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Enumerative Combinatorics

Organized by
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11 December – 17 December 2022

Abstract. Enumerative Combinatorics focuses on the exact and asymptotic counting of combinatorial objects. It has fruitful connections to several disciplines, including statistical physics, algebraic combinatorics, probability theory, graph theory and computer science. This workshop brought together experts from all these various fields with the goal of promoting cooperation and interaction among researchers with largely varying backgrounds. It was the third workshop on this topic (the first two editions dating back to 2014 and 2018) and this time the main focus was on interactions with algebraic combinatorics.

Mathematics Subject Classification (2010): Primary: 05A; Secondary: 05C80, 05E, 60C05, 60J, 68R, 82B.

Introduction by the Organizers

Eight years ago, in March 2014, and four years ago, in May 2018, Mireille Bousquet-Mélou (Bordeaux), Michael Drmota (Vienna), Christian Krattenthaler (Vienna), and Marc Noy (Barcelona) organised two *Workshops on "Enumerative Combinatorics"* at the Mathematische Forschungsinstitut at Oberwolfach. The present workshop, organised by a renewed team (Mireille Bousquet-Mélou (Bordeaux), Guillaume Chapuy (Paris), Michael Drmota (Vienna), and Sergi Elizalde (Dartmouth), continues these (short) series of workshops that were the first ones of their kind.

Four years after the previous meeting, it was time to meet again and assess the developments which had taken place since then; in particular, to examine the impact of the previous workshops, and to witness and discuss the recent trends and most exciting developments in Enumerative Combinatorics. This need was all the more felt after more than two years of the covid pandemic, which put to a halt a large number of scientific meetings. Among the participants of the last two workshops there was of course some intersection. However, many "new" and younger researchers participated this time; in fact, almost half of the participants (25 out of 53) were new to this workshop.

It was especially important to be able to meet in person after two years of pandemic. The scientific value of an in-person meeting cannot be overstated, particularly for younger participants who got to meet their pears and colleagues for the first time in real life. Moreover, even though this workshop maintains a wide range of connections to many areas of mathematics, it also builds on a feeling of community around Enumerative Combinatorics as a field. For us it was a great pleasure to bring this community together for a week.

The impact of the last workshops could be felt in several ways. As expected, the presentations and discussions from 2014 and 2018 resulted in several new collaborations, and in several papers, as could be witnessed on the arχiv. As an example, Tony Guttmann opened his talk by saying "At the open problem session in the workshop that took place four years ago, I learnt about pattern-avoiding permutations. Thanks to that, they have become one of the main topics of my research for the past four years. Today I will present some of the progress that I have made since then." Other speakers made similar statements.

This workshop took place in December 11–17, 2022. There were 46 on-site participants coming from the US, Canada, Australia, Japan, and various European countries. Furthermore, 7 participants attended the workshop online. The program consisted of 9 one-hour lectures, accompanied by 19 shorter contributions and a special session of presentations by the four Oberwolfach Leibniz graduate fellows. There was also an extensive and inspiring problem session which gave rise to exciting discussions, and we are confident that it will lead to new collaborations and to new results in the forthcoming years. Three of the one-hour lectures were designated "keynote lectures" — delivered by Sylvie Corteel, Olivier Bernardi, and Peter Winkler. They provided overviews of recent exciting developments on cylindric partitions, hyperplane arrangements, and permutons, respectively. Three talks (including one keynote talk) were delivered remotely.

As a whole, the lecturers in this workshop presented the state of the art in various areas related to Enumerative Combinatorics, together with relevant new results. The lectures and short talks ranged over a wide variety of topics including classical enumerative problems, algebraic combinatorics, asymptotic and probabilistic methods, statistical physics, and methods from computer algebra. Special attention was paid to providing a platform for younger researchers to introduce themselves and their results to the community. This report contains extended abstracts of the talks, as well as the statements of the problems that were posed during the problem session.

The goal of the workshop was to bring together researchers from different fields with a common interest in enumeration, whether from an algebraic, analytic, probabilistic, geometric or computational angle, in order to enhance collaboration and new research projects. The organizers believe this goal was amply achieved, as demonstrated by the strong interaction among the participants and the lively discussions in and outside the lecture room during the whole week.

The working environment was perfect, and in particular the tools offered to manage the hybrid workshop were impressively efficient. On behalf of all participants, the organizers would like to thank the staff and the director of the Mathematisches Forschungsinstitut Oberwolfach for providing such a stimulating and inspiring atmosphere.

We also would like to thank two young participants: Umberto De Ambroggio, who was the video conference assistant (VCA) during the conference, and Sergey Yurkevich, who helped with the collection of abstracts and the editing of this document.

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Workshop: Enumerative Combinatorics

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Abstracts

Sign clusters in the Infinite-Ising weighted triangulations.

Marie Albenque
(joint work with Laurent Ménard, Gilles Schaeffer)

A planar map is an embedding of a planar graph in the sphere, considered up to deformations. A triangulation is a planar map, where all the faces are triangles. In 2003, in order to define a model of *generic* planar geometry, Angel and Schramm studied the limit of random triangulations on the sphere [3]. They proved that this model of random maps converges for the Benjamini-Schramm topology, or local topology, towards the now famous Uniform Infinite Planar Triangulation (or UIPT), a probability distribution on infinite triangulations.

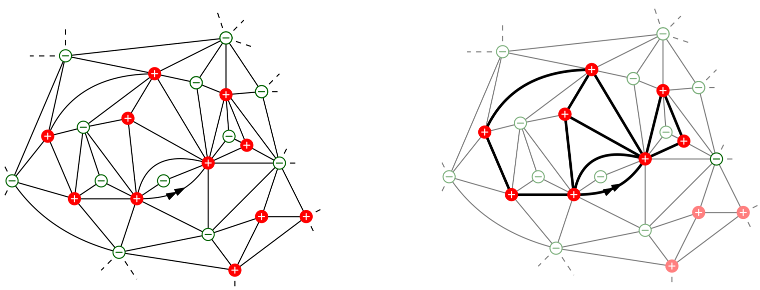

Figure 1. Example of a triangulation endowed with a spin configuration (left), and its root spin cluster (right).

With Laurent Ménard and Gilles Schaeffer [2], we extend the result of Angel and Schramm to triangulations endowed with a *spin configuration*. More precisely, we consider triangulations where vertices carry either a $+$ spin or a $-$ spin, see Figure 1. For $n \in \mathbb{N}$ and $\nu > 0$, we define the probability distribution $\mathbb{P}_n^\nu$ on spin-decorated triangulations with n vertices as:

$$\mathbb{P}_n^\nu\left(\{(t,\sigma)\}\right) \propto \nu^{\mathrm{mono}(t,\sigma)}, \tag{1}$$

where t is a triangulation with n vertices, σ is a spin configuration and $\mathrm{mono}(t,\sigma)$ is the number of monochromatic edges in (t,σ). We establish that this model converges in the local weak limit:

Theorem 1. *For every $\nu > 0$, the sequence of probability measures $\mathbb{P}_n^\nu$ converges weakly for the local topology to a limiting probability measure $\mathbb{P}_\infty^\nu$ supported on one-ended infinite triangulations endowed with a spin configuration.*

We call a random triangulation distributed according to this limiting law the Infinite Ising Planar Triangulation with parameter *ν or ν-*IIPT*.*

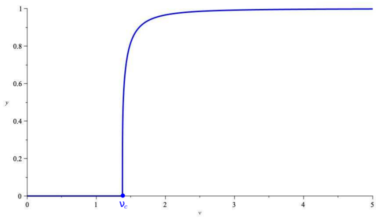

Figure 2. Probability that the root spin cluster is infinite in the ν-IIPT.

In a subsequent work with Laurent Ménard [1], we exhibit a phase transition for the geometric properties of the ν-IIPT, depending on the value of ν. The fact that such a transition exists at the critical value $\nu_c := 1 + \sqrt{7}/7$ has been established in previous works [4, 5, 6], but only enumerative evidence of this phase transition were obtained. To get geometric evidence of it, we focus on *the root spin cluster* of a spin-decorated triangulation, which is defined as the connected component of the root vertex when keeping only monochromatic edges between + spin vertices, see Figure 1. We prove that the root spin cluster undergoes a phase transition, in the following sense:

Theorem 2. *The root spin cluster of the ν-IIPT is almost surely finite when $\nu \leq \nu_c$ and infinite with positive probability when $\nu > \nu_c$.*

In addition, we have an explicit parametric formula for the probability that the root spin cluster is infinite, see Figure 2.

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2-core Littlewood identities

Seamus Albion

Littlewood-type identities are, broadly speaking, summations for symmetric functions which admit closed-form expressions in terms of products, determinants or Pfaffians. The first example of such an identity actually appeared in an exercise of Schur [7], and gives an expression for the sum of Schur functions over all partitions

$$\sum_{\lambda} s_\lambda(x) = \prod_{i\geq 1} \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_ix_j}. \tag{1a}$$

Littlewood derived two similar identities, with the only difference being that the sum is restricted to partitions whose Young diagram has only even rows or even columns, written 'λ even' and 'λ' even' respectively [4, p. 238]

$$\sum_{\substack{\lambda \\ \lambda \text{ even}}} s_\lambda(x) = \prod_{i\geq 1} \frac{1}{1-x_i^2} \prod_{i<j} \frac{1}{1-x_ix_j} \tag{1b}$$

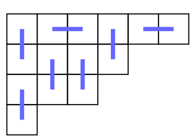
$$\sum_{\substack{\lambda \\ \lambda' \text{ even}}} s_\lambda(x) = \prod_{i<j} \frac{1}{1-x_ix_j}. \tag{1c}$$

The identities (1) have since afforded many far-reaching generalisations and have found applications in areas such as combinatorics, representation theory and elliptic hypergeometric series; see [6] for comprehensive references to the literature. Of particular interest are bounded Littlewood identities, where the sum is restricted to partitions whose Young diagram fits inside an $m \times n$ box, which we write as $\lambda \subseteq (m^n)$. The first example of such a bounded identity is Macdonald's formula

$$\sum_{\lambda\subseteq(m^n)} s_\lambda(x_1,\dots,x_n) = \frac{\det_{1\leq i,j\leq n}\big(x_i^{m+2n-j} - x_i^{j-1}\big)}{\prod_{i=1}^n (x_i-1)\prod_{1\leq i<j\leq n}(x_i-x_j)(x_ix_j-1)},$$

which he used to prove MacMahon's conjecture for the generating function of symmetric plane partitions in an $n\times n\times m$ box.

In their work on branching rules for Macdonald polynomials, Lee, Rains and Warnaar were led to conjecture a swathe of curious identities such as integral evaluations, branching formulae, Littlewood identities and hypergeometric summations [3]. What unifies their conjectures are partitions with empty 2-core, i.e., partitions whose Young diagrams can be tiled by dominoes. For example the partition $(6,4,3,1)$ has empty 2-core since it admits the tiling



In [1], we were able to prove the Schur case of a particular pair of vanishing integrals conjectured by Lee, Rains and Warnaar [3, Conjecture 9.2]. Following the virtual Koornwinder integral approach to Littlewood identities of Rains and Warnaar [6], these vanishing integrals imply a pair of bounded Littlewood identities. Taking the large-m limit then yields a pair of unbounded identities, generalising both (1b) and (1c).

To state these unbounded forms we need some notation. Denote the multiset of hook lengths of a partition λ by $\mathcal{H}_\lambda$, which we refine by writing $\mathcal{H}_\lambda^{\mathrm{e/o}}$ for the submultiset of even/odd hook lengths respectively. We also need a statistic

$$b(\lambda) := \sum_{(i,j)\in\lambda} (-1)^{\lambda_i+\lambda'_j-i-j+1}(\lambda_i - i),$$

where Young diagram of $\lambda = (\lambda_1, \dots, \lambda_n)$ is interpreted as the set of integer points (i, j) such that $1 \le i \le n$ and $1 \le j \le \lambda_i$. Finally, $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$ is the infinite q-shifted factorial. Then the identities of [1, Theorem 1.1] are

$$\sum_{\substack{\lambda\\ \text{2-core}(\lambda)=0}} q^{b(\lambda)} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(1-q^h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(1-q^h)} s_\lambda(x) = \prod_{i\ge1} \frac{(qx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty} \prod_{i<j} \frac{1}{1-x_ix_j}, \tag{2a}$$

and

$$\sum_{\substack{\lambda\\ \text{2-core}(\lambda)=0}} q^{b(\lambda')} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(1-q^h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(1-q^h)} s_\lambda(x) = \prod_{i\ge1} \frac{(q^2x_i^2;q^2)_\infty}{(qx_i^2;q^2)_\infty} \prod_{i<j} \frac{1}{1-x_ix_j}. \tag{2b}$$

The condition 2-core$(\lambda) = 0$ generalises both the even row and even column conditions of (1). In fact, $b(\lambda) = 0$ if and only if λ is even, so that for $q = 0$ (2a) and (2b) reduce to (1b) and (1c) respectively. In this sense these identities are in the spirit of Kawanaka's identity [2, Theorem 1.1]

$$\sum_\lambda \prod_{h\in\mathcal{H}_\lambda} \left(\frac{1+q^h}{1-q^h}\right) s_\lambda(x) = \prod_{i\ge1} \frac{(-qx_i;q)_\infty}{(x_i;q)_\infty} \prod_{i<j} \frac{1}{1-x_ix_j},$$

since this reduces to (1a) when $q = 0$. Unlike Kawanaka's identity one can make sense of the $q \to 1$ limit of (2a) and (2b). In either case, the following Littlewood-type identity is obtained [1, Corollary 1.2]

$$\sum_{\substack{\lambda\\ \text{2-core}(\lambda)=0}} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}} h}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}} h} s_\lambda(x) = \prod_{i\ge1} \frac{1}{(1-x_i^2)^{1/2}} \prod_{i<j} \frac{1}{1-x_ix_j}.$$

At the level of Macdonald polynomials, all of the conjectures of [3] remain wide open.

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Jack polynomials as generating series of non-oriented bipartite maps

Houcine Ben Dali
(joint work with Maciej Dołęga)

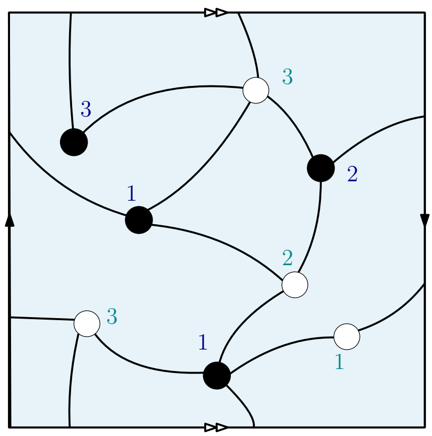

Figure 1. A 3-layered map on the Klein bottle, with face-type $[4, 4, 2, 2]$.

Layered maps. A *bipartite map* is a 2-cell embedding of a bipartite graph into a surface, orientable or not [9]. A map is oriented if it is embedded on an orientable surface. The *face-type* of a bipartite map M, denoted by $\nu_\diamond(M)$, is the integer partition obtained by reordering the half-sizes of the faces degrees. Let k be a positive integer. A bipartite map M is k-layered if

- each black vertex has a label in $1, 2, \ldots, k$.
- each white vertex is labelled by the maximal label among the labels of its black neighbors.

See Figure 1 for an example of a layered map. Schur functions have a well known expansion in the power-sum basis in terms of layered orientable maps. This expansion can be obtained using Young symmetrizers [7] and the correspondence between oriented maps and pairs of permutations [2].

Jack polynomials. Jack polynomials $J_\lambda^{(\alpha)}$ are symmetric functions which depend on a parameter α, and which give Schur functions for $\alpha = 1$ [8]. We also consider the shifted parameter $b := \alpha - 1$ introduced by Goulden and Jackson [6]. In 1988, Hanlon conjectured that Jack polynomials, can be expressed in the power-sum basis as a sum of weighted oriented maps. In joint work with Maciej Dołęga, we obtain an explicit formula of the power-sum expansion of Jack polynomials in terms of weighted non-oriented maps.

We call a *statistic of non-orientability* on k-layered maps a statistic which associates to each k-layered map M a non-negative integer such that $\vartheta(M) = 0$ if and only if M is oriented [6].

Theorem 1. *Let λ be a partition of size n. There exists an explicit statistic of non-orientability ϑ, such that*

$$J_\lambda^{(\alpha)} = (-1)^n \sum_{\substack{\ell(\lambda)-\text{layered maps}\\ M \text{ with } n \text{ edges}}} \frac{p_{\nu_\diamond(M)} b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}} \prod_{1\leq i\leq \ell(\lambda)} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

where $cc(M)$ is the number of connected components of M[1]*, $|\mathcal{V}_\bullet(M)|$ is the number of black vertices of M, $|\mathcal{V}_\circ^{(i)}(M)|$ is the number of white vertices of M labelled by i, and $z_{|\nu_\bullet^{(i)}(M)|}$ is a normalization factor.*

In addition to the case $\alpha = 1$ which corresponds to Schur function, this result has been proved in [3, 1] for partitions λ with rectangular shape.

Jack characters. The Jack character associated to a partition μ is the function on Young diagrams defined by

$$\theta_\mu^{(\alpha)}(\lambda) := \begin{cases} 0, & \text{if } |\lambda| < |\mu|. \\ \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)}[p_{\mu,1^{|\lambda|-|\mu|}}]J_\lambda^{(\alpha)}, & \text{if } |\lambda| \geq |\mu|. \end{cases}$$

where $m_1(\mu)$ is the number of parts of size 1 in μ. We prove that Jack characters also have a combinatorial interpretation in terms of weighted maps.

Theorem 2. *For every partition μ, there exists a statistic of non-orientability ϑ*

$$\theta_\mu^{(\alpha)}(\lambda) = (-1)^{|\mu|} \sum_{\substack{\text{layered maps} M\\ \text{of face-type } \mu}} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)}\alpha^{cc(M)}} \prod_{i\geq 1} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}.$$

This result is a generalization of a well-known formula for the normalized characters of the symmetric group conjectured by Stanley [11] and proved by Féray [4], which corresponds the previous result for $\alpha = 1$. A similar result for $\alpha = 2$ has been established by Féray and Śniady [5].

Lassalle's conjecture Let $k \geq 1$ and let $s_1 \geq s_2 \cdots \geq s_k \geq 1$ and $r_1, \dots r_k$ be two sequences of non negative integers. We say that $(s_1, \dots, s_k)$ and $(r_1, \dots, r_k)$ are Stanley coordinates for a partition λ if $\lambda = [s_1^{r_1}, \dots, s_k^{r_k}]$. We prove the following result which has been conjectured by Lassalle in 2008 [10].

[1]These maps are also rooted, i.e they have some marked corners. This rooting is not detailed in this abstract.

Theorem 3. *Let λ be a partition, and let $(r_1, \ldots, r_k)$ and $(s_1, \ldots, s_k)$ be Stanley coordinated for λ. The normalized Jack character $(-1)^{|\mu|} z_\mu \theta_\mu^{(\alpha)}$ is a polynomial in $(b, r_1, \ldots, r_k, s_1, \ldots, s_k)$ and Stanley's coordinates, with non-negative integer coefficients.*

The positivity part is obtained using Theorem 2. However, we use a different approach to prove the integrality part.

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Bijections for hyperplane arrangements of Coxeter type

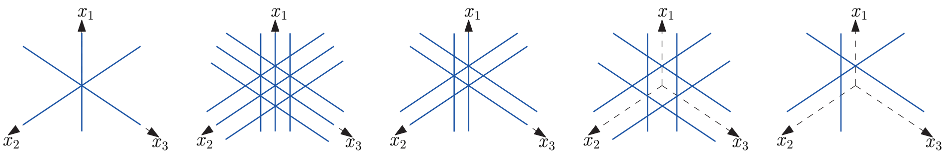
Olivier Bernardi

In this abstract we report on bijective results about classical families of hyperplane arrangements.

Consider real hyperplane arrangements made of a finite number of hyperplanes of the form

$$H_{i,j,s} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = s\},$$

with $i, j \in \{1, \ldots, n\}$ and $s \in \mathbb{Z}$. We shall call them *braid-type arrangements*. In particular, given an integer n and a finite set of integers S, the *S-braid arrangement in dimension n*, denoted $\mathcal{A}_S(n)$, is the arrangement made of the hyperplanes $H_{i,j,s}$ for all $1 \leq i < j \leq n$ and all $s \in S$. Classical examples include the *braid*, *Catalan*, *Shi*, *semiorder*, and *Linial* arrangements, which correspond to the sets $S = \{0\}$, $\{-1, 0, 1\}$, $\{0, 1\}$, $\{-1, 1\}$, and $\{1\}$ respectively. These arrangements are represented in Figure 1.



Braid: $\mathcal{A}_{\{0\}}(3)$ *Catalan:* $\mathcal{A}_{\{-1,0,1\}}(3)$ *Shi:* $\mathcal{A}_{\{0,1\}}(3)$ *Semiorder:* $\mathcal{A}_{\{-1,1\}}(3)$ *Linial:* $\mathcal{A}_{\{1\}}(3)$

FIGURE 1. The braid, Catalan, Shi, semiorder, and Linial arrangements in dimension $n = 3$ (seen from the direction $(1, 1, 1)$).

There is an extensive literature on counting regions of braid-type arrangements. Important seminal results were established by Zaslavsky [6], Stanley [5], Postnikov and Stanley [4], and Athanasiadis [1].

About a decade ago, an interesting pattern was observed by Ira Gessel. In an unpublished manuscript, Gessel obtained an equation for the generating function of labeled binary trees counted according to ascents and descents along left or right edges. Gessel observed a that each of the five classical arrangements families $\mathcal{A}_S$ defined above (braid, Catalan, Shi, semi-order, Linial) can be associated to a simple family $\mathcal{T}_S$ of binary trees (characterized by some ascent and descent conditions), in such a way that the regions of $\mathcal{A}_S(n)$ are equinumerous to trees with n nodes in $\mathcal{T}_S$. This opened the question of explaining these mysterious enumerative identities.

In [2] we give a bijective framework for braid-type arrangements explaining these identities and many more. The bijective approach we establish applies to braid-type arrangements satisfying a certain "transitivity condition".

Definition 1. *A braid-type arrangement $\mathcal{A} \subset \mathbb{R}^n$ is* transitive *if for all $s, t \in \mathbb{N}$ the following holds: if $H_{i,j,s} \notin \mathcal{A}$, $H_{j,k,t} \notin A$ (for some $i, j, k \in [n]$, with $i > j$ if $s = 0$ and $j > k$ if $t = 0$), then $H_{i,k,s+t} \notin \mathcal{A}$.*

We now state the main bijective result of [2]. A *n-tree* is a rooted plane tree with n nodes labeled with distict labels in $[n]$ (leaves have no label). For a braid-type arrangement $\mathcal{A} \subseteq \mathcal{A}_{[-m:m]}(n)$, we call *$\mathcal{A}$-tree* a $(m + 1)$-ary n-tree satisfying the following local conditions: for all $i, j \in [n]$ and $s \in \mathbb{N}$ such that $H_{i,j,s} \notin \mathcal{A}$, if i is sth child of j, then i has at least one right sibling which is a node.

Theorem 1. *If a braid-type arrangement $\mathcal{A} \subseteq \mathcal{A}_{[-m:m]}(n)$ is transitive, then the regions of $\mathcal{A}$ are in bijection with the $\mathcal{A}$-trees (via a simple, explicit, bijection).*

Theorem 1 applies to the braid, Catalan, Shi, semi-order and Linial arrangements which are all transitive. As an example, the bijection for the Shi arrangement in dimension 3 is shown in Figure 2.

Recently we extended the bijective framework in two directions. One extension is about the lower dimensional faces of braid-type arrangements. The bijective approach for lower dimensional faces applies to braid-type arrangements which are strongly transitive.

