we also obtained new existence and partial regularity results for harmonic maps from higher-dimensional manifolds to a larger class of targets containing the round spheres: in particular, while the results of [EeSa64] deliver a satisfactory existence theory for harmonic maps from higher-dimensional manifolds into targets of *nonpositive* sectional curvature, our results show that the space of harmonic maps from higher-dimensional manifolds into targets satisfying certain curvature *positivity* conditions can be fruitfully explored via Morse-theoretic methods.

Building on work of Lin [Lin99] and Hsu [Hsu05], our work begins with the observation that, for any closed manifold N^k containing *no stable minimal two-spheres*–e.g., 3-manifolds N of positive Ricci curvature, or $4 \leq k$-manifolds N^k with positive isotropic curvature–the space of stationary harmonic maps satisfying simultaneous bounds on *energy* $E(u)$ and *Morse index* $ind_E(u)$ is strongly compact in the $W^{1,2}$ sense. Moreover, similar compactness results hold for certain relaxations of the harmonic map problem of the kind appearing in [CS], allowing us to produce stationary harmonic maps with Morse index bounds by min-max methods. As an application, we obtain the following existence result.

Theorem 1. [KS22] Let (M^n, g) be an arbitrary closed Riemannian manifold of dimension $n \geq 3$, and let (N, h) be a closed Riemannian manifold containing no stable minimal 2-spheres. Then if $\pi_\ell(N) \neq 0$ for some $\ell \geq 3$, there exists a nonconstant stationary harmonic map $u : M \to N$ of Morse index $ind_E(u) \leq \ell+1$, smooth away from a singular set $Sing(u)$ of dimension

$$\dim(Sing(u)) \leq n - m \leq n - 3,$$

where m is the smallest integer for which there exists a nonconstant stable 0-homogeneous harmonic map $\phi : \mathbb{R}^m \to N$.

In the special case where $N = S^k$ for $k \geq 3$, work of Schoen-Uhlenbeck [SchUhl84] and Lin-Wang [LW] gives improved partial regularity results for locally stable stationary harmonic maps. Combining their work with the preceding theorem in the case $N = S^k$, $\ell = k$, we obtain the following result, generalizing the construction of [KS20] to higher dimension.

Theorem 2. [KS22] On any closed Riemannian manifold (M^n, g) of dimension $n \geq 3$, for every $k \geq 3$ there exists a nonconstant stationary harmonic map $u_k : M \to S^k$ of Morse index $ind_E(u) \leq k+1$, smooth away from a singular set $Sing(u)$ of dimension

$$\dim(Sing(u_k)) \leq n - k - 1 \text{ for } 3 \leq k \leq 5,$$
$$\dim(Sing(u_k)) \leq n - 6 \text{ for } 6 \leq k \leq 9,$$

and

$$\dim(Sing(u_k)) \leq n - 7 \text{ for } 10 \leq k.$$

In particular, on manifolds (M^n, g) of dimension $3 \leq n \leq 5$, these maps are smooth as soon as $k \geq n$. Moreover, in these dimensions, we can indeed relate these harmonic maps to an intrinsic variational problem for certain elliptic operators on (M^n, g), generalizing the conformal maximization of $\bar{\lambda}_1$ in the surface case. By work of Grigor'yan-Netrusov-Yau [GNY] and Grigor'yan-Nadirashvili-Sire [GNS], on any Riemannian manifold (M^n, g), the integral $\int V$ of a bounded function $V \in L^\infty(M)$ can be bounded above in terms of the geometry (M, g) and the index $ind(L_V)$ of the associated Schrödinger operator $L_V = \Delta_g - V$, where $\Delta_g = d^*d$ is the positive-spectrum Laplacian on (M, g). In other words, for every $m \in \mathbb{N}$, the quantity

$$\mathcal{V}_m(M, g) := \sup\{\int_M V dvol_g \mid ind(L_V) \leq m\}$$

is a finite geometric invariant. In dimension two, one can see using the conformal covariance of the Laplacian that

$$\mathcal{V}_m(M, g) = \sup_{\tilde{g} \in [g]} \bar{\lambda}_m(M, \tilde{g}),$$

and the existence of potentials realizing $\mathcal{V}_m$ is equivalent to the existence of conformally maximizing metrics for $\bar{\lambda}_m(M, g)$.

After a delicate analysis showing that the harmonic maps $u_k : M^n \to S^k$ constructed above stabilize in a certain sense as $k \to \infty$, we are able to show the following.

Theorem 3. [KS22] On any closed manifold (M^n, g) of dimension $3 \leq n \leq 5$, there exists a Schrödinger operator L_V of smooth, nonnegative potential $V \in C^\infty(M)$ with index $ind(L_V) = 1$ such that $\mathcal{V}_1(M, g) = \int V$. For k sufficiently large, we in fact have $V = |du_k|_g^2$, where $u_k : M^n \to S^k$ is the harmonic map of index $ind_E(u_k) \leq k+1$ constructed in the previous theorem, and any other Schrödinger operator realizing $\mathcal{V}_1(M, g)$ must arise in the same way from the energy density of a smooth sphere-valued harmonic map.

References

[CS] Y.M. Chen, M. Struwe, Existence and partial regularity results for the heat flow for harmonic maps, *Math. Z.* **201**:1 (1989), 83–103.

[EeSa64] J. Eells, J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., **86** (1964), 109–160.

[ESI08] A. El Soufi, S. Ilias, *Laplacian eigenvalues functionals and metric deformations on compact manifolds*, J. Geom. Phys. **58**:1 (2008), 89–104.

[GNS] A. Grigor'yan, N. Nadirashvili, Y. Sire, *A lower bound for the number of negative eigenvalues of Schrödinger operators*, J. Differential Geom., **102**:3, 395–408.

[GNY] A. Grigor'yan, Y. Netrusov, S. T. Yau, *Eigenvalues of elliptic operators and geometric applications*, Surveys in differential geometry, **9**:1 (2004), 147–217.

[Hsu05] D. Hsu, *An approach to the regularity for stable-stationary harmonic maps*, Proc. Amer. Math. Soc., **133**:9 (2005), 2805–2812

[KNPP20] M. Karpukhin, N. Nadirashvili, A. Penskoi, I. Polterovich, *Conformally maximal metrics for Laplace eigenvalues on surfaces*, Surveys in Differential Geometry **24**:1 (2019), 205 – 256. Preprint `arXiv:2003.02871`.

[KNPS] M. Karpukhin, M. Nahon, I. Polterovich, D. Stern, *Stability of isoperimetric problems for Laplace eigenvalues on surfaces*, Preprint `arXiv:2106.15043`

[KS20] M. Karpukhin, D. Stern, *Min-max harmonic maps and a new characterization of conformal eigenvalues*, Preprint `arXiv:2004.04086`.

[KS21] M. Karpukhin, D. Stern, *From Steklov to Laplace: free boundary minimal surfaces with many boundary components*. Preprint `arXiv:2109.11029`.

[KS22] M. Karpukhin, D. Stern, *Harmonic maps and extremal Schrödinger operators on higher-dimensional manifolds.*, to appear

[Lin99] F.H. Lin, *Gradient estimates and blow-up analysis for stationary harmonic maps*, Ann. of Math. **149**:3 (1999), 785–829.

[LW] F.H. Lin, C-Y Wang, *Stable stationary harmonic maps to spheres*, Acta Math. Sinica **22** (2006), 319–330.

[Na96] N. Nadirashvili, *Berger's isoperimetric problem and minimal immersions of surfaces*, Geom. Func. Anal. **6**:5 (1996), 877–897.

[Pet14] R. Petrides, *Existence and regularity of maximal metrics for the first Laplace eigenvalue on surfaces*, Geom. Funct. Anal., **24**:4 (2014), 1336–1376.

[SaUhl77] J. Sacks, K. Uhlenbeck, *The existence of minimal immersions of 2-spheres.* Annals of Mathematics (1981), 1–24.

[SchUhl84] R. Schoen, K. Uhlenbeck, *Regularity of minimizing harmonic maps into the sphere,* Invent. Math. **78**:1 (1984), 89–100.

Prescribed Mean Curvature Min-Max Theory in Some Non-compact Manifolds

Liam Mazurowski

Let (M, g) be a Riemannian manifold and let $h\colon M \to \mathbb{R}$ be a smooth function. Let Ω be an open region in M with smooth boundary Σ. The hypersurface Σ is said to have mean curvature prescribed by h provided $H_\Sigma(x) = h(x)\nu(x)$ for every

$x \in \Sigma$. Here H_Σ denotes the mean curvature vector of Σ and ν is the inward pointing normal as determined by Ω.

A basic problem in differential geometry is to construct prescribed mean curvature surfaces in a given manifold M. One approach to this problem is variational: a hypersurface $\Sigma = \partial\Omega$ has mean curvature prescribed by h provided Ω is a critical point of the functional

$$A^h(\Omega) = \text{Area}(\partial\Omega) - \int_\Omega h.$$

Thus to find prescribed mean curvature surfaces in M, it is enough to find (smooth) critical points of the A^h functional. In the minimal case $h \equiv 0$, the Almgren-Pitts min-max theory is a powerful tool for finding critical points of the area functional. It was first developed in the early 1980s by Almgren [1], Pitts [5], and Schoen-Simon [6] and has since been extensively developed by Marques, Neves, and others (see, for example, [2] and [3]). Their combined work gives a very detailed understanding of the Morse theory of the area functional on a closed manifold.

