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Mathematical Advances in Geophysical Fluid Dynamics

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Abstract. The workshop "Mathematical Advances in Geophysical Fluid Dynamics" addressed recent advances in modeling, analytical, computational and stochastical studies of geophysical flows. Of particular interest were contributions concerning modeling and analysis of sea ice models, well-posedness results for the primitive equations and boundary layers, stratified flows and models for moist atmospheric dynamics including phase transitions.

Mathematics Subject Classification (2010): 76-XX, 86A10, 35-XX, 60Hxx.

Introduction by the Organizers

This workshop was fostering the investigation of certain classes of geophysical models by models stemming from analysis, computation, stochastics and modeling. The complexity of fluid models taking into account geophysical considerations showed the need for reliable reduced models.

The mathematical investigation of geophysical flows involves many modern techniques from analysis, stochastics and computation. Of special interest are local and global well-posedness properties of the associated systems of equations, such as the primitive equations and boundary layers, their rigorous justification, the development of numerical and computational schemes, the incorporation of stochastic forces and stochastic boundary conditions as well as non-uniqueness results by convex integration. A new development in this context is the rigorous mathematical understanding of various sea-ice models. The complexity of these models requires new analytical and computational methods. A particular challenge is the investigation of coupled atmosphere-sea ice-ocean models.

Models from moist atmospheric dynamics including phase transitions also ask for new analytical and computational tools in order to treat fast time saturation effects.

A main characteristic of this workshop was bringing together leading experts form diverse scientific backgrounds such as analysis, modeling, numerics and computations, stochastic analysis and convex integration. The meeting ignited lively discussions and exchange of ideas. The presence of early career participants and gender diversity was very visible during the meeting. The workshop also aimed to encourage early career participants to play an important role in this area of research.

The lectures presented took 40 minutes which were followed by lively and interactive discussions for about 15 minutes.

All together, the workshop brought together an excellent mixture of various communities and several leaders from different disciplines met in person for the first time. Evening sessions attracted special attention, where graduate students as well as postdoctoral fellows gave excellent presentations about their research work. We are convinced that the scientific exchange between the participants will lead to many exciting new developments and collaborations.

Workshop: Mathematical Advances in Geophysical Fluid Dynamics

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Abstracts

The effect of concavity on the stability of boundary layer flows

David Gerard-Varet
(joint work with S. Iyer, Y. Maekawa, N. Masmoudi)

We investigate the role of concavity in three stability problems related to boundary layer flows:

- the stability of Prandtl expansions in 2D Navier-Stokes
- the well-posedness of hydrostatic Navier-Stokes equations
- the well-posedness of the triple deck model

On one hand, it is well-known that in the absence of further structural assumptions, positive results (stability, local well-posedness) are only possible in the analytic framework. On the other hand, in the absence of diffusion, concavity allows for Sobolev stability. We show that when considering both diffusion and concavity, one can obtain results in the intermediate Gevrey setting.

Reduced models for tropical climate dynamics: Influence of the tropics on extra-tropics

Boualem Khouider
(joint work with H. Shin)

The tropics receive the majority of Earth's solar energy intake at its surface. To maintain a balanced climate state this energy is transported poleward through various atmospheric and oceanic patterns. The complexity of tropical climate dynamics however, which harbours a wide spectrum of waves and vortices, hinders the clear understanding of the complex interactions between the tropics and extra-tropics and limits weather and climate predictability. In my talk, I showcase via the use of basic examples how simple mathematical models of reduced dynamics can help shed light on some of such interactions. In particular, when the hydrostatic primitive equations, that govern the planetary atmospheric flow are projected onto the barotropic, and first baroclinic modes of vertical structure, a coupled set of two systems of PDE's, representing the two modes emerges. The two systems are two-way coupled to each other through non-linear cross advection terms that allow them to exchange kinetic energy. The barotropic system which in effect reduces to the 2d incompressible Euler equations is transparent to thermal perturbations while the baroclinic mode is fully coupled to the temperature/energy equation. When taken separately, the barotropic mode equations have the vortical-type planetary Rossby waves as their exact/fundamental solutions while the barolinic mode equations harbour an infinite spectrum of equatorially trapped waves including the Kelvin, the mixed Rossby-gravity, and the inertia-gravity waves. The barotropic Rossby waves propagate westward and poleward and as such they can transport energy from the tropical to the extra-tropics while the equatorially trapped waves remain in the vicinity of the tropics and propagate in both the eastward and westward directions along the equator. As it has been amply demonstrated (through both satellite observations and climate models), the equatorially trapped waves are the main modes through which the tropical atmosphere responds to both sensible and latent heating from clouds and convection in the tropics. As a proof of concept, it is demonstrated here, through numerical experiments, that when the barotropic equations are taken alone as a forced system, forced by freely-moving equatorial waves (which is of course a severe simplification, as the feedback onto those waves is ignored–in reality the two systems are two-way coupled), a barotopic response consistent of a phase locked response that propagates with the wave and a multitude of planetary-Rossby waves that can carry this response to the extra-tropics [1]. As a demonstration of the relevance of this barotropic response to tropical wave forcing for the climate system, we considered the case of a tropical cyclone that evolves in this barotropic response background. It is found that the tropical waves in general can have a significant–stochastic effect of the path of the cyclone which may strongly influence tropical cyclone predictability [2]. Given that the forecast of tropical cyclones such as hurricanes and typhoons can be sometimes chalenging for weather prediction centres around the world (examples are plentiful) and that typically climate models (and in particular the numerical weather prediction models used for long range hurricane forecasting) are notoriously known to misrepresent tropical waves in particular, and tropical weather and climate variability in general, the work presented here calls for a route for possible improvement of tropical cyclone forecasts through the improvement of tropical wave representation and of convection parametrization in particular in those models.

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Multi-scale modeling of Arctic sea ice and floes

Samuel N. Stechmann

(joint work with Q. Deng and N. Chen [1], and with A. Davis, D. Giannakis, and G. Stadler)

As computational power increases, it is becoming more feasible to model sea ice as a collection of ice floes, as opposed to the more traditional approach of treating sea ice as a material, as a continuum, with various rheological models proposed. In this talk, I will discuss approaches for floe-based (i.e., particle-based) models of sea ice, including some approaches to multi-scale modeling that aim to bridge the gap between floe-based and continuum-based models.

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Numerical modeling of viscous-plastic sea-ice dynamics

Carolin Mehlmann
(joint work with S. Danilov, P. Korn, T. Richter, G. Stadler)

Subject of this talk are the mathematical challenges and the numerical treatment of large scale sea-ice problems. The model under consideration goes back to Hibler [1] and is based on a viscous-plastic description of sea ice.

In the model, sea ice is considered as a two dimensional fluid which is located between ocean and atmosphere. Sea ice is characterized by three variables: the sea-ice concentration A (the percentage of a grid cell that is covered with ice), the mean ice thickness H and the sea-ice velocity $\mathbf{v}$. The sea-ice concentration and the sea-ice thickness are advected in time by transport equations, whereas the sea-ice velocity is determined by the following momentum equation:

$$m\partial_t \mathbf{v} = F + div(\boldsymbol{\sigma}), \tag{1}$$

where m is the sea-ice mass and F are the external forces (e.g. wind ocean/drag). The tensor $\boldsymbol{\sigma}$ describes internal sea-ice stresses. The ice stresses are related to the strain rate tensor $(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$ by the viscous-plastic rheology

$$\sigma = \frac{1}{2}\zeta(\mathbf{v})\Big(\nabla\mathbf{v} + \nabla\mathbf{v}^T\Big) + \frac{3}{4}\zeta(\mathbf{v})\mathrm{tr}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)I - \frac{P}{2}I,$$

where the viscosity ζ and the ice strength P is chosen as

$$\zeta := \frac{P}{2\max(\Delta, 2\cdot 10^{-9})}, \quad P := H\exp(20(1-A)),$$
$$\Delta := \sqrt{1.25(\mathbf{v}_{1,x}^2 + \mathbf{v}_{2,y}^2) + 0.5(\mathbf{v}_{1,y} + \mathbf{v}_{2,x})^2 + 1.5(\mathbf{v}_{1,x}\mathbf{v}_{2,y})}.$$

We reformulate the sea-ice momentum equation (1) in order to find a suitable presentation for applying numerical analysis and modern approximation techniques. Key is the application of a proper regulariziation of the maximum in the denominator of ζ. As suggested by Kreyscher et al. in [2], we use

$$\zeta = \frac{P}{2\sqrt{\Delta^2 + 4\cdot 10^{-18}}}.$$

In a second step, we apply the decomposition of the strain rate tensor

$$\frac{1}{2}\big(\nabla\mathbf{v} + \nabla\mathbf{v}^T\big) =: \boldsymbol{\epsilon} = \boldsymbol{\epsilon}' + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\epsilon})I \tag{2}$$

and write the stress tensor in its weak formulation as

$$\Big(\boldsymbol{\sigma}, \nabla\phi\Big) = 2\Big(\zeta\boldsymbol{\tau}(\boldsymbol{\epsilon}), \boldsymbol{\tau}(\boldsymbol{\phi})\Big) + \Big(\frac{P}{2}, \mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{\phi})\Big) = \mathcal{A}(\mathbf{v})(\boldsymbol{\phi}) + \Big(\frac{P}{2}, \mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{\phi})\Big), \tag{3}$$

with $\boldsymbol{\tau}(\boldsymbol{\epsilon}) := \frac{1}{2}\boldsymbol{\epsilon}' + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\epsilon})I$. $\mathcal{A}(\mathbf{v})(\boldsymbol{\phi})$ has similarities to the regularized p-Laplacian and the time-dependent minimal surface probelm [9]. Furthermore the Jacobian of the momentum equation (1) is symmetric and positive definite [8], such that the time-discretized problem can be written as convex function which is bounded from below [12]. The viscous-plastic rheology introduces a strong nonlinearity to the sea-ice model. Thus, solving the sea-ice momentum equation with increasing spatial resolution is extremely difficult. The analysis based on the reformulated stress tensor (3) leads to the development of efficient Newton-type solvers for the sea-ice momentum equation such as [8] or [12].