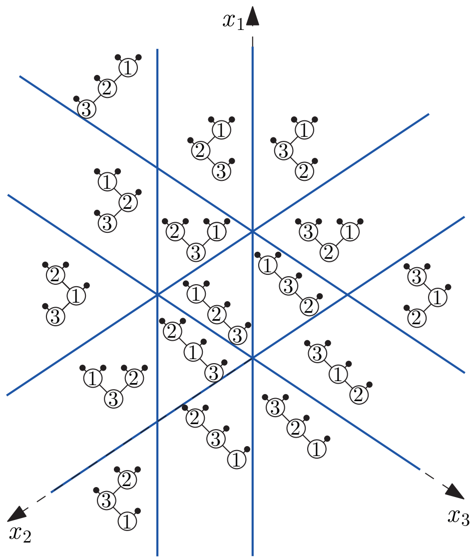


FIGURE 2. The bijection between the regions of the Shi arrangement $\mathcal{A} = \mathcal{A}_{\{0,1\}}(3)$ and the $\mathcal{A}$-trees (these are the binary 3-trees with no right ascents).

Definition 2. *A braid-type arrangement $\mathcal{A} \subset \mathbb{R}^n$ is* strongly transitive *if for all $s, t \in \mathbb{N}$ the following holds: if $H_{i,j,s} \notin \mathcal{A}$, $H_{j,k,t} \notin A$ (for some $i, j, k \in [n]$), then $H_{i,k,s+t} \notin \mathcal{A}$.*

For strongly transitive braid-type arrangements one can again establish an explicit bijection between the faces (of a given dimension) and some family of (decorated) n-trees. We recover in particular the results of Levear for the Catalan and Shi arrangements [3].

Another extension, established in collaboration with Te Cao, concerns the enumeration of regions in arrangements of *Coxeter-type* (in which hyperplanes are of the form $x_i - x_j = s$ or $x_i + x_j = s$). This extension allows in particular to obtain bijections for the type-C versions of the braid, Catalan, Shi, Semi-order and Linial arrangements, as well as the type-B versions of the braid, Catalan, Shi and Linial arrangements, and the type-D versions of the braid, Catalan and Shi arrangements.

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On maps with tight boundaries

Jérémie Bouttier

(joint work with Emmanuel Guitter, Grégory Miermont)

Maps, in the combinatorial sense, are cellular embeddings of graphs into surfaces, considered up to homeomorphism. In this talk we consider orientable maps, whose topology is characterized by a pair of integers (g, n), where g is the genus and n the number of boundaries. For a given topology we consider the problem of counting maps according to the distribution of their face degrees: this is a classical problem first investigated by Tutte, who solved it for genus $g = 0$ and even face degrees [1]. The case of other topologies can be treated via the formalism of *topological recursion*, reviewed for instance in [2].

We have initiated a long-term project whose goal is to develop a bijective approach to topological recursion, by extending the method of *slice decomposition*. After reviewing the basics of this method, following essentially [3, Section 2.2], I report on our two first steps in our project.

In [4], we obtain a very simple formula for the generating function of planar maps with three *tight* boundaries. Here, by tight boundary we mean that its contour is a path of minimal length in its homotopy class. Our derivation is bijective and involves new objects called bigeodesic diangles and triangles. It is reminiscent of hyperbolic geometry, hinting that our approach may be universal in a sense yet to be determined.

In [5], we consider quasi-polynomials counting so-called tight maps (maps whose all faces are tight boundaries). Such quasi-polynomials were previously encountered by Norbury in the context of the enumeration of lattice points in the moduli space of curves. We give a fully explicit expression for these quasi-polynomials in the genus 0 case. On the way, we obtain an extension of Tutte's slicings formula to the case of maps with arbitrary (odd or even) face degrees.

Some future directions are discussed at the end of the talk.

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Tree bijections for hyperbolic surfaces with boundaries

Timothy Budd
(joint work with Thomas Meeusen, Bart Zonneveld)

Whereas this workshop is primarily devoted to counting problems, various methods that emerged in the combinatorial literature have analogues that apply to integration problems as well. In this talk we focus on an integration problem involving the computation of the Weil–Petersson volumes of moduli spaces of hyperbolic surfaces. To be precise, we let $\mathcal{M}_{g,n}(\mathbf{L})$ with $\mathbf{L} = (L_1, \ldots, L_n) \in \mathbb{R}^n_{\geq 0}$ be the moduli space of hyperbolic surfaces of genus g with n labeled geodesic boundaries of lengths $L_1, \ldots, L_2$ viewed up to isometry. It constitutes an orbifold of dimension $6g - 6 + 2n$, and it is equipped with a natural volume form μ_{WP} arising from its Weil–Petersson symplectic structure. The total volume of $\mathcal{M}_{g,n}(\mathbf{L})$ is called the Weil–Petersson volume

$$V_{g,n}(\mathbf{L}) = \int_{\mathcal{M}_{g,n}(\mathbf{L})} \mu_{\mathrm{WP}}. \tag{1}$$

It is known since the celebrated work of Mirzakhani [5] that the dependence of the volume on the boundary lengths is polynomial, more precisely $V_{g,n}(\mathbf{L}) \in \mathbb{Q}[L_1^2, \ldots, L_n^2, \pi^2]$. The main goal of this talk is to provide a combinatorial explanation of this fact and precise expressions for the generating functions in the spherical case $g = 0$.

One can think of $V_{g,n}(\mathbf{L})$ as analogous to the enumeration problem of combinatorial genus-g maps with n labeled faces of prescribed even degrees. This enumeration problem in the planar case $g = 0$ is conveniently approached through tree bijections. In particular, the Bouttier–Di Francesco–Guitter bijection [1] provides a relation between such pointed maps (i.e. maps equipped with a marked vertex) and certain labeled trees, called mobiles. The mapping is based on a labeling of the vertex set of the map by graph distance to the marked vertex.

It turns out that hyperbolic surfaces admit a similar encoding by labeled trees, arising from considering the hyperbolic distance from each point of the surface towards a marked cusp (boundary of length 0). More precisely, given a hyperbolic surface $S \in \mathcal{M}_{0,n+1}(0, L_1, \ldots, L_n) = \mathcal{M}_{0,n+1}(0, \mathbf{L})$ we consider the extended surface $\bar{S}$ obtained from S by gluing an infinite hyperbolic cylinder with geodesic boundary of length L_i to the ith boundary for all $i = 1, \ldots, n$. See Figure 1. Then we consider the spine of $\bar{S}$ with respect to the marked cusp (the first boundary with length $L_0 = 0$) in the sense of [2], which is the subset of points of $\bar{S}$ that have more than one shortest geodesic to this cusp. The spine has the structure of a plane forest built from geodesic segments meeting at ("red") vertices. Introducing an additional ("white") vertex type that connects the ends of all segments that run into the same boundary cylinder, one ends up with a combinatorial plane tree $\mathfrak{t}$

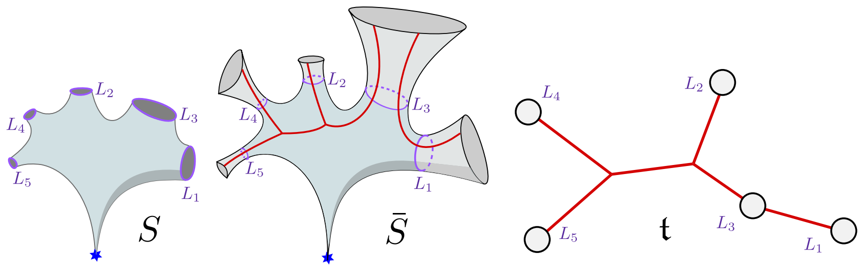

FIGURE 1. A surface $S \in \mathcal{M}_{0,1+5}(0, \mathbf{L})$, its extension $\bar{S}$ together with the spine, and the corresponding tree $\mathfrak{t}$ with 5 labeled white vertices and two red vertices.

with white vertices labeled $1, \ldots, n$. We decorate this tree with certain real labels: for each end a of an edge at a red vertex we record the angle $\phi_a \in (0, \pi)$ between the corresponding segment in $\bar{S}$ and the shortest geodesic towards the cusp. For each of the white vertices v_i, $i = 1, \ldots, n$, of degree $k = \deg v_i$ we record certain hyperbolic distances $t_{i,1}, \ldots, t_{i,k} \geq 0$ and $w_{i,1}, \ldots, w_{i,k} > 0$ that characterize the configuration of segments running into the corresponding hyperbolic cylinder. Then we can formulate our main bijective result as follows [4].

Theorem 1. *For $n \geq 2$ and $L_1, \ldots, L_n > 0$ this spine construction determines a bijection*

$$\mathcal{M}_{0,n+1}(0, \mathbf{L}) \to \bigsqcup_{\mathfrak{t}} \mathcal{A}_{\mathfrak{t}}(\mathbf{L}),$$

where the disjoint union is taken over all plane trees $\mathfrak{t}$ with n labeled white vertices of arbitrary degree and red vertices of degree at least 3, and the set $\mathcal{A}_{\mathfrak{t}}(\mathbf{L}) \subset \mathbb{R}^{6n-6}$ is the convex polytope of labels determined by the constraints

- $\sum_a \phi_a = \pi$ *where the sum runs over ends a that meet at a single red vertex;*
- $\phi_a = 0$ *for each end a incident to a white vertex;*
- $\phi_a + \phi_b < \pi$ *if a and b are two ends of the same edge;*
- $\sum_{j=1}^{\deg v_i} t_{i,j} = \sum_{j=1}^{\deg v_i} w_{i,j} = \frac{L_i}{2}$ *for $i = 1, \ldots, n$.*

This is useful for the computation of $V_{0,n+1}(0, \mathbf{L})$ because the Weil–Petersson volume form takes on a very simple form in terms of the tree labels.

Theorem 2. *The polytopes $\mathcal{A}_{\mathfrak{t}}(\mathbf{L})$ of maximal dimension $2n - 4$ are those for which all red vertices of $\mathfrak{t}$ are of degree 3. The push forward of the Weil–Petersson volume form μ_{WP} to such a polytope $\mathcal{A}_t(\mathbf{L})$ agrees with the $(2n-4)$-dimensional Euclidean volume form, up to a constant that depends only on n.*

It is straightforward to convince oneself that the Euclidean volume $|\mathcal{A}_{\mathfrak{t}}(\mathbf{L})|$ of the polytope $\mathcal{A}_{\mathfrak{t}}(\mathbf{L})$ of maximal dimension is a rational multiple of

$$\pi^{2r} \prod_{i=1}^{n} L_i^{2 \deg v_i - 2},$$

where $r = 2n - 2 - \sum_{i=1}^{n} \deg v_i$ is the number of red vertices of $\mathfrak{t}$. Since there are only finitely many trees, this explains why

$$V_{0,n+1}(0,\mathbf{L}) = \sum_{\mathfrak{t}} |\mathcal{A}_{\mathfrak{t}}(\mathbf{L})| \in \mathbb{Q}[L_1^2, \dots, L_n^2, \pi^2].$$

Just like in the planar map case, the labels on the trees carry information about distances to the marked cusp. Analysis of the trees with control on the labels, therefore provides access to distance statistics of random hyperbolic surfaces sampled from $\mathcal{M}_{0,n+1}(0,\mathbf{L})$ with density proportional to μ_{WP}. In the case $L_1, \dots, L_n = 0$, this is used in a work with Curien [3] to establish convergence in distribution in the Gromov–Prokhorov topology of the random hyperbolic surface with n cusps rescaled by $n^{-1/4}$ to (a multiple of) the Brownian sphere as $n \to \infty$.

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Cylindric partitions

Sylvie Corteel

Cylindric partitions are cylindric analogues of reverse plane partitions. They were introduced by Gessel and Krattenthaler in 1993. Since 2016, following ideas of Foda and Welsh, they have been studied due to a connection to Rogers-Ramanujan identities. If the profile of the cylindric partitions has r up steps (level r), it is believed that those partitions will produce A_{r-1} versions of the Andrews-Gordon identities. In particular there should exist C_r (r^{th} Catalan number) identities of level r and rank $r+1$. Results are now well understood for $r = 1, 2$ and much progress has been made in the last couple of years for $r = 3$ by Bridges, Corteel, Dousse, Kanada, Russell, Tsuchioka, Uncu, Welsh and Warnaar. The hope is that the case $r = 4$ can be attacked using the combinatorics of cylindric partitions.

Unusually large components in near-critical Erdős-Rényi graphs via ballot theorems

Umberto De Ambroggio
(joint work with Matthew I. Roberts)

The Erdős-Rényi random graph, denoted by $\mathbb{G}_{n,p}$, is obtained from the complete graph with vertex set $[n]$ by independently retaining each edge with probability $p \in [0,1]$ and deleting it with probability $1-p$. We are interested in the size of largest components $\mathcal{C}_{\max}$. It is well known (see e.g [2] and [6] for more details) that, if $p = p(n) = c/n$ for constant c, then $\mathbb{G}_{n,p}$ undergoes a phase transition in the size of $\mathcal{C}_{\max}$ as c passes 1. Specifically, with probability tending to one we know that:

(1) if $c < 1$ (the *subcritical* case), then $|\mathcal{C}_{\max}|$ is of order $\log n$;
(2) if $c = 1$ (the *critical* case), then $|\mathcal{C}_{\max}|$ is of order $n^{2/3}$;
(3) if $c > 1$ (the *supercritical* case), then $|\mathcal{C}_{\max}|$ is of order n.

Motivated by the lack of a simple proof of (2), Nachmias and Peres [10] used a martingale argument to prove that for any $A > 8$ and for all sufficiently large n,

$$\mathbb{P}(|\mathcal{C}(V_n)| > An^{2/3}) \leq 4n^{-1/3}\exp\{-A^2(A-4)/32\}$$

and

$$\mathbb{P}(|\mathcal{C}_{\max}| > An^{2/3}) \leq \frac{4}{A}\exp\{-A^2(A-4)/32\}$$

(with V_n denoting a vertex selected uniformly at random from $[n]$). The best known bound on the latter quantity is due originally to Pittel [11] who showed that for p of this form,

$$\lim_{n\to\infty} A^{3/2} e^{\frac{A^3}{8} - \frac{\lambda A^2}{2} + \frac{\lambda^2 A}{2}}\mathbb{P}(|\mathcal{C}_{\max}| > An^{2/3})$$

converges as $A \to \infty$ to a specific constant, which is stated to be $(2\pi)^{-1/2}$ but should be $(8/9\pi)^{1/2}$, as remarked by Roberts [12] (who established a stronger result that allows both A and λ to depend on n).

Both Pittel [11] and Roberts [12] heavily relied on a combinatorial formula for the expected number of components with exactly k vertices and $k+\ell$ edges, which is specific to Erdős-Rényi graphs and appears difficult to adapt to other random graph models.

We introduced a new proof methodology to compute the precise asymptotics for $\mathbb{P}(|\mathcal{C}_{\max}| > An^{2/3})$ (up to multiplicative constants in the polynomial term) that combines the strengths of the results mentioned above, namely:

- it gives accurate bounds for large A as $n \to \infty$;
- it allows A and λ to depend on n;
- it uses only robust probabilistic tools and therefore has the potential to be adapted to other models of random graphs.

Our main result is the following (see [4])

Theorem 1. *There exists $A_0 > 0$ such that if $A = A(n)$ satisfies $A_0 \leq A = o(n^{1/30})$ and $p = p(n) = 1/n + \lambda/n^{4/3}$ with $\lambda = \lambda(n)$ such that $|\lambda| \leq A/3$, then for sufficiently large n we have*

$$\frac{c_1}{A^{3/2}} e^{-\frac{A^3}{8}+\frac{\lambda A^2}{2}-\frac{\lambda^2 A}{2}} \leq \mathbb{P}(|\mathcal{C}_{\max}| > An^{2/3}) \leq \frac{c_2}{A^{3/2}} e^{-\frac{A^3}{8}+\frac{\lambda A^2}{2}-\frac{\lambda^2 A}{2}} \tag{1}$$

for some constants $0 < c_1 < c_2 < \infty$.

The methodology which lead to (1) was later adapted to study the (near-critical) $\mathbb{G}_{n,d,p}$ model; this is the random graph on n vertices obtained by first drawing uniformly at random a d-regular simple graph on $[n]$ and then performing independent p-bond percolation on it. For this model, we have the following (see [3])

Theorem 2. *There exist $A_0 > 0$ such that, if $A_0 \leq A = A(n)$ satisfies $A_0 \leq A \ll n^{1/30}$ and $p = p(d,n) = (d-1)^{-1}(1+\lambda n^{-1/3})$ with $\lambda = \lambda(n)$ such that $|\lambda| \leq A(1-2/d)[3(d-1)]^{-1}$ and $d \geq 3$ is fixed, then for all sufficiently large n we have*

$$\frac{c_1}{A^{3/2}} e^{-G_\lambda(A,d)} \leq \mathbb{P}(|\mathcal{C}_{\max}| > An^{2/3}) \leq \frac{c_2}{A^{3/2}} e^{-G_\lambda(A,d)} \tag{2}$$

for some constants $0 < c_1 = c_1(d) < c_2(d) = c_2$, where

$$G_\lambda(A,d) := \frac{A^3(d-1)(d-2)}{8d^2} - \frac{\lambda A^2(d-1)}{2d} + \frac{\lambda^2 A(d-1)}{2(d-2)}.$$

We briefly describe the main ideas for the proof of the upper bound stated in (1) when $p = 1/n$; the lower bound is more technical.

To establish the upper bound in (1), we use an exploration process to reveal the components of $\mathbb{G}_{n,1/n}$, which allows us to rewrite $\mathbb{P}(|\mathcal{C}(V_n)| > An^{2/3})$ as the probability that an integer-valued random process R_t (with $R_0 = 1$) stays positive for $k := An^{2/3}$ steps. Then we couple such a process with a (mean-zero) random walk S_t (with $S_0 = 1$) having independent increments $\mathrm{Bin}(n, 1/n) - 1$. Such a coupling allows us to bound from above the probability that R_t stays positive for k steps by the probability that S_t stays positive for k steps *and* finishes somewhere above $k^2/(2n)$. (Note that $\mathrm{Var}(S_k) \approx k$ and $k^2/(2n) = A^2 n^{1/3}/2 = A^{3/2}\sqrt{k}/2$, hence we are asking a mean zero random walk with *well-behaved* increments to be far above $\sqrt{k}$ at time k; this fact already suggests the appearing of an exponential upper bound.) Summarizing, by means of an exploration process and a coupling we show that

$$\mathbb{P}(|\mathcal{C}(V_n)| > k) = \mathbb{P}(R_t > 0 \ \forall t \in [k]) \leq \mathbb{P}(S_t > 0 \ \forall t \in [k], S_k > k^2/(2n)). \tag{3}$$

In order to bound from above the we use the following result (see [4]).

Theorem 3. *Fix $n \in \mathbb{N}$ and let $(X_i)_{i\geq 1}$ be independent and identically distributed random variables taking values in $\mathbb{Z}$, whose distribution may depend on n. Let $h \in \mathbb{N}$, and suppose that $\mathbb{P}(X_1 = h) > 0$. Define $W_t = \sum_{i=1}^t X_i$ for $t \in \mathbb{N}_0$. Then for any $j \geq 1$ we have*

$$\mathbb{P}(h + W_t > 0 \ \forall t \in [n], \, h + W_n = j) \leq \mathbb{P}(X_1 = h)^{-1} \frac{j}{n+1} \mathbb{P}(W_{n+1} = j).$$

Applying Theorem 3 to our random walk $S_t = 1 + \sum_{i=1}^t (\mathrm{Bin}_i(n,1/n) - 1)$ we obtain something like

$$\mathbb{P}(S_t > 0\ \forall t \in [k], S_k > k^2/(2n)) \lesssim \sum_{j > k^2/(2n)} \frac{j}{k} \mathbb{P}(\sum_{i=1}^k \mathrm{Bin}_i(n,1/n) = k+j). \tag{4}$$

Since $\mathbb{E}\left[\sum_{i=1}^k \mathrm{Bin}_i(n,1/n)\right] = k$ and, as we have seen, $j > k^2/(2n) \gg \sqrt{k}$, we expect the probabilities in the sum on the right-hand side of (4) to decay exponentially fast. By means of analytic estimates for binomial point probabilities (and recalling that $k = An^{2/3}$) it is not difficult to show that

$$\sum_{j > k^2/(2n)} \frac{j}{k} \mathbb{P}(\sum_{i=1}^k \mathrm{Bin}_i(n,1/n) = k+j) \lesssim \frac{1}{A^{1/2} n^{1/3}} e^{-A^3/8}$$

so that

$$\mathbb{P}(|\mathcal{C}(V_n)| > An^{2/3}) \leq \frac{1}{A^{1/2} n^{1/3}} e^{-A^3/8}. \tag{5}$$

The bound concerning $|\mathcal{C}_{\max}|$ is easily obtained from (5) by means of Markov's inequality applied to the (non-negative) random variable counting the number of vertices contained in connected components formed by more then $An^{2/3}$ nodes; the details can be found e.g. in [10] and [4].

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Triangular Ice: Combinatorics and Limit Shapes

Philippe Di Francesco
(joint work with Bryan Debin and Emmanuel Guitter)

Two-dimensional integrable lattice models such as the Six Vertex (6V) model have a long history first rooted in the physics of spin systems with local interaction, for which exact bulk thermodynamic properties were derived [1], including continuum descriptions via Coulomb Gas [2] or Conformal Field Theory [3]. More recently, these models also entered the realm of combinatorics (by considering domains with "domain-wall" boundary conditions (DWBC) [4] and best illustrated by the correspondence between 6V-DWBC and Alternating Sign Matrices (ASM) [5], probability theory (by interpreting the configurations in terms of particle trajectories, see e.g. [6]), and algebraic geometry (by interpreting partition functions as K-theoretic characters of certain varieties, see e.g. [7]).

It was observed that in the presence of DWBC, the models behave quite differently and may display interesting scaling behavior, such as the arctic phenomenon. The latter is the emergence of sharp phase separations between ordered regions (crystal-like, near the boundaries of the domain) and disordered (liquid) ones.

1. The 20V model: enumerative results and conjectures

The 6V model with DWBC is defined on an $n \times n$ square grid by picking orientations of the lattice edges in such a way that at each inner vertex there are as many entering and outgoing edges. This condition is called the "ice rule", as it allows to rephrase the configurations in terms of a two-dimesnional ice crystal. Here there are $\binom{4}{2} = 6$ allowed local vertex configurations, hence the name. Moreover, configurations receive local weights a, b, c invariant under global orientation reversal. A suitable choice of weights obeying the celebrated Yang-Baxter equation ensures the integrability of the system. In particular there is a nice determinantal formula for the partition function $Z_n^{6V}(a, b, c)$, i.e. the sum over all configurations weighted by the product of all local weights.

When considered on a triangular lattice, the ice rule gives rise to $\binom{6}{3} = 20$ different local vertex environments, hence the name 20V model [8, 9, 10]. The simplest DWBC1 can be imposed on an $n \times n$ square grid of square lattice with a single diagonal NW-SE edge on each face (a convenient way of thinking of the triangular lattice), in such a way that all edges of the W and E vertical borders are pointing towards the domain while all edges of the N and S horizontal border point out. Again, a suitable choice of local vertex weights ensures the integrability of the model, and eventually to relate the partition function $Z_n^{20V_1}$ to that of the 6V model.

Theorem 1. *The number of configurations of the 20V model with DWBC1 on an $n \times n$ grid is given by*

$$Z_n^{20V_1} = Z_n^{6V}(1, \sqrt{2}, 1) = \det_{0 \le i,j \le n-1} \left\{ [u^i v^j] \frac{1}{1-uv} + \frac{2u}{(1-u)(1-u-v-uv)} \right\}$$
$$= 1, 3, 23, 433, 19705, 2151843, 561696335, 349667866305, 518369549769169, \ldots$$

The latter is easily expressed in the osculating path formulation of the 6V model as weighting each "straight" portion of path by a factor of $\sqrt{2}$, leading to a special kind of 2-enumeration of Alternating Sign Matrices. Remarkably, the above formula matches that for the enumeration of domino tilings of a specific domain [10].

