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Low-dimensional Topology

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Abstract. The workshop brought together experts from across all areas of low-dimensional topology, including knot theory, computational topology, three-manifolds and four-manifolds. In addition to the standard research talks we had two survey talks by Marc Lackenby and Joel Hass, leading to discussions of open problems. Furthermore we had three sessions of five-minute talks by a total of roughly thirty participants.

Mathematics Subject Classification (2020): 57K10, 57K20, 57K30, 57K40.

Introduction by the Organizers

The workshop *Low-dimensional Topology* (2023) was organized by Stefan Friedl (Regensburg), Yoav Moriah (Haifa), Jessica Purcell (Melbourne) and Saul Schleimer (Coventry). The workshop was attended by nearly 50 researchers from countries including Australia, Canada, France, Germany, India, Israel, UK, and USA.

This was a follow up workshop to one that was held in February of 2020, just before Oberwolfach shut down due to the Covid-19 pandemic. For almost all of our participants, the 2020 workshop was the last that they attended before lockdowns across the globe. For many participants in 2023, this was the first workshop they attended in person since lockdowns ended. Everyone appreciated being back in person. There were many mathematical discussions over meals, in corridors, meeting rooms, in the library and during hikes. Many people commented on the excellent atmosphere, and the ability to be immersed in interesting mathematical research again with a community.

We had twenty-two research talks across multiple areas of low-dimensional topology. Also, we had two survey talks on 3-manifolds and computational low-dimensional topology. Finally we had three lively sessions of five-minute talks. The five minute talks gave a venue in which all participants could share their recent results, open problems, and mathematical challenges.

Although the talks covered a wide range of topics, the speakers established connections between the various subfields of low-dimensional topology.

Workshop: Low-dimensional Topology

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Abstracts

Lens Space Recognition and Seifert Fiber Spaces

Kate Petersen
(joint work with Neil Hoffman)

The Lens space recognition problem is the problem of deciding if a given 3–manifold is a lens space (including $\mathbb{S}^3$). A decision problem is said to lie in NP if an affirmative solution can be verified via certificate in polynomial time relative to the input size (of a triangulation in this case) and we say that a problem lies in coNP if a negative solution can be verified by such a certificate. That is, the Lens space recognition problem lies in coNP if given a manifold M that is not a lens space, there is a certificate (for example, an explicit homomorphism to a non-cyclic group that can be written down from a triangulation of M) which can be checked in polynomial time.

Lackenby and Schleimer [2] proved that Lens space recognition is in NP. Zentner [5, Theorem 11.2] proved that the $\mathbb{S}^3$ recognition problem is in coNP provided the Generalized Riemann Hypothesis (GRH) is true. Our work shows that if the input is a Seifert fiber space, then Lens space recognition is in coNP, unconditionally.

Theorem 1. *For a Seifert fiber space M with non-abelian fundamental group, the* Lens space recognition *problem lies in coNP. In particular, there is a polynomial time verifiable certificate to distinguish M from $\mathbb{S}^3$.*

All of our certificates are either (non-trivial) non-abelian representations or non-cyclic abelian representations, we also distinguish these manifolds from $\mathbb{S}^1 \times \mathbb{S}^2$ and so we state the following direct corollary.

Corollary 2. *For a Seifert fiber space M with non-abelian fundamental group, the* $\mathbb{S}^1 \times \mathbb{S}^2$ recognition *problem lies in coNP.*

There are no non-orientable 3–manifolds with finite fundamental groups. There is a polynomial time algorithm (relative to the size of the triangulation) to determine if a triangulation represents an orientable or non-orientable 3–manifold. Therefore, we can distinguish non-orientable 3–manifolds from lens spaces (including $\mathbb{S}^3$) in polynomial time. Using work of Haraway and Hoffman [1] and an understanding of Seifert fiber spaces and their groups, we reduce the scope of the problem to distinguishing small, prime Seifert fiber spaces with non-cyclic fundamental groups from lens spaces. The only non-prime Seifert fiber space is $\mathbb{RP}^3 \# \mathbb{RP}^3$. Since its fundamental group is isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, it surjects the dihedral group of order 6 and we use this as our certificate in this case.

The small, prime, non-cyclic Seifert fiber space groups surject triangle groups T_{n_1,n_2,n_3} where $n_k > 1$ are integers for $k = 1, 2, 3$. We demonstrate that for most of these triangle groups there is a particularly nice integral representation into $\mathrm{PSL}(2, K)$ where K is a number field which is "almost" the cyclotomic field

$\mathbb{Q}(\zeta_{2n_1n_2n_3})$. The trace field of this representation has degree $\frac{1}{2}\phi(2n_1n_2n_3)$. We then use Linnik's theorem (as improved by Xylouris [4, 3]) to find a "small" prime that splits completely in K and show that in the natural quotient the representation stays non-abelian. This gives us a non-abelian representation of these Seifert fiber space groups into $\mathrm{PSL}(2,\mathbb{F})$ where $|\mathbb{F}|$ is bounded above by a polynomial function of $n_1n_2n_3$; explicitly there is a constant c so that $|\mathcal{F}| \leqslant c\,\mathrm{lcm}(n_1,n_2,n_3)^{10}$. (This covers most cases, and for the remaining cases we get a compatible bound.)

We then convert this upper bound to an upper bound in terms of t, the number of tetrahedra in a triangulation. We accomplish this in two main steps. First, we show that for any 3–manifold with a triangulation with t tetrahedra, there is a presentation for $\pi_1(M)$ where the number of generators, relations, and their length is governed by t. Then we translate those complexity bounds for $\pi_1(M)$ into upper bounds for the degree of the trace field of a 0-dimensional component of the $\mathrm{PSL}(2,\mathbb{C})$ character variety in terms of t. We use this to show that the degree of the trace field is at most $2^{t-1}3^{6t}$.

Finally, we reconcile these bounds. Our explicit representations of the triangle groups have trace fields of degree $\frac{1}{2}\phi(2n_1n_2n_3)$ and we use this to translate our upper bounds for $|\mathbb{F}|$ from a dependence on $n_1n_2n_3$ to a dependence on t, as needed. (This plan covers most cases, and we handle the remaining cases separately.) Explicitly, we show the following which gives our certificate when M is an orientable, closed, small, non-cyclic Seifert fiber space.

Theorem 3. *Assume that M is an orientable, closed, small, non-cyclic Seifert fiber space admitting a triangulation with t tetrahedra. Then either $\pi_1(M)$ surjects a non-cyclic abelian group of order at most $2^{4t}3^{24t}$ or $\pi_1(M)$ surjects a (non-trivial) non-abelian subgroup of $PSL(2,\mathbb{F})$ where $|\mathbb{F}| < c(2^{20t}3^{120t})$ for some effectively computable $c > 0$.*

Although not Seifert fiber spaces, we point out that only one Sol manifold has cyclic homology. In the one case of cyclic homology, the double cover has homology $\mathbb{Z}/5\mathbb{Z}\times\mathbb{Z}$. So if M admits Sol geometry, the homology of the manifold or its double cover serves as a certificate that M is neither a lens space nor $\mathbb{S}^1 \times \mathbb{S}^2$.

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A proposal for sweep-outs and the Hempel distance of a trisection

Alex Zupan
(joint work with Jeffrey Meier)

Trisections were introduced by Gay and Kirby [GK16] as a novel way to study smooth 4-manifolds, a 4-dimensional analogue of Heegaard splittings of 3-manifolds. A general motivating problem in the theory of trisections is to adapt 3-dimensional tools to the context of dimension four. The narrative we seek to adapt here is that of the Hempel distance of a Heegaard splitting. In dimension three, the story goes back to Haken's Lemma, which asserts that every Heegaard splitting of a reducible 3-manifold Y is reducible [Hak68]. In other words, if Y contains a 2-sphere S that does not bound a 3-ball in Y and if $Y = H_0 \cup_\Sigma H_1$ is a Heegaard splitting of Y, then there exists an essential simple closed curve in Σ that bounds disks in both H_0 and H_1.

For an arbitrary 3-manifold Y with Heegaard splitting $Y = H_0 \cup_\Sigma H_1$, the notion of reducibility was extended to define the *Hempel distance* of the splitting [Hem01] via the curve complex $\mathcal{C}(\Sigma)$ of Σ. Vertices in $\mathcal{C}(\Sigma)$ represent essential simple closed curves in Σ, and edges represent pairs of disjoint curves. The Hempel distance $d(\Sigma)$ is then defined as the length of a shortest path in the curve complex $\mathcal{C}(\Sigma)$ connecting the disk sets $\mathcal{D}_0$ and $\mathcal{D}_1$, where $\mathcal{D}_i$ is the subcomplex of $\mathcal{C}(\Sigma)$ spanned by those curves bounding compressing disks in H_i. In this setting, a Heegaard surface Σ is reducible if and only if $d(\Sigma) = 0$.

Haken's Lemma was generalized to tori by Hempel, who proved that if Y contains an essential torus, then for any Heegaard surface Σ in Y, we have $d(\Sigma) \leqslant 2$ [Hem01]. This bound was further generalized in the following theorem of Hartshorn.

Theorem 1. [Har02] *Let Y be a 3-manifold containing an essential surface S. If Σ is any Heegaard surface for Y, then $d(\Sigma) \leqslant 2g$.*

An alternate proof of this theorem provided by Li uses the idea of a *sweep-out* arising from a Heegaard splitting $Y = H_0 \cup_\Sigma H_1$ [Li07]. A sweep-out is a map $h : Y \to [0,1]$ with the following properties:

(1) $h^{-1}(0)$ is a spine for H_0;
(2) $h^{-1}(1)$ is a spine for H_1; and
(3) For any $t \in (0,1)$, $h^{-1}(t)$ is a surface isotopic to Σ.

Here, we provide a sketch of Li's proof of Hartshorn's Theorem by listing several main ingredients: First, assume Y is irreducible (otherwise, Haken's Lemma would apply) and observe that if S is an essential surface in Y, then $h(S) = [0,1]$ (if not, S would be isotopic into a handlebody and as such would be compressible). Second, note that for all $t \in (0,1)$, the level set $h^{-1}(t)$ must intersect S in at least one essential curve. Finally, after perturbing S so that it meets the spines $h^{-1}(0)$ and $h^{-1}(1)$ transversely and so that $h|_S$ is Morse, we can use S to construct a path in $\mathcal{C}(\Sigma)$ from some curve c_0 in $h^{-1}(\varepsilon) \cap S$ to some curve $c_1 \in h^{-1}(1-\varepsilon)$, where $c_i \in \mathcal{D}_i$ and the length of this path is at most $2g(S)$.

It is this story we seek to modify to see what we can learn in the context of 4-manifold trisections. A *trisection* of a smooth 4-manifold X is a decomposition of X into $X_0 \cup X_1 \cup X_2$, where each X_i is a 4-dimensional 1-handlebody, each pairwise intersection $H_i = X_i \cap X_{i+1}$ (with indices modulo 3) is a 3-dimensional handlebody, and the triple intersection $\Sigma = X_0 \cap X_1 \cap X_2$ is a surface. A key feature is that X is uniquely determined up to diffeomorphism by the union of the 3-dimensional components, $H_0 \cup H_1 \cup H_2$. As such, trisections provide a natural venue for importing techniques from Heegaard splittings into dimension four. For example, the authors used the Wave Theorem [HOT80], a combinatorial assertion about Heegaard splittings of S^3, as a key ingredient in their proof that any 4-manifold admitting a trisection of genus two is among a small finite family of standard examples [MZ17].

The most straightforward attempt to generalize Haken's Lemma to trisections encounters immediate problems. The unresolved Schönflies conjecture leaves open the possibility that even the standard 4-ball could contain a 3-sphere Y that does not bound a ball. If this were the case, we would have no hope of understanding this notion of reducibility via trisections, since X could have a trisection in which Y and the union $H_0 \cup H_1 \cup H_2$ fail to interact. In order to get a better grasp of what might be possible, we first adapt the idea of a sweep-out to dimension four via trisections.

Let Δ be a 2-simplex, with vertices v_i and edges e_i connecting v_i and v_{i+1}, where $i \in \{0, 1, 2\}$. Given a trisection of X with components notated as above, a compatible *sweep-out* is a map $h : X \to \Delta$ with the following properties:

(1) For each i, $h^{-1}(v_i)$ is a spine for X_i;
(2) For any $t \in \mathrm{int}(e_i)$, $h^{-1}(t)$ is a spine for H_i; and
(3) For any $s \in \mathrm{int}(\Delta)$, $h^{-1}(s)$ is a surface isotopic to Σ.

In addition, we require that for $t_i \in \mathrm{int}(e_i)$ and $t_{i+1} \in \mathrm{int}(e_{i+1})$ sufficiently close to the vertex v_i, the spines $h^{-1}(t_i)$ and $h^{-1}(t_{i+1})$ are standard spines, in the sense that the boundaries of two collections of disks dual to each spine yield a standard Heegaard diagram for ∂X_{i+1}, which is $\#^k(S^1 \times S^2)$ for some k. We prove

Proposition 2. *Let X be a smooth 4-manifold with a trisection $X = X_0 \cup X_1 \cup X_2$. There exists a compatible sweep-out $h : X \to \Delta$.*

Analogous to the three-dimensional setting, for $i \in \{0, 1, 2\}$ we let $\mathcal{D}_i$ denote the disk set in $\mathcal{C}(\Sigma)$ associated to the handlebody H_i. In order to circumvent the immediate issues discussed above, we replace the essential surface from the 3-dimensional setting with a *non-separating* 3-manifold Y embedded in X. This allows us to deduce

Theorem 3. *Let X be a smooth 4-manifold with a trisection $X = X_0 \cup X_1 \cup X_2$ and compatible sweep-out $h : X \to \Delta$, and let $Y \subset X$ be a non-separating 3-manifold. Then the following are true:*

(1) $h(Y) = \Delta$;
(2) For every $s \in \mathrm{int}(\Delta)$, the intersection $h^{-1}(s) \cap Y$ contains a curve that is essential in Σ;

(3) For every $s \in int(\Delta)$ sufficiently close to e_i, the intersection $h^{-1}(s) \cap Y$ contains a curve in $\mathcal{D}_i$; and

(4) Y induces a loop in $\mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$ that meets all three disk sets.

The proof uses many ingredients of the 3-dimensional argument, with the additional observation that if Y avoids a surface isotopic to Σ, then the inclusion of Y into X factors through a 2-complex, contradicting the fact that this inclusion induces an injection $H_3(Y) \hookrightarrow H_3(X)$. Moreover, if Y meets some surface $h^{-1}(s)$ in only a collection of trivial curves, these intersections can be surgered away by ambient 2-handle attachments to produce a 3-manifold Y' homologous to Y in X but such that $Y' \cap h^{-1}(s) = \varnothing$, another contradiction to the above argument.

The theorem, of course, motivates the titular "proposal for Hempel distance" associated to a trisection. For a given trisection of X with components labeled as above, define the distance $d(\Sigma)$ to be the length of a shortest loop in $\mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$ meeting all three disk sets. As in the 3-dimensional case, the trisection is reducible if and only if $d(\Sigma) = 0$. We conclude with a problem and a conjecture for further examination.

Problem 4. *Suppose that X is a smooth 4-manifold containing a non-separating 3-manifold Y, and let Σ be the central surface of a trisection of X. Use the topology of Y and the induced loop in $\mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$ to give an upper bound on $d(\Sigma)$.*

Conjecture 5. *If X is a smooth 4-manifold containing a non-separating 3-sphere, then every trisection of X is reducible.*

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$\mathbf{Homeo}_+(S^1)$ representations and the L-space conjecture

Steven Boyer
(joint work with Cameron McA. Gordon and Ying Hu)

Throughout we take M to be a closed, connected, oriented, irreducible rational homology 3-sphere.

In this talk we discussed how $\mathrm{Homeo}_+(S^1)$-representations of $\pi_1(M)$ arising from foliations and flows on M can be used to verify the left-orderability of $\pi_1(M)$ in a number of interesting situations. We also discussed the extent to which the analogous results for the existence of taut foliations on M hold, as predicted by the L-space conjecture. Here are the details.

The obstruction to lifting a non-trivial representation $\rho : \pi_1(M) \to \mathrm{Homeo}_+(S^1)$ to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) = \{f \in \mathrm{Homeo}_+(\mathbb{R}) \mid f(x+1) = f(x) + 1\}$, the universal covering group of $\mathrm{Homeo}_+(S^1)$, is an element $e(\rho) \in H^2(M)$ called the *Euler class* of ρ.

$$\begin{tikzcd}
 & \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \arrow[d] \arrow[r, phantom, "\leqslant"] & \mathrm{Homeo}_+(\mathbb{R}) \\
\pi_1(M) \arrow[ur, "\exists\,?"] \arrow[r, "\rho"] & \mathrm{Homeo}_+(S^1) &
\end{tikzcd}$$

The Euler class is non-zero in general, though if zero, Theorem 1.1 of [BRW] implies that $\pi_1(M)$ is *left-orderable*, that is, it admits a total order which is invariant under left-multiplication. This is a property of special interest as the *L-space conjecture* contends that the following conditions are equivalent for M:

- M is LO. That is, $\pi_1(M)$ is non-trivial and is a left-orderable group.
- M is CTF. That is, M admits a co-oriented, taut foliation.
- M is NLS. That is, M is not a Heegaard Floer L-space.