For the spatial discretization we suggest the use of a nonconforming finite element, the Crouzeix-Raviart (CR) element. The element places the degrees of freedom at edge-midpoints of a cell. Therefore the CR element allows for a direct coupling to ocean models with the same type of staggering (e.g ICON-O [3]). The CR element needs a stabilization. The instability of the element has its origin in the discretization of the symmetric strain rate tensor in the rheology. Korn's inequality is not uniformly satisfied by the CR element [4]. In order to circumvent this instability we follow the idea of Hansbo and Larson [6] and introduce a stabilization of the Crouzeix-Raviart element, see [7]. To show that the stabilized momentum equation is qualitatively consistent with the solution of the sea-ice equations an energy estimate is derived. From an evaluation in a numerical experiment we infer that the derived energy functional stays bounded as in the estimate for the continuous case, provided the stabilization is applied. Without stabilization the energy functional grows with increasing mesh resolution showing a qualitatively different behavior compared to the solution of the continuous sea-ice equations.

In the global ocean model ICON-O we demonstrate that the CR discretization can capture the large-scale sea-ice drift [11]. Compared to other low order discretizations, the CR element has appealing resolution properties [10]. At high spatial mesh resolution the CR discretization resolves more deformation characteristics on grids with less degrees of freedom compared to all the other low order approximations.

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Kelvin-Helmholtz instability and relaxed solutions by space-time optimization

Yann Brenier

The Euler model of a homogeneous incompressible fluid moving in a compact domain $D \subset \mathbb{R}^d$ without external force, during a time interval $[0,T]$, can be described by the trajectories $t \in [0,T] \to X(t,a) \in D$ of its fluid parcels labelled by $a \in A$, where (A,μ) is a suitable probability space, which are accelerated by a *common* pressure field $p = p(t,x)$ and equally occupy the available volume, say

$$\partial_{tt}X(t,a) = -(\nabla p)(t,X(t,a)), \quad \int_A \phi(X(t,a))\mu(da) = \int_D \phi(x)dx, \quad \forall \phi \in C(\mathbb{R}^d)$$

(where the volume of D is normalized to 1). This can be written in "Vlasov" style, by introducing the nonnegative measure (denoted as a function by abuse)

$$f(t,x,\xi) = \int_A \delta(x - X(t,a))\delta(\xi - \partial_t X(t,a))\mu(da).$$

At least assuming $\nabla_x p$ to be continuous, one easily obtain:

$$(1) \qquad \partial_t f + \nabla_x \cdot (\xi f) - \nabla_\xi \cdot (\nabla_x p(t,x) f) = 0, \quad \int_{\xi \in \mathbb{R}^d} f(t,x,\xi) = 1,$$

including, integrating in $\xi \in \mathbb{R}^d$ with weight $(1,\xi)$, the moment conditions

$$(2) \qquad \nabla_x \cdot (\int_\xi \xi f) = 0, \quad \partial_t(\int_\xi \xi f) + \nabla_x \cdot (\int_\xi \xi \otimes \xi f) + \nabla_x p = 0.$$

Among special solutions of this "kinetic formulation" of the Euler model, we immediately recover the smooth solutions $(t,x) \in [0,T] \times D \to v(t,x) \in \mathbb{R}^d$ of the classical Euler's equations, just by setting $f(t,x,\xi) = \delta(\xi - v(t,x))$.

More surprisingly we recover in the case $d = 1$, *without any limit process*, the "hydrostatic limit" (in the sense of [6]) of the 2D Euler equations with $(x, z) \in D \times [0, 1]$

$$\partial_t u + \partial_x(u^2) + \partial_z(wu) + \partial_x p = 0, \ \ \partial_z p = 0, \ \ \partial_x u + \partial_z w = 0, \quad (u, w)(t, x, z) \in \mathbb{R}^2,$$

just by setting $f(t, x, \xi) = \int_0^1 \delta(\xi - u(t, x, z))dz$.

The "Vlasov formulation" of the Euler model goes back (at least) to [1] and can also be seen as the singular limit, as $\beta \to 0$, of the Vlasov-Poisson model for which $\beta\Delta_x p(t, x) = \int_{\xi\in\mathbb{R}^d} f(t, x, \xi) - 1$, which describes Coulombian interactions in Plasma Physics, or Newtonian interactions in Cosmology, according to the sign of constant β. In [2] we proposed a strategy to solve initial value problems by space-time convex optimization for various evolution PDEs, including the classical Euler equations. It is therefore tempting to extend this method to Vlasov equations. This has been done, with Ivan Moyano [3], for the gravitational Vlasov-Poisson equations, but our approach seems to fail in the limit case $\beta = 0$. In the present talk, we limit ourself to the less ambitious problem

$$\inf_{f\geq 0} \int_{(t,x,\xi)\in[0,T]\times D\times\mathbb{R}^d} |\xi|^2 f(t, x, \xi)/2$$

where f is only subject to the moment conditions (2), $\int_\xi \xi f$ being prescribed at $t = 0$ as a given divergence-free vector field v_0 over D. This reads as the infinite dimensional "linear program"

$$I = \inf_{f\geq 0} \sup_{A,r,q} \int_{(t,x)\in[0,T]\times D} (v_0(x) \cdot \partial_t A(t, x) + r(t, x))dxdt$$

$$+ \int_{t,x,\xi} \left(|\xi|^2/2 - \xi \cdot \partial_t A(t, x) - \nabla_x A(t, x) \cdot \xi \otimes \xi - r(t, x) - \nabla q(t, x) \cdot \xi\right) f(t, x, \xi)$$

where $r = r(t, x) \in \mathbb{R}$, $q = q(t, x) \in \mathbb{R}$, $A = A(t, x) \in \mathbb{R}^d$ are Lagrange multipliers for the moment conditions (2) and initial condition v_0 (all written in weak form), the vector field A being divergence-free, tangent to ∂D (in order to eliminate the pressure gradient in (2)) and vanishing at $t = T$. By convex duality, we may exchange the *inf* and the *sup*, giving

$$I = \sup_{A,r,q} \int_{(t,x)\in[0,T]\times D} (v_0 \cdot \partial_t A + r)dxdt$$

subject to the pointwise inequality: $\forall(t, x, \xi) \in [0, T] \times D \times \mathbb{R}^d$,

$$|\xi|^2/2 - \xi \cdot \partial_t A(t, x) - \nabla_x A(t, x) \cdot \xi \otimes \xi - \nabla q(t, x) \cdot \xi \geq r(t, x).$$

which is quadratic in ξ over $\mathbb{R}^d$, leading to the concave maximization problem

$$I = \sup_{A,q} \int \left(v_0 \cdot \partial_t A - \frac{1}{2} \left(\mathbb{I} - \nabla_x A - (\nabla_x A)^t\right)^{-1} \cdot (\partial_t A + \nabla q)^{\otimes 2} \right) dxdt$$

where A is a divergence-free vector field, tangent to ∂D, vanishing at $t = T$ and subject to $\mathbb{I} \geq \nabla_x A + (\nabla_x A)^t$ in the sense of symmetric matrices, pointwise. This problem is *exactly* the one addressed in [2] (which was obtained without any reference to the "Vlasov formulation" (1)), where it is established that any smooth solution v of the classical Euler equation provides a maximizer through $A(t,x) = (t-T)v(t,x)$ (and an analogous formula for q) as long as

$$(3) \qquad -\nabla_x v(t,x) - (\nabla_x v(t,x))^t \leq (T-t)^{-1}\mathbb{I}, \quad \forall (t,x) \in [0,T] \times D.$$

This condition looks appealing (and reminiscent of the Ponce criterion [8]) and numerical computations performed by Andrea Natale [7] confirm the result. However, in case of a shear flow $v(t,x) = v(t,x_1,x_2) = (U(x_2),0)$ (where D should be taken as $\mathbb{T} \times [-L,L]$), this condition means $|U'(x_2)| \leq T$ which is quite restrictive. In particular, this rules out the extreme case $U(x_2)$ =sign(x_2), which is typical of the Kelvin-Helmholtz instability. Nevertheless, as shown by Helge Dietert [4], in that precise case, the concave maximization problem can be solved and one obtain (provided $T \leq L$)

$$A(t,x) = (t-T)(U_T(x_2),0), \quad U_T(x_2) = \inf\left(1, \sup(-1, x_2/T)\right).$$

This looks as a very bad result since the maximization problem completely misses the given initial condition and provides a different shear flow, depending on the time interval $[0,T]$! However, we observe that $(U_T(x_2),0)$ viewed as a function of (T,x) belongs to the one-parameter family of "relaxed solutions" obtained by László Székelyhidi [9] through convex integration methods [5], which is supposed to give a good description of turbulent layers. As a matter of fact, our concave optimization problem fits very well to the concept of "subsolution" since, as shown in [2] by convex duality (at least when v_0 is continuous), it reads

$$I = \inf_{W,v,p} \int_{(t,x)\in[0,T]\times D} \mathrm{trace}(W(t,x))$$

where W is a symmetric matrix-valued measure subject to $W \geq v \otimes v$ in the sense of symmetric matrices and $\partial_t v + \nabla_x \cdot W + \nabla_x p = 0$, for some fields $v = v(t,x) \in \mathbb{R}^d$, $p = p(t,x) \in \mathbb{R}^d$, $v = v(t,x)$ being an L^2 divergence-free vector field tangent to ∂D with initial value v_0 at time $t=0$ in the weak sense.