Theorem 2. *The number of configurations $Z_n^{20V_1}$ of the 20V model with DWBC1 on an $n \times n$ grid is identical to that, Z_n^{DT}, of quarter-turn symmetric domino tilings of the holey Aztec quasi-square of size $2n$.*

The latter is indeed directly enumerated by the above determinant formula. Indeed, one writes $Z_n^{DT} = \det(I+M) = \sum_{k=0}^{n} \sum_{0 \le i_1 < \ldots < i_k \le n-1} |M|_{i_1,\ldots,i_k}^{i_1,\ldots,i_k}$, where the matrix elements of $I + M$ are identified with the coefficients in the u, v series expansion of Theorem 1, and the diagonal minors count non-intersecting lattice paths with horizontal, diagonal and vertical steps, in bijection with domino tilings of the conic fundamental domain of the holey Aztec square under quarter-turn symmetry.

Other DWBC3 boundary conditions consist in having first the 6V DWBC on the horizontal and vertical edges of the triangular lattice, together with diagonal boundary edges all pointing up/left. The partition function $Z_n^{20V_3}$ has no direct connection to Z_n^{6V}, but the number of configurations seems to be identical to that of domino tilings of a triangle introduced by Pachter [13], obtained by halving a $2n \times 2n$ square with a diagonal staircase of steps of length 2. The sequence of numbers reads for the uniformly weighted $Z_n^{20V_3}$:

$$1, 3, 29, 901, 89893, 28793575, 29607089625, 97725875584681,$$
$$1035449388414303593, \ldots$$

Conjecture 3. *The number of configurations of the 20V model with DWBC3 on an $n \times n$ grid is equal to that of domino tilings of Pachter's triangle of size $2n$.*

In fact an extension of the 20V domain to a rectangle $(n+k) \times n$ with horizontal outgoing edge orientations was also considered in [10]. Parallelly, the Pachter triangle can be enhanced by raising the "roof" that constains its non-intersecting lattice path formulation by an amount of k.

Conjecture 4. *The number of configurations of the extended 20V model with DWBC3 on an $(n+k) \times n$ grid is equal to that of domino tilings of the enhanced Pachter's triangle of size $2n$ with displaced roof by an amount k, for all $k \ge 0$.*

In the case of maximal raised roof/enhancement corresponding to $k = n - 1$, the sequence of numbers of configurations $Z_n^{20V_e}$ was obtained in terms of that $Z_n^{6V_U}$ of the so-called U-turn boundary DWBC 6V model [14]:

Theorem 5.

$$\begin{aligned}
Z_n^{20V_e} &= \det_{0\le i,j\le n-1}\left\{[u^iv^j]\frac{(1+u^2)(1+2u-u^2)}{(1-u^2v)\Big((1-u)^2-v(1+u)^2\Big)}\right\}\\
&= 2^{n(n-1)/2}\prod_{j=0}^{n-1}\frac{(4j+2)!}{(n+2j+1)!}\\
&= 1, 4, 60, 3328, 678912, 508035072, 1392439459840, 13965623033856000, \dots
\end{aligned}$$

The last formula was conjectured in [14], and proved automatically by C. Koutschan. It is very reminiscent of the celebrated ASM number formula $\prod_{j=0}^{n-1}\frac{(3j+1)!}{(n+j)!}$.

2. Limit shapes

The 6V model in its disordered phase, while including a free fermion (non-intersecting lattice path) case, is generically a model of *interacting* fermions: it admits an "osculating path" description, in which paths are non-intersecting, but are allowed to interact by "kissing" i.e. sharing a vertex at which they bounce against each-other.

The 6V model on a square domain with DWBC exhibits an arctic phenomenon in its disordered phase, which was predicted via non-rigorous methods [15, 16], the latest of which being the Tangent Method introduced by Colomo and Sportiello [17]: in a nutshell, one displaces the end of the outermost path of the configurations to find a family of tangent lines to the arctic curve. This method was validated in a number of cases, mostly in free fermion situations [12, 19, 20, 21, 22, 23]. Beyond free fermions and the case of the 6V-DWBC model (see also [24] for the case of U-turn reflective boundaries), the method was applied successfully to the 20V model [11, 24]. The new feature in non-free fermion cases is that the arctic curves are generically *no longer analytic*, but rather *piecewise analytic*. For instance, the arctic curve for large Alternating Sign Matrices (uniformly weighted 6V-DWBC) is made of four pieces of different ellipses as predicted in [15] and later proved in [18].

In our two 20V cases (corresponding to Theorems 1 and 5, there is an intriguing correspondence between arctic curves of both tilings and Vertex models: the former is the analytic continuation of the topmost part of the latter, while other pieces of the arctic curve are obtained from this analytic continuation via a shear transformation. The same phenomenon was observed when comparing the limit shape of large ASMs to that of Descending Plane Partitions in the form of 1/3-turn symmetric rhombus tilings of a holey hexagon [25], with a single ellipse as arctic curve.

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b-Hurwitz numbers and enumeration of maps

Maciej Dołęga
(joint work with Valentin Bonzom, Guillaume Chapuy)

The (weighted) Hurwitz numbers are one of the primary objects in enumerative geometry. From the combinatorial point of view one can approach them by studying the associated generating series τ^G given by the following explicit partition function of the Schur symmetric functions s_λ:

$$\tau^G \equiv \tau^G(t,u;\mathbf{p},\mathbf{q}) := \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}) \prod_{\square\in\lambda} G(u\cdot c(\square)).$$

Here G is a weight function, and $c(\square)$ is the "content" of the box $\square \in \lambda$, i.e. the difference between its x and y-coordinates. Various choices of the function G correspond to many classical cases such as (double) Hurwitz numbers, (double) monotone Hurwitz numbers, or Grothendieck's dessins d'enfants.

Jack polynomials $J_\lambda^{(1+b)}$ are famous one-parameter deformations of Schur symmetric functions, and (up to normalization) they specialize to them at $b = 0$. It seems that when we replace Schur functions by their Jack deformations in this generating series, the positivity and integrality is mysteriously not affected. To be more precise, consider the following generating series:

$$\tau_b^G \equiv \tau_b^G(t,u;\mathbf{p},\mathbf{q};b) := \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{J_\lambda^{(1+b)}(\mathbf{p}) J_\lambda^{(1+b)}(\mathbf{q})}{\|J_\lambda^{(1+b)}\|^2} \prod_{\square\in\lambda} G(u\cdot c_b(\square)),$$

where $c_b(\square)$ is a Jack-deformed content of the box ([3]). This series is a one-parameter deformation of the classical tau function for the (weighted) Hurwitz numbers ($\tau^G \equiv \tau^G_{b=0}$), and similarly as before, for various choices for G we recover functions that naturally appear in the context of β-deformations of matrix models such as the partition function for the Gaussian β-Ensembles, or for the β-deformed HCIZ integral (see [3, 1]).

Our main result provides a combinatorial and geometric interpretation of this b-deformed tau function. We proved in [3] that τ_b^G is a generating series of certain graphs embedded into not necessarily orientable surfaces. Such an embedded graph M (called a *map*) is counted with the "weight" $b^{\mathrm{MON}(M)}$, where the exponent $\mathrm{MON}(M)$ measures the "non-orientability" of this embedding.

In the orientable case, Goulden and Jackson used in [4] the integrability of τ^G (for a specific choice of G) to prove a surprisingly simple recurrence relation satisfied by the number $t_{n,g}$ of triangulations of the surface of genus g into n triangles. Integrability in this context means that there exists a certain infinite family of PDEs (called the *KP hierarchy*) with the solution given by τ^G. Since then, several other models for orientable maps were studied using integrability in order to find similar recurrence formulas.

In [2] we were interested in obtaining efficient recurrence formulas to count maps on non-oriented surfaces (according to their genus and size parameters).

Using relations between $\tau^G_{b=1}$ and matrix models we showed that the generating series of maps and bipartite maps on non-oriented surfaces are tau functions of the *BKP hierarchy* of Kac and van de Leur [5]. There is a certain structural difference between the KP and the BKP hierarchies, therefore during elimination of variables in the BKP equation that we needed to perform in order to obtain recurrence formulas to count maps, we end up with a differential equation for the generating function that involves shifts of the main variable, hence it is *not* an ODE in their main variable. As a consequence, the recurrence obtained by us does not have polynomial coefficients, in contrast with the analogous result for the orientable models. Nevertheless, they are incredibly short compared to any alternative, and they allow one to count non-oriented maps much faster than using the standard Tutte decomposition. Moreover, their relatively simple form suggests that they might have a direct combinatorial interpretation, which deserves further studies.

Finally, we found out that if we use the first three equations from the BKP hierarchy instead of only the first one, we can eliminate variables in a way that we produce the ODE for the generating function of non-oriented maps and, in return, recurrences with polynomial coefficients. The price we pay is that these recurrences are much larger than the ones with non-polynomial coefficients. Nevertheless, if we restrict to maps with only one face, the recurrence formulas became very short and to our surprise, by doing this we recovered Ledoux's recurrence for the number of genus g, one-face non-oriented maps, and proved a non-oriented analogue of the recurrence of Adrianov for the number of genus g, one-face bipartite maps.

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Grounded partitions and characters of infinite dimensional Lie algebras

Jehanne Dousse
(joint work with Isaac Konan)

The character of a Lie algebra module can be seen as a generating function for the dimensions of certain submodules. More precisely, if $\mathfrak{g}$ be an affine Kac–Moody Lie algebra and $L(\lambda)$ is an irreducible highest weight $\mathfrak{g}$-module of highest weight λ, the character of $L(\lambda)$ is defined as

$$\mathrm{ch}(L(\lambda)) := \sum_{\mu} \dim V_\mu \cdot e^{\mu},$$

where e is a formal exponential, and $\dim V_\mu$ is the dimension of the weight space V_μ in the weight space decomposition of $L(\lambda)$. Background on Lie algebras and characters can be found in [5].

By definition, the character $e^{-\lambda}\mathrm{ch}(L(\lambda))$ is a series with positive coefficients in $\mathbb{Z}[[e^{-\alpha_0}, \dots, e^{-\alpha_n}]]$, where $\alpha_0, \dots, \alpha_n$ are the simple roots of $\mathfrak{g}$. However, finding explicit formulas that exhibit this positivity can be difficult, and is an important problem in representation theory.

Several connections between integer partitions and characters have emerged over the years, starting with the work of Lepowsky, Milne and Wilson [7, 8]. They showed that both sides of the Rogers–Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

once multiplied by the factor $(-q;q)_\infty$, are expressions for the principal specialisation (i.e. for all i, $e^{-\alpha_i} \mapsto q$) of the character of level 3 standard modules of the affine Lie algebra $A_1^{(1)}$. The product side comes from the Weyl–Kac character formula while the sum side comes from constructing a basis of the modules using the theory of vertex operators.

In a related but different approach, Primc [9] expressed the principal specialisation of characters of level 1 standard modules of $A_1^{(1)}$ and $A_2^{(1)}$ in two different ways to conjecture and prove two partition identities on coloured partitions. In his approach, the product side still comes from the Weyl–Kac character formula, but the sum side now comes from the theory of perfect crystals. Intuitively, a perfect crystal is a directed graph $\mathcal{B}$ together with a weight function wt satisfying certain conditions encoding the structure of the modules under consideration (basics on crystals can be found in [4]). An important character formula, which was used by Primc together with the principal specialisation, is the $(\mathrm{KMN})^2$ character formula, due to Kang, Kashiwara, Misra, Miwa, Nakashima and Nakayashiki [6]:

$$\mathrm{ch}(L(\lambda)) = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{\mathrm{wt}\,\mathfrak{p}}.$$

It expresses the character $\mathrm{ch}(L(\lambda))$ as a series indexed by so-called "λ-paths", which are infinite sequences of vertices of the crystal graph which are ultimately equal to a particular path called the "ground state path".

We use the theory of perfect crystals and a bijection with a new kind of integer partitions called "grounded partitions" to obtain purely combinatorial character formulas. Let us define these objects more precisely. Let $\mathcal{C}$ be a set of colours and $c_g \in \mathcal{C}$. Let $\succ$ be a binary relation defined on the coloured integers $\mathbb{Z}_{\mathcal{C}} = \{k_c : k \in \mathbb{Z}, c \in \mathcal{C}\}$. A *grounded partition* with ground c_g and relation $\succ$ is a finite sequence $(\pi_0, \ldots, \pi_s)$ of coloured integers, such that

- for all $i \in \{0, \ldots, s-1\}$, $\pi_i \succ \pi_{i+1}$,
- $\pi_s = 0_{c_g}$,
- $\pi_{s-1} \neq 0_{c_g}$.

Let $\mathcal{P}_{c_g}^{\succ}$ denote the set of such partitions.

We first show [2] that in the case where the ground state path is constant equal to $\cdots \otimes g \otimes g \otimes g$, the set of λ-paths $P(\lambda)$ is in bijection with the set $\mathcal{P}_{c_g}^{\succ}$ of grounded partitions, where the binary relation $\succ$ is defined via the so-called energy function of the perfect crystal. Thus, writing $q = e^{-\delta/d_0}$ where $\delta = d_0\alpha_0 + d_1\alpha_1 + \cdots + d_{n-1}\alpha_{n-1}$ is the null root, and $c_b = e^{\mathrm{wt}\, b}$ for all $b \in \mathcal{B}$, we have an expression for the character as a generating function for grounded partitions:

$$e^{-\lambda}\mathrm{ch}(L(\lambda)) = \sum_{\pi \in \mathcal{P}_{c_g}^{\succ}} C(\pi) q^{|\pi|},$$

where $C(\pi)$ is the product of the colours of the parts of π and $|\pi|$ is the sum of the values of the parts of π. This allows us to show that the purely combinatorial generalisation of Primc's identity which we proved in [1] actually gives an expression with obviously positive coefficients for level 1 standard modules of $A_n^{(1)}$ for all n.

Then we generalise this idea to all ground state paths (not necessarily constant) by introducing multi-grounded partitions [3]. This gives an expression for characters of all standard modules as generating functions for multi-grounded partitions. Note that we do not need to perform the principal specialisation, and obtain manifestly positive expressions for the characters.

By explicitly computing generating functions for multi-grounded partitions corresponding to the level 1 modules of all classical types except $C_n^{(1)}$, we give simple manifestly positive formulas for their characters. For example, in type $B_n^{(1)}$ with $n \geq 3$, let $\Lambda_0, \ldots, \Lambda_n$ be the fundamental weights, $\alpha_0, \ldots, \alpha_n$ be the simple roots of $B_n^{(1)}$ and $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 \cdots + 2\alpha_n$ be the null root. Setting

$$q = e^{-\delta/2}, \quad c_0 = 1, \quad \text{and} \quad c_i = e^{\alpha_i + \cdots + \alpha_{n-1} + \alpha_n} \text{ for all } i \in \{1, \ldots, n\},$$

we have

$$e^{-\Lambda_0}\mathrm{ch}(L(\Lambda_0)) = \frac{1}{2}\Big((-c_0q;q^2)_\infty \prod_{k=1}^{n}(-c_kq;q^2)_\infty(-c_k^{-1}q;q^2)_\infty + (c_0q;q^2)_\infty \prod_{k=1}^{n}(c_kq;q^2)_\infty(c_k^{-1}q;q^2)_\infty\Big).$$

Ideally, our goal would be to find explicit expressions for the characters of standard modules of all types at all levels. But even if our method is in theory applicable in all cases, computing the generating function explicitly can be difficult. We plan to return to this in future work.

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Recursions and proofs in Coxeter-Catalan combinatorics

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(joint work with Matthieu Josuat-Vergès)

Some of the most fascinating results about the symmetric group S_n are special cases of theorems that hold for all (finite) Coxeter groups W, or more generally complex reflection groups; this is the world of Coxeter combinatorics. Coxeter-Catalan combinatorics study in particular the poset of W-*noncrossing partitions* $NC(W) := [1,c]_{\leq_R}$ which consists of elements $w \in W$ that lie below a Coxeter element c under the absolute order $\leq_R$ of W. It is a lattice, it has many applications outside of combinatorics –in particular it encodes $K(\pi,1)$ spaces for the braid group $B(W)$– and when $W = S_n$, it is isomorphic to the lattice of noncrossing partitions due to Kreweras which are enumerated by the Catalan numbers.

The number $MC(W)$ of *maximal* chains of the noncrossing partition lattice $NC(W)$ is given by the *Deligne-Arnold-Bessis formula* $MC(W) = h^n n!/|W|$

where n is the rank and h the Coxeter number of W. In [6] we gave the first Coxeter theoretic proof of this formula by solving the Deligne-Reading recursion on $MC(W)$, counting maximal chains with respect to their last element. The success of this approach led us to apply this idea towards a more general setting.

One of the main open problems in Coxeter-Catalan combinatorics (for more than twenty years [2, §7]) has been to explain, i.e. give type-independent proofs for, the remarkable product formula below for the enumeration of chains in $NC(W)$. Our main contribution in the area is the first such proof, for which we developed a framework of techniques in [7, 8, 9], and which is finally presented at [10].

Theorem 1 (Athanasiadis-Reiner [2] via the classification, [10] uniformly)**.** *For any real reflection group W, the number $\mathrm{Krew}_{W,[X]}(m)$ of length-m chains in $NC(W)$ whose first element has parabolic type $[X]$ (given as a W-orbit of flats $[X] \in \mathcal{L}_W/W$) is given by the formula*

$$\mathrm{Krew}_{W,[X]}(m) = \frac{\prod_{i=1}^{\dim(X)}(mh+1-b_i^X)}{[N(X):W_X]}, \tag{1}$$

where $N(X)$ and W_X are the setwise and pointwise normalizers of X, and b_i^X its Orlik-Solomon exponents.

There is a natural way to define parking functions associated to m-chains of $NC(W)$; they carry a natural W-action and the resulting module is called the *m-Fuss noncrossing parking space* $\mathrm{Park}_W^{NC}(m) := \oplus_{[X]} \mathrm{Krew}_{W,[X]}(m) \cdot \uparrow_{W_X}^W \mathbf{1}$. A sibling object to this space is the so called *algebraic parking space* $\mathrm{Park}_W^{\mathrm{alg}}(m) := \mathbb{C}[V]/(\boldsymbol{\Theta})$ introduced in [1] as the quotient of the ambient polynomial ring over a system of parameters $\boldsymbol{\Theta} := (\theta_1, \dots, \theta_n)$ of homogeneous degrees $\deg(\theta_i) = mh+1$ that carry the reflection representation of W (the existence of such h.s.o.p. relies on Rouquier's shift functors for rational Cherednik algebras). The numbers given by the product formula of Theorem 1 are naturally structure coefficients for $\mathbb{C}[V]/(\boldsymbol{\Theta})$ so that the theorem can be equivalently phrased as the W-isomorphism between the two parking spaces; that is, we prove that $\mathrm{Park}_W^{NC}(m) \cong_W \mathrm{Park}_W^{\mathrm{alg}}(m)$.

A comparison of recursions. We prove Theorem 1 by expanding the ideas of [6]. There is a natural recursion on the numbers $\mathrm{Krew}_{W,[X]}(m)$ if one counts length-m chains with respect to the parabolic type of their k-th element. The main ingredient of our proof is to show that the same recursion is satisfied by the right hand side of (1). Phrased in terms of the algebraic parking spaces, this becomes the following theorem, which we prove by comparing the characters of the two representations.

Theorem 2 ([9])**.** *For any natural numbers m, k, r such that $m = k + r$, we have the expansion formula*

$$\mathrm{Park}_W^{\mathrm{alg}}(m) = \bigoplus_{[X]\in\mathcal{L}_W/W} \mathrm{Krew}_{W,[X]}(k) \cdot \Big\uparrow_{W_X}^W \mathrm{Park}_{W_X}^{\mathrm{alg}}(r). \tag{2}$$

The proof of Theorem 2 relies on our work on arrangement Laplacians and their spectrum. We showed in [5] that the characteristic polynomial of the $\mathcal{A}$-Laplacian $L_{\mathcal{A}}(\boldsymbol{\omega})$, for arbitrary weights $\boldsymbol{\omega}$, is given in terms of the Laplacians of the localizations $\mathcal{A}_Y$:

$$\det\big(t\cdot \mathrm{Id} + L_{\mathcal{A}}(\boldsymbol{\omega})\big) = \sum_{Y\in\mathcal{L}_{\mathcal{A}}} \mathrm{qdet}\big(L_{\mathcal{A}_Y}(\boldsymbol{\omega}_Y)\big)\cdot t^{\dim(Y)}. \tag{3}$$

In the setting of Theorem 2, we considered the restricted reflection arrangements $\mathcal{A}^X$ and a special selection of weights. For any hyperplane $Z\in\mathcal{A}^X$, the *relative Coxeter number* $h(X,Z)$ is defined as the Coxeter number of the unique irreducible component of W_Z that does not belong to W_X. We prove in [9] that the recursion (3) for the arrangement $\mathcal{A}^X$ with weights $\omega_Z := h(X,Z)$, gives essentially the equality of characters for the two sides of (2).

From a different perspective, the equality between the structure coefficients in the two sides of (2) can be seen as a relation between Coxeter numbers and Orlik-Solomon exponents of a reflection arrangement $\mathcal{A}$ and its flats. We prove that the following positive expansion theorem is equivalent to the parking space recursion of Theorem 2. In [9] we give a conjectural interpretation for it in terms of special multi-derivation modules for the arrangements $\mathcal{A}^X$.

Theorem 3 ([9])**.** *For an irreducible real reflection arrangement $\mathcal{A}$ and a flat $X\in\mathcal{L}_{\mathcal{A}}$, we have that*

$$\prod_{i=1}^{\dim(X)}(t+mh+b_i^X) = \sum_{Y\in\mathcal{L}_{\mathcal{A}^X}} t^{\dim(Y)}\cdot \prod_{i=1}^{\dim(X)-\dim(Y)}\big(mh_i(X,Y)+b_i^{X,Y}\big),$$

where h and $h_i(X,Y)$ are Coxeter numbers and $b_i^X, b_i^{X,Y}$ Orlik-Solomon exponents for $\mathcal{A}$, $\mathcal{A}^X$, and $\mathcal{A}^X_Y$.

Future directions. The two previous theorems are very suggestive of further research. Our proof of Theorem 2 did not make use of the graded module structure of the parking spaces. It is natural to ask for a q-version:

Problem 1. *Give a q-version of Theorem 2, for instance via Rouquier's shift functors for Cherednik algebras, or by generalizing the Lie-theoretic q-Kreweras numbers of* [12]*, or via the freeness conjecture of* [9]*.*

In Theorem 3 and when the flat X is the whole ambient space V, the left hand side agrees with the Poincare polynomial of the m-Fuss-Catalan deformation $\mathcal{A}^{[-m,m]}$ of $\mathcal{A}$ (which is non central and adds for each hyperplane $H\in\mathcal{A}$ an extra $2m$-many, parallel, equally spaced copies of it). In [9] we give a separate Ehrhart theoretic proof of Theorem 3 for $X=V$ relying on reciprocity theorems of Athanasiadis [4]; it suggests the following question.

Problem 2. *In Weyl groups W, generalize Athanasiadis' works* [3, 4] *and construct deformations $\mathcal{A}^{X,\boldsymbol{m}}$ of the restricted arrangements $\mathcal{A}^X$ so that their resulting*

Poincare polynomials are given by the formulas

$$P(\mathcal{A}^{X,\boldsymbol{m}},t)=\prod_{i=1}^{\dim(X)}(t+mh+b_i^X).$$

In recent work [11] a *rational* version of W-noncrossing partitions has been introduced, resolving another old open problem in the area. The authors gave type-independent proofs for the enumeration of these objects that recover the special case $X=V$ of our Theorem 1 (but their proof naturally produced a q-version for that case as well). The algebraic recursions we prove in Theorem 2 do generalize to that setting and it is natural to try to combine the two techniques.

Problem 3. *Refine the combinatorial models for rational Catalan objects and generalize Theorem 1 in that setting.*

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Counting lattice walks by winding number

Andrew Elvey Price

The behaviour of the winding of long random walks around a given point has been well understood for over 30 years [1, 2], yet until the recent work of Budd [3] there were no exact results. Budd enumerated simple walks on the square lattice by winding angle with certain fixed endpoints by analyzing the Eigenvalues of certain infinite matrices counting paths in the lattice - a method unfamiliar to the combinatorics of lattice paths. Subsequently the author derived similar results for several step-sets, including Kreweras walks using methods more familiar to the combinatorics of lattice paths [5]. In this work we generalise this further: for any (non-trivial) step-set $S \in \{-1,0,1\}^2 \setminus \{(0,0)\}$ we enumerate walks by winding number using steps in S. We say a step set is non-trivial if $(0,0)$ is on the interior of the convex hull of S. To derive these results we first characterise generating series counting the walks using functional equations, which we solve analytically taking inspiration from methods used for the enumeration of walks restricted to a cone [6, 7, 4].