In the case of functions h, an analogous min-max theory was developed only comparatively recently by Zhou and Zhu (see [7], [8]). They proved the following result.

Theorem 1 (Zhou and Zhu)**.** Assume M is a closed Riemannian manifold of dimension between 3 and 7. Then for an admissible function $h\colon M \to \mathbb{R}$, there exists a smooth, almost-embedded hypersurface Σ in M with mean curvature prescribed by h.

Here the class of admissable functions includes the constant functions as well as functions which are never zero. Moreover, a generic choice of function h is admissable.

In this talk, we discuss a technique for applying Zhou and Zhu's min-max theory on certain non-compact manifolds. As an application, we give the following min-max theorem for constructing constant mean curvature surfaces in an asymptotically flat manifold [4].

Theorem 2. Assume (M^3, g) is a complete, asymptotically flat manifold with no boundary. Fix a constant $c > 0$. Assume the one parameter min-max value ω satisfies

$$\omega < \frac{4\pi}{3}\left(\frac{2}{c}\right)^2. \tag{1}$$

Then there exists a smooth, closed, almost-embedded hypersurface Σ of constant mean curvature c in M.

Here the one parameter min-max value ω is defined by

$$\omega = \inf_{\{\Omega_t\}_{t\in[0,1]}} \left[\sup_{t\in[0,1]} A^c(\Omega)\right]$$

where the infimum is taken over all smoothly varying families of open sets $\{\Omega_t\}_{t\in[0,1]}$ with $\Omega_0 = \emptyset$ and $A^c(\Omega_1) < 0$.

Finally we discuss some geometric conditions on M which may imply that hypothesis (1) holds. In particular, we conjecture that (1) holds whenever M has non-negative scalar curvature and is not isometric to Euclidean $\mathbb{R}^3$.

References

[1] F. Almgren, *The Theory of Varifolds*, mimeographed notes, 1965.

[2] F.C. Marques and A. Neves, *Min-max theory and the Willmore conjecture*, Ann. of Math., 179(2):683–782, 2014.

[3] F.C. Marques and A. Neves, *Morse index of multiplicity one min-max minimal hypersurfaces*, Adv. Math., 378, Paper No. 107527, 2021.

[4] L. Mazurowski, *Prescribed Mean Curvature Min-Max Theory in Some Non-compact Manifolds*, ArXiv:2204.07493, 2022.

[5] J. Pitts, *Existence and regularity of minimal surfaces on Riemannian manifolds*, Princeton University Press, 1981.

[6] R. Schoen and L. Simon, *Regularity of stable minimal hypersurfaces*, Comm. pure appl. math, 34:741–797, 1981.

[7] X. Zhou and J. Zhu, *Min-max theory for constant mean curvature hypersurfaces*, Invent. Math., 218(2):441–490, 2019.

[8] X. Zhou and J. Zhu, *Existence of hypersurfaces with prescribed mean curvature I - generic min-max*, Cambridge Journal of Mathematics, 8(2): 311–362, 2020.

On ALF Gravitational Instantons

Olivier Biquard

(joint work with Paul Gauduchon)

A *gravitational instanton* is a 4-dimensional Riemannian manifold (M^4, g), which is complete, Ricci flat, and has finite energy:

$$\int_M |R_m|^2 < \infty.$$

By Cheeger-Tian [3] one must have $|R_m| = O(r^{-2})$, where r is the distance to a base point.

There are four classes of gravitational instantons, depending on the dimension of asymptotic cones: dimensions 4, 3, 2, 1 correspond respectively to ALE (Asymptotically Locally Euclidean), ALF (Asymptotically Locally Flat), ALG and ALH gravitational instantons.

Kähler gravitational instantons (which is almost the same as hyperKähler gravitational instantons) are classified or almost classified: this began more than 30 years ago with the classification by Kronheimer [6] of ALE hyperKähler gravitational instantons and is being finished only nowadays after lots of works, see in particular the recent work of Sun and Zhang [8].

In this work we are interested in non Kähler gravitational instantons. The first examples are older and come from physics: the Schwarzschild metric, the Kerr metrics (of which Schwarzschild is a special case), the Taub-bolt metric. These metrics are ALF or even AF (Asymptotically Flat) in the following sense: we say that a Riemannian manifold (M^4, g) is *ALF* if

- it has one end diffeomorphic to $(A, +\infty) \times L$, where L is $S^1 \times S^2$ (in this case we say that (M^4, g) is AF), or S^3, or a finite quotient;
- there is a triple (η, T, γ) defined on L, where η is a 1-form, T a vector field such that $\eta(T) = 1$ and $T \lrcorner d\eta = 0$, and γ is a T-invariant metric on the distribution $\ker \eta$;
- the transverse metric γ has constant curvature $+1$;
- the metric g has the behaviour

$$g = dr^2 + r^2\gamma + \eta^2 + h, \text{ with } |\nabla^k h| = O(r^{-1-k}), \tag{1}$$

  where γ is extended to the whole TL be deciding that $\ker \gamma$ is generated by T, and the covariant derivative ∇ and the norm are with respect to the asymptotic model $dr^2 + r^2\gamma + \eta^2$.

The meaning of formula (1) is that at infinity the metric g looks locally like a product $\mathbb{R}^3 \times S^1$. But this is only a local picture, in particular the 'collapsed direction' T (this is locally the factor S^1) does not need to have closed orbits.

A breakthrough in the subject was made by Chen and Teo [5] who found a new family of AF gravitational instantons by finding explicit formulas, and our work originated from an attempt to understand better this example, and maybe construct other examples.

The Kerr metrics are defined on $\mathbb{R}^2 \times S^2$, and the Chen-Teo metrics are defined on a (complex) blowup of this manifold at a point. (The Taub-bolt metric is defined on the complex blowup of $\mathbb{R}^4$ at the origin). They are all toric, that is have an isometric action of a 2-torus.

Let us say that a metric is *Hermitian* if it is conformal to a Kähler metric. All the classical examples (Kerr, Taub-bolt) are *Type D*, which means that they are Hermitian for both orientiations. It was observed by Andersson and Aksteiner [1] that the Chen-Teo metrics are also Hermitian, but only for one orientation. We can also add to this list the Taub-NUT metric, which is an hyperKähler ALF metric on $\mathbb{R}^4$, but it is also Hermitian non-Kähler with respect to the opposite orientation.

In [2] we classify these metrics and prove that no new smooth example can exist; nevertheless we also produce new examples with conical singularities along 2-spheres:

Theorem A.

1) Suppose (M^4, g) is a smooth Hermitian, toric, ALF gravitational instanton. If g is not Kähler, then g is a Kerr metric, a Chen-Teo metric, the Taub-NUT metric (with the orientation opposed to the hyperKähler orientation) or the Taub-bolt metric.

2) There exist an infinite number of Hermitian, toric, ALF gravitational instantons (with infinitely many different topologies), with conical singularities along 2-spheres.

There is a curious duality between this classification of toric, Hermitian, Ricci flat metrics, and that of compact, Hermitian, Einstein 4-manifolds with positive Einstein constants [7, 4]: the examples are $\mathbb{C}P^2$ (with the reverse orientation), the Page metric on the blowup of $\mathbb{C}P^2$ at one point, and the Chen-LeBrun-Weber metric on the blowup of $\mathbb{C}P^2$ at two points. This is exactly similar to the Taub-NUT, Kerr, and Chen-Teo metrics, but we have found no explanation for this unexpected duality. (The Taub-bolt does not seem to fit in the picture but is actually a degeneration of the Chen-Teo family).

The proof of the Theorem is based on two reductions:

(1) A reduction to toric Kähler geometry: it turns out that the conformal Kähler metric has to be extremal and Bach flat, both conditions having nice interpretations in the setting of toric Kähler geometry and the formalism of Delzant polytopes, symplectic potential, etc. conversely, a suitable extremal, Bach flat, toric Kähler metric will give rise to a gravitational instanton, so the problem is completely reduced to a problem in toric Kähler geometry.

(2) Despite everything known in Kähler geometry on the extremal metric problem (Yau-Tian-Donaldson conjecture), it turns out that it is not possible to use this theory in our case; fortunately the Bach flat equation in addition to the extremal Kähler equation is so strong that solutions can be given in terms of an ansatz of Tod [9], of which we give a proof in terms of older ansatz of LeBrun and Ward. It then turns out that the solutions are generated from an axisymmetric harmonic function on $\mathbb{R}^3$, and the singularity on the axis gives a convex, piecewise, linear function on $\mathbb{R}$. From this combinatorial object we can reconstruct the whole structure and in particular find when the corresponding Ricci flat metric is smooth, or has conical singularities. This leads to the classification theorem.

References

[1] S. Aksteiner and L. Andersson. Gravitational instantons and special geometry. arXiv:2112.11863.

[2] O. Biquard and P. Gauduchon. On toric Hermitian ALF gravitational instantons. arXiv:2112.12711.

[3] J. Cheeger and G. Tian. Curvature and injectivity radius estimates for Einstein 4-manifolds. *J. Am. Math. Soc.*, 19(2):487–525, 2006.