Various structures on M determine a non-trivial acton of $\pi_1(M)$ on the circle. Of particular interest to us are

- Thurston's universal circle action $\rho_{\mathcal{F}} : \pi_1(M) \to \mathrm{Homeo}_+(S^1)$ associated to a co-oriented taut foliation $\mathcal{F}$ on M ([CD]);
- the asymptotic circle actions $\rho_\Phi : \pi_1(M) \to \mathrm{Homeo}_+(S^1)$ associated to a pseudo-Anosov flow Φ on M ([CD], [Ca], [Fe]).

In these cases the Euler class of the associated representations are given by

$$e(\rho) = \begin{cases} e(T\mathcal{F}) & \text{if } \rho = \rho_{\mathcal{F}} \text{ and } T\mathcal{F} \text{ is the tangent bundle of } \mathcal{F} \\ e(\nu_\Phi) & \text{if } \rho = \rho_\Phi \text{ and } \nu_\Phi \text{ is the normal plane bundle to } \Phi \end{cases}$$

Given a non-trivial ρ as above, $e(\rho)$ automatically vanishes when M is an integer homology 3-sphere and therefore M is LO. Conjecturally then, M is also NLS and CTF. Regarding this case, we have the following conjecture of Ozsváth and Szabó.

Conjecture (Ozsváth-Szabó). *If M is an integer homology 3-sphere then M is NLS if and only if M is neither S^3 nor the Poincaré homology sphere $\Sigma(2,3,5)$.*

This is known to hold in many cases, for instance Eftekhary showed that M is NLS if it is toroidal ([Eft]). Our first result uses Thurston's universal circle representation to prove that it is also LO, as predicted by the L-space conjecture.

Theorem 1 ([BGH1]). *A toroidal integer homology 3-sphere M is LO.*

To prove the theorem, express M as the union $M_1 \cup_T M_2$ of two knot manifolds M_1, M_2 along their common boundary T. (A *knot manifold* is a compact, connected, orientable, irreducible 3-manifold whose boundary is an incompressible torus.) Since M is an integer homology 3-sphere, both M_1 and M_2 are integer homology solid tori and their longitudinal slopes λ_1, λ_2 are of distance 1 on T.

Certain slopes on the boundary of a knot manifold, called *LO-detected slopes*, are singled out using left-orders ([BC2]). For instance, the longitudinal slope of a knot manifold is always LO-detected, so in particular λ_2 is LO-detected in M_2. One of the main results of [BGH1] shows that it is also LO-detected in M_1.

Theorem 2 ([BGH1]). *Suppose that M_1 is a knot manifold integer homology solid torus. Then any slope of distance 1 from its longitudinal slope is LO-detected.*

The proof of Theorem 2 depends on a detailed analysis of a universal circle action of the fundamental group of M_1 related to a finite depth foliation on M_1. Theorem 1 is now a consequence of the following LO-gluing theorem:

Theorem 3 ([BC2]). *Suppose that $M = M_1 \cup_T M_2$ is the union of two knot manifolds M_1, M_2 along their common boundary T. If there is a slope on T which is LO-detected in both M_1 and M_2, then M is LO.*

A notion of slope detection, called *CTF-detection*, can also be defined using co-oriented taut foliations ([BGH1]), and there is a gluing theorem analogous to Theorem 3:

Theorem 4 ([BGH1]). *Suppose that $M = M_1 \cup_T M_2$ is the union of two knot manifolds M_1, M_2 along their common boundary T. If there is a slope on T which is CTF-detected in both M_1 and M_2, then M is CTF*

Since longitudinal slopes are CTF-detected, if we knew that the analogue of Theorem 2 held for CTF-detection, then toroidal integer homology 3-spheres would be CTF. Such an analogue has been proven when M_1 is fibred ([BGH1]), but is open in general.

Conjecture 5. *Suppose that M_1 is a knot manifold which is an integer homology solid torus. Then any slope of distance 1 from its longitudinal slope is CTF-detected.*

There is also a notion of NLS-detection ([BC1], [RR]) together with a gluing theorem analogous to Theorems 3 and 4 ([HRW]). Moreover, the NLS analogue of Theorem 2 holds, so arguing as above leads to a proof that toroidal integer homology spheres are NLS. (This argument is due to Hanselman-Rasmussen and Watson [HRW]). The following conjecture is closely related to the L-space conjecture for toroidal manifolds. See §2.3 of [BGH1].

Conjecture 6. *The sets of LO-detected slopes, CTF-detected slopes, and NLS-detected slopes on the boundary of a knot manifold coincide.*

Here is an application of the detection-gluing method to surgery on satellite knots.

Theorem 7 ([BGH2])**.** *If the JSJ graph of a satellite knot K is not an interval and $r \in \mathbb{Q}$ is not the slope of a cabling of K, then $K(r)$ is NLS and LO. Moreover, if meridional slopes of non-trivial knots in the 3-sphere are CTF-detected, then $K(r)$ is CTF.*

It is a result of Krcatovic that L-space knots are prime ([Krc]). We recover this result as a corollary of Theorem 7 and deduce LO and CTF analogues.

Corollary 8. *All rational surgeries on a composite knot are LO and NLS. They would also be CTF if the meridional slopes of non-trivial knots in the 3-sphere are CTF-detected.* □

The LO results discussed above depend on the existence of $\mathrm{Homeo}_+(S^1)$-representations of the fundamental groups of knot manifolds obtained through Thurston's universal circle construction. In [BGH3] we prove LO results using representations $\rho_\Phi : \pi_1(M) \to \mathrm{Homeo}_+(S^1)$ associated to pseudo-Anosov flows. Here is an example of what we prove.

Theorem 9 ([BGH3])**.** *Let L be a hyperbolic link in an integer homology 3-sphere M whose complement admits a cusped pseudo-Anosov flow none of whose degeneracy loci are meridional. Then given any orientation on L and $n \geqslant 2$, the standard n-fold cyclic branched cover $\Sigma_n(L)$ has a left-orderable fundamental group.*

The proof of this result, which applies to much more general cyclic branched covers, uses the cusped pseudo-Anosov flow to construct an infinite family of $\mathrm{Homeo}_+(S^1)$-representations of $\pi_1(M \backslash L)$ with prescribed peripheral values. More precisely, let $K_1, K_2, \ldots, K_m$ be the components of L and $\mu_1, \mu_2, \ldots, \mu_m \in \pi_1(M \backslash L)$ their meridional classes. We show that there are representations $\rho_{r_*} : \pi_1(M \backslash L) \to \mathrm{Homeo}_+(S^1)$ with infinite, non-cyclic image parameterised by a dense subset $D \subset [0,1]^m$ of points $r_* = (r_1, r_2, \ldots, r_m)$ with rational coordinates, where $\rho_{r_*}(\mu_i)$ is conjugate to a rotation of angle $2\pi r_i$ for each i.

Corollary 10. *All cyclic branched covers of fibred strongly quasipositive hyperbolic knots in the 3-sphere are LO. In particular, this holds for hyperbolic L-space knots.* □

It is known that the n-fold cyclic branched cover $\Sigma_n(K)$ of a hyperbolic L-space knot K is NLS if $n \geqslant 4$ by [BBG] and $n = 3$ by [FRW]. It is expected that $\Sigma_n(K)$ is NLS when $n = 2$ (cf. the L-space conjecture), but this is still open. The results for CTF are much weaker, the best to date being that $\Sigma_n(K)$ is CTF if $n \geqslant 2(2g(K) - 1)$, due essentially to Rachel Roberts. See the discussion in the introduction of [BH].

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How not to study low-dimensional topology

Sarah Blackwell

(joint work with Robion Kirby, Michael Klug, Vincent Longo, Benjamin Ruppik)

A correspondence, by way of Heegaard splittings, between closed oriented 3-manifolds and pairs of surjections from a surface group to a free group has been studied by Stallings, Jaco, and Hempel [12, 7, 5]. A *Heegaard splitting* of a closed 3-manifold M^3 is a pair of handlebodies H_1 and H_2 embedded inside of M with boundaries a common genus g surface Σ_g such that $M = H_1 \cup_{\Sigma_g} H_2$. Every such 3-manifold admits a Heegaard splitting. By choosing a basepoint on Σ_g, we then obtain the following pushout diagram between fundamental groups, where the maps are induced by inclusion.

$$\begin{tikzcd}
\pi_1(\Sigma_g,*) \arrow[r] \arrow[d] & \pi_1(H_1,*) \arrow[d] \\
\pi_1(H_2,*) \arrow[r] & \pi_1(M,*)
\end{tikzcd}$$

Note that $\pi_1(H_1,*)$ and $\pi_1(H_2,*)$ are both free groups of rank g, and the maps are surjections. Jaco proved that given a surjective homomorphism $\phi\colon \pi_1(\Sigma_g,*) \to F_g$, there is a unique handlebody $H(\phi)$ with $\partial H(\phi) = \Sigma_g$ such that the map induced on π_1 by inclusion of Σ_g as the boundary agrees with ϕ [6]. From this, it follows that a pair of surjective homomorphisms (ϕ_1, ϕ_2) with $\phi_i\colon \pi_1(\Sigma_g,*) \to F_g$ determines a 3-manifold $H(\phi_1) \cup_{\Sigma_g} H(\phi_2)$, and that every closed 3-manifold arises in this way. Jaco referred to these pairs of maps as *splitting homomorphisms*.

One concrete application of this is the following group-theoretic recasting of the 3-dimensional Poincaré conjecture. Writing $\pi_1(\Sigma_g,*) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1,b_1]\cdots[a_g,b_g] = 1\rangle$, there is a surjective homomorphism

$$\begin{aligned}
\pi_1(\Sigma_g,*) &\to \langle x_1,\ldots x_g\rangle \times \langle y_1,\ldots y_g\rangle \\
a_i &\mapsto (x_i, 1) \\
b_j &\mapsto (1, y_j).
\end{aligned}$$

The Poincaré conjecture is equivalent to the statement that this is the unique surjective homomorphism of these groups modulo pre-composing with automorphisms and post-composing with products of automorphisms [5]. Thus by Perelman's work [11] this result follows, and we are left in the state where the only known proof of this perhaps innocent-looking group-theoretic result involves a careful analysis of Ricci flow.

In addition to the observation that every closed 3-manifold admits a Heegaard splitting, there is a corresponding uniqueness theorem called the *Reidemeister-Singer theorem*, which states that any two Heegaard splittings of a fixed 3-manifold differ by a sequence of simple inverse geometric operations called stabilization and destabilization. Jaco proposed a way of incorporating the Reidmeister-Singer theorem into the construction of 3-manifolds from appropriate pairs (ϕ_1, ϕ_2) to obtain a bijective correspondence [7].

More recently, a 4-dimensional analogue of Heegaard splittings, called *trisections*, together with a corresponding uniqueness theorem has been introduced by Gay and Kirby [3]. A trisection of a closed 4-manifold X^4 is a decomposition $X = X_1 \cup X_2 \cup X_3$ into 4-dimensional 1-handlebodies X_i, which pairwise intersect in genus g handlebodies H_g, and with triple intersection a genus g surface Σ_g. Every smooth, closed, connected, oriented 4-manifold admits a trisection, which is unique up to a stabilization operation [3]. The inclusion maps between the various components of a trisection of a 4-manifold induce maps between their fundamental groups, which produces the following commutative diagram, where every face is a pushout and every homomorphism is surjective.

$$
\begin{tikzcd}
 & \pi_1(H_g,*) \arrow[rr] \arrow[dd] & & \pi_1(X_1,*) \arrow[dd] \\
\pi_1(\Sigma_g,*) \arrow[ur] \arrow[rr] \arrow[dd] & & \pi_1(H_g,*) \arrow[ur] \arrow[dd] & \\
 & \pi_1(X_3,*) \arrow[rr] & & \pi_1(X^4,*) \\
\pi_1(H_g,*) \arrow[ur] \arrow[rr] & & \pi_1(X_2,*) \arrow[ur] &
\end{tikzcd}
$$

In [1], Abrams, Gay, and Kirby noticed that the analogue of being able to recover a 3-manifold from a pair of surjective homomorphisms (ϕ_1, ϕ_2) holds in dimension four via trisections. Namely, given three surjective homomorphisms (ϕ_1, ϕ_2, ϕ_3) with $\phi_i \colon \pi_1(\Sigma_g, *) \to F_g$ such that the pairwise pushout of any pair ϕ_i and ϕ_j is a free group F_k, then since $\#^k(S^1 \times S^2)$ is the unique closed, orientable 3-manifold with fundamental group F_k (by Perelman's work [11]), we obtain a closed 4-manifold by realizing three handlebodies $H(\phi_i)$, gluing them along their common boundary Σ_g, and filling in their pairwise unions, which are diffeomorphic to $\#^k(S^1 \times S^2)$, with three 4-dimensional 1-handlebodies (uniquely by [8]). They called these triple of maps a *group trisection*, where the object being trisected is the group resulting from pushing out the three maps into a cube (in this case, $\pi_1(X^4, *)$).

Additionally in [1], Abrams, Gay, and Kirby use the uniqueness theorem for tisections to obtain results analogous to those previously mentioned in dimension three. Namely, they obtain a group-theoretic statement that is equivalent to the smooth 4-dimensional Poincaré conjecture and, by modding out the set of such triples (ϕ_1, ϕ_2, ϕ_3), they obtain a bijection between a group-theoretically defined set and the set of all smooth, closed, connected, oriented 4-manifolds.

Not only can every 3-manifold be split into a union of two handlebodies, but additionally, given a link $L \subset M$ we have a Heegaard splitting $M = H_1 \cup_{\Sigma_g} H_2$ such that the tangles $T_1 = L \cap H_1$ and $T_2 = L \cap H_2$ are trivial (that is, consist of arcs that can all be simultaneously isotoped in H_i into Σ_g). This is called a *bridge splitting* of $L \subset M$. Note that the complement of L in each handlebody is again a handlebody and hence has free fundamental group.

One dimension up, a similar story emerges. A *knotted surface* is a closed (potentially non-orientable or disconnected) surface smoothly embedded in a 4-manifold. Meier and Zupan showed that given a knotted surface in a trisected 4-manifold, it can always be isotoped to be in *bridge position*, meaning that it intersects the trisected 4-manifold in such a way that the surface inherits its own trisection, called a *bridge trisection* [9, 10]. This is unique up to a stabilization operation [9, 4]. Given the existence and uniqueness of such a decomposition in this setting, it is natural to wonder whether knotted surfaces in 4-manifolds can also be given such a group-theoretic framework.

In this talk, which is a report on a recent preprint of the authors [2], we unify the previous work done for closed 3- and 4-manifolds, and generalize this correspondence to the case of links in closed, oriented 3-manifolds and links of knotted surfaces in smooth, closed, connected, oriented 4-manifolds. Just as the cases of 3- and 4-manifolds are facilitated by Heegaard splittings and trisections, respectively, our result for links in 3-manifolds and surfaces in 4-manifolds use bridge splittings and bridge trisections, respectively. The algebraic manifestations of these four subfields of low-dimensional topology (3-manifolds, 4-manifolds, knot theory, and knotted surface theory) are all strikingly similar, and this correspondence perhaps elucidates some unique character of low-dimensional topology.

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Highly twisted diagrams

Nir Lazarovich
(joint work with Yoav Moriah, Tali Pinsky)

As knot theorists we aim to extract topological information about a knot directly from its diagram. One instance of this is the following theorem of Menasco about the hyperbolicity of alternating knots.

Theorem 1 (Menasco [5])**.** *Let K be a prime knot with an alternating diagram, then K is either a torus knot, or a hyperbolic knot.*

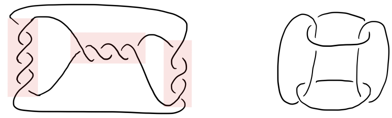

FIGURE 1. A 3-highly twisted diagram with its twist-regions highlighted in red (left), and a non-hyperbolic link with a 2-highly twisted diagram (right).