Given a walk w, starting at $(1,0)$ and not passing through $(0,0)$, we define the winding angle of w as follows: let x be a variable point that moves continuously along the path w, and let $v_x = \frac{x}{|x|}$ be a variable unit vector pointing towards x. The *winding angle* of w, which we denote by $\theta(w)$, is the total anticlockwise angle that v_x spins around 0. Note that if two walks start and end at the same vertices as each other, then their winding angles must differ by a multiple of 2π. We define $a(w)$ and $b(w)$ to be the final x and y coordinates of w respectively and $n(w)$ to be the number of steps in w. Our aim in this work is then to determine the generating function

$$\mathsf{W}(x,y;s,t) = \sum_{w\in\mathcal{W}} x^{a(w)} y^{a(w)} s^{\lfloor \frac{\theta(w)}{2\pi} \rfloor} t^{n(w)}.$$

Our expressions are in terms of the Jacobi theta function

$$\vartheta(z,\tau) := \sum_{n=0}^{\infty} (-1)^n e^{i\pi\tau(n+\frac{1}{2})^2} \left(e^{(2n+1)iz} - e^{-(2n+1)iz}\right),$$

defined for $z, \tau \in \mathbb{C}$, with τ in the upper half-plane. Our results also rely on values $\alpha, \beta, \gamma, \delta, \epsilon, \tau, x_c, y_c \in \mathbb{C}$ which depend only on the step set and t and functions $X, Y : \mathbb{C} \to \mathbb{C}$ defined in [4] to satisfy

$$X(z) = x_c \frac{\vartheta(z-\alpha,\tau)\vartheta(z+\gamma+\alpha,\tau)}{\vartheta(z-\delta,\tau)\vartheta(z+\gamma+\delta,\tau)}$$

$$Y(z) = y_c \frac{\vartheta(z-\beta,\tau)\vartheta(z-\gamma+\beta,\tau)}{\vartheta(z-\epsilon,\tau)\vartheta(z-\gamma+\epsilon,\tau)}$$

$$\sum_{(j,k)\in S} X(z)^j Y(z)^k = \frac{1}{t}.$$

Our exact expression for $\mathsf{W}(x,y;s,t)$ is quite complicated, however for $\mathsf{E}(s,t) := [x^1y^0]\mathsf{W}(x,y;s,t)$, which counts excursions starting and ending at $(1,0)$, we have the following expression

$$\begin{aligned}\mathsf{E}(t,s) = \frac{x_c}{t(1-e^{-2i\kappa})} \frac{Y(\alpha)Y(-\gamma-\alpha)}{Y(\alpha)-Y(-\gamma-\alpha)} \frac{\vartheta(\delta-\alpha,\tau)\vartheta(\delta+\gamma+\alpha,\tau)}{\vartheta(\gamma+2\delta,\tau)\vartheta(\kappa,\tau)} \\ \left(-\frac{\vartheta(\alpha+\gamma+\delta+\kappa,\tau)}{\vartheta(\alpha+\gamma+\delta,\tau)} + e^{-2i\kappa}\frac{\vartheta(\alpha-\delta+\kappa,\tau)}{\vartheta(\alpha-\delta,\tau)}\right. \\ \left. + \frac{\vartheta(\alpha-\delta-\kappa,\tau)}{\vartheta(\alpha-\delta,\tau)} - e^{-2i\kappa}\frac{\vartheta(\alpha+\delta+\gamma-\kappa,\tau)}{\vartheta(\alpha+\delta+\gamma,\tau)}\right)\end{aligned}$$

as long as $(-1,-1)\in S$, with similar expressions in other cases. In the special case of simple walks, that is $S=\{(-1,0),(0,1),(1,0),(0,-1)\}$, our results are more explicit:

Theorem 1. *Let $q=e^{\frac{i\pi\tau}{4}}=e^{i\gamma}=$ be the unique series satisfying*

$$e^{-i\frac{\gamma}{2}}\frac{\vartheta\left(\frac{\gamma}{2},\tau\right)}{\vartheta\left(\frac{3\gamma}{2},\tau\right)}=\frac{\sqrt{1+4t}-1}{2\sqrt{t}},$$

and define

$$X(z)=e^{-i\gamma}\frac{\vartheta(z,\tau)\vartheta(z+\gamma,\tau)}{\vartheta(z-\gamma,\tau)\vartheta(z+2\gamma,\tau)}\ \text{ and } Y(z)=e^{-i\gamma}\frac{\vartheta(z,\tau)\vartheta(z-\gamma,\tau)}{\vartheta(z+\gamma,\tau)\vartheta(z-2\gamma,\tau)}.$$

there are unique series $C_0(x;t,s)$ and $C_1(x;t,s)$ satisfying for $s=e^{2i\kappa}$

$$C_0(X(z))=\frac{1}{1-Y(z)^2}e^{i\gamma}\frac{\vartheta(2\gamma)}{\vartheta(\kappa)}\left(-\frac{Y(z)^2\vartheta(z+2\gamma+\kappa)+e^{-2i\kappa}\vartheta(z+2\gamma-\kappa)}{\vartheta(z+2\gamma)}\right.$$
$$\left.+\frac{e^{-2i\kappa}Y(z)^2\vartheta(z-\gamma+\kappa)+\vartheta(z-\gamma-\kappa)}{\vartheta(z-\gamma)}\right),$$
$$C_1(X(z))=\frac{Y(z)}{1-Y(z)^2}e^{i\gamma}\frac{\vartheta(2\gamma)}{\vartheta(\kappa)}\left(\frac{\vartheta(z+2\gamma+\kappa)+e^{-2i\kappa}\vartheta(z+2\gamma-\kappa)}{\vartheta(z+2\gamma)}\right.$$
$$\left.-\frac{e^{-2i\kappa}\vartheta(z-\gamma+\kappa)+\vartheta(z-\gamma-\kappa)}{\vartheta(z-\gamma)}\right).$$

Then the generating function $\mathsf{W}(x,y;t,s)$ counting simple walks sarting at $(1,0)$ by winding angle is given by

$$\mathsf{W}(x,y;t,s)=-\frac{C_0(x;t,s)+y^{-1}C_1(x;t,s)}{1-t(x+y+x^{-1}+y^{-1})}.$$

Moreover, the generating function $\mathsf{E}(t,s)$ for walks ending at $(1,0)$ is given by

$$t\mathsf{E}(t,s)=\frac{e^{i\gamma}}{e^{-2i\kappa}-1}\frac{\vartheta(2\gamma)}{\vartheta(\kappa)}\left(\frac{\vartheta(2\gamma+\kappa)+e^{-2i\kappa}\vartheta(2\gamma-\kappa)}{\vartheta(2\gamma)}\right.$$
$$\left.-\frac{e^{-2i\kappa}\vartheta(\gamma-\kappa)+\vartheta(\gamma+\kappa)}{\vartheta(\gamma)}\right).$$

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Combinatorial models and bijections in Parabolic Cataland, type A and B

Wenjie Fang
(joint work with Cesar Ceballos, Henri Mühle, Jean-Christophe Novelli)

The well-studied Tamari lattice can be seen as a quotient of the weak order of the symmetric group, which is also a Coxeter group. With this point of view, Reading introduced Cambrian lattices in [8], generalizing the Tamari lattices to all irreducible Coxeter groups. A further generalization to parabolic quotients of irreducible Coxeter groups was given by Mühle and Williams in [6], giving *parabolic Tamari lattices*.

In this talk, we consider combinatorial models of parabolic Tamari lattices of type A and B, which can be considered as a generalization of Catalan objects. These models are also crucial for the study of lattice structures, as we can see in [8] and in the following.

In type A (the symmetric group $\mathfrak{S}_n$), some combinatorial models of elements in parabolic Tamari lattices such as bounce pairs and $(\alpha, 231)$-avoiding permutations were already given along with bijections in [6]. However, the bijections are quite complicated. In [2], we proposed a new combinatorial model called *left-aligned colored tree*, or simply *LAC tree*, which relates all previously given models with simple bijections (see Figure 1).

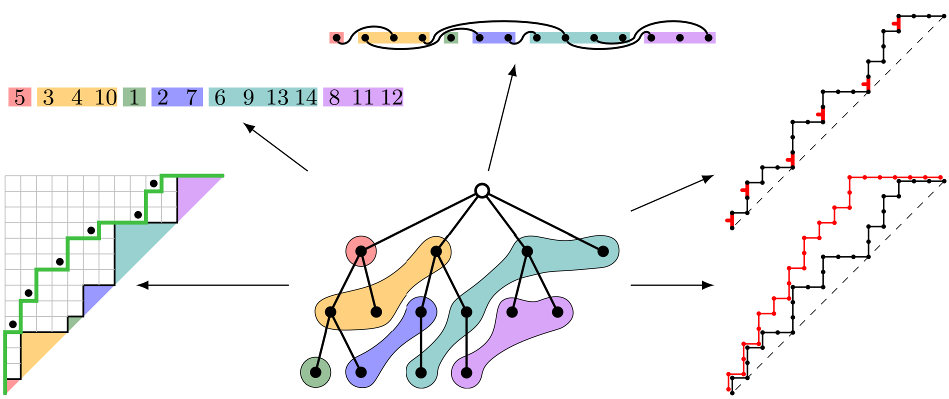

Figure 1. Combinatorial models for elements in parabolic Tamari lattices of type A, along with bijections.

Furthermore, we give a bijection from LAC trees to steep pairs proposed in [1], solving a conjecture therein on bijection between steep pairs and bounce pairs.

This bijection also relate LAC trees to some walks in $\mathbb{N}^2$ studied in [5], giving enumeration of LAC trees. Using the bijections, we proved that parabolic Tamari lattices of type A are isomorphic to certain ν-Tamari lattices defined in [7]. We also recover the famous zeta map in q,t-Catalan combinatorics (see [4]) and its labeled version, leading to a simple reformulation of the zeta map.

For type B (the hyperoctahedral group $\mathfrak{H}_n$ of sign-symmetric permutations), an initial combinatorial model of type-B $(\alpha, 231)$-avoiding permutations was given in our work [3], in which we constructed explicitly parabolic Tamari lattices of type B and proved that it has many nice lattice properties. In a work in progress, we define *type-B LAC trees*, which are type-A LAC trees with some *switch nodes*. We also give simple bijections analogous to the type-A ones, connecting type-B LAC trees with other type-B parabolic objects, such as type-B $(\alpha, 231)$-avoiding permutations, where switch nodes are used to give signs to elements in permutations. Using these bijections, we recover the (labeled) type-C zeta map defined in [9]. We believe that our type-B LAC trees may lead to new understanding of type-B q,t-Catalan combinatorics.

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Components in meandric systems and the infinite noodle

Valentin Féray

(joint work with Paul Thévenin)

This is an extended abstract of the paper [FT22]. We study *meandric systems* of size n, which are collections of non-crossing loops intersecting the horizontal axis exactly at the points $0, \dots, 2n-1$ (up to continuous deformation fixing the horizontal axis). Combinatorially, meandric systems can be uniquely represented as a pair of non-crossing pair-partitions ; in particular there are Cat_n^2 meandric systems

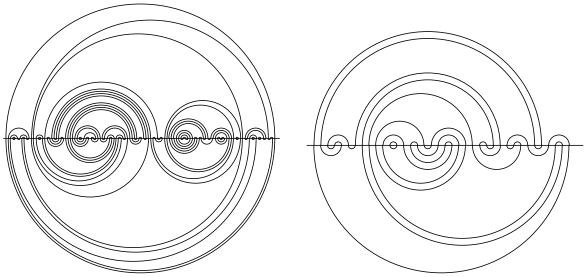


FIGURE 1. Left: a uniform random meandric system of size 60. Right: a uniform random meandric system of size 20, with 4 connected components.

of size n, where Cat_n is the n-th Catalan number. See Fig. 1 for simulations of uniform meandric system of size 60 and 20, respectively.

Main result. In this paper, we discuss the following question: what is the number of connected components $\mathrm{cc}(M_n)$ of a uniform random (unconditioned) meandric system M_n of size n? This question has been raised recently, independently by Kargin [Kar20] and Goulden–Nica–Puder [GNP20]. Both sets of authors prove (through different methods) a linear lower bound for $\mathbb{E}[cc(M_n)]$ and conjecture that the quotient $\mathbb{E}[\mathrm{cc}(M_n)]/n$ converges to a constant. We show here a stronger version of this conjecture, proving the convergence in probability of $cc(M_n)/n$ towards a constant.

Theorem 1. *Let M_n be a uniform random meandric system of size n. Then there exists a constant $\kappa \in (0,1)$ such that $\mathrm{cc}(M_n)/n \to \kappa$ in probability.*

We note that Kargin [Kar20] further conjectures that $\mathrm{cc}(M_n)$ is asymptotically normal with a variance linear in n, but we leave this problem open.

Proof stategy. We start with the following lemma, obtained through a basic double counting argument. If M is a meandric system of size n and i an integer in $\{0,\dots,2n-1\}$, we denote by $|C_i(M)|$ the size of the connected component $C_i(M)$ of M containing i.

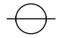
Lemma 1. *Let M be a meandric system of size n and i_n a uniform random integer in $\{0,\dots,2n-1\}$. Then $\frac{\mathrm{cc}(M)}{n} = \mathbb{E}\big[\frac{2}{|C_{i_n}(M)|}\big]$.*

The advantage of this lemma is that $|C_i(M)|$ is a local quantity, in the sense that, if we know the meandric system M in a sufficiently big window around i, one may be able to determine $|C_i(M)|$. Hence to study $\mathbb{E}\big[\frac{2}{|C_{i_n}(M)|}\big]$, we first look for the local limit of a uniform random meandric system M. This turns out to be the so-called *infinite noodle* or *uniform infinite meandric system* (UIMS), studied in the papers [CKST19] and [BGP22]. With a continuity argument, not detailed here, we can prove that $\frac{\mathrm{cc}(M)}{2n}$ converges in distribution to the expected inverse size of the component of 0 in the UIMS.

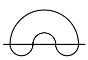
A combinatorial formula for κ. Looking at the "shape" S_0 of the component of 0, one can prove the following formula for κ:

$$\kappa = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{C \in \mathcal{M}_k} p_C$$

where the second sum is taken over meanders (i.e. connected meandric systems) of size k and p_C is an explicit infinite sum of product of Catalan numbers. For example:

- for $C =$  (which is the only meander of size 2), we have

$$p_C = \frac{1}{8} \sum_{\ell=0}^{\infty} \mathrm{Cat}_\ell^2 2^{-4\ell} = \frac{2}{\pi} - \frac{1}{2} \approx 0.137$$

- for $C =$  (which is the only meander of size 4, up to vertical symmetry), we have

$$p_C = \frac{1}{64} \cdot \left(\sum_{\ell_2 \geq 0} \mathrm{Cat}_{\ell_2} 2^{-2\ell_2} \right) \cdot \left(\sum_{\ell_1, \ell_3 \geq 0} \mathrm{Cat}_{\ell_1} \mathrm{Cat}_{\ell_3} \mathrm{Cat}_{\ell_1+\ell_3} 2^{-4\ell_1 - 4\ell_3} \right)$$

$$= \frac{1}{64} \cdot 2 \cdot \left(8 - \frac{64}{3\pi} \right) = \frac{1}{4} - \frac{2}{3\pi} \approx 0.038. \tag{1}$$

It is conjectured (see the report on the open problem sessions) that p_C is always a polynomial in $1/\pi$ with rational coefficients.

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New Littlewood-type identities and the sequence $1, 4, 60, 3328\ldots$

Ilse Fischer

Littlewood's identity reads as

$$\sum_{\lambda} s_\lambda(X_1,\ldots,X_n) = \prod_{i=1}^{n} \frac{1}{1-X_i} \prod_{1\le i<j\le n} \frac{1}{1-X_iX_j}, \tag{1}$$

where $s_\lambda(X_1,\ldots,X_n)$ denotes the Schur polynomial associated with the partition λ and the sum is over all partitions λ. In recent papers [Fis19a, Fis19b, Hön22], where "alternating sign matrix objects" have been connected to certain "plane partition objects", a very similar identity played the crucial role to establish this still mysterious connection. All these proofs are not combinatorial and involve complicated calculations, and so the study of the combinatorics of our Littlewood-type identities (2) and (3) is very likely to lead to a better understanding of the combinatorics of these relations.

In order to formulate the identity, we rewrite (1) using the bialternant formula for the Schur polynomial as follows.

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\sum_{0\le k_1<k_2<\ldots<k_n} X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)} = \prod_{i=1}^{n}\frac{1}{1-X_i}\prod_{1\le i<j\le n}\frac{1}{1-X_iX_j}$$

We have used the following identity for $w=0$ in [Fis19a, Fis19b]. There it is proved by induction with respect to n.

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(1+wX_i+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n} X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)} = \prod_{i=1}^{n}\frac{1}{1-X_i}\prod_{1\le i<j\le n}\frac{1+X_i+X_j+wX_iX_j}{1-X_iX_j} \tag{2}$$

Both sides have combinatorial interpretations, and the one of the LHS is as follows. An *arrowed Gelfand-Tsetlin pattern* is a Gelfand-Tsetlin pattern, where entries may be decorated with elements from $\{\nwarrow, \nearrow, \nwarrow\!\!\!\!\!\nearrow\}$ such that the following is satisfied: *Suppose an entry a is equal to its $\nearrow$-neighbour (resp. $\nwarrow$-neighbour) and a is decorated with either $\nearrow$ or $\nwarrow\!\!\!\!\!\nearrow$ (resp. $\nwarrow$ or $\nwarrow\!\!\!\!\!\nearrow$), then the entry right (resp. left) of a in the same row is also equal to a and decorated with $\nwarrow$ or $\nwarrow\!\!\!\!\!\nearrow$ (resp. $\nearrow$ or $\nwarrow\!\!\!\!\!\nearrow$).*

To associate a sign with these decorated Gelfand-Tsetlin patterns, suppose the pattern contains an entry a that is equal to its $\swarrow$-neighbor b as well as to its $\searrow$-neighbor c, and b is decorated with $\nearrow$ or $\nwarrow\!\!\!\!\!\nearrow$ and c is decorated with $\nwarrow$ or $\nwarrow\!\!\!\!\!\nearrow$. Such a configuration is said to be a *special little triangle*. Then the sign of A is $(-1)^{\#\text{ of special little triangles in } A}$. The weight of an arrowed Gelfand-Tsetlin pattern is

$$\mathbf{sgn}\cdot t^{\#\emptyset}u^{\#\nearrow}v^{\#\nwarrow}w^{\#\nwarrow\!\!\!\!\!\nearrow}\prod_{i=1}^{n} X_i^{i\text{-th rowsum}-(i-1)\text{-st rowsum}+\#\nearrow\text{in row } i\ -\#\nwarrow\text{in row } i}.$$

Then the LHS is essentially the generating function of arrowed Gelfand-Tsetlin patterns with strictly increasing non-negative bottom row. Note the analogy with the ordinary Littlewood identity, where the LHS is the generating function of (ordinary) Gelfand-Tsetlin patterns.

Now the bounded version of (2) reads as

(3)
$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(1+wX_i+X_j+X_iX_j)\sum_{0\le k_1<\ldots<k_n\le m}X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$

(4)
$$=\frac{\det_{1\le i,j\le n}\left(X_i^{j-1}(1+X_i)^{j-1}(1+wX_i)^{n-j}-X_i^{m+2n-j}(1+X_i^{-1})^{j-1}(1+wX_i^{-1})^{n-j}\right)}{\prod_{i=1}^{n}(1-X_i)\prod_{1\le i<j\le n}(1-X_iX_j)(X_j-X_i)}.$$

The LHS is now essentially the generating function of arrowed Gelfand-Tsetlin patterns as above but where the entries are bounded by m. We have a combinatorial interpretation of the RHS in terms of families of lattice paths. In special cases they can nicely be translated into pairs of symplectic tableaux and totally symmetric self-complementary plane partitions.

The sequence

$$1,4,60,3328\ldots=2^{n(n-1)/2}\prod_{j=0}^{n-1}\frac{(4j+2)!}{(n+2j+1)!}$$

appears in recent work of Di Francesco [DF21]. In our work it appears when specializing all $X_i=1$, $w=-1$ and $m=n-1$ in (3). A proof of a generalization of this will be provided in a forthcoming paper in collaboration with Florian Schreier-Aigner.

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Enumeration of rectangulations and corner polyhedra

Éric Fusy

(joint work with Erkan Narmanli, Gilles Schaeffer)

A *rectangulation* is a tiling of a rectangle by rectangles, it is called *generic* if there is no point where 4 rectangles meet. For counting, we consider generic rectangulations with n regions under equivalence relations, in two different forms.

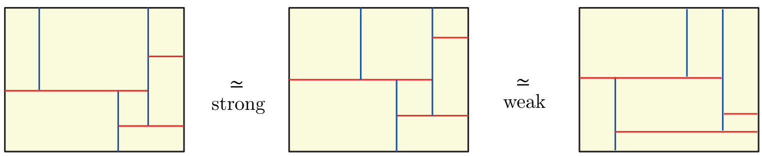

Figure 1. Examples of rectangulations: the left one is strongly equivalent to the middle one, which is weakly equivalent to the right one.

In the strong form of equivalence, we retain the adjacencies between regions (and whether the adjacency is left-to-right or bottom-to-top). This is dually encoded by a triangulation of the 4-gon endowed with a *transversal structure* (inner edges are partitioned into two transversal bipolar orientations, the red one "from bottom to top" and the blue one "from left to right"), as shown in Figure 2 (middle drawing). Note that for each maximal horizontal (or vertical) segment S, the total order (from left to right if S is horizontal, from bottom-to-top if S is vertical) of contacts with other segments matters. In the weak form of equivalence, we only retain the adjacency-system of maximal segments (the contacts on both sides of a maximal segment can be arbitrarily shuffled). This is encoded by a plane bipolar orientation, where vertices correspond to maximal horizontal segments, and faces correspond to maximal vertical segments (considering the outer face is split into a left outer face and a right outer face), as shown in Figure 2 (right drawing).

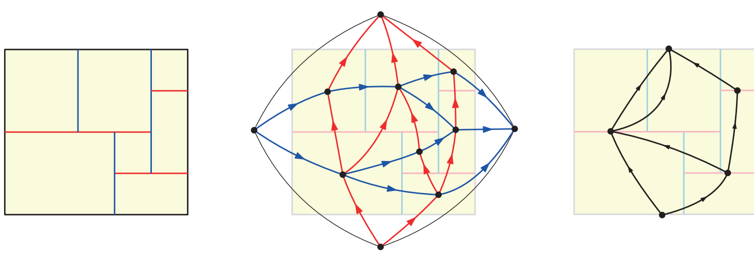

Figure 2. A rectangulation, the transversal structure corresponding to its strong equivalence class, and the plane bipolar orientation corresponding to its weak equivalence class.

Let w_n and s_n denote respectively the number of weak equivalence classes and strong equivalence classes of rectangulations with n regions. The combinatorics for weak equivalence classes is well known. In each weak equivalence class, there is a canonical geometric representation, called the *diagonal representation*, where the outer rectangle is the square $[0,n]^2$, and each region intersects the diagonal $\{x+y=n\}$ at a segment of the form $\{x \in [i,i+1]\}$ (for some $i \in \{0,\ldots,n-1\}$) on that diagonal. Cutting along the diagonal, one obtains a *twin pair* of binary trees, which can be encoded by a non-intersecting triple of walks [3], so that by Gessel-Viennot w_n is the so-called Baxter number

$$w_n = \frac{2}{n(n+1)^2}\sum_{r=0}^{n-1}\binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}.$$

Alternatively, there exist bijective encodings of plane bipolar orientations by non-intersecting triples of walks. A more recent bijection [7], the KMSW bijection, encodes a plane bipolar orientation by a so-called *tandem walk* (walk for the infinite step-set $(1,-1)\cup\{(-i,j),\ i,j\geq 0\}$) staying in the quadrant. This encoding has the advantage that it contains the information on the types of the faces of the bipolar orientation (each inner face of type (i,j), i.e., with left boundary of length $i+1$ and right boundary of length $j+1$, corresponds to a step $(-i,j)$, called a face-step), and therefore it can serve as a black box to obtain bijections for models of decorated planar maps that involve plane bipolar orientations.

