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Resolutions in Local Algebra and Singularity Theory

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Abstract. Commutative algebra is a vast subject, with connections to many different areas of mathematics, and beyond. The focus of this workshop was on three areas, all concerned with resolutions in various forms. One is the resolution of singularities of algebraic varieties, which remains a vibrant topic of research. The second is the theory of noncommutative resolution of singularities. Introduced two decades ago, this subject has witnessed remarkable growth developing connections to algebraic geometry, commutative algebra, cluster algebras, and the representation theory of algebras, both commutative and noncommutative, among others. The third intended meaning of the world "resolution" is as in free resolutions of algebras and modules in commutative algebra. There is another sense in which the title is appropriate: recently three long standing open problems in commutative algebra have been resolved. This workshop brought together experts and early career researchers in these various fields, to facilitate exchange of ideas and to explore potential collaborations.

Mathematics Subject Classification (2020): 13XX, 14XX, 16XX, 18XX.

Introduction by the Organizers

Commutative algebra has been evolving at a fast pace in the past few years, driven in part by an influx of ideas and techniques from number theory. This made for a lively meeting, with participants from most of the major research centers in commutative algebra across the world. The workshop was also well-represented demographically, for it had a number of early-career researchers, many of who are driving the research in our area, as well as more senior researchers. The timing of workshop at Oberwolfach was propitious, for in Spring 2024, MSRI will host a semester-long program on Commutative Algebra. Though independent of this event, our workshop played a complementary role, by providing an opportunity for the community to connect and set the tone for the major themes of the MSRI semester. Here are a few topics that covered at workshop. Some are classical topics in commutative algebra that yet remain the focus of active research; all have close ties to the subject, and to each other.

Mixed and positive characteristic. Recently Bhatt (2020) proved that if R is a complete noetherian local domain of mixed characteristic, its integral closure in the algebraic closure of its field of fraction—called the absolute integral closure of R—is big Cohen-Macaulay. The possibility that such a statement might be true arose when Hochster and Huneke (1992) proved the analogous result when R is a local ring containing a field of positive characteristic. The importance of these results is that they provide big Cohen-Macaulay R-algebras; the adjective "big" indicates that these need not be finite as R-modules. This result comes in the wake of André's (2016) of Hochster's Direct Summand Conjecture for mixed characteristic local rings, and Bhatt's own simplification of André's proof. The proofsof these results are inspired by, and make critical use of, Scholze's perfectoid techniques; Bhatt's recent work also builds on his recent work with Lurie on the p-adic versions of the Riemann-Hilbert correspondence. These developments have already begun to have a huge impact on commutative algebra and algebraic geometry, in part by paving the way to extending classical constructions and invariants, hitherto available only in the equi-characteristic case, to the world of mixed characteristic. This was apparent from many of the results presented in the workshop.

Linquan Ma talked about his work with Cai, Lee, Schwede and Tucker on mixed-characteristic analogues of various numerical invariants, like the Hilbert-Kunz multiplicity and F-signature, that are based on the Frobenius map in positive characteristic. To this end, they develop a mixed-characteristic analog of Falting's normalized length, using the perfectoidzation functor of Bhatt and Scholze. Shunsuke Takagi explained joint work with Tatsuki Yamaguchi concerning big Cohen-Macaulay test ideal. As an application, they give an alternate proof of a recent theorem of Zhang showing that the property of KLT type descends along a pure local homomorphism.

There has also been significant progress on phenomena specific to local rings of positive characteristic. Thomas Polstra presented his striking new results, with Aberbach and Huneke, on the long-standing conjecture that any weakly F-regular ring is already strong F-regular. Alessandro De Stefani discussed joint work with Jonathan Montaño and Luis Núñez-Betancourt on the F-purity and strong F-regularity of blowup algebras of various determinantal rings, which yields new bounds on degrees of defining equations for these algebras, among other things.

Closely related to this last topic was Claudia Polini talk on duality and Rees algebra, based on her joint work with Yairon Cid-Ruiz, Bernd Ulrich, and Matthew Weaver. Polini discussed a new method for finding implicit equations for graphs and images of rational maps between projective varieties. A key new ingredient is a weaker notion of the Gorenstein property, and a generlization of a duality theorem of Jouanolou to this much larger family of rings.

Homological aspects. Homological algebra has played an important role in commutative algebra, ever since Serre's proof that the regularity property of rings is inherited by localizations. Since then there have been homological characterizations of many other properties in local algebra. At the workshop, Benjamin Briggs spoke about his spectacular proof of Vasconcelos' conjecture characterizing the complete intersection property in terms of the finiteness of the projective dimension of conormal modules. He also discussed his subsequent work with Iyengar on the cotangent complex that significant strengthen Avramov's result, conjectured by Quillen, characterizing locally complete intersection maps in terms of cotangent complexes. The work of Briggs is part of a bigger story involving support varieties for modules over commutative rings. Introduced first in the context of modules over local complete intersection rings it now encompasses modules over any local ring, thanks to work of Avramov, Jorgensen, Pollitz, and many others. Nevertheless basic questions remain. The talk of Eloísa Grifo discussed joint work with Briggs and Pollitz on one such: Given an local ring and an appropriately chosen variety when can it be realized as the variety associated to some module the ring? One of the main conclusion is that this is not always possible, and that there are lower bounds on the dimension of the given variety. Grifo et. al. also draw connections to, and extend, earlier work of Avramov, Buchweitz, Iyengar, and Miller, on Loewy lengths of the homology of finite free complexes.

Claudiu Raicu presented his work on the cohomology of line bundles over flag varieties defined a field of positive characteristic. The characteristic zero case is the content of the celebrated Borel-Weil-Bott theorem. He also explained some implications of his calculations to statements about Castelnuovo-Mumford regularity for Koszul modules an determinantal ideals.

The search for structure theorems for resolutions in small codimension has always been central topic with the classical Hilbert–Burch theorem (for perfect ideals of codimension 2) and Buchsbaum–Eisenbud theorem (for Gorenstein ideals of codimension 3) as main results. Jerzy Weyman discussed his new contributions establishing connections between the structure of perfect ideals of codimension 3 and Gorenstein ideals of codimension 4 with root systems and Schubert varieties in homogeneous spaces. Roser Homs Pons spoke about her results on canonical Hilbert-Burch matrices for power series extending work of Conca and Valla, and Constantinescu on homogenous ideals in polynomial rings. A related topic is the problem of transferring free resolutions along surjective maps of commutative rings. A few years ago Burke used A_∞-rings and modules to solve this problem. Janina Letz spoke about ongoing work with Briggs, Cameron, and Pollitz where they build on Burke's work to obtain a recipe for transferring resolutions along Koszul homomorphisms, yielding more optimal resolutions than Burke's methods.

Chow rings of matriods are a topic of active research interest. They enter, for example, in the proof of the Heron-Rota-Welsh Conjecture by Adiprasito, Huh, and Katz on the log-concavity of the coefficients of the chromatic polynomial of matroids), and also in the recent proof of the Top Heavy Conjecture by Braden, Huh, Matherne, Proudfoot, and Wang. It is known that these Chow rings are commutative, artinian, Gorenstein algebras. And that they are defined by quadratic relations, leading Dotsenko to conjecture that they are in fact Koszul. This was proved recently by Jason McCullough and Matthew Mastroeni, and McCullough presented their work at our workshop. The Koszul property places restrictions on possible Hilbert Series of Chow rings and implies that their Poincare series are rational. Continuing on this combinatorial theme, Hema Srinivasan reported on her joint work with Philippe Gimenez on the problem presenting a semigroup as a gluing of sub semigroups.

Differential operators remain a topic of active interest in commutative and algebraic geometry. Jeffries presented his recent work with Josep Àlvarez Montaner, Daniel J. Hernández, Luis Núñez-Betancourt, Pedro Teixeira, and Emily E. Witt, wherein they develop the theory of holonomic D-modules for rings of invariants of finite groups in characteristic zero, and for strongly F-regular finitely generated graded algebras with FFRT in prime characteristic. Differential operators were also the focus of Claudia Miller's talk, describing the operators of low order for certain isolated hypersurface singularities. This was based on joint work with Rachel Diethorn, Jack Jeffries, Nick Packauskas, Josh Pollitz, Hamid Rahmati, and Sophia Vassiliadou.

Non-commutative resolutions and singularity categories. The motivation for commutative resolutions of singularities (NCRs) is to find a smooth (in some sense) noncommutative ring that encompasses the geometry of a traditional resolution of singularities and provides a more transparent way to understand the initial singularity. NCRs are given as endomorphism rings of certain modules, usually of maximal Cohen–Macaulay modules, over the commutative coordinate ring of an algebraic variety. The study of these rings opens up new connections between algebraic geometry, commutative algebra, tilting theory, and representation theory. A further object of study arising from a noncommutative viewpoint is the singularity category of a ring, which is equivalent to the stable category of maximal Cohen–Macaulay modules for a Gorenstein ring. Singularity categories provide a homological measure for singularities of (non)commutative rings.

Following Van den Bergh's introduction of noncommutative crepant resolutions nearly 20 years ago, there has been a lot of activity in constructing noncommutative resolutions and further studying singularity categories of commutative rings. Very recently, noncommutative singularity theory has become a new development in order to understand the birational geometry in particular of 3-folds. Michael Wemyss gave a vibrant account of his joint work with Gavin Brown about classifying noncommutative hypersurfaces à la Arnol'd. The goal is to find normal forms, similar to the commutative ADE-classification. In the noncommutative case there is a much richer structure of normal forms, which have interesting applications to the homological minimal model programme of 3-folds.

Singularity categories have been introduced by Buchweitz and they are a homological invariant for (non)commutative rings. The main question of Martin

Kalck's talk was, how fine this invariant is: the classical example is Knörrer's correspondence which allows to determine the equivalence of singularity categories of hypersurfaces. In his talk he showed several implications of equivalent singularity categories and gave an interesting new example for an equivalence of dimension 2 and 3 singularities for which the parity of Krull-dimensions (as in Knörrer's theorem) is not preserved.

Hilbert schemes play a prominent role in the resolution of singularities, in particular in the 3-dimensional McKay correspondence considered by Bridgeland–King–Reid. Špela Špenko talked about the structure of noncommutative Hilbert schemes and their connection to combinatorics, in particular parking functions and Fuss-Catalan numbers. In her joint work with Lunts and Van den Bergh she could further determine a tilting bundle of the noncommutative Hilbert scheme of cyclic modules of dimension n over the free algebra $\mathbb{C}\langle x_1, \ldots, x_m\rangle$.

Singularities and resolution of singularity in positive and mixed characteristic. The problem of local uniformization, which is a local form of resolution of singularities, was introduced by Zariski. Suppose that S is a local domain with quotient field L, and that ω is a valuation of L which dominates S. Then S admits a local uniformization along ω if there exists a birational extension $S \to S_1$ such that S_1 is dominated by ω and S_1 is a regular local ring. Zariski proved that local uniformization always holds in algebraic function fields over an arbitrary field of characteristic zero, and used this to prove resolution of singularities for projective varieties of dimension ≤ 3 and characteristic zero. Hironaka (proved the stronger global theorem that resolution of singularities of projective varieties of characteristic zero is true in all dimensions. As of this time, resolution of singularities is known for excellent reduced schemes in dimension ≤ 3 and local uniformization is known for excellent local rings in dimension ≤ 3; this is due to work of Abhyankar, Lipman, and Cossart and Piltant, and many others. Recently there has been significant progress in understanding these problems in positive and mixed characteristic. The significant new understanding is that the only obstruction to local uniformization in positive and mixed characteristic is the defect of the extension $K \to L$.

In another direction, there is a remarkable dictionary between singularities defined in positive characteristic that have good properties with respect to the Frobenius mapping and the singularities which occur in characteristic zero in the minimal model program.

Anna Bravo presented joint work with Angélica Benito, Santiago Encinas and Javier Guillán-Rial on an application of the classical Samuel function to resolution of singularities. From the Samuel function, they define the Samuel slope of a local ring and show that it computes key invariants of resolution of singularity.

Franz-Viktor Kuhlmann reported on joint work with Steven Dale Cutkosky and Anna Rzepka on ramification of valued fields. He explained the essential role of extensions of degree p and his classification of such extensions into those with dependent and independent defect. He presented their joint theorem that such an extension has independent defect if and only if the Kähler differentials of the extension of valuation rings are zero, and explained the relationship of this result to deep ramification of fields.

Hussein Mourtada presented some of his results, obtained in joint work with Bruschek and Schepers, and Afsharijoo, linking integer partitions to singularity theory. He computes the Arc Hilbert-Poincaré series of some rational double points, and has also found new Rogers-Ramanujan identity.

Parangama Sarkar presented joint work with Cutkosky where they introduce and study analytic spread for filtrations of noetherian local rings. One of the key results is that the analytic spread is bounded above the dimension of the ring, but, unlike for ideals, there are no positive lower bounds.

Workshop: Resolutions in Local Algebra and Singularity Theory

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Abstracts

The asymptotic Samuel function: some properties and invariants of singularities

Ana Bravo

(joint work with Angélica Benito and Santiago Encinas; Santiago Encinas and Javier Guillán-Rial)

Let A be a commutative ring with 1, and let $J \subset A$ be a proper ideal. The *asymptotic Samuel function with respect to* J is defined as:

$$\begin{array}{rrcl} \bar{\nu}_J : & A & \longrightarrow & \mathbb{R} \cup \{\infty\} \\ & a & \longmapsto & \lim_{n\to\infty} \frac{\nu_J(a^n)}{n}, \end{array}$$

where $\nu_J(a) := \sup\{m \in \mathbb{N} : a \in J^m\}$. When $(A, \mathfrak{m}, k)$ is a regular local ring, then $\bar{\nu}_{\mathfrak{m}}$ is the usual order function $\nu_{\mathfrak{m}}$. And if A is a Noetherian ring and $f \in A$, then $\bar{\nu}_J(f) \geq \frac{a}{b}$ if and only if $f^b \in \overline{I^a}$ (see [17, Corollary 6.9.1]).

The *asymptotic Samuel function* was first introduced by Samuel in [15] and studied afterwards by D. Rees in a series of papers ([11], [12], [13], [14]). Thorough expositions on this topic can be found in [10] and [17]. A generalization for arbitrary filtrations of ideals can be found in [7].

We have studied this function from two points of view. On the one hand, we have explored some of its properties mainly comparing it with the usual order function on regular local rings. This is work in progress with S. Encinas nad J. Guillán-Rial ([6]). On the other, we have established some connections with some invariants of singularities that appear in constructive resolution. This is joint with A. Benito and S. Encinas ([1], [2]).

Some properties of the asymptotic Samuel function. When $(A, \mathfrak{m}, k)$ is regular then the usual order function satisifies the following properties:

(1) [8, Theorem 2.11] For $a \in A$ and $\mathfrak{p} \subset \mathfrak{m}$ a prime ideal, $\nu_{\mathfrak{p}A_{\mathfrak{p}}}(a) \leq \nu_{\mathfrak{m}}(a)$.
(2) For $a, b \in A$, $\nu_{\mathfrak{m}}(a \cdot b) = \nu_{\mathfrak{m}}(a) + \nu_{\mathfrak{m}}(b)$.
(3) There is a natural graded ring $\mathrm{Gr}_{\mathfrak{m}}(A) = \bigoplus_{n\in\mathbb{N}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ associated to $\nu_{\mathfrak{m}}$.

We have found that for the asymptotic Samuel function the following analogous results hold.

Theorem 1 ([6])**.** *Let $(A, \mathfrak{m})$ be equicharacteristic, equidimensional, excellent and reduced. Let $\mathfrak{p} \subset A$ be a prime such that the multiplicity of $(A_{\mathfrak{p}}, \mathfrak{p}A_{\mathfrak{p}})$ equals that of A. Let $a \in A$. Then $\bar{\nu}_{\mathfrak{p}A_{\mathfrak{p}}}(a) \leq \bar{\nu}_{\mathfrak{m}}(a)$.*

Theorem 2 ([1])**.** *Let $(A, \mathfrak{n}) \to (B, \mathfrak{m})$ be a finite morphism of Noetherian local rings, with B reduced, A regular and so that $\mathfrak{n}B$ is a reduction of $\mathfrak{m}$. Then for all $a \in A$ and all $b \in B$, $\bar{\nu}_{\mathfrak{m}}(ab) = \bar{\nu}_{\mathfrak{m}}(a) + \bar{\nu}_{\mathfrak{m}}(b)$.*

For $J \subset A$, the order $\bar{\nu}_J$ defines a graduation on A: for $t \in \mathbb{R}_{\geq 0}$, $J^{(\geq t)} := \{a \in A : \bar{\nu}_J(a) \geq t\}$, and $J^{(>t)} := \{a \in A : \bar{\nu}_J(a) > t\}$. The graded ring: $\mathrm{Gr}_{\bar{\nu}_J}(A) = \oplus_t J^{(\geq t)}/J^{(>t)}$ is graded over $\mathbb{R}$.

Theorem 3 ([6])**.** *Let $(A, \mathfrak{m})$ be reduced, excellent, equicharacteristic, and equidimensional. Then there is some $\ell \in \mathbb{Z}_{>0}$ such that $Gr_{\bar{\nu}_{\mathfrak{m}}}(A)$ is graded over $\frac{1}{\ell}\mathbb{Z}_{\geq 0}$.*

The Samuel slope of a local ring. [1, §3] Let $(A, \mathfrak{m}, k)$ be a Noetherian ring. There is a natural map of k-vector spaces, $\lambda_{\mathfrak{m}} : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^{(>1)}$. If $(A, \mathfrak{m}, k)$ is singular of dimension d and if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d + t$, then $0 \leq \dim_k \ker(\lambda_{\mathfrak{m}}) \leq t$. With this notation, the following holds for the *Samuel slope of A*, $\mathcal{S}\text{-sl}(A)$.

- When $\dim_k \ker(\lambda_{\mathfrak{m}}) < t$, then $\mathcal{S}\text{-sl}(A) := 1$.
- When $\dim_k \ker(\lambda_{\mathfrak{m}}) = t$ select $\gamma_1, \ldots, \gamma_t \in \mathfrak{m}$, inducing basis of $\ker(\lambda_{\mathfrak{m}})$. For $\mathcal{B} = \{\gamma_1, \ldots, \gamma_t\}$ define $\mathrm{sl}(\mathcal{B}) := \min\{\bar{\nu}_{\mathfrak{m}}(\gamma_i) : i = 1, \ldots, t\}$. Then
$$\mathcal{S}\text{-sl}(A) := \sup_{\mathcal{B}}\{\mathrm{sl}(\mathcal{B}) : \mathcal{B} \text{ induces a basis of } \ker(\lambda_{\mathfrak{m}})\}.$$

Theorem 4 ([6])**.** *Let $(A, \mathfrak{m}, k)$ be equidimensional, equicharacteristic, excellent, and reduced. Then $\mathcal{S}\text{-}sl(A) \in \mathbb{Q}$.*

The asymptotic Samuel function and resolution of singularities. Let X be an equidimensional singular algebraic variety of dimension d defined over a perfect field k. Then the set of points of maximum multiplicity, $\underline{\mathrm{Max}}\ \mathrm{mult}_X$, is a closed proper set in X. Let $\max \mathrm{mult}_X$ be the maximum value of the multiplicity at points of X. A *simplification of the multiplicity* of X is a finite sequence of blow ups,

$$X = X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_{L-1}} X_{L-1} \xleftarrow{\pi_L} X_L \tag{1}$$

with $\max \mathrm{mult}_{X_0} = \max \mathrm{mult}_{X_1} = \cdots = \max \mathrm{mult}_{X_{L-1}} > \max \mathrm{mult}_{X_L}$, where $\pi_i : X_i \to X_{i-1}$ is the blow up at a regular center contained in $\underline{\mathrm{Max}}\ \mathrm{mult}_{X_{i-1}}$.

Simplifications of the multiplicity exist if the characteristic of k is zero (see [18]), and resolution of singularities follows from there. Recall that Hironaka's line of approach to resolution makes use of the Hilbert-Samuel function instead of the multiplicity [9]. The centers in the sequence (1) are determined by resolution functions. These are upper semi-continuous functions $f_{X_i} : X_i \to (\Gamma, \geq)$, $i = 0, \ldots, L-1$, and their maximum value, $\mathrm{Max} f_{X_i}$, achieved in a closed regular subset $\underline{\mathrm{Max}} f_{X_i} \subseteq \mathrm{Maxmult}_{X_i}$, selects the center to blow up. Hence, a simplification of the multiplicity of X, $X \leftarrow X_L$, is defined as a sequence of blow ups at regular centers.

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_L. \tag{2}$$

so that $\mathrm{Max} f_{X_0} > \mathrm{Max} f_{X_1} > \ldots > \mathrm{Max} f_{X_L}$, where $\mathrm{Max} f_{X_i}$ denotes the maximum value of f_{X_i} for $i = 0, 1, \ldots, L$.