We recall that a knot is *hyperbolic* if its complement can be endowed with a complete Riemannian metric of constant negative sectional curvature, and a knot diagram is *alternating* if when following the knot the crossings alternate between over and under crossing. In the same spirit, we proved:

Theorem 2 (Lazarovich-Moriah-Pinsky [4] for 3-highly twisted diagrams, and Futer-Purcell [2] for 6-highly-twisted)**.** *Let K be a connected, prime, twist-reduced, 3-highly twisted link diagram with at least 2 twist regions, then K is hyperbolic.*

Let P be the projection plane, and consider the diagram of a knot K on P. A *twist region* is a maximal subdiagram D in which $K \cap D$ is a simple twisting of two strands of the knot, as the highlighted regions in Figure 1. The diagram is *k-highly twisted* if every twist region has at least k crossings. The definition of prime and twist-reduced is better summarized in a picture than in words, see Figure 2. Note that one can always assume that a diagram is twist-reduced by performing flypes. We remark that Theorem 2 is sharp as there are 2-highly twisted links which are not hyperbolic, such as the one in Figure 1.

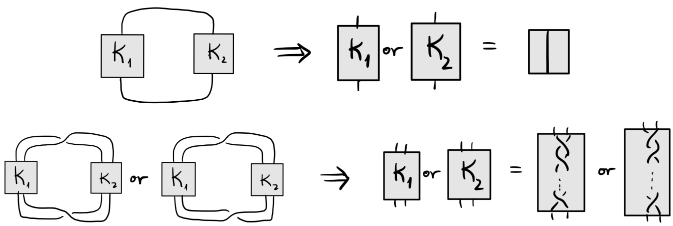

FIGURE 2. The diagram conditions: prime (top) and twist-reduced (bottom).

While the proof of Futer-Purcell uses the 6-surgery Theorem of Agol [1] and Lackenby [3], our proof is closer to Menasco's proof and relies on studying the intersection of surfaces with the plane P. We outline it here:

Let K be a knot as in the theorem, and let $M = \mathbb{S}^3 - \mathcal{N}(K)$ be its complement. To prove that M is hyperbolic we will use the following theorem of Thurston.

Theorem 3 (Thurston [6])**.** *If a 3-manifold M is connected, compact, oriented, irreducible, atoroidal, with non-empty incompressible, acylindrical torus boundaries, then M is hyperbolic.*

The underlined properties are all related to the non-existence of proper essential spheres, tori, disks and annuli in M (respectively). Let us focus on proving that M is irreducible and atoroidal. That is we have to show that every sphere in M is inessential, and every incompressible torus in M is boundary parallel. By contradiction assume that S is an essential sphere or a torus in $M = \mathbb{S}^3 - K$.

For every crossing of the knot K consider a little sphere, "bubble", centered at the point of crossing on the plane P. Let P^+ (resp. P^-) be the plane P in which we replace the small disks cut by the bubbles by their upper (resp. lower) hemispheres. The knot can be placed on $P^+ \cup P^-$, and we may assume that the surface S intersects each of P^+ and P^- transversely. Since S is a closed embedded surface, the intersection $S \cap P^\pm$ is a collection of disjoint simple closed curves on $P^\pm$.

Using the assumption that S is essential, we show that, up to homotoping S, one can compute the Euler characteristic of S using the intersection curves as follows: For each curve c in $S \cap P^\pm$ define its *contribution* to be $\chi_+(c) = 1 - \frac{1}{4}J(c)$ where $J(c)$ is the number of bubbles c meets (counted with multiplicities). Then,

$$\chi(S) = \sum_c \chi_+(c) \tag{1}$$

where the sum is taken over all curves in $S \cap P^+$ and $S \cap P^-$.

In view of (1), it would have been extremely convenient if $\chi_+(c) \leqslant 0$ for all curves. This is indeed the case if the diagram is alternating, however in our setting there are curves for which $J(c) = 2$, and so $\chi_+(c) = \frac{1}{2} > 0$. The strategy of proof is therefore to redistribute the positive contributions of "positively contributing" curves among the "negatively contributing" curves. To do so, we define a new contribution $\chi'(c)$ for all intersection curves c, such that

$$\chi(S) = \sum_c \chi'(c) \quad \text{and} \quad \chi'(c) \leqslant 0. \tag{2}$$

The precise definition is too involved for this abstract. However, the mere existence of such a function already suffices to deduce that $\chi(S) \leqslant 0$ and hence S is not a sphere. When S is a torus, (2) implies that $\chi'(c) = 0$ for all intersection curves. By analyzing the possible cases in which this happens one arrives to the conclusion that the torus S is, in fact, boundary parallel.

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Triangulations of Seifert fibered spaces

Adele Jackson

One common approach to algorithmic problems in a 3-manifold M is to show that a certain surface or curve of interest has a small normal representative in any triangulation of M. We present some results in this vein from a recent paper [1].

Theorem 1. *Let M be a Seifert fibered space with non-empty boundary and let $\mathcal{T}$ be a (material) triangulation of M. The collection of singular fibres of M that are not of multiplicity two have disjoint simplicial representatives in $\mathcal{T}^{(79)}$, the 79^{th} barycentric subdivision of $\mathcal{T}$. Furthermore, in $\mathcal{T}^{(82)}$, these simplicial singular fibres have disjoint simplicial solid torus neighbourhoods such that there is a simplicial meridian curve of length 48 for each such neighbourhood.*

We sketch the proof of this result. First, replace the triangulation of M by its dual handle structure. Take normal surface representatives of a disjoint collection of annuli that separate neighbourhoods of the singular fibres from M. We introduce the idea of the *parallelity bundle*, which is the collection of the regions lying between parallelity elementary discs, and naturally has an I-bundle structure. We analyse this bundle for cutting along one of the separating annuli to conclude that it consists of I-bundles over discs, annuli, and Möbius bands. Show that I-bundles over Möbius bands are impossible when the singular fibres is not of multiplicity two, remove the bundles over annuli and some of those over discs, and replace the remaining I-bundles over discs with 2-handles. We then invoke a result of Lackenby to give a singular fibre representative which runs only through the 0- and 1-handles, so avoids the parallelity bundle entirely, and has controlled intersection with these handles [2]. Finally, we use these bounds to make this core curve simplicial in the 79^{th} barycentric subdivision.

This result is the main technical tool in the proof of a theorem about the triangulation complexity of a Seifert fibered space. The *triangulation complexity* $\Delta(M)$ of a 3-manifold M is the minimal number of tetrahedra in a triangulation of M. It is known for some specific families such as the lens spaces $L(2n, 1)$ and prism manifolds, and to within multiplicative bounds for elliptic and sol geometries and hyperbolic mapping tori, and there is a lower bound on it from the simplicial volume of M. In general, it is quite hard to determine.

Definition 1. For $0 < q < p$, let

$$\frac{q}{p} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}$$

be the unique continued fraction expansion with $a_i > 0$. The *continued fraction norm* $\|q/p\|$ is $\sum_i a_i$.

Theorem 2. *There exists $k > 0$ such that for any Seifert fibered manifold M with non-empty boundary other than the solid torus, whose Seifert data is*

$[\Sigma, p_1/q_1, \ldots, p_n/q_n]$ *with (without loss of generality)* $0 < q_i < p_i$ *for each* i,

$$\frac{1}{k}\left(|\chi(\Sigma)| + \sum_{i=1}^{n} \|q_i/p_i\| + 1\right) \leqslant \Delta(M) \leqslant k\left(|\chi(\Sigma)| + \sum_{i=1}^{n} \|q_i/p_i\| + 1\right).$$

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Selectively searching the Pachner graph

Alexander He

(joint work with Ben Burton)

Lackenby and Schleimer [1] recently used **core curves**—curves γ embedded in a lens space such that removing a regular neighbourhood of γ leaves behind a solid torus—to show (among other things) that lens space recognition is in NP. Roughly, the idea is to find a core curve in the 1-skeleton of a given triangulation $\mathcal{T}$; however, they need to perform 86 barycentric subdivisions on $\mathcal{T}$ before they can guarantee the existence of such a **simplicial** core curve.

It is natural to wonder whether it is really necessary to perform so many subdivisions. Even more optimistically, we might hope that if we are given any one-vertex triangulation $\mathcal{T}$ of a lens space, then we can find a simplicial core curve in $\mathcal{T}$ without needing to subdivide even once. In other words, could every one-vertex triangulation $\mathcal{T}$ of a lens space contain at least one **core edge**—an edge that realises a core curve? The answer turns out to be "no": we used a targeted search algorithm to find a 20-tetrahedron one-vertex 3-sphere with no core edges (for the 3-sphere, another way to say this is that we have a one-vertex triangulation whose edges are all non-trivially knotted).

Before we describe how our search algorithm works, it is worth emphasising that the "targeted" part of this algorithm is critical to making it feasible to find this counterexample. To illustrate why, consider the census of all triangulations with up to 10 tetrahedra, which contains 422 533 279 one-vertex triangulations of the 3-sphere. We needed over 22 hours of wall time, running on 12 threads in parallel, to certify that all of these 3-spheres have at least one core edge (we also performed a more exhaustive test that required over 53 hours to certify that all of these 3-spheres have at least *two* core edges). Since the number of possible triangulations will grow exponentially as we increase the number of tetrahedra beyond 10, there is little hope that it would be feasible to simply use a brute-force search to find a one-vertex 3-sphere with no core edges.

Our search algorithm uses **2-3** and **3-2 moves** to generate new triangulations; these moves are illustrated in Figure 1. Notice that a 3-2 move begins with a configuration of three distinct tetrahedra arranged around an edge e; moreover, if

e happens to be a core edge, then notice that performing this 3-2 move has the effect of removing this core edge, but otherwise leaving the 1-skeleton untouched.

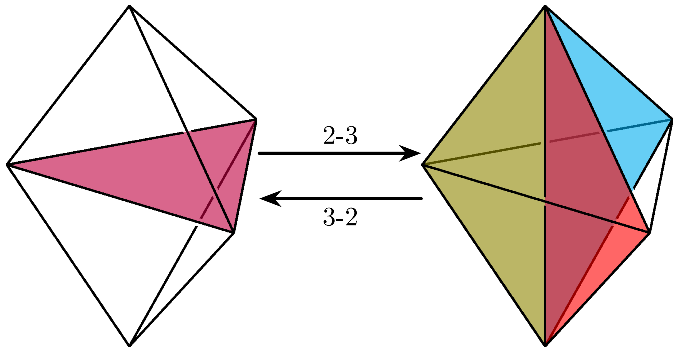

FIGURE 1. The 2-3 and 3-2 moves.

With this in mind, the main idea of our algorithm is to search for a triangulation in which one of the core edges can be removed using a 3-2 move. Provided that the search never uses an intermediate 2-3 move that introduces a *new* core edge, we can (at least in principle) run this search repeatedly until we have removed all of the core edges from our initial triangulation.

For the reasons we have already mentioned, running such a search using an exhaustive approach is unlikely to be feasible. What we do instead is perhaps obvious in hindsight: since a core edge e can be removed using a 3-2 move if and only if e has degree 3 and actually meets three distinct tetrahedra, we can try to minimise how "far away" the following two quantities are from what we want:

(1) the **degree** $d(e)$ of e (the number of tetrahedra that meet e, counted with multiplicity); and
(2) the number $n(e)$ of *distinct* tetrahedra that meet e.

More precisely, if we are given a t-tetrahedron triangulation containing core edges $e_1, \ldots, e_k$, where $k \geqslant 1$, then our search algorithm seeks to minimise the following **complexity** with respect to the lexicographical ordering:

$$\left(k,\ \max_{1\leqslant i\leqslant k} \left(d(e_i) - n(e_i)\right),\ \max_{1\leqslant i\leqslant k} |d(e_i) - 3|,\ t\right).$$

This targeted approach proved to be remarkably effective: our search algorithm only needed to enumerate 2256 triangulations to find a 22-tetrahedron one-vertex 3-sphere with no core edges (we were subsequently able to simplify this to a 20-tetrahedron triangulation without introducing any core edges). For contrast, recall that even up to just 10 tetrahedra, we already have over 400 million one-vertex triangulations of the 3-sphere, and it is almost an understatement to say that the number of such triangulations up to 22 tetrahedra would be much larger than this. Because our algorithm was able to hone in on such a small portion of the search space, it terminated very quickly: even running on a laptop with an Intel Core i5-7200U processor, which has just two physical cores divided into four logical processors, we only needed about 94 seconds of walltime to find our counterexample.

To conclude, we note that the key ideas of our targeted search extend beyond core edges. A relatively immediate extension is that we can replace core edges with some other type of interesting edge. In particular, we have adapted our search algorithm to find:

(1) ideal triangulations of various knots with tunnel number equal to one, such that none of the edges of these triangulations realise tunnel arcs; and
(2) a one-vertex triangulation of a small Seifert fibre space such that none of the edges of this triangulation are isotopic to Seifert fibres.

More ambitiously, the success of our targeted search suggests that, with the right heuristics, it may be possible to dramatically speed up other applications of 2-3 and 3-2 moves, such as:

- using such moves to improve triangulations with respect to important measures of complexity, like the number of tetrahedra or the treewidth; or
- finding sequences of such moves that connect two given triangulations, in order to prove that these triangulations are homeomorphic.

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Strongly quasipositive knots are concordant to infinitely many strongly quasipositive knots

Paula Truöl

This talk provides context on the following result.

Theorem 1 ([16]). *Every strongly quasipositive knot other than the unknot is smoothly concordant to infinitely many pairwise non-isotopic strongly quasipositive knots.*

A similar result holds for links, but for the purposes of this presentation we focus on knots. A *knot* in the 3-sphere S^3 is a non-empty, connected, oriented, closed, smooth 1-dimensional submanifolds of S^3, considered up to ambient isotopy. Two knots K and J are called *concordant* if there exists an annulus $A \cong S^1 \times [0,1]$ smoothly and properly embedded in $S^3 \times [0,1]$ such that $\partial A = K \times \{0\} \cup J \times \{1\}$ and such that the induced orientation on the boundary of the annulus agrees with the orientation of K, but is the opposite one on J. *Slice* knots are those knots that are concordant to the unknot. Knots up to concordance form a group, the *concordance group* $\mathcal{C}$, with the group operation induced by connected sum. Isotopic knots are concordant, but the converse is in general not true as any nontrivial slice knot shows.

We are particularly interested in families of knots for which concordance does imply isotopy. Consider the following set of inclusions [7, 13, 11]:

$$\{\text{positive torus knots}\} \subset \{\text{algebraic knots}\} \subset \{\text{positive knots}\}$$
$$\subset \{\text{strongly quasipositive knots}\} \subset \{\text{quasipositive knots}\}.$$

The superordinate set of quasipositive knots considered here arises in algebraic geometry as the set of those knots that are transverse intersections of complex algebraic curves in $\mathbb{C}^2$ with the 3-sphere $S^3 \subset \mathbb{C}^2$. This provides a geometric characterization of these knots [14, 5]. Litherland [10] showed that *algebraic knots* — knots of isolated singularities of complex algebraic curves in $\mathbb{C}^2$ — are isotopic if they are concordant. In particular, every concordance class in $\mathcal{C}$ contains at most one algebraic knot. Positive torus knots $T_{p,q}$ for coprime positive integers p, q form a well-known example of algebraic knots; they arise as $V(f) \cap S^3_\varepsilon \subset S^3_\varepsilon \subset \mathbb{C}^2$, where $V(f)$ denotes the zero-set of $f\colon \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto y^p - x^q$, and S^3_ε a 3-sphere of radius $\varepsilon > 0$ centered at the origin in $\mathbb{C}^2$. Moreover, Baader, Dehornoy, and Liechti [3] showed that every concordance class in $\mathcal{C}$ contains at most finitely many *positive knots*, which are those knots that admit a diagram in which all crossings are positive. It is an open question whether concordance implies isotopy for the set of positive knots.

Theorem 1 stands in contrast to these results as it states that each equivalence class in the concordance group $\mathcal{C}$ of a non-trivial strongly quasipositive knot contains infinitely many such knots. A knot is called *strongly quasipositive* if it is the closure of a strongly quasipositive braid $\beta \in B_n$ for some $n \geqslant 1$. Here B_n denotes the braid group on n strands which can be presented by $n - 1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| \geqslant 2 \quad \text{and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad [2].$$

An n-braid is called *strongly quasipositive* if it is a (finite) product of certain conjugates of the positive Artin generators σ_i of B_n, namely of the *positive band words* $\sigma_{i,j}$, where

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\, \sigma_{j-1}\, (\sigma_i \cdots \sigma_{j-2})^{-1} \qquad \text{for } 1 \leqslant i < j \leqslant n.$$

Each knot K that is the closure of a strongly quasipositive braid β comes equipped with a canonical Seifert surface associated to β [12] and by work of Bennequin [6] and Rudolph [15] — the latter building on Kronheimer and Mrowka's proof of the local Thom conjecture [9] — these surfaces realize the genus and the smooth 4-genus of K. Since the 4-genus is a concordance invariant and the unknot is the only knot of genus zero, this implies that there is only one strongly quasipositive slice knot and thus the nontriviality assumption in Theorem 1 is necessary.