[4] X. Chen, C. LeBrun, and B. Weber. On conformally Kähler, Einstein manifolds. *J. Am. Math. Soc.*, 21(4):1137–1168, 2008.

[5] Y. Chen and E. Teo. A new AF gravitational instanton. *Phys. Lett. B*, 703(3):359–362, 2011.

[6] P. B. Kronheimer. A Torelli-type theorem for gravitational instantons. *J. Differential Geom.*, 29(3):685–697, 1989.

[7] C. LeBrun. Einstein metrics on complex surfaces. In *Geometry and physics. Proceedings of the conference at Aarhus University, Aarhus, Denmark, 1995*, pages 167–176. New York, NY: Marcel Dekker, 1997.

[8] S. Sun and R. Zhang. Collapsing geometry of hyperkaehler 4-manifolds and applications. arXiv:2108.12991.

[9] P. Tod. One-sided type-D Ricci-flat metrics. arXiv:2003.03234.

Conformally Invariant Random Geometry on Manifolds of Even Dimension

Karl-Theodor Sturm
(joint work with Lorenzo Dello Schiavo, Ronan Herry, Eva Kopfer)

For large classes of *even-dimensional* Riemannian manifolds (M, g), we construct and analyze conformally invariant random fields. These centered Gaussian fields $h = h_g$, called *co-polyharmonic Gaussian fields*, are characterized by their covariance kernels k which exhibit a precise logarithmic divergence: $\left|k(x,y) - \log \frac{1}{d(x,y)}\right| \leq C$. They share the fundamental quasi-invariance property under conformal transformations: if $g' = e^{2\varphi} g$, then

$$h_{g'} \overset{(\mathrm{d})}{=} \mathrm{e}^{n\varphi}\, h_g - C \cdot \mathrm{Vol}_{g'}$$

with an appropriate random variable $C = C_\varphi$.

In terms of the co-polyharmonic Gaussian field h, we define the *Liouville Quantum Gravity measure*, a random measure on M, heuristically given as

$$d\mu_g^h(x) := e^{\gamma h(x) - \frac{\gamma^2}{2k(x,x)}}\, d\mathrm{Vol}_g(x)\,,$$

and rigorously obtained as almost sure weak limit of the right-hand side with h replaced by suitable regular approximations $h_\ell, \ell \in \mathbb{N}$. These measures share a crucial quasi-invariance property under conformal transformations: if $g' = e^{2\varphi} g$, then

$$d\mu_{g'}^{h'}(x) \overset{(\mathrm{d})}{=} \mathrm{e}^{F^h(x)}\; d\mu_g^h(x)$$

for an explicitly given random variable $F^h(x)$.

In terms on the Liouville Quantum Gravity measure, we define the *Liouville Brownian motion* on M and the *random GJMS operators*. Finally, we present an approach to a conformal field theory in arbitrary even dimensions with an ansatz based on Branson's Q-curvature: we give a rigorous meaning to the *Polyakov–Liouville measure*

$$d\boldsymbol{\nu}_g^*(h) = \frac{1}{Z_g^*} \exp\left(-\int \Theta\, Q_g h + m e^{\gamma h} d\mathrm{Vol}_g\right) \exp\left(-\frac{a_n}{2}\mathfrak{p}_g(h,h)\right) dh$$

for suitable positive constants Θ, m, γ and a_n, and we derive the corresponding *conformal anomaly*.

The set of *admissible* manifolds is conformally invariant. It includes all compact 2-dimensional Riemannian manifolds, all compact non-negatively curved Einstein manifolds of even dimension, and large classes of compact hyperbolic manifolds of even dimension. However, not every compact even-dimensional Riemannian manifold is admissible.

Our results concerning the logarithmic divergence of the kernel k — defined as the Green kernel for the GJMS operator on (M, g) — rely on new sharp estimates for heat kernels and higher order Green kernels on arbitrary compact manifolds.

References

[1] Dello Schiavo, L., Kopfer, E., and Sturm, K.-T. A Discovery Tour in Random Riemannian Geometry. *arXiv:2012.06796*, 2020.
[2] Dello Schiavo, L., Herry, R., Kopfer, E., and Sturm, K.-T. Conformally Invariant Random Fields, Liouville Quantum Gravity measures, and Random Paneitz Operators on Riemannian Manifolds of Even Dimension. *arXiv:2105.13925*, 2021.
[3] Dello Schiavo, L., Herry, R., Kopfer, E., and Sturm, K.-T. Polyharmonic Fields and Liouville Quantum Gravity Measures on Tori of Arbitrary Dimension: from Discrete to Continuous. *Preprint*, 2022.

The geometry of metrics with $\lambda_1(-\Delta + \frac{1}{2}R) > 0$ and applications

Chao Li
(joint work with Otis Chodosh)

Consider a smooth Riemannian manifold (M^n, g) (possibly with nonempty smooth boundary). Let R denote the scalar curvature of g. We study the condition

$$\lambda_1(-\Delta + \tfrac{1}{2}R) > 0 \tag{1}$$

on (M, g). Here λ_1 denotes the first eigenvalue of the self-adjoint elliptic operator, with Dirichlet boundary condition if $\partial M \neq \emptyset$.

One motivation of investigating (1) comes from the study of two-sided stable minimal hypersurfaces in manifolds with positive scalar curvature. Indeed, suppose $M^n \to (X^{n+1}, g_X)$ is a two-sided stable minimal immersion (here stablility means $\delta^2|M|(f) \geq 0$ for any normal variation f vanishing on ∂M). A well-known trick due to Schoen–Yau implies that

$$\int_M |\nabla f|^2 + \frac{1}{2}R_M f^2 \geq \int_M \frac{1}{2}R_X f^2, \quad \forall f \in C^1_0(M).$$

Thus, the variational characterization of the first eigenvalue implies

$$\lambda_1(-\Delta + \tfrac{1}{2}R_M) \geq \inf_M R_X,$$

and hence (1) holds if $R_X > 0$ everywhere.

A quick observation here is that when $n \geq 3$, the conformal Laplacian of (M^n, g) is $-\Delta + \frac{n-2}{4(n-1)}R$. Since $\frac{n-2}{4(n-1)} < \frac{1}{2}$, we have that

$$\lambda_1(-\Delta + \tfrac{1}{2}R) > 0 \Rightarrow \lambda_1(-\Delta + \tfrac{n-2}{4(n-1)}R) > 0.$$

Therefore, (1) implies that (M, g) is conformal to a manifold with positive scalar curvature. This motivates us to study condition (1) using techniques from positive scalar curvature.

Indeed, joint with Chodosh [1], we showed the following metric property for 3-manifolds with (1), which extends earlier results of Gromov [7].

Theorem 1. Suppose (M^3, g) is a 3-manifold, possibly with compact nonempty boundary, such that

$$\lambda_1(-\Delta + \tfrac{1}{2}R) \geq \lambda > 0, \tag{2}$$

and suppose that there exists $p \in M$ such that $d_g(p, \partial M) \geq \frac{5\pi}{\sqrt{\lambda}}$. Then there exists an open set Ω containing ∂M, $\Omega \subset B_{\frac{5\pi}{\sqrt{\lambda}}}(\partial M)$, such that each connected component of $\partial\Omega \setminus \partial M$ is a 2-sphere with area at most $\frac{8\pi}{\lambda}$ and intrinsic diameter at most $\frac{2\pi}{\sqrt{\lambda}}$.

Previously, Theorem 1 was been applied to study the topology of manifolds with positive scalar curvature [1, 4]. We now apply Theorem 1 to an entirely different problem - the stable Bernstein conjecture for minimal surfaces.

Conjecture 2. A complete, two-sided, stable minimal immersion $M^n \to (\mathbf{R}^{n+1}, \delta)$ is flat.

It is well-known that the conjecture is falst when $n \geq 7$. Previous, the case when $n = 2$ was solved independently by Fischer-Colbrie–Schoen [6], do Carmo–Peng [5] and Pogorelov [8]. The case when $n = 3$ was conjectured in the affirmative by Schoen, and was recently verified by Chodosh and the author [2]. When $2 \leq n \leq 5$, Conjecture 2 was proved by Schoen–Simon–Yau [9] with the additional assumption that M has intrinsic Euclidean volume growth, that is,

$$\sup_{\rho > 0} \frac{|B_M(0, \rho)|}{\rho^n} < \infty.$$

In [3], we studied the $n = 3$ case of Conjecture 2 using Theorem 1, and showed the following result.

Theorem 3. Let $M^3 \to \mathbf{R}^4$ be a complete, two-sided, simply connected, stable minimal immersion, $0 \in M$. Then

$$|B_M(0, \rho)| \leq \left(\frac{32\pi}{3}\right)^{\frac{3}{2}} \frac{e^{\frac{30\pi}{\sqrt{3}}}}{6\sqrt{\pi}} \rho^3,$$

for all $\rho > 0$.