Usually, f_X is defined at each point as a sequence of rational numbers. The first coordinate of f_X is the multiplicity, and the second is what we refer to as *Hironaka's order function in dimension d*, $\mathrm{ord}_X^{(d)}$. The function $\mathrm{ord}_X^{(d)}$ is a positive rational number. The remaining coordinates of $f_X(\zeta)$ can be shown to depend on $\mathrm{ord}_X^{(d)}(\zeta)$. The function $\mathrm{ord}_X^{(d)}$ can always be defined if k a perfect field, but falls short to define a resolution function when the characteristic of the field is positive. This motivated the papers [4] and [5]. There, the function $\text{H-ord}_X^{(d)}$ was introduced by A. Benito and O. Villamayor. In [3], this function played a role in the proof of desingularization of two dimensional varieties. For a point $\zeta \in X$ of multiplicity greater than 1, $1 \leq \text{H-ord}_X^{(d)}(\zeta) \leq \mathrm{ord}_X^{(d)}(\zeta)$.

The definition of the $\text{H-ord}_X^{(d)}$ and $\mathrm{ord}_X^{(d)}$ is done locally, in an étale neighborhood of a singular point of X. This requires the selection of suitable embeddings and certain finite projections to smooth schemes, together with the application of different techniques that involve elimination and saturation by differential operators. Hence, part of the work in resolution consists on proving that these functions are actually invariants of the singularities (i. e., it has to be proven that that their definition does not depend on the may different choices that have been made for their construction). With A. Benito and S. Encinas, we have proven that the Samuel slope of the local ring $\mathcal{O}_{X,\zeta}$ captures the same information as the previous functions:

Theorem 5 ([1], [2]). *Let X be an equidimensional algebraic variety defined over a perfecti field k and let $\zeta \in X$ be a singular point. Then: if char$(k) > 0$, $\mathcal{S}$-sl$(\mathcal{O}_{X,\zeta}) =$ H-ord$^{(d)}(\zeta)$; if char$(k) = 0$, then $\mathcal{S}$-sl$(\mathcal{O}_{X,\zeta}) =$ ord$^{(d)}(\beta(\zeta))$.*

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Perfectoid signature and applications

Linquan Ma
(joint work with Hanlin Cai, Seungsu Lee, Karl Schwede, Kevin Tucker)

Let $(R, \mathfrak{m})$ be a Noetherian F-finite domain of characteristic $p > 0$. The F-signature of R is defined as follows:

$$s(R) := \lim_{e\to\infty} \frac{\mathrm{frk}(R^{1/p^e})}{\mathrm{rk}(R^{1/p^e})}$$

where $\mathrm{frk}(R^{1/p^e})$ is the largest integer a such that one can write $R^{1/p^e} \cong R^a \oplus M$ for some M. Quite obviously, $0 \le s(R) \le 1$. There are three fundamental theorems of F-signature:

- Existence: the limit above exists [7].
- Characterizing regularity: $s(R) = 1$ if and only if R is regular [6].
- Characterizing strong F-regularity: $s(R) > 0$ if and only if R is strongly F-regular [1].

Furthemore, it is proved in [4] that $s(R)$ satisfies certain transformation rule under finite étale in codimension one map. As a consequence, it is shown in [4] that $\pi_1^{et}(\mathrm{Spec}(R) \setminus \{\mathfrak{m}\})$ is finite for every Noetherian complete local strongly F-regular ring $(R, \mathfrak{m}, k)$ with $k = R/\mathfrak{m}$ algebraically closed.

We use Bhatt-Scholze's perfectoidization functor [2] and Faltings' normalized length function [5] to define a mixed characteristic version of F-signature, which we call perfectoid signature. Let $(R, \mathfrak{m}, k)$ be a Noetherian complete local domain such that $k = R/\mathfrak{m}$ is perfect and has characteristic $p > 0$. By Cohen-Gabber theorem, we can find a complete regular local ring $(A, \mathfrak{m}_A, k)$ and a finite extension $A \to R$ that is generically étale. Fix an isomorphism $A \cong k[[x_1, \ldots, x_d]]$ in positive characteristic and $A \cong W(k)[[x_2, \ldots, x_d]]$ in mixed characteristic (and set $x_1 = p$). Let $A_\infty := A[x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^\wedge$ where the completion is p-adic. Then A_∞ is a perfectoid ring. Further, let $R^{A_\infty}_{\mathrm{perfd}} := (A_\infty \otimes_A R)_{\mathrm{perfd}}$ be the perfectoidization of $A_\infty \otimes_A R$, and let

$$I_\infty := \{z \in R^{A_\infty}_{\mathrm{perfd}} \mid \text{the map } R \to R^{A_\infty}_{\mathrm{perfd}} \text{ sending 1 to } z \text{ is not split}\}.$$

Finally, define the perfectoid signature of R with respect to $\underline{x} = x_1, \ldots, x_d$ to be

$$s^{\underline{x}}_{\mathrm{perfd}}(R) := \lambda_\infty(R^{A_\infty}_{\mathrm{perfd}}/I_\infty)$$

where $\lambda_\infty(-)$ denotes the normalized length computed over A_∞. We prove the following in [3]:

- In positive characteristic, $s^{x}_{\text{perfd}}(R) = s(R)$.
- $0 \leq s^{x}_{\text{perfd}}(R) \leq 1$.
- $s^{x}_{\text{perfd}}(R) = 1$ if and only if R is regular.
- If R is Q-Gorenstein, then $s^{x}_{\text{perfd}}(R) > 0$ if and only if R is BCM-regular: that is, $R \to B$ is split for all perfectoid big Cohen-Macaulay algebras B.

We also prove that $s^{x}_{\text{perfd}}(R)$ satisfies the same transformation rule as for $s(R)$ under finite étale in codimension one map. As a main application of these results, we obtain that $\pi_1^{et}(\text{Spec}(R) \setminus \{\mathfrak{m}\})$ is finite for every Noetherian complete local Q-Gorenstein BCM-regular rings $(R, \mathfrak{m}, k)$ with $k = R/\mathfrak{m}$ algebraically closed, see [3] for more details and more general statements.

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Duality and Rees Algebras

Claudia Polini

(joint work with Yairon Cid-Ruiz and Bernd Ulrich, Yairon Cid-Ruiz, Bernd Ulrich and Matt Weaver)

In this talk we will discuss a method to find the implicit equations defining the graphs and images of rational maps between projective varieties. The problem amounts to identifying the torsion in the symmetric algebra of an ideal, and one technique to achieve this is based on a duality statement due to Jouanolou that expresses the torsion of a graded algebra in terms of a graded dual of this algebra [1]. Unfortunately, Jouanlou duality requires the algebra to be Gorenstein, a rather restrictive hypothesis for symmetric algebras.

In this talk, we introduce a generalized notion of Gorensteinness, which we call weakly Gorenstein, and prove that Jouanolou duality generalizes to this larger class of algebras. Surprisingly, the weak Gorenstein property is rather common and is satisfied, for instance, by determinantal rings and by symmetric algebras assuming that the latter are Cohen-Macaulay. This leads to the solution of the implicitization problem for new classes of rational maps.

Our approach to showing that symmetric algebras have the weakly Gorenstein property is by computing explicitly the canonical module. The formula for the canonical module of the symmetric algebras coincides with a formula for the canonical module of certain Rees algebras (see [3]), which leads to interesting applications. Our main tool to compute $\omega_{\mathrm{Sym(I)}}$ is a newly defined complex that mends one of the main drawbacks of the approximation complex $\mathcal{Z}_\bullet$ [2]. The approximation complex $\mathcal{Z}_\bullet$ is ubiquitous in the study of blowup algebras, and it provides a resolution of the symmetric algebra in many cases of relevance. However, the fact that is made up of Koszul syzygies, which are typically non-free modules, can be a non trivial obstacle. To remedy this problem we introduce a halfway resolution that refines $\mathcal{Z}_\bullet$. We introduce a new complex that consists of free modules in the last $g-1$ positions and that coincides with $\mathcal{Z}_\bullet$ in the remaining positions. This new complex is acyclic when $\mathcal{Z}_\bullet$ is. Furthermore, these halfway free resolutions lead to actual free resolutions of the symmetric algebra for special families of ideals such as almost complete intersections and perfect ideals of deviation two. The usefulness of this new complex came as surprise, since computing free resolutions of symmetric algebras is a problem of tall order.

As an application of our generalized duality we provide the explicit solution of the implicitization problem for maps from $\mathbb{P}^n$ to $\mathbb{P}^{n+1}$ when the ideal generated by the forms parametrizing the map is either Gorenstein of codimension three or Cohen-Macaulay of codimension 2. The talk is based on joint works with Yairon Cid-Ruiz, Bernd Ulrich, and Matthew Weaver.

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Cohomology and regularity in characteristic p

Claudiu Raicu

(joint work with Zhao Gao, Jennifer Kenkel)

One of the fundamental invariants in commutative algebra and algebraic geometry is the Castelnuovo–Mumford regularity of a homogeneous ideal or a graded module over a polynomial ring. It is closely related to many classical homological invariants such as Tor and Ext groups or local cohomology, and it provides a versatile measure of the complexity of an ideal or module. For instance, it bounds the generating degree of a homogeneous ideal, and also gives a bound for when the Hilbert function of a graded module becomes a polynomial [6, Chapter 4].

One of the celebrated results about regularity in commutative algebra (see [4], [12]) asserts that the regularity of powers of a homogeneous ideal is asymptotically computed by a linear function:

$$\operatorname{reg}(I^d) = ad + b \text{ for } d \gg 0,$$

but explicit values for a, b and effective bounds for d are often difficult to determine. For generic determinantal ideals, we showed the following in [16].

Theorem 1. *Over a field of characteristic zero, if I_t denotes the ideal of $t \times t$ minors of a generic $m \times n$ matrix, then*

$$\operatorname{reg}(I_t^d) = td + \left\lfloor \frac{t-1}{2} \right\rfloor \cdot \left\lceil \frac{t-1}{2} \right\rceil \qquad \textit{for } d \geq \min(m,n) - 1.$$

In positive characteristic, such a formula is not known beyond the special cases $t = 1$ (where I_t is the maximal homogeneous ideal), and the case $t = \min(m, n)$ of maximal minors treated in [1]. The main tool used in the proof of Theorem 1 is an explicit calculation of Ext (or local cohomology) by establishing direct relationships with cohomology of line or vector bundles on full or partial flag varieties. In characteristic zero, such cohomology calculations can then be performed using the Borel–Weil–Bott theorem, but in characteristic $p > 0$ the description of cohomology is much less understood.

In the case of symbolic powers, in [3, Chapter 10] we were able to extend the results of [16] to characteristic $p > 0$, by establishing some new vanishing results for cohomology of special line bundles on flag varieties.

Theorem 2. *If $I_t^{(d)}$ denotes the d-th symbolic power of the ideal of $t \times t$ minors of a generic $m \times n$ matrix, then*

$$\operatorname{reg}(I_t^{(d)}) = td \qquad \textit{for } d \gg 0.$$

In characteristic zero or when $t = 2$, one can take $d \geq \min(m, n) - 1$.

Beyond generic determinantal varieties, there are many examples of classical varieties (Segre or Veronese varieties, binary forms etc.) where invariants of homological nature (syzygies, local cohomology, Ext modules, Castelnuovo–Mumford regularity) admit a description relating them to cohomology on a flag variety. This motivates, from a commutative algebra standpoint, the following fundamental question.

Problem 3. *Determine the cohomology groups of line bundles on a flag variety in characteristic $p > 0$. In particular, characterize the (non)vanishing behavior of such cohomology groups.*

A characterization of (non)vanishing of cohomology is given in the case of H^0 by Kempf's vanishing theorem [9, 8], and for H^1 by Andersen [2], but general results for intermediate cohomology remain unknown. Following up on work of Liu [14] and Liu–Polo [15], we consider in [7] the case of the incidence correspondence

$$X = \{(p, H) \in \mathbb{P} \times \mathbb{P}^\vee : p \in H\},$$

which is a partial flag variety parametrizing pairs of a point in projective space and a hyperplane containing it. The line bundles on X are parametrized by $\mathbb{Z}^2$, and beyond some well-understood cases, the cohomology of such line bundles translates into cohomology for the sheaves $\mathrm{D}^d\,\mathcal{R}(e)$ on projective space, where D^d denotes the divided power functor, and $\mathcal{R} = \Omega^1_{\mathbb{P}}(1)$ is the universal sub-bundle on projective space. In [7] we characterize (non)vanishing of cohomology of line bundles on X by reducing it to a formula for Castelnuovo–Mumford regularity, this time for an appropriate sheaf on projective space.

Theorem 4. *Suppose $\mathbb{P} = \mathbb{P}^{n-1}$ is the projective $(n-1)$ space over a field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = p > 0$. For $d \geq 1$, let $q = p^r$ and $1 \leq t < p$ such that $tq \leq d < (t+1)q$. We have that the Castelnuovo–Mumford regularity of the sheaf $\mathrm{D}^d\,\mathcal{R}$ is given by*

$$\mathrm{reg}(\mathrm{D}^d\,\mathcal{R}) = (t + n - 2)q - n + 2.$$

The explicit description for cohomology of line bundles on X (or for the sheaves $\mathrm{D}^d\,\mathcal{R}(e)$ on $\mathbb{P}$) remains an open problem in general, but one which is closely related to the study of determinantal ideals. To illustrate this with an example, suppose that $\mathbb{P} = \mathbb{P}^2$, and let I denote the ideal of 2×2 minors of the generic matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

If $S = \mathbf{k}[x_1, x_2, x_3, y_1, y_2, y_3]$ then a special case of [3, Equation (10.26)] implies

$$\mathrm{Ext}^j_S(I^d/I^{d+1}, S)_{e-2d-4} \cong H^{j-2}(\mathbb{P}^2, \mathrm{D}^d\,\mathcal{R}(e-1)) \otimes_{\mathbf{k}} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e)).$$

The explicit calculations in [7, Theorem 1.9] can then be used to conclude that if $\mathrm{char}(\mathbf{k}) = 2$ then

$$\lim_{d\to\infty} \frac{\ell(\mathrm{Ext}^3_S(I^d/I^{d+1}, S))}{d^5} \quad \text{does not exist!}$$

It is still an open question to understand the existence of the related limit

$$\lim_{d\to\infty} \frac{\ell(\mathrm{Ext}^3_S(S/I^d, S))}{d^6}, \tag{1}$$

which is an instance of the higher ϵ-multiplicities from [5]. In characteristic zero the limit (1) is known to exist, as well as its natural generalization to maximal minors of a generic matrix of any size [10], [13]. With Jenny Kenkel we are investigating the behavior of $\ell(\mathrm{Ext}^3_S(S/I^d, S))$ by employing a related cohomological interpretation of Ext groups:

$$\mathrm{Ext}^3_S(S/I^d, S)_{-6-e} \cong H^0(\mathbb{P}^2, \mathrm{D}^{2d-4-e}(\mathcal{R} \oplus \mathcal{R})(d-2)).$$

We can compute the above quantities explicitly when $e = 0$, which proves that the bound in [11, Theorem 5.1] is in fact an equality, and illustrates the fractal behavior of cohomology in characteristic $p > 0$. Whether this fractal behavior that occurs in a fixed degree of Ext^3 smoothens out when considering all degrees (in order to give a well-defined limit in (1)) remains an open question for now.

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Noncommutative Singularity Theory

Michael Wemyss

(joint work with Gavin Brown)

Singularity theory (à la Arnold) seeks to classify all $f \in \mathbb{C}[\![x_1,\ldots,x_d]\!]$, up to specified isomorphism, satisfying some fixed numerical criteria, and produce theory for when classification is not possible.

There are many choices. The first is the numerical criteria. Perhaps the most famous is to consider those f which satisfy the numerical condition that

$$\#\{I \mid I \text{ proper ideal of } \mathbb{C}[\![x_1,\ldots,x_d]\!] \text{ with } f \in I^2\} < \infty.$$

It is a theorem that the only such f, after relabelling the variables $z_1, \dots z_{d-2}, x, y$, and up to formal change of variables, is one of

$$\begin{array}{lll} A_n & \mathbf{z}^2 + x^2 + y^{n+1} & n \geq 1 \\ D_n & \mathbf{z}^2 + x^2 y + y^{n-1} & n \geq 4 \\ E_6 & \mathbf{z}^2 + x^3 + y^4 & \\ E_7 & \mathbf{z}^2 + x^3 + xy^3 & \\ E_8 & \mathbf{z}^2 + x^3 + y^5 & \end{array}$$

where above and below we will write $\mathbf{z}^2 = z_1^2 + \ldots + z_{d-2}^2$.

Other rings exist, e.g. the noncommutative power series ring $\mathbb{C}\langle\langle x_1, \dots, x_d\rangle\rangle$. Asking similar classification questions in such rings is valuable in its own right, and turns out to have perhaps unexpected consequences.

To set notation, for $f \in \mathbb{C}\langle\langle x_1, \dots, x_d\rangle\rangle$ consider the Jacobi algebra

$$\mathrm{Jac}(f) = \frac{\mathbb{C}\langle\langle x_1, \dots, x_d\rangle\rangle}{(\!(\delta_1 f, \dots, \delta_d f)\!)},$$

where the denominator is the closure of the two sided ideal generated by the cyclic derivatives of f. We consider two elements f and g to be equivalent if $\mathrm{Jac}(f) \cong \mathrm{Jac}(g)$, remarking that the naive analogue of the Tjurina algebra is not well defined in this context.

It is the main theorem of [1] that the only f satisfying the numerical condition $\dim_{\mathbb{C}} \mathrm{Jac}(f) < \infty$, after relabelling the variables $z_1, \dots z_{d-2}, x, y$, is equivalent to one of

$$\begin{array}{lll} A_n & \mathbf{z}^2 + x^2 + y^n & n \geq 2 \\ D_{n,m} & \mathbf{z}^2 + x^2 y + y^{2n} + y^{2m-1} & n, m \geq 2, m \leq 2n-1 \\ D_{n,\infty} & \mathbf{z}^2 + x^2 y + y^{2n} & n \geq 2 \\ E_{6,n} & \mathbf{z}^2 + x^3 + xy^3 + y^n & n \geq 4 \\ E_7, E_8 & \mathbf{z}^2 + x^3 + \mathcal{O}_4 & \text{(various cases)} \end{array}$$

Remarkably, taking the limit $n \to \infty$ turns out to give normal forms that classify those Jacobi algebras of growth rate one, suitably interpreted[1].

There are two justifications for the naming of the ADE families. The first is intrinsic to noncommutative singularity theory, via quotients by generic central elements. The second is via birational geometry.

The applications of the above noncommutative singularity theory are to birational geometry, specifically to the classification of smooth 3-fold flops, and to divisor-to-curve contractions. The theory presented here, which is specific to the noncommutative power series ring, translates into statements regarding single irreducible curves. The multi-curve situation requires the ring $\mathbb{C}\langle\langle x_1, \dots, x_d\rangle\rangle$ to be replaced by $\mathbb{C}\langle\langle Q\rangle\rangle$, where Q is a symmetric quiver.

[1] finite dimensional algebras have growth rate zero.

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Canonical Hilbert-Burch matrices for power series

Roser Homs

(joint work with Anna-Lena Winz)

The goal of this presentation is to provide a parametrization of all deformations of a codimension 2 monomial ideal that preserve the Hilbert function based on the results of [1].

We first fix the setup and review a few basic facts about Hilbert functions of local rings. Let $R = \mathbf{k}[\![x,y]\!]$ be the ring of formal power series in two variables over a field $\mathbf{k}$ of characteristic 0, let $P = \mathbf{k}[x,y]$ be the polynomial ring and let I be a 0-dimensional ideal in R. Note that its initial ideal I^* is a homogeneous ideal in P and its leading term ideal $\mathrm{Lt}_\tau(I^*)$ with respect to some monomial term ordering is a monomial ideal, again in P. Both these deformations of the ideal preserve the Hilbert function by definition:

$$\mathrm{HF}_{R/I} := \mathrm{HF}_{P/I^*} = \mathrm{HF}_{P/\mathrm{Lt}_\tau(I^*)} = \mathrm{HF}_{P/\mathrm{Lex}(I)}.$$

Moreover, the lex-segment ideal $\mathrm{Lex}(I)$ is a special monomial ideal: it is the generic initial ideal with respect to a given ordering, see [2, Section 1.4] for a discussion on this notion in the local setting. This property implies that if we can parametrize deformations of a lex-segment ideal that preserve the Hilbert function, we are actually giving a parametrization of all ideals with a given Hilbert-function up to generic change of coordinates. We can thus weaken our original goal.

<u>Goal 1</u>: Parametrize all deformations of a lex-segment ideal that preserve the Hilbert function.

Definition 1. The monomial ideal $L = (x^t, x^{t-1}y^{m_1}, x^{t-2}y^{m_2} \dots, y^{m_t})$ is *lex-segment* iff $0 < m_1 < m_2 < \cdots < m_t$.

In [3], Rossi and Sharifan provide specific deformations of L by looking at its syzygies instead of its generators. By Hilbert-Burch, $L = I_t(H)$ is the ideal of t-minors of H, where

$$0 \longrightarrow P^t \xrightarrow{H} P^{t+1} \longrightarrow P \longrightarrow P/L \longrightarrow 0.$$

In the context of obtaining the Betti numbers of any ideal from the Betti numbers of its associated lex-segment ideal, the authors prove in [3, Remark 4.7] that there exist matrices N with entries 0 or 1 such that $I = I_t(H+N)$ realizes any admissible number of generators of $\mathrm{HF}_{P/L}$.