Theorem 1 shows that the following conjecture by Baker is not true in a very strong sense if the assumption of fiberedness is dropped. A knot K is called *fibered* if its complement in S^3 is the total space of a locally trivial fiber bundle whose fiber is a Seifert surface for K.

Conjecture 2 ([4])**.** *Concordance implies isotopy for strongly quasipositive, fibered knots.*

Baker showed that either Conjecture 2 is true or the slice-ribbon conjecture is false. The slice-ribbon conjecture goes back to a question asked by Fox in the 1960s [8] and asserts that every slice knot is *ribbon*, i.e. bounds an immersed disk in S^3 with only ribbon singularities.

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End-periodic homeomorphisms and volumes of mapping tori

Marissa Loving

(joint work with Elizabeth Field, Heejoung Kim, Autumn Kent, and Christopher Leininger)

An incredibly fruitful relationship in low dimensional topology is the interplay between the dynamics and topology of surface homeomorphisms, and the geometric structure of their associated mapping tori. For example, a fundamental theorem of Thurston established that if $f : \Sigma \to \Sigma$ is a homeomorphism of a finite-type surface and M_f is its associated mapping torus, then M_f is hyperbolic if and only if f is pseudo Anosov. It is natural to ask how much this connection can be refined. In particular, given finer dynamical data about $f : \Sigma \to \Sigma$, can we obtain finer geometric data about M_f, e.g. its volume?

Some examples of finer dynamical data come from the action of the *mapping class group* $\mathrm{Map}(\Sigma)$ (the group of isotopy classes of orientation preserving homeomorphisms of Σ) on Teichmüller space $\mathcal{T}(\Sigma)$ (the space of equivalence classes of marked hyperbolic structures on Σ). Depending on the metric we place on $\mathcal{T}(\Sigma)$ we obtain two different pieces of dynamical data related to a homeomorphism f: the Teichmüller translation distance of f, $\log(\lambda(f))$, and the Weil–Petersson translation distance of f, $\tau_{\mathrm{WP}}(f)$. When $f : \Sigma \to \Sigma$ is a pseudo Anosov homeomorphism, both $\log(\lambda(f))$ and $\tau_{\mathrm{WP}}(f)$ yield information about the volume of M_f. There are many results of this kind, but here are two particularly illustrative ones.

First is a theorem proven independently by Kojima–McShane [KM18] and Brock–Bromberg [BB16] which relates the volume of M_f to the Teichmüller translation distance of f. Second is a theorem of Brock relating the volume of M_f to the Weil–Petersson translation distance of f.

Theorem 1 (Brock–Bromberg, Kojima–McShane)**.** *If $f : \Sigma \to \Sigma$ is a pseudo-Anosov, then*

$$\log(\lambda(f)) \geqslant \frac{1}{3\pi|\chi(\Sigma)|}\mathrm{Vol}(M_f).$$

Theorem 2 (Brock)**.** *Let $f : \Sigma \to \Sigma$ be pseudo Anosov. Then there is some $K > 0$ so that*

$$\frac{1}{K}\tau(f) \leqslant \mathrm{Vol}(M_f) \leqslant K\tau(f),$$

where K depends only on the surface Σ.

A key step in the proof of Brock's theorem is showing that when $\mathcal{T}(\Sigma)$ is equipped with the Weil–Petersson metric, $\mathcal{T}(\Sigma)$ is quasi-isometric to the pants graph $\mathcal{P}(\Sigma)$ with the standard word metric. Here, we define $\mathcal{P}(\Sigma)$ to be the graph with vertices given by isotopy classes of pants decompositions of Σ (e.g. maximal collections of pairwise disjoint simple closed curves), and edges defined by so-called *elementary moves*, described in Figure 1.

The motivation question for my work is to ask if similar relationships between the dynamics and topology of surface homemorphisms and the geometry of their associated mapping tori hold for other "interesting" surface homeomorphisms. Namely, for dynamically rich homeomorphisms of infinite-type surfaces. One class of such homeomorphisms arise in the setting of depth-one foliations of 3-manifolds.

Consider a taut, depth-one foliation $\mathcal{F}$ of a closed, hyperbolic 3-manifold M. Any noncompact leaf S of $\mathcal{F}$ is an infinite-type surface with finitely many ends all accumulated by genus. This leaf is dense in an open submanifold $M_f \subseteq M$, which is the mapping torus of an *end-periodic homeomorphism* $f : S \to S$. See Figure 2 for an example of such a homeomorphism. This mapping torus M_f is the interior of a compact irreducible 3-manifold $\overline{M}_f$ with incompressible boundary [FKLL23, Proposition 3.1]. When f is *irreducible* (a notion introduced in unpublished work

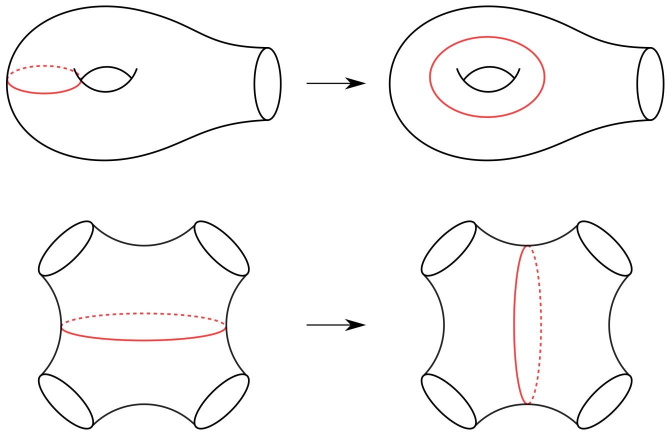

Figure 1. Elementary moves on pants decompositions come in two types depending on whether the pants curve being flipped lives in a one-holed torus or four-holed sphere

of Handel-Miller), $\overline{M}_f$ is atoroidal [FKLL23, Proposition 3.4]. When f is *strongly irreducible* M_f is also acylindrical [FKLL23, Lemma 3.5].

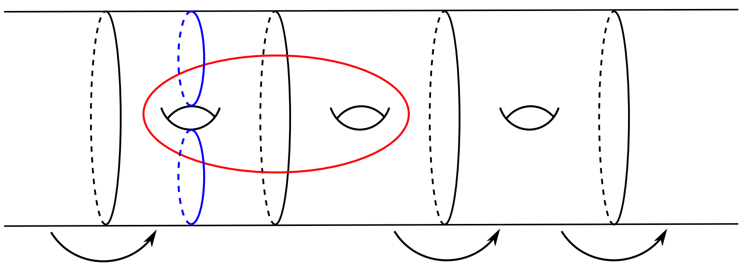

Figure 2. The homeomorphism given by the composition of the Dehn twists about the blue and red curves with a *handle shift* is an irreducible end-periodic homeomorphism [Fen97].

Thus, in many ways irreducible and strongly irreducible end-periodic homeomorphisms serve as the infinite-type analogue of pseudo-Anosov homeomorphisms of finite-type surfaces. Other important connections along these lines has been developed in work of Handel–Miller (unpublished), Fenley [Fen89, Fen92, Fen97], Cantwell–Conlon [CC16], Cantwell–Conlon–Fenley [CCF21], and Landry–Minsky–Taylor [LMT21].

We extend Theorem 2 to the infinite-type setting as follows:

Theorem 3 (in progress)**.** *For $f : S \to S$ a strongly irreducible end-periodic homeomorphism with finitely many ends, all accumulated by genus, we have*

$$K\tau(f) \leqslant \mathrm{Vol}(\overline{M}_f) \leqslant V_{\mathrm{oct}}\tau(f),$$

where K is an integer depending only on f and V_{oct} is the volume of a regular ideal octahedron.

The upper bound is proved in joint work with Field, Kim, and Leininger [FKLL23] while the lower bound is proved in forthcoming joint work with Field, Kent, and Leininger.

Our strategy to prove the lower bound is to adapt Brock's proof to the infinite-type setting, which requires developing some new machinery that we will mention briefly. We first discuss some key ideas from Brock's proof. The overarching mantra is to control the volume by controlling the existence and location of bounded length curves. This yields a lower bound on volume, as the *Margulis Lemma* tells us that each of these bounded length curves makes a definite contribution to volume. The main tool to control the existence and location of bounded length curves is by building an interpolation of the infinite cyclic cover of the mapping torus through simplicial hyperbolic surfaces (Canary [Can96], Bonahon [Bon86]) with bounded length curves corresponding to a path in the pants graph. This argument makes extensive use of Brock's work on volumes of convex cores of quasi-fuchsian 3-manifolds [Bro03], as well as work of Masur–Minsky [MM99, MM00] to relate the number of bounded length curves to distance in the pants graph.

So what breaks in this proof when f is end-periodic? We need a notion of *pleated surfaces* and *simplicial hyperbolic surfaces* in the infinite-type setting. We also need a way to control the action of f on its cores (i.e. where the action happens). Our first step in this direction is to introduce the notion of a *well-pleated surface*, which is essentially a pleated surface that interacts well with some core for f (or a power of f).

This work represents one piece of progress towards the authors' larger goal to bound volumes of 3-manifolds that admit depth-one foliations in terms of the structure of their foliations (e.g. the translation length of their associated monodromy).

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On canonical knots in tangent bundles

Andrew Yarmola
(joint work with Tommaso Cremaschi, Jake Intrater, José Andrés Rodriguez-Migueles)

We study 3-manifolds that arise as knot and link complements in unit and projective tangent bundles $UT(S)$ and $PT(S)$ of an orientable surface S, where the link $\widehat{\kappa}$ is the set of tangents to a collection of closed curves κ on S. We call these *canonical* knots and links. In recent years, there has been a lot of interest in such links as they include the class of links realized by periodic orbits of a geodesic flow on the unit tangent bundle of a surface. Classically, Ghys [Ghy07] showed that the periodic orbits of the geodesic flow over the modular surface Σ_{mod} correspond to Lorenz links in $\mathbb{S}^3$, which are periodic orbits of the Lorenz attractor. Lorenz links enjoy lots of nice properties, they are prime, positive, fibered, hence amphicherical, and moreover both the genus and their braid index can be studied combinatorially, see [BW83] for more details. Our class of interest is dramatically larger than this. Indeed, in the $PT(S)$ setting, we allow of piecewise-smooth closed curves κ on S where the non-smooth points are cusps. We call such closed curves *cusp-smooth*. Such loops can be interepreted as Legendrian diagrams on S, since $PT(S)$ has a canonical contact structure. Our first observation about such link complements is that they are just as abundant as links in $\mathbb{S}^3$.

Proposition 1. *Every closed orientable connected 3-manifold M is a Dehn filling of $PT(\Sigma_{0,n})\backslash\widehat{\kappa}$ for some n and a collection κ of smooth closed curves on $\Sigma_{0,n}$. Allowing κ to have cusps, one can take $n = 1$.*

In the case of knots, there are also analogues of classical theorems. In particular, a version of the Gordon–Luecke theorem holds in our context. Let $M_\kappa = PT(S)\backslash\widehat{\kappa}$, then:

Theorem 1. *Let S_1, S_2 be orientable surfaces and $\chi(S_1) < -1$. Consider cusp-smooth loops $\kappa_1 \subset S_1$, $\kappa_2 \subset S_2$. If $M_{\kappa_1} \cong^+ M_{\kappa_2}$ where the κ_1 end maps to the κ_2 end, then $S = S_1 \cong S_2$ and κ_1, κ_2 are equivalent under* $\mathrm{Diffeo}^+(S)$*, Legendrian Reidemeister moves, and adding/removing cusps.*

A version of the above theorem was first proven by Rodriguez-Migueles [RM1], where the κ_i are in minimal position, smooth, and one of them is homologically non-trivial on S. Our proof makes use of both algebra and topology, requiring several results about one-relator surface groups and uniqueness of $\mathbb{S}^1$-fibrations for circle bundles over hyperbolic surfaces.

Another interesting aspect of this construction is how it relates to geometry. For example, when κ is a single closed curve, an application of geometrization quickly gives that M_κ is hyperbolic if and only if κ is filling on S, see [FH13] for details. In particular, we should expect a connection between invariants of κ and invariants of M_κ. Let $\mathrm{vol}(M_\kappa)$ denote the volume of the hyperbolic pieces of the JSJ decomposition.

Theorem 2. [CM19] *Let v_3 denote the volume of the regular ideal hyperbolic tetrahedron, then*

$$\mathrm{vol}(M_\kappa) \leqslant 8v_3 i(\kappa, \kappa).$$

Theorem 3. [BPS17] *Assume $\chi(S) < 0$ and fix a hyperbolic metric X on S. Then there is a constant C_X such that for any minimal position smooth closed curve κ on S one has*

$$\mathrm{vol}(M_\kappa) \leqslant C_X \ell_X(\kappa).$$

While the first result generalized a classical fact, the second result is quite surprising. One important technicality, which forces the minimal position hypothesis above, is that M_κ is not invariant under classical Reidemeister moves, but is invariant under Legendrian ones.

Lower bounds on volume can also benefit from this surface perspective. Some recent results include:

Theorem 4. [RM1] *Given any simple multicurve η on S and collection κ of closed curves,*

$$vol(M_\kappa) \geqslant \frac{v_3}{2}(\#\{\text{isotopy classes of } \kappa\text{-arcs on } S\backslash\eta\} - 3).$$

Theorem 5. [CKMV] *Assume $\chi(S) < 0$ and fix a hyperbolic metric X on S. There is a notion of randomness for geodesics on X, such that* $\mathrm{vol}(M_\gamma) \asymp \sqrt{\ell_X(\gamma)}$ *for a random geodesic γ.*

Allowing for links that arrise from collections of simple multicurves, we can do even better. Similar to results of Brock for volumes of convex cores, one can show that the volume of our link complements is linearly asymptotic to both distance in the pants graph and Weil-Petersson distance on $\overline{\mathrm{Teich}(S)}$.

Theorem 6. [CMY21] *Let $\kappa = \{\alpha, \beta\}$ a filling pair of simple multicurves, then*

$$vol(M_\kappa) \asymp \inf_{P_\alpha, P_\beta} d_{\mathcal{P}(S)}(P_\alpha, P_\beta) \asymp d_{WP}(\mathcal{T}_\alpha, \mathcal{T}_\beta),$$

where $\mathcal{P}(S)$ is the pants graph, P_γ denotes a pants decomposition containing the simple multicurve γ, d_{WP} is Weil-Petersson distance, and $\mathcal{T}_\gamma \subset \partial\,\mathrm{Teich}(S)$ is the boundary strata with γ pinched.

Many outstanding challenges remain. For example, are cusp-shapes of M_κ dense? An approach to this problem could be to study Lorenz knots in more detail. Connecting back to Dehn surgery, another interesting challenge if to see if our knot complements admit characterising slopes, see [Lac19] for details in the $\mathbb{S}^3$ case? The most difficult question appears to be the search for a combinatorial estimate for $\mathrm{vol}(M_\kappa)$ when κ is a closed curve. One approach could be to look for polynomial invariants where κ is treated as knot diagram on a surfaces and connecting them to volume.

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Comparing Legendrian knots: a general algorithm and practical solutions

Ivan Dynnikov

The present work is motivated by an attempt to extend the monotonic simplification theorem of [2] to general links. The theorem says that any rectangular diagram of the unknot can be transformed into the simplest possible one by a sequence of elementary moves not including stabilizations. For a non-trivial topological link type, there may exist many different representations by rectangular diagrams that do not admit any simplification. However, it could be true that the number of such representations is always finite.

To understand which rectangular diagrams admit a simplification it turns out to be very useful to explore the strong connection between the rectangular diagram formalism and contact topology. Namely, with every rectangular diagram of a link one associates two equivalence classes of Legendrian links, one is Legendrian with respect to the standard contact structure ξ_+, and the other with respect to the mirror image ξ_- of ξ_+. We denote these classes by $\mathscr{L}_+(R)$ and $\mathscr{L}_-(R)$, respectively, where R is the given rectangular diagram. Figure 1 illustrates the idea.

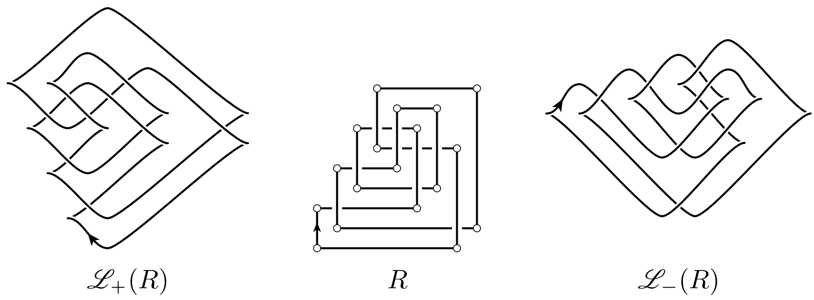

FIGURE 1. Legendrian types $\mathscr{L}_\pm(R)$

Elementary moves preserving the rectangular diagram's complexity, which are called *exchange moves*, also preserve the associated Legendrian types $\mathscr{L}_\pm$. We call the set of all diagrams obtained from R by a sequence of exchange moves *the exchange class of* R.