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Compressible Navier–Stokes equations with potential temperature transport: Analysis and Numerics

Mária Lukáčová-Medviďová
(joint work with A. Schömer)

We have introduced dissipative measure-valued (DMV) solutions to the compressible Navier-Stokes system with potential temperature transport motivated by the concept of Young measures, see [1] for the concept of DMV solutions and their relation to weak and strong solutions. This model is often used in meteorological applications, but its analysis for any $\gamma > 1$ was not available in literature. In [2] we have proved global-in-time existence of DMV solutions by means of convergence analysis of a mixed finite element-finite volume method. Our results hold for the full range of adiabatic indices including the physically relevant cases in which the existence of global-in-time weak solutions was open. In [3] we have presented a DMV-strong uniqueness result for the compressible Navier–Stokes system with potential temperature transport. This implies that strong solutions are stable in the class of DMV solutions. Consequently, if a strong solution to the compressible Navier-Stokes system with potential temperature transport exists, we obtain the strong convergence of numerical solutions.

As an application of the DMV-strong uniqueness principle we have also derived a priori error estimates for a mixed finite element-finite volume method. The numerical solutions are computed on polyhedral domains that approximate a sufficiently a smooth bounded domain, where the exact solution exists. Novel consistency estimates were presented that allow to compare a strong solution on a smooth domain Ω with numerical solutions computed on polygonal domains Ω_h, $\Omega \subset \Omega_h$. Here, we only assume that $\mathrm{dist}(\mathbf{x}, \partial\Omega) = \mathcal{O}(\mathrm{h})$ for all $\mathbf{x} \in \partial\Omega_h$, see [3] for further details.

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From micro to macro in the fluid dynamics of sea ice

Kenneth M. Golden

Figure 1. Multiscale structure of sea ice. From left to right: X-ray tomography image of the salty fluid inclusions (brine) in sea ice from the Arctic Ocean [9]. Brine volume fraction, geometry, and connectedness depend strongly on temperature, and control fluid transport processes that are central to the role of sea ice in climate and polar ecosystems, such as the evolution of ponds on the surface of melting Arctic sea ice, and nutrient replenishment processes for algae and other microbes living in the brine inclusions; cross-polarized image of the polycrystalline microstructure of sea ice from the Ross Sea, Antarctica (Golden, Langhorne), where each individual crystal has complex fluid microstructure. Polycrystalline statistics and geometrical properties, and related fluid flow properties, depend strongly on conditions in the ocean during formation; pancake ice with a granular polycrystalline microstructure forming in a wave field in the Southern Ocean (Golden); the sea ice pack as a granular composite of ice floes in a sea water host, which displays large-scale fluid-like dynamics when viewed from space (NASA); satellite image of the Arctic Ocean with its sea ice cover (NASA). For perspective, rough image sizes from left to right are 1 cm, 5 cm, 50 m, 100 km, 10,000 km.

Sea ice exhibits complex composite structure on length scales ranging over many orders of magnitude. From the millimeter scale brine inclusion and centimeter scale polycrystalline microstructures, to the evolution of melt ponds on the surface of Arctic sea ice and the ice pack itself, fluid dynamics is central to understanding sea ice behavior. A principal challenge in modelling sea ice and its role in climate and polar ecosystems is how to use information on smaller scale structure to find the effective or homogenized behaviour on larger scales relevant to climate and ecosystem models. In other words, how do we predict macroscopic behavior from microscopic laws and information? We'll give an overview of recent results on modelling macroscopic behaviour in the sea ice system, with a focus on novel mathematical approaches, and the central role that fluid behavior over a tremendous range of length and time scales plays in studying sea ice, and the climate system more broadly. Percolation theory for fluid flow, fractal geometry of the brine microstructure, and Stieltjes integral representations for homogenized parameters of two phase and polycrystalline composites, as well as for advection diffusion processes and ocean surface waves in the sea ice cover would be considered. Spectral analysis of these representations leads to a random matrix theory

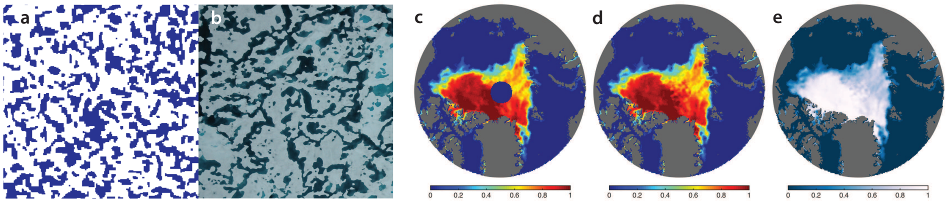

Figure 2. (a) Ising model simulation of melt ponds [7]; (b) photo of real melt ponds (Perovich); (c) example of the polar data gap (blue disc) on 30 August 2007 with shading indicating sea ice concentration; (d) and (e) show the data fill in [11], with the shading in (e) similar to that used by the National Snow and Ice Data Center (http://nsidc.org).

picture of connectedness processes in sea ice, with parallels to Anderson localization and semiconductor physics. Melt pond connectedness and complexification can also be viewed through the lens of Morse theory and persistent homology in topological data analysis, and the Euler characteristic curve as a function of pond water level in particular. Related heterogeneity in the parameters of nonlinear algal bloom models is addressed through polynomial chaos methods in uncertainty quantification, to analyse effective bloom dynamics when the local parameters are random variables, which is the case for the highly heterogeneous fluid microstructure of sea ice. Finally, we consider the application of homogenization ideas to the large scale dynamics of the marginal ice zone (MIZ), the transitional region between the dense inner core of pack ice and open ocean. We have developed a rather simple "mushy layer" model that quantitatively explains the dramatic annual cycle of MIZ width and location. Our work is helping to advance how sea ice is represented in global climate models, and improve projections of the fate of Earth's sea ice packs and the polar ecosystems they support.

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On a compatibility condition for the Navier-Stokes solutions in maximal L^p-regularity class

Hideo Kozono
(joint work with S. Shimizu)

We consider a compatibility condition on the initial data and the external force for the initial-boundary value problem of the Navier-Stokes equations with no-slip condition in bounded domains. Our class of solutions is based on that of maximal L^s-regularity in the domain of fractional powers of the Stokes operator in L^p. We show that if the solution belongs to some spaces with higher ordered derivatives, then the compatibility condition is necessarily satisfied.

Dynamics of jet variability

Peter H. Haynes

A canonical process in atmosphere-ocean dynamics is the spontaneous formation of zonal jets in turbulent flow on a β-plane. (The β-plane is a mathematical construction that includes the important dynamical effect of spherical geometry on a fluid on the surface of a rotating planet, which is that the vertical component of rotation varies with latitude.) This process is important in the formation of jets in the ocean, in the atmospheres of solar system planets and in the Earth's atmosphere. A comprehensive set of review papers covering this subject can be found in the book [1]. Whilst there has been much previous work on formation mechanisms for jets, there has been less attention paid to their time variability. This variability is, in particular, relevant to week-to-week variation of weather in mid-latitudes and also, because of the link between variability and response to forcing (e.g. the fluctuation-response theorem), to the potential systematic changes in the mid-latitude circulation as a response to increasing greenhouse gases.

Part of my talk will review some of the aspects of β-plane jet variability described in a recent PhD thesis by Laura Cope [2]. The system being considered, following much previous work, is doubly-periodic in x and y, with turbulence eddies driven by a stochastic forcing. Important parameters are β (the y-gradient of the vertical component of rotation), μ (linear friction) and ϵ (rate of energy input). In certain parameter regimes jets form and vary strongly in time. Three types of variability (see Figure 1, which shows the x-mean over the x-component of velocity, $\overline{u}(y,t)$) are (a) random 'wandering', (b) successive formation-merger (a new jet forms and then merges with another) and, surprisingly, (c) coherent translation in y, which is a symmetry breaking since the system is (statistically) invariant under reflection in y. The translation from time to time changes direction – a sort of 'rare event'. Various reduced models of the system have been studied,

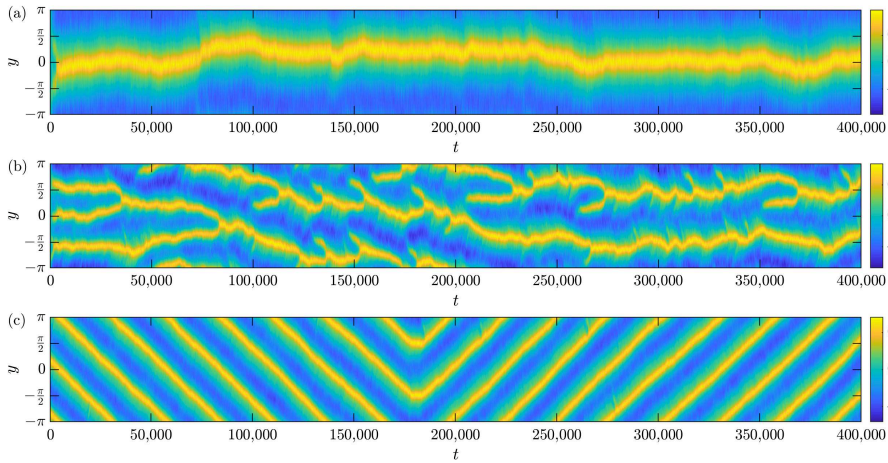

Figure 1. (From [2]). The mean in x of the x-component of velocity, $\overline{u}(y,t)$, shown as a function of y and time t, showing different types of jet variability. (a) random wandering, (b) successive formation-merger, (c) translation with random changes in direction.

one being a quasi-linear model in which nonlinear interactions between waves are omitted and the other a 'CE2' model for statistical average quantities, based on equations which neglect third- and higher order moments. Some aspects of the variability can be reproduced by the reduced models; some cannot be reproduced. See [2] for further details. Note in particular that the quasi-linear model does not show translation of the type shown in Figure 1(c). Analysis suggests that in the nonlinear simulations the nonlinear interactions between different x-wavenumbers generate long waves, which play an important organising role in maintaining the structure of turbulent eddies and x-mean flow required for translation. These nonlinear interactions are absent in the quasilinear simulations, no long waves are generated and translation cannot be sustained.