The main idea to achieve Goal 1 is to extend the notion of canonical Hilbert-Burch matrix from lex-segment ideals to any of its deformations. This approach had already been exploited by Conca-Valla [4] and Constantinescu [5] for the lexicographical and degree lexicographical ordering, respectively, to parametrize

$$\textit{Gröbner cells} \quad V_\tau(L) := \{I \subset P : \mathrm{Lt}_\tau(I) = L\}.$$

More generally, it follows from results on smooth varieties with a torus action (here the weight vector that represents the term ordering) by Bialynicki-Birula [6], that these Gröbner cells are affine spaces for any term ordering in codimension 2.

However, there is an obstacle in terms of the preservation of the Hilbert function: $\mathrm{Lt}_\tau(I) \neq \mathrm{Lt}_\tau(I^*)$ in general. This can be solved by considering an ordering which is compatible with the local structure:

Definition 2. A term ordering τ in P induces a reverse-degree ordering $\overline{\tau}$ in R such that for any monomials m, m' in R, $m >_{\overline{\tau}} m'$ if and only if

$$\deg(m) < \deg(m') \text{ or } \deg(m) = \deg(m') \text{ and } m >_\tau m'.$$

We call $\overline{\tau}$ the **local term ordering** induced by the global ordering τ.

Note that $\overline{\tau}$-enhanced standard basis are the local analogue of Gröbner bases in the polynomial ring; similarly, Grauert's division and the tangent cone algorithm are the analogues of Buchberger division and Buchberger's algorithm. Another key tool is the lifting of syzygies in the local case, see [2, Theorem 1.10] for a formulation that fits our notation.

Local term orderings are compatible with the local structure in the sense that $\mathrm{Lt}_{\overline{\tau}}(I) = \mathrm{Lt}_{\overline{\tau}}(I^*)$. Therefore, Gröbner cells $V_{\overline{\tau}}(L) := \{I \subset P : \mathrm{Lt}_{\overline{\tau}}(I) = L\}$ only contain ideals with Hilbert function $\mathrm{HF}_{P/L}$.

Definition 3. We denote by $T(L)$ the set of matrices $N = (n_{i,j})$ of size $(t+1) \times t$ with entries in $\mathbf{k}[y]$ such that

- $n_{i,j} = 0$ for any $i \leq j$,
- $u_{i,j} \leq \mathrm{ord}(n_{i,j}) \leq \deg(n_{i,j}) < m_j - m_{j-1}$ for any $i > j$.

Theorem 4. [1, Theorem 5.7] *Given a zero-dimensional lex-segment monomial ideal $L \subset R$ with canonical Hilbert-Burch matrix H, the map*

$$\begin{array}{rccc} \Phi_L : & T(L) & \longrightarrow & V(L) \\ & N & \longmapsto & I_t(H+N) \end{array}$$

is a bijection.

In the presentation we used as a running example the lex-segment ideal $L = (x^4, x^3y, x^2y^5, xy^8, y^{10})$ from [3, Example 4.8] to illustrate how to obtain Gröbner cells in practise. In particular, $V(L) \simeq \mathbb{A}^{20}$ and $V_{\mathrm{CI}}(L)$ is the quasi-affine variety obtained by substracting the union of 3 hyperplanes from $\mathbb{A}^{20}$.

Therefore, Theorem 4 allows us to completely characterize — up to generic change of coordinates — any ideal with a given Hilbert function and any admissible number of generators by looking into the local Gröbner cell $V(L)$ and certain special quasi-affine varieties.

However, for some tasks a parametrization up to generic change of coordinates is not enough: the computation of Gorenstein covers (see [1, Section 6]) or celullar decompositions of the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbf{k}[\![x,y]\!])$ that are compatible with the local structure require Gröbner cells $V(E)$ for any monomial ideal $E = (x^t, x^{t-1}y^{m_1}, x^{t-2}y^{m_2} \dots, y^{m_t})$, with $0 < m_1 \leq m_2 \leq \cdots \leq m_t$.

Goal 2: Parametrize all deformations of any monomial ideal that preserve the Hilbert function.

Definition 5. We denote by $T(E)$ the set of matrices $N = (n_{i,j})$ of size $(t+1)\times t$ with entries in $\mathbf{k}[y]$ such that

- $u_{i,j} + 1 \leq \mathrm{ord}(n_{i,j}) \leq \deg(n_{i,j}) < m_i - m_{i-1}$ for any $i \leq j$,
- $u_{i,j} \leq \mathrm{ord}(n_{i,j}) \leq \deg(n_{i,j}) < m_j - m_{j-1}$ for any $i > j$.

Conjecture 6. [1, Conjecture 5.14] *For any zero-dimensional monomial ideal $E \subset P$, the map*

$$\begin{array}{rccc} \Phi_E : & T'(E) & \longrightarrow & V(E) \\ & N & \longmapsto & I_t(H+N) \end{array}$$

is a bijection.

We highlight the main obstacle in the proof of the conjecture: injectivity of Φ_E. For lex-segment ideals, the reduced $\overline{\tau}$-enhanced standard basis is a lex-Gröbner basis with the same leading term ideal. This allows us to use the results for the global term ordering lex from [4]. In the case of deglex [5], injectivity was already an issue for general monomial ideals. Surjectivity is also not proven but it seems plausible that the approach by "moves" already found in [4, 5, 1], although very technical, should succeed.

Our evidence for the conjecture is twofold:

- It holds for extremal cases: lex-segment ideals (and a slighlty more general class) and complete intersections.
- The dimensions of the cells of the celullar decomposition of $\mathrm{Hilb}^n(\mathbf{k}[\![x,y]\!])$ resulting from the conjecture are correct for $n \leq 30$. More precisely, by [7] the number of cells of dimension l in any celullar decomposition of the punctual Hilbert scheme is equal to the $2l$-Betti numbers of $\mathrm{Hilb}^n(\mathbf{k}[\![x,y]\!])$, namely the OEIS sequence A058398.

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On equivalences between singularity categories of commutative rings

Martin Kalck

For simplicity, all rings are assumed to be of the form $\mathbb{C}[\![z_1, \ldots, z_n]\!]/I$.

Let R be a ring and M a finitely generated R-module. Its free resolution

$$\begin{tikzcd}[column sep=small] & & & \Omega^d(M) \arrow[dr] & & & & & & \\ \cdots \arrow[r] & F_{d+1} \arrow[r] & F_d \arrow[ur] \arrow[rrr] & & & F_{d-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0, \end{tikzcd} \tag{1}$$

$$\underbrace{\cdots \longrightarrow F_{d+1} \longrightarrow F_d}_{\text{'stable range'}} \qquad \underbrace{F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0}_{\text{'initial noise'}}$$

starts with some 'initial noise' but becomes 'stable' after finitely many steps. While the initial part mainly depends on M, the 'stable range' often reveals structural properties of the ring R. For example, for Cohen–Macaulay rings R of Krull dimension d, the Depth Lemma implies that $\Omega^d(M)$ is a maximal Cohen–Macaulay R-module. Often, (minimal) free resolutions behave more regular after this point, e.g. for the rings in items (P1) & (P2) below. Moreover, for Gorenstein rings, the syzygy functor Ω becomes invertible on maximal Cohen–Macaulay modules.

Natural categorical framework to study 'stable ranges'. (Buchweitz [1]) The following triangulated quotient category is called *singularity category* of R.

$$\underset{\circlearrowright\,[1]}{D^{sg}(R)} := \frac{\mathsf{Hot}^{-,b}(\text{proj } R)}{\mathsf{Hot}^{b}(\text{proj } R)} \cong \frac{D^b(\text{mod } R)}{\mathsf{Perf}(R)} \tag{2}$$

More precisely, it is a Verdier quotient of the homotopy category $\mathsf{Hot}^{-,b}(\text{proj } R)$ of bounded below complexes of finitely generated projective R-modules with *bounded* cohomologies by the subcategory $\mathsf{Hot}^{b}(\text{proj } R)$ of bounded complexes. The category $\mathsf{Hot}^{-,b}(\text{proj } R)$ contains, in particular, all free resolutions (1). Modulo the 'initial noise' in $\mathsf{Hot}^{b}(\text{proj } R)$, we see that $D^{sg}(R)$ captures the 'stable range'. The shift functor [1] of $D^{sg}(R)$ is an auto-equivalence induced by shifting complexes.

Question 1. *What do singularity categories know about singularities?*

We list some examples of properties that are detected by singularity categories.

- (P1) $\mathsf{Spec}(R)$ is non-singular $\Leftrightarrow$ all 'stable ranges' $= 0$ $\Leftrightarrow$ $D^{sg}(R) = 0$. [Auslander-Buchsbaum; Serre]
- (P2) $R \cong \mathbb{C}[\![z_0, \ldots, z_d]\!]/(f)$ is a hypersurface singularity $\Leftrightarrow$ all 'stable ranges' are 2-periodic: $\cdots \xrightarrow{A} F \xrightarrow{B} F \xrightarrow{A} F \xrightarrow{B} \cdots$ $\Leftrightarrow$ $[1] \circ [1] \cong \text{id}$ in $D^{sg}(R)$. [Eisenbud]
- (P3) R has Gorenstein isolated singularities $\Leftrightarrow$ $D^{sg}(R)$ is Hom-finite over $\mathbb{C}$. [Auslander; Avramov–Veliche]

Remark 2.

(1) This uses Buchweitz's equivalence [1]: $D^{sg}(R) \cong \underline{\mathsf{CM}}(R)$, translating the statements to stable categories of maximal Cohen–Macaulay modules.

(2) The 2-periodic 'stable ranges' in (P2) are induced by matrix factorizations of f.

Question 3. *How fine is the invariant $D^{sg}(R)$, i.e. when are two rings R, S* singular equivalent*: $D^{sg}(R) \cong D^{sg}(S)$?*

To explain what is known about this question, we introduce some notation.

Notation. For a primitive nth root of unity $\epsilon_n \in \mathbb{C}$ and a tuple $(a_1, \ldots, a_m) \in \mathbb{Z}^m_{>0}$, we define a cyclic subgroup of order n in $\mathsf{GL}(m, \mathbb{C})$

$$\frac{1}{n}(a_1, \ldots, a_m) = \langle \mathsf{diag}\,(\epsilon_n^{a_1}, \ldots \epsilon_n^{a_m}) \rangle \subset \mathsf{GL}(m, \mathbb{C}). \tag{3}$$

The invariant rings under the diagonal action on $\mathbb{C}[\![z_1, \ldots, z_m]\!]$ are denoted by

$$\mathbb{C}[\![z_1, \ldots, z_m]\!]^{\frac{1}{n}(a_1,\ldots,a_m)}. \tag{4}$$

Complete list of *known* singular equivalences. [1]

(E0) $D^{sg}(R) \cong 0 \cong D^{sg}(S)$ for R, S regular [Auslander–Buchsbaum; Serre].

(E1) $D^{sg}\left(\frac{\mathbb{C}[\![z_0,\ldots,z_d]\!]}{(f)}\right) \cong D^{sg}\left(\frac{\mathbb{C}[\![z_0,\ldots,z_d,x_1,\ldots,x_{2m}]\!]}{(f+x_1^2+\ldots+x_{2m}^2)}\right)$, for $0 \neq f \in \mathbb{C}[\![z_0, \ldots, z_d]\!]$. [Knörrer 1987, [10]]

(E2) $D^{sg}\left(\mathbb{C}[\![y_1, y_2]\!]^{\frac{1}{n}(1,1)}\right) \cong D^{sg}\left(\frac{\mathbb{C}[z_1,\ldots,z_{n-1}]}{(z_1,\ldots,z_{n-1})^2}\right)$ [Yang 2015 [12][2], Kawamata 2015 [9], Kalck–Karmazyn 2017 [6][3]]

(E3) $D^{sg}\left(\mathbb{C}[\![x_1, x_2, x_3]\!]^{\frac{1}{2}(1,1,1)}\right) \cong D^{sg}\left(\mathbb{C}[\![y_1, y_2]\!]^{\frac{1}{4}(1,1)}\right)$ [Kalck 2021 [3]]

Remark 4.

(a) In contrast to the equivalences in (E1) & (E2), the singular equivalence in (E3) does not preserve the parity of the Krull dimension.

(b) There are several ways to prove (E2) & (E3). One option is to use (non-commutative) resolutions of singularities & relative singularity categories building on [7, 8]. Another possibility is to show that up to taking syzygies & direct summands there are *unique* maximal Cohen-Macaulay modules M_R and M_S over the rings R and S, respectively, cf. [3]. More precisely, the modules M_R, M_S are direct summands of $\Omega(\omega_R)$ respectively $\Omega(\omega_S)$, where ω_R, ω_S are the canonical modules.

(c) If we allow S in Question 3 to be non-commutative, then there are many further examples: indeed, for every ADE-surface singularity R except E_8, we construct an uncountable family of complete Noetherian $\mathbb{C}$-algebras S_R, which are singular equivalent to R and have pairwise non-isomorphic centers, see [5].

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[1] That is, singular equivalences between commutative complete local Noetherian $\mathbb{C}$-algebras $R \ncong S$, that we could find in the literature, cf. also the older list in [11, Example 4.1.].

[2] Based on our joint works [7, 8].

[3] This is a generalization, yielding singular equivalences between explicit finite dimensional algebras $K_{n,a}$ and all cyclic quotient surface singularities $\frac{1}{n}(1, a)$. The $K_{n,a}$ are commutative iff $a = 1$ or $a = n - 1$. They appear as building blocks of derived categories of toric surfaces [13]

(d) Only the equivalences in (E0) & (E1) involve Gorenstein rings. The next result indicates, why it might be difficult to find more examples in the Gorenstein case.

Proposition 5. *Let R be a Gorenstein isolated singularity and let S be isomorphic to $\mathbb{C}[\![z_1, \ldots, z_n]\!]/I$.*

(a) If there is an equivalence of triangulated categories

$$D^{sg}(R) \cong D^{sg}(S), \tag{$\star$}$$

then S is a Gorenstein isolated singularity, by (P3) above. If, in addition, $\dim R \neq \dim S$*, then both R and S are isolated* hypersurface *singularities, cf.* [4, 14].

(b) If the $\mathbb{C}$-linear equivalence ($\star$) admits a differential graded enhancement, then ($\star$) can be realized as a Knörrer equivalence[4] *(E1), see* [4].

Remark 6.

(1) If R is a 3-dimensional isolated hypersurface singularity admitting a small resolution of singularities (cf. M. Wemyss's talk), then the condition in (b) holds by recent work of Jasso–Muro as observed by Keller, [2].

(2) Examples of singular equivalences without dg enhancement[5] are given by

$$D^{sg}(\mathbb{Z}/p^2) \cong D^{sg}\left(\frac{(\mathbb{Z}/p)[x]}{(x^2)}\right),$$

for all primes $p \neq 2$, see [15] which uses higher K-theory.

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Deeply ramified fields and independent defect

Franz-Viktor Kuhlmann

(joint work with Steven Dale Cutkosky and Anna Rzepka)

The *defect* of finite extensions of valued fields plays a crucial role in

- local uniformization, and
- the model theory of valued fields

in positive characteristic. Both can be attacked via the structure theory of valued algebraic function fields. In both, *elimination of ramification* is a crucial goal, and the defect is a main obstacle.

The defect is defined as follows. By $(L|K,v)$ we denote an extension of valued fields, where v is a valuation on L and K is endowed with its restriction. We denote by vK the value group and by Kv the residue field of (K,v). For a finite extension $(L|K,v)$ where the extension of v from K to L is unique, the *Lemma of Ostrowski* says

$$[L:K] \;=\; \tilde{p}^{\ell}\cdot(vL:vK)[Lv:Kv]$$

where ℓ is a non-negative integer and $\tilde{p}$ is the *characteristic exponent* of Kv, that is, $\tilde{p}=\operatorname{char}Kv$ if it is positive, and $\tilde{p}=1$ otherwise. The factor $d(L|K,v):=\tilde{p}^{\ell}$ is the *defect* of the extension $(L|K,v)$. We call $(L|K,v)$ a *defect extension* if $d(L|K,v)>1$. Nontrivial defect only appears when $\operatorname{char}Kv=p>0$, in which case $\tilde{p}=p$. When we say that "(K,v) does not admit defect extensions", we will actually mean that its henselization has this property (which gives us the reduction to extensions with unique extension of the valuation). Background and a large collection of examples of defect extensions can be found in [7].

Using the structure theory of valued algebraic function fields, I proved in joint work with Hagen Knaf:

- *All Abhyankar places of algebraic function fields admit local uniformization* ([3]). (*Abhyankar places* are places that satisfy equality in the *Abhyankar inequality.*)
- *Every place of an algebraic function field $F|K$, where $F|K$ is separable, admits local uniformization in a finite separable extension of F* ([4]).

While this result also follows from de Jong's resolution by alteration, the presence of the place allows us to say more about the extension of F; moreover, our proof reveals the connection with the structure theory of valued algebraic function fields and the phenomenon of the defect.

The proof of the first theorem uses the "Generalized Stability Theorem" ([5]) which states that *if $F|K$ is an algebraic function field with an Abhyankar place* *under which K does not admit defect extensions, then the same is true for F.* The proof of the second theorem uses the first theorem, applied to subfunction fields of maximal transcendence degree on which the place is Abhyankar, together with the "Henselian Rationality Theorem" ([9]). These two main tools are also used to prove model theoretic results for the classes of tame and separably tame fields ([8, 10]). These fields do not admit any defect extensions (or separable defect extensions, respectively). They play a crucial role in the proof of our local uniformization by alteration, in which defects are essentially "killed" by separable alteration.

The question arises whether the above results can be generalized by studying situations where (certain) defects are allowed. In [6] I introduced a classification of defects of Galois extensions of prime degree of valued fields of positive characteristic. Those that can be derived by some transformation from purely inseparable defect extensions are *dependent*, and the others *independent*; examples for such defects and the mentioned transformation are given in my talk. Independent defects appear to be more harmless than the former. This seems to be witnessed by Temkin's Inseparable Local Uniformization, which proves local uniformization after a finite purely inseparable extension of the function field. Only the dependent defect can be killed by purely inseparable extensions of the function field, so this indicates that independent defect can be handled without alteration. This leads to the question: what are the valued fields that admit only independent defects?

It is obvious from the classification that the perfect valued fields of positive characteristic are among the fields that we are looking for. Also the perfectoid fields of positive characteristic are such fields. But what about perfectoid fields in *mixed characteristic* where the field has characteristic 0 and its residue field has positive characteristic? In this case, the above classification does not work, as there are no nontrivial purely inseparable extensions. In joint work with Anna Rzepka ([11]), we have generalized the classification to the mixed characteristic case, and we have studied the valuation theory of *deeply ramified fields* and related classes of valued fields. They encompass the perfectoid fields, which are the complete deeply ramified fields whose value groups are archimedean, i. e., can be embedded in the reals. In positive characteristic, deeply ramified fields are those that lie dense in their perfect hull, which includes the perfect valued fields. In mixed characteristic, we consider the even larger class of *roughly deeply ramified fields* (in short: rdr fields), which are valued fields (K, v) that satisfy the following two conditions:

(DRvp) vp is not the smallest positive element in vK,

(DRvr) the homomorphism

$$\mathcal{O}_K/p\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K/p\mathcal{O}_K$$

is surjective, where $\mathcal{O}_K$ denotes the valuation ring of (K, v).

In positive characteristic, we let rdr fields coincide with deeply ramified fields.

Among other things, we show in [11] (by purely valuation theoretical proofs):

(i) rdr fields do not admit defect extensions with dependent defect,
(ii) all algebraic extensions of rdr fields are again rdr fields,

(iii) if $(L|K,v)$ is finite and (L,v) is an rdr field, then so is (K,v),

(iv) the tame fields are exactly the henselian rdr fields with p-divisible value groups that do not admit defect extensions.

It is an open question whether also the converse of (i) holds, i. e., whether every valued field that admits only independent defects is roughly deeply ramified.

We denote by K^{sep} the separable-algebraic closure of K and extend v from K to K^{sep}. By (K^r,v) we denote the ramification field of the extension $(K^{\mathrm{sep}}|K,v)$. In [11], we also prove:

(v) (K,v) is an rdr field if and only if (K^r,v) is (in which case (K^r,v) is even deeply ramified).

In their book [2], Gabber and Ramero present the condition

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} \;=\; 0\,, \tag{1}$$

where $\Omega_{B|A}$ denotes the module of relative differentials when A is a ring and B is an A-algebra. They show that it is equivalent to property (DRvr) together with

(DRvg) if $\Gamma_1 \subset \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete).

Note that (DRvg) implies (DRvp).