For instance, we can address the problem of counting strong equivalence classes of rectangulations (a different approach based on a certain shelling order of regions has been developed in [6]), which can be encoded as the red bipolar orientation of the transversal structure, with a weight $\binom{i+j-2}{i-1}$ for each inner face of type (i,j) to account for the number of ways to triangulate the face by transversal blue edges. In the associated tandem walk, this means that each face-step $(-i,j)$ is weighted by $\binom{i+j-2}{i-1}$.

We can also address a tricolored analog of the problem, as shown in Figure 3: we consider contact systems of curves that are either red, blue, or green, where contact points (except for the 3 external ones) are made of the tips of two curves that meet at an interior point of another curve, on the same side of that curve, the 3 curves being of different colors, and the color order around the contact-point being red,blue,green in clockwise order. Here we can also consider equivalence classes either in a strong form (isotopy), or a weak form where the contacts on both sides of a curve can slide independently.

Weak contact-systems correspond to so-called corner polyhedra (a "topological" version of plane partitions where no more than 3 flats meet at a point) and to certain orientations on Eulerian triangulations [4]. Strong contact-systems correspond to rigid orthogonal surfaces and to 3-connected Schnyder woods [5]. Let w'_n and s'_n denote respectively the number of weak equivalence classes and strong equivalence classes of such contact-systems with $2n$ regions (regions can be considered as pseudo-triangles that point either downward or upward). Then tricolored

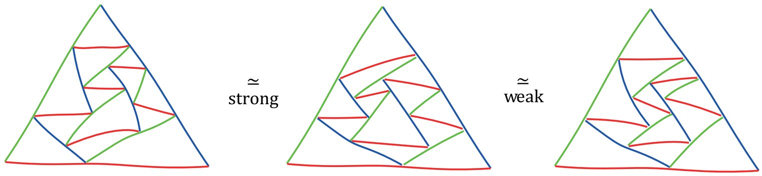

Figure 3. Examples of tricolored contact-systems of curves: the left one is strongly equivalent to the middle one, which is weakly equivalent to the right one.

sequence	w_n	s_n	w'_n	s'_n
γ	8	27/2	9/2	16/3
ξ	1/2	7/8	9/16	22/27

Table 1. Values of the constants appearing in the asymptotic estimates $c\,\gamma^n n^{-1-\pi/\arccos(\xi)}$ of the sequences w_n, s_n, w'_n, s'_n. A rigorous proof of the values of ξ for w'_n, s'_n would require extending the results of [2] to a bimodal setting.

weak equivalence classes can be encoded by certain bipartite plane bipolar orientations for which the associated tandem walks exhibit a bimodal behaviour: they are quadrant tandem walks on the lattice $\{x+y \text{ even}\}$, with the restriction that steps starting from points of even x, y (resp. odd x, y) can not be horizontal (resp. vertical). Strong contact systems have a similar encoding by tandem walks with bimodal conditions, this time with a binomial weight on face-steps (similarly as for the bicolored version).

Based on these quadrant walk encodings, we can obtain recurrences for fast computation of s_n, w'_n, s'_n (a reduction to small step walks can be done in each case to make the computation faster), and via [1, 2] an asymptotic estimate of the form $c\,\gamma^n\, n^{-1-\pi/\arccos(\xi)}$, where remarkably the constants γ, ξ are rational in all the 4 cases w_n, s_n, w'_n, s'_n, as given in Table 1. Using results from [1], the exponent $\alpha = 1 + \pi/\arccos(\xi)$ can be shown to be non-rational for s_n, w'_n, s'_n, so that the generating function is not D-finite (during the meeting Alin Bostan made some computations based on the LLL algorithm suggesting that the exponent α is not algebraic of degree 2 for the sequence s_n, it seems likely that it is transcendental for s_n, w'_n, s'_n), whereas for w_n (where $\alpha = 4$) the generating function is known to be D-finite.

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Counting occurrences of patterns in permutations

Anthony Guttmann
(joint work with Andrew Conway)

1. Introduction

Let π be a permutation on $[n]$ and τ be a permutation on $[k]$. τ is said to occur as a *pattern* in π if for some sub-sequence of π of length k all the elements of the sub-sequence occur in the same relative order as do the elements of π. If the permutation τ does not occur in π, then this is said to be a *pattern-avoiding permutation* or PAP.

If the permutation τ occurs r times, it is said to be an r*-occurrence* of the pattern. Clearly, pattern-avoidance corresponds to the case $r = 0$.

Let $s_n(\tau)$ denote the number of permutations of $[n]$ that avoid the pattern τ. Stanley and Wilf conjectured, and Marcus and Tardos subsequently proved, that for any pattern τ in $[k]$ the limit $\lim_{n\to\infty} s_n(\tau)^{\frac{1}{n}} = \lambda$ exists and is finite. This means that the number of PAPs grows exponentially with n, whereas of course the number of permutations of n grows factorially.

For the more general problem of r-occurrences of a given pattern τ, a similar result holds, and the exponential growth rate λ is independent of r, proved by Mansour, Rastegar and Roitershtein. There are 6 possible permutations of length three, and the number of permutations of length n avoiding any of these 6 patterns is given precisely by $C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n^3}}$, where C_n denotes the n^{th} Catalan number. That is to say, all 6 possible patterns have the same exponential growth-rate as PAPs. Alternatively expressed, there is only one Wilf class for length-3 PAPs.

For length-4 PAPs there are three Wilf classes. Typical representatives of the three classes are 1234, 1342 and 1324. The generating function for the first two classes is known.

These three Wilf classes have generating functions that are D-finite, algebraic, and (almost certainly) non-D-finite respectively.

Here we are studying the more general question of the behaviour of the generating function of r-occurrences of a given pattern in a permutation of length n. The problem of pattern-avoiding permutations thus corresponds to the case $r = 0$.

For this more general problem, it is known that, for patterns of length-3, there are two Wilf classes, one corresponding to the two patterns 123 and 321, and the other corresponding to the remaining four permutations of length 3.

Surprisingly, the r-dependence of the two classes is quite different. Let $\psi_r(n)$ denote the number of permutations of length n containing exactly r occurrences of the nominated pattern. Then for the class 123 and 321 one has

$$\psi_r(n) = \frac{Q_{2r}(n)(2n-r)!}{(n+2r+1)!(n-r)!} \sim \frac{C_r \cdot 4^n}{n\sqrt{n\pi}} \tag{1}$$

where $Q_{2r}(n)$ is a polynomial with integer coefficients of degree $2r$. For small values of r, the amplitude coefficient C_r appears to increase exponentially, growing seemingly like λ^r, where the growth constant $\lambda \approx 2.67$. That is to say, the asymptotics remain unchanged, and only the amplitude, or premultiplying constant changing with r. However this exponential increase in the amplitude cannot continue indefinitely, and indeed must decline toward zero as r becomes large, as we explain below.

For the second class, corresponding to patterns 132, 231, 213, 312, Mansour and Vainshtein show that

$$\psi_r(n) = \frac{Q_r(n)(2n-3r)!}{n!r!(n-r-2)!} \sim \frac{4^{n-3r/2} \cdot n^{r-3/2}}{r!\sqrt{\pi}} \tag{2}$$

where $Q_r(n)$ is a polynomial of degree $3(r-1)$, whose leading-order term is precisely $n^{3(r-1)}$. Here the asymptotics are quite different. The amplitude changes in a very regular way, but the subdominant power-law exponent increases by 1 as r increases by 1.

Another noteworthy property of these r-occurrences is that, while the counting sequences $\psi_r(n)$ are known or conjectured to be Stieltjes moment sequences for the case $r = 0$, as discussed in Bostan, Elvey Price, Guttmann and Maillard, this is not the case for $r > 0$, for all the cases we have studied. We discuss this further below.

Aspects of this problem have been previously studied by several authors. For the class 132, Noonan and Zeilberger conjectured the result for $r = 1$, subsequently proved by Bóna. Bóna also proved that the number of r-occurrences of the pattern 132 is P-recursive in the size. Equivalently, the ordinary generating function is D-finite. He then proved the stronger statement that the generating function is algebraic.

Mansour and Vainshtein proved the corresponding result for $r = 2$ and then gave conjectural results for $r \leq 6$, and conjectured the structure of the general formula.

For the increasing subsequence 123, Noonan proved the result for $r = 1$, and Noonan and Zeilberger conjectured the result for $r = 2$. This was subsequently proved by Fulmek, who also gave conjectured results for $r = 3$ and $r = 4$. These were subsequently proved by Callan. Nakamura and Zeilberger developed a Maple package implementing a functional equation that readily generated terms for the cases $r \leq 7$, and gave the corresponding expressions for $\psi_r(n)$ for $r \leq 7$.

In 2007 Bóna proved that, for the monotone pattern $1234 \cdots k$, the distribution function of r-occurrences is asymptotically normal. Indeed, this is true for *any* classical pattern, a result first claimed to be proved by Bóna in a later paper[1]. It was unequivocally proved by Janson et al. in 2013, and can also be proved, perhaps even more easily, by the methods developed by Hofer in 2018.

As far as we are aware, the general situation for patterns of length 4 has not previously been studied, due in large part to the difficulty of generating coefficients of the underlying generating functions, though there have been some series generated by Zeilberger, Nakamura and collaborators. We have developed an algorithm for this purpose, and find that of the $4! = 24$ permutations of length-4, there are now seven effective Wilf classes. They are as follows:

$$\begin{aligned}
I: &\quad 1234, 4321\\
II: &\quad 1243, 2134, 3421, 4312,\\
III: &\quad 1432, 2341, 3214, 4123\\
IV: &\quad 2143, 3412,\\
V: &\quad 1324, 4231\\
VI: &\quad 1342, 1423, 2314, 2431, 3124, 3241, 4132, 4213\\
VII: &\quad 2413, 3142
\end{aligned}$$

By *effective Wilf class* in this context we mean that the number of r-occurrences of any pattern in the class in a permutation of length n is the same, irrespective of the value of r. For pattern-avoiding permutations, the first four classes above correspond to a single Wilf class, so these will all have coefficient growth 9^n. The fifth entry, with coefficient growth μ^n, where $\mu \approx 11.598$ (due to Conway,Guttmann and Zinn-Justin) corresponds to another Wilf class, and the third Wilf class for PAPs comprises the patterns in class VI and VII above, with coefficient growth 8^n.

2. Generating the number of occurrences of a given pattern

In 2011 Minato came up with an innovative and general algorithm for counting pattern avoiding permutations. It involved a straightforward algorithm operating upon sets of permutations that generates all permutations containing the pattern, and then crucially an efficient computational representation of said sets taking up space and time much smaller than the number of elements in the set. This makes the algorithm much more efficient in practice than other known general algorithms that examine each pattern avoiding permutation individually. In 2017 Inoue improved the set representation to make the algorithm even more efficient in practice.

[1] Miklós Bóna tells me that this arXiv was never proceeded to publication, as he found an error in the proof.

We use the same algorithm operating upon sets, except apply it to multisets instead of sets. That is, each element in the set has a multiplicity. The union of two multisets in our context sums the multiplicities of each element. A consequence of the construction algorithm is that the resulting set contains each permutation containing the pattern, with a multiplicity equal to the number of occurrences of the given pattern. It is then an efficient operation on the multiset to obtain the number of elements with each multiplicity. This is the desired enumeration.

We generalise the back end set representation of Inoue to handle multisets. This actually is a potentially generally useful multiset representation for combinatorics. In practice, it ends up being reasonably efficient. As a more complex structure containing more information than the simple set it takes somewhat more time and memory than the simple set, but is still vastly more efficient than techniques that consider elements individually.

3. Counting occurrences of 132, 231,312,213

Let $\psi_r(n)$ denote the number of permutations of length n containing exactly r occurrences of the nominated pattern.
Let $\Psi_r(x)$ be the ordinary generating function for $\psi_r(n)$.
The generating function was shown by Bóna to behave as

$$\Psi_r(x) = \frac{1}{2}\left(P_1(x) + P_2(x)(1-4x)^{-r+1/2}\right). \tag{3}$$

Results are given for $r \le 5$ by Mansour and Vainshtein. They find quite generally,

$$\psi_r(n) = \frac{Q_r(n)(2n-3r)!}{n!r!(n-r-2)!} \sim \frac{4^{n-3r/2}\cdot n^{r-3/2}}{r!\sqrt{\pi}}$$

where $Q_r(n)$ is a polynomial of degree $3(r-1)$ whose leading-order term is precisely $n^{3(r-1)}$.

From the results of Mansour and Vainshtein it appears that the polynomials in eqn. (3), $P_1(x)$ and $P_2(x)$, are polynomials with integer coefficients, of degree r and $2r+1$ respectively.

4. Counting occurrences of 123 and 321

Let $C(x) = \frac{1}{2}(1-\sqrt{1-4x})$ be the generating function for Catalan numbers. Then it is well known that

$$\psi_0(n) = \frac{(2n)!}{(n+1)!n!} \sim \frac{4^n}{n\sqrt{n\pi}}$$

$$\Psi_0(x) = \frac{1}{2x}(1-\sqrt{1-4x}).$$

With 1 occurrence of the pattern, Noonan proved that

$$\psi_1(n) = \frac{3}{n}\binom{2n}{n-3} = \frac{6(2n-1)!}{(n+3)!(n-3)!} = \frac{6(n^2-3n+2)(2n-1)!}{(n+3)!(n-1)!} \sim \frac{3\cdot 4^n}{n\sqrt{n\pi}},$$

From this we derive

$$\Psi_1(x) = \frac{1}{2x^3}\left((1-6x+9x^2-2x^3) - (1-4x+3x^2)\sqrt{1-4x}\right).$$

The general situation seems to be

$$\psi_r(n) = \frac{Q_{2r}(n)(2n-r)!}{(n+2r+1)!(n-r)!} \sim \frac{C_r \cdot 4^n}{n\sqrt{n\pi}}$$

where $Q_{2r}(n)$ is a polynomial with integer coefficients of degree $2r$. The amplitude coefficient C_r appears to increase exponentially, growing seemingly like λ^r, where the growth constant $\lambda \approx 2.67$. That is to say, the asymptotics remain unchanged, and only the amplitude, or premultiplying constant changing with r. However this exponential increase in the amplitude cannot continue indefinitely, as the histogram plotting $\psi_r(n)$ against r is asymptotically normally distributed. So given that, asymptotically, only the amplitude changes as r changes, it must first increase and then decrease, reflecting the heights of the various histogram entries.

The generating function is conjectured to behave as

$$\Psi_r(x) = \frac{1}{2x^{2r+1}} \left(P_1(x) - P_2(x)\sqrt{1-4x} \right),$$

where $P_1(x)$ and $P_2(x)$ are polynomials with integer coefficients whose degree depends on the parity of r. If r is even, both polynomials are of degree $5r/2$. If r is odd, $P_1(x)$ is of degree $(5r+1)/2$, and $P_2(x)$ is of degree $(5r-1)/2$.

The conjectured form of the generating function has also been given previously by Fulmek in 2002, though without comment on the degree of the polynomials.

5. Length 4 patterns

In this section we investigate some properties of length-4 sequences. We have been able to generate data for permutations up to size 14, for all values of r, but to go further requires greater computing resources than we have. The principal limitation is memory. We had 2TB at our disposal, but even that is insufficient to go beyond $n = 14$. As a consequence, we have been unable to conjecture any exact results for $\psi_r(n)$ for $n > 0$ for any pattern, though we have been able to conjecture quite a lot about the asymptotics.

For classes I and II we find that $\psi_r(n) = \frac{C_r \cdot 9^n}{n^4}$ for all r, just as is the case for the shorter pattern 123. That is to say, as r changes, only the amplitude C_r changes. The exponential growth remains (provably) the same, and the sub-dominant power law term is also (conjecturally) unchanged. The pre-multiplicative amplitude is also of course just conjectured, rather than proved.

For classes III and IV we find

$$\psi_r(n) \sim \frac{C_r \cdot 9^n}{n^{4-r}},$$

where C_0 is known, C_1 is conjectured, and C_2 is estimated.

For class V we find the behaviour of the r-occurrences is qualitatively similar to that of 132-avoiders, in that the power-law exponent apparently increases by 1 as r increases by 1. Accordingly, we conjecture that

$$\psi_r(n) \sim C_r \cdot n^r \cdot \psi_0(n).$$

For class VI and VII we find the behaviour of the r-occurrences is qualitatively similar to that of 132-avoiders, in that the power-law exponent apparently increases by 1 as r increases by 1. Accordingly, we conjecture that

$$\psi_r(n) \sim C_r \cdot n^r \cdot \psi_0(n)$$

for both classes.

6. Stieltjes moment sequences

Bostan, Elvey Price, Guttmann and Maillard showed that for all known PAP generating functions, the coefficients were Stieltjes moment sequences (SMS), and for Av(1324) the Hankel determinants were monotonically increasing with the determinant size, leading them to confidently conjecture that it too was a SMS. However for r-occurrences of patterns, for all patterns and for all $r > 0$ we have investigated, none are SMSs.

Blitvic and Steingrímmson suggested the following: For a given pattern of length m, form the generating function $G(q)$, with coefficients

$$g_n = \sum_{k=0}^{\binom{n}{m}} q^k \psi_k(n).$$

Then ask for what values of q, which we call q_c, do these coefficients form a Stieltjes moment sequence? Clearly it is true for $q = 0$, as that corresponds to the the observation that the ogf for PAPs is an SMS. For $q > 0$ the Hankel determinants, observationally, become larger as q becomes larger. So this only leaves $q < 0$ to explore.

For patterns of length $m = 3$ the answer appears to be for $q_c = 0$. That is, the Hankel determinants are negative for $q < 0$. For patterns of length $m = 4$ the answer appears to be $q_c < 0$, and is possibly pattern dependent.

This work is ongoing.

Weighted walks in the quadrant-invariants, decoupling and differential algebraicity

Charlotte Hardouin
(joint work with Michael F. Singer)

The enumeration of planar lattice walks with small steps confined to the north-east quadrant has attracted a considerable amount of interest over the past fifteen years. For the lattice $\mathbb{Z}^2$, a lattice path model is comprised of a finite set $\mathcal{D}$ of lattice vectors called the *step set*. The combinatorial question boils down to the count $q_{i,j}(n)$ of n-step walks, i.e., of polygonal chains, that remain in the first quadrant, start from the origin, end at (i, j) and consists of n oriented line segments whose associated translation vectors belong to $\mathcal{D} \subset \{-1, 0, 1\}^2$. Many algebraic and analytic properties of the combinatorial sequence $(q_{i,j}(n))$ of a lattice walk are embodied in the algebraic nature of the associated generating function $Q(x,y,t) = \sum_{i,j,n\geq 0} q_{i,j}(n)x^iy^jt^n$.

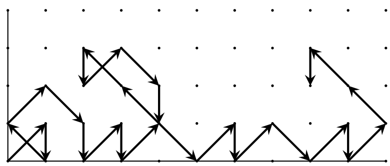


FIGURE 1. An example of walk with step set $\mathcal{D} = \{NW, NE, SE, S\}$.

Of the $2^8 - 1$ possible choices of step sets it is shown in [BMM10] that taking symetries into account and eliminating trivial sets, one need only consider 79 of these models. Doing this, one only eliminates models whose generating series are algebraic. Inspired by the work of [FIM17] for stationary distribution of random walks, Bousquet-Mélou and Mishna attached to any model of walk with small steps in the quadrant, an algebraic curve defined over $\mathbb{Q}(t)$ of genus zero or one and a group of automorphisms of the curve called the *group of the walk*. Bousquet-Mélou and Mishna conjectured that the generating series of the model is D-finite, that is, satisfies a linear differential equation in x, y if and only if the group of the walk is finite. Of these 79 models, 23 have D-finite (in all variables) generating series ([BMM10, BvHK10]) of which 4 are algebraic. The remaining 56 models were shown to have non-D-finite generating series with respect to various variables in [KR12, MR09, MM14, BRS14]. In [BBMR17], the authors give new uniform proofs of the 4 algebraic cases and also show that 9 of the 56 non-D-finite models in fact have differentially algebraic [1] generating functions by constructing explicit algebraic differential equations. For the remaining 47 models, the generating series is differentially transcendental in the variables x, y by [DHRS18, DHRS19] and in t by [DH19]. These last results whose proofs rely on the Galois theory of difference equations, achieve the classification of the algebraic nature of the generating series with respect to the variables x and y.

Unfortunately, the aforementioned Galoisian criteria for differential transcendence lack of combinatorial interpretation whereas Bernardi, Bousquet-Mélou and Raschel discovered, by a case by case analysis, that the nine differentially algebraic models shared a more algebraic property, that they called *decoupling*. The consideration of weighted models shed a new light on the relation of the notion of *decoupled model* and the differential algebraicity of the series. Weighted models with small steps are planar lattice walks whose steps have been endowed with rational weights, denoted here $(d_{i,j})_{i,j\in\{-1,0,1\}^2}$, whose sum equals one. When the model is unweighted, that is, when the non-zero weights are all equal, one can assume, up to a rescaling, that they are all equal to one which corresponds to the initial counting problem. Fixing a transcendental real value of the parameter t in $\mathbb{C}$, one

[1] that is, they satisfy a polynomial differential equation in each of the variables x, y, t.

can associate to any weighted model an algebraic curve E_t, which is the Zariski closure in $\mathbf{P}^1 \times \mathbf{P}^1$ of the set of zeros of $K(x,y,t) = xy(1 - t\sum_{(i,j)\in\mathcal{D}} d_{i,j}x^iy^j)$, the kernel polynomial attached to the weighted model. When the polynomial $K(x,y,t)$ is irreducible, the curve E_t is of genus zero or one. Analogously to the unweighted case, one can define the group of the walk as a certain subgroup of automorphism of the curve E_t. In [DHRS19], we proved that, in the genus zero case, the group of the walk is always infinite and that the generating series is differentially transcendental in x, y. The article [DH19] allows to conclude to the differential transcendence in t. When the curve is of genus one, the situation is more subtle and the group of the walk might be finite or not.

Indeed, the finitness of the group of the walk is equivalent to the existence of invariants, that is, rational fractions $f(x) \in \mathbb{Q}(t,x), g(y) \in \mathbb{Q}(t,y)$ not in $\mathbb{Q}(t)$ such that

$$0 = f(x) + g(y) + R(x,y)K(x,y,t),$$

where $R(x,y) \in \mathbb{Q}(t)(x,y)$ is a rational fraction whose denominator is not divisible by $K(x,y,t)$ (see [BBMR17, Theorem 7]). When the weights of the model are fixed, the Sage package combwalks developed in [BCJPL20] allows to test the finitness of the group. If the group of the walk is finite then the generating series is always D-finite in x and y by [DR19, Theorem 42]. It is algebraic over $\mathbb{C}(x,y)$ if and only if the model is decoupled, that is, there exists $F(x) \in \mathbb{Q}(t,x), G(y) \in \mathbb{Q}(t,y)$ such that

$$xy = F(x) + G(y) + R(x,y)K(x,y,t),$$

where $R(x,y) \in \mathbb{Q}(t)(x,y)$ is a rational fraction whose denominator is not divisible by $K(x,y,t)$. For a finite group, Theorem 7 in [BBMR17] gives an algorithmic criteria based on orbit sums computations to test the existence of the decoupling and produce explicitely $F(x)$ and $G(y)$ for any decoupled model.

In the case of a genus one weighted model with an infinite group of the walk, the approach developed in [HS21] allows to relate the Galoisian criteria of [DHRS18] and the notion of decoupled model of [BBMR17] as follows.