The types of variability illustrated in Figure 1 are interesting but not very relevant to the variability of the real mid-latitude atmospheric circulation. Future work is aimed at introducing new ingredients to the model that may give variability behaviour that is closer to something realistic. This may then provide a route to understanding the perplexing 'signal-to-noise' paradox (e.g. [3]) in seasonal weather prediction.

A further line of research, recently started, is to investigate 'data-driven' models of jet variability, where the future evolution of $\overline{u}(y,t)$ might be predicted on the basis of knowledge of the current state of, or the recent history of $\overline{u}(y,t)$ but not of the turbulence field. This has raised the question of the extent to which the variability illustrated in Figure 1 can be understood as the chaotic behaviour of an essentially deterministic nonlinear dynamical system, in which the primary role of the stochastic forcing to 'activate' the nonlinear dynamics, or whether it is fundamentally stochastically driven, so that the time evolution depends on certain details of the time-history of the stochastic forcing. It seems likely that the 'wandering' behaviour (a) is fundamentally stochastically driven and the transitions in direction of translation (c) may be considered as 'rare event' transitions analogous to those in the number of jets considered in [4]. But the 'formation-merger' behaviour (b) might combine elements of deterministic chaos and stochastic driving.

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The primitive equations with stochastic boundary conditions

Amru Hussein

(joint work with T. Binz, M. Hieber, M. Saal)

The primitive equations are a fundamental geophysical model. Here, these equations subject to *stochastic wind driven boundary conditions* are studied. These boundary conditions on the atmosphere-ocean interface describe the balance of the shear stress of the ocean and the horizontal wind force. In contrast to previous works on deterministic wind driven boundary conditions described by Lions, Temam and Wang in [2], we investigate in [1] for the first time stochastic wind driven boundary conditions.

These stochastic wind driven boundary conditions are modeled by a cylindrical Wiener process. We adapt an approach by Da Prato and Zabczyk for stochastic boundary value problems to define a notion of solutions. Then a rigorous treatment of these stochastic boundary conditions, which combines stochastic and deterministic methods, yields that these equations admit a unique, local pathwise solution within the anisotropic L^q_t-$H^{-1,p}_z L^p_{xy}$-setting. This solution is constructed in critical spaces.

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Fluid structure interaction models

Igor Kukavica

(joint work with M. Ignatova, I. Lasiecka, W.S. Ożański, and A. Tuffaha.)

In the first part of the talk, we address a system of partial differential equations modeling a motion of an elastic body inside an incompressible fluid. The fluid is modeled by the incompressible Navier-Stokes equations while the structure is represented by the wave equation. We will review known local and global existence theorems.

In the second part, we consider a system describing the interaction of an incompressible inviscid fluid, modeled by the Euler equations, and an elastic plate, represented by a fourth-order hyperbolic PDE. We provide a priori estimates for the existence of solutions with a sharp regularity for the Euler initial data. We also construct solutions with initial data in the same regularity class.

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A model of convection in two miscible layers

Michio Yamada
(joint work with T. Ishikawa, S. Takehiro)

Thermal convection in two horizontal layers has been a subject of research from theoretical and applied points of view [1-7]. We consider thermal convection in two horizontal layers of miscible fluids, where the flow continues to change until the concentration becomes uniform over the layers, This unsteadiness makes it difficult to apply standard stability analysis to the miscible case.

A room experiment of thermal convection in two miscible layers by Yanagisawa and Kurita [8] showed that there are two convection patterns according to the width of the transition layer between the upper and the lower layers: viscous coupling convection for thin transition layer and thermal coupling convection for thick transition layer. In both the convection patterns, a horizontal vortex street is formed in each of the upper and the lower layers, where the adjacent vortices rotates in the oppsite direction. In the viscous coupling pattern, the vertically adjacent vortices rotates in the opposite direction, while the directions are the same in the thermal coupling pattern.

These two convection patterns are reproduced in a numerical simulation of 2D convection governed by the following nondimensional equations for momentum, temperature and concentration,

$$\partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi) = -RaPr\partial_x(T - BS) + Pr\nabla^2\nabla^2\psi, \tag{1}$$

$$\partial_t T + J(\psi, T) = \nabla^2 T, \tag{2}$$

$$\partial_t S + J(\psi, S) = \frac{1}{Le}\nabla^2 S, \tag{3}$$

on $\{(x, z) \in [0, L_x]\times[-1, 1]\}$, where ψ is the stream function, $J(a, b) = (\partial_z a)(\partial_x b)-(\partial_x a)(\partial_z b)$ and Ra, Pr, B, Le are Rayleigh, Prandtl, Buoyancy and Lewis numbers, respectively. The boundary conditions are $\boldsymbol{u} = 0, \partial_z S = 0$ at $z = \pm 1$, $T = 0\,(z = 1)$, $1\,(z = -1)$, and ψ, T, S are periodic in x with period L_x.

Starting from the initial condition that $\boldsymbol{u} =$ small disturbance, $T = T_0(z) = (1 - z)/2$, and $S = 0\,(z > 0), 1\,(z < 0)$, the convection takes the viscous coupling form in the early stage, but it changes soon into the thermal coupling form as the width of the transition layer increases, and the convection becomes disordered finally. In this time evolution, both the viscous and the thermal coupling state are not stationary as well as the initial state, which makes analysis of the convection difficult.

We propose a new model in which the horizontally averaged width of the transition layer is kept constant in time evolution. Decomposition of the concentration S into the conduction part $S_{\delta(t)} = (1/2)\mathrm{erfc}(z/\delta(t))$ and deviation part Σ, gives

$$\partial_t \Sigma + J(\psi, S_{\delta(t)}) + J(\psi, \Sigma) = \frac{1}{Le}\nabla^2\Sigma, \quad \left(\delta(t) = 2\sqrt{t/Le}\right). \tag{4}$$

Observing that the horizontal average of this equation is

$$\partial_t\langle\Sigma\rangle + \langle J(\psi,\Sigma)\rangle = \frac{1}{Le}\partial_z^2\langle\Sigma\rangle, \tag{5}$$

we discard the term $J(\psi,\Sigma)$, and then $\langle\Sigma\rangle$ obeys the diffusion equation, and so we further assume that $\langle\Sigma\rangle$ vanishes initially to assure that $\langle\Sigma\rangle$ always vanishes. The horizontally averaged concentration $\langle S\rangle$ is then equal to $S_0(t)$, and our final assumption is that $\delta(t)$ is a constant. Under these assumptions, our model is obtained as

$$\partial_t\nabla^2\psi + J(\psi,\nabla^2\psi) = -RaPr\partial_x(T - BS) + Pr\nabla^2\nabla^2\psi, \tag{6}$$

$$\partial_t T + J(\psi,T) = \nabla^2 T, \tag{7}$$

$$\partial_t\Sigma + J(\psi,S_\delta) = \frac{1}{Le}\nabla^2\Sigma, \quad (S = S_\delta + \Sigma), \tag{8}$$

where δ is a constant parameter denoting the width of the transition layer.

The model has the steady solutions, $\boldsymbol{u} = 0$, $T = T_0(z)$, $\Sigma = 0$ for any positive δ, and so its linear stability analysis is possible. Stability eigenvalue problem is solved numerically. According to the stability neutral curves obtained, the steady state becomes unstable at a critical Rayleigh number which increases as the width δ increases. The critical mode is viscous coupling when δ is small, but it is thermal coupling when δ is larger than a certain value, in harmony with the room experiment and the numerical simulation of the original system. However, the marginal modes are associated with eigenvalues with nonzero imaginary part (Hopf). This is considered to be an aritifact not observed in the original system, while the real unstable eigenvalues also take part in as the Rayleigh number increases.

Numerical simulation of the time evolution in the model system shows long time asymptotic behavior of solutions. The viscous coupling convection appearing as a travelling wave solution keeps its viscous coupling form and continues for a long time. The viscous coupling convection has a clear symmetry and is invariant to $\mathscr{P}_{vis}[f] = (1/2)(f(x,z) - f(x,-z))$. Also the thermal coupling convection appears as a traveling wave solution and continues for a long time, and is invariant to $\mathscr{P}_{therm}[f] = (1/4)(f(x,z) - f(x+L_x/4,-z) + f(x+2L_x/4,z) - f(x+3L_x/4,-z))$. The ranges of these projection operators (viscous coupling and thermal coupling spaces) are invariant set of the time evolution of the model system, and it is verified that the asymptotic solutions are also lies in these spaces. However, the intersection of these spaces is not simply $\{0\}$, but a nonzero dimensional space. Extraction of the orthogonal components to these spaces shows that the asymptotic solutions are coexisting attractors in some parameter region, and the attractors have different domains of attraction. Although the relation of the model system to the original system is not yet clear, these properties suggests saddle-like structures in solution space of the original system.

This talk is based on the paper: Ishikawa, Takehiro and Yamada (2022), Model system for the transient behavior of double diffusive convection in two miscible layers, JJIAM, 39-2, DOI: 10.1007/s13160-022-00540-z (open access).