If (K,v) satisfies condition (1), then they call it a deeply ramified field. The proof in [2] of the equivalence is quite involved (to say the least). The goal of recent joint work with Steven Dale Cutkosky is to give a down-to-earth alternative to this proof. In [1], we have already succeeded to show that given a Galois defect extension $(L|K,v)$ of prime degree p, the following assertions are equivalent:

a) the extension has independent defect,

b) $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$,

c) $I^p=I$, where $I \;=\; \left(\dfrac{\sigma b-b}{b}\;\middle|\; b\in L^\times\right)$ for any generator σ of $\mathrm{Gal}\,L|K$ is the ramification ideal of the extension.

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Blowup algebras of determinantal ideals in prime characteristic

Alessandro De Stefani

(joint work with Jonathan Montaño and Luis Núñez-Betancourt)

A Noetherian F-finite ring R of prime characteristic $p > 0$ is said to be F-pure if the Frobenius homomorphism splits as a map of R-modules. Equivalently, there exists $\varphi \in \operatorname{Hom}_R(R^{1/p}, R)$ such that $\varphi(1) = 1$, where $R^{1/p}$ denotes R viewed as a module over itself via Frobenius.

Let $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ be a filtration, i. e. $I_0 = R$, $I_1 \subseteq R$ is a proper ideal, $I_{n+1} \subseteq I_n$ for all $n \geq 0$ and $I_m \cdot I_m \subseteq I_{m+n}$ for all $m, n \geq 0$. Consider the blowup algebras

$$\mathcal{R}(\mathcal{I}) = \bigoplus_{n\geq 0} I_n T^n \subseteq R[T] \quad \text{and} \quad \operatorname{gr}(\mathcal{I}) = \bigoplus_{n\geq 0} I_n/I_{n+1}$$

associated to $\mathcal{I}$. The main goal it to study F-singularities and, in particular, F-purity, of blowup algebras associated to some interesting filtrations. To this end, we make the following definition.

Definition 1. Let $\varphi \in \operatorname{Hom}_R(R^{1/p}, R)$ be a map such that $\varphi(1) = 1$. We say that $\mathcal{I} = \{I_n\}$ is F-pure (with respect to φ) if $\varphi(I_{np+1}^{1/p}) \subseteq I_{n+1}$ for all $n \geq 0$.

If $\mathcal{I}$ is F-pure with respect to $\varphi \in \operatorname{Hom}_R(R^{1/p}, R)$, then φ induces maps $\varphi_{\mathcal{R}} \in \operatorname{Hom}_{\mathcal{R}(\mathcal{I})}(\mathcal{R}(\mathcal{I})^{1/p}, \mathcal{R}(\mathcal{I}))$ and $\varphi_{\mathrm{gr}} \in \operatorname{Hom}_{\operatorname{gr}(\mathcal{I})}(\operatorname{gr}(\mathcal{I})^{1/p}, \operatorname{gr}(\mathcal{I}))$ both sending $1 \mapsto 1$. It follows that blowup algebras associated to F-pure filtrations are F-pure.

We observe that if one is only interested in F-purity of $\mathcal{R}(\mathcal{I})$, then it suffices to require that $\varphi(I_{np}^{1/p}) \subseteq I_n$ for all $n \geq 1$. Moreover, if $\psi \in \operatorname{Hom}_R(R^{1/p}, R)$ is a surjective map such that $\psi(I_{np+1}^{1/p}) \subseteq I_{n+1}$ for all $n \geq 0$, then $\mathcal{I}$ is F-pure with respect to a map φ defined as follows: there exists $c \in R$ such that $\psi(c^{1/p}) = 1$; set $\varphi(-) = \psi(c^{1/p} \cdot -)$.

One of our main results about general F-pure filtrations is the following.

Theorem 2. [2, Theorem 4.10] *Let R be either local or a non-negatively graded $R_0 = K$-algebra, where K is a field. If $\mathcal{I} = \{I_n\}$ is an F-pure filtration, then*

(1) $\lim_{n\to\infty} \operatorname{depth}(I_n) = \min\{\operatorname{depth}(I_n)\}$.

(2) In the graded setup, $\lim_{n\to\infty} \frac{\operatorname{reg}(I_n)}{n}$ *exists.*

Symbolic F-purity. From now on, let K be a perfect field, $S = K[x_1, \dots, x_n]$ with the standard grading, and let $\mathfrak{m} = (x_1, \dots, x_n)$. Recall that if $I \subseteq S$ is homogeneous and $R = S/I$, then any map $\varphi \in \mathrm{Hom}_R(R^{1/p}, R)$ can be described very explicitly thanks to the work of Fedder [3], based on Kunz's Theorem [6]. First of all, such a map corresponds to $\varphi \in \mathrm{Hom}_S(S^{1/p}, S)$ such that $\varphi(I^{1/p}) \subseteq I$. Since S is Gorenstein, we have that $\mathrm{Hom}_S(S^{1/p}, S) \cong \omega_{S^{1/p}} \cong S^{1/p}$ as $S^{1/p}$-modules, and we can explicitly identify a generator Φ, called trace map. In particular, given φ as above there exists $f \in S$ such that $\varphi(-) = \Phi(f^{1/p} \cdot -)$, and because $S^{1/p}$ is a free S-module the condition that $\varphi(I^{1/p}) \subseteq I$ translates into $fI \subseteq I^{[p]}$. It follows that to each map φ as above corresponds an element $f \in I^{[p]} : I$, and such an element is unique modulo $I^{[p]}$. From now on we will assume that φ is graded, and then f can be chosen to be homogeneous. The same principle used above finally shows that φ is surjective if and only if $f \notin \mathfrak{m}^{[p]}$.

With this correspondence at hand, we have the following:

Lemma 3. *Given a filtration $\mathcal{I} = \{I_n\}$ of homogeneous ideals of S we have that $\mathcal{I}$ is F-pure if and only if $\bigcap_{n \geq 0} \left(I_{n+1}^{[p]} : I_{np+1}\right) \not\subseteq \mathfrak{m}^{[p]}$.*

We will now focus on the case $\mathcal{I} = \{I^{(n)}\}$, where I is a radical ideal of pure height h. If such a filtration is F-pure we also simply say that I is symbolic F-pure. Since $I^{(h(p-1))} \subseteq (I^{(n+1)})^{[p]} : I^{(np+1)}$ for all $n \geq 0$ [5, 4], in order to show F-purity of $\mathcal{I}$ it suffices to find $f \in I^{(h)}$ such that $\mathrm{in}_<(f)$ is square free for some monomial order $<$ of S; in fact, in this case $f^{p-1} \in I^{(h(p-1))} \setminus \mathfrak{m}^{[p]}$.

We show how this can be done for determinantal ideals.

Determinantal ideals. Let X be an $m \times n$ generic matrix with $m \leq n$, and $t \leq m$ be an integer. Let I_t be the ideal generated by the t-minors of X inside $S = K[X]$, where K is a perfect field of characteristic $p > 0$. For $1 \leq i < j \leq m$ and $1 \leq a < b \leq n$ we let $X_{[a,b]}^{[i,j]}$ be the submatrix of X obtained by taking the rows $i, i+1, \dots, j$ and columns $a, a+1, \dots, b$ of X. Consider the polynomial:

$$f_t = \prod_{\ell=t}^{m-1} \left(\det\left(X_{[1,\ell]}^{[1,\ell]}\right) \det\left(X_{[n-\ell+1,n]}^{[m-\ell+1,m]}\right)\right) \cdot \prod_{\ell=1}^{n-m+1} \det\left(X_{[\ell,m+\ell-1]}^{[1,m]}\right).$$

If $<$ is any anti-diagonal order on X (e.g. LEX with $x_{1,n} > x_{1,n-1} > \dots > x_{1,1} > x_{2,n} > x_{2,n-1} > \dots > x_{m,1}$), then $\mathrm{in}_<(f_t)$ is a square-free monomial. Furthermore, using that $I_j \subseteq I_i^{(j-i+1)}$ for all $1 \leq i \leq j$, we have that $f_t \in I_t^{(h)}$, where $h = (m-t+1)(n-t+1)$ is the height of the prime ideal I_t. As an immediate consequence, the symbolic blowup algebras $\mathcal{R}^s(I_t)$ and $\mathrm{gr}^s(I_t)$ are F-pure. Moreover, one can show that also the ordinary Rees algebra $\mathcal{R}(I_t)$ is F-pure if $p > \min\{t, m-t\}$. This is because we just showed that I_j is symbolic F-pure for every $1 \leq j \leq m$, and because of a primary decomposition for ordinary powers $I_t^n = \bigcap_{1 \leq \ell \leq t} I_\ell^{(n(t-\ell+1))}$, which holds for $p > \min\{t, m-t\}$ [1].

Finally, set $\Delta = \det(X_{[1,t-1]}^{[1,t-1]})$. Since $\mathrm{in}_<(f_t)$ does not involve any of the variables of $\mathrm{in}_<(\Delta)$, one can see that there exists a monomial $u \in S$ such that the map $\psi(-) = \Phi((f_t^{p-1}u)^{1/p} \cdot -)$ sends $\Delta^{1/p} \mapsto 1$. Using that $(S/I_t)_\Delta$ is regular, one gets that both $\mathcal{R}^s(I_t)_\Delta$ and $\mathrm{gr}^s(I_t)_\Delta$ are strongly F-regular. As a consequence, we obtain our main theorem concerning symbolic blowup algebras of ideals of minors.

Theorem 4. [2, Theorem 6.7] *The symbolic blowup algebras $\mathcal{R}^s(I_t)$ and $\mathrm{gr}^s(I_t)$ are strongly F-regular.*

In [2] we obtain some further results for filtrations associated to initial ideals of symbolic and ordinary powers, as well as for filtrations associated to other determinantal objects such as Pfaffians of a generic skew-symmetric matrix, minors of a generic symmetric matrix and of a generic Hankel matrix.

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Free resolutions of perfect ideals

Jerzy Weyman

This is the report on the ongoing projects on the finite free resolutions of perfect ideals of codimension 3 and Gorenstein ideals of codimension 4. They are a result of collaborations with many persons, notably Ela Celikbas, Lars Christensen, Lorenzo Guerrieri, Jai Laxmi, Xianglong Ni and Oana Veliche.

Perfect ideals of codimension 3. It was established in [15] that the structure of free resolutions of length 3 is related to the root systems of type $T_{p,q,r}$ where $(p, q, r) = (r_1 + 1, r_2 - 1, r_3 + 1)$, where r_i denotes the rank of the differential d_i in our free resolution for $i = 1, 2, 3$. This means that the formats corresponding to finite root systems of type ADE play special role. It turns out that for perfect ideals the connection goes much deeper. In particular one has

Conjecture 1 (LICCI Conjecture, [6]). *Every perfect ideal of I in a local ring R codimension 3 with R/I having a resolution of ADE format is LICCI (i. e. in the linkage class of a complete intersection).*

Recall that ADE types are the resolutions of formats

$$D_n : (1, n, n, 1) \text{ and } (1, 4, n, n-3),$$
$$E_6 : (1, 5, 6, 2),$$
$$E_7 : (1, 5, 7, 3) \text{ and } (1, 6, 7, 2),$$
$$E_8 : (1, 5, 8, 4) \text{ and } (1, 7, 8, 2).$$

In dealing with the LICCI Conjecture, it turns out it is related to the generators of generic ring $\hat{R}_{\text{gen}}$ established in [15]. These generators correspond to three irreducible representations

$$W(d_3) = F_2^* \otimes V(\omega_z), W(d_2) = F_2 \otimes V(\omega_y), W(d_1) = \mathbf{C} \otimes V(\omega_x)$$

where $V(\omega_x), V(\omega_y), V(\omega_z)$ are three irreducible fundamental representations of $\underline{g}(T_{p,q,r})$ corresponding to three extremal nodes at the end of three arms. It turns out that the LICCI property is related to the appearance of units in the entries of higher structure maps corresponding to homogeneous components of these representations. This is true for the known type D_n and we can establish this in [7] for the smallest unknown type $(1, 5, 6, 2)$ for E_6. This is done by analyzing the higher structure maps for the linked ideal, extending in this case the procedure from [1]. In general we expect

Conjecture 2. *Let I be a perfect ideal of codimension 3 over a local Gorenstein ring R. Then I is LICCI if and only if there exists a higher structure map in the representation $W(d_1)$ whose entry is a unit. Similarly, the existence of a unit in $W(d_2)$ and $W(d_3)$ should be equivalent to the fact that I can be linked to an ideal I' with a resolution of a smaller format.*

Moreover, one can construct a uniform family of generic examples of perfect ideals I with R/I having resolutions of ADE formats which are expected to provide generic models for these resolutions. Such resolutions can be viewed from three angles:

(1) From bigradings of Lie algebras of type $T_{p,q,r}$, with uniform description of all differentials, see [10],
(2) As defining ideals of affine pieces of Schubert varieties in certain homogeneous spaces of groups of type $T_{p,q,r}$, see [12],
(3) As partial Jacobian ideals of sporadic invariants of prehomogeneous representations from Sato-Kimura list, see [12], in this case the $SL(6)$-invariant Δ_4 of degree 4 in $\mathrm{Sym}(\bigwedge^3 \mathbf{C}^6)$, the $SL(7)$-invariant Δ_7 of degree 7 in $\mathrm{Sym}(\bigwedge^3 \mathbf{C}^7)$, and the $SL(8)$-invariant Δ_{16} of degree 16 in $\mathrm{Sym}(\bigwedge^3 \mathbf{C}^8)$.

In the case of root systems of type D_n one recovers the Buchsbaum-Eisenbud generic Gorenstein ideals of codimension 3 [4] and the almost complete intersections of codimension 3 constructed by Anne Brown [2].

The most recent progress is that we can prove (with L. Guerrieri and X. Ni)

Theorem 3. *Let I be a perfect ideal of codimension 3 over a local Gorenstein ring R, with the resolution of R/I having ADE format. Then*

- *Conjecture 2 is true,*
- *If I is LICCI, then (R, I) is a specialization of the corresponding generic example described above.*

The conjecture 2 seems to be within reach, one just has to be sure that manipulation of units works for Kac-Moody groups.

Gorenstein ideals of codimension 4. Similar theory for Gorenstein ideals of codimension 4 was started in [16]. For the ideals with n minimal generators the role of the root system $T_{p,q,r}$ is played by the root system E_n. Starting with resolution of the type

$$0 \to R \xrightarrow{d_4} F \otimes R \xrightarrow{d_3} G \otimes R \cong G^* \otimes R \xrightarrow{d_3^{tr}} F^* \otimes R \xrightarrow{d_4^{tr}} R,$$

where R is a $\mathbf{C}$-algebra, $F = \mathbf{C}^n$ and $G = \mathbf{C}^{2n-2}$ is an orthogonal module with a nondegenerate quadratic form (assumed to be in hyperbolic form), one first construct "the generic complex" over a ring $A(n)_1$ of this type and then introduces the cycle killing procedure to construct a generic ring $A(n)_\infty$ akin to $\hat{R}_{\mathrm{gen}}$ from [15]. There are, however, two nonobvious conditions. First, one has to introduce from the beginning the spinor coordinates of the isotropic subspace $\mathrm{Im}(d_3)$ (see [5]). Second (and this was the least obvious), one treats the above resolution as a complex of length 3, dropping d_4^{tr}. Then one is able to lift cycles and construct a generic object $A(n)_\infty$ which is a multiplicity free $\underline{sl}(F) \times \underline{g}(E_n)$ representation. The ring $A(n)_\infty$ is Noetherian if and only if $n \leq 8$, suggesting that for $n \geq 9$ the problem of classifying the Gorenstein ideals of codimension 4 might be very complicated. Still, one gets many benefits from this theory. The first immediate application is

Theorem 4. *The spinor coordinates of the module $Im(d_3)$ in the free resolution of the module R/J where J is a Gorenstein ideals of codimension 4 are in J.*

Moreover, one gets very interesting examples of Gorenstein ideals with n generators for $4 \leq n \leq 8$. They are obtained by uniform construction and are candidates for generic types in these cases. For $n = 4$ one gets a complete intersection on 4 independent variables. For $n = 5$ one gets a nonminimal resolution which is a complete intersection in 4 variables plus a split summand. For $n = 6$ one gets a generic hyperplane section in a generic Gorenstien ideal of codimension 3. For $n = 7, 8$ one gets two most interesting new examples. The corresponding resolutions have similar properties to the ones constructed for perfect ideals of codimension 3. They can be viewed from three angles:

(1) From bigradings of Lie algebras of type E_n, with uniform description of all differentials, see [10],
(2) As defining ideals of affine pieces of Schubert varieties in certain homogeneous spaces of groups of type E_n, see [16],
(3) As partial Jacobian ideals of sporadic invariants from Sato-Kimura list, see [16], [7] in this case the invariants in half-spinor representations: Spin(12)-invariant Θ_4 of degree 4 in $\mathrm{Sym}(V(\omega_5, D_6))$, and the Spin(14)-invariant Θ_8 of degree 8 in $\mathrm{Sym}(V(\omega_6, D_7))$.

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Catalan numbers and noncommutative Hilbert schemes

Špela Špenko

(joint work with Valery Lunts and Michel Van den Bergh)

We first state some purely combinatorial results which give a new interpretation of parking functions and Fuss-Catalan numbers [9, 10] in terms of lattice points in a certain polytope related to the permutahedron. Afterwards we give the motivation behind these results.

Polytopes and Catalan numbers. Let $m, n \in \mathbb{N}$ and let $(e_i)_{i=1,\dots,n}$ be the standard basis for $\mathbb{Z}^n \subset \mathbb{R}^n$. We let the symmetric group S_n act on $\mathbb{Z}^n$ and $\mathbb{R}^n$

by permutations. Let $\Delta^{m,n}$ be the S_n-invariant zonotope which is the Minkowski sum of the intervals

$$[0,e_i]\,,\ 1\le i\le n,\qquad \left[0,\frac{m}{2}(e_i-e_j)\right],\ 1\le i\ne j\le n.$$

For $\nu := \sum_{i=1}^n e_i$ and $\tau\in\mathbb{R}$ put $\Delta^{m,n}_\tau = \Delta^{m,n}+\tau\nu$.

Proposition 1. *Assume that $\tau - m(n-1)/2$ is not a rational number with denominator $\le n$.*[1] *Let $L = (mn+1)\mathbb{Z}^n + \mathbb{Z}\nu$. Then*

$$\Delta^{m,n}_\tau \cap \mathbb{Z}^n \to \mathbb{Z}^n/L : a\mapsto \bar{a}$$

is an S_n-equivariant bijection.

This result follows very quickly from the fact that $\Delta^{m,n}_\tau$ is equivalent, in a suitable sense, to the permutahedron and hence is space tiling.

Proposition 1 allows one to relate the lattice points in $\Delta^{m,n}_\tau$ for admissible τ to *parking functions*. Recall that an (m,n)-parking function is a sequence of natural numbers[2] $a=(a_1,\ldots,a_n)\in\mathbb{N}^n$ such that its weakly increasing rearrangement $a_{i_1}\le a_{i_2}\le\cdots\le a_{i_n}$ satisfies $a_{i_j}\le m(j-1)$. Note that S_n acts on parking functions by permuting indices. Below we denote the set of (m,n)-parking functions by $\mathcal{Q}^m$. According to [8, NOTE in §3] or [1, §5.1] the map

$$\mathcal{Q}^m \to \mathbb{Z}^n/L : a\mapsto \bar{a}$$

is an S_n-equivariant bijection. Combining it with Proposition 1 yields the following corollary.

Corollary 2. *Assume that $\tau - m(n-1)/2$ is not a rational number with denominator $\le n$. Then there is an explicit S_n-equivariant bijection between lattice points in $\Delta^{m,n}_\tau$ and (m,n)-parking functions.*

If $a=(a_1,\ldots,a_n)\in\mathbb{N}^n$ is weakly increasing then we say that a is an (m,n)-*Dyck path* if $a_j\le(m-1)(j-1)$. The number of (m,n)-Dyck paths is

$$A_n(m,1) := \frac{1}{mn+1}\binom{mn+1}{n} = \frac{1}{(m-1)n+1}\binom{mn}{n}$$

and is called the (m,n)-*Fuss-Catalan number*.

There is a bijection between regular orbits of (m,n)-parking functions and (m,n)-Dyck paths which sends the orbit representative $a=(a_1,\ldots,a_n)\in\mathbb{N}^n$ with $a_1<\ldots<a_n$ to $(a_1,a_2-1,\ldots,a_n-(n-1))$.

We thus obtain a new interpretation of the Fuss-Catalan numbers.

Corollary 3. *Assume that $\tau - m(n-1)/2$ is not a rational number with denominator $\le n$. There is an explicit S_n-equivariant bijection between regular S_n-orbits in $\Delta^{m,n}_\tau\cap\mathbb{Z}^n$ and (m,n)-Dyck paths. In particular the number of such regular orbits is $A_n(m,1)$.*

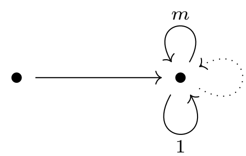
[1] This condition is equivalent to $\partial(\Delta^{m,n}_\tau)\cap\mathbb{Z}^n=\emptyset$.

[2] We assume $0\in\mathbb{N}$.