Theorem(Theorem 3.8 and Proposition 3.9 in [HS21]) For a genus one walk with infinite group of the walk, the following statement are equivalent

- the generating series $Q(x,y,t)$ is differentially algebraic in x and y
- the weighted model is decoupled.

In addition, [HS21] produces an algorithm which associates to any set of directions the necessary and sufficient conditions on the weights that correspond to the differential algebraicity of the generating series. This algorithm is based on the theory of Mordell-Weil lattices as developed by Shioda in [Shi90] and on a height computation of certain sections in the elliptic fibration attached to the pencil $(E_t)_{t\in\mathbf{P}^1}$ of elliptic curves. This algorithm gives in particular the necessary and sufficient conditions for the differential algebraicity of the weighted model attached to the nine unweighted differentially algebraic models studied in [BBMR17] as follows.

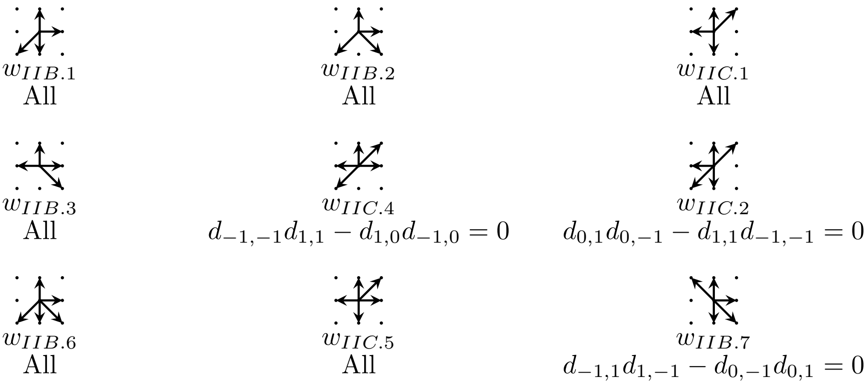
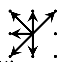


Figure 2. Nine weighted models which are differentially algebraic when unweighted

The above conditions are automatically met when the weights are all equal, that is, in the unweighted case. On the contrary, Figure 3 describes a weighted model whose weight conditions for differentially algebraicity are not met when the weights are equal to one. Thus, the generating series of the corresponding unweighted model is differentially transcendental but, for a convenient choice of the weights on the step set, this series might become differentially algebraic.



$w_{IIB.6}$

$$d_{-1,1}d_{0,1}^2 - d_{0,1}d_{-1,-1}d_{0,-1} + d_{1,1}d_{-1,-1}^2 = 0$$

Figure 3. A weighted model which is differentially transcendental when unweighted

The aforementioned examples show that there is no straightforward relation between the structure of the set of directions of a model and the algebraic relations among its weights characterizing the differential algebraicity of the generating series. These algebraic relations among weights are in fact connected to the relative position of the base points of the pencil of curves $(E_t)_{t\in\mathbf{P}^1}$.

The notion of invariants and decoupling are essentially related to the biquadratic polynomial $K(x, y, t)$. Therefore, these notion might be useful in the more general situation of a linear equation with two catalytic variables and a biquadratic kernel.

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Proofs of Borwein Conjectures

Christian Krattenthaler
(joint work with Chen Wang)

In 1993, at a workshop at Cornell University, George Andrews delivered a two-part lecture on *"AXIOM and the Borwein Conjecture"*. He presented — first of all — three conjectures that had been communicated to him by Peter Borwein (the first of which became known as "the Borwein Conjecture"), and then reported the lines of attack that he had tried, all of which had failed to give a proof, stressing (quoting from [1], which contains Andrews' findings in printed form) that *"this is the sort of intriguing simply stated problem that devotees of the theory of partitions* *love."* Indeed, the statement of the first conjecture, dubbed the "First Borwein Conjecture" in [1], is the following.

Conjecture 1 (P. Borwein). *For all positive integers n, the sign pattern of the coefficients in the expansion of the polynomial $P_n(q)$ defined by*

$$P_n(q) := (1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1}) \tag{1}$$

is $+--+--+--\cdots$, with a coefficient 0 being considered as both $+$ and $-$.

Many people tried to prove this conjecture (including the speaker), in the first place using combinatorial arguments or q-series identities. The only "progress" consisted in further conjectures generalising the Borwein Conjecture, due to Bressoud [4] and to Ismail, Kim and Stanton [5, Conj. 1 in Sec. 7], and partial results about Bressoud's conjecture by Berkovich and Warnaar [2, 3, 8, 9]. However, none of these attempts came anything close to progress concerning the original Borwein Conjecture, Conjecture 1. It took almost 30 years until Chen Wang succeeded in proving this conjecture in [6], using saddle point approximation techniques.

As I already indicated, "the" Borwein Conjecture (Conjecture 1) is actually the first of *three* conjectures made by Peter Borwein. The *Second Borwein Conjecture* from [1] predicts the same sign behaviour of the coefficients for the square of the "Borwein polynomial".

Conjecture 2 (P. Borwein). *For all positive integers n, the sign pattern of the coefficients in the expansion of the polynomial $P_n^2(q)$, where $P_n(q)$ is defined by* (1)*, is $+--+--+--\cdots$, with the same convention concerning zero coefficients.*

The *Third Borwein Conjecture* from [1] is an assertion on the sign behaviour of the coefficients of a polynomial similar to $P_n(q)$, where however the involved modulus is 5 instead of 3.

There is even more. Chen Wang observed recently that a cubic version of the conjecture also appears to hold, which both Borwein and Andrews missed.

Conjecture 3 (C. Wang). *For all positive integers n, the sign pattern of the coefficients in the expansion of the polynomial $P_n^3(q)$, where $P_n(q)$ is defined by* (1)*, is $+--+--+--\cdots$, with the same convention concerning zero coefficients as before.*

This raises an obvious question: *Is Wang's proof just an isolated instance, or can similar ideas also lead to proofs of the other conjectures?*

From the outset, it seemed that Wang's proof cannot be adapted in any way to attack these other conjectures since his proof was crucially based on a formula of Andrews from [1] which has no analogues in these other cases.

In the meantime, however, we realised that, instead of relying on Andrews' sum representations for the decomposition polynomials, we should apply saddle point approximations directly to $P_n(q)$ and its powers. When doing this, surprisingly the quantities that have to be approximated are very similar to those that were at stake in [6]. There is a price to pay though: while in [6] the (dominant) saddle points were located on the real axis, with this new approach we have to deal with (dominant) saddle points located at complex points. This makes the estimations that have to be performed more delicate. On the positive side, it allows one to proceed in a more streamlined fashion. In fact, it allows us to provide a *uniform* proof of the First *and* Second Borwein Conjecture (Conjectures 1 and 2), *as well as* a proof of "two thirds" of Wang's "Cubic Borwein Conjecture" (Conjecture 3), and altogether this is not longer than the proof of "just" the First Borwein Conjecture in [6].

All this is written up in [7]. Moreover, based on computer experiments, we came up with further conjectures of the "Borwein type", namely with similar conjectures for the modulus 4 (instead of 3 as in Conjectures 1–3) and for the modulus 7, see again [7]. It is clear that the proposed approach also allows an attack on these new conjectures, and we are convinced that it will also be able to provide a proof of "three fifth" of the Third Borwein Conjecture (pertaining to modulus 5). Further refinements will be necessary to come up with full proofs of the Third Borwein Conjecture and of Wang's Cubic Borwein Conjecture in Conjecture 3.

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Hurwitz numbers and the RSK algorithm (and random partitions)

Baptiste Louf

(joint work with Guillaume Chapuy, Harriet Walsh)

We start with a very simple enumeration problem, let

$$H_{n,\ell} := \#\{\tau_1, \tau_2, \ldots, \tau_\ell \text{ transpositions of } \mathfrak{S}_n | \tau_1 \cdot \tau_2 \cdot \ldots \cdot \tau_\ell = 1\} \tag{1}$$

(note that if ℓ is odd, then $H_{n,\ell} = 0$).

Simply put, $H_{n,\ell}$ counts factorizations of the identity into transpositions. This type of enumeration problem finds its roots in geometry: $H_{n,\ell}$ is the simplest example of *Hurwitz numbers* (see e.g. [4]). It enumerates ramified covers of the sphere by a (non-necessarily connected) surface, with only simple ramifications. Alternatively, $H_{n,\ell}$ also counts *walks in the Cayley graph* of $\mathfrak{S}_{\mathfrak{n}}$.

It turns out that there exists a closed formula for $H_{n,\ell}$:

$$H_{n,\ell} = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 C_\lambda^\ell \tag{2}$$

where the sum spans over all partitions λ of n, with f_λ being the number of *standard Young tableaux* of λ (or alternatively the dimensions of the irreducible representation of $\mathfrak{S}_n$ associated to λ), and C_λ is the *content sum* of λ (we refer to [7] for definitions).

The proof of this identity uses the *Frobenius formula* (a classical representation theoretic formula to enumerate factorizations in groups). However, since all the quantities in (2) are both combinatorially defined and positive, it is natural to ask for a combinatorial proof.

Open problem 1. *Prove* (2) *bijectively.*

Since $H_{n,0} = 1$, the case $\ell = 0$ boils down to showing bijectively that

$$\sum_{\lambda \vdash n} f_\lambda^2 = n! \tag{3}$$

which is exactly the *Robinson-Schensted-Knuth algorithm* (RSK algorithm in short, see e.g. [7]). One can then hope to solve Problem 1 by finding a generalization of the RSK algorithm that would take as input a permutation and a factorization of the identity.

Besides the interest of finding a natural generalization of such an important bijection, solving problem 1 would help us better understand random partitions under the *Plancherel-Hurwitz measure*: given n, ℓ, sample a random $\lambda \vdash n$ with probability $\mathbb{P}_{n,\ell}(\lambda) = \frac{f_\lambda^2 C_\lambda^\ell}{n! H_{n,\ell}}$ (taking $\ell = 0$, one recovers the usual *Plancherel measure*, see e.g. [6]). In [2], we studied the typical shape of a partition under $\mathbb{P}_{n,\ell}$ as $n, \ell \to \infty$ with $\frac{\ell}{n} \to \theta \in (0, \infty)$.

Theorem 1. *Let $\lambda = \lambda_{n,\ell}$ be sampled under the Plancherel measure $\mathbb{P}_{n,\ell}$. For $n, \ell \to \infty$ with $\frac{\ell}{n} \to \theta \in (0, \infty)$ (and conditionally on $C_\lambda \geq 0$), we have*

$$\lambda_1 \left(\frac{2\ell}{\log n} \right)^{-1},$$

and the limit shape of $\lambda \setminus \lambda_1$ is the classical 'VKLS' limit shape [8, 5]

We now wish to understand the behaviour of partitions under the Plancherel-Hurwitz measure in other regimes for n, ℓ. In particular, this demands a good asymptotic approximation of the quantity $H_{n,\ell}$, a bijection explaining (2) might make it easier.

Finally, it is also an important question to obtain asymptotic enumeration for **connected** Hurwitz numbers (that is, with a definition similar to the geometric definition of $H_{n,\ell}$, but requiring the surface above to be connected), both as the genus (of the surface above) and the size (here, the degree of the cover) go to infinity, as it was done for triangulations of surfaces in [1]. One way to achieve this goal would be to obtain precise enough asymptotics for unconnected numbers $H_{n,\ell}$ in all regimes of n, ℓ.

We finish by mentioning that a first step towards a bijective proof might be a combinatorial understanding of the quantity C_λ^2 (what does it count on a partition?), and that the case $\ell = 2$ has a proof with a *signed-bijection* in an article of Fang [3].

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Enumeration of vertices of generalized permutahedra and Pitman-Stanley polytopes

Alejandro Morales

(joint work with Laura Colmenarejo, Greta Panova, Jacob Matherne, and Jesse Selover)

The goal of this talk is to give two results about the generating function of the number of vertices of two different families of polytopes where the transfer-matrix method is used. Generalized permutahedra is an important family of polytopes defined by Postnikov with very interesting face structure and volume formulas. The combinatorics of their faces were further studied by Postnikov, Reiner and Williams. We study a generalized permutahedra that is the Minkowski sum of certain simplices motivated from a problem in machine learning. We give generating functions and asymptotics for the number of vertices (arxiv:2209.14978). The Pitman-Stanley polytope is a well known polytope motivated from probability and with connections to Dyck paths and parking functions. Its lattice points correspond to plane partitions with entries at most 1 and has the same face structure as a hypercube. We study the face structure of a generalized Pitman-Stanley polytope whose lattice points correspond to plane partitions with entries at most m. We realize this polytope as a flow polytope and give generating functions and polynomial formulas with interesting positivity properties for the number of vertices (preprint will be out soon). Linear relations and Lorentzian property of chromatic symmetric functions

Chromatic symmetric functions (CSF) are well-studied symmetric functions in algebraic combinatorics that generalize the chromatic polynomial and are related to Hessenberg varieties and LLT polynomials. Motivated by the Stanley–Stembridge conjecture of e-positivity of CSFs of indifference graphs of Dyck paths, we show an identity of Guay-Paquet relating them to q-hit numbers of Ferrers' diagrams that implies e-positivity for "abelian" Dyck paths. We also show that the allowable coloring weights for indifference graphs of Dyck paths are the lattice points of a permutahedron. Furthermore, we conjecture that such CSFs are Lorentzian, a property by Brändén and Huh as a bridge between discrete convex analysis and concavity properties in combinatorics, and we prove this conjecture for abelian Dyck paths. The first part is joint work with Colmenarejo and Panova and the second part is joint work with Matherne and Selover.

Applications of the minor-summation formula to combinatorics and representation theory

Soichi Okada

The minor-summation formula of Ishikawa and Wakayama [5] expresses a weighted summation of maximal minors of a given $n \times N$ matrix $T = (t_{i,j})_{1 \le i \le n, 1 \le j \le N}$ ($n \le N$) in terms of a single Pfaffian in the case where the weights are given by the subPfaffians of an $N \times N$ skew-symmetric matrix $A = (a_{i,j})_{1 \le i,j \le N}$:

$$\sum_I \mathrm{Pf} A(I) \cdot \det T([n]; I) = \mathrm{Pf}\left(T A\, {}^t T\right),$$

where n is even, $I = \{i_1, \dots, i_n\}$ ($i_1 < \cdots < i_n$) runs over all n-element subsets of $[N] = \{1, 2, \dots, N\}$ and

$$A(I) = \left(a_{i_p, i_q}\right)_{1 \le p,q \le n}, \quad T([n]; I) = \left(t_{p, i_q}\right)_{1 \le p,q \le n}.$$

In this talk we present the following three applications of this formula.

1. The first application is a proof of Hopkins' conjecture on the enumeration of shifted plane partitions. By using the minor-summation formula, we can prove a character identity involving intermediate symplectic characters, and then obtain the following theorem by specializing all the variables to 1. Let $\delta_r = (r, r-1, \dots, 2, 1)$ be the staircase partition.

Theorem 1. *(Hopkins' conjecture, Hopkins-Lai* [3]*, Okada* [8]*) The number of shifted plane partitions of double staircase shape*

$$\delta_n + \delta_k = (n+k, n+k-2, \dots, n-k+2, n-k, n-k-1, \dots, 2, 1)$$

with entries bounded by m is given by

$$\prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j}.$$

Hopkins and Lai [3] proves this theorem by appealing to the "free boundary" Kuo condensation due to Ciucu for lozenge tilings.

2. The second application is a derivation of identities for classical group characters of nearly rectangular shape. (See [7] for rectangular-shaped characters.) As an example, here we give the branching rule from the odd orthogonal Lie algebra $\mathfrak{so}_{2n+1}$ to the general linear Lie algebra $\mathfrak{gl}_n$. Let $\mathrm{so}^B_\lambda(x_1, \dots, x_n)$ denote the irreducible character of $\mathfrak{so}_{2n+1}$ with highest weight λ. Then we apply the minor summation formula to obtain a decomposition formula for $\mathrm{so}^B_{(r^{n+1})}(x_1, \dots, x_n, u)$ in terms of Schur functions $s_\lambda(x_1, \dots, x_n)$. From this decomposition formula we can derive the following branching rule.

Theorem 2. *(Krattenthaler* [6]*, Okada* [9]*) Let r be a nonnegative integer or half-integer and p a nonnegative integer such that $p \le r$. Then we have*

$$\mathrm{so}^B_{(r^{n-1}, r-p)}(x_1, \dots, x_n) = (x_1 \cdots x_n)^{-r} \sum_{\lambda \subset ((2r)^n)} c_{\lambda,p}\, s_\lambda(x_1, \dots, x_n),$$

where λ runs over all partitions whose diagrams fit inside the $n \times (2r)$ rectangle, and the multiplicity $c_{\lambda,p}$ is equal to the number of partitions μ of length $\le n+1$ satisfying

- *μ/λ is a horizontal strip of size $2r-p$;*
- *$\mu_1 = 2r$ and $\mu_1 + 2\mu_2 + \cdots + 2\mu_{i-1} + \mu_i \ge 2\lambda_1 + \cdots + 2\lambda_{i-1}$ for $2 \le i \le n+1$.*

Note that Krattenthaler [6] used a tableaux combinatorics to prove this theorem and gave a different combinatorial description for $c_{\lambda,p}$.

3. The third application is an affine analogue of the Gordon–Bender–Knuth identities. This application is a joint work with JiSun Huh, Jang Soo Kim and Christian Krattenthaler. Let $\mathrm{Par}(m, w)$ be the set of all partitions λ satisfying $l(\lambda) \le m$ and $\lambda_1 - \lambda_m \le w$, and $s^{[m,w]}_\lambda(\boldsymbol{x})$ the cylindric Schur function associated to such a partition λ. We denote by $e_k(\boldsymbol{x})$ the kth elementary symmetric function.

Theorem 3. *(Huh–Kim–Krattenthaler–Okada* [4]*) For positive integers h and w, we have*

$$\sum_{\lambda \in \mathrm{Par}(2h+1, w)} s^{[w, 2h+1]}_{{}^t\lambda}(\boldsymbol{x}) \\ = \sum_{k \ge 0} e_k(\boldsymbol{x}) \cdot \det\left(F_{-i+j, 2h+1+w}(\boldsymbol{x}) - F_{i+j, 2h+1+w}(\boldsymbol{x})\right)_{1 \le i,j \le h},$$

and

$$\sum_{\lambda\in\mathrm{Par}(2h,w)} s^{[w,2h]}_{{}^t\lambda}(\boldsymbol{x}) = \det\left(\overline{F}_{-i+j,2h+w}(\boldsymbol{x}) + \overline{F}_{i+j-1,2h+w}(\boldsymbol{x})\right)_{1\le i,j\le h},$$

where ${}^t\lambda$ *is the conjugate partition of* λ *and*

$$F_{r,N}(\boldsymbol{x}) = \sum_{k\in\mathbb{Z}} f_{r+kN}(\boldsymbol{x}),\ \ \overline{F}_{r,N}(\boldsymbol{x}) = \sum_{k\in\mathbb{Z}}(-1)^k f_{r+kN}(\boldsymbol{x}),\ \ f_r = \sum_{k\in\mathbb{Z}} e_k(\boldsymbol{x})e_{k+r}(\boldsymbol{x}).$$

In the $w\to\infty$ limit, we recover the dual version of the Gordon–Bender–Knuth identities ([2], [1]).

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Correlation inequalities for the numbers of Young tableaux

Igor Pak

The numbers of standard Young tableaux of straight and skew shapes are both classic and extremely well-studied. Yet, they satisfy a number of inequalities, some of which are more mysterious than others. In this talk I intend to survey known results and make sense of them all. I give no proofs, but explain where the inequalities come from: the combinatorial atlas technology [1, 3], the Ahlswede–Daykin inequality [4], the direct injective proofs [2], etc.

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Poset inequalities

Greta C. Panova
(joint work with Swee Hong Chan and Igor Pak)

Partially ordered sets (posets) are central objects in enumerative combinatorics and graph theory, as well as in Computer Science. A linear extension is just a bijection $P \to \{1, \ldots, |P|\}$ which respects the partial order and a main example, from algebraic combinatorics, are the Standard Young Tableaux. The flagship problem in the area is the 50+ year old $\frac{1}{3} - \frac{2}{3}$ conjecture of Kislitsyn, Fredman and Linial, which states that in every not totally ordered finite poset there exist two elements x and y, such that the probability x occurs before y in a total order is within $[\frac{1}{3}, \frac{2}{3}]$. The conjecture is related to optimal sorting algorithms for partially ordered data.

While the above conjecture remains widely open in general, it has prompted the creation and understanding of sorting probabilities and various poset inequalities to understand the behavior of restricted linear extensions. The "simplest" of these inequalities is Stanley's inequality. Fix an element $x \in P$ and let $N(k) = |\{L : P \to [n], \text{ s.t. } L(x) = k\}|$. Then $N(k)^2 \geq N(k-1)N(k+1)$ – the sequence $N(k)$ is log-concave. Despite the simple combinatorial formulation, Stanley's proof uses the powerful but nontransparent Alexandrov-Fenchel inequalities for log-concavity of mixed volumes. No elementary injective combinatorial proof is known. A more general inequality, developed for the $\frac{1}{3} - \frac{2}{3}$ conjecture, is the Kahn-Saks inequality. Fix $x, y, z \in P$ and set $F(k) = |\{L : P \to [n], \text{ s.t. } L(y) - L(x) = k\}|$. Then $F(k)^2 \geq F(k-1)F(k+1)$, which is again only proven using the Alexandrov-Fenchel inequalities. And the most general, the Cross-product conjecture, states that $F(i,j)F(i+1,j+1) \leq F(i,j+1)F(i+1,j)$, where $F(i,j) = |\{L : P \to [n], \text{ s.t. } L(y) - L(x) = i, L(z) - L(y) = j\}|$. This conjecture can no longer be approached via Alexandrov-Fenchel.

In [1] we proved the Cross-product conjecture for posets of width two using two methods – a recursive, linear algebraic approach, and a combinatorial – interpretation of linear extensions as lattice paths in a region, and proving the inequality via an explicit injection of these lattice paths. In [4] we used these ideas to give injective proofs of Stanley and Kahn-Saks inequality, which also enabled us to understand explicitly the nontrivial equality cases. The injective proofs on lattice paths and ideas generalize to a big class of random walks in monotone regions. In [3] we show that the probability $p(k)$ a random walk exits the right (vertical) boundary at height k is actually log-concave, $p(k)^2 \geq p(k+1)p(k-1)$. This result holds for regions of arbitrary (monotonous) boundaries, and thus could not be derived through explicit computation.

Inspired by such inequalities, in [2] we prove via injections various new inequalities on the order polynomial $\Omega(P, t)$ of the poset P, that is the function which counts the number of order preserving maps $P \to [1, \ldots, t]$, alternatively the integer points in the order polytope associated to P. We also define a group action on restricted linear extensions and use it to prove some simple criteria for the existence of such linear extensions, as well as uniqueness. This group action leads to a possible approach for an injective proof of Stanley's log-concavity inequality. With this approach we rediscovered conjectures on the monotonocity and log-concavity of normalized order polynomials. The most notable is the, originally stated by Kahn and Saks for integer values of t, monoticity conjecture, namely that $\Omega_P(t)/t^{|P|}$ is a decreasing function of t for $t > 1$.

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Combinatorial Exploration: An algorithmic framework for enumeration

Jay Pantone

(joint work with Michael H. Albert, Christian Bean, Anders Claesson, Émile Nadeau, Henning Ulfarsson)

Since 2016, the Combinatorial Exploration project has sought to develop a rigorous, algorithmic framework to discover combinatorial specifications—from which one can obtain counting formulas, generating functions, random sampling routines, and more—for sets of combinatorial objects. While this work was originally focused specifically on permutation classes, it broadened into a domain-agnostic approach that can be effectively applied to many different kinds of combinatorial objects. We have recently released the first article of a series about Combinatorial Exploration [1].

This talk will give a broad summary of Combinatorial Exploration and describe how it's being applied to several types of combinatorial objects. We will also introduce our new website, the Permutation Pattern Avoidance Library (PermPAL) [2], available at https://permpal.com, which provides a reference database for our enumerative results.