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Free boundary problems of the incompressible Navier-Stokes equations in the critical Besov space

Senjo Shimizu

(joint work with T. Ogawa)

We consider free boundary problems of the incompressible Navier-Stokes equations in $\mathbb{R}^n$ with $n \geq 2$. Let the initial velocity vector $u_0(y)$, $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and the initial boundary $\eta_0(y')$ be given. We consider the problem to find the domain

$$\Omega_t := \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ y_n > \bar{\eta}(t, y')\},$$

the velocity vector $\bar{u}(t, y)$ and the pressure $\bar{p}(t, y)$ for $y \in \Omega_t$ satisfy the Navier–Stokes equations:

$$(1) \qquad \begin{cases} \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} - \operatorname{div} T(\bar{u}, \bar{p}) = 0, & t > 0, \quad y \in \Omega_t, \\ \operatorname{div} \bar{u} = 0, & t > 0, \quad y \in \Omega_t, \\ T(\bar{u}, \bar{p})\nu_t = 0, & t > 0, \quad y \in \partial\Omega_t, \\ \dfrac{\partial_t \bar{\eta}}{\sqrt{1 + |\nabla' \bar{\eta}|^2}} = -\bar{u} \cdot \nu_t, & t > 0, \quad y \in \partial\Omega_t, \\ \bar{u}(0, y) = u_0(y), & \qquad\quad y \in \Omega_t, \\ \bar{\eta}(0, y') = \eta_0(y'), & \qquad\quad y' \in \partial\Omega_t. \end{cases}$$

Here, $\partial\Omega_t$ denotes the boundary of Ω_t, ν_t is the unit outward normal at a point $y \in \partial\Omega_t$ given by $\nu_t = (\nabla' \bar{\eta}, -1)^{\mathsf{T}} / \sqrt{1 + |\nabla' \bar{\eta}|^2}$, $T(\bar{u}, \bar{p})$ is the stress tensor defined by $T(\bar{u}, \bar{p}) = (\nabla \bar{u} + (\nabla \bar{u})^{\mathsf{T}}) - \bar{p}I$, where I is the $n \times n$ identity matrix, $(\nabla_y u)_{i,j} = \big(\partial u_j / \partial y_i\big)_{(1 \leq i,j \leq n)}$, $(\nabla \bar{u})^{\mathsf{T}}$ denotes the transposed matrix of $\nabla \bar{u}$, where $\nabla = \nabla_y = (\partial_{y_1}, \partial_{y_2}, \cdots, \partial_{y_n})$, $\nabla' = (\partial_{y_1}, \partial_{y_2}, \cdots, \partial_{y_{n-1}})$.

At the first step, we transform the free boundary problem on $\partial\Omega_t$ into the fixed boundary problem $\partial\Omega_0$ by using the Lagrangian transformation. As the next step, we transform the Lagrangian transformed problem into a problem with flat boundary. In order to do so, we extension of the boundary function $\eta_0 \in \dot{B}^{1+(n-1)/q}_{q,1}(\mathbb{R}^{n-1})$ to $E(x',x_n) = e^{-|\nabla'|x_n}\eta_0(x')$ for $x_n > 0$. Transformed problem in the half-space $\mathbb{R}^n_+$ is the following quasi-linear variable coefficient problem:

$$(2)\quad \begin{cases} \partial_t u - \Delta u + \nabla p = f(u,p,E) + F(u,p,E), & t>0, \quad x \in \mathbb{R}^n_+, \\ \operatorname{div}\, u = g(u,E) + (1+\partial_n E) G_{\mathrm{div}}(u,E), & t>0, \quad x \in \mathbb{R}^n_+, \\ T(u,p)e_n = h(u,p,E) + H(u,p,E), & t>0, \quad x \in \partial\mathbb{R}^n_+, \\ u(0,x',x_n) = u_0(x', x_n - E(x',x_n)), & x \in \mathbb{R}^n_+, \end{cases}$$

where $e_n = (0,\cdots,0,-1)^\mathsf{T}$, $\{D_E u\}_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial_j E}{1+\partial_n E}\frac{\partial u_i}{\partial x_n}$, $f(u,p,E)$, $g(u,E)$ and $h(u,p,E)$ are the linear variable coefficient terms, and the nonlinear terms are given by

$$\begin{aligned} F_u(u,p,E) &:= \Pi^{2n-2}\Big(\int_0^t D_E u ds\Big) D_E^2 u + \Pi^{n-1}\left(\int_0^t D_E u ds\right)\nabla_E p, \\ G_{\mathrm{div}}(u,E) &:= \operatorname{div}\Big(\Pi^{n-1}\Big(\int_0^t D_E u ds\Big) u\Big), \\ H_u(u,p,E) &:= \Pi^{2n-2}\Big(\int_0^t D_E u ds\Big)\nabla_E u\, e_n, + \Pi^{n-1}\Big(\int_0^t D_E u ds\Big) p\, e_n. \end{aligned}$$

We denote $\Pi^m(A)$ polynomials of A of order at most m.

Theorem 1 (Global well-posedness for the flat boundary problem (2))**.** *Let $n \geq 2$, $n/2 < p < 2n-1$ and $1 \leq q < pn/|p-n|$. If the initial data $u_0 \in \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ and the initial boundary $\eta_0 \in \dot{B}^{1+(n-1)/q}_{q,1}(\mathbb{R}^{n-1})$ satisfy for some small $\varepsilon_0 > 0$ that*

$$(3)\qquad \|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})} \leq \varepsilon_0,$$

then the initial boundary value problem (2) *admits a unique global solution*

$$\begin{aligned} &u \in C_b(\overline{\mathbb{R}_+}; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)), \\ &\Delta u, \nabla p \in L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)), \\ &p|_{x_n=0} \in \dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\partial\mathbb{R}^n_+)) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{n-1}{p}}_{p,1}(\partial\mathbb{R}^n_+)) \end{aligned}$$

with the estimate

$$\begin{aligned} &\big\|\partial_t u\big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \big\|D^2 u\big\|_{L^1(\mathbb{R}_+;\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \big\|\nabla p\big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\ &\quad + \big\|p|_{x_n=0}\big\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\partial\mathbb{R}^n_+))} + \big\|p|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\partial\mathbb{R}^n_+))} \leq \varepsilon_1, \end{aligned}$$

where $\varepsilon_1 = \varepsilon_1(n,p,\varepsilon_0)$ is a constant.

The proof is based on maximal L^1-regularity of the Stokes system with stress free boundary condition in the half-space with inhomogeneous data, and combination of bilinear estimates on the boundary and sharp trace estimates.

Corollary 2 (Global well-posedness of the free boundary problem (1)). *Let $n \geq 2$, $n/2 < p < 2n-1$ and $1 \leq q < pn/|p-n|$. For the same ε_0 in Theorem 1 and u_0 and η_0 satisfying* (3)*, let (u,p) be the global solution of* (2) *obtained in Theorem 1. Then the pull-back $(\bar{u},\bar{p})$ of (u,p) given uniquely solves the original problem* (1).

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On energy conservation for the hydrostatic Euler equations: an Onsager conjecture

Daniel W. Boutros
(joint work with S. Markfelder and E. Titi)

Onsager's conjecture was first posed in [5], it states that weak solutions of the incompressible Euler equations conserve kinetic energy (the L^2 norm in space) if the velocity field is Hölder continuous in space with exponent bigger than $\frac{1}{3}$. In case the exponent is less than $\frac{1}{3}$ energy dissipation can occur. We consider an analogue of Onsager's conjecture for the hydrostatic Euler equations (also known as the inviscid primitive equations of oceanic and atmospheric dynamics). We recall that these equations are given by

$$\partial_t \mathbf{u}_h + (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h + w\partial_z \mathbf{u}_h + \nabla p = 0, \tag{1}$$

$$\nabla \cdot \mathbf{u}_h + \partial_z w = 0, \quad \partial_z p = 0, \tag{2}$$

where $\mathbf{u}_h$ is the horizontal velocity field and w is the vertical velocity component. We consider system (1)-(2) in the three-dimensional periodic channel

$$M = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq z \leq L, (x,y) \in \mathbb{T}^2\},$$

where the vertical velocity component w is assumed to satisfy the no-normal flow boundary condition, that is

$$w(x,y,0,t) = w(x,y,L,t) = 0.$$

For classical solutions, the stated physical boundary-value problem in the periodic channel can be equivalently formulated as a problem in the three-dimensional torus $\mathbb{T}^3$ with the requirement that the vertical velocity w is an odd function with respect to z and the horizontal velocities $\mathbf{u}_h$ are even functions with respect to z.

The hydrostatic Euler equations arise from the Euler equations under the assumption of the hydrostatic balance, as well as the small aspect ratio limit (in which the vertical scale is much smaller compared to the horizontal scales).

Formally, solutions of the hydrostatic Euler equations with the boundary conditions stated above conserve the spatial L^2 norm of $\mathbf{u}_h$. Therefore the analogous formulation of Onsager's conjecture for these equations concerns the critical threshold for weak solutions to conserve this quantity.

Unlike the Euler equations, in the case of the hydrostatic Euler equations the vertical velocity is one degree spatially less regular compared to the horizontal velocities. The fact that the equations are anisotropic in regularity and nonlocal makes it possible to prove a range of sufficient criteria for energy conservation, which are independent of each other. This means that there probably is a 'family' of Onsager conjectures for these equations.