The noncommutative Hilbert scheme. The Hilbert scheme of length n sheaves on $\mathbb{A}^m$ may be viewed as the moduli space of cyclic modules of dimension n over the polynomial algebra $\mathbb{C}[x_1, \ldots, x_m]$. It is then natural to define the corresponding *noncommutative Hilbert scheme* $H_{m,n}$ as the moduli space of cyclic modules of dimension n over the free algebra $\mathbb{C}\langle x_1, \ldots, x_m\rangle$. While the Hilbert scheme is in general very singular, the noncommutative Hilbert scheme is smooth. Moreover, it has an affine stratification.

Proposition 4 ([6, 11])**.** *$H_{m,n}$ has a stratification consisting of affine spaces and the number of strata is given by the Fuss-Catalan number $A_n(m, 1)$.*

Moreover, $H_{m,n}$ can be described as the moduli space of stable (or equivalently semi-stable) representations with dimension vector $(1, n)$ and stability condition $(-n, 1)$ [3, Definition 1.1] of the following quiver $Q_{m,n}$:



It follows from loc. cit. that $H_{m,n}$ can also be described as a GIT quotient for the group $(\mathbb{C}^* \times \mathrm{GL}_n(\mathbb{C}))/\{\text{center}\} \cong \mathrm{GL}_n(\mathbb{C})$. More precisely we get $H_{m,n} = W^{ss,\chi}/G$ where $G = \mathrm{GL}_n(\mathbb{C})$, $W = \mathrm{End}(\mathbb{C}^n)^{\oplus m} \oplus \mathbb{C}^n$ and $W^{ss,\chi} \subset W$ is the semi-stable locus associated to the determinant character χ.

Using the GIT description $H_{m,n}$ we show using [2, 7] that $H_{m,n}$ admits a family of tilting bundles. Let $\Delta_\tau^{m,n} \subset \mathbb{R}^n$ be as previously defined. We identify $\mathbb{Z}^n$ with the character group of the diagonal torus $(\mathbb{C}^*)^n$ in $\mathrm{GL}_n(\mathbb{C})$. Let $(\mathbb{Z}^n)^+$ be the "dominant" part of $\mathbb{Z}^n$, i.e. those $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $a_1 \leq \cdots \leq a_n$. For $\xi \in (\mathbb{Z}^n)^+$ let $V(\xi)$ be the irreducible $\mathrm{GL}_n(\mathbb{C})$ representation with highest weight ξ and let $\mathcal{V}(\xi)$ be the equivariant vector bundle on $H_{m,n}$ corresponding to the $\mathrm{GL}_n(\mathbb{C})$-equivariant vector bundle $V(\xi) \otimes_k \mathcal{O}_{W^{ss,\chi}}$ on $\mathcal{O}_{W^{ss,\chi}}$. Put

$$\hat{\rho} = \frac{1}{2}\sum_{i>j}(e_i - e_j) + \frac{1}{2}(n-1)\nu = (0, 1, \ldots, n-2, n-1).$$

Proposition 5. [4] *Assume that $\tau - m(n-1)/2$ is not a rational number with denominator $\leq n$. Then*

$$\mathcal{T}_\tau := \bigoplus_{\xi \in (\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})} \mathcal{V}(\xi)$$

is a tilting bundle on $H_{m,n}$.

Comparing the ranks of $K_0(H_{m,n})$ obtained from Propositions 4 and 5 yields the identity

$$|(\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})| = A_n(m, 1).$$

Sending $a \mapsto a - \hat{\rho}$ defines a bijection between the regular orbits in $\mathbb{Z}^n \cap \Delta_\tau^{m,n}$ and $(\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})$. This yields a "geometric" proof of the claim about $A_n(m, 1)$ in Corollary 3.

Further directions. In [4] we further show that the derived category of the noncommutative Hilbert scheme admits a semi-orthogonal decomposition.

Pădurariu and Toda [5] construct a finer decomposition for $H_{3,n}$. Moreover, they construct a semi-orthogonal decomposition of the category of matrix factorisations on $H_{3,n}$ with a super-potential whose critical locus is the (classical) Hilbert scheme. These provide categorifications of Donaldson-Thomas invariants.

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On the Weak Implies Strong Conjecture

Thomas Polstra

(joint work with Ian Aberbach and Craig Huneke)

An introduction to the weak implies strong conjecture. Let $(R, \mathfrak{m}, k)$ be an excellent Cohen-Macaulay normal domain of prime characteristic $p > 0$. For each natural number e let $F^e(-) : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ be the base change functor along the eth iterate of the Frobenius $F^e : R \to R$. If $N \subseteq M$ are R-modules then an element $\eta \in M$ belongs to N^*_M, the tight closure of N in M, if there exists a $0 \neq c \in R$ such that η belongs to the kernel of the following composition of maps for all $e \gg 0$:

$$M \to M/N \to F^e(M/N) \xrightarrow{\cdot c} F^e(M/N).$$

The finitistic tight closure of $N \subseteq M$ is denoted by N^{*fg}_M and is the union of $(N \cap M')^*_{M'}$ where M' runs through all finitely generated submodules of M. For example, if $\mathfrak{a} \subseteq R$ is an ideal then $F^e(R/\mathfrak{a}) = R/\mathfrak{a}^{[p^e]}$ and therefore $x \in \mathfrak{a}^*_R$ if there exists $0 \neq c \in R$ such that $cx^{p^e} \in \mathfrak{a}^{[p^e]}$ for all $e \gg 0$. Tight closure was introduced and developed by Hochster and Huneke in the late 80's and early 90's. See [9] for an introduction of tight closure and its applications.

A ring R is said to be *weakly F-regular* if $N_M^{*fg} = N$ for all modules $N \subseteq M$, *F-regular* if R_P is weakly F-regular for all $P \in \operatorname{Spec}(R)$, and strongly F-regular if $N_M^* = N$ for all modules $N \subseteq M$. Direct summands of regular rings are strongly F-regular. It is conjectured that all three notions of F-regularity coincide.

Conjecture 1 (The weak implies strong conjecture). *If R is an excellent weakly F-regular ring of prime characteristic $p > 0$ then R is strongly F-regular.*

Weak and strong F-regularity can be determined by studying the behavior the 0-submodule of the local cohomology module $H_{\mathfrak{m}}^{\dim(R)}(\omega_R)$.

Theorem 2 ([9],[16]). *Let $(R, \mathfrak{m}, k)$ be an excellent normal domain of prime characteristic $p > 0$ and of Krull dimension d. Let ω_R be a canonical module of R. The ring R is weakly F-regular if and only if $0^{*fg}_{H^d_{\mathfrak{m}}(\omega_R)} = 0$ and R is strongly F-regular if and only if $0^*_{H^d_{\mathfrak{m}}(\omega_R)} = 0$.*

History and progress on the weak implies strong conjecture. There has been incremental progress on the weak implies strong conjecture since the inception of tight closure theory.

- Conjecture 1 was settled for the class of Gorenstein rings by Hochster and Huneke, [10].
- Williams settled the weak implies strong conjecture for rings of dimension at most 3, [17]. Williams methodology is an interplay of commutative algebra techniques and the theory of the birational geometry of surfaces, namely resolutions of singularities and the theory of rational/F-rational singularities, [13, 12, 15].
- Murthy equated the classes of weakly F-regular rings and F-regular rings for finite type algebras over an uncountable field. See [11].
- If the anti-canonical algebra of R is Noetherian and R is weakly F-regular, or more generally a splinter, then R is strongly F-regular [5, 14].
- Every F-regular ring of dimension at most 4 and of prime characteristic $p > 5$ is strongly F-regular by [2]. The methodology requires the theory of birational geometry of 3-folds, namely resolution of singularities and results of the prime/mixed characteristic minimal model program in dimension 3, [6, 7, 3].

Main results and an inductive program. A novel insight to Conjecture 1 is that the conjecture can be solved by establishing annihilation properties of local cohomology modules of quotients of R by symbolic powers of an anti-canonical ideal.

Theorem 3 (Aberbach-Huneke-Polstra, [1]). *Let $(R, \mathfrak{m}, k)$ be an excellent Cohen-Macaulay normal domain of prime characteristic $p > 0$, of Krull dimension d, and*

$I \subseteq R$ an anti-canonical ideal. Suppose that there exists $m \geq 1$ so that for each $1 \leq j \leq d-2$ there exists an ideal $\mathfrak{a}_j$ of height $d-j+1$ such that

$$\mathfrak{a}_j^{p^e} H^j_{\mathfrak{m}}\left(\frac{R}{I^{(mp^e)}}\right) = 0$$

for every $e \in \mathbb{N}$. If R is weakly F-regular then R is strongly F-regular.

The modules $H^j_{\mathfrak{m}}(R/I^{(mp^e)})$ are annihilated by an ideal of height $d-j+1$. The criterion of Theorem 3 is therefore reasonable as it is natural to anticipate that the annihilators of $H^j_{\mathfrak{m}}(R/I^{(mp^e)})$ are of linear comparisons as $e \to \infty$. We present a characteristic-free problem in commutative algebra that if solved would settle the weak implies strong conjecture:

Open Problem: Let $(R, \mathfrak{m}, k)$ be a complete Cohen-Macaulay normal domain of arbitrary characteristic. Let $I \subseteq R$ be an ideal of pure height h. For each $1 \leq j \leq d-h-1$ does there exist an ideal $\mathfrak{a}_j$ of height $d-j+1$ such that

$$\mathfrak{a}_j^n H^j_{\mathfrak{m}}(R/I^{(n)}) = 0$$

for every $n \in \mathbb{N}$? It suffices to solve the open problem for ideals of pure height 1 in a weakly F-regular ring to settle Conjecture 1.

Properties described in Theorem 3 can be established under mild hypotheses.

Theorem 4 (Aberbach-Huneke-Polstra, [1])**.** *Let $(R, \mathfrak{m}, k)$ be an excellent weakly F-regular ring of prime characteristic $p > 0$, of Krull dimension d, and $I \subseteq R$ an anti-canonical ideal. Suppose that the anti-canonical algebra of R is Noetherian on the punctured spectrum. There exists $m \in \mathbb{N}$ so that for each $1 \leq j \leq d-2$ there exists an ideal $\mathfrak{a}_j$ of height $d-j+1$ such that*

$$\mathfrak{a}_j^{p^e} H^j_{\mathfrak{m}}\left(\frac{R}{I^{(mp^e)}}\right) = 0$$

for every $e \in \mathbb{N}$. In particular, the ring R is strongly F-regular by Theorem 3.

Results of the complex minimal program and methods of reduction to prime characteristic, e.g. [4, 8, 15], inspire the following conjecture.

Conjecture 5. *If $(R, \mathfrak{m}, k)$ is a strongly F-regular ring then the anti-canonical algebra of R is Noetherian.*

Conjecture 5 has been established for all rings of dimension 2, [12, 13, 15], as well as 3-dimensional F-regular rings of characteristic $p > 5$.

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Bernstein's inequality for certain singular rings

Jack Jeffries

(joint work with Josep Àlvarez Montaner, Daniel Hernández, Luiz Núñez-Betancourt, Pedro Teixeira, Emily Witt, David Lieberman)

Let $S = \mathbb{C}[x_1, \ldots, x_n]$. The ring of differential operators D_S is the ring generated by S and the partial derivatives with respect to the variables. The ring S, its principal localizations S_f, and local cohomology modules $H^i_I(S)$ are all naturally left modules over D_S, and the action of D_S on these modules has a range of applications and connections to commutative algebra. For example:

(1) This action is applied by Lyubeznik [4] to show that every local cohomology module $H^i_I(S)$, though usually not finitely generated as an S-module, nonetheless has finitely many associated primes as an S-module.

(2) Bernstein [3] and Sato [6] show that for every element $f \in S$, there is an operator $\delta(s) \in D_S[s]$ and a nonzero complex polynomial $s \in \mathbb{C}[s]$ such that $\delta(t) \cdot f^{t+1} = b(t) f^t$ for all $t \in \mathbb{N}$; the minimal such $b(s)$ is called the Bernstein-Sato polynomial of f, and its roots are intricately tied to the

singularities of $V(f)$, e.g., in terms of jumping numbers and multiplier ideals.

Both applications follow somewhat easily from the following fact.

Theorem 1 (Bernstein's inequality [3]). *Every nonzero D_S-module has dimension at least* $\dim(S)$.

Both of the consequences above are known to hold for various additional classes of rings, but to fail in general; for example, see [1, 7, 8]. The new results discussed in this talk provide a version of Bernstein's inequality for certain singular rings that recovers the consequences above.

For the remainder of the abstract,

- K denotes a field of arbitrary characteristic, and
- R denotes an $\mathbb{N}$-graded finitely generated K-algebra with $R_0 = K$.

Our notion of Bernstein's inequality is for the ring of K-linear differential operators, denoted D_R. This is a graded K-algebra, but is not necessarily finitely generated or Noetherian, and we avoid any specific finite generation hypotheses in our investigations. Instead, to obtain a meaningful notion of dimension we use *generalized Bernstein filtrations*

$$B^i = K\{\delta \text{ homogeneous} \mid \deg(\delta) + w\mathrm{ord}(\delta) \leq i\},$$

for some w greater than the generation degree of R as a K-algebra. For a D_R-module M, equipped with a $B^\bullet$-compatible filtration $F^\bullet$, we define

$$\dim(M, F^\bullet) = \inf\{\lambda \mid \dim_K(F^i) = o(\lambda^i)\} \text{ and } e(M, F^\bullet) = \limsup_{i\to\infty} \frac{\dim_K(F^i)}{i^{\dim(M,F^\bullet)}}.$$

We then have the following.

Theorem 2. *Let R be either*

(0) An invariant ring of a polynomial ring under the action of a finite group in characteristic zero,
(p) A graded strongly F-regular ring of finite F-representation type in positive characteristic, or
(#) The coordinate ring of a Segre product $\mathbb{P}^m \times \mathbb{P}^n$ in characteristic zero.

Then, for any D_R-module M and any filtration $F^\bullet$ compatible with $B^\bullet$, one has

(1) $\dim(M, F^\bullet) \geq \dim(R)$ *and when equality holds* $e(M, F^\bullet) > 0$.
(2) If $\dim(M, F^\bullet) = \dim(R)$ *and* $e(M, F^\bullet) < \infty$*, then M has finite length as a D_R-module.*
(3) The D_R-modules R, R_f for $f \in R$, and $H^i_I(R)$ for $I \subseteq R$ and $i \in \mathbb{N}$ each admit filtrations $F^\bullet$ compatible with $B^\bullet$ with dimension $= \dim(R)$ *and multiplicity* $< \infty$.

We think of the first part as a version of Bernstein's inequality, the second as saying that holonomic module have finite length, and the third as saying that the motiving classes of D-modules are indeed holonomic. This result in the case of polynomial rings of positive characteristic is due to Bavula [2] as reinterpreted by

Lyubeznik [5]. Cases (0) and (p) above are from the first listed project in the first paragraph, and case (#) is from the second listed work in progress.

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Semigroup Rings and Gluing Operation

Hema Srinivasan

(joint work with Philippe Gimenez)

Introduction. Let $\mathbb{N}$ denote the set of natural numbers. We denote by $\mathbb{N}^n$ the semigroup (a monoid) under addition. All our semigroups will be monoids. Let $\langle A\rangle$ denote the semigroup minimally generated by a subset $A = \{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ of $\mathbb{N}^n$. Let k be an arbitrary field and $\phi_A : k[x_1, \dots, x_p] \to k[t_1, \dots, t_n]$ be the ring homomorphism given by $\phi_A(x_j) = t^{\mathbf{a}_j} = \prod_{i=1}^n t_i^{a_{ij}}$ where $\mathbf{a}_j = (a_{1j}, \dots, a_{nj}) \in \mathbb{N}^n$. The kernel of ϕ_A, denoted I_A, is a binomial prime ideal and the semigroup ring $k[A]$ is isomorphic to $k[x_1, \dots, x_p]/I_A$. We will also denote by A the $n \times p$ integer matrix whose columns are the elements in A.

The concept of Gluing was introduced by Rosales in 1990's perhaps inspired by the classical construction by Delorme (1976) for the study and characterization of complete intersection numerical semigroups. For a semigroup $\langle C\rangle$, when the set of generators of the semigroup splits into two disjoint parts, $C = A \cup B$, such that $I_C = I_A + I_B + \langle\rho\rangle$ where ρ is a binomial whose first, respectively second, monomial involves only variables corresponding to elements in A, respectively B, we say that $\langle C\rangle$ is a *gluing* of $\langle A\rangle$ and $\langle B\rangle$. Let the two semigroups $\langle A\rangle$ and $\langle B\rangle$ in $\mathbb{N}^n$ with $A = \{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ and $B = \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$ have corresponding semigroup rings denoted by $k[A] \simeq k[x_1, \dots, x_p]/I_A$ and $k[B] \simeq k[y_1, \dots, y_q]/I_B$ respectively.

Definition 1. Given an integer $n \geq 1$ and two subsets $A = \{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ and $B = \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$ in $\mathbb{N}^n$, we say that the semigroups $\langle A\rangle$ and $\langle B\rangle$ *can be glued* if there exist two integers $k_1, k_2 \in \mathbb{N}$ such that for $C = k_1A \cup k_2B$, the semigroup

$\langle C\rangle$ is a gluing of $\langle k_1A\rangle$ and $\langle k_2B\rangle$, i.e., $I_C = I_A + I_B + \langle\rho\rangle$ for some binomial $\rho = \underline{x}^\alpha - \underline{y}^\beta$ with $\alpha \in \mathbb{N}^p$ and $\beta \in \mathbb{N}^q$.

When this occurs, we will say that $\langle C\rangle$ is a gluing of $\langle A\rangle$ and $\langle B\rangle$ instead of saying that it is a gluing of $\langle k_1A\rangle$ and $\langle k_2B\rangle$ and write $C = k_1A \bowtie k_2B$. In the definition of gluing, one can always assume that k_1 and k_2 are relatively prime, if needed. Such a semigroup $\langle C\rangle$ is called *decomposable* or that it *decomposes*) as $\langle C\rangle = \langle A\rangle \sqcup \langle B\rangle$.

Questions. *Given two semigroups $\langle A\rangle$ and $\langle B\rangle$ in $\mathbb{N}^n$, can $\langle A\rangle$ and $\langle B\rangle$ be glued? When it is possible to glue them, what should the integers k_1 and k_2 be, so that for $C = k_1A \cup k_2B$, $\langle C\rangle$ is a gluing of $\langle A\rangle$ and $\langle B\rangle$?*

The case of numerical semigroups, where $n = 1$, this is well understood. Moreover, it is well known that given two arbitrary numerical semigroups $\langle A\rangle$ and $\langle B\rangle$, if one chooses $k_1 \in \langle B\rangle$ and $k_2 \in \langle A\rangle$, then for $C = k_1A \cup k_2B$, one has that $I_C = I_A + I_B + \langle\rho\rangle$ for some binomial $\rho = \underline{x}^\alpha - \underline{y}^\beta$ with $\alpha \in \mathbb{N}^p$ and $\beta \in \mathbb{N}^q$. One can thus answer to the above questions when $n = 1$: Two numerical semigroups can always be glued and one knows how to glue them (choosing $k_1 \in \langle B\rangle$ and $k_2 \in \langle A\rangle$). Moreover, if $\langle C\rangle$ is a gluing of $\langle A\rangle$ and $\langle B\rangle$, the semigroup rings $k[A]$, $k[B]$ and $k[C]$ are always Cohen-Macaulay in this case.

Main Theorems. The main results, theorems 2, 10, and 11, and the examples can be found in [4].

Theorem 2. *If $C = k_1A \bowtie k_2B$, then*

(1) depth $k[C]$ = depth $k[A]$ + depth $k[B] - 1$,
(2) $\dim k[C] = \dim k[A] + \dim k[B] - 1$.

Now, it can be shown that A is the $n\times p$ matrix over $\mathbb{N}$ whose columns minimally generate the semigroup $\langle A\rangle$, then the dimension of the semigroup ring $k[A]$ is precisely the rank of A. Hence, we say A or $\langle A\rangle$ is degenerate if the rank $A < n$. We get the following immediate consequences of Theorem 2.

Corollary 3. *If $n \geq 2$, then $\langle A\rangle$ and $\langle B\rangle$ can not be glued unless at least one of $\langle A\rangle$ and $\langle B\rangle$ is degenerate.*

Corollary 4. *If $C = k_1A \bowtie k_2B$, then*

(1) $k[C]$ is Cohen-Macaulay if and only if $k[A]$ and $k[B]$ are,
(2) $k[C]$ is Gorenstein if and only if $k[A]$ and $k[B]$ are,
(3) $k[C]$ is a complete intersection if and only if $k[A]$ and $k[B]$ are.

Thus, we see that one necessary condition for when two semigroups A and B can be glued to obtain a semigroup C is that $\operatorname{rank} C = \operatorname{rank} A + \operatorname{rank} B - 1$. This somewhat partially explains when we can hope to glue two semigroups A and B.

Moreover, the converse is true for the complete intersection ([1], [2]), but not for the other two properties, namely Gorenstein or Cohen-Macaulay semigroups. There exist, even in the case of numerical semigroups, examples of Gorenstein semigroups that are not a gluing of semigroups of smaller embedding dimension.