When combined with the classical result due to Schoen–Simon–Yau [9], this gives an alternative proof of our recent stable Bernstein theorem when $n = 3$. Moreover, the approach is flexible enough to handle stable solutions to anisotropic functional. Precisely, consider $\Phi : \mathbf{R}^4 \setminus \{0\} \to (0, \infty)$ a 1-homogeneous C^3_{loc} function. For $M^3 \to \mathbf{R}^4$ a two-sided immersion with a chosen unit normal vector field $\nu(x)$, consider the anisotropic area functional

$$\mathbf{\Phi}(M) = \int_M \Phi(\nu(x)) dx.$$

Theorem 4. Assume that Φ satisfies

$$|v|^2 \leq D^2\Phi(\nu)(v, v) \leq \sqrt{2}|v|^2,$$

for any $v \in \nu^\perp$. Consider $M^3 \to \mathbf{R}^4$ a complete, two-sided, $\mathbf{\Phi}$-stationary and stable immersion. Suppose $0 \in M$, and M is simply connected. Then there exist explicit constants $V_0 = V_0(\|\Phi\|_{C^1(S^3)}), Q > 0$ such that

$$|B_M(0, \rho)| \leq V_0 \rho^3, \text{ for all } \rho > 0.$$

Combined with a previous of Winklmann [10], Theorem 4 implies the stable Bernstein theorem for anisotropic minimal hypersurfaces in $\mathbf{R}^4$, as long as Φ is C^4-close to the area functional (with explicit numerical estimates on the closeness).

References

[1] O. Chodosh and C. Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, arXiv 2008.11888, 2020.
[2] O. Chodosh and C. Li, *Stable minimal hypersurfaces in* $\mathbf{R}^4$, arXiv 2108.11462, 2021.
[3] O. Chodosh and C. Li, *Stable anisotropic minimal hypersurfaces in* $\mathbf{R}^4$, arXiv 2206.06394.
[4] O. Chodosh, C. Li and Y. Liokumovich, *Classifying sufficiently connected psc manifolds in 4 and 5 dimensions*, Geom. Topo., to appear.
[5] M. do Carmo and C.K. Peng, *Stable complete minimal surfaces in* $\mathbf{R}^3$ *are planes*, Bull. Amer. Math. Soc. (N.S.), 1(6): 903–906, 1979.
[6] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math., 33(2): 199–211, 1980.
[7] M. Gromov, *Metric inequalities with scalar curvature*, Geom. Funct. Anal., 28 (3): 645–726, 2018.
[8] A.V. Pogorelov, *On the stability of minimal surfaces*, Soviet Math. Dokl., 24: 274–276, 1981.
[9] R. Schoen, L. Simon and S.T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math., 134 (3-4): 275–288, 1975.
[10] S. Winklmann, *Pointwise curvature estimates for F-stable hypersurfaces*, Ann. Inst. H. Poincaré Anl. Non Linéaire, 22 (5): 543–555, 2005.

Short-time existence for Ricci flow

Peter Topping
(joint work with Hao Yin, Man-Chun Lee)

It has been understood since the beginning of Ricci flow theory [4] that given a smooth Riemannian metric on a closed smooth manifold, there exists a unique smooth Ricci flow continuation. However for many applications it is necessary to have existence for *noncompact* underlying manifolds, and/or with initial data that is rough or uncontrolled in one sense or another. In the talk I explained several instances of this.

It is a long-standing open problem as to whether one can start the Ricci flow with an arbitrary complete three-dimensional Riemannian manifold with nonnegative Ricci curvature. This problem became well known starting in the 1980s because being able to do so would give a classification of complete three-manifolds with non-negative Ricci curvature via an approach of Hamilton and Shi. Recently, with Man-Chun Lee, we proved much more than this if we also assume a Ricci pinching condition. More precisely, we proved:

Theorem 1. Suppose (M^3, g_0) is a complete non-compact three-dimensional Riemannian manifold with $\mathrm{Ric}(g_0) \geq \varepsilon\mathcal{R}(g_0) \geq 0$ for some $\varepsilon > 0$, where $\mathcal{R}$ is the scalar curvature. Then there exists $a = a(\varepsilon) > 0$ such that the Ricci flow has a complete long-time solution $g(t)$ starting from g_0 so that

(a) $\mathrm{Ric}(g(t)) \geq \varepsilon \mathcal{R}(g(t)) \geq 0$;
(b) $|\mathrm{Rm}(g(t))| \leq a t^{-1}$;

for all $(x,t) \in M \times (0,\infty)$.

As a corollary we were able to extend the applicability of earlier work of Lott [5] and Deruelle-Schulze-Simon [2] in order to fully settle Hamilton's pinching conjecture without additional hypotheses:

Corollary 2 (Hamilton's pinching conjecture)**.** Suppose (M^3, g_0) is a complete (connected) three-dimensional Riemannian manifold with $\mathrm{Ric} \geq \varepsilon \mathcal{R} \geq 0$ for some $\varepsilon > 0$. Then (M^3, g_0) is either flat or compact.

In the second part of the talk I focussed on two-dimensional Ricci flow. With Hao Yin we have proved an existence theorem starting with a (possibly noncompact) Riemann surface equipped with a nonatomic Radon measure that extends earlier work, especially [3].

Theorem 3 ([7])**.** Suppose M is any (connected, possibly noncompact) Riemann surface and μ is any (nonnegative) nontrivial Radon measure on M that is nonatomic in the sense that

$$\mu(\{x\}) = 0 \text{ for all } x \in M.$$

Define

$$T := \begin{cases} \frac{\mu(M)}{4\pi} & \text{if } M = \mathbb{C} \simeq \mathbb{R}^2 \\ \frac{\mu(M)}{8\pi} & \text{if } M = S^2 \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

Then there exists a smooth complete conformal Ricci flow $g(t)$ on M, for $t \in (0,T)$, such that the Riemannian volume measure $\mu_{g(t)}$ converges weakly to μ as $t \searrow 0$.

In the cases that $T < \infty$, as $t \uparrow T$ we have

$$\mathrm{Vol}_{g(t)}(M) = (1 - \tfrac{t}{T})\mu(M) \to 0.$$

Moreover, if μ has no singular part then $\mu_{g(t)} \to \mu$ in $L^1_{loc}(M)$. More generally, if Ω is the complement of the support of the singular part of μ, then $\mu_{g(t)} \llcorner \Omega \to \mu \llcorner \Omega$ in $L^1_{loc}(\Omega)$.

Two application areas of this theory were described. Because the theory is general enough to allow initial data with certain scale invariance, we are able to obtain a large new family of expanding Ricci solitons. The new examples change our intuition of what an expanding soliton typically looks like. They include the first known nongradient Kähler Ricci soliton.

As a second application, we show how to construct an example of nonuniqueness in Ricci flow when the initial data is only expected to be attained in a metric sense. Combining work with Yin [7] and with Lee [6], we construct a flow starting with the Euclidean plane $(\mathbb{R}^2, g_0)$ other than the static solution $g(t) \equiv g_0$, and with explicit curvature decay:

Theorem 4. There exists a smooth complete conformal Ricci flow $g(t)$ on $\mathbb{R}^2$, for $t > 0$, with $|K_{g(t)}| \leq \frac{1}{2t}$ such that

$$d_{g(t)} \to d_{g_0} \quad \text{locally uniformly on } \mathbb{R}^2 \times \mathbb{R}^2 \text{ as } t \searrow 0, \tag{2}$$

(where d_g denotes the Riemannian distance of a metric g) but so that $g(t)$ is not identically g_0.

This example settles the time zero regularity question of whether a smooth Ricci flow for positive time that attains smooth initial data in the metric sense (2) must necessarily be smooth down to time zero. It does not. On the other hand, Deruelle-Schulze-Simon [1] have proved that one does obtain time zero regularity if the flow satisfies both a uniform lower Ricci bound and C/t curvature decay. In particular, our example must not satisfy a lower Ricci bound.

Following on from these observations, with M.-C. Lee [6] we prove that a lower Ricci bound is sufficient for time zero regularity in dimensions 2 and 3, and give an analogous higher dimensional result. In particular, a flow that is PIC1 will necessarily have time zero regularity. As an application we prove that smooth Gromov-Hausdorff limits of (weakly) PIC1 manifolds are necessarily also (weakly) PIC1.

References

[1] A. Deruelle, F. Schulze and M. Simon, *On the regularity of Ricci flows coming out of metric spaces.* J. Eur. Math. Soc. **24** (2022) 2233–2277.

[2] A. Deruelle, F. Schulze and M. Simon, *Initial stability estimates for Ricci flow and three dimensional Ricci-pinched manifolds.* Preprint (2022). arXiv:2203.15313

[3] G. Giesen and P. M. Topping, *Existence of Ricci flows of incomplete surfaces.* Comm. Partial Differential Equations, **36** (2011) 1860–1880.

[4] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982) 255–306.

[5] J. Lott, *On 3-manifolds with pointwise pinched nonnegative Ricci curvature.* Preprint (2019). arXiv:1908.04715

[6] P. M. Topping and M.-C. Lee, *Time zero regularity of Ricci flow.* Preprint (2022). arXiv:2202.12564

[7] P. M. Topping and H. Yin, *Smoothing a measure on a Riemann surface using Ricci flow.* Preprint (2021). arxiv:2107.14686

Five-point Toponogov theorem

Anton Petrunin

(joint work with Nina Lebedeva)

Toponogov theorem provides an if-and-only-if condition on a metric on four-point space that admits an isometric embedding into a nonnegatively curved Riemannian manifold. The only-if part is proved by Toponogov, and the if part follows from a result of Abraham Wald [1, §7].

We show that the so-called Lang–Schroeder–Sturm inequality is the analogous condition for five-point spaces.