In the next section of this abstract, we give an abbreviated[1] list of permutation classes for which Combinatorial Exploration is able to rigorously compute a combinatorial specification. In the section that follows that, we show heatmaps derived by sampling uniformly at random from each of the 55 non-finite 2×4 classes.

[1] A more complete list, including citations to the work mentioned here, can be found in Section 2.4 of [1].

Successes of Combinatorial Exploration in Permutation Patterns.

- We can find specifications automatically for six out of the seven symmetry classes of permutations avoiding one pattern of length 4, all but Av(1324). These include the first *direct* enumerations of Av(1342) and Av(2413), as previous enumerations were via bijections to each other and to other objects. The final class, Av(1324), currently remains out of reach, but we are optimistic that several not-yet-implemented strategies may lead to progress.
- We can find specifications for all 56 symmetry classes of permutations avoiding two patterns of length 4. 53 have specifications that allow us to derive their algebraic generating functions. The remaining three are conjectured to be non-D-finite, and for these we can derive polynomial-time counting algorithms.
- Out of the 317 symmetry classes of permutations avoiding three patterns of length 4, again we can find specifications for all of them. One is conjectured to be non-D-finite; for the remaining 316 we find algebraic generating functions.
- Similarly, we can find specifications and generating functions for all symmetry classes avoiding n patterns of length 4 for $4 \leq n \leq 24$. We have not yet done a comprehensive search for specifications for classes avoiding only length 5 patterns, although we have found specifications for around 200 of these avoiding between one and forty patterns.
- Bevan, Brignall, Elvey Price, and Pantone found improved lower and upper bounds on the exponential growth rate of Av(1324) by considering a set of gridded permutations that they called "domino permutations". The enumeration of these was challenging, requiring a bijection to a type of arch systems and several pages of work to enumerate these arch systems. We can find a specification and the algebraic generating function for the domino permutations.
- Defant recently studied the preimage of various permutation classes under the West-stack-sorting operation, derived that the preimage of Av(321) is Av$(34251, 35241, 45231)$, and gave rough bounds on its exponential growth rate, but is unable to enumerate it. We find a specification that permits us to compute 636 terms in the counting sequence. We are unable to conjecture the generating function from these terms, and thus we predict that it is non-D-finite. We estimate that the growth rate is $6 + 2\sqrt{5}$.
- Bóna and Pantone used Combinatorial Exploration to assist with the study of five classes avoiding four patterns of length 5, and one class avoiding five patterns of length 6.
- Egge conjectured that a group of permutation classes defined by avoiding two patterns of length 4 and one of length 6 are all counted by the Schröder numbers. Burstein and Pantone proved one of these conjectures, and then Bloom and Burstein proved the remainder. We are able to find specifications and generating functions for all of these classes.

- Guo and Kitaev explore the notion of "partially ordered permutations". We are able to find specifications for many of the classes they consider.
- Alland and Richmond recently showed that for a permutation π, the Schubert variety X_π has a complete parabolic bundle structure if and only if $\pi \in \mathrm{Av}(3412, 52341, 635241)$. We are able to find a specification with a property that guarantees that this class has an algebraic generating functions, but the system is too large for us to solve. We can, however, compute the first 400 terms of the counting sequence and conjecture a value for the generating function; it appears to be algebraic with a minimal polynomial of order 6.

Heatmaps for Permutation Classes. When Combinatorial Exploration finds a specification for a permutation class, it typically allows us to sample permutations from that class uniformly at random. For each of the 55 non-finite 2×4 classes (up to symmetry), we have sampled one million permutations of length 300 and drawn their plots on top each other to form a *heatmap*. Darker areas of the heatmap indicate that many of the sampled permutations have entries in this location, while lighter areas indicate that few do.

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Chromatic symmetric functions and sign-reversing involutions

Bruce Sagan

(joint work with Zachary Hamaker and Vincent Vatter)

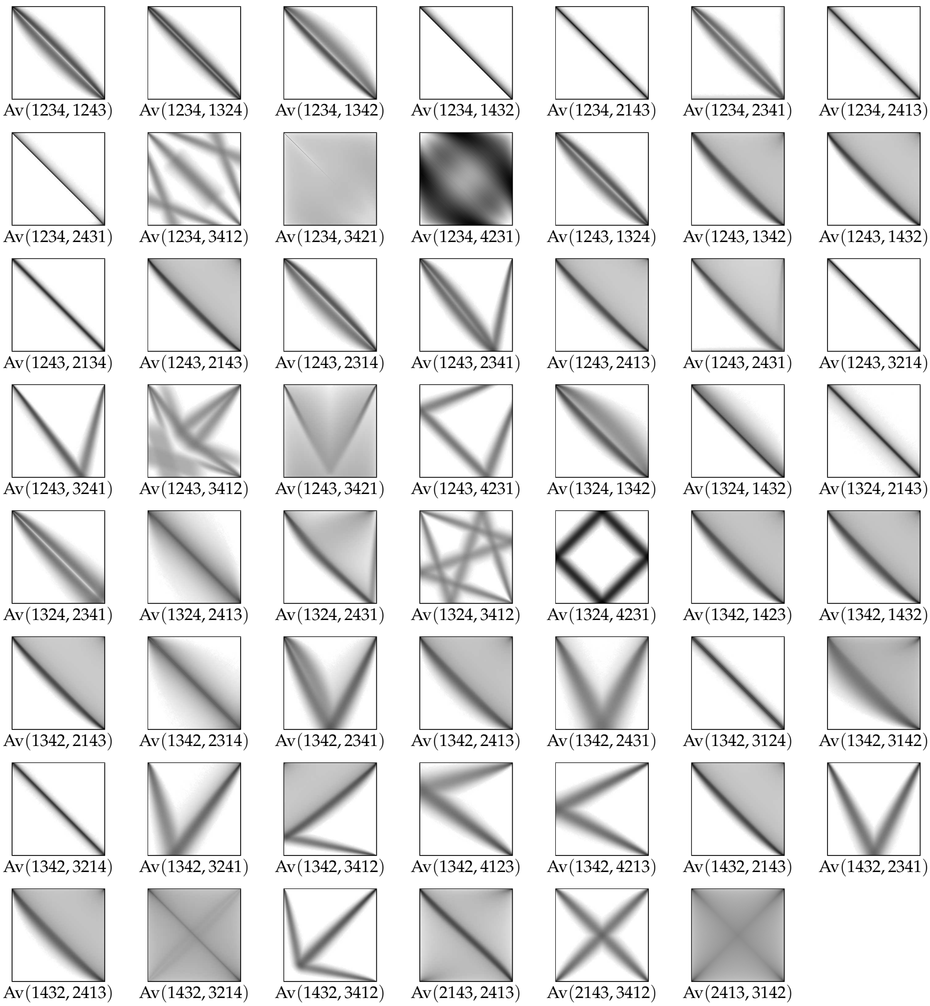
Let $G = (V, E)$ be a finite graph with vertices V and edges E. Given a set S, a function $\kappa : V \to S$ is a *proper coloring* if, for any edge $uv \in E$, we have

$$\kappa(u) \neq \kappa(v).$$

The study of proper colorings has a long history with one of the most famous results being the Four Color Theorem of Appel and Haken [1, 2].

Stanley associated a symmetric function with proper colorings [6]. Let $\mathbb{P}$ denote the positive integers and consider a set of variables $\mathbf{x} = \{x_1, x_2, x_3, \ldots\}$ indexed by $\mathbb{P}$. Associate with any proper coloring $\kappa : V \to \mathbb{P}$ a monomial

$$\mathbf{x}^\kappa = \prod_{v \in V} x_{\kappa(v)}.$$



The *chromatic symmetric function* of G is the generating function

$$X(G) = X(G;\mathbf{x}) = \sum_{\text{proper } \kappa:V\to\mathbb{P}} \mathbf{x}^{\kappa}.$$

It is clear that $X(G;\mathbf{x})$ is a symmetric function since permuting the colors of a proper coloring results in a proper coloring. So one can ask about the expansion of $X(G;\mathbf{x})$ in the various standard bases for the algebra $\Lambda(\mathbf{x})$ of symmetric functions. Bases for $\Lambda(\mathbf{x})$ are indexed by integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Let $\{e_\lambda\}$ and $\{s_\lambda\}$ denote the elementary and Schur bases, respectively. Given a basis $\{b_\lambda\}$ and $f(\mathbf{x}) \in \Lambda(\mathbf{x})$ we say that $f(\mathbf{x})$ is *b-positive* if the nonzero coefficients in the expansion $f(\mathbf{x}) = \sum_\lambda c_\lambda b_\lambda$ are all positive.

One particular question about the positivity concept which has received considerable attention is the following. Let P be a finite poset (partially ordered set) with partial order $\leq_P$. The *incomparability graph* of P is the graph $i(P)$ whose vertex set is P and with an edge uv if u and v are incomparable in P. We say that P is *(*3+1*)-free* if it has no induced subposet isomorphic to 3+1 which is the disjoint union of a three-element chain and a one-element chain. The following conjecture was made by Stanley and Stembridge in their work on Jacobi-Trudi immanants.

Conjecture 1 ([7]). *Let P be a (*3+1*)-free poset. They $X(i(P))$ is e-positive.*

The purpose of this talk is to present an approach to this conjecture which we believe has been overlooked by the combinatorial community. In fact, it has only been used twice before in papers of Cho and Huh [4] and Cho and Hong [3] where they prove the conjecture for posets of height 2 and 3, respectively. (The *height* of a poset is the number of elements in a longest chain.) This method consists of three steps.

(1) Express $X(i(P))$ in the Schur basis s_λ using the P-tableaux of Gasharov [5].
(2) Expand the s_λ in terms of the elementary basis using dual Jacobi-Trudi determinants.
(3) Cancel the signs introduced in the second step using a sign-reversing involution.

Most of the work on the (3+1)-free Conjecture has concentrated on looking at a family of posets P and then proving that $X(i(P))$ is e-positive for each of these posets. The method outlined in the previous paragraph permits one to approach the conjecture from a new perspective: fix a partition λ and then show that the coefficient of e_λ is nonnegative in all (3+1)-free posets P. We have been able to do this for the Young diagram of λ being a row, or having at most two columns. As a bonus, we can give new interpretations to these coefficients in terms of P-tableaux.

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Absolute Root Separation

Bruno Salvy

(joint work with Yann Bugeaud, Andrej Dujella, Wenjie Fang, Tomislav Pejković)

The absolute separation of a polynomial is the minimum nonzero difference between the absolute values of its roots:

$$\operatorname{abs\,sep}(P) := \min_{\substack{P(\alpha)=P(\beta)=0,\\ |\alpha|\neq|\beta|}} \big||\alpha| - |\beta|\big|.$$

Having good lower bounds on this quantity for polynomials with integer coefficients is of interest in the asymptotic analysis of linear recurrent sequences. We improve the known bounds for this problem and related ones and report on extensive experiments in low degree.

If $\mathrm{H}(P)$ denotes the *height* of a polynomial, i.e., the maximum of the absolute values of its coefficients, our bounds are as follows.

Theorem 1. *Let $P \in \mathbb{Z}[X]$ be a polynomial of degree d and let α and β be two of its roots such that $|\alpha| \neq |\beta|$, then*

(1) *if α and β are real, then $\big||\alpha| - |\beta|\big| \gg \mathrm{H}(P)^{-(d-1)}$;*
(2) *if α is real and β is not, then $\big||\alpha| - |\beta|\big| \gg \mathrm{H}(P)^{-2(d-1)(d-2)}$;*
(3) *if neither of them is real, then $\big||\alpha| - |\beta|\big| \gg \mathrm{H}(P)^{-(d-1)(d-2)(d-3)/2}$.*

In this result, the constant implicit in the $\gg$ sign depends only on the degree d.

Only the first of these bounds has been proved to be optimal. The proof of this theorem is based on constructing auxiliary polynomials with integer coefficients of controlled height whose roots contain the desired difference. The bounds are obtained by combining Cauchy's lower bound on the nonzero roots of a polynomial and an effective version of the fundamental theorem of symmetric functions.

Through extensive experiments, we obtain the following bounds in the other direction.

Theorem 2. *For each $d \in \{3, 4, 5, 6\}$, there exists a sequence $(P_{d,M})$ of polynomials of degree d in $\mathbb{Z}[X]$, such that as $M \to \infty$, the polynomial $P_{d,M}$ has two roots α_M, β_M with $|\alpha_M| \neq |\beta_M|$ and*

$$\big||\alpha_M| - |\beta_M|\big| \ll \mathrm{H}(P_{d,M})^{-d-1}, \quad M \to \infty.$$

In particular, when $d = 3$, this bound matches the lower bound from the previous theorem so that in that case the bound 4 is tight. This case is actually nontrivial, the polynomial family we found being given by the following.

Proposition 1. *The family of cubic polynomials*

$$P_n(X) = p_n(3X^3 - 2X^2 + 4X - 6) + 6q_n(X^3 - X^2 + 1) \in \mathbb{Z}[X],$$

where $(p_n/q_n)_n$ is the sequence of convergents of the continued fraction expansion of $\sqrt{3}$, has the property that

$$\operatorname{abs\,sep}(P_n) \ll \mathrm{H}(P_n)^{-4}, \quad n \to \infty.$$

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Pipe dream perspectives and algebraic aspects of plane partitions and alternating sign matrices

Jessica Striker

(joint work with Daoji Huang and Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, and Joshua P. Swanson)

Plane partitions are three-dimensional analogues of ordinary partitions; plane partitions in an $a \times b \times c$ box are enumerated by MacMahon's product formula $\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$. In a 1986 [11], Stanley considered symmetry operations on plane partitions, namely, reflection (transpose), rotation, and complementation. This yielded 10 symmetry classes of plane partitions consisting of plane partitions invariant under combinations of these operations. The plane partitions invariant under all three operations are called *totally symmetric self-complementary* (*TSSCPP*). As in the case of all plane partitions, each symmetry class has a nice enumeration. TSSCPP inside a $2n \times 2n \times 2n$ box was shown in 1994 by Andrews [1] to be counted by $\prod_{j=0}^{n-1}\frac{(3j+1)!}{(n+j)!}$. This was, at the time, the conjectured [10] number of $n \times n$ *alternating sign matrices* (*ASM*); square matrices with entries in $\{0, 1, -1\}$ such that the rows and columns each sum to 1 and the nonzero entries alternate in sign along each row or column. The 1996 proofs of this conjecture [13, 7] sparked a search for a natural, explicit bijection between TSSCPP and ASM. Partial bijections have been found on small subsets, including the permutation case [12], the case of two monotone triangle diagonals [5, 4], and the 312-avoiding case [2].

This work, joint with Daoji Huang, interprets TSSCPP as pipe dreams to extend the bijection of [12] to a larger subset than any previous partial bijection. Pipe dreams give combinatorial formulas for Schubert polynomials $\mathfrak{S}_\pi$, $\pi \in S_n$, which are important in the study of Schubert calculus [3]. Bumpless pipe dreams are other objects shown more recently [9] to also give a combinatorial expansion of Schubert polynomials and are in natural bijection with alternating sign matrices. Thus, a key component of our partial ASM-TSSCPP bijection is the recent bijection of [6] between reduced pipe dreams and reduced bumpless pipe dreams.

Theorem 1. *Let $\pi \in S_n$. Let $TSSCPP^{red}(\pi)$ denote the set of TSSCPP whose associated pipe dream is reduced and has permutation π, and let $ASM^{red}(\pi)$ denote the set of ASM whose associated bumpless pipe dream is reduced and has permutation π. There is an explicit weight-preserving injection φ from $TSSCPP^{red}(\pi)$ to $ASM^{red}(\pi)$. If π avoids 1432, then φ is a bijection.*

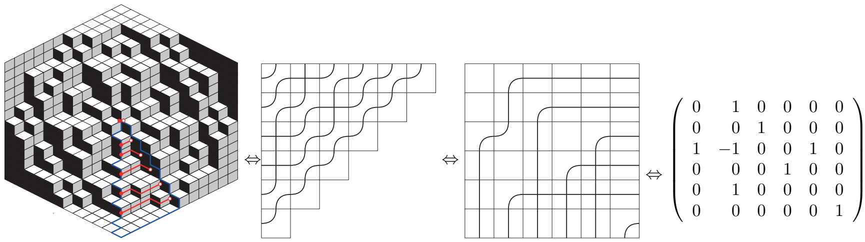

FIGURE 1. An example of the bijection of Theorem 1. From left to right, the objects are: TSSCPP, pipe dream, bumpless pipe dream, ASM.

The second project discussed in this talk involves a surprising algebraic manifestation of alternating sign matrices and plane partitions and is joint with Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, and Joshua P. Swanson.

The irreducible representations of the symmetric group S_n are the *Specht modules* S^λ indexed by integer partitions λ. For the case of 3-row rectangles, Kuperberg [8] famously introduced a diagrammatic "web" basis of the Specht module $S^{3\times b}$ (and more generally for other spaces of invariant tensors). Kuperberg's web basis has many important applications to quantum link invariants, cluster algebras, and algebraic geometry.

We introduce *hourglass plabic graphs* and prove that equivalence classes of these graphs index an *SL_4-web basis*, a structure that has been sought since Kuperberg's introduction of the SL_3-web basis in 1996.

Theorem 2. *Equivalence classes of hourglass plabic graphs with $n = 4d$ boundary vertices of one color give a web basis for the Specht module $S^{(d,d,d,d)}$.*

A lovely feature of this basis, which is also true in the 2- and 3- row case, is that promotion on $4 \times d$ standard Young tableaux corresponds to rotation of hourglass plabic graph equivalence classes. These equivalence classes decompose as the disjoint union of regions corresponding to alternating sign matrices and plane partitions.

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Grass(mannian) trees and forests: Variations of the exponential formula, with applications to the momentum amplituhedron

Lauren Williams
(joint work with Robert Moerman)

The Exponential Formula allows one to enumerate any class of combinatorial objects built by choosing a set of connected components and placing a structure on each connected component which depends only on its size. There are multiple variants of this result, including Speicher's result for noncrossing partitions, as well as analogues of the Exponential Formula for series-reduced planar trees and forests. In this paper we use these formulae to give generating functions for *contracted Grassmannian trees* and *forests*, certain graphs whose vertices are decorated with a *helicity*. Along the way we enumerate bipartite planar trees and forests, and we apply our results to enumerate various families of permutations: for example, bipartite planar trees are in bijection with separable permutations.

It is postulated by Livia Ferro, Tomasz Łukowski and Robert Moerman (2020) that contracted Grassmannian forests are in bijection with boundary strata of the momentum amplituhedron, an object encoding the tree-level S-matrix of maximally supersymmetric Yang-Mills theory. With this assumption, our results give a rank generating function for the boundary strata of the momentum amplituhedron, and imply that the Euler characteristic of the momentum amplituhedron is 1.

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Sets that Support a Joint Distribution

PETER WINKLER
(joint work with Chris Coscia and Martin Tassy)

Given a closed set $S \subseteq \mathbb{R}^2$ and probability distributions μ and ν on the real line, when are there random variables distributed as μ and ν the support of whose joint distribution is precisely S?

In the discrete case we characterize such sets; in the continuous case, we provide a partial characterization. The discrete case is equivalent to asking when there is a full, nowhere-zero flow in a given (possibly infinite) vertex-weighted bipartite graph; we also consider the case where edge-capacities are present. The continuous case amounts to asking what sets can support a permuton, that is, a probability distribution on the unit square with uniform marginals.

We address the fundamental question of what can happen when two random variables with given real distributions interact. This issue has of course arisen many times, and re-inspired among the authors by recent interest in limit objects for permutations, known as *permutons*.

A permuton is a Borel probability measure on the unit square with uniform marginals; the set of all permutons is the closure of the set $\bigcup_{n=1}^{\infty} S_n$ where S_n is the set of permutations of $\{1, 2, \ldots, n\}$, under a topology given by pattern densities. Each permuton γ determines a probability measure γ_n on S_n for each n, by sampling n times from γ and comparing the orders of the x- and y-coordinates of the sampled points.

Permutons were introduced in [2] and enjoy a powerful variational principle (see [5, 4, 3, 1]) that facilitates understanding of large sets of permutations, such as those that share a particular pattern density. In particular, if there are $n!e^{-c}$ permutations of order n in such a class then the entropy of the class is the non-positive number $-c$, obtainable as the maximum value of a certain integral over the permutons that represent that class.

We ask: given a closed set in $[0, 1]$, when is it the support of a permuton? For example, Fig. 1 shows a variety of sets; which of them can support a permuton?

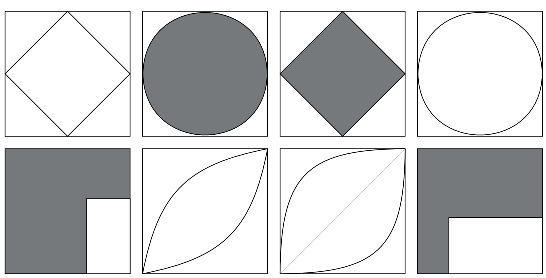

FIGURE 1. Which of these eight subsets of the unit square support a permuton?

The question of support for joint discrete distributions is equally interesting (and easier, but still tricky in the infinite case). We address this in the next section.

1. The Discrete Case

Let X and Y be the (finite or countably infinite) discrete supports of μ and ν. We assume X and Y are disjoint, thus we can unambiguously use absolute value signs to denote both μ-measure on X and ν-measure on Y. A joint measure γ supported by $S \subseteq X \times Y$ with marginals μ and ν can be thought of as a flow on a vertex-weighted bipartite graph $G = \langle X \cup Y, S\rangle$ in which each $x \in X$ is a source of volume $|x|$ and each $y \in Y$ a sink of volume $|y|$, and each edge in S has positive flow. We say that such a flow on G is *full* and *nowhere-zero*.

It is well known (and easily demonstrated, e.g. via max-flow–min-cut) that a full flow exists if the following "Hall condition" is satisfied:

A1 For every $U \subset X$, $|N(U)| \geq |U|$

where $N(U)$ denotes the neighborhood of U in Y, that is, $N(U) = \{y \in Y : (x, y) \in S \text{ for some } x \in U\}$.

To get a full, nowhere-zero flow, we need a second condition:

A2 For every $U \subset X$, if $|N(U)| = |U|$ then $|N(N(U))| = |U|$.

The necessity of **A2** is clear, since if $|N(U)| = |U|$ then all the flow into $N(U)$ must come from U, so any edge leading from $N(U)$ to $X \setminus U$ will be left flowless. Sufficiency is easy in the finite case, and holds in general:

Theorem 1. *A set $S \subseteq X \times Y$ is the support of a joint distribution of given discrete distributions iff conditions* **A1** *and* **A2** *are both satisfied.*

Note that the conditions **A1** and **A2**, despite appearances, are symmetric in X and Y.

The flow interpretation suggests a generalization of Theorem 1 in which the edges have capacities. Let a bipartite, edge-weighted graph $G = \langle X \cup Y, S\rangle$ be given with weights that sum to 1 on each bipart, and suppose that in addition a positive real capacity $c(x, y)$ is associated with each edge $(x, y) \in S$. Given $U \subset X$ and $y \in N(U)$, we must now be concerned with $\sum_{x\in U} c(x, y)$ as well as $|y|$; let $|y|_U$ denote the minimum of $|y|$ and $\sum_{x\in U} c(x, y)$, and $|N(U)|_U$ the sum of $|y|_U$ over $y \in N(U)$. Then **A1** and **A2** become:

B1 For every $U \subset X$, $|N(U)|_U \geq |U|$

and

B2 For every $U \subset X$, if $|N(U)|_U = |U|$ then for every $y \in N(U)$ with a neighbor outside U, $|y|_U < |y|$.

We then have

Theorem 2. *The countable, vertex-weighted, edge-capacitated, bipartite graph G has a full, nowhere-zero flow that respects its edge capacities iff conditions* **B1** *and* **B2** *are both satisfied.*

2. The Continuous Case

If both distributions μ and ν are continuous (with respect to Lebesgue measure) on $\mathbb{R}$, and random variables $\mathbf{X}$ and $\mathbf{Y}$ (respectively) respect these distributions, then the support of their joint distribution—that is, the intersection of all closed sets of measure 1—is a closed set in $\mathbb{R}^2$. By transforming each of μ and ν to Lebesgue measure on the unit interval [0,1], we lose no generality in assuming both are already uniform on [0,1] and therefore their support is a closed set in the unit square.