For example, one has that a numerical semigroup $\langle C\rangle$ minimally generated by an arithmetic sequence of length $n \geq 3$, $C = \{a, a+d, \ldots, a+(n-1)d\}$, is a gluing if and only if $n = 3$, a is even, and d is odd and relatively prime to $a/2$. Note that these are also the only numerical semigroup minimally generated by an arithmetic sequence that are a complete intersection. In fact, a numerical semigroup minimally generated by an arithmetic sequence of length $n \geq 4$ is never a gluing of two smaller numerical semigroups. Thus, for all $n \geq 4$ and $1 \leq t \leq n-1$, there is a numerical semigroup of embedding dimension n and Cohen-Macaulay type t which is not a gluing of two smaller numerical semigroups.

It is not true that the gluing is not determined by the minimal number of generators of the glued semigroup.

Example 5. Consider these two semigroups $\langle A\rangle$ and $\langle B\rangle$ in $\mathbb{N}^3$ given by

$$A = \{(1,6,7),\ (1,4,5),\ (2,5,7),\ (5,5,10)\} \quad \text{and}$$
$$B = \{(1,1,6),\ (2,2,7),\ (3,3,8),\ (10,10,20)\}.$$

Here, I_A is minimally generated by 3 binomials, $I_A = \langle x_2^2x_3^3 - x_1^3x_4, x_3^5 - x_2^5x_4, x_2^7 - x_1^3x_3^2\rangle$, $I_B = \langle y_2^2 - y_1y_3, y_3^4 - y_1^2y_4\rangle$ and I_C is minimally generated by 6 elements but it is not a gluing. $I_C = I_A + I_B + \langle x_4^2 - y_4\rangle + \langle x_4y_1 - y_3^2\rangle$. In this case, the second minimal generator of I_B is not minimal in I_C.

Theorem 6 ([3, Thm. 6.1, Cor. 6.2])**.** *Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ and $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_q\}$ be two finite subsets of $\mathbb{N}^n$ and assume that $\langle C\rangle$ is a gluing of $\langle A\rangle$ and $\langle B\rangle$, i. e., $C = A \cup B$ and $I_C = I_A + I_B + \langle\rho\rangle$ for some $\rho = \underline{x}^\alpha - \underline{y}^\beta$ with $\alpha \in \mathbb{N}^p$ and $\beta \in \mathbb{N}^q$. Consider F_A and F_B, minimal graded free resolutions of $k[A]$ and $k[B]$.*

(1) A minimal graded free resolution of $k[C]$ can be obtained as the mapping cone of $\rho : F_A \otimes F_B \to F_A \otimes F_B$ where ρ is induced by multiplication by ρ.

(2) The Betti numbers of $k[A]$, $k[B]$ and $k[C]$ are related as follows. $\forall i \geq 0$,

$$\begin{aligned}\beta_i(k[C]) &= \textstyle\sum_{i'=0}^{i} \beta_{i'}(k[A])[\beta_{i-i'}(k[B]) + \beta_{i-i'-1}(k[B])] \\ &= \textstyle\sum_{i'=0}^{i} \beta_{i'}(k[B])[\beta_{i-i'}(k[A]) + \beta_{i-i'-1}(k[A])]\,.\end{aligned}$$

(3) The relation between the projective dimensions of $k[A]$, $k[B]$ and $k[C]$ is

$$\mathrm{pd}(k[C]) = \mathrm{pd}(k[A]) + \mathrm{pd}(k[B]) + 1\,.$$

Using the last part of the previous result, one can easily show that the only nondegenerate semigroups whose semigroup ring is Cohen-Macaulay that can be glued are the numerical semigroups.

Theorem 7. *Let $\langle A\rangle$ and $\langle B\rangle$ be nondegenerate semigroups in $\mathbb{N}^n$ such that $k[A]$ and $k[B]$ are nondegenerate. Then $\langle A\rangle$ and $\langle B\rangle$ can be glued if and only if $n = 1$.*

Example 8. For $n \geq 2$, some degeneracy is necessary in order to glue two semigroups. If $\langle S\rangle \subset \mathbb{N}^2$ is the semigroup generated by $S = \{(3,0),(2,1),(1,2),(0,3)\}$, the ideal I_S is the defining ideal of the twisted cubic which is known to be Cohen-Macaulay. By theorem 7, $\langle S\rangle$ can not be glued with itself in $\mathbb{N}^2$. But one can consider the two degenerate semigroups $\langle A\rangle$ and $\langle B\rangle$ of $\mathbb{N}^3$ generated respectively by $A = \{(4,0,0),(3,1,0),(2,2,0),(1,3,0)\}$ and $B = \{(3,3,0),(3,2,1),(3,1,2),(3,0,3)\}$, whose defining ideals $I_A \subset k[x_1,\dots,x_4]$ and $I_B \subset k[y_1,\dots,y_4]$ are both the defining ideal of the twisted cubic. In other words, $k[A] \simeq k[B] \simeq k[S]$ and $\langle A\rangle$ and $\langle B\rangle$ can be thought as two copies of $\langle S\rangle$ in $\mathbb{N}^3$ where they are degenerate. However, $\langle A\rangle$ and $\langle B\rangle$ can be glued because if $C = A\cup B$, then $I_C = I_A + I_B + \langle y_1^2 - x_1x_4^2\rangle$.

Towards understanding what happens when C is a gluing of A and B, we prove the following condition.

Lemma 9. *Given $A = \{\mathbf{a}_1,\dots,\mathbf{a}_p\}$ and $B = \{\mathbf{b}_1,\dots,\mathbf{b}_q\}$ in $\mathbb{N}^n$ satisfying the rank conditions $\operatorname{rank} A|B = n$ and $\operatorname{rank} A + \operatorname{rank} B = n+1$, let $\mathbf{u} = \mathbf{u}(A,B) \in \mathbb{Z}^n$ be the gluable lattice point of A and B. Then, the following are equivalent:*

(1) the system $A\cdot X = B\cdot Y$ has a nontrivial solution (X,Y) such that $X\in\mathbb{N}^p$ and $Y\in\mathbb{N}^q$;

(2) there exist positive integers a and b such that $a\mathbf{u}\in\langle A\rangle$ and $b\mathbf{u}\in\langle B\rangle$.

The $\mathbf{u}$ in Lemma 9 is called the gluable lattice point.

Theorem 10. *Let A and B be two finite sets in $\mathbb{N}^n$ satisfying the rank conditions*

$$\operatorname{rank} A|B = n \quad \textit{and} \quad \operatorname{rank} A + \operatorname{rank} B = n+1$$

and let $\mathbf{u} = \mathbf{u}(A,B)\in\mathbb{Z}^n$ be the gluable lattice point of A and B. Then,

$$\textit{(a)} \Longrightarrow \textit{(b)} \Longrightarrow \textit{(c)} \iff \textit{(d)}$$

for the following four conditions:

(a) there exist relatively prime positive integers k_1, k_2 such that $k_2\mathbf{u}\in\langle A\rangle$ and $k_1\mathbf{u}\in\langle B\rangle$;

(b) $\langle A\rangle$ and $\langle B\rangle$ can be glued;

(c) there exists positive integers k_1,k_2 such that $k_2\mathbf{u}\in\langle A\rangle$ and $k_1\mathbf{u}\in\langle B\rangle$;

(d) The system $A\cdot X = B\cdot Y$ has a nontrivial solution (X,Y) with $X\in\mathbb{N}^p$ and $Y\in\mathbb{N}^q$.

There are examples showing that *(c)* does not imply *(b)* in Theorem 10. At the same time, there are also examples showing that if *(b)* is not satisfied, then *(c)* may not be true either. Thus, in some sense, these are the best possible implications.

Finally, we can ask if given two non degenerate semigroups $\langle A\rangle$ and $\langle B\rangle$ in $\mathbb{N}^n, n\geq 2$, how does one embed them in $\mathbb{N}^m, m>n$ in order to glue them?

Theorem 11. *Any two homogeneous nondegenerate semigroups in dimension 2 can be glued after an appropriate embedding in dimension 3.*

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Commutative Algebra and Integer Partitions

Hussein Mourtada

An integer partition λ of an integer number n is a decreasing sequence of integer numbers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 + \cdots + \lambda_r = n$; the λ_i's are called the parts of λ and r is its size. An important direction in the theory of integer partitions ([4]) is the study of partition identities: this amounts to find a family A of integer partitions defined by some restrictions on the partitions and another family B determined by different restrictions, such that the number of partitions of n in A is equal to the number of partitions of n in B for every integer number n. The following example of a partition identity is due to Euler.

For every integer number n, the number of partitions of n whose parts are odd is equal to the number of partitions of n whose parts are distinct.

One of the most famous partition identities are the (two) Rogers-ramanujan identities. The first one is given by:

Let n be an integer number. Let $T(n)$ be the number of partitions of n such that the difference between consecutive parts is at least 2. Let $E(n)$ be the number of partitions of n into parts congruent to 1 or 4 $\mod 5$. Then we have

$$T(n) = E(n).$$

The fame of these identities is probably due to the fact that their proofs are difficult and that they appear in many domains such as combinatorics, statistical mechanics, number theory, representation theory or algebraic geometry (see the references in [8, 2]). Another version of the first Rogers-Ramanujan identity can be stated in terms of q-series:

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q)\cdots(1-q^k)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \tag{1}$$

the empty products obtained when one puts $k = 0$ on the left-hand sides are taken to be 1. One can prove that the left member of (1) is actually the generating series of $T(n)$ (*i.e.* is equal to $1 + \sum_{n\in \mathbf{Z}_{>0}} T(n)q^n$) and that the right member of (1) is equal to the generating series of $E(n)$.

In this note, we report on some results linking partition identities to singularity theory. Let (X, O) be a singularity defined over a field $\mathbf{K}$ of characteristic 0 (O being a closed point on X that we assume affine). Let $X_\infty^O = \mathrm{Spec} A_\infty^O$ be the space of arcs centered at the point O : this is the moduli space of formal unibranched curves centered at O; it is not difficult to prove that it hase a structure of a scheme. Moreover, it has a natural cone structure which induces a grading on A_∞^O (*i.e.*,

$A_\infty^O = \oplus_{h\in\mathbf{N}} A_{\infty,h}^O$) and one can consider its Hilbert-Poincaré series that we call the **Arc-Hilbert-Poincaré series** of the singularity [8, 6]:

$$\mathrm{AHP}_{X,0}(q) = \sum_{h\in\mathbf{N}} \dim_{\mathbf{K}} A_{\infty,h}^O \; q^h.$$

This series is an invariant of singularities (it detects regularity) and it contains different ingredients which motivate its study from the viewpoint of singularity theory [8]. But it is in general very difficult to compute, even though sometimes for mild singularities this is possible:

Theorem 1 ([7]). *Let (X,O) be a rational double point surface singularity (i. e. an ADE surface singularity). We have*

$$AHP_{X,O}(q) = \frac{1}{(1-q)^3}\left(\prod_{i\geq 2}\frac{1}{(1-q^i)^2}\right).$$

Question 2. *Does the Arc Hilbert-Poincaré series characterize ADE surface singularities?*

Let us go back to the link to integer partitions. It happened that for the simplest possible singularities, the Arc Hilbert-Poincaré series is related to the Rogers-Ramanujan identities:

Theorem 3 ([5]). *For $X = \mathrm{Spec}\,\frac{\boldsymbol{K}[x]}{(x^2)}$, we have*

$$\mathrm{AHP}_{X,O}(q) = \prod_{i\equiv 1,4 \pmod 5}\frac{1}{1-q^i}.$$

Notice that the power series in the theorem is the right hand side of the first Rogers-Ramanujan identity. The proof uses the fact that A_∞^O has a differential structure which allows using differential calculus to compute a Groebner basis (with respect to some order) of the infinitely generated ideal defining the space of arcs (centered at O) in a natural infinite dimensional affine space. Moreover, using simple commutative algebra [6], the proof allows to find in a very natural way a sequence of q-series which converges (for the q-adic topology) to the two q-series appearing in (1); this gives a proof of the Rogers-Ramnujan identities, which was found by Andrews and Baxter in an empirical way. Again using simple commutative algebra, differential calculus and Groebner basis computations with respect to some monomial orderings, we were able to guess and prove a family of partition identities indexed by an integer number k; for $k = 1$, this adds another member to the Rogers-Ramanujan identities.

Theorem 4 ([3]). *Let $n \geq k$ be positive integers. The number of partitions λ of n whose parts are larger or equal to k and whose size is less than or equal to $s(\lambda) - (k-1)$ is equal to the number of partitions of n with parts larger or equal to k and without neither consecutive nor equal parts.*

Studying the Arc Hilbert-Poincaré series of Spec$\frac{\mathbf{K}[x]}{(x^n)}, n \geq 2$, we found in [6] a link with another famous family of identities (Gordon's identities) which generalises Rogers-Ramanujan identities. In her thesis [1], using similar ideas to those of [3] and the results of [6], Afsharijoo conjectured a large family of exciting partition identities. These conjectures were recently proved in [2] using commutative algebra, combinatorial methods from the theory of partitions and q-series calculus.

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Pure subrings of KLT singularities and BCM test ideals

Shunsuke Takagi

(joint work with Tatsuki Yamaguchi)

Throughout this abstract, all rings are commutative rings with unity. First we recall the definition of pure homomorphisms.

Definition 1. A ring extension $R \hookrightarrow S$ is said to be *pure* if the induced map $M = M \otimes_R S \to M \otimes_R S$ is injective for every R-module M.

For example, if a linearly reductive group G acts on a polynomial ring $S = \mathbb{C}[X_1, \dots, X_n]$, then $S^G \hookrightarrow S$ is a pure homomorphism.

Boutot proved the following important property of rational singularities.

Theorem 2 ([2]). *Let $R \hookrightarrow S$ be a pure local homomorphism of local rings essentially of finite type over $\mathbb{C}$. If S is a rational singularity, then so is R.*

It is then natural to ask, "What about other classes of singularities in birational geometry?". We focus on KLT singularities in this talk. Classical KLT singularities are $\mathbb{Q}$-Gorenstein by definition, but R is not necessarily $\mathbb{Q}$-Gorenstein even if S is regular. Therefore, being (classical) KLT does not descend under pure extensions. On the other hand, de Fernex-Hacon [3] generalized the definition of KLT singularities to the non-$\mathbb{Q}$-Gorenstein setting: a normal local ring R essentially of finite type over $\mathbb{C}$ is said to be of *KLT type* if there exists an effective $\mathbb{Q}$-Weil divisor Δ on $\operatorname{Spec} R$ such that $(\operatorname{Spec} R, \Delta)$ is a (classical) KLT pair.

Very recently, Z. Zhuang proved that being of KLT type descends under pure extensions.

Theorem 3 ([11]). *Let $R \hookrightarrow S$ be a pure local homomorphism of local rings essentially of finite type over $\mathbb{C}$. If S is of KLT type, then so is R.*

Remark 4. Schwede-Smith [9] conjectured that R is of KLT type if and only if its reduction modulo p is strongly F-regular for all large p. Since strong F-regularity descends under pure extensions, one may think that the conjecture implies Theorem 3. However, it is not clear at all, because purity is not preserved under reduction modulo p in general (see [5]).

We give an alternative proof of Theorem 3, using BCM test ideals. Let $(R, \mathfrak{m}, k)$ be an excellent Noetherian local domain with dualizing complex.

Definition 5 ([7], [6]). Let B be a Big Cohen-Macaulay R-algebra.

(1) For a (not necessarily finitely generated) R-module M, the submodule 0_M^B of M is defined as the kernel of the map $M \to M \otimes_R B$ sending x to $x \otimes 1$.
(2) The *BCM test ideal* $\tau_B(R)$ of R associated to B is defined as

$$\tau_B(R) = \bigcap_M \operatorname{Ann}_R 0_M^B,$$

where M runs through all R-modules.

Let R^+ denote an absolute integral closure of R, that is, the integral closure of R in an algebraic closure $\overline{K}$ of the fractional field K of R.

Theorem 6 ([4], [1]). *Suppose that the residue field k has characteristic $p > 0$. Then p-adic completion $\widehat{R^+}^p$ of R^+ is a big Cohen-Macaulay R-algebra.*

However, when k has characteristic zero, R^+ is not a big Cohen-Macaulay R-algebra in general. We construct a big Cohen-Macaulay algebra in equal characteristic zero, following Schoutens [8].

From now on, we assume that $(R, \mathfrak{m}, k)$ is a normal local ring essentially of finite type over $\mathbb{C}$. Let P be the set of all prime numbers, and fix a non-principal ultrafilter on P. Then we have a non-canonical isomorphism

$$\operatorname{ulim}_{p\in P} \overline{\mathbb{F}_p} \cong \mathbb{C},$$

where $\overline{\mathbb{F}_p}$ is an algebraic closure of the prime field $\mathbb{F}_p$ of characteristic $p > 0$. We can construct an $\overline{\mathbb{F}_p}$-algebra R_p from R via this isomorphism. R_p is a normal local ring essentially of finite type over $\overline{\mathbb{F}_p}$ for almost all $p \in P$.

Theorem 7 ([8]). *$\mathcal{B}(R) := \operatorname{ulim}_{p\in P} R_p^+$ is a big Cohen-Macaulay R^+-algebra.*

de Fernex-Hacon [3] generalized the definition of multiplier ideals too. Their multiplier ideal $\mathcal{J}_{\mathrm{dFH}}(R)$ of R is characterized as

$$\mathcal{J}_{\mathrm{dFH}}(R) = \sum_\Delta \mathcal{J}(X, \Delta),$$

where $\mathcal{J}(X,\Delta)$ is the classical multiplier ideal associated to $(X := \operatorname{Spec} R, \Delta)$ and Δ runs through all effective $\mathbb{Q}$-Weil divisors on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. By definition, R is of KLT type if and only if $\mathcal{J}_{\mathrm{dFH}}(R) = R$.

Now our main result is stated as follows.

Theorem 8 ([10])**.** *If the anti-canonical ring of R is Noetherian, then $\tau_{\mathcal{B}(R)}(R) = \mathcal{J}_{\mathrm{dFH}}(R)$.*

Thanks to the above theorem, we can study the behavior of multiplier ideals under pure extensions using BCM test ideals.

Corollary 9. *Let $R \hookrightarrow S$ be a pure local homomorphism of normal local rings essentially of finite type over $\mathbb{C}$, and suppose that the anti-canonical ring of R is Noetherian. Then the following holds.*

(1) $\mathcal{J}_{\mathrm{dFH}}(S) \cap R \subseteq \mathcal{J}_{\mathrm{dFH}}(R)$.
(2) If $R \hookrightarrow S$ is faithfully flat, then $\mathcal{J}_{\mathrm{dFH}}(S) \subseteq \mathcal{J}_{\mathrm{dFH}}(R)S$.

Remark 10. For simplicity, we only consider the no boundary case in this talk. However, we can actually prove the following: let D be a prime divisor on $X := \operatorname{Spec} R$, $\mathfrak{a}$ be an ideal of R not contained in any minimal prime ideals of $R(-D)$ and $t \geq 0$ be a real number. Assume that the cycle-theoretic pullback D_S of D by the morphism $\operatorname{Spec} S \to X$ is a prime divisor on $\operatorname{Spec} S$ and that the log anti-canonical ring $\bigoplus_{n \geq 0} R(-n(K_X + D))$ of (X, D) is Noetherian. Then the following holds.

(1) $\operatorname{adj}_{I_{D_S}}(S, D_S, (\mathfrak{a}S)^t) \cap R \subseteq \operatorname{adj}_{I_D}(R, D, \mathfrak{a}^t)$.
(2) If $R \hookrightarrow S$ is faithfully flat, then $\operatorname{adj}_{I_{D_S}}(S, D_S, (\mathfrak{a}S)^t) \subseteq \operatorname{adj}_{I_D}(R, D, \mathfrak{a}^t)S$.

Here, $\operatorname{adj}_{I_D}(R, D, \mathfrak{a}^t)$ (resp. $\operatorname{adj}_{I_{D_S}}(S, D_S, (\mathfrak{a}S)^t)$) is a generalization of the classical adjoint ideal to the case where $K_{\operatorname{Spec} R} + D$ (resp. $K_{\operatorname{Spec} S} + D_S$) is not necessarily $\mathbb{Q}$-Cartier.

Theorem 3 is easily obtained from Corollary 9 (1), because if S is of KLT type, then the anti-canonical rings of S and R are both Noetherian. Similarly, the following result is an immediate consequence of Remark 10: suppose that $R \hookrightarrow S$ is a pure local homomorphism of local rings essentially of finite type over $\mathbb{C}$ and D is a prime divisor on $\operatorname{Spec} R$, and let D_S denote the cycle-theoretic pullback of D on $\operatorname{Spec} S$. If (S, D_S) is of plt type, that is, there exists an effective $\mathbb{Q}$-Weil divisor Δ on $\operatorname{Spec} S$ such that $(\operatorname{Spec} S, D + \Delta)$ is a classical plt pair, then so is (R, D).