Namely, we prove that *a five-point metric space F admits an isometric embedding into a complete nonnegatively curved Riemannian manifold if and only if all Lang–Schroeder–Sturm inequalities hold in F.* The only-if part of this statement is well known, but the if part is new.

References

[1] A. Wald. *Begründung einer Koordinatenlosen Differentialgeometrie der Flächen.* Ergebnisse eines mathematischen Kolloquium **6** (1935), 24–46.

Classification of noncollapsed translators in $\mathbb{R}^4$

Robert Haslhofer
(joint work with Kyeongsu Choi, Or Hershkovits)

In the analysis of mean curvature flow it is crucial to understand ancient noncollapsed flows. We recall that a mean curvature flow M_t is called ancient if it is defined for all $t \ll 0$, and noncollapsed if it is mean-convex and there is an $\alpha > 0$ such that every point $p \in M_t$ admits interior and exterior balls of radius at least $\alpha/H(p)$. In particular, thanks to the work of White [14] it is known that all blowup limits of mean-convex mean curvature flow are ancient noncollapsed flows.

In a recent breakthrough, Brendle-Choi [2, 3] and Angenent-Daskalopoulos-Sesum [1] classified all ancient noncollapsed flows in $\mathbb{R}^3$ (and similarly in $\mathbb{R}^{n+1}$ under a uniform two-convexity assumption). Specifically, they showed that any such flow is either a flat plane, a round shrinking sphere, a round shrinking cylinder, a translating bowl soliton, or an ancient oval. This in turn has been generalized in our recent proof of the mean-convex neighborhood conjecture [5, 9]. In stark contrast, the classification of ancient noncollapsed flows in higher dimensions without two-convexity assumption has remained a widely open problem.

As an important first step towards overcoming this dimension barrier, we recently classified all ancient noncollapsed flows in $\mathbb{R}^4$ assuming self-similarity:

Theorem (Choi-H.-Hershkovits [7, 8])**.** Every noncollapsed translator in $\mathbb{R}^4$ is either $\mathbb{R}\times$2d-bowl, or the 3d round bowl, or belongs to the one-parameter family of 3d oval-bowls $\{M_k\}_{k\in(0,1/3)}$ constructed by Hoffman-Ilmanen-Martin-White [12].

As a corollary we obtain a classification of certain blowup limits in $\mathbb{R}^4$:

Corollary (Choi-H.-Hershkovits [7, 8])**.** For mean-convex mean curvature flow in $\mathbb{R}^4$ (or more generally in any 4-manifold), every type I blowup limit (ala Huisken) is either a round shrinking S^3, or a round shrinking $\mathbb{R} \times S^2$, or a round shrinking $\mathbb{R}^2 \times S^1$, and every type II blowup limit (ala Hamilton) is either $\mathbb{R}\times$2d-bowl, or the 3d round bowl, or belongs to the one-parameter family of 3d oval-bowls.

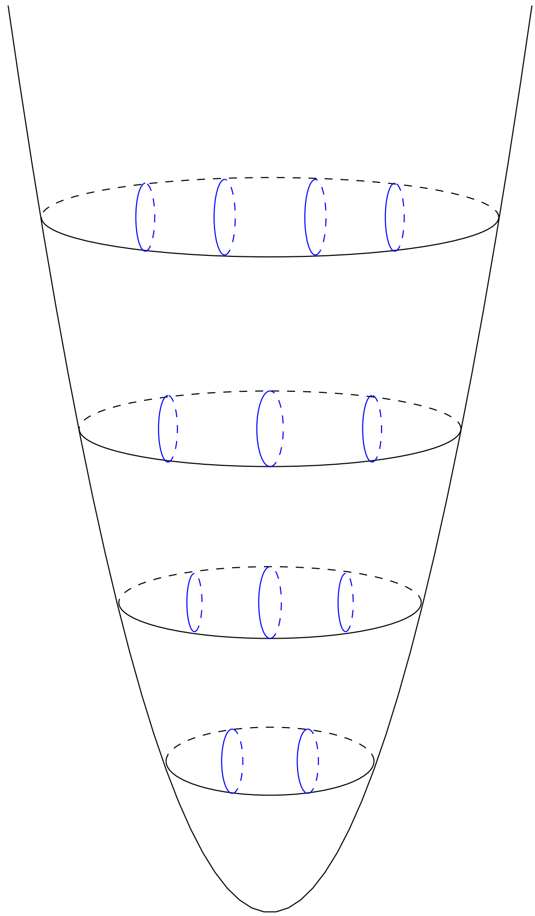

Figure 1. The oval-bowls $\{M_k\}_{k\in(0,1/3)}$ are 3-dimensional translators in $\mathbb{R}^4$, whose level sets look like 2d ovals in $\mathbb{R}^3$. They are parametrized in terms of the smallest principal curvature at the tip, and interpolate between the 3d round bowl and $\mathbb{R}\times$2d-bowl.

To sketch the main steps of the proof given a noncollapsed translator $M\subset\mathbb{R}^4$, that is neither $\mathbb{R}\times$2d-bowl nor 3d-bowl, we normalize without loss of generality such that $\mathbf{H}=e_4^\perp$. To begin with, by our no-wings theorem from [6], we have

$$\lim_{\lambda\to 0}\lambda M=\{\mu e_4|\mu\geq 0\}. \tag{1}$$

In particular, together with a recent result of Zhu [15] this yields SO(2)-symmetry. Hence, the level sets $\Sigma^h=M\cap\{x_4=h\}$ can be described by a renormalized profile function $v(y,\tau)$, where $\tau = -\log h$, whose analysis is governed by the one-dimensional Ornstein-Uhlenbeck operator $\mathcal{L} = \partial_y^2 - \frac{y}{2}\partial_y + 1$. Next, we show that $v(y,\tau)$ satisifies similar sharp asymptotics as the 2d ancient ovals in $\mathbb{R}^3$. We then establish a spectral uniqueness theorem, which says that if for two (suitably normalized) translators the difference of the profile functions $v_1 - v_2$ is perpendicular to the unstable and neutral eigenspace of $\mathcal{L}$, then the translators agree. We arrange this spectral condition using a delicate continuity argument. Finally, we relate the eccentricity at high levels and the tip curvature using a Rado-type argument and Lyaponov-Schmidt reduction and linearized variants of our estimates.

The result is part of a larger classification program for ancient noncollapsed flows in $\mathbb{R}^4$ that I recently introduced in joint work with Choi-Hershkovits [6] and Du [10]. In particular, in another paper with Du [11] we constructed a one-parameter family of $\mathbb{Z}_2^2\times \mathrm{O}(2)$-symmetric ancient ovals in $\mathbb{R}^4$, which can be viewed as compact counterpart of the HIMW-family. In forthcoming work we prove:

Theorem (Choi-Daskalopoulos-Du-H.-Sesum [4])**.** Every bubble-sheet oval for the mean curvature flow in $\mathbb{R}^4$, up to scaling and rigid motion, either is the $\mathrm{O}(2)\times\mathrm{O}(2)$-symmetric ancient oval from [14], or belongs to the one-parameter family of $\mathbb{Z}_2^2\times \mathrm{O}(2)$-symmetric ancient ovals constructed in [11].

Finally, it is tempting to conjecture that similar results hold for κ-solutions in 4d Ricci flow. In particular, concerning self-similar solutions I believe:

Conjecture. Every noncollapsed 4d steady Ricci soliton with nonnegative curvature operator is either $\mathbb{R}\times$3d-Bryant soliton, or the 4d Bryant soliton, or belongs to the one-parameter family of noncollapsed examples constructed by Lai [13].

References

[1] S. Angenent, P. Daskalopoulos, N. Sesum, *Uniqueness of two-convex closed ancient solutions to the mean curvature flow*, Ann. of Math. **192** (2020), 353–436.

[2] S. Brendle, K. Choi, *Uniqueness of convex ancient solutions to mean curvature flow in* $\mathbb{R}^3$, Invent. Math. **217** (2019), 35–76.

[3] S. Brendle, K. Choi, *Uniqueness of convex ancient solutions to mean curvature flow in higher dimensions*, Geom. Topol. **25** (2021), 2195–2234.

[4] B. Choi, P. Daskalopoulos, W. Du, R. Haslhofer, N. Sesum, *Classification of bubble-sheet ovals in* $\mathbb{R}^4$, in preparation.

[5] K. Choi, R. Haslhofer, O. Hershkovits, *Ancient low entropy flows, mean convex neighborhoods, and uniqueness*, Acta Math. (to appear)

[6] K. Choi, R. Haslhofer, O. Hershkovits, *A nonexistence result for wing-like mean curvature flows in* $\mathbb{R}^4$, preprint.

[7] K. Choi, R. Haslhofer, O. Hershkovits, *Classification of noncollapsed translators in* $\mathbb{R}^4$, preprint.

[8] K. Choi, R. Haslhofer, O. Hershkovits, *The linearized translator equation in* $\mathbb{R}^4$, in preparation.

[9] K. Choi, R. Haslhofer, O. Hershkovits, B. White, *Ancient asymptotically cylindrical flows and applications*, Invent. Math. **229** (2022), 139–241.