We reformulate the abovementioned conditions as follows. Here, U is a measurable subset of the unit interval (thought of as the horizontal axis of $[0,1]^2$), and its neighborhood $N(U)$ is now $\{y \in [0,1] : (x,y) \in S \text{ for some } x \in U\}$. Absolute value signs now denote Lebesgue measure.

> **C1** For every measurable $U \subset X$, $|N(U)| \geq |U|$;

and

> **C2** For every measurable $U \subset X$, if $|N(U)| = |U|$ then $|N(N(U))| = |U|$.

We then have

Theorem 3. *A closed set $S \subseteq [0,1]^2$* contains *the support of a permuton iff it satisfies condition* **C1**.

Theorem 4. *If a closed set $S \subseteq [0,1]^2$* is *the support of a permuton, it satisfies conditions* **C1** *and* **C2**.

What of the permutons in Fig. 1? The four subsets on the left side of the figure support a permuton, the rest do not.

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Algebraic solutions of continuous and discrete differential equations

Sergey Yurkevich

(joint work with Alin Bostan, Jacques-Arthur Weil, and Hadrien Notarantonio)

A function $f(x)$ is called *algebraic* if there exists a non-zero polynomial $P(x,y) \in \mathbb{Q}[x,y]$ such that $P(x, f(x)) = 0$. Here x can be a single variable, or a vector of unknowns: $x = (x_1, \ldots, x_n)$. Justifying algebraicity of solutions of functional equations is often a difficult task which frequently appears in combinatorics, number theory, physics, and other disciplines. In my talk in Oberwolfach I focused on the one hand on linear ordinary differential equations

$$p_n(t)y^{(n)}(t) + \cdots + p_0(t)y(t) = 0, \qquad (\mathbf{ODE})$$

where $p_0(t), \dots p_n(t) \in \mathbb{Q}[t]$, and on the other hand on (systems of) discrete differential equations in one catalytic variable:

$$F(t,u) = f(u) + t \cdot Q(F(t,u), \Delta_a F(t,u), \dots, \Delta_a^k F(t,u), t, u), \qquad (\mathbf{DDE})$$

where $f(t) \in \mathbb{Q}[t], Q \in \mathbb{Q}[x_1, \dots, x_{k+2}, t, u]$, and Δ_a^ℓ denotes the ℓth iteration of the discrete differential operator $\Delta_a : \mathbb{Q}[u][[t]] \to \mathbb{Q}[u][[t]]$ defined by

$$\Delta_a F(t,u) := \frac{F(t,u) - F(t,a)}{u-a}.$$

A power series $f(t) \in \mathbb{Q}[[t]]$ that is a solution of a (non-trivial) linear ODE is called D-finite. Given the differential equation (**ODE**), it is often useful to study the assiciated differential operator

$$p_n(t)\partial^n + \cdots + p_0(t) \in \mathbb{Q}[t]\langle\partial\rangle,$$

where $\partial = \frac{\mathrm{d}}{\mathrm{d}t}$. Thanks to the work [5] by Michael Singer from 1979, proving that a linear differential operator L has an algebraic solution is decidable in theory. Moreover, given a D-finite function $f(t)$ (by an annihilating operator L and initial conditions) is possible to compute a differential operator L_f^{alg} which is the right factor of L whose solution space is spanned by all algebraic solutions of L. This shows that proving that a D-finite function is algebraic is a decidable problem.

However, in practice this question can still be very challenging. In this talk I show, based on the example of Dubrovin-Yang-Zagier numbers, how it can be attacked using methods from differential Galois theory, arithmetic conjectures and numerical calculations. The sequence $(c_n)_{n\geq 0}$ defined by

$$\begin{aligned}
&80352000n(5n-1)(5n-2)(5n-4)c_n + \\
25\left(2592000n^4 - 16588800n^3 + 39118320n^2 - 39189168n + 14092603\right) &c_{n-1} + \\
&20\left(4500n^2 - 18900n + 19739\right)c_{n-2} + c_{n-3} = 0,
\end{aligned}$$

with initial conditions $c_0 = 1, c_1 = -161/(2^{10}3^5)$ and $c_2 = 26605753/(2^{23}3^{12}5^2)$ is mentioned by Don Zagier in [7, p. 769] and corresponds to the case $r = 5$ of the one-point r-spin intersection numbers $\langle\tau_{s,m}\rangle$ introduced by Witten. Boris Dubrovin, Di Yang and Don Zagier noticed and proved [2] the surprising fact that $a_n, b_n \in \mathbb{Z}[1/30]$, where

$$a_n = (3/5)_n(4/5)_n \cdot c_n \quad \text{and} \quad b_n = (2/5)_n(9/10)_n \cdot c_n.$$

The generating functions $f_a(t), f_b(t)$ of a_n, b_n are D-finite and, while $f_b(t)$ was proven to algebraic by Zagier, it was initially unclear [7, p. 769] whether $f_a(t)$ is algebraic or transcendental. In the joint work with Alin Bostan and Jacques-Arthur Weil we found 7 other sequences of same type and showed by a uniform approach that both $f_a(t)$ and $f_b(t)$ are algebraic functions. We use the theory of differential invariants introduced by Liouville in order to show that the differential Lie algebra associated to the corresponding differential operator vanishes.

In the case of discrete differential equations in one catalytic variable the situation is very different. These equations (or systems of such) frequently arise in combinatorial problems. For example

$$F(t,u) = 1 + tuF(t,u) + t\frac{F(t,u) - F(t,0)}{u}$$

describes the generating function $F(t,u)$ of walks in $\mathbb{N}^2$ which have n steps in $\{\nearrow, \searrow\}$ and end at level (height) k. Hence, the unique solution $F(t,0) = \frac{1-\sqrt{1-4t}}{2t}$ is the generating function of the Catalan numbers.

Due to a theorem of Dorin Popescu [4] from 1986 the unique solution of the equation (**DDE**) is always algebraic. In 2006 Mireille Bousquet-Mélou and Arnaud Jehanne [1] provided an elementary and effective proof of this result. In my talk I presented a recent effective proof of algebraicity of solutions of systems of DDEs in one catalytic variable, i.e. solutions of equations of the form:

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \ldots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \ldots, \nabla^k F_n, t, u). \end{cases}$$

This part is based on joint work with Hadrien Notarantonio [3].

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Problem Session

Collected by Nathan Williams
(joint work with Andrew Elvey Price, Jay Pantone, Bruce Sagan,
Theo Douvropoulos, Philippe Di Francesco)

1. Andrew Elvey Price: 3/4 plane walks from $(2,0)$ to $(-1,0)$

Definition 1. *A* small step *in $\mathbb{Z}^2$ is a step of the form*

$$\{(1,0),(-1,0),(0,1),(0,-1),(1,1),(1,-1),(-1,1),(-1,-1)\}.$$

Fix a multiset $\widehat{S}$ of small steps, and consider all walks $W(\widehat{S})$ from $(2,0)$ to $(-1,0)$ using the collection of steps in $\widehat{S}$ that avoid the negative x and y axes, except at the final point of the path (drawn in red in Figure 1). Let $X(\widehat{S})$ be the subset of walks in $W(\widehat{S})$ that do not touch the line segment $\{(1,y) : y \leq 0\}$(drawn in green in Figure 1), and let $Y(\widehat{S}) = W(\widehat{S}) \setminus X(\widehat{S})$. An example path is drawn in blue in Figure 1.

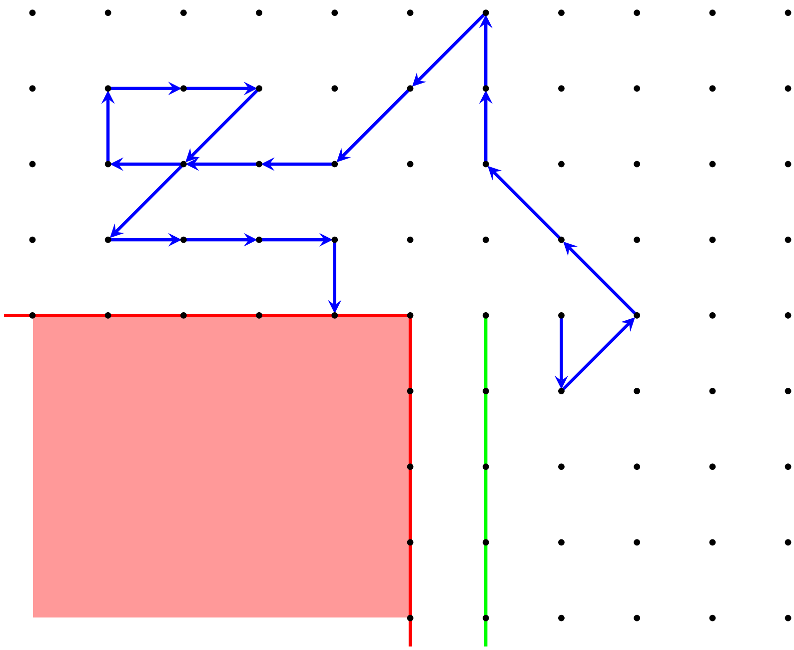

Figure 1. A walk in $X(\widehat{S})$ for $\widehat{S} = \{\nearrow, 2\times\downarrow, 2\times\nwarrow, 3\times\uparrow, 3\times\leftarrow, 4\times\swarrow, 5\times\rightarrow\}$.

Theorem 1 (A. Elvey Price)**.** *For any multiset $\widehat{S}$ of small steps, $|X(\widehat{S})| = |Y(\widehat{S})|$.*

The existing proof of this surprising fact uses elliptic functions.

Problem 1. *Find a bijection between $X(\widehat{S})$ and $Y(\widehat{S})$.*

2. Nathan Williams: Parking functions via Deodhar subwords

Taking all indices modulo $n+1$, the affine symmetric group has the familiar presentation

$$\widetilde{S}_{n+1} = \left\langle s_0, s_1, \ldots, s_n : s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e \right\rangle,$$

where $i \neq j$ and $i \neq j \pm 1$.

Theorem 2 (P. Galashin, T. Lam, N. Williams)**.** *Let P_n be the set of subwords of $(s_0, s_1, \ldots, s_n)^n$ of length $n(n-1)$ whose product is the identity and whose consecutive products decrease in weak order whenever possible. Then $|P_n| = (n+1)^{n-1}$.*

The interesting condition on consecutive products comes from the Deodhar decomposition of braid varieties. The 4^2 subwords in P_3 are given in Figure 2. The proof uses a trace formula for the affine Hecke algebra due to Opdam and a Tessler matrix identity due to Haglund. (See [11] for some related problems.)

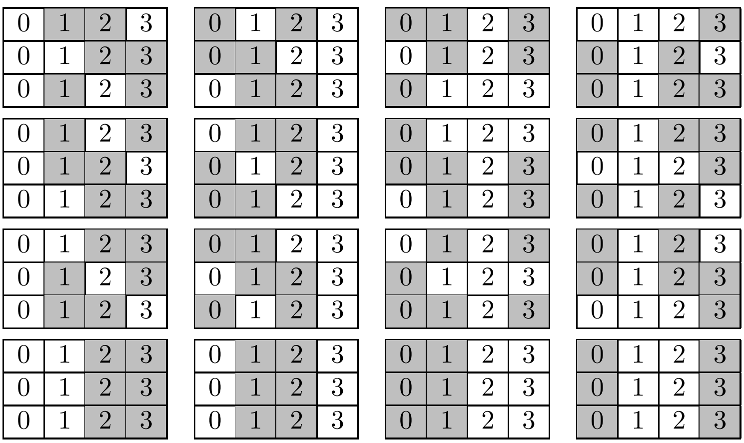

Figure 2. The letters in each of the 4^2 subwords in P_3 are indicated by a gray background. For example, the subword in the top left should be read as $(e, s_1, s_2, e, e, e, s_2, s_3, e, s_1, e, s_3)$.

Problem 2. *Find a bijection between P_n and parking functions.*

It might be useful to aim for noncrossing parking functions [9]—there is some progress using a refinement coming from the "root configuration" of a subword.

3. Jay Pantone: 321-avoiding permutations obeying parity

We make the following two definitions regarding the one-line notation of a permutation.

Definition 2. *A permutation* π contains the pattern 321 *if it has three (not-necessarily-consecutive in position or value) decreasing entries. Otherwise,* π avoids 321.

Definition 3. *A permutation* π obeys parity *if* $\pi(i) = i \mod 2$.

Problem 3 (P. Alexandersson [1]). *Let* $P^{321}(n)$ *be the number of permutations of length* n *that both avoid* 321 *and obey parity. Is the generating function* $\sum_{n=1}^{n} P^{321}(n)q^n$ *algebraic, D-finite, or D-algebraic? What are the asymptotics of* $P^{321}(n)$*?*

Example 1. *There are six permutations of length* 6 *that both avoid* 321 *and obey parity:*

$$\{123456, 341256, 145236, 125634, 561234, 345612\}.$$

- This sequence appears as A354298 in the OEIS [2].
- From 65 terms, it appears the growth rate is approximately $(2.31)^n n$.
- The terms demonstrate an even-odd behavior.

4. Bruce Sagan: Log-concavity of q-Stirling numbers and their type B analogues

The *Stirling numbers of the second kind* $S(n, k)$ count set partitions of $[n] := \{1, 2, \ldots, n\}$ with k blocks $B_1, B_2, \ldots, B_k$. They satisfy the recursion

$$S(n, k) = S(n-1, k-1) + kS(n-1, k) \text{ with } S(0, k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For π a set partition of $[n]$ with k blocks, we put π in standard form $[B_1, B_2, ..., B_k]$ so that $1 = \min B_1 < \min B_2 < \cdots < \min B_k$. An *inversion* of π is a pair (b, B_j) where $b \in [n]$ and B_j is a block such that

- $b \in B_i$ for some $i < j$, and
- $b > \min B_j$.

The *inversion number* $\mathrm{inv}(\pi)$ of a set partition is its number of inversions. Carlitz's *q-Stirling numbers of the second kind* $S_q[n, k]$ count set partitions of $[n]$ with k blocks by inversion number [5, 8]:

$$S_q[n, k] = \sum_{\pi \text{ a set partition of } [n] \text{ with } k \text{ blocks}} q^{\mathrm{inv}(\pi)}.$$

Writing $[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$, these q-numbers satisfy the recursion

$$S_q[n, k] = S_q[n-1, k-1] + [k]_q S_q[n-1, k] \text{ with } S_q[0, k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4. *A polynomial* $\sum_{i\geq 0} a_i q^i$ *is* log-concave *if* $a_i^2 \geq a_{i-1}a_{i+1}$ *for all* $i \geq 1$.

Conjecture 3 ([3, Conjecture 7.4])**.** *The polynomials* $S_q[n,k]$ *are log-concave.*

- Have checked by computer all $n, k \leq 50$.
- For $k \geq 2$, have confirmed that the coefficients of $S_q[n,k]$ are asymptotically normal as $n \to \infty$ ([3, Theorem 7.7]).
- Have tried standard techniques, like Lorentzian polynomials [7]. Haven't tried the theory of atlases [6].
- Q: What about q-log concavity? A: This is a application of ideas in [10].

There is a natural extension of q-Stirling numbers from set partitions to the signed set partitions of type B_n. Let

$$S_q^B[n,k] = S_q^B[n-1,k-1] + [2k+1]_q S_q^B[n-1,k] \text{ with } S_q^B[0,k] = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the polynomials $S_q^B[n,k]$ are *not* log-concave.

Definition 5. *A polynomial* $\sum_{i\geq 0} a_i q^i$ *is* parity log-concave *if* $\sum_{i\geq 0} a_{2i} q^i$ *and* $\sum_{i\geq 0} a_{2i+1} q^i$ *are log-concave.*

Conjecture 4 ([3, Conjecture 7.5])**.** *The polynomials* $S_q^B[n,k]$ *are parity log-concave.*

5. Theo Douvropoulos: Deformations of braid arrangements

Figure 3 displays the braid arrangement, the Shi arrangement, and the Catalan arrangement for the symmetric group S_3, along with their defining set of hyperplanes and their characteristic polynomials.

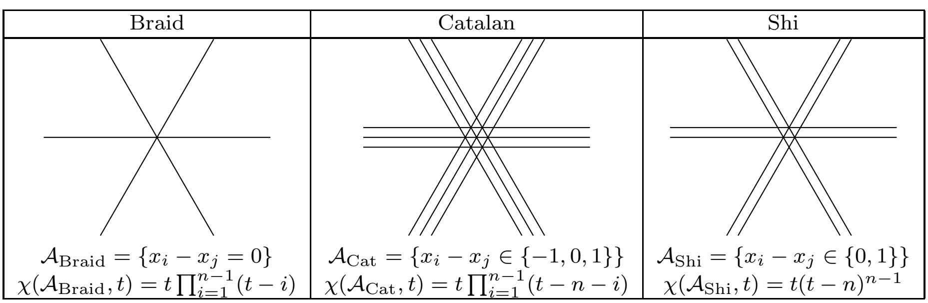

Figure 3. The $n = 3$ braid arrangement (*left*), Catalan arrangement (*middle*), and Shi arrangement (*right*).

We generalize the last two kinds of arrangements as follows. Pick n positive integers $\mathbf{k} = (k_1, \ldots, k_n)$ and define

$$\mathcal{A}^{\mathbf{k}}_{\mathrm{Cat}} = \big\{x_i - x_j \in \{-k_i - k_j, \dots, k_i + k_j\}\big\} \text{ and}$$
$$\mathcal{A}^{\mathbf{k}}_{\mathrm{Shi}} = \big\{x_i - x_j \in \{-k_i - k_j + 1, \dots, k_i + k_j\}\big\}.$$

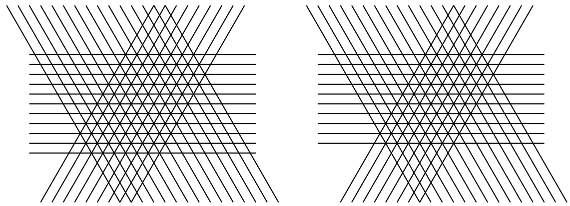

Figure 4. The arrangements $\mathcal{A}^{\mathbf{k}}_{\mathrm{Cat}}$ and $\mathcal{A}^{\mathbf{k}}_{\mathrm{Shi}}$ for $\mathbf{k} = (1, 3, 4)$.

Theorem 5. *For any n positive integers $\mathbf{k} = (k_1, \dots, k_n)$, we have*

$$\chi(\mathcal{A}^{\mathbf{k}}_{\mathrm{Cat}}, t) = t \prod_{i=1}^{n-1} (t - 2(k_1 + \cdots + k_n) - i) \textit{ and}$$
$$\chi(\mathcal{A}^{\mathbf{k}}_{\mathrm{Shi}}, t) = t(t - 2(k_1 + \cdots + k_n))^{n-1}.$$

There are other versions, but they are not quite as clean.

Problem 4. *Find a combinatorial interpretation for the regions of $\mathcal{A}^{\mathbf{k}}_{\mathrm{Cat}}$ and $\mathcal{A}^{\mathbf{k}}_{\mathrm{Shi}}$, à la Athanasiadis–Linusson* [12]*, Stanley–Pak* [13, Section 5]*, or Bernardi* [14]*.*

6. Valentin Féray: Catalan numbers and polynomials in $1/\pi$.

Here are two interesting sums that arise from work that V. Féray will present later in the week [4].

Theorem 6. *Write* $\mathrm{Cat}_\ell = \frac{1}{\ell+1}\binom{2\ell}{\ell}$*. Then*

$$\sum_{\ell=0}^{\infty} \mathrm{Cat}_\ell^2 \cdot 16^{-\ell} = \frac{16}{\pi} - 4$$
$$\sum_{\ell_1,\ell_2 \geq 0}^{\infty} \mathrm{Cat}_{\ell_1} \cdot \mathrm{Cat}_{\ell_2} \cdot \mathrm{Cat}_{\ell_1+\ell_2} \cdot 16^{-\ell_1-\ell_2} = 8 - \frac{64}{3\pi}.$$

The general form of such sums is as follows. Let τ_1, τ_2 be two set partitions of $[k]$ such that $([k], \tau_1 \uplus \tau_2)$ is a connected hypertree with vertex degrees exactly 2—that is, the join $\tau_1 \vee \tau_2$ is the maximal partition $\{[k]\}$ into one part and $\#(\tau_1) + \#(\tau_2) = k + 1$, where $\#(\tau_i)$ is the number of parts of τ_i.

Problem 5. *Is*

$$\sum_{\ell_1,\dots,\ell_k \geq 0} \left(\prod_{B \in \tau_1 \uplus \tau_2} \mathrm{Cat}_{\sum_{i \in B} \ell_i} \right) 16^{-\sum_{i=1}^{k} \ell_i} \in \mathbb{Q}\left[\frac{1}{\pi}\right]?$$

- There are examples where the sum gives a degree 2 polynomial in $1/\pi$.
- Q: Is there a connection to restricted meanders? A: Perhaps—the problem arises from a meander question.
- Q: Is there a relation to the Green function for $\mathbb{Z}^2$? A: Perhaps (not always affine).

7. Philippe Di Francesco: Enumeration of planar bicubic maps

This is a problem with an apparently similar complexity to problems on meanders — the enumeration of edge-rooted Hamiltonian cycles in genus 0 planar bicubic (that is, bipartite and trivalent) maps. An example is given on the left of Figure 5.

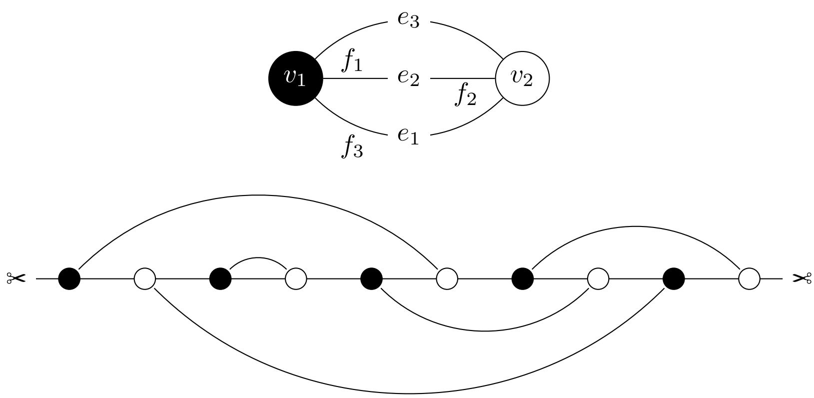

Figure 5. *Left:* an example of a vertex bicolored trivalent map of genus zero—here $F - E + V = 3 - 3 + 2 = 2 = 2 - 2g$, so that $g = 0$. *Right:* A redrawn map, where the path is the given Hamiltonian cycle and the rooted edge is indicated by scissors.

We now cut the rooted edge and use the given Hamiltonian cycle to redraw the map as a path of the vertices in the order visited by the Hamiltonian cycle, with some additional noncrossing arcs connecting vertices. An example is given on the right of Figure 5.

Let H_{2n} be the number of such maps for a fixed number n of vertices. Then it is predicted by physics, and verified to 3 significant digits (using, for example, the transfer matrix method), that

$$H_{2n} \sim c\frac{\mu^{2n}}{n^\gamma}, \text{ where } \gamma = \frac{13+\sqrt{13}}{6} \text{ and } \log(\mu^2) = 2.313.$$

Problem 6. *Is there some mathematical (probabilistic or even combinatorial) approach to proving this prediction?*

As a case study, if we remove the bicoloring of the vertices, then the quantity in question is easily computed to be

$$\sum_{m=0}^{n} \binom{2n}{2m} \mathrm{Cat}_m \mathrm{Cat}_{n-m} = \mathrm{Cat}_n \mathrm{Cat}_{n+1} \sim \frac{4}{\pi} \frac{4^{2n}}{n^3},$$

so that in this case we have $c = 4/\pi$, $\gamma = 3$, and $\mu^2 = 16$—and the same machinery from physics predicts these constants.

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