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Transferring resolutions along a Koszul homomorphism

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(joint work with Benjamin Briggs, James C. Cameron, Josh Pollitz)

The Koszul property for algebras over a field was introduced by Priddy [8], see also [7, 4]. A finite dimensional augmented (graded) k-algebra K is *Koszul*, if K admits a grading (compatible with the grading on K)

$$K_{(\star)} = \coprod_{w \geqslant 0} K_{(w)} \quad \text{with } K_{(0)}\,,$$

called the *weight grading*, such that the minimal resolution of k is linear with respect to the weight grading.

Classically, for a graded algebra K the weight grading coincides with the given grading on K. We make a point of allowing the weight grading to be different from the given one.

It is well-known, that any Koszul algebra is quadratic; that means

$$K_{(\star)} \cong \mathrm{T}^a_{(\star)}(V)/(W)$$

for $V = K_{(1)}$ and $W \subseteq V \otimes V$, where $\mathrm{T}^a_{(\star)}(V)$ is the tensor algebra on V.

Let Q be a commutative noetherian local ring with residue field k.

Definition 1. A finite local homomorphism $\varphi\colon Q \to R$ is *Koszul* if

(1) $R \otimes^{\mathbf{L}}_Q k$ is formal, that is $R \otimes^{\mathbf{L}} k \cong \mathrm{Tor}^Q(R,k)$ as dg algebras, and
(2) $\mathrm{Tor}^Q(R,k)$ is Koszul as a k-algebra.

Note, that the weight grading need not coincide with the homological grading.

This definition generalizes the Koszul property for k-algebra: A k-algebra K is Koszul if and only if $k \to K$ is Koszul.

We can construct a quadratic presentation for a local Koszul homomorphism. Let $\varepsilon\colon A \to R$ be a surjective semi-free resolution of R over Q. The algebra structure on R lifts to A. While the induced multiplication need not be associative, it is associative up to homotopy. More precisely, by [2, Proposition 3.6], the algebra structure on R lifts to an A_∞-algebra structure on A such that ε is a quasi-isomorphism of A_∞-algebras. A_∞-structures where first introduced by Stasheff [9, 10], for an overview see [5].

If φ is Koszul, then we can choose the A_∞-structure on A such that there exists a free, non-negatively graded module V and a summand $W \subseteq V \otimes V$ such that

$$A \cong \mathrm{T}^a(V)/(W) \quad \text{with} \quad m_2 \otimes k = \mu \otimes k \quad \text{and} \quad m_n \otimes k = 0 \text{ for } n \neq 2\,.$$

Here the m_n's are the higher multiplications of the A_∞-structure and μ denotes the multiplication of the tensor algebra.

Example 2. Any local homomorphism $Q \to Q[x]/(x^2 - ax - b) =: R$ with $a, b \in \mathfrak{m}$ is Koszul. A free resolution of R is given by $A = R$ and it has a quadratic presentation with

$$V = Qx \quad \text{and} \quad W = V \otimes V\,.$$

The induced multiplication on A and the tensor algebra multiplication do not coincide when $a \neq 0$ or $b \neq 0$; explicitly

$$m_2(x \otimes x) = ax + b \neq 0 = \mu(x \otimes x)\,.$$

Example 3. Any surjective complete intersection map $\varphi\colon Q \to R = Q$ is Koszul. The Koszul complex $A = \mathrm{Kos}^Q(f_1, \dots, f_c)$ on a minimal generating set of the kernel of φ is a free resolution of R, and it has a quadratic presentation with

$$V = A_1 \quad \text{and} \quad W = \langle \{x \otimes x\} \cup \{x \otimes y + y \otimes x\}_{x \neq y} \rangle\,.$$

Further examples of local Koszul homomorphisms include Golod maps and surjective Gorenstein maps of projective dimension 3; for the former see [2] and the latter [1].

Let A be an A_∞-algebra with $\bar{A}$ the cokernel of its unit. The bar construction $\mathsf{B}(A)$ is the tensor coalgebra on $\bar{A}$ with a curved dg coalgebra structure induced by the A_∞-algebra structure of A; see [6, Section 2.2] for details. If φ is Koszul and $A \cong \mathrm{T}^a(V)/(W)$ a quadratic presentation, then we define

$$\mathsf{C}_{(0)}(V, W) := Q\,, \quad \mathsf{C}_{(1)}(V, W) := V \quad \text{and}$$
$$\mathsf{C}_{(n)}(V, W) := \bigcap_{i+2+j=n} V^{\otimes i} \otimes W \otimes V^{\otimes j}\,.$$

By construction this is a sub-coalgebra of $\mathsf{B}(A)$. When the curved dg coalgebra structure on $\mathsf{B}(A)$ restricts to a curved dg coalgebra structure on $\mathsf{C}(V, W)$, then we say φ is *special Koszul*. In this case, the inclusion $\mathsf{C}(V, W) \to \mathsf{B}(A)$ is a quasi-isomorphism of curved dg coalgebras.

All the examples mentioned above are special Koszul. We do not know whether that always holds.

Question 4. *Is any local Koszul homomorphism special Koszul?*

For a local Koszul homomorphism $\varphi\colon Q \to R$ we can construct R-resolutions from Q-resolutions: Let M be an R-complex and $G \to M$ a semi-free resolution of M over Q. Then there is an A_∞-module structure on G over A. From this data we can construct a semi-free resolution of M over R:

$$R \otimes^\tau \mathsf{C}(V, W) \otimes^\tau G = (R \otimes \mathsf{C}(V, W) \otimes G, \partial^\tau) \to M\,.$$

The twisted differential ∂^τ consists of the differentials of R and G, and the A_∞-structures on A and G. This generalizes the construction for surjective complete intersection maps of Eisenbud an Shamash [3], and for Golod maps of Burke [2].

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Chow rings of matroids are Koszul

Jason McCullough

(joint work with Matthew Mastroeni)

Chow rings of matroids have been integral in the recents proofs of the Heron-Rota-Welsh Conjecture [1] and the Top Heavy Conjecture [2]. They have their origins in the cohomology rings of wonderful compactifications of complex hyperplane arrangement complements ala de-Concini and Procesi [3]. They were generalized to arbitrary graded lattices and building sets by Feichtner and Yuzvinsky [6], where a not necessarily quadratic Gröbner basis is given.

For an arbitrary matroid M, the Chow ring of M is

$$\underline{CH}(M) = \frac{\mathbb{Q}[x_F \mid F \in \mathcal{L}(M) \setminus \{\emptyset\}]}{(x_F x_{F'} \mid F, F' \text{ incomparable}) + (\sum_{G \supseteq F} X_G \mid \mathrm{rk}(F) = 1)}.$$

In [1] Adiprasito, Huh, and Katz show that $\underline{CH}(M)$ has the Kähler package: Poincarè duality, the hard Lefschetz property, and the Hodge-Riemann relations. Thus it is an Artinian, Gorenstein, quadratic ring and it is natural to ask if it is

Koszul. This was explicitly conjecture by Dotsenko [5] who showed that a related Chow ring (with respect to a non-maximal building set) of a certain matroid known to be isomorphic to the cohomology of the compactification of the moduli space of n marked points on the projective line, was Koszul. This answered a question of Manin.

Via a Koszul filtration argument, Mastreoni and the author prove Dotsenko's conjecture for all matroids:

Theorem 1 (Mastroeni and McCullough [8]). *For any matroid M, $\underline{CH}(M)$ is Koszul.*

The filtration is defined by means of a new notion of a total coatom ordering on a graded lattice. We proved that the lattice of flats of any matroid (equivalently any geometric lattice) has a total coatom ordering. Similar results have been proved by Delucchi [4].

As corollaries, we show that the Chow ring of a matroid always has a rational Poincarè series, which is not true even for all quadratic Gorenstein algebras. It also follows that the Hilbert series of $\mathrm{CH}(M)$ is at least one real root. There is an outstanding conjecture, due to Huh (see [7]), that the Hilbert series of $\mathrm{CH}(M)$ is real-rooted, which would impose severe restrictions on the possible coefficients of the Hilbert series. Thus our result can be viewed a step toward Huh's conjecture.

We similarly show that the augmented Chow ring introduced in [2] is Koszul. While it can be realized as the Chow ring of a certain matroid with respect to a non-maximal building set, not all building sets give rise to Koszul, or even quadratic, algebras. It would be nice to have a way to unify these two results and so we ask the following question:

Question 2. *Is every Artinian, quadratic, Gorenstein algebra with the Kähler package Koszul?*

Examples show that the hard Lefschetz property is not sufficient.

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The dimension of cohomological support varieties

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(joint work with Ben Briggs, Josh Pollitz)

Given a local ring R and an R-module M, the cohomological support variety of M is a geometric object encoding homological information about M and R. These were first defined by Avramov in 1989 over complete intersection rings [1], and later extended to the general setting by Jorgensen.

Let us fix some notation. Throughout, R will denote a noetherian local ring, and we will fix a minimal Cohen presentation $\widehat{R} \cong Q/I$, where $(Q, \mathfrak{m}, k)$ is a regular local ring and $I \subseteq \mathfrak{m}^2$ is minimally generated by n elements $f_1, \ldots, f_n$. Under the isomorphism $I/\mathfrak{m}I \cong \mathbb{A}^n_k$, for each nonzero point $a = (a_1, \ldots, a_n)$ fix lifts b_i of each a_i to Q, and let $Q_a := Q/(b_1 f_1 + \cdots + b_n f_n)$.

Definition 1. Given a finitely generated R-module $M \neq 0$, the **cohomological support variety** of M is given by

$$\mathrm{V}_R(M) := \{a = (a_1, \ldots, a_n) \in I/\mathfrak{m}I \cong \mathbb{A}^n_k \mid a = 0 \text{ or } \operatorname{pdim}_{Q_a}(\widehat{M}) = \infty\}.$$

One can show that this is the cone over a projective variety in $\mathbb{P}^{n-1}$, and that the definition is independent of the choices made. Moreover, the definition can be extended to any M in the the derived category $\mathrm{D}^b(R)$, of bounded complexes of finitely generated R-modules up to quasi-isomorphism. For details, see [9].

The easiest example to compute is the support of the residue field k: since Q_a is not regular for any a, we must have $\operatorname{pdim}_{Q_a}(k) = \infty$ for all a, and thus $\mathrm{V}_R(k) = I/\mathfrak{m}I$. In a previous paper, [7], we showed that if $J \supseteq I$ is a complete intersection, meaning it is generated by a regular sequence, then the support of Q/J is given by the minimal generators of I that are not minimal generators of J. More precisely, the map induced by the inclusion of I into J gives

$$\mathrm{V}_R(Q/J) = \ker\left(I/\mathfrak{m}I \to J/\mathfrak{m}J\right).$$

In general, one can explicitly compute $\mathrm{V}_R(M)$ for any finite complex of finitely generated modules using the Macaulay2 package `ThickSubcategories`, written by the author, Janina Letz, and Josh Pollitz. But for a general module M, it is not feasible to give explicit formulas that one can compute by hand. Moreover, the examples above are all linear, but one can construct examples where even $\mathrm{V}_R(R)$, is not a finite union of linear spaces.

Cohomological supports detect complete intersections (ci). Pollitz showed [9] that R is a complete intersection if and only if $\mathrm{V}_R(R) = \{0\}$.

Cohomological support varieties can be used as auxiliary tools to prove homological results. For example, they are the key piece in the Avramov and Buchweitz' proof [2] that for all finitely generated modules M and N over a complete intersection ring R,

$$\operatorname{Ext}^{\geqslant 0}_R(M, N) = 0 \iff \operatorname{Ext}^{\geqslant 0}_R(N, M) = 0.$$

In fact, in [2] Avramov and Buchweitz apply cohomological support varieties to study asymptotic properties of betti and bass numbers of modules over complete intersections; for further developments on this that also apply cohomological support varieties techniques, see the recent work of Briggs, Pollitz, and McCormick.

Support varieties also encode information about the triangulated structure of $\mathrm{D}^b(R)$, and how complexes can be built out of each other. One important construction in $D^b(R)$ is the notion of building. We say M **builds** N in $D^b(R)$ if we can construct N from M via finitely many of the following building operations:

- Shifting: if M builds X, then it builds any complex obtained by shifting the homological grading on X.
- Direct summands: if M builds X, then it builds any direct summand of X.
- Cones: given a short exact sequence of complexes

$$0 \to A \to B \to C \to 0,$$

if M builds two of A, B, and C, then M also builds the third.

We will write $\langle M \rangle$ to denote the collection of all the objects that M finitely builds, which in the language of triangulated categories is the thick closure of M.

We can reinterpret the Auslandar-Buchsbaum-Serre characterization of regular rings in this context: R is a regular ring if and only if $\langle R \rangle = \mathrm{D}^b(R)$. Here $\langle R \rangle$, the complexes built by R, play the role of modules of finite projective dimension. The Auslandar-Buchsbaum-Serre theorem solved the Localization Problem: if R is a regular local ring then R_P must also be regular for all primes P. The theorem gives us a structural characterization of regularity that allows us to focus on a property of modules that easily localizes. In contrast, the Localization Problem for complete intersections was solved by Avramov in the 1970s, but his solution does not provide such a characterization of the ci property. Using his characterization of cis in terms of $\mathrm{V}_R(R)$, Pollitz showed that R is a complete intersection if and only if for every $M \in \mathrm{D}^b(R)$ there exists $P \in \langle M \rangle \cap \langle R \rangle$ with the same support (in the classical sense) as M. If M satisfies these properties, we say M is **proxy small**; this is a notion that localizes well, giving a new solution to the Localization Problem. Roughly speaking, M is proxy small if it is finitely many steps away from having finite projective dimension.

But Pollitz' characterization was in terms of all $M \in \mathrm{D}^b(R)$; the problem of whether this characterization can be restricted to finitely generated R-modules remains open. In previous work [7], we solved this problem for certain classes of rings by explicitly constructing modules that are not proxy small:

Theorem 2 (Briggs-Grifo-Pollitz, [7])**.** *Let R be an equipresented local ring or such that $\mathrm{V}_R(R) = \mathbb{A}^n$. If R is not ci, there exists a finitely generated R-module that is not proxy small. If moreover the residue field of R is infinite, there exists a quotient $R \to S$ to an artinian hypersurface that is not proxy small.*

The key ingredient in the proof is once more cohomological support varieties; we construct finitely generated R-modules M with $\mathrm{V}_R(R) \subsetneq \mathrm{V}_R(M)$. This prompts the question of when is $\mathrm{V}_R(R) = \mathbb{A}^n$, or more generally of how small can $\mathrm{V}_R(M)$ be. A related question, of interest for other reasons as well, is the following:

Problem 3. *Fix R. Given a conical subvariety V of $\mathbb{A}^n$, is there $M \in \mathrm{D}^b(R)$ such that $\mathrm{V}_R(M) = V$?*

When R is a complete intersection, the answer is all of them, by independent work of Bergh [5] and Avramov and Jorgensen.

Theorem 4 (Briggs-Grifo-Pollitz, [8]). *For any $M \in \mathrm{D}^b(R)$,*

$$\dim(\mathrm{V}_R(M)) \geqslant n - \textit{embdim}(R) + \textit{depth}(R).$$

and the inequality is strict if R is not a complete intersection.

When R is a Cohen-Macaulay ring, the quantity on the right is $\mu(I) - \mathrm{height}(I)$, also known as the **ci-defect** of R, which measures of how far R is from being ci.

Corollary 5. *If R is Cohen-Macaulay but not ci, $\dim(\mathrm{V}_R(M)) > \mu(I) - \textit{height}(I)$ for all $M \in \mathrm{D}^b(R)$, so not every conical subvariety of $\mathbb{A}^n$ can be realized as $\mathrm{V}_R(M)$.*

Our work in [8] also contains other bounds on $\dim(\mathrm{V}_R(M))$; one of those bounds recovers and strengthens a result from [3] on the Loewy length of finite free complexes, while also relating the codimension of $\mathrm{V}_R(M)$ with an invariant coming from the homotopy Lie algebra of R. The homotopy Lie algebra $\pi^*(R)$ is a graded Lie algebra with graded Lie bracket $[-,-]$ and such that $\pi^2(R) \cong (I/\mathfrak{m}I)^\vee$. An element α in $\pi^i(R)$ is called **central** if $\mathrm{ad}(\alpha) := [\alpha, -] = 0$, and **radical** if $\mathrm{ad}(\alpha)^p = 0$ for some p. The radical elements in $\pi^2(R)$, $\rho^2(R)$, play a key role in Briggs' recent solution of Vasconcelos' conjecture [6]; Avramov and Halperin showed that R is ci if and only if $\rho^2(R) = \pi^2(R)$. Our bounds on $\dim(\mathrm{V}_R(M))$ recover this result.

We say R has an **embedded deformation** if $\widehat{R} \cong S/(f)$ for some quotient S of Q and some f regular on S. In the 1980s, Avramov showed that embedded deformations of R give rise to central elements in $\pi^2(R)$, and asked if the converse holds. Pollitz showed that embedded deformations give rise to hyperplanes containing $\mathrm{V}_R(R)$. In [8], we show that elements in $\rho^2(R)$ gives rise to a hyperplane containing $\mathrm{V}_R(R)$, giving a new avenue to study Avramov's question: by studying whether hyperplanes containing $\mathrm{V}_R(R)$ give rise to embedded deformations of R. The advantage is that $\mathrm{V}_R(R)$ can often be computed explicitly.

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Analytic spread of filtrations

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(joint work with Steven Dale Cutkosky)

Let R be a Noetherian local ring and $\mathfrak{m}$ denote the unique maximal ideal of R. A collection $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of ideals in R is called a filtration if

$$I_0 = R,\ I_m \subset I_n \text{ for all } m \geq n \text{ and } I_m I_n \subset I_{m+n} \text{ for all } m, n \in \mathbb{N}.$$

A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ is said to be Noetherian if the graded ring $R[\mathcal{I}] = \bigoplus_{n\in\mathbb{N}} I_n$ is a finitely generated R-algebra. Otherwise, we say $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ is a non-Noetherian filtration.

For an ideal I in R, the filtration $\mathcal{I} = \{I^n\}$ is a Noetherian filtration. The Krull dimension of the ring $R[\mathcal{I}]/\mathfrak{m}R[\mathcal{I}]$ is called the analytic spread of I and it is denoted by $\ell(I)$. Geometrically, $\ell(I) = \delta + 1$ where δ is the dimension of the closed fiber $f^{-1}(\{\mathfrak{m}\})$ and $f : X \to \operatorname{Spec} R$ is the blow up of I. By upper semicontinuity of fiber dimension, we have $\ell(I_P) \leq \ell(I_{P'})$ for all $P, P' \in V(I)$ with $P \subset P'$. If $\ell(I) = 0$ then $I^n = 0$ for all $n \gg 0$. Analytic spread of I is bounded above by $\dim R$ and bounded below by the height of I, i.e., $\operatorname{height}(I) \leq \ell(I) \leq \dim R$. One interesting question is when the above-mentioned bounds are achieved.

The equality $\operatorname{height}(I) = \ell(I)$ holds if and only if all fibers of

$$f_0 : f^{-1}(\operatorname{Spec}(R/I)) \to \operatorname{Spec}(R/I) \quad (\text{restriction of } f)$$

have the same dimension. In 1980, McAdam proved the following fundamental theorem which gives the necessary and sufficient conditions for the equality $\ell(I) = \dim R$ to hold true in terms of the prime divisors of integral closures of powers of I [5].

Theorem 1. *Let R be a formally equidimensional local ring and I be an ideal in R. Then $\mathfrak{m} \in Ass(R/\overline{I^n})$ for some n if and only if $\ell(I) = \dim R$.*

Using the above result and the persistence property of integral closures of powers of ideals, one can immediately produce a result due to Burch [3] which says $\ell(I) = \dim R$ implies $\mathfrak{m} \in \operatorname{Ass}(R/\overline{I^n})$ for all $n \gg 0$.

In [1] Brodmann proved that $\ell(\mathcal{I}) \leq \dim R - \liminf_n \operatorname{depth} R/I^n$ where $\mathcal{I} = \{I^n\}$. If R has infinite residue field then Burch improved the result of Brodmann for the filtration $\mathcal{I} = \{\overline{I^n}\}$ and proved that $\ell(\mathcal{I}) \leq \dim R - \liminf_n \operatorname{depth} R/\overline{I^n}$ [3]. This result was generalized to the filtration $\mathcal{I} = \{I^{(n)}\}$ if the Symbolic Rees algebra of I is finitely generated [2].

Motivated by the above results, we define analytic spread of a filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ where the filtration is not necessarily Noetherian. We first show that the Krull dimension of the ring $R[\mathcal{I}]/\mathfrak{m}R[\mathcal{I}]$ is bounded above by $\dim R$ and define $\ell(\mathcal{I}) := \dim R[\mathcal{I}]/\mathfrak{m}R[\mathcal{I}]$. We define $\operatorname{height} \mathcal{I} = \operatorname{height} I_n$ for any $n \geq 0$.

An easy example of a non-Noetherian filtration $\mathcal{I} = \{I_n = \mathfrak{m}\}$ in a Noetherian local ring $(R, \mathfrak{m})$ with $\dim R \geq 1$ shows that $\ell(\mathcal{I}) = 0 < \operatorname{height} \mathcal{I} = \dim R$ and $I_n \neq 0$ for all $n \geq 0$. We show some of the above mentioned classical results of analytic spread of an ideal extend to analytic spread of discrete valued filtrations and also illustrate examples to show the differences.