In [1] we consider three different notions of weak solutions to the hydrostatic Euler equations:

- A type I weak solution is the 'canonical' weak solution to the hydrostatic Euler equations. For such a weak solution it is assumed that $w \in L^2(\mathbb{T}^3 \times (0,T))$ and $\mathbf{u}_h \in L^\infty((0,T); L^2(\mathbb{T}^3))$.
- For a type II weak solution it is assumed that $w \in L^2((0,T); L^2(\mathbb{T}; B^{-s}_{2,\infty}(\mathbb{T}^2)))$ and $\mathbf{u}_h \in L^\infty((0,T); L^2(\mathbb{T}^3)) \cap L^4((0,T); L^4(\mathbb{T}; B^{\sigma'}_{4,2}(\mathbb{T}^2)))$ for $\sigma' > s$ (where we have separately specified the regularity in the z-variable), for $0 < s < \frac{1}{2}$.
- For a type III weak solution we assume that $w \in L^2((0,T); B^{-s}_{2,\infty}(\mathbb{T}^3))$ and $\mathbf{u}_h \in L^\infty((0,T); L^2(\mathbb{T}^3)) \cap L^4((0,T); B^{\sigma'}_{4,2}(\mathbb{T}^3))$ for $0 < s < \frac{1}{2}$ and $s < \sigma'$.

In the above, we have denoted the Besov spaces (as well as their dual spaces) by $B^s_{p,q}$. In order to make sense of the vertical part of the nonlinearity (i.e. the product $\mathbf{u}_h w$) for both type II and type III weak solutions, we use the Bony decomposition as well as the paradifferential calculus. One of the differences between types II and III are the way in which the boundary conditions are attained.

In the case of type I weak solutions, we prove that each of the following conditions are sufficient for energy conservation:

- If $\mathbf{u}_h \in L^4((0,T); B^{1/2+}_{4,\infty}(\mathbb{T}^3))$ and $w \in L^2((0,T); L^2(\mathbb{T}^3))$
- If $w \in L^3((0,T); C^\beta(\mathbb{T}^3))$ and $\mathbf{u}_h \in L^3((0,T); C^\alpha(\mathbb{T}^3))$ with $\alpha > \frac{1}{2} - \frac{1}{2}\beta$ and $\alpha \geq \beta$
- If $w \in L^2((0,T); L^2(\mathbb{T}^3))$ and $\mathbf{u}_h$ has Besov regularity $B^\alpha_{3,\infty}(\mathbb{T})$ with respect to the vertical variable and $B^\beta_{3,\infty}(\mathbb{T}^2)$ with respect to the horizontal variables, such that $\alpha > \frac{1}{3}, \beta > \frac{2}{3}$ and $\beta + 2\alpha > 2$

The global existence of type I weak solutions was established in [2], using techniques from convex integration.

In the case of type III weak solutions, we show that each of the following conditions are sufficient for energy conservation:

- If $w \in L^2((0,T); B^{-s}_{2,\infty}(\mathbb{T}^3))$ and $\mathbf{u}_h \in L^4((0,T); B^{s+1/2+}_{4,\infty}(\mathbb{T}^3))$
- If $\mathbf{u}_h \in L^3((0,T); B^{3/4+}_{4,\infty}(\mathbb{T}^3))$

These results are proven by first establishing an equation of local energy balance, which contains a 'defect term' which captures the potential lack of regularity of the weak solution (this approach was first introduced in [4]). Then one can derive sufficient conditions for energy conservation of weak solutions by deriving sufficient conditions for the defect term to be zero. It is also possible to prove sufficient conditions for energy conservation by using commutator estimates (which was done for the first time in [3]), see appendix B in [1].

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Hibler's viscous-plastic sea ice model: rigorous analysis, time periodic solutions and interaction with a rigid body

Felix Brandt

(joint work with T. Binz, K. Disser, R. Haller and M. Hieber)

Sea ice has attracted much attention in climate science due to its role as a hot spot in global warming. As a material, it exhibits a complex mechanical and thermodynamical behaviour. In 1979, Hibler introduced the governing equations of large-scale sea ice dynamics in a seminal paper [4]. The model has been investigated numerically by various communities, whereas rigorous analysis has started only very recently by the works of Brandt, Disser, Haller-Dintelmann and Hieber [2] as well as Liu, Thomas and Titi [5].

For a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary of class C^2 and a time intervall $J = (0,T)$, $0 < T \leq \infty$, we consider the horizontal ice velocity $u \colon \Omega \times J \to \mathbb{R}^2$, the mean ice thickness $h \colon \Omega \times J \to [\kappa, \infty)$, where $\kappa > 0$ is a small parameter indicating the transition to open water, and the ice compactness $a \colon \Omega \times J \to (0,1]$. In addition, the ice mass m is given by $m = \rho_{\text{ice}} h$, where $\rho_{\text{ice}} > 0$ denotes the ice density, $c_{\text{cor}} > 0$ represents a Coriolis parameter, H is the sea surface dynamic height, $\tau_{\text{ocean}}(u)$ as well as τ_{atm} represent the oceanic forces and atmospheric wind forces, respectively, and for the ice growth rate f, the thermodynamic terms are given by $S_h = f\left(\frac{h}{a}\right) a + (1-a)f(0)$ and $S_a = \frac{f(0)}{\kappa}(1-a)\chi_{f(0)>0} + \frac{a}{2h} S_h \chi_{S_h<0}$.

Following [4], we assume that the viscous-plastic rheology is given by a constitutive law relating the internal ice stress σ and the deformation tensor $\varepsilon = \varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^{\mathrm{T}}\right)$ via an internal ice strength $P = p^* h \exp(-c(1-a))$ for $p^* > 0$ and $c > 0$ constant and nonlinear bulk and shear viscosities ζ and η. As the viscosities become singular, we introduce a suitable regularization, and the regularized stress tensor then takes the shape

$$\sigma_\delta = 2\eta_\delta \varepsilon + [\zeta_\delta - \eta_\delta]\mathrm{tr}(\varepsilon)\mathrm{I} - \frac{P}{2}\mathrm{I},$$

with $\zeta_\delta = e^2 \eta_\delta = \frac{P}{2\triangle_\delta(\varepsilon)} = \frac{P}{2\sqrt{\delta + \triangle^2(\varepsilon)}}$ for $e > 1$ denoting the ratio of the long axis to the short axis of the elliptical yield curve and $\delta > 0$, and with

$$\triangle^2(\varepsilon) = \left(\varepsilon_{11}^2 + \varepsilon_{22}^2\right)\left(1 + \frac{1}{e^2}\right) + \frac{4}{e^2}\varepsilon_{12}^2 + 2\varepsilon_{11}\varepsilon_{22}\left(1 - \frac{1}{e^2}\right).$$

The system of equations on $\Omega \times J$, completed by boundary and initial conditions, is then given by

$$(1) \qquad \begin{cases} m(u_t + u \cdot \nabla u) = \mathrm{div}\sigma_\delta - mc_{\mathrm{cor}} n \times u - mg\nabla H + \tau_{\mathrm{atm}} + \tau_{\mathrm{ocean}}(u), \\ h_t + \mathrm{div}(uh) = S_h + d_h \Delta h, \\ a_t + \mathrm{div}(ua) = S_a + d_a \Delta a, \\ u = 0, \quad \partial_\nu h = 0, \quad \partial_\nu a = 0, \\ u(0) = u_0, \quad h(0) = h_0, \quad a(0) = a_0. \end{cases}$$

Denoting the principle variable of the system by $v = (u, h, a)$, we rewrite (1) as a quasilinear evolution equation of the shape

$$(2) \qquad v' + A(v)v = F(v), \quad t > 0, \quad v(0) = v_0$$

on the ground space $X_0 = \mathrm{L}^q(\Omega; \mathbb{R}^2) \times \mathrm{L}^q(\Omega) \times \mathrm{L}^q(\Omega)$ and with regularity space

$$X_1 = \{v \in \mathrm{H}^{2,q}(\Omega; \mathbb{R}^2) \times \mathrm{H}^{2,q}(\Omega) \times \mathrm{H}^{2,q}(\Omega) : u = 0,\ \partial_\nu h = \partial_\nu a = 0 \text{ on } \partial\Omega\}.$$

Solutions are considered in the maximal regularity space with time weights, i.e., in $\mathbb{E}_1(J) := \mathrm{H}_\mu^{1,p}(J; X_0) \cap \mathrm{L}_\mu^p(J; X_1)$. The trace space of this class is given by $X_{\gamma,\mu} = (X_0, X_1)_{\mu - 1/p, p}$, and we observe that $X_{\gamma,\mu} \hookrightarrow \mathrm{B}_{qp}^{2(\mu - 1/p)}(\Omega)^4 \hookrightarrow \mathrm{C}^{1,\alpha}(\overline{\Omega})^4$ if $\frac{1}{2} + \frac{1}{p} + \frac{1}{q} < \mu \le 1$. Furthermore, we consider an open subset $V_\mu \subset X_{\gamma,\mu}$ such that $(u, h, a) \in V_\mu$ satisfy $h \ge \kappa$ and $a \in (0, 1]$.

Verifying maximal regularity of the linearized operator matrix $A(v_0)$ by showing that the linearized operator corresponding to $\mathrm{div}\sigma_\delta$ exhibits certain ellipticity properties and by using the respective results for the Neumann Laplacian on bounded domains in conjunction with the upper triangular structure of the operator matrix, and showing Lipschitz estimates of the nonlinear terms, we apply local existence theory to prove local strong well-posedness. More precisely, we get the following result, see [2, Theorem 2.1]:

Assume that $1 < p, q < \infty$ and $\mu \in (1/p, 1]$ satisfy $\frac{1}{2} + \frac{1}{p} + \frac{1}{q} < \mu \leq 1$, and let $v_0 \in V_\mu$. Then there exist $\tau = \tau(v_0) > 0$ and $r = r(v_0) > 0$ with $\overline{B}_{X_{\gamma,\mu}}(v_0, r) \subset V_\mu$ such that (2), i.e., (1), admits a unique solution $v(\cdot, v_1) \in \mathbb{E}_1(0, \tau) \cap \mathrm{C}([0, \tau); V_\mu)$ for each initial value $v_1 \in \overline{B}_{X_{\gamma,\mu}}(v_0, r)$.