[10] W. Du, R. Haslhofer, *Hearing the shape of ancient noncollapsed flows in* $\mathbb{R}^4$, Comm. Pure Appl. Math. (to appear)

[11] W. Du, R. Haslhofer, *On uniqueness and nonuniqueness of ancient ovals*, preprint.

[12] D. Hoffman, T. Ilmanen, F. Martin, B. White, *Graphical translators for mean curvature flow* Calc. Var. Partial Differential Equations **58** (2019), Art. 117.
[13] Y. Lai, *A family of 3d steady gradient solitons that are flying wings*, J. Differential. Geom. (to appear)
[14] B. White, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **16** (2003), 123–138.
[15] J. Zhu, SO(2)*-symmetry of translating solitons of the mean curvature flow in* $\mathbb{R}^4$, preprint.

A modern take on the Bochner technique

Matthias Wink
(joint work with Jan Nienhaus, Peter Petersen)

The Bochner technique provides a fundamental connection between the geometry and the topology of a closed Riemannian manifold. In particular, in 1946 Bochner [Boc46] proved that manifolds with positive Ricci curvature have vanishing first Betti number. The key observation is that every harmonic form satisfies the elliptic equation

$$\Delta \frac{1}{2}|\omega|^2 = |\nabla \omega|^2 + g(\mathrm{Ric}_L(\omega), \omega).$$

For 1-forms, the curvature term of the Lichnerowicz Laplacian Ric_L is precisely Ricci curvature. However, for p-forms, $p \geq 2$, it depends on the entire curvature operator of the Riemannian manifold. Until recently, results based on the Bochner technique that control p-forms for $p \geq 3$ required assumptions on the lowest eigenvalue of the curvature operator. In particular, this applies to the results of D. Meyer [Mey71], Gallot-Meyer [GM75] and Gallot [Gal81].

In this talk we explain how the holonomy representation on tensors provides structural insights into the curvature term of the Lichnerowicz Laplacian. This generalizes an idea of Poor [Poo80]. As an application, we show that n-dimensional manifolds with $\lceil \frac{n}{2} \rceil$-positive curvature operators are rational homology spheres, [PW21a]. We note that this curvature condition is different from positive isotropic curvature and is not invariant under the Ricci flow ODE. The theorem is a consequence of the fact that the p-th Betti number vanishes provided the curvature operator is $(n-p)$-positive. We also establish the corresponding rigidity and estimation results.

Recall that the curvature operator vanishes on the orthogonal complement of the holonomy algebra. Therefore, the above results are relevant for manifolds with holonomy $SO(n)$. For manifolds with reduced holonomy, we consider the restriction of the curvature operator to the holonomy algebra. In the case $\mathrm{Hol} = U(m)$ this is the Kähler curvature operator. Let $\mu_1 \leq \ldots \leq \mu_{m^2}$ denote its eigenvalues. We show that a Kähler manifold of complex dimension m with $\mu_1+\mu_2+\left(1-\frac{2}{m}\right)\mu_3 > 0$ has the rational cohomology ring of complex projective space, [PW21b]. Note that this curvature condition does not imply positive orthogonal complex bisectional curvature.

The theorem relies on vanishing theorems for the individual Hodge numbers. For example, if the Kähler curvature operator is $(m+1-p)$-positive, then $h^{p,p} = 1$.

Furthermore, our methods also provide the corresponding rigidity and estimation results.

For applications of the techniques to curvature tensors and generalizations of Tachibana's theorem [Tac74], please refer to [PW22].

In the last part of the talk we explain recent results on Nishikawa's conjecture [Nis86] on manifolds with nonnegative curvature operator of the second kind. Note that the curvature tensor induces a map on symmetric $(0,2)$-tensors via $h_{ij} \mapsto \sum_{k,l} R_{iklj} h_{kl}$. The induced map on trace-free, symmetric $(0,2)$-tensors is called curvature operator of the second kind. Extending work of Cao-Gursky-Tran [CGT21], X. Li [Li22] proved that manifolds with 3-nonnegative curvature operator of the second kind are either flat, diffeomorphic to a spherical space form, or their universal cover is isometric to a compact irreducible symmetric space. We exclude the symmetric spaces from this list and moreover show that manifolds with $\lfloor \frac{n+2}{n} \rfloor$-nonnegative curvature operator of the second kind are rational homology spheres or flat, [NPW22]. More generally, unless the manifold is flat, we obtain vanishing of the p-th Betti number provided the curvature operator of the second kind is $C(p,n)$-nonnegative. For large n we have $C(p,n) \sim \frac{3n}{2} \frac{p}{p+2}$. In particular, for $p \geq 5$ this curvature assumption does not imply nonnegative Ricci curvature anymore. Our techniques again rely on the Bochner technique, and extend work of Ogiue-Tachibana [OT79].

References

[Boc46] S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. **52** (1946), 776–797.

[CGT21] Xiaodong Cao, Matthew J. Gursky, and Hung Tran, *Curvature of the second kind and a conjecture of Nishikawa*, arXiv:2112.01212 (2021).

[Gal81] Sylvestre Gallot, *Estimées de Sobolev quantitatives sur les variétés riemanniennes et applications*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 6, 375–377.

[GM75] S. Gallot and D. Meyer, *Opérateur de courbure et laplacien des formes différentielles d'une variété riemannienne*, J. Math. Pures Appl. (9) **54** (1975), no. 3, 259–284.

[Li22] Xiaolong Li, *Manifolds with nonnegative curvature operator of the second kind*, arXiv:2112.08465 (2022).

[Mey71] Daniel Meyer, *Sur les variétés riemanniennes à opérateur de courbure positif*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A482–A485.

[Nis86] Seiki Nishikawa, *On deformation of Riemannian metrics and manifolds with positive curvature operator*, Curvature and topology of Riemannian manifolds (Katata, 1985), Lecture Notes in Math., vol. 1201, Springer, Berlin, 1986, pp. 202–211.

[NPW22] Jan Nienhaus, Peter Petersen, and Matthias Wink, *Betti numbers and the curvature operator of the second kind*, arXiv:2206.14218 (2022).

[OT79] Koichi Ogiue and Shun-ichi Tachibana, *Les variétés riemanniennes dont l'opérateur de courbure restreint est positif sont des sphères d'homologie réelle*, C. R. Acad. Sci. Paris Sér. A-B **289** (1979), no. 1, A29–A30.

[Poo80] W. A. Poor, *A holonomy proof of the positive curvature operator theorem*, Proc. Amer. Math. Soc. **79** (1980), no. 3, 454–456.

[PW21a] Peter Petersen and Matthias Wink, *New curvature conditions for the Bochner technique*, Invent. Math. **224** (2021), no. 1, 33–54.

[PW21b] ———, *Vanishing and estimation results for Hodge numbers*, J. Reine Angew. Math. **780** (2021), 197–219.

[PW22] ———, *Tachibana-type Theorems and special Holonomy*, Ann. Global Anal. Geom. **61** (2022), 847–868.
[Tac74] Shun-ichi Tachibana, *A theorem on Riemannian manifolds of positive curvature operator*, Proc. Japan Acad. **50** (1974), 301–302.

Stability of Einstein metrics

Frieder Jäckel
(joint work with Ursula Hamenstädt)

The question which manifolds admit an Einstein metric has a long history. Related to the existence question is the question of *stability*, that is, if a manifold M admits a metric $\bar{g}$ that is almost Einstein (in a suitable sense), does there exist an Einstein metric on M that is close to $\bar{g}$?

Using the convergence theory of Riemannian manifolds one can easily show that under upper diameter and lower injectivity radius bounds, it suffices that $||\mathrm{Ric}(\bar{g}) - \lambda \bar{g}||_{C^0}$ is sufficiently small (see [3, Theorem 11.4.16]). Namely, for all $n \geq 2$, $\lambda \in \mathbb{R}$, $D \geq 0$, and $i_0 > 0$ there exists $\varepsilon_0 = \varepsilon_0(n, \lambda, D, i_0) > 0$ so that if M^n admits a Riemannian metric $\bar{g}$ such that

$$||\mathrm{Ric}(\bar{g}) - \lambda \bar{g}||_{C^0} \leq \varepsilon_0, \quad \mathrm{diam}(M, \bar{g}) \leq D, \quad \text{and} \quad \mathrm{inj}(M, \bar{g}) \geq i_0,$$

then M admits an Einstein metric with Einstein constant λ that is $C^{1,\alpha}$-close to $\bar{g}$. Therefore, it is important to find stability results that do *not* assume an upper bound on the diameter, nor a lower bound on the injectivity radius.

In [4] Tian proved a stability result for Einstein metrics of negative sectional curvature that does not even assume an upper bound on the volume, though it still assumes a lower bound on the injectivity radius. In [2] we extend Tian's result to 3-manifolds without a lower injectivity radius bound. Our proof builds on earlier ideas of Bamler [1] and Tian [4].