It was proved in [3] that a rectangular diagram of a link R admits a simplification if and only if one of the Legendrian types $\mathscr{L}_\pm(R)$ admits a Legendrian destabilization. It was also shown that, for any two rectangular diagrams R_1, R_2 representing topologically equivalent links, there is a diagram R such that $\mathscr{L}_+(R) = \mathscr{L}_+(R_1)$ and $\mathscr{L}_-(R) = \mathscr{L}_-(R_2)$. Using our recent results we can now characterize the set of all such diagrams viewed up to exchange moves as follows.

Denote by $\Lambda(R_1, R_2)$ the set of all exchange classes of rectangular diagrams R with $\mathscr{L}_+(R) = \mathscr{L}_+(R_1)$ and $\mathscr{L}_-(R) = \mathscr{L}_-(R_2)$. Denote also by $\mathrm{Mor}(R_1, R_2)$ the set of morphisms from R_1 to R_2, where by a *morphism* we mean a connected component of the space orientation-preserving self-homeomorphisms of $\mathbb{S}^3$ taking the link represented by R_1 to the one represented by R_2. The *symmetry group* $\mathrm{Mor}(R, R)$ is denoted by $\mathrm{Sym}(R)$, and the two subgroups of $\mathrm{Sym}(R)$ consisting of elements realized by a ξ_+- or ξ_--Legendrian isotopy by $\mathrm{Sym}_+(R)$ and $\mathrm{Sym}_-(R)$, respectively.

Theorem 1. *There is a bijection between* $\Lambda(R_1, R_2)$ *and the following set*

$$\mathrm{Sym}_-(R_2)\backslash\mathrm{Mor}(R_1, R_2)/\mathrm{Sym}_+(R_1).$$

Since the set $\Lambda(R_1, R_2)$ is always finite and consists of (exchange classes of) diagrams of equal complexity, this theorem can be used to distinguish Legendrian links for which the symmetry group is known. In particular, it allows to resolve in the positive most of the conjectures in the Atlas of Legendrian knots by W. Chongchitmate and L. Ng [1]. Even when the symmetry group is unknown, one can still compute algorithmically a generating set for it, which allows to establish the following result.

Theorem 2. *There is an algorithm to decide, given two rectangular diagrams of a link R_1 and R_2, whether or not $\mathscr{L}_+(R_1) = \mathscr{L}_+(R_2)$.*

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Orbits of homeomorphisms of T^2 and fibrations of the Whitehead link

Tali Pinsky

(joint work with Bronek Wajnryb, Eiko Kin)

Our aim is to present a curious example relating the idea of a forcing relation from dynamical systems and three manifolds having different fibrations over S^1.

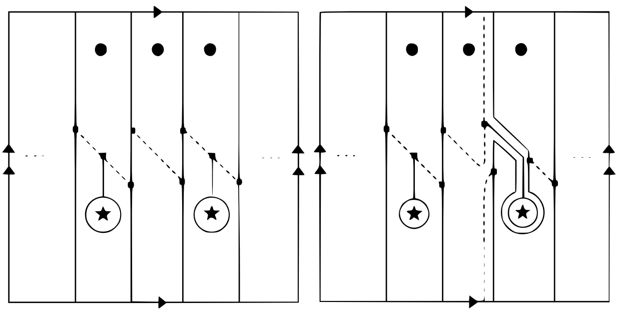
Given a surface X, a periodic point x_0 for a self homeomorphism f_0 and a periodic point x_1 for a homeomorphism f_1, we say that x_0 is isotopic to x_1 if f_0 is isotopic to f_1 and for any $t \in [0,1]$ f_t has a periodic point x_t so that x_t is a continuous arc in X. It follows that x_1 and x_2 have isotpoic braids defined in the suspension $X \times [0,1]/\sim$, or equivalently that the actions of f_0 and f_1 on the complements of the orbits are isotopic and thus represent the same group element in the punctured surface that is the complement of each orbit.

Thus specifying an orbit up to isotopy is equivalent of specifying a mapping class group element up to isotopy, which is equivalent of specifying the action of the diffeomorphism on a graph that is a spine for the surface punctured at the orbit. We are interested in *forcing relations*, i.e. when does the existence of an orbit x imply the existence of another orbit y, generalizing the Sharkovskii order for continuous maps on $\mathbb{R}$ to two dimensional systems [1].

For $X = T^2$ we define a *simple pair* to be a pair of orbits, with the action of the diffeomorphism class on the complement given by the graph map in the following figure:

For each such orbit we can define a rational rotation number, and in a joint work with Bronek Wajnryb we have shown that for x and y to be a simple pair their rotation numbers have to be Farey neighbors, and the mapping class group element defined by the above graph map is pseudo-Anosov. Furthermore, denoting the simple pair corresponding to Farey neighbors p and q by $p \vee q$, we prove that $p \vee q$ forces $r \vee s$ whenever r and s are Farey neighbors between p and q [2].

In the Whitehead link complement, there is a fibration corresponding to any surface $S_{n,m} = (m\lambda_1 + n\lambda_2)$ where λ_1 and λ_2 represent each a once punctured torus



with boundary a longitude on one of the link components [3]. In a recent work with Eiko Kin we show that the fiber corresponding to $\lambda_1 + \lambda_2$ is a twice punctured torus that is exactly the $0 \vee 1$ simple pair (i.e., the action of the monodromy on a graph in the torus minus the two punctures is exactly as in the figure above).

We use this to inductively show that the fiber corresponding to $m\lambda_1 + n\lambda_2$ has the monodromy of the simple pair $s \vee t$ where s and t are the Farey parents of m/n. Thus we have an explicit representation for all monodromies of all different fibrations of the Whitehead link complement.

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In search of the Margulis constant

David Futer

(joint work with David Gabai and Andrew Yarmola)

Consider a hyperbolic 3–manifold $M = \mathbb{H}^3/\Gamma$. The *Margulis number* $\mu(M)$ is the largest ϵ such that for every non-commuting pair $\varphi, \psi \in \Gamma$, and for every $x \in \mathbb{H}^3$, one of φ, ψ moves x by distance at least ϵ. In symbols,

$$\mu(M) = \sup\{\epsilon \;:\; \forall \varphi, \psi \in \Gamma,\ \text{if } d(x, \varphi x) < \epsilon \text{ and } d(x, \psi x) < \epsilon,\ \text{then } [\varphi, \psi] = 1\}.$$

This number controls the thick–thin decomposition of M. Here, the *ϵ–thin part*, denoted $M^{<\epsilon}$, is the set of all points lying on an essential loop of length less than ϵ. The *ϵ–thick part* is the complementary set $M^{\geqslant \epsilon} = M \setminus M^{<\epsilon}$. Then $\mu(M)$ is the largest number ϵ such that $M^{<\epsilon}$ is a disjoint union of cusps and tubes.

The Kazhdan–Margulis theorem [13] is the profound insight that the infimal value

$$\epsilon_3 = \inf\{\mu(M) : M \text{ is a hyperbolic 3–manifold}\}$$

is strictly positive. This value ϵ_3 is called the (3–dimensional) *Margulis constant.*

The Kazhdan–Margulis result is a fundamental tool in structural theorems about hyperbolic manifolds. The fact that $M^{<\epsilon}$ has standard topology for every $\epsilon \leqslant \epsilon_3$ leads to the result of Jørgensen and Thurston that all hyperbolic manifolds of volume less than V are Dehn fillings of finitely many surgery parents. The thick–thin decomposition is also a critical tool in the construction of combinatorial bi-Lipschitz models for the geometry of 3–manifolds based on fibrations [16, 4] and Heegaard splittings [3, 17]. Making the results effective, with explicits estimates on geometry, requires some estimate on ϵ_3.

Indeed, most quantitative results in hyperbolic geometry require some estimate on the Margulis constant. A sample of recent results in this vein includes the work of Cooper, Futer and Purcell on unknotting tunnels [5], the work of Detcherry and Kalfagianni relating hyperbolic volume to Turaev–Viro invariants [6], and the work of Aougab, Patel, and Taylor on curve complexes [2]. All of these papers use an estimate of Meyerhoff [14] from 1987, namely that $\epsilon_3 \geqslant 0.104$. Knowing the value of ϵ_3, or even substantially improving Meyerhoff's estimate on it [14], would substantially improve these quantitative results.

In this ongoing project with Gabai and Yarmola, we are seeking to prove

Conjecture 1. *The 3–dimensional Margulis constant ϵ_3 is uniquely realized by the Weeks manifold M_W. In particular, $\epsilon_3 = \mu(M_W) = 0.774\ldots$.*

The method of attack is to first bound the possibilities to a compact domain, and then rigorously analyze that domain by computer.

Parameter methods. Over the past 25 years, mathematicians have developed considerable expertise in proving difficult theorems about hyperbolic 3–manifolds using extensive computer assistance. For instance, Gabai, Meyerhoff, and N. Thurston [10] proved that any homeomorphism $f : M \to N$ between hyperbolic 3–manifolds is isotopic to an isometry (whereas Mostow rigidity only says f is homotopic to an isometry). Gabai, Meyerhoff, and Milley [11, 15] showed that the Weeks manifold M_W is the hyperbolic 3–manifold of smallest volume, meaning $\mathrm{vol}(M) \geqslant \mathrm{vol}(M_W)$ for every M. Gabai, Haraway, Meyerhoff, N. Thurston, and Yarmola [9] proved that every cusped M (apart from the figure–8 knot complement) has at most 8 exceptional Dehn fillings. In all of these projects, the technique has been to first prove the result for all M above a certain explicit level of topological or combinatorial complexity, and then to perform a parameter space analysis that covers all M below that level of complexity. The analysis might directly produce a list of explicit manifolds that optimize some property, or provide further geometric and/or topological restrictions that allow one to prove the final desired result.

The parameter space analysis works as follows. One considers all 2–generator subgroups $\langle \varphi, \psi \rangle \subset \mathrm{Isom}^+(\mathbb{H}^3)$, where the matrices representing φ and ψ are constrained to lie in a compact domain. (In certain contexts, e.g. [11], a third generator is also present.) The bounds on the compact domain come from the previous work on manifolds above a particular complexity. The compact domain is then subdivided into millions of tiny boxes, where each coordinate of φ and ψ is very tightly constrained within each box. Within each tiny box, some particular word in $\Gamma = \langle \varphi, \psi \rangle$ shows that the group is not discrete (contradiction), or contains torsion (contradiction), or that the desired result holds anyway. The entire computer-assisted search runs with rigorous arithmetic. See Gabai, Meyerhoff, Thurston, and Yarmola [12] for an excellent survey of the method.

Conjecture 1 is well adapted to this kind of analysis. Since the manifold M_W is known to have $\mu(M_W) < 0.775$, the search for the Margulis constant can be restricted to non-commuting isometries φ and ψ that translate some chosen basepoint $x \in \mathbb{H}^3$ by distance $\leqslant 0.775$. For a given basepoint x, the set of such isometries is compact. However, this set has a needlessly large dimension (even after quotienting by symmetries, the dimension is 9).

A more efficient parametrization runs as follows. One can show using the work of Adams on waist sizes [1] that for small Margulis numbers, the isometries φ, ψ must be loxodromic. Thus it suffices to know the (complex) translation lengths of φ, ψ along their invariant axes, in addition to the (complex) distance between the axes. This was the parametrization used in [10]. With some effort, we can bound the distance between the axes, constraining the search to a compact 6–dimensional domain. We currently have code running on the 240–core Polar cluster at Princeton that adaptively breaks up the parameter space into tiny boxes and tries to find a contradiction in each box.

The symmetric case. The parameter space analysis turns out to be particularly tractable in the presence of an additional symmetry, where φ and ψ are assumed conjugate in $\mathrm{Isom}^+(\mathbb{H}^3)$. Given $M = \mathbb{H}^3/\Gamma$, the *symmetric Margulis number of M* is

$$\begin{aligned}\mu^{\mathrm{sym}}(M) = \sup\{\epsilon \ : \forall \varphi, \psi \in \Gamma &\text{ that are conjugate in } \mathrm{Isom}^+(\mathbb{H}^3), \\ &\text{if } d(x, \varphi x) < \epsilon \text{ and } d(x, \psi x) < \epsilon \text{ then } [\varphi, \psi] = 1\}.\end{aligned}$$

We prove the following symmetric case of Conjecture 1:

Theorem 2. *The lowest value of the symmetric Margulis number is uniquely realized by the Weeks manifold M_W. In particular, $\mu^{\mathrm{sym}}(M) = \mu(M_W) = 0.774\ldots$.*

The symmetric setting of Theorem 2 is easier than the general case for two reasons. First, the parameter space has a lower dimension (real dimension 4 instead of 6). Second, in many boxes in the parameter space, the program finds a relation in φ, ψ. If φ, ψ are conjugate in $\mathrm{Isom}^+(\mathbb{H}^3)$ and generate a discrete group Γ, then they are conjugate in a finite extension of Γ. Common conjugation turns one relation into a *second* relation. With two generators and two relations, we are looking at a point in the character variety of a closed 3–manifold group, which turns out to be isomorphic to $\pi_1(M_W)$. Using the volume rigidity theorem for representations [7, 8], and the enumeration of lowest-volume manifolds [9], we can conclude that this is actually the discrete-faithful representation of M_W.

Toward the general case. To identify the Margulis constant ϵ_3, we need to study a 6–dimensional parameter space of pairs of isometries φ, ψ that are not necessarily conjugate. Here, the symmetric result of Theorem 2 provides a powerful elimination criterion. Indeed, suppose that φ, ψ generate an indiscrete group. (This will typically be the case, but we need a rigorous certificate of indiscreteness.) In this situation, there must be a word $w \in \Gamma$ such that φ and $w\varphi w^{-1}$ have axes that are extremely close. Both axes are moved by a small distance, so a common point in between is moved by a small distance. By Theorem 2, we know that either Γ is a known group (so we are happy), or Γ must be indiscrete (providing a certificate that eliminates a box in the search). The conjugating word w tends to be fairly short in practice, enabling extensive use.

Completely apart from its application toward Conjecture 1, Theorem 2 provides very strong estimates on the radius of an embedded tube about a short geodesic. The best estimates available to date, due to Meyerhoff [14], only work for geodesics shorter than about 0.114. By contrast, the proof of Theorem 2 gives results for geodesics up to length about 0.845.

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Canonical decompositions and algorithmic recognition of spatial graphs

José Pedro Quintanilha
(joint work with Stefan Friedl, Lars Munser and Yuri Santos-Rego)

Spatial graphs extend the classical notions of knots and links by allowing not only embedded unions of circles in a 3-sphere, but arbitrary compact 1-dimensional complexes (throughout, all spaces live in the piecewise-linear category). Specifically, a **spatial graph** is a triple $\Gamma = (\mathcal{S}, V, E)$, with $\mathcal{S}$ an oriented 3-sphere, $V \subset \mathcal{S}$ a finite set (of "vertices") and E a finite set of "edges", each edge being a subspace of $\mathcal{S}$ homeomorphic to an interval or a circle and satisfying natural conditions on its intersection with V. An **isomorphism** of spatial graphs is then an orientation-preserving homeomorphism of the ambient 3-spheres inducing bijections on the sets of vertices and edges.

The aim of my talk was to present the proof of the following result [1]:

Theorem 1 (Algorithmic recognition of spatial graphs)**.** *There exists an algorithm that takes as input two spatial graphs and decides whether they are isomorphic.*

The basic idea is to encode each spatial graph as a **manifold with boundary pattern**, that is, a compact 3-manifold together with a compact 1-dimensional sub-complex of its boundary, and then appeal to the following theorem, written down in detail by Matveev building on work of Haken [2, Theorem 6.1.6]:

Theorem 2 (Recognition Theorem)**.** *There is an algorithm that takes as input two Haken manifolds with boundary pattern and decides whether they are homeomorphic (respecting boundary patterns).*

Before explaining the crucial "Haken" condition, let us sketch the construction of a manifold with boundary pattern encoding a given spatial graph Γ. We remove a small open neighborhood of the vertices and edges of Γ in its ambient 3-sphere $\mathcal{S}$, and mark the boundary of the resulting 3-manifold X_Γ with a pattern that allows for reconstructing Γ; see Figure 1 for the basic idea.