Let R be a local domain of dimension d with quotient field K. Let ν be a discrete valuation of K with valuation ring $\mathcal{O}_\nu$ and maximal ideal m_ν. Suppose that $R \subset \mathcal{O}_\nu$. For $n \in \mathbb{N}$, consider the valuation ideals

$$I(\nu)_n = \{f \in R \mid \nu(f) \geq n\} = m_\nu^n \cap R.$$

A discrete valued filtration of R is a filtration $\mathcal{I} = \{I_n\}$ of ideals such that there exist discrete valuations $\nu_1, \ldots, \nu_r$ and $a_1, \ldots, a_r \in \mathbb{Z}_{>0}$ such that for all $n \in \mathbb{N}$,

$$I_n = I(\nu_1)_{na_1} \cap \cdots \cap I(\nu_r)_{na_r}.$$

A divisorial valuation of R is a valuation ν of K such that if $\mathcal{O}_\nu$ is the valuation ring of ν with maximal ideal $\mathfrak{m}_\nu$, then $R \subset \mathcal{O}_\nu$ and if $\mathfrak{p} = \mathfrak{m}_\nu \cap R$ then $\mathrm{trdeg}_{\kappa(\mathfrak{p})}\kappa(\nu) = \mathrm{ht}(\mathfrak{p}) - 1$, where $\kappa(\mathfrak{p})$ is the residue field of $R_\mathfrak{p}$ and $\kappa(\nu)$ is the residue field of $\mathcal{O}_\nu$. A divisorial valuation is a discrete valuation.

A divisorial filtration of R is a discrete valued filtration $\mathcal{I} = \{I_n\}$ such that there exist divisorial valuations $\nu_1, \ldots, \nu_r$ and $a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}$ such that for all $n \in \mathbb{N}$,

$$I_n = I(\nu_1)_{na_1} \cap \cdots \cap I(\nu_r)_{na_r}.$$

If $\mathcal{I} = \{I_n\}$ is a discrete valued filtration then $I_n = \overline{I_n}$ for all $n \geq 1$. We first prove the following.

Theorem 2. [4] *If $(R, \mathfrak{m})$ is a d-dimensional excellent local domain and $\mathcal{I} = \{I_n = I(\nu_1)_{na_1} \cap \cdots \cap I(\nu_r)_{na_r}\}$ is a divisorial filtration of $\mathfrak{m}$-primary ideals on R then $\ell(\mathcal{I}) = d$.*

The main ingredient of the proof is the following. We first consider the filtration $\{J_n = J(\nu_1)_{a_1 n} \cap \cdots \cap J(\nu_r)_{a_r n}\}$ in the normalization S of R where $J(\nu_i)_m = \{f \in S \mid \nu_i(f) \geq m\}$ for all $i = 1, \ldots, r$. Then we show that $\oplus_{n \geq 0} J_n$ is integral over $R[\mathcal{I}]$ and construct a chain of distinct prime ideals $C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_d$ in $\oplus_{n \geq 0} J_n$ with $\mathfrak{m}R[\mathcal{I}] \subset C_0 \cap R[\mathcal{I}]$. Therefore $C_0 \cap R[\mathcal{I}] \subset C_1 \cap R[\mathcal{I}] \subset C_2 \cap R[\mathcal{I}] \subset \cdots \subset C_d \cap R[\mathcal{I}]$ is a chain of distinct prime ideals in $R[\mathcal{I}]$ and hence $\ell(\mathcal{I}) = d$.

Next we deal with the "if" condition of Theorem 1 and prove the following.

Theorem 3. [4] *Suppose that $(R, \mathfrak{m})$ is a local domain and $\mathcal{I} = \{I_n\}$ is a discrete valued filtration in R. Let $I_n = I(\nu_1)_{na_1} \cap \cdots \cap I(\nu_r)_{na_r}$ for $n \geq 1$, some valuations ν_i and some $a_1, \ldots, a_r \in \mathbb{Z}_{>0}$. Suppose that $\ell(\mathcal{I}) = \dim R$. Then for some ν_i, the center $m_{\nu_i} \cap R = \{f \in R \mid \nu_i(f) > 0\}$ is $\mathfrak{m}$. Moreover, there exists a positive integer n_0 such that $\mathfrak{m}$ is an associated prime of $I_n = \overline{I_n}$ for all $n \geq n_0$.*

As a consequence of the above result, we get if R is a local domain, $\dim R \geq 1$ and $\mathcal{I} = \{I_n = I(\nu_1)_{na_1} \cap \cdots \cap I(\nu_r)_{na_r}\}$ is a discrete valued filtration in R with either $\mathfrak{m} \in \mathrm{Ass}(R/I_t)$ for some $t \geq 1$ or $\dim R/m_{\nu_i} \cap R = 1$ for all $i = 1, \ldots, r$ then $\ell(\mathcal{I}) \leq \dim R - \liminf_n \operatorname{depth} R/I_n$ [4].

We construct examples of 3-dimensional regular local rings and height two prime ideals p such that $\ell(\mathcal{I}) = 0, 1, 2$ can occur where $\mathcal{I} = \{p^{(n)}\}$ and for $\ell(\mathcal{I}) = 0, 1$ cases upper semicontinuity of fiber dimension fails [4].

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Differential operators of low order for an isolated hypersurface singularity

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(joint work with Rachel Diethorn, Jack Jeffries, Nick Packauskas, Josh Pollitz, Hamid Rahmati, and Sophia Vassiliadou)

The history of differential operators goes far back to understanding the theory of differential equations. In the past half century, it played an important role in algebraic geometry due to the development of the theory of D-modules. The ring of differential operators is defined as follows.

Definition 1 (Grothendieck [3])**.** Let $\phi\colon k \to R$ be a homomorphism of commutative rings. The module of k-linear differential operators on R of order at most i, denoted by $D^i_{R|k}$, is defined inductively as follows:

- $D^0_{R|k} = \operatorname{Hom}_R(R,R) \cong R\,;$
- $D^i_{R|k} = \{\delta \in \operatorname{Hom}_k(R,R) \mid \delta\circ\mu - \mu\circ\delta \in D^{i-1}_{R|k} \text{ for all } \mu \in D^0_{R|k}\}\,.$

One has the following order filtration

$$D^0_{R|k} \subseteq D^1_{R|k} \subseteq D^2_{R|k} \subseteq \cdots$$

of their union, which is defined to be the ring $D_{R|k}$ of k-linear differential operators on R with product given by composition of operators. It is an R-module, but not an R-algebra as $D^0_{R|k} = R$ is not in the center of $D_{R|k}$.

To motivate this classic inductive definition, we see the operators of order 1 are exactly the k-linear derivations on R. Indeed, note that a k-linear map $\delta\colon R \to R$ is a derivation if it satisfies

$$\delta(ab) = \delta(a)b + a\delta(b) \qquad \text{for all } a, b \in R$$

Rearranging this and setting μ_a to be multiplication by a, one gets

$$(\delta\circ\mu_a - \mu_a\circ\delta)(b) = \delta(a)b$$

so that the commutator $[\delta, \mu_a]$ is the same as $\mu_{\delta(a)} \in D^0_{R|k}$. In fact, one has that

$$D^1_{R|k} = \operatorname{Der}_k(R) \oplus R.$$

Similarly, the higher order operators are those that satisfy a higher order analogue of the product rule.

We assume that k is a field of characteristic zero for the remainder of this report, including the historical portions.

The quintessential example of a ring of differential operators is that of the operators for the polynomial ring $R = k[x_1, \ldots, x_n]$. This is the Weyl algebra

$$D_{R|k} = R\langle \partial_1, \ldots, \partial_n \rangle = k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

where ∂_i is the operators given by partial differentiation with respect to x_i. Here the differential operators of order i are the R-linear combinations of partial derivatives of order at most i. More generally, Grothendieck showed that $D_{R|k}$ is generated as an R-algebra by the derivations under composition whenever R is smooth. In 1961, Nakai [6] conjectured that the converse should hold; this is now known as *Nakai's Conjecture* and implies the well-known *Lipman-Zariski Conjecture* [5]. Nakai's conjecture is still wide open outside of a handful of cases.

We consider the non-smooth case, where the phenomena are radically different. Nevertheless, studying $D_{R|k}$ when R is singular is an old and interesting problem that has seen a revival of interest lately, especially with its connections with simplicity of D-modules.

Let k be a field of characteristic zero and R a finitely generated standard graded k-algebra, so

$$R = Q/I \text{ where } Q = k[x_1, \ldots, x_n] \text{ and } I \text{ homogeneous.}$$

It is straightforward to see that the k-linear differential operators on R are the elements of the Weyl algebra for which the ideal I is invariant, more precisely:

$$D_{R|k} \cong \frac{\{\delta \in D_{Q|k} \mid \delta(I) \subseteq I\}}{ID_{Q|k}}.$$

Now Kantor showed that, for quotient singularities R, the ring $D_{R|k}$ is still quite tame, and in fact a finitely generated k-algebra. For example, this includes normal 2-dimensional rational hypersurfaces, such as $k[x, y, z]/(x^2 + y^2 + z^2)$. However, beyond this situation or that for rings coming from combinatorial objects, the ring of differential operators seems much wilder.

For isolated singularity hypersurface rings of the form $R = k[x, y, z]/(f)$, Vigué [7] established that $D_{R|k}$ is not generated by the operators of any bounded order and has no differential operators of negative degree when $R = k[x, y, z]/(f)$ is an isolated singularity hypersurface with f homogeneous of degree at least 3; this generalizes the work of Bernstein, Gel'fand, and Gel'fand on the cubic cone $k[x, y, z]/(x^3 + y^3 + z^3)$ in [1]. Moreover, Vigué showed that in each order i, the module $D^i_{R|k}$ has at least 3 generators that are not in the R-subalgebra of $D_{R|k}$ generated by lower order operators. Existence of these operators was determined abstractly by an analysis of sheaf cohomology.

However, when R is singular, even in specific examples, it is extremely difficult to determine the differential operators of each order that are not compositions of lower order operators. We develop a new homological approach for finding these operators, by discovering that the resolutions of each $D^i_{R|k}$ have an unexpectedly beautiful structure, as described in the result below; this finding was aided in part by the computer algebra system *Macaulay2* [2]. Once we developed a coherent surmise for the matrix factorizations of these resolutions, we were able to work "forwards" in the resolution to find the generating operators.

Once we found the generating operators, we were also able to express them in terms of the Euler derivation,

$$E = x\partial_x + y\partial_y + z\partial_z,$$

and the Hamiltonian derivations,

$$H_{yz} = f_z\partial_y - f_y\partial_z, \quad H_{zx} = f_x\partial_z - f_z\partial_x, \quad \text{and} \quad H_{xy} = f_y\partial_x - f_x\partial_y.$$

These four operators form a minimal generating set for the module of derivations, whose minimal resolution was found by Herzog and Martsinkovsky [4].

Our main results give the generators and minimal R-free resolutions of $D^i_{R|k}$ for $i = 2, 3$. We present the $i = 2$ case here as the other case is similar but more complex to describe.

Theorem 2. *Assume $R = k[x,y,z]/(f)$ is an isolated hypersurface singularity where f is homogeneous of degree $d \geqslant 3$ and k is a field of characteristic zero.*

1. *A minimal set of generators for $D^2_{R|k}$ is given by*

$$\{1,\, E, H_{yz},\, H_{zx},\, H_{xy},\, E^2,\, EH_{yz},\, EH_{zx},\, EH_{xy},\, A_x,\, A_y,\, A_z\},$$

with
$$A_x = \frac{1}{x}\left[H_{yz}^2 + \frac{1}{(d-1)^2}\Delta_{xx}E^2 + \frac{d-2}{(d-1)^2}\Delta_{xx}E\right]$$

where Δ_{xx} is the 2×2-minor obtained by deleting the 1st row and 1st column of the Hessian matrix of f, and similar formulas hold for A_y, A_z.

2. *The minimal free resolution is 2-periodic supported on the matrix factorization obtained from the free complex over Q that is the mapping cone of a chain map from the Koszul complex* $\operatorname{Kos}(x,y,z)$ *into the totalization of the following diagram*

$$\begin{tikzcd}
\operatorname{Kos}(f_x,f_y,f_z) \arrow[d, "\bigwedge \mathrm{Hessian}(f)"] & Q \arrow[r, "D_3"] \arrow[d, "\alpha_3"] & Q^3 \arrow[r, "D_2"] \arrow[d, "\alpha_2"] & Q^3 \arrow[r, "D_1"] \arrow[d, "\alpha_1"] & Q \arrow[d, equal, "\alpha_0"] \\
\operatorname{Kos}(x,y,z) & Q \arrow[r, "\partial_3"] & Q^3 \arrow[r, "\partial_2"] & Q^3 \arrow[r, "\partial_1"] & Q
\end{tikzcd}$$

where the rows are the dg algebras give by Koszul complexes and the chain map between the rows is induced by the map induced by the map on generators of these algebras by the Hessian matrix of second partial derivatives.

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Syzygies of the Cotangent Complex

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(joint work with Srikanth J. Iyengar)

Jacobian criteria for classical invariants. We shall assume throughout that $\varphi\colon R \to S$ is a homomorphism of commutative noetherian rings, essentially of finite type, so that

$$S = U^{-1}R[x_1, \ldots, x_m]/(f_1, \ldots, f_n).$$

The module of differentials. The module of differentials can be presented as the cokernel of the Jacobian matrix

$$\Omega^1_{S/R} = \operatorname{coker}\left(\tfrac{\partial f_j}{\partial x_i}\right).$$

In the case that $R = k$ is a field, the classical Jacobian criterion asserts that S is a smooth k algebra exactly when $\Omega^1_{S/k}$ is a projective S-module and $\operatorname{rank}_{S_\mathfrak{p}} \Omega^1_{S/k} = \dim S_\mathfrak{p}$ for all primes $\mathfrak{p} \subseteq S$.

It is natural to ask how weaker homological conditions on $\Omega^1_{S/k}$, or on related modules, correspond to weaker constraints on the singularities of S. It is the philosophy of Avramov and Herzog that results of this form can be thought of as *higher Jacobian criteria* [2]. The first such example we state is independently due to Ferrand [6] and Vasconcelos [10]: Assuming that k has characteristic zero, S is a reduced complete intersection ring if and only if $\Omega^1_{S/k}$ has projective dimension no more than 1 over S.

Vasconcelos made the stronger conjecture that, when k has characteristic zero, if $\operatorname{projdim}_S \Omega^1_{S/k}$ is finite then S is a reduced complete intersection. Various cases of this conjecture have been established, notably the graded case by Avramov and Herzog [2]. We show in [4] that Vasconcelos' conjecture holds contingent on the Eisenbud-Mazur conjecture (which was known to hold more generally).

An interesting example is the coinvariant algebra for the symmetric group S_n acting by permuting the coordinates of k^n, having a presentation

$$S = k[x_1, \ldots, x_n]/(x_1 + \cdots + x_n,\ \ldots\ , x_1^n + \cdots + x_n^n),$$

with k a field of characteristic zero. Up to rescaling, the Jacobian matrix is the VanderMonde matrix, and this explains the right-hand part of the sequence below:

$$\cdots \longrightarrow S^n \xrightarrow{\begin{pmatrix} x_1^n & \cdots & x_n^n \\ \vdots & & \vdots \\ x_1 & \cdots & x_n \end{pmatrix}} S^n \xrightarrow{\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix}} S^n \longrightarrow \Omega^1_{S/k} \longrightarrow 0.$$

The ring S is complete intersection, but it is not reduced, and it follows that $\mathrm{projdim}_S(\Omega^1_{S/k}) > 1$. However, the Jacobian presentation of $\Omega^1_{S/k}$ can be continued as shown into a 2-periodic resolution. This resolution comes from a matrix factorisation of the symmetric polynomial $x_1^n + \cdots + x_n^n$; see [3] for an explanation and generalisation to other reflection groups. This example raises the question of the meaning of the (necessarily polynomial) rate of growth of the resolution of $\Omega^1_{S/k}$, for non-reduced complete intersection rings.

The conormal module. Assume that the homomorphism $\varphi\colon R \to S$ is surjective with kernel $I = (f_1, \ldots, f_n)$. The S-module

$$I/I^2$$

is called the conormal module of φ. It controls the deformation theory of $\mathrm{Spec}(S)$ inside $\mathrm{Spec}(R)$. Ferrand [6] and Vasconcelos [11] independently established the following Jacobian criterion for I/I^2: Assuming that $\mathrm{projdim}_R\, S$ is finite (for example, if S is regular), the map φ is locally complete intersection if and only if I/I^2 is a projective S-module.

Vasconcelos later conjectured that one can weaken the condition on I/I^2 to having finite projective dimension over S [12]. This is a rigidity statement: it means that $\mathrm{projdim}_S\, I/I^2$ can only be 0 or ∞. A number of authors made progress on this and related problems (cf. [2]), and we establish the conjecture in [4]:

Theorem 1. *Assume that* $\mathrm{projdim}_R\, S$ *is finite. Then φ is locally complete intersection if and only if I/I^2 has finite projective dimension over S.*

The first Koszul homology module. Assuming still that $\varphi\colon R \to S$ is surjective, we consider the degree one homology of the associated Koszul complex:

$$\mathrm{H}^1(I; R) = \mathrm{H}^1[\mathrm{Kos}^R(f_1, \ldots, f_n)].$$

After Buchsbaum's classical characterisation of regular sequences in terms of the vanishing of Koszul homology, it was Gulliksen [7] who established the first higher analogue of the Jacobian criterion for Koszul homology: Assuming that $I \subseteq R$ has finite projective dimension, I is locally generated by a regular sequence if and only if $\mathrm{H}^1(I; R)$ is a projective module over $S = R/I$. Vasconcelos and others considered the weaker condition that $\mathrm{projdim}_R\, \mathrm{H}^1(I; R)$ is finite [12, 2]. We prove the corresponding rigidity statement in [4]:

Theorem 2. *Assume that $I \subseteq R$ has finite projective dimension. Then I is locally generated by a regular sequence if and only if* $\mathrm{projdim}_S\, \mathrm{H}^1(I; R) < \infty$.

The cotangent complex. The cotangent complex is a complex of projective S-modules

$$\mathbb{L}_{S/R} = L_0 \leftarrow L_1 \leftarrow L_2 \leftarrow \cdots$$

defined by Quillen to be the total left derived functor of the module of differentials $\Omega^1_{-/R}$, applied to S; see [9] for more information on how to compute this object.

Let us say that $\operatorname{projdim}(\mathbb{L}_{S/R}) = n$ if n is the smallest integer such that $\mathbb{L}_{S/R}$ is quasi-isomorphic to a complex of projective S modules concentrated in degrees n and below. With this the cotangent complex detects geometric conditions: φ is étale exactly when $\mathbb{L}_{S/R} \simeq 0$; φ is smooth exactly when $\operatorname{projdim}(\mathbb{L}_{S/R}) \leq 0$; and φ is locally complete intersection exactly when $\operatorname{projdim}(\mathbb{L}_{S/R}) \leq 1$; see [9].

Quillen conjectured, when $\operatorname{projdim}_R S < \infty$, that if $\operatorname{projdim}(\mathbb{L}_{S/R})$ is finite then φ is locally complete intersection. This was proven by Avramov [1]. At the same time, Quillen also conjectured that for *any* homomorphism φ, if $\operatorname{projdim}(\mathbb{L}_{S/R})$ is finite then $\operatorname{projdim}(\mathbb{L}_{S/R}) \leq 2$. This second conjecture remains largely open.

The higher cotangent modules. In [2, 5] the ith cotangent module of φ is defined as a syzygy of the cotangent complex:

$$\mathrm{C}_i(S/R) = \operatorname{coker}(L_{i+1} \xrightarrow{\partial} L_i).$$

The first few cotangent modules are familiar: $\mathrm{C}_0(S/R) = \Omega^1_{S/R}$ is the module of differentials; $\mathrm{C}_1(S/R) = I/I^2$ is the conormal module of the surjection $U^{-1}R[x_1, \ldots, x_m] \to S$; and $\mathrm{C}_2(S/R) = \mathrm{H}^1(I; U^{-1}R[x_1, \ldots, x_m])$ is the Koszul homology. In [5] we generalise Theorems 1 and 2 to all of the higher $\mathrm{C}_i(S/R)$.

Theorem 3. *Assume that* $\operatorname{projdim}_R S$ *is finite. If there exists* $i \geq 1$ *such that* $\operatorname{projdim}_S \mathrm{C}_i(S/R) < \infty$*, then* φ *is locally complete intersection (and then all* $\mathrm{C}_i(S/R)$ *are projective).*

If $\operatorname{projdim}(\mathbb{L}_{S/R}) \leq n$ then $C_i(S/R) = 0$ for all $i > n$, and Theorem 3 implies that φ is locally complete intersection. Therefore, we obtain a new proof of Quillen's first conjecture on the cotangent complex. In fact we obtain the stronger statement that the cotangent complex cannot be zero in any positive degree, analogous the Halperin's Theorem on the nonvanishing of deviations [8].

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