Observing that $(0, h_*, a_*)$, $h_* > 0$, $a_* \in (0, 1]$ constant, are trivial equilibria for (2) and employing the generalized principle of linearized stability, we show stability of $(0, h_*, a_*)$ in $X_{\gamma,\mu}$ and global existence of the unique solution to (2) for initial data close to the equilibrium provided δ is chosen small enough and the external forces vanish, see [2, Theorem 2.3].

It is natural to ask for time periodicity of solutions to Hibler's sea ice model in view of time dependence of e.g. the ice growth rate $f = f(t)$. Employing maximal periodic regularity for dealing with the linear problem, verifying Lipschitz estimates of the nonlinear terms and using the contraction mapping principle, we show in [3] that (2), i.e., (1), admits a unique time periodic strong solution close to constant equilibria under certain smallness assumptions and subject to time periodic forces. This is especially valid for time periodic wind forces of the shape $|U_{\mathrm{atm}}|U_{\mathrm{atm}} = c(t)h$, with $c(t)$ time periodic, and time periodic ice growth rate $f(t) = f(t + T)$.

The interaction problem of sea ice with a rigid body is studied in [1], where Newton's law is used for describing the equations for the rigid body. Transforming the problem to a fixed domain, using a "monolythic" approach, a suitable decoupling technique as well as a similarity transform, showing Lipschitz estimates, especially using nonlinear complex interpolation for estimates of the stress tensor, and applying a variant of a classical local-in-time existence theorem, local strong well-posedness of the coupled system is shown provided the collision case is excluded.

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A rigorous proof of the validity of the point vortex description for a class of inviscid gSQG models

Carina Geldhauser

(joint work with Marco Romito)

Point vortex models are a classical approach in 2D turbulence. We characterize a point vortex by its position X_j in $\mathbb{R}^2$, and its intensity $\gamma_j \in \mathbb{R}$. Point vortex models describe the evolution of vortex positions according to the system of equations

$$\begin{cases} \dot{X}_j = \sum_{k \neq j} \gamma_k \nabla^\perp G_m(X_j, X_k), \\ X_j(0) = x_j, \end{cases} \qquad j = 1, 2, \ldots, N, \tag{1}$$

where γ_k are real numbers and G_m is the Green function of the operator $(-\Delta)^{\frac{m}{2}}$. Here, we consider the case $m \in [1,2]$, where $m = 2$ is the Euler case and $m = 1$ the surface-quasigeostrophic case. In [GR20] we prove that the above system (1) has, for fixed N, a global solution for a. e. initial condition, under the assumption that $\sum_{j \in J} \gamma_j \neq 0$ for all $J \subset \{1, 2, \ldots, N\}$.

An interesting question is whether point vortices provide an approximation of solutions to the generalized surface quasigeostrophic equation on $\mathbb{R}^2$, i.e. the case $m \in (1,2)$ of

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ (-\Delta)^{\frac{m}{2}} \psi = \theta, \\ u = \nabla^\perp \psi. \end{cases} \tag{2}$$

To be more precise, we aim at a statement of the form: if an initial condition is approximated, in the sense of measures, by point vortices as $N \uparrow \infty$, then solutions to gSQG are approximated, again in the sense of measures, by the evolution of the point vortex measure $\sum_{j=1}^N \gamma_j \delta_{X_j(t)}$..

We note, first of all, that these are measure valued solution, so should interpret gSQG in the sense of distributions, this is not enough to include measures with atoms. In the case of Euler equations ($m = 2$), a symmetrisation [Del91] (see also [Sch95, Sch96]) allows to tame the singularity of the Biot-Savart kernel. In this context, writing the equation against a test function φ only in terms of θ yields

$$\int \int \theta(t,x) \varphi(t,x)\, dx\, dt + \\ + \int \iint k_m(x-y) \cdot (\nabla \varphi(t,x) - \nabla \varphi(t,y)) \theta(t,x) \theta(t,y)\, dx\, dy\, dt = 0,$$

where $k_m = \nabla^\perp G_m$. However, in our more singular setting, the new kernel $k_m(x - y) \cdot (\nabla \varphi(t,x) - \nabla \varphi(t,y))$ is not bounded on the diagonal.

To overcome these difficulties, in [GR20] we approximate point vortices by vortex blobs of radius ϵ, and we are able to prove that for values of the parameter m not too small ($\sqrt{3} < m < 2$), a sequence of vortex blobs solutions to gSQG converges, as the size of the blobs goes to 0, to the configuration of point masses that obeys to (1). Also, we prove localisation of vortices, namely, if θ is initially a vortex blob, then it remains a vortex blob of comparable size. A similar result in a slightly different setting, which avoids the technical lower bound of $\sqrt{3}$ was proven recently in [Ros20].

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Free boundaries in the atmosphere: quasi-geostrophic equations with moisture

Antoine Remond-Tiedrez

(joint work with L. Smith, S. Stechmann)

To describe the atmosphere on a synoptic scale (the scale at which high- and low-pressure systems are apparent on a weather map, for example) one may use the quasi-geostrophic equations, which are derived as a limit of the classical Boussinesq system under the assumptions of fast rotation and strong stratification. When incorporating the dynamics of water content in the atmosphere, a.k.a. moisture, one may then study the moist Boussinesq equations and its limit, the precipitating quasi-geostrophic equations.

These models are important for atmospheric scientists in light of the role that the water cycle plays in atmospheric dynamics, notably through energy budgeting (such as for example when atmospheric circulations are driven by latent heat release in storms). Mathematically, these models present interesting challenges due to the presence of boundaries, whose locations are a priori unknown, between phases saturated and unsaturated in water (schematically: boundaries between clouds and their surroundings).

In particular, while the (dry) quasi-geostrophic equations rely on the inversion of a Laplacian, this becomes a much trickier adversary in the presence of free boundaries. In this short talk we will discuss how this nonlinear equation underpinning the precipitating quasi-geostrophic equations can be characterized using a variational formulation and we will describe the many benefits one may derive from this formulation.

Motion of small rigid bodies in a viscous incompressible fluid

Arnab Roy
(joint work with E. Feireisl, A. Zarnescu)

In this talk, we consider a finite number of rigid bodies immersed in a viscous incompressible fluid and discuss the effect of the rigid bodies on the fluid flow as their diameters are asymptotically tending to zero.

There are some results in the literature regarding the impact of a small rigid body immersed in a viscous fluid on the fluid motion. But a general approach used so far is based on the idea that if the mass density is large then the velocity can be controlled. So, it deals with the scenario when the body is small but “heavy” and the resulting situation is therefore close to the rigid obstacle problem.

But these results are slightly at odds with a physically relevant hypothesis that the body density should be at least bounded. We want to understand the small but “light” (bounded density) particle case. In the talk we present a recent result [1] where we prove that the effect of a finite number of rigid bodies is negligible as soon as their diameters are small whereas their mass densities are irrelevant. We need the following condition on the shape of the bodies:

$$D_\varepsilon \equiv \max_{i=1,\cdots,N} \{\mathrm{diam}[\mathcal{S}^i_\varepsilon]\} \to 0 \text{ as } \varepsilon \to 0,$$

$$0 < \lambda D_\varepsilon^\beta \leq |\mathcal{S}_\varepsilon| \text{ as } \varepsilon \to 0,\ d{\leq}\beta < \begin{cases} 15 \text{ if } d = 3, \\ \\ \text{arbitary finite if } d = 2, \end{cases}$$

for some $\lambda > 0$ independent of ε. But this allows different bodies to shrink to zero in different order of scaling.

Our approach is based on a new *restriction operator* that assigns a given function its “projection” on the space of rigid motions of the bodies. Considering the case of several bodies needs a nontrivial modification of the construction of restriction operator presented in [2].

The new restriction operator improves considerably the error estimates necessary to perform the asymptotic limit. Another new ingredient is that we use the dissipation energy rather than the energy itself to obtain suitable bounds on the translation velocity pf the rigid body. This is why the result is independent of the mass densities of the bodies.

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Non-uniqueness of the Vanishing Viscosity Limit for Passive Scalar Transport along an Incompressible Flow

Lucas Huysmans
(joint work with E. Titi)

In this talk we consider the linear transport equation for a passive scalar $f(x,t) \in \mathbb{R}$ along a background incompressible flow $u(x,t) \in \mathbb{R}^2$, with $\nabla \cdot u = 0$ in the distributional sense. We work on the two-torus $x \in \mathbb{R}^2/\mathbb{Z}^2$, with $t \geq 0$. The corresponding equation for f is $\frac{\partial f}{\partial t} + \nabla \cdot (uf) = 0$ when written in divergence form, and when understood in the weak sense permits solutions with u, f bounded without further regularity.

Such an equation is a hyperbolic conservation law [3], and as such may be regularised with the addition of viscosity $\nu > 0$ acting on f. In the context of passive transport this is often instead named diffusion. The resulting equation is $\frac{\partial f}{\partial t} + \nabla \cdot (uf) = \nu \Delta f$ – the advection-diffusion equation for f along a background incompressible flow u.

Here, and for many other hyperbolic conservation laws, the additional viscosity causes the equation to be well-posed (e.g. with bounded background flow u [2]). Meanwhile the non-viscous problem is ill-posed [4, 6] (unless u is sufficiently regular, e.g. [1, 5]).

This phenomenon is common for hyperbolic conservation laws and leads to the study of admissible solutions in the absence of viscosity. As such the study of the limit $\nu \to 0$ is of great interest, in particular for the compressible and incompressible Navier-Stokes equations.

We provide an example of an incompressible background flow for which the weak solutions to the advection-diffusion equation may not converge as $\nu \to 0$. Moreover we show that for different sub-sequences of ν the weak solutions may converge to different weak solutions of the non-viscous problem – the transport equation. This phenomena occurs for a carefully constructed background flow u, exploiting a novel mechanism contingent on the flow u having a structure bearing semblance to intermittency in turbulence. Correspondingly, non-uniqueness is created through a novel kind of turbulent transport relying on the following effect.