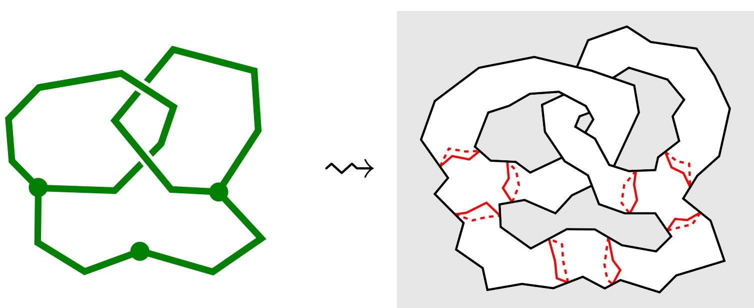

Figure 1. A spatial graph, and its exterior marked with a boundary pattern.

The complete boundary pattern P_Γ (see Section 6.1 of the manuscript) also indicates which regions of $\partial X_\Gamma \backslash P_\Gamma$ correspond to vertices/edges, as well as the orientation of X_Γ induced from $\mathcal{S}$. The resulting (X_Γ, P_Γ), which we call the **marked exterior** of Γ, thus gives a perfect encoding:

Proposition 2 (Encoding via marked exteriors)**.** *Two spatial graphs are isomorphic if and only if their marked exteriors are homeomorphic.*

Were it not for the "Haken" hypothesis in Theorem 2, and Theorem 1 would follow immediately. A manifold with boundary pattern (M, P) is **Haken** if it satisfies three conditions – a technical one that is easily dealt with in our setting, and two requiring more attention: irreducibility and boundary-irreducibility. (M, P) is **irreducible** if M itself is irreducible, that is, every embedded 2-sphere in M bounds an embedded 3-ball. It is **boundary-irreducible** if for every properly embedded disk $D \subset M$ disjoint from P, its boundary ∂D bounds a disk in ∂M disjoint from P.

Given a spatial graph Γ, irreducibility of X_Γ is equivalent to Γ being **non-split**, that is, there being no embedded 2-sphere S in the ambient 3-sphere $\mathcal{S}$ with vertices of Γ on both sides. This suggests implementing a topological analogue of the disjoint union operation on abstract graphs, leading to the definition of the **disjoint union** of spatial graphs (here we use the orientations of the ambient 3-spheres). Designating non-empty, non-split spatial graphs by **pieces**, we have:

Proposition 3 (Unique factorization into pieces)**.** Γ *can be expressed as a disjoint union of finitely-many pieces* $\Gamma \cong \bigsqcup_{i\in I} \Lambda_i$*. Any two such decompositions have pairwise-isomorphic pieces.*

Proposition 3 reduces the task of testing if two spatial graphs are isomorphic, to decomposing them as a disjoint union of pieces (which one can do using standard machinery for finding reducing spheres in 3-manifolds) and testing whether those are pairwise-isomorphic. Since pieces have irreducible exteriors, we are one step closer to being able to apply Theorem 2.

Next we ask when the marked exterior of a piece Γ is boundary-irreducible, and this condition again translates into a natural property. Call a vertex v of Γ **cut** if there is an embedded sphere S in the ambient sphere $\mathcal{S}$, containing v but otherwise disjoint from the vertices and edges of Γ, such that there are edges on both sides of S. It turns out that (X_Γ, P_Γ) is boundary-irreducible precisely if Γ has no cut vertices and Γ does not consist of two vertices joined by one edge.

Inspired by Proposition 3, we now seek an operation for which every piece uniquely factorizes into pieces without cut vertices. Indeed one can define such an operation, the **vertex sum**, which takes as input two spatial graphs with a distinguished vertex $(\Gamma_1, v_1), (\Gamma_2, v_2)$, and whose output consists of Γ_1 and Γ_2 "glued along the distinguished vertices".

In pursuing a unique factorization result for the vertex sum, one is faced with the difficulty of even specifying iterated vertex sums, as one needs to package the combinatorics of what vertices are to be glued. To that end, we introduce the notion of a **tree of spatial graphs** $\mathcal{T}$, which consists of:

(1) a tree T with vertices in two alternating colors, green and blue, with all blue vertices having degree at least 2,
(2) for each green vertex, a spatial graph, and
(3) for each edge of T, a vertex of the spatial graph at its green endpoint, with no two edges corresponding to the same vertex.

This data gives a blueprint for putting together the spatial graphs at the green vertices of T by gluing two such vertices whenever they correspond to edges of T meeting at a blue vertex ("blue rhymes with glue!"). The resulting spatial graph is called the **realization** of $\mathcal{T}$. We illustrate in Figure 2.

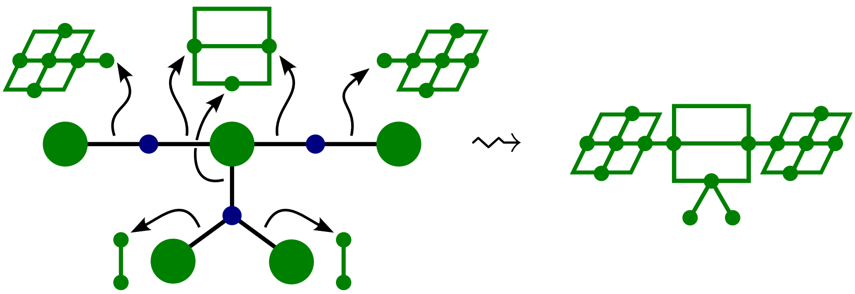

FIGURE 2. A tree of spatial graphs and its realization.

Define a **block** to be a piece without cut vertices and which does not consist of one vertex. By a **tree of blocks**, we mean a tree of spatial graphs whose spatial graphs at the green vertices are blocks. We then have:

Proposition 4 (Unique factorization into blocks)**.** *Every piece* Γ *other than a one-vertex graph is isomorphic to the realization of a tree of blocks. In any two trees of blocks for* Γ*, the underlying trees are isomorphic and the blocks are pairwise-isomorphic via isomorphisms respecting the vertices to be glued.*

Using Proposition 4, we can further reduce the isomorphism problem from pieces to blocks. Since their marked exteriors are Haken (except for a few easy special cases), we can apply the Recognition Theorem (Theorem 2), proving Theorem 1.

Using a more sophisticated boundary pattern P_Γ, we can prove Theorem 1 even for spatial graphs decorated with vertex/edge colorings or with edge orientations. Vertex colorings are in fact used in an essential way for encoding the requirement that block isomorphisms respect the combinatorics of the trees of blocks.

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On a structure theorem of Legendrian knots

Rima Chatterjee
(joint work with John Etnyre, Hyunki Min and Anubhav Mukherjee)

A contact structure ξ on a 3-manifold M is a no-where integrable 2-plane field. A contact 3-manifold (M, ξ) is a 3-manifold equipped with a contact structure ξ. There are two types of contact structures−tight and overtwisted. If one can find an overtwisted disk (a disk is overtwisted if the contact planes are tangential along its boundary) embedded in a contact manifold, we call it overtwisted. While tight contact structures still remain a mystery, overtwisted contact structures are completely classified by Eliashberg's fundamental result.

Theorem 1 (Eliashberg [3])**.** *Overtwisted contact structures (up to isotopy) are in one-to-one correspondence with* 2*-plane fields (up to homotopy).*

A knot is Legendrian if it is everywhere tangent to the contact planes. A Legendrian knot has two classical invariants − the Thurston-Bennequin invariant(also known as `tb`) and the rotation number. A knot is called *Legendrian simple* if it can be completely classified by its classical invariants. For this talk we focused on Legendrian knots in an overtwisted manifold with tight complements (also known as non-loose or exceptional knots).

It is natural to ask how a Legendrian knot behaves under certain topological constructions. For the purpose of this talk we stick to cabling. A (p, q) cable $K(p, q)$ of a knot type $\mathcal{K}$ is a (p, q) curve on the tubular neighborhood of the knot representing $\mathcal{K}$. Etnyre–Honda proved the following statement in tight manifolds.

Theorem 2 (Etnyre-Honda [5])**.** *Let $\mathcal{K}$ be a knot type which is Legendrian simple and satisfies the UTP. Then $K(p, q)$ is Legendrian simple and admits a classification in terms of the classification of $\mathcal{K}$.*

UTP or the Uniform Thickness Property is a special property for knots in tight manifolds. A knot is said to satisfy UTP if its maximum `tb` is same as its contact width. For details check [5]. It turns out that UTP is not very common among knot types. As an example, the unknot does not satisfy UTP. UTP is necessary for the above theorem as there are Legendrian simple knot types whose cables are not Legendrian simple as they do not satisfy UTP [6]. Later Tosun [7], Etnyre-LaFountain-Tosun [6] and recently Chakraborty-Etnyre-Min [1] relaxed this condition. All of these results are in tight contact manifolds.

The motivation of our project was to have a similar structural result for non-loose Legendrian knots in any overtwisted 3-manifold. The following is out main result:

Theorem 3 ([2])**.** *Suppose $\mathcal{K}$ is a knot type in an overtwisted contact* 3*-manifold (M, ξ). Suppose L be a Legendrian representative of $\mathcal{K}$ in (M, ξ). Then for $\frac{q}{p} >$ $\mathtt{tb}(L)$, the standard (p, q) cable $L(p, q)$ of L is non-loose if and only if L is non-loose.*

As a standard neighborhood of a cable is contained in the standard neighborhood of the underlying knot, one direction is obvious. To prove the other direction we mainly relied on Colin and Honda's state transition technique.

Our next theorem states a condition on non-looseness when $\frac{q}{p} < \mathtt{tb}(L)$.

Theorem 4 ([2])**.** *Let (M, ξ) be an overtwisted contact 3-manifold and L be a Legendrian knot in it. Suppose $\frac{q}{p} \in (\mathtt{tb}(L) - 1, \mathtt{tb}(L))$. Then the standard cable $L(p, q)$ is non-loose if and only if $S_\pm(L)$ is non-loose. Here $S_\pm(L)$ denotes the positive (resp. negative) stabilization of L.*

Notice, this tells us that a non-loose left handed trefoil can never arise as the $(-2, 3)$ cable of the non-loose unknot in the same contact structure. Note that, non-loose unknot can only live in (S^3, ξ_{-1}) [4] where ξ_{-1} denotes the unique overtwisted contact structure on S^3 with Hopf invariant -1.

While we shed some light on cabling of non-loose knots, our future goal will be to answer similar questions on other topological constructions. For example:

Question: Suppose K_1 and K_2 be two non-trivial non-loose knots in an overtwisted contact 3-manifold (M, ξ). Is $K_1 \# K_2$ non-loose?

Currently, we have examples that if one of K_i's is trivial then the connected sum operation does not preserve tightness of the complement.

A similar question can also be asked for other satellite constructions as well.

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Endperiodic maps via pseudo-Anosov flows

Michael P. Landry

(joint work with Yair N. Minsky, Samuel J. Taylor)

Let L be an infinite-type surface with finitely many ends and without boundary. A homeomorphism $g \colon L \to L$ is *endperiodic* if each end of L is either attracting or repelling under a power of g, and *atoroidal* if it fixes no finite essential multicurve up to isotopy.

Such maps appear naturally in Thurston's work [Thu86] on fibered compact 3-manifolds: a fiber S can be "spun" around a sufficiently nice surface Σ yielding a foliation in which Σ is a compact leaf and its complement is fibered by parallel copies of a non-compact surface L, so that the monodromy of this fibering is an endperiodic map of L, which must be atoroidal when M is hyperbolic.

In this work we reverse this process, obtaining any atoroidal endperiodic map from some fibration by a spinning operation in a suitable hyperbolic fibered 3-manifold. More importantly, the construction can be performed so that the resulting foliation is transverse to the canonical pseudo-Anosov flow associated to the fibration (see [Fri79]), and the stable and unstable foliations of this flow induce a similar structure on L. We call the return map of such a construction a *spun pseudo-Anosov (spA) map.*

This construction has the following consequences:

- We recover, directly from the pseudo-Anosov structure, the dynamical laminations of Handel–Miller theory [CCF19].
- We identify dynamical growth rates of the spun pseudo-Anosov map: the spA map minimizes the exponential growth rate of periodic points among all homotopic endperiodic maps. Further, this rate is equal to the exponential growth rate of intersection numbers of curves under iteration and its log is the topological entropy (suitably defined) of the spA map.
- The compactified mapping torus of the endperiodic map is a manifold N with boundary, which can admit a variety of depth one foliations whose compact leaves are ∂N. These foliations are parameterized by the *foliation cones* of Cantwell–Conlon [CC99, CC17], which are analogous to the cones on fibered faces of Thurston's norm. This analogy can be made explicit by the spinning construction, and we show that the foliation cones are exactly the pullbacks of Thurston fibered cones by the inclusion of N into a certain fibered manifold M. From this we can show that topological entropy defines a continuous, convex function on each foliation cone. This mirrors the corresponding picture, due to Fried and McMullen, for Thurston's fibered cones [Fri82, McM00].

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Short decompositions of graphs embedded on surfaces

Arnaud de Mesmay

Given two graphs G_1 and G_2 embedded on a compact surface, the joint crossing number of G_1 and G_2 is the minimum number of crossings of these two graphs over all possible homeomorphic reembeddings of one of them. Intuitively, it quantifies the best way to simultaneously embed both graphs on the same surface so as to minimize the number of crossings. If, say, the first graph, is interpreted as a discrete metric, this amounts to looking for the shortest embedding of the second graph. Computing the joint crossing number is NP-hard [7], and this talk focused on the problem of merely finding simultaneous embeddings with an upper bound on the number of crossings, which is a problem that arises naturally in various settings:

- In applied settings, when one is given two surface meshes, it is often useful to compute a parameterization, i.e., a homeomorphism between them. One natural way to do this involves first cutting them both along a specific auxiliary graph so as to obtain disks, which can then be put in correspondence. Controlling the deformations occurring during this process involves controlling the intersections between the mesh and the cutting graph.
- In order to represent a graph embedded on a surface, a common way is to first cut it into a disk and represent the resulting planar drawing with the boundary identifications of the polygon. The readability of this representation will be directly tied to the intersections of the graph with this cutting graph (see, e.g., Duncan, Goodrich and Kobourov [2]).
- By graph duality, controlling the intersections of two one-vertex, one-face graphs can be recast as the problem of controlling the size of words involved in switching between different one-relator presentations of the fundamental group of a surface.
- Some variants of this problem on surface with boundaries also occur in algorithm design: for instance in the problem of deciding whether a given 2-complex embeds into $\mathbb{R}^3$ [10, 11], as well as in the problem of finding explicit upper bounds on the algorithms arising from graph minor theory [5].

A seminal result of Lazarus, Pocchiola, Vegter and Verroust [9] shows that if the surface is orientable and one of the graphs is a *canonical system of loops*, i.e., a one-vertex, one-face graph whose boundary identifications are of the form $a_1b_1a_1^{-1}b_1^{-1}\ldots a_gb_ga_g^{-1}b_g^{-1}$, one can always embed both graphs so that each pair of edges cross at most four times. Furthermore, the proof is readily algorithmic.

Strikingly, such a bound is unknown for more general graphs, and a conjecture of Negami [12] posits that the joint crossing number can always be upper bounded by $O(|E(G_1)||E(G_2)|)$, where the constant in the $O(\cdot)$ is independent of the genus. The best known bound on this general problem is $O(g|E(G_1)||E(G_2)|)$, also due to Negami [12]. See also [1] and [14] for more results around this conjecture.

In this talk, I first presented a recent result obtained with Niloufar Fuladi and Alfredo Hubard [4], where we proved an analogue of the orientable theorem of [9] in the case of non-orientable surfaces, for the non-orientable canonical system of loops $a_1a_1 \dots a_ga_g$:

Theorem 1. *There exists a polynomial time algorithm that, given a graph cellularly embedded on a non-orientable surface, computes a non-orientable canonical system of loops such that each loop in the system intersects any edge of the graph in at most* 30 *points.*

The techniques involved differ significantly from those of [9] and rely instead on an embedding technique of Schaefer and Stefankovič [15], as well as an a priori unexpected connection with the problem of computing the *signed reversal distance* between two signed permutations, which is a very well known problem in computational genomics [6].

A second part of the talk surveyed an earlier attempt to attack the conjecture of Negami using geometric techniques. For a given surface S, a *universal shortest path metric* is a (Riemannian) metric on S such that any simple graph embeddable on S can be embedded so that the edges are *shortest paths*. Since shortest paths cross generically at most once, the existence of such a metric would provide a satisfying geometric proof of the conjecture of Negami. In a joint work with Alfredo Hubard, Vojtěch Kaluža and Martin Tancer [8], we provided such metrics for the sphere, the projective plane, the torus and the Klein bottle, but proved that asymptotically, as the genus goes to infinity, a *random* (for the Weil-Petersson metric) hyperbolic metric on an orientable surface is *not* a universal shortest path metric.