Theorem 1. For all $\alpha \in (0,1)$, $\Lambda \geq 0$, $\delta \in (0,2)$, $b > 1$ and $\eta > 2$ there exist $\varepsilon_0 = \varepsilon_0(\alpha, \Lambda, \delta, b, \eta) > 0$ and $C = C(\alpha, \Lambda, \delta, b, \eta) > 0$ with the following property. Let M be a 3-manifold that admits a complete Riemannian metric $\bar{g}$ satisfying the following conditions for some $\varepsilon \leq \varepsilon_0$:

(1) $\mathrm{vol}(M, \bar{g}) < \infty$;
(2) $-1 - \varepsilon \leq \sec_{(M,\bar{g})} \leq -1 + \varepsilon$;
(3) For all $x \in M_{\text{thin}}$ it holds

$$\max_{\pi \subseteq T_x M} |\sec(\pi) + 1|,\ |\nabla Rm|(x),\ |\nabla^2 Rm|(x) \leq \varepsilon e^{-\eta d(x, M_{\text{thick}})};$$

(4) For all $x \in M$ it holds

$$e^{bd(x,M_{\text{thick}})} \int_M e^{-(2-\delta)d(x,y)} |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}}(y)\, d\mathrm{vol}_{\bar{g}}(y) \leq \varepsilon^2;$$

(5) $||\nabla \mathrm{Ric}(\bar{g})||_{C^0(M,\bar{g})} \leq \Lambda$.

Then there exists a hyperbolic metric g_{hyp} on M so that

$$||g_{\mathrm{hyp}} - \bar{g}||_{C^{2,\alpha}(M,\bar{g})} \le C\varepsilon^{1-\alpha}.$$

Moreover, if additionally $\bar{g}$ is already hyperbolic outside a region $U \subseteq M$, and if

$$\int_U |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}}\, d\mathrm{vol}_{\bar{g}} \le \varepsilon^2,$$

then for all $x \in M_{\mathrm{thick}}$ it holds

$$|g_{\mathrm{hyp}} - \bar{g}|_{C^{2,\alpha}}(x) \le C\varepsilon^{1-\alpha} e^{-(1-\frac{1}{2}\delta)\mathrm{dist}_{\bar{g}}(x, U \cup \partial M_{\mathrm{thick}})}.$$

It is important to note that in (4) the weight $e^{-(2-\delta)d(x,y)}$ inside the integral has the chance to absorb the weight $e^{bd(x,M_{\mathrm{thick}})}$ outside the integral (if $2-\delta > b$).

Applications of Theorem 1 include a new analytic proof for the drilling and filling of hyperbolic 3-manifolds, and effective hyperbolization in large Hempel distance.

References

[1] R. H. Bamler. *Construction of Einstein metrics by generalized Dehn filling.* Journal of the European Mathematical Society, 14:887–909, 2012.

[2] U. Hamenstädt, and F. Jäckel. *Stability of Einstein metrics and effective hyperbolization in large Hempel distance.* arXiv:2206.10438, 2022.

[3] P. Petersen. *Riemannian Geometry.* Springer, 3. edition, 2016.

[4] G. Tian. *A Pinching Theorem on Manifolds with Negative Curvature.* unpublished manuscript.

Positive curvature, Torus symmetry, and Matroids

Michael Wiemeler

(joint work with Lee Kennard and Burkhard Wilking)

A long standing open problem in Riemannian Geometry is the classification of simply connected, positively curved, closed Riemannian manifolds. What makes this problem difficult is the lack of both, examples and obstructions: In dimensions $n > 24$ all known examples of such manifolds are diffeomorphic to S^n, $\mathbb{C}P^{n/2}$, or $\mathbb{H}P^{n/4}$. Moreover, for simply connected, non-negatively curved manifolds there is no known obstruction to admitting positive sectional curvature.

To make the classification problem simpler Karsten Grove in 1992 suggested to first classify positively curved manifolds with large isometry group. This was partly motivated by the fact that all known constructions of positively curved examples involved some sort of symmetry. Since then this has been a successful approach (see e.g. [4], [9], [5], [3]).

Recently we proved the following two theorems:

Theorem 1 ([6])**.** If T^d, $d \ge 5$, acts effectively by isometries on a connected, closed, orientable, positively curved Riemannian manifold, then every component of the fixed-point set has the rational cohomology of a sphere or a complex or quaternionic projective space.

Corollary 1. If M is an even-dimensional, connected, closed, positively curved Riemannian manifold whose isometry group has rank at least five, then $\chi(M) > 0$.

Note that by the corollary the Hopf Conjecture holds for manifolds with T^5-action. Theorem 1 has strong implications on the topology of the one-skeleton $M_1 = \{x \in M;\ \dim Tx \leq 1\}$ of an isometric action of a torus of dimension at least seven on a positively curved manifold. So by combining this theorem with the Chang–Skjelbred Lemma [2] one can show:

Theorem 2 ([6])**.** If T^d, $d \geq 7$, acts effectively by isometries on a connected, closed, orientable, positively curved, even-dimensional manifold M^n with vanishing odd-degree cohomology, then M^n has the rational cohomology of S^n, $\mathbb{C}P^{\frac{n}{2}}$, or $\mathbb{H}P^{\frac{n}{4}}$.

Note that the condition of $H^{\mathrm{odd}}(M;\mathbb{Q}) = 0$ is natural in view of the Bott Conjecture which says that a simply connected, non-negatively curved manifold is rationally elliptic. Together with the Hopf Conjecture it would imply that $H^{\mathrm{odd}}(M;\mathbb{Q}) = 0$ holds for positively curved, even-dimensional manifolds.

Note, moreover, that by work of Nienhaus [7] the conclusion of Theorems 1 and 2 also hold for $d = 4$ and $d = 6$, respectively.

Since the Bott Conjecture is still wide open, it is natural to ask whether one can replace the topological condition, $H^{\mathrm{odd}}(M;\mathbb{Q}) = 0$, in Theorem 2 by a more geometric condition. In view of the proof of Theorem 1 it is natural to require that near a fixed point of the action all isotropy groups are connected. An investigation of this condition leads to the following unpublished result.

Theorem 3 (Kennard, Wiemeler, Wilking, 2022)**.** Let M^n be a closed, oriented, positively curved Riemannian manifold. Assume T^d acts effectively by isometries with a fixed point and the property that all isotropy groups in a neighborhood of the fixed point have an odd number of components.

(1) If $d = 9$, then M has the rational cohomology of $S^n, \mathbb{C}P^{n/2}$ or $\mathbb{H}P^{n/4}$.
(2) If $d = 6$, then M has the rational cohomology of $S^n, \mathbb{C}P^{n/2}$ or $\mathbb{H}P^{n/4}$ up to degree $\frac{n}{3}$.

The proof of this theorem consists of two steps. The first step is to find circle groups $S^1 \subset T^d$ such that the codimension of $M^{S^1} \subset M$ is low in comparison to $\dim M$. The second step is then to use Theorem 1, Nienhaus' results [7] and Wilking's Connectedness Lemma [9] to get the conclusion.

There are two main observations in the proof of the first step.

(1) The weights of a faithful torus representation $\rho : T^d \to SO(V)$ without isotropy groups with an even number of components form a combinatorial object called regular matroid $M(\rho)$ of rank d.
(2) $\min_{S^1 \subset T^d} \frac{\mathrm{codim}(V^{S^1} \subset V)}{\dim V} = g^*(M(\rho), \lambda)$ where $g^*(M(\rho), \lambda)$ denotes the co-girth of $M(\rho)$ weighted with the dimensions of the weight spaces of ρ.

Here the co-girth of a weighted matroid is the girth of its dual matroid, i.e. the length of the shortest cycle in the dual matroid.

We use Seymour's classification [8] of regular matroids to compute sharp upper bounds for $g^*(M,\lambda)$ for all weighted regular matroids of rank less than or equal to nine which only depend on the rank of M.

For ranks less than or equal to six these bounds easily follow from known results [1]. However for ranks between seven and nine our bounds seem to be new. The important estimates which make the proof of Theorem 3 possible are $g^*(M,\lambda) \leq \frac{1}{3}$ for weighted, regular matroids (M,λ) of rank six and $g^*(M,\lambda) \leq \frac{1}{4}$ for weighted, regular matroids (M,λ) of rank nine.

References

[1] B. Bolobás and E. Szemerédi, *Girth of sparse graphs*, J. Graph Theory **39** (2002), 194–200.

[2] T. Chang and T. Skjelbred, *The topological Schur lemma and related results*, Ann. of Math. (2) **100** (1974), 307–321.

[3] O. Goertsches and M. Wiemeler, *Positively curved GKM-manifolds* Int. Math. Res. Not. IMRN **2015** (2015), 12015–12041.

[4] K. Grove and C. Searle, *Positively curved manifolds with maximal symmetry-rank*, J. Pure Appl. Algebra **91** (1994), 137–142.

[5] L. Kennard, *On the Hopf conjecture with symmetry*, Geom. Topol. **17** (2013), 563–593.

[6] L. Kennard, M. Wiemeler, and B. Wilking *Splitting of torus representations and applications in the Grove symmetry program*, arXiv-Preprint, arXiv:2106.14723 (2021).

[7] J. Nienhaus, *PhD-Thesis*, Universität Münster (2022).

[8] P. D. Seymour, *Decomposition of regular matroids*, J. Combin. Theory Ser. B **28** (1980), 305–359.

[9] B. Wilking, *Torus actions on manifolds of positive sectional curvature*, Acta Math. **191** (2003), 259–297.