This led us to the third part of the talk, where we considered the following question. Given an orientable surface S of genus g, we say that a family of closed curves Γ *realizes all types of pants decompositions* if for any pants decomposition P of S, there is a homeomorphism sending the curves of P to a subset of the curves in Γ. In a forthcoming paper with Niloufar Fuladi and Hugo Parlier [3], we investigate the minimum possible size of such a family of curves, as well as other variants of this question. The connection to the previous parts is that due to polynomial upper bounds on the size of families of curves crossing pairwise at most k times [13], obtaining an exponential lower bound for the size of a family of curves realizing all types of pants decompositions would show that asymptotically, no universal shortest path metric exists. We show that while the number of types of pants decompositions is superexponential in the genus, one can find a family of curves of singly exponential size realizing all types of pants decompositions. On the other hand, the only known lower bound is that such a family of curves has at least superlinear size. This leaves open the innocuous-looking problem of bridging the gap between the superlinear lower bound and the exponential upper bound.

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Mapping class groups of 1-connected 4-manifolds

Mark Powell

(joint work with Patrick Orson)

I will report on joint work with Patrick Orson on the mapping class groups of compact, simply-connected 4-manifolds.

Given an oriented, topological manifold X, with (possibly empty) boundary, we consider $\mathrm{Homeo}^+(X, \partial X)$, the topological group of orientation preserving self-homeomorphisms that restrict to the identity on the boundary ∂X, with the compact-open topology. The set of connected components $\pi_0 \mathrm{Homeo}^+(X, \partial X)$ is the *topological mapping class group* of X, the group of isotopy classes of orientation preserving self-homeomorphisms that fix the boundary pointwise. We study topological mapping class groups for X a compact, oriented, simply connected 4-manifold.

Let $\lambda_X\colon H_2(X)\times H_2(X)\to\mathbb{Z}$ be the intersection pairing of X. When $\partial X=\varnothing$, it was shown by Perron and Quinn [Qui86, Per86] (cf. Kreck [Kre79, Theorem 1]), that if two orientation preserving self-homeomorphisms of X induce the same isometry of the intersection form then they are isotopic. Freedman [Fre82, Theorem 1.5, Addendum] showed that every automorphism of the intersection form is induced by a homeomorphism. Therefore the results of Perron, Quinn and Freedman combine to compute the mapping class group of every closed, simply connected 4-manifold, in the sense of reducing the problem to algebra: $\pi_0\operatorname{Homeo}^+(X)\xrightarrow{\cong}\operatorname{Aut}(H_2(X),\lambda_X)$; $F\mapsto F_*$.

When X has nonempty boundary, we need to consider a refinement of $\operatorname{Aut}(H_2(X),\lambda_X)$ to capture the algebraic data of a homeomorphism. A map $F\in\operatorname{Homeo}^+(X,\partial X)$ determines a homomorphism $\Delta_F\colon H_2(X,\partial X)\to H_2(X)$ called a *variation* [Lam75, DK75, Kau74], defined by $[x]\mapsto[x-F(x)]$. Using that X has Poincaré-Lefschetz duality, Saeki [Sae06] showed that Δ_F satisfies an additional condition, making it what we call a *Poincaré variation.* There is a binary operation on the set of Poincaré variations, together with which they form a group $\mathcal{V}(H_2(X),\lambda_X)$. The map $F\mapsto F_*$ factors through this group via homomorphisms:

$$\pi_0\operatorname{Homeo}^+(X,\partial X)\xrightarrow{F\mapsto\Delta_F}\mathcal{V}(H_2(X),\lambda_X)\xrightarrow{\Delta\mapsto\operatorname{Id}-\Delta\circ j}\operatorname{Aut}(H_2(X),\lambda_X),$$

where $j\colon H_2(X)\to H_2(X,\partial X)$ is the quotient map. In general Δ_F contains more information than F_*. Saeki [Sae06] used $\mathcal{V}(H_2(X),\lambda_X)$ to describe the smooth stable mapping class group for simply connected 4-manifolds with nonempty, connected boundary.

When ∂X has more than one connected component and X admits a spin structure, there is a further invariant that does not appear in the closed case nor when the boundary is connected. For $F\in\operatorname{Homeo}^+(X,\partial X)$ we may compare a (topological) spin structure $\mathfrak{s}$ on X with the induced spin structure $F^*\mathfrak{s}$. The two agree on ∂X because F fixes the boundary pointwise. There is a free, transitive action of $H^1(X,\partial X;\mathbb{Z}/2)$ on the set of isomorphism classes of spin structures on X that agree on ∂X, and we denote by $\Theta(F)\in H^1(X,\partial X;\mathbb{Z}/2)$ the class representing the difference between $\mathfrak{s}$ and $F^*\mathfrak{s}$.

Our main result shows that these invariants describe the entire topological mapping class group.

Theorem 1 (Orson-Powell)**.** *Let $(X,\partial X)$ be a compact, simply connected, oriented, topological 4-manifold.*

(1) When X is spin, the map $F\mapsto(\Theta(F),\Delta_F)$ induces a group isomorphism

$$\pi_0\operatorname{Homeo}^+(X,\partial X)\xrightarrow{\cong}H^1(X,\partial X;\mathbb{Z}/2)\times\mathcal{V}(H_2(X),\lambda_X).$$

(2) When X is not spin, the map $F\mapsto\Delta_F$ induces a group isomorphism

$$\pi_0\operatorname{Homeo}^+(X,\partial X)\xrightarrow{\cong}\mathcal{V}(H_2(X),\lambda_X).$$

Our key contribution is injectivity of the maps in Theorem 1. Let us outline the proof strategy. First recall that a *topological pseudo-isotopy* is a homeomorphism $F\colon X \times I \to X \times I$ such that $F|_{\partial X \times I} = \mathrm{Id}_{\partial X \times I}$. The restrictions $F_0 = F|_{X\times\{0\}}$ and $F_1 := F|_{X\times\{1\}}$ are said to be *topologically pseudo-isotopic*. In this article we will classify homeomorphisms of simply connected 4-manifolds with boundary, up to topological pseudo-isotopy. The strategy builds on that of [Kre79, Proposition 2]. In broad strokes, if we can find a 6-manifold with boundary the (capped off) mapping torus of F, such that the 6-manifold is a rel. boundary h-cobordism from $X \times [0,1]$ to itself, then it follows that F is pseudo-isotopic to the identity. Our proof consists of an analysis of the obstructions to finding such an h-cobordism, and uses Kreck's modified surgery theory [Kre99] as the main technical tool in its construction. With the pseudo-isotopy classification in hand, the proof that the maps in Theorem 1 are injective concludes by appealing to Quinn's result [Qui86, Theorem 1.4] that topological pseudo-isotopy implies topological isotopy for homeomorphisms of simply connected, compact 4-manifolds.

Of course, the injectivity in Theorem 1 can be applied to diffeomorphisms of smooth 4-manifolds, yielding a topological isotopy. This is a important step in the hunt for exotic diffeomorphisms, which is currently a topic of considerable interest. For example Theorem 1 was applied in this way by Iida-Konno-Mukherjee-Taniguchi [IKMT22].

When X has nonempty, connected boundary, surjectivity of the map $\pi_0 \operatorname{Homeo}^+(X, \partial X) \to \mathcal{V}(H_2(X), \lambda_X)$ was already known, and is a consequence of Boyer's classification of simply connected compact 4-manifolds with connected boundary, and a subsequent result of Saeki [Boy86, Boy93, Sae06]. To show that the map in Theorem 1 (1) is surjective, in particular to realise the Θ invariants topologically, requires a novel geometric construction, again in combination with Boyer and Saeki's results [Boy86, Boy93, Sae06].

Dehn twists. An important type of self-homeomorphism of 4-manifolds is the *Dehn twist*, which arises as follows. Let $\phi_t \in \pi_1(\mathrm{SO}(4))$ be a generator based at the identity matrix, represented by a smooth map $S^1 \to \mathrm{SO}(4)$ that is constant near the basepoint. This induces a smooth loop of self-diffeomorphisms of S^3, which generates $\pi_1(\mathrm{Diffeo}^+(S^3)) \cong \mathbb{Z}/2$, and thence a self-diffeomorphism

$$\Phi\colon S^3 \times I \xrightarrow{\cong} S^3 \times I; \qquad (x,t) \mapsto (\phi_t(x), t).$$

Given an embedding of $S^3 \times I$ into a 4-manifold, one can extend the map Φ by the identity to obtain a self-homeomorphism of the entire 4-manifold, and we call any self-homeomorphism obtained this way a *Dehn twist*. If X is smooth to begin with, and $S^3 \times I$ is smoothly embedded, then the Dehn twist is a self-diffeomorphism.

Now let X be a closed, simply connected 4-manifold and decompose $X \setminus \mathring{D}^4$ as the union $N \cup_{S^3\times\{1\}} S^3 \times I$ of a collar neighbourhood of $\partial(X\setminus\mathring{D}^4)$ and the closure of its complement. The diffeomorphism Φ induces a Dehn twist homeomorphism

$$t_X\colon X\setminus\mathring{D}^4 \to X\setminus\mathring{D}^4; \qquad y \mapsto \begin{cases} \Phi(x,t) & y = (x,t) \in S^3 \times I, \\ y & y \in N. \end{cases}$$

Corollary 2. *For every closed, simply connected, topological manifold X, the Dehn twist t_X is topologically isotopic to* $\mathrm{Id}_{X\setminus\mathring{D}^4}$.

An explicit geometric argument of Giansiracusa shows that $t_{\mathbb{CP}^2}$ is smoothly isotopic to the identity [Gia08]. This result can be extended to show that t_X is smoothly isotopic to the identity for any non-spin, smooth, simply connected, closed 4-manifold X; this argument was communicated to us by Auckly, Kronheimer, and Ruberman. On the other hand, it was shown independently by Baraglia-Konno [BK22] and Kronheimer-Mrowka [KM20] that t_{K3} is not smoothly isotopic to the identity. This prompts the obvious question.

Question 3. *For which closed, spin, simply connected, smooth manifolds X is t_X smoothly isotopic to the identity?*

Homeomorphisms not restricting to the identity on the boundary. We consider the implications of our results when we relax the assumption that homeomorphisms must fix the boundary pointwise. Let X be a compact, oriented, simply connected 4-manifold. There is a fibre sequence

$$\mathrm{Homeo}^+(X, \partial X) \to \mathrm{Homeo}^+(X) \to \mathrm{Homeo}^+(\partial X).$$

Consequently there is an exact sequence in homotopy groups, extending to the left,

$$\pi_1 \mathrm{Homeo}^+(\partial X) \to \pi_0 \mathrm{Homeo}^+(X, \partial X) \to \pi_0 \mathrm{Homeo}^+(X) \to \pi_0 \mathrm{Homeo}^+(\partial X).$$

Here, the first arrow can be defined by inserting the loop of diffeomorphisms of ∂X (based at $\mathrm{Id}_{\partial X}$) into a collar of the boundary, and extending by the identity. Taking the basepoint of each group of homeomorphisms to be the respective identity map, the sequence is an exact sequence of groups. Here the π_0 terms are also groups because they are connected components of topological groups. The sequence suggests that the problem of whether two homeomorphisms $F_1, F_2 \colon (X, \partial X) \to (X, \partial X)$ are isotopic in $\mathrm{Homeo}^+(X)$ can be decomposed into two stages, as follows.

The first-stage question is purely about 3-manifolds: are $F_1|_{\partial X}$ and $F_2|_{\partial X}$ isotopic? This is a highly nontrivial question in general, but thanks to the modern spectacular understanding of 3-manifolds, we have a good chance of being able to decide. Self-homeomorphisms of ∂X must respect the prime decomposition [Kne29, Mil62] and the JSJ decomposition [JS79, Joh79]; see also [Hat07]. Restricting to geometric pieces it often suffices to understand the isometry groups (in the sense of Riemannian geometry), by [Gab01, HKMR12, BK21, BK17] and the references therein. For simple 3-manifolds their mapping class groups were known earlier. For lens spaces the mapping class groups were computed by Bonahon [Bon83], while for Seifert fibred spaces in general see e.g. [BO91]. For Haken 3-manifolds, Hatcher and Ivanov [Hat76, Iva79] showed that the mapping class group equals the group of homotopy self-equivalences. So with enough work, the first-stage question can in principle be answered with our current knowledge.

If there is no isotopy between $F_1|_{\partial X}$ and $F_2|_{\partial X}$, then certainly F_1 and F_2 are not isotopic. So let us assume that the 3-manifold question has been solved affirmatively. Then, after an isotopy of F_1 supported in a collar of ∂X we can assume that $F_1|_{\partial X} = F_2|_{\partial X}$. We may ask the second-stage question: is $G := F_2 \circ F_1^{-1} \in \operatorname{Homeo}^+(X, \partial X)$ in the image of $\pi_1 \operatorname{Homeo}^+(\partial X)$?

In some cases, $\pi_1 \operatorname{Homeo}^+(\partial X) = 0$ and so it causes no additional complications. A general condition for this, using work of Gabai, Hatcher, Ivanov, and Waldhausen [Gab01, Hat76, Iva79, Wal67], is as follows.

Proposition 4. *Let X be a compact, simply connected, oriented, topological 4-manifold and suppose that every connected component of ∂X is irreducible but not Seifert fibred. Then $\pi_1 \operatorname{Homeo}^+(\partial X) = 0$ and so there is exact sequence of groups*

$$0 \to \pi_0 \operatorname{Homeo}^+(X, \partial X) \to \pi_0 \operatorname{Homeo}^+(X) \to \pi_0 \operatorname{Homeo}^+(\partial X).$$

Theorem 1 describes the left group. The image of the right hand map was described precisely by Boyer [Boy86, Boy93], for all 3-manifolds. So in the case that every connected component of ∂X is irreducible but not Seifert fibred, the combination of our work with Boyer's results can be employed to complete the two-stage process discussed above.

We considered Seifert fibred 3-manifold boundary components, and studied the problem of realising the invariants in Theorem 1 using loops of diffeomorphisms in a boundary collar. For S^3, lens spaces, and $S^1 \times S^2$ we found some success, showing that for X spin and ∂X a disjoint union of 3-manifolds of Heegaard genus at most one, every element of $0 \times H^1(X, \partial X; \mathbb{Z}/2)$ can be obtained by collar insertion. In addition, if the dimension of $H_1(\partial X; \mathbb{Q})$ is at most one, then we can identify $\mathcal{V}(H_2(X), \lambda_X)$ with a subgroup $\operatorname{Aut}^{\mathrm{fix}}_{\partial}(H_2(X), \lambda_X)$ of $\operatorname{Aut}(H_2(X), \lambda_X)$. We obtain the following corollary.

Corollary 5. *Let X be a compact, simply connected, orientable, topological 4-manifold. Suppose that every connected component of ∂X has Heegaard genus at most 1, and at most one of the connected components is $S^1 \times S^2$. Then there is an exact sequence of groups*

$$0 \to \operatorname{Aut}^{\mathrm{fix}}_{\partial}(H_2(X), \lambda_X) \to \pi_0 \operatorname{Homeo}^+(X) \to \pi_0 \operatorname{Homeo}^+(\partial X).$$

Note that this statement is independent of whether or not X admits a spin structure. Let me end by setting the following challenge.

Challenge. *Compute the collar insertion map*

$$\pi_1 \operatorname{Homeo}^+(\partial X) \to \pi_0 \operatorname{Homeo}^+(X, \partial X)$$

when ∂X consists of more general Seifert fibred spaces.

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Polynomially many surfaces in a hyperbolic 3-manifold in terms of volume

Anastasiia Tsvietkova
(joint work with Marc Lackenby)

For a low-dimensional manifold, one often tries to understand its intrinsic topology through its submanifolds, in particular of co-dimension 1. This is reflected, for example, in the study of trisections of 4-manifolds by Kirby, Gay and others; study of surfaces in 3-manifolds by Waldhausen, Haken, and others; study of curves on surfaces by Mirzakani and others. Among natural questions that arise is the question about a number of such submanifolds, up to isotopy or homotopy. For 3-manifolds, a further question, following the Geometrization Theorem by Thurston and Perelman, is how the topological data, and number of surfaces in particular, relates to the geometry of the 3-manifold.

The question about number of essential surfaces in a 3-manifold has been studied in recent years. One can fix the 3-manifold and investigate how the surface count changes with surface genus or Euler characteristic. The earliest work in this direction is due to Haken, for embedded normal surfaces, and Kneser, for surfaces simultaneously embedded in a 3-manifold. Later a related question, about immersed surfaces, was studied by Masters [10], and Kahn and Markovic [8]. There, it is shown that in a closed hyperbolic 3-manifold, the number of immersed closed connected surfaces (up to homotopy) of genus g grows like g^{2g}. For embedded surfaces, the count can be smaller (up to isotopy). In recent work [3] by Dunfield, Garoufalidis, Rubinstein, it is proved that for a class of hyperbolic 3-manifolds, the count for closed embedded surfaces is quasi-polynomial in genus for all but finitely many its values. In all these results, one needs to fix the 3-manifold to obtain the exact expression for the bound or count.

If one instead fixes the genus g or Euler characteristic of the surfaces rather than fixing a 3-manifold, then upper bounds can be obtained that are *universal* and *polynomial.* By universal we mean that the expression and constants are given by an explicit general formula for all 3-manifolds. For complements of prime alternating links in S^3, Hass, Thompson and Tsvietkova obtain universal bounds that are polynomial in n, the number of crossings of a link. This holds for orientable and non-orientable surfaces, closed surfaces and surfaces with meridianal boundary [6], as well as spanning surfaces [7]. The result for closed surfaces also holds in all but finitely many Dehn fillings of alternating links [6]. More recently, these techniques [11] and results [12] have been extended by Purcell and Tsvietkova to a broad class of cusped 3-manifolds, called weakly generalized alternating links. These are links in an arbitrary irreducible 3-manifold, with an alternating projection on some (not necessarily incompressible, connected or orientable) embedded surface. This work gives a universal polynomial bound for embedded essential surfaces that are not just closed or spanning, but also have other types of boundary.

Here, we give a universal polynomial upper bound that holds for all hyperbolic 3-manifolds. The number of surfaces is at most polynomial in $Vol(M)$. The connection between the intrinsic topology of a hyperbolic 3-manifold (the number submanifolds embedded) and its geometry (volume) is perhaps surprising.

Theorem. *Let M be an orientable hyperbolic 3-manifold of finite hyperbolic volume $Vol(M)$, closed or with cusps. The number of π_1-injective surfaces with Euler characteristic χ, up to isotopy, embedded in M, is of the order $Vol(M)^{\chi}$. Moreover, there is an explicit universal upper bound for all hyperbolic 3-manifolds that involves only χ and $Vol(M)$.*

The above bound is stated explicitly in our preprint [9]. The theorem includes surfaces that are closed or have boundary on the cusp boundary of M.

The proof uses a blend of techniques from differential geometry and low-dimensional topology, including Delaunay [2] and thick triangulations [1], ideas from normal surface theory [5] and stable minimal surfaces [4, 13], and was initially inspired by the proofs from [6, 7].

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Slice obstructions from genus bounds in definite 4-manifolds

Maggie Miller
(joint work with Paolo Aceto, Nickolas A. Castro, JungHwan Park, András Stipsicz)

In this talk, I present an alternate proof of a recent theorem of Dai–Kang–Mallick–Park–Stoffregen [5]. This will appear in an upcoming preprint [1].

Theorem 1 (Dai–Kang–Mallick–Park–Stoffregen [5])**.** *The (2,1)-cable of the figure eight knot is not smoothly slice.*

This theorem is motivated in part by the Slice–Ribbon Conjecture.

Definition 1. A knot in S^3 is *slice* if it bounds a smooth disk into B^4. A knot in S^3 is *ribbon* if it bounds a smooth disk into B^4 with the property that radial height of B^4 restricts to the disk as a Morse function with no local maxima.

Conjecture 2 (Slice–Ribbon Conjecture (Fox [6]))**.** *Every slice knot is ribbon.*

I believe the common consensus in the 4-dimensional topology community is that the Slice–Ribbon Conjecture is likely false. However, there are not many potential obstructions to a slice knot being ribbon, as most invariants that could obstruct a knot from being ribbon also obstruct the knot from being slice. One notable possible obstruction comes from work of Casson–Gordon [4] for fibered knots.

Theorem 3 (Casson–Gordon [4])**.** *Let K be a fibered knot in S^3, so $S^3\backslash\nu(K) = \mathring{\Sigma}_g \times I/(x,1) \sim (\phi(x),0)$ for some automorphism $\phi : \mathring{\Sigma}_g \to \mathring{\Sigma}_g$ restricting to the identity on $\partial\mathring{\Sigma}_g$. Let $\widehat{\phi}$ denote the natural extension of ϕ to the closed surface Σ_g.*

If K is ribbon, then $\widehat{\phi}$ extends over a 3-dimensional genus-g handlebody with boundary Σ_g.

In the statement of Theorem 3, the map ϕ is known as the *monodromy* of the fibered knot K.

Casson–Gordon's work is actually more general, implying that some fibered knots are not even *homotopy-ribbon*, a less restrictive condition than being ribbon. Nevertheless, there is currently no reason to believe that the above theorem applies to arbitrary fibered, slice knots. There are many situations in which we can compute the monodromy of a fibered knot K and prove that it does not extend over a handlebody, thus implying that K is not ribbon without proving that K is not slice. The simplest such family comes from a paper of Miyazaki [10].

Theorem 4 ([10])**.** *Let K be a fibered knot with irreducible Alexander polynomial. Then no cable of K is ribbon.*

Miyazaki points out that if K is negative-amphicheiral (e.g. K is the figure eight knot), then for any natural number n the $(2n,1)$-cable $K^{2n,1}$ of K is algebraically slice yet not ribbon.

On the other hand, while the figure eight knot K_4 is not slice, it does bound a smooth disk into a smooth rational homology ball W with $\pi_1(W) = \mathbb{Z}/2\mathbb{Z}$ [8], and the connected sum $K_4\#K_4$ is slice. Thus, the knot $K_4^{2,1}$ "looks" slice (in an extremely vague sense). Before the work of [5], the knot $K_4^{2,1}$ was probably the most promising potential counterexample to the Slice–Ribbon Conjecture.

Instead of summarizing the Floer-theoretic techniques of Dai–Kang–Mallick–Park–Stoffregen, I give an alternate argument making use of minimum-genus bounds for integral second homology classes in definite 4-manifolds coming from gauge theory. The key fact in this particular proof is due to Bryan [3].

Theorem 5 (Bryan [3])**.** *A smooth surface representing the integral second homology class* $(2,6)$ *of* $\mathbb{CP}^2\#\mathbb{CP}^2$ *has genus at least* 10*.*

Bryan's work is a slight improvement over the bound coming from Furuta's "10/8+2" theorem [7], whose preprint served as motivation for Bryan. The key idea is to observe that the homology class $(2,6)$ is twice a characteristic class. Therefore, if Σ is a smooth surface representing $(2,6)$, the 2-fold cover X of $\mathbb{CP}^2\#\mathbb{CP}^2$ branched along Σ is a smooth, spin 4-manifold. It is simple to work out that

$$b_2(X) = 2b_2(\mathbb{CP}^2\#\mathbb{CP}^2) + 2g(\Sigma) = 4 + 2g(\Sigma).$$

An application of the G-signature theorem [2] yields

$$\begin{aligned}\sigma(X) &= 2\sigma(\mathbb{CP}^2\#\mathbb{CP}^2) - \frac{1}{2}\left([\Sigma]\cdot[\Sigma]\right)\\ &= 4 - \frac{1}{2}(6^2+2^2)\\ &= -16.\end{aligned}$$

Thus, the "10/8+2" theorem immediately gives us

$$\begin{aligned}b_2(X) &\geqslant \frac{10}{8}|\sigma(X)| + 2\\ 4 + 2g(\Sigma) &\geqslant 20 + 2\\ g(\Sigma) &\geqslant 9.\end{aligned}$$

Bryan analyzed the covering involution on X to further restrict the genus of Σ to be at least 10. The techniques in Bryan's paper are out of the scope of this talk. Using Theorem 5, we are now in position to prove Theorem 1.

Alternate proof of Theorem 1. In Figure 1 we illustrate an annulus A properly embedded in $\mathbb{CP}^2\#\mathbb{CP}^2\setminus(\mathring{B}^4 \sqcup \mathring{B}^4)$. We view the ambient manifold as a self-cobordism of S^3 that is built by attaching two 4-dimensional 2-handles to $S^3\times I$ along a 2-component unlink (each with framing $+1$). In the left of the figure, we see one boundary C_0 of A, which is the $(2,1)$-cable of the figure eight knot, i.e. $K_4^{2,1}$. Moving to the right, we attach the two 2-handles along curves that each link A geometrically 6 times. We include arrows showing the orientation of A to see that one of these curves has linking number 6 with A; the other links A algebraically twice. Moving to the right, the effect of attaching the 2-handles is to

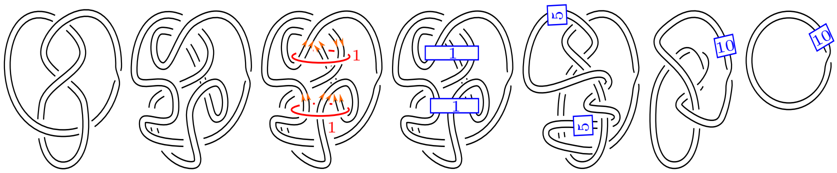

FIGURE 1. The annulus A discussed in the given proof of Theorem 1. The numbered twist-boxes indicate numbers of whole, negative twists.

introduce negative twists to the cross-sections of A. We perform further isotopy to see that the other boundary C_1 of A is the (mirror image of, if we are careful with orientations) torus knot $T(2,-19)$. Note that $T(2,-19)$ has Seifert genus $(2-1)\cdot(19-1)/2 = 9$.

Now suppose that $K_4^{2,1}$ is slice. We obtain a smooth, closed surface in $\mathbb{CP}^2\#\mathbb{CP}^2$ by capping off $\mathbb{CP}^2\#\mathbb{CP}^2\backslash(\mathring{B}^4 \sqcup \mathring{B}^4)$ with two 4-balls while capping off the annulus A with a slice disk bounded by C_0 in one ball and a genus-9 surface bounded by C_1 in the other ball. The resulting closed, genus-9 surface Σ intersects standard $\mathbb{CP}^1$s in each summand algebraically 2 and 6 times respectively, so Σ represents the homology class $(2,6) \in H_2(\mathbb{CP}^2\#\mathbb{CP}^2;\mathbb{Z})$. This contradicts Theorem 5, so we conclude that the (2,1)-cable of the figure eight knot is not slice. □

We end with some remaining open questions.

Question 6. *Is the (2,1)-cable of the figure eight knot topologically slice? That is, does it bound a topological, locally flat disk into the 4-ball?*

None of our techniques nor those used by [5] can obstruct topological sliceness. We remark again that Casson–Gordon [4] and Miyazaki [10] actually show that the (2,1)-cable of the figure eight knot is not homotopy-ribbon; a topological version of the Slice–Ribbon Conjecture asks whether every topologically slice knot is homotopy-ribbon, so an answer to Question 6 would be of great interest.

Question 7. *For $n > 1$, is the $(2n,1)$-cable of the figure eight knot slice?*

Again, neither our techniques nor those of [5] happen to obstruct sliceness for higher cables, but at least in principal one could hope to use either set of ideas. The construction in the presented proof of Theorem 1 can be repeated for the $(2n,1)$ cable, yielding an annulus in $\mathbb{CP}^2\#\mathbb{CP}^2\backslash(\mathring{B}^4 \sqcup \mathring{B}^4)$ cobounded by $K_4^{2n,1}$ and the (mirror of the) torus knot $T(2n, 1-20n)$, which has Seifert genus

$$\frac{(2n-1)\cdot(20n-2)}{2} = 20n^2 - 12n + 1.$$

Assuming that $K_4^{2n,1}$ is slice, we obtain a smooth genus-$(20n^2 - 12n + 1)$ surface in $\mathbb{CP}^2\#\mathbb{CP}^2$ representing the homology class $(2n, 6n)$. Observe that the obvious surface representing this homology class, obtained from connect-summing complex surfaces of degrees $2n, 6n$ in either summand, has genus

$$\frac{(2n-1)(2n-2)}{2}+\frac{(6n-1)(6n-2)}{2} = 2n^2-3n+1+18n^2-9n+1 = 20n^2-12n+2.$$

This prompts the following question.

Question 8. *For $n > 1$, is the connected sum of complex surfaces of degrees $2n$ and $6n$ in two copies of $\mathbb{CP}^2$ a smoothly minimum-genus surface in $\mathbb{CP}^2\#\mathbb{CP}^2$?*

Theorem 5 answers Question 7 affirmatively for $n = 1$. The above discussion implies that if the answer to Question 8 is "yes," then the answer to Question 7 is "no."

Question 8 is a more specific version of the following well-known question, which is natural in light of the Thom Conjecture [9].

Question 9. *If $n, m > 0$, is the connected sum of complex surfaces of degrees n and m in two copies of $\mathbb{CP}^2$ a smoothly minimum-genus surface in $\mathbb{CP}^2\#\mathbb{CP}^2$?*

For $n > 2$, it is a simple exercise to check that a degree-n surface in $\mathbb{CP}^2$ does not give a minimum-genus surface representing the homology class $(n, 0)$ when included into $\mathbb{CP}^2\#\mathbb{CP}^2$. The answer to Question 9 is known to be "yes" for some specific small n, m; see work of Nouh [11] for discussion and possible directions toward a "no."

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Approaches to the SSC4

David Gabai

The 4-dimensional smooth Schoenflies conjecture (SSC4) asserts that every embedded smooth 3-sphere in the 4-sphere bounds a smooth 4-ball.

Various 3-dimensional approaches and partial results towards this conjecture including Property R [G1], generalized Property R [GST], [MZ], and a theorem of Scharlemann [Sc] were discussed. We then explained an approach via pseudo-isotopy that has its origins in Barry Mazur's theorem [Ma] that a smooth 3-sphere in the 4-sphere bounds a topological 4-ball and the following biproduct of its proof using [Ce] Cerf that was known to topologists in the 1960's. Details for what follows can be found in the recent preprint [G2].

Theorem 1. *If SSC4 is false, then there exists a diffeomorphism* $\phi : S^1 \times S^3 \to S^1 \times S^3$ *such that* ϕ *is homotopic to* id *but* $\phi(x_0 \times S^3)$ *is not isotopic to* $x_0 \times S^3$*, even after lifting to any finite sheeted covering of* $S^1 \times S^3$*.*

We stated the following characterization of Schoenflies balls which uses pseudo-isotopy theory.

Theorem 2. *Every Schoenflies ball has a carving/surgery presentation.*

By Schoenflies ball we mean a closed complementary region of a smooth 3-sphere Σ in S^4. A carving/surgery presentation for a Schoenflies ball means that it is obtained by a finite process starting with the 4-ball, attaching finitely many 2-handles, then carving finitely many 2-handles, then attaching finitely many 2-handles, etc., with every step happening in the 4-sphere. An attached or carved 2-handle may nest a previously carved or attached 2-handle and so on. Actually we show that the presentation can be chosen to be of a special type called an *optimized* $F|W$*-carving/surgery presentation.* See §9 [G2] for details. A key feature of a $F|W$-carving/surgery presentation is that when viewed as a 3-dimensional surgery presentation of the boundary, our Σ is *obviously* the 3-sphere. On the other hand, there are many compact 4-manifolds in S^4 with carving/surgery presentations.

A diffeomorphism arising from Theorem 1 is pseudo-isotopic to id by [LS] and [Sa]. Work of Hatcher and Wagoner [HW] and Quinn [Qu] shows that the pseudo-isotopy has a nested eye structure such that all the data is contained in the middle middle level. We explained how the flexibility of passing to finite sheeted covers allows us to construct a one parameter family having a middle middle level such that the finger and Whitney discs coincide near their boundaries. Theorem 2 relies on this result. We then indicated that SSC4 is equivalent to a certain *interpolation problem* in the universal cover between the Whitney disc family and the finger disc family, i.e. in the cover there is a third family of *Whitney discs* that agrees with the original Whitney discs near $-\infty$ and agrees with the finger discs near $+\infty$.

We closed by stating the following *slice missing slice disc* problem, which was introduced in an earlier 5 minute talk.

Problem 3. *(Slice missing slice disc problem). The knot* $K \subset S^1 \times S^2 = \partial S^1 \times B^3$ *shown in Figure 1 bounds two obvious ribbon discs* D_1 *and* D_2 *such that the simple*

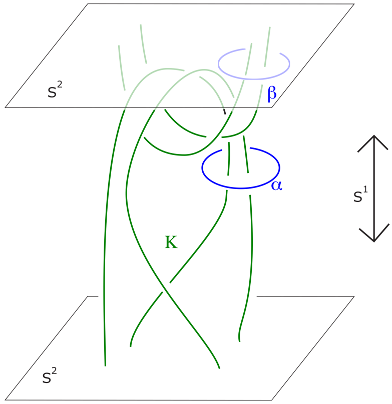

Figure 1. Slice Missing Slice Disc

closed curve $\alpha \subset S^1 \times S^2 \backslash K$ (resp. β) slices in $S^1 \times B^3$ with a slice disc disjoint from D_1 (resp. D_2). Is it true that for any smooth disc D bounded by K, one of α or β slices in the complement of D?

A positive solution may lead to the introduction of new techniques to address the interpolation problem. Conversely, a concrete counter example may suggest new methods for constructing interesting discs and spheres in 4-manifolds.

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