Decomposing 4-manifolds with positive scalar curvature

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(joint work with Richard H. Bamler, Chao Li)

Recall the following well-known theorem of Schoen–Yau ([8]) and Gromov–Lawson ([3]) that exhibits the richness of the class of manifolds that can carry Riemannian metrics with positive scalar curvature. For convenience and brevity, we will adopt the convention of writing "PSC" for "positive scalar curvature," and will call a manifold "topologically PSC" if it can carry PSC Riemannian metrics.

Theorem. *Let M be a topologically PSC n-manifold with $n \geq 3$. Any manifold obtained from M by performing a sequence of* 0-, 1-, ..., *and/or* $(n-3)$*-surgeries is also topologically PSC.*

The theorem above naturally leads one to ask:

Question. *Can all topologically PSC n-manifolds be built out of "simple" topologically PSC n-manifolds by performing codimension ≥ 3 surgeries?*

Our understanding of 3-manifold topology yields a conclusive answer when $n = 3$. The following theorem follows from the combined work of Schoen–Yau ([8, 9]), Gromov–Lawson ([2]), and Perelman ([5, 6, 7]):

Theorem. *Every closed, oriented, topologically PSC* 3*-manifold can be obtained by performing* 0*-surgeries on a disjoint union of spherical space forms (i.e.,* $\mathbf{S}^3/\Gamma$*'s, where the* Γ*'s are finite subgroups of* $SO(4)$ *acting freely on* $\mathbf{S}^3$*).*

The following new result was presented, obtained by the speaker, R. H. Bamler, and C. Li:

Main Theorem ([1]). *Every closed, oriented, topologically PSC* 4*-manifold* M *can be obtained from a possibly disconnected, closed, oriented, topologically PSC* 4*-orbifold* M' *with isolated singularities such that* $b_1(M') = 0$ *and* $b_2(M') \leq b_2(M)$ *by performing* 0*- and* 1*-surgeries. All* 1*-surgeries are standard manifold ones, but* 0*-surgeries may occur at orbifold points.*

Recall that the jth Betti number $b_j(M')$ of the orbifold M' is defined to be the jth Betti number of M' viewed as a topological space; in our case, this is equivalent to the jth Betti number of the regular part $M'_{\mathrm{reg}} \subset M'$. In the connected case, $b_1(M')$ is the same as the rank of the abelianization of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(M')$. Finally, a 0-surgery occurring at two orbifold points both modeled on $\mathbf{R}^4/\Gamma$ means that the corresponding connected sum operation is performed with a $\mathbf{S}^3/\Gamma$ neck.

Our proof of the Main Theorem relies on the flexibility of two-sided stable minimal hypersurfaces in PSC 4-manifolds manifolds due to the speaker and C. Li ([4]). Specifically, the following metric preparation lemma was necessary:

Metric Preparation Lemma. *Let* Σ *be a two-sided, closed, embedded, stable, minimal hypersurface inside an oriented PSC* 4*-manifold* (M, g)*. Then:*

(a) Σ *must be topologically PSC and thus obtained by performing* 0*-surgeries on a disjoint union of spherical space forms.*
(b) *Given any auxiliary PSC metric* σ *on* Σ*, there exists a new PSC metric* $\tilde{g}$ *on* M*, which:*
  - *is isometric to a product cylinder* $(\Sigma, \sigma) \times (-2, 2)$ *in the distance-*2 *tubular neighborhood of* Σ*, and*
  - *coincides with* g *outside a larger tubular neighborhood of* Σ*.*

Let now us outline the proof of the Main Theorem. Endow M with an arbitrary PSC metric. We "exhaust" the codimension-1 homology of M with a two-sided, stable minimal hypersurface Σ. By a now-standard argument of Schoen–Yau, the metric induced on Σ is conformal to a PSC metric. Thus, Σ is topologically the result of 0-surgeries on spherical space forms. With the help of the aforementioned flexibility theory, we locally modify the metric on M to another PSC metric that is locally a product near Σ and induces a "model" PSC metric on Σ. If Σ is merely the disjoint union of spherical space forms $\mathbf{S}^3/\Gamma$, with no 0-surgeries, then our model metrics are all round and simple 3-surgeries on M along the components of Σ yield a 4-orbifold whose b_2 is unchanged and b_1 is trivial (assuming Σ suitably "exhausted" the codimension-1 homology of M). If Σ does involve 0-surgeries, we first undo these using 2-surgeries on M near Σ's 0-surgery neck regions; this may decrease b_2. We have only performed 3- and 2-surgeries on M to get to the orbifold, so M can be obtained from the orbifold via 0- and 1-surgeries, respectively.

We conclude our introduction by posing the following:

Question. *Let M be a closed, oriented, topologically PSC 4-manifold. Can one obtain M from a closed, oriented, topologically PSC 4-orbifold M' with isolated singularities and the property that each component has finite orbifold fundamental group $\pi_1^{orb}(M')$ by performing 0- and 1-surgeries?*

References

[1] R. H. Bamler, C. Li and C. Mantoulidis, *Decomposing 4-manifolds with positive scalar curvature*, arXiv Mathematics e-prints (2022), math/2206.09335.

[2] M. Gromov and H. B. Lawson, Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. (1983), no. 58, 83–196 (1984). MR 720933

[3] M. Gromov and H. B. Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **111** (1980), no. 3, 423–434. MR 577131

[4] C. Li and C. Mantoulidis, *Metrics with $\lambda_1(-\Delta + kR) \geq 0$ and flexibility in the riemannian penrose inequality*, arXiv Mathematics e-prints (2021), math/2106.15709.

[5] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv Mathematics e-prints (2002), math/0211159.

[6] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv Mathematics e-prints (2003), math/0303109.

[7] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv Mathematics e-prints (2003), math/0307245.

[8] R. Schoen, S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), no. 1-3, 159–183. MR 535700

[9] R. Schoen, S.-T. Yau, *The structure of manifolds with positive scalar curvature*, Directions in partial differential equations (Madison, WI, 1985), Publ. Math. Res. Center Univ. Wisconsin, vol. 54, Academic Press, Boston, MA, 1987, pp. 235–242. MR 1013841

Fundamental Gap Estimate for Convex Domains

Guofang Wei

The fundamental (or mass) gap refers to the difference between the first two eigenvalues of the Laplacian or more generally for Schrödinger operators. It is a very interesting quantity both in mathematics and physics as the eigenvalues are possible allowed energy values in quantum physics. Naturally one looks for optimal upper and lower estimates for the gap. For convex domains with Neumann boundary condition, this is well studied and optimal lower bound has been obtained awhile back. Here we concentrate on the Dirichlet boundary condition.

In their celebrated work, B. Andrews and J. Clutterbuck [1] proved the fundamental gap conjecture that difference of first two eigenvalues of the Laplacian with Dirichlet boundary condition on convex domain with diameter D in the Euclidean space is greater than or equal to $3\pi^2/D^2$. In several joint works with X. Dai, Z. He, S. Seto, L. Wang (in various subsets) [7, 4, 3] the estimate is generalized, showing the same lower bound holds for convex domains in the unit sphere. The key is to prove super log-concavity of the first eigenfunction.

In sharp contrast, in joint work with T. Bourni, J. Clutterbuck, X. Nguyen, A. Stancu and V. Wheeler [2], we prove that there is no lower bound at all for the fundamental gap of convex domains in hyperbolic space in terms of the diameter. Recently, jointed with X. Nguyen, A. Stancu [6], we show that even for horoconvex (which is much stronger than convex) domains in the hyperbolic space, the product of their fundamental gap with the square of their diameter has no positive lower bound.

Many questions remain open, especially for manifolds with variable curvature. In a joint work in progress with G.Khan, X. Nguyen, M. Tuerkoen [5], we obtain a log-concavity estimate of the first eigenfunction for convex domains in surfaces with variable curvature. Namely given Ω a convex domain in a Riemann surface (M^2, g) with positive sectional curvature κ, denote $\lambda_1(\Omega), u_1$ be its first Dirichlet eigenvalue and eigenfunction, if $\Delta \log \kappa - 5\kappa > -4\lambda_1(\Omega)$, then $\mathrm{Hess}\,(\log u_1) < -\frac{\kappa}{2} g$.

References

[1] B. Andrews and J. Clutterbuck, *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. **24** (2011), no. 3, 899–916.

[2] T. Bourni, J. Clutterbuck, X.-H. Nguyen, A. Stancu, G. Wei, and V.-M. Wheeler, *The vanishing of the fundamental gap of convex domains in* $\mathbb{H}^n$, Annales Henri Poincare **23** (2022), no. 2, 595–614.

[3] X. Dai, S. Seto, and G. Wei, *Fundamental gap estimate for convex domains on sphere–the case* $n = 2$, Comm. in Analysis and Geometry **29**, No. 5 (2021), 1095–1125.

[4] C. He and G. Wei, *Fundamental gap of convex domains in the spheres (with appendix B by Qi S. Zhang)*, Amer. Journal of Math, **142** (2020), no. 4, 1161–1192.

[5] G. Khan, X.H. Nguyen, M. Tuerkoen and G. Wei, in preparation.

[6] X. H. Nguyen, A. Stancu and G. Wei, *The fundamental gap of horoconvex domains in* $\mathbb{H}^n$, to appear in International Mathematics Research Notices.

[7] S. Seto, L. Wang, and G. Wei. *Sharp fundamental gap estimate on convex domains of sphere*, Journal of Differential Geometry **112** (2019), no. 2, 347–389.

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