Through two elementary lemmas we can show that the effect of small viscosity is to 'blur' the transport of f along the highly oscillatory parts of the background flow u to zero, whilst transport of f along the less oscillatory regions of u remains mostly unchanged despite diffusion. Thus it is possible to find a sequence of viscosities ν_n such that viscous transport of f includes the more turbulent regions of u one at a time. If the effect on f of each subsequent region exactly cancels the previous, the sequence of solutions will alternate between two different behaviours. This is achieved explicitly with background shear flows, and leads to different vanishing viscosity limits along the even and odd sub-sequences $\nu_{2n} \to 0$, $\nu_{2n+1} \to 0$.

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Well-posedness of Hibler's dynamical sea-ice model

Xin Liu
(joint work with M. Thomas and E. Titi)

We establish the local-in-time well-posedness of solutions to an approximating system constructed by mildly regularizing the dynamical sea-ice model of *W.D. Hibler, Journal of Physical Oceanography, 1979.* Our choice of regularization has been carefully designed, prompted by physical considerations, to retain the original coupled hyperbolic-parabolic character of Hibler's model. Various regularized versions of this model have been used widely for the numerical simulation of the circulation and thickness of the Arctic ice cover. However, due to the singularity in the ice rheology, the notion of solutions to the original model is unclear. Instead, a simplified, approximating system, which captures current numerical study, is proposed. The well-posedness theory of such a system provides a first-step groundwork in both numerical study and future analytical study.

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Analysis of a rotationally constrained convection model

Yanqiu Guo
(joint work with C. Cao, E. Titi)

This talk is about the analysis of an asymptotically reduced system for rotationally constrained convection. The presence of a dominant balance in equations for fluid flow can be exploited to derive a simpler set of governing equations that permits analytical explorations. For rotation dominated flows, the geostrophic balance occurs: the pressure gradient force is balanced by the Coriolis effect. The Taylor-Proudman constraint suggests that the dominant Coriolis force leads to flows that are organized into vertical plumes or columns whose horizontal scale is small compared to the layer height. Applying the asymptotic theory for small Rossby number and tall columnar structures, Julien and Knobloch derived a closed set of reduced equations from the three-dimensional Boussinesq equations. Also, Sprague et al. numerically simulated this reduced model to study the equal populations of cyclonic and anticyclonic structures in rapid rotating convection. This reduced system is interesting yet challenging for analytical study. On the one hand, the nonlinear convection term has a reduced complexity since it contains only the horizontal gradient. On the other hand, the physical domain remains three dimensional, while the regularizing viscosity acts in the horizontal direction only, creating a major difficulty for establishing the global existence theory. Another difficulty arises due to a linear term involving the vertical derivative of the stream function, reflecting the balance of the Coriolis force by the pressure. I will present some of our results motivated by the global regularity problem. We show that the model is globally well-posed if regularized by a very weak dissipation. I will also discuss the case of infinite Prandtl number convection, and the situation when both of the Prandtl and Rayleigh numbers approach infinity. This is a joint project with Chongsheng Cao and Edriss Titi.

The primitive equations with transport noise

Antonio Agresti

(joint work with M. Hieber, A. Hussein, M. Saal)

The primitive equations are one of the fundamental models for geophysical flows used to describe oceanic and atmospheric dynamics. In this talk I will discuss some recent results on the primitive equations with noise of transport type. Such noise is often used in fluid mechanics to model turbulent flows. In addition to transport noise, we also consider non-isothermal turbulent pressure. From a modeling point of view, the temperature dependence of the turbulent pressure can be seen as a large scale effect of an additive noise acting on the small vertical dynamics. For the primitive equations with transport noise and non-isothermal turbulent pressure, we provide a physical derivation and we discuss the global well-posedness for data in the critical spaces H^1. The latter result gives a non-trivial extension of the celebrated work by C. Cao and E.S. Titi [3] on the deterministic model. Our approach is based on recent developments of maximal regularity techniques in the context of stochastic parabolic PDEs. Finally, starting from these results, we discuss some open problems.

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Radiative transfer in fluids: from analysis to numerical simulations

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(joint work with C. Bardos, O. Pironneau, E. Titi)

Radiative transfer describes the interaction of electromagnetic radiation, viewed as a gas of photons, with a background medium such as a gas, a plasma or a liquid. Photons interact with the atoms, ions, electrons or nanoparticles in the medium in essentially two different manners, viz. (a) scattering and (b) absorption/emission. Scattering corresponds to "collisions" between a photon and the particles in the medium, involving a sudden change in the direction of the photon. Absorption and emission are quantum effects: a photon can be absorbed by one electron in an atom of the background medium, exciting it to a higher energy level; conversely, an electron can fall back to a lower energy level, thereby emitting one photon.

Radiation is described by the radiative intensity $I_\nu(t,\vec{x},\vec{\omega}) := ch\nu f(t,\vec{x},\vec{\omega},\nu)$, where $f(t,\vec{x},\vec{\omega},\nu)$ is the number density of photons of frequency ν at the position $\vec{x}$ and in the direction $\vec{\omega}$ at time t, while c is the speed of light and h the Planck constant. An important example is the radiative intensity for a black body at temperature T, given by the Planck function $B_\nu(T) := 2h\nu^3/c^2(e^{h\nu/kT}-1)$, where k is the Boltzmann constant. One easily checks that

$$\int_0^\infty B_\nu(T)d\nu = \tfrac{2\pi^4k^4}{15c^2h^3}T^4 \quad \text{(Stefan-Boltzmann's law).}$$

Assuming isotropic scattering for simplicity, the radiative intensity satisfies the radiative transfer equation

$$(RT) \qquad \left(\tfrac{1}{c}\partial_t + \vec{\omega}\cdot\nabla_{\vec{x}}\right) I_\nu + \kappa_\nu I_\nu = \kappa_\nu a_\nu J_\nu + \kappa_\nu(1-a_\nu)B_\nu(T)\,.$$

In (RT), $\kappa_\nu := \rho\bar{\kappa}_\nu$, where ρ is the density of the background medium and $\bar{\kappa}_\nu$ the specific extinction coefficient for radiation of frequency ν, and $a_\nu \in [0,1]$ is the scattering albedo, while

$$J_\nu(t,\vec{x}) := \tfrac{1}{4\pi}\int_{\mathbf{S}^2} I_\nu(t,\vec{x},\vec{\omega})d\vec{\omega} \quad \text{(mean radiative intensity).}$$

Finally $T(t,\vec{x})$ is the temperature at the point $\vec{x}$ at time t.

The specific extinction coefficient $\bar{\kappa}_\nu$ depends strongly on the frequency ν, see `www.gemini.edu/observing/telescopes-and-sites/sites#Transmission`; this is important in the understanding of the greenhouse effect. For liquid water, see `en.wikipedia.org/wiki/Electromagnetic_absorption_by_water` .

The radiative transfer equation is coupled to the energy balance equation in the fluid dynamics system, as follows:

$$(H)\qquad \begin{aligned}(\partial_t + \vec{u}(t,\vec{x})\cdot\nabla_{\vec{x}})T(t,\vec{x}) =& \tfrac{c_P}{c_V}\kappa_T\Delta_{\vec{x}}T(t,\vec{x})\\ &+\tfrac{4\pi}{\rho c_V}\int_0^\infty \kappa_\nu(1-a_\nu)(J_\nu(t,\vec{x})-B_\nu(T(t,\vec{x})))d\nu\end{aligned}$$

where $\vec{u}$ is the fluid velocity field, while κ_T is the (constant) heat diffusivity. The fluid is assumed to be incompressible with constant density ρ, while c_V and c_P are respectively the specific heats of the fluid at constant volume and at constant pressure. Viscous heating is neglected in equation (H) — for instance if $|\vec{u}| \ll 1$.

1. Stratified Radiative Transfer

In (RT), let $\vec{x} = (x,y,z) \in \Omega = \mathcal{O}\times(0,Z)$ where $z\in(0,Z)$ is the height, while $(x,y)\in\mathcal{O}$ are the horizontal variables. Consider a steady problem where

$$|\partial_x I_\nu(t,\vec{x},\vec{\omega})| + |\partial_y I_\nu(t,\vec{y},\vec{\omega})| \ll |\partial_z I_\nu(t,\vec{x},\vec{\omega})|\,,$$

Setting $\mu := \omega_z$, one can average out the variables ω_x,ω_y, and define

$$\mathcal{I}_\nu(\vec{x},\mu) := \int_0^{2\pi} I_\nu(\vec{x},(\sin\theta\cos\alpha,\sin\theta\sin\alpha,\cos\theta))\tfrac{d\alpha}{2\pi}\,,\quad J_\nu(\vec{x}) = \int_{-1}^1 \mathcal{I}_\nu(\vec{x},\mu)\tfrac{d\mu}{2}\,.$$

The steady, stratified Radiative Transfer with heat convection system is
$(SSRT)$

$$\begin{cases}(\mu\partial_z+\kappa_\nu)\mathcal{I}_\nu(\vec{x},\mu) = \kappa_\nu a_\nu J_\nu(\vec{x}) + \kappa_\nu(1-a_\nu)B_\nu(T(\vec{x}))\,,\\ (\vec{u}(\vec{x})\cdot\nabla_{\vec{x}} - \tfrac{c_P}{c_V}\kappa_T\Delta_{\vec{x}})T(\vec{x}) = \tfrac{4\pi}{\rho c_V}\displaystyle\int_0^\infty \kappa_\nu(1-a_\nu)(J_\nu(\vec{x})-B_\nu(T(\vec{x})))d\nu\,.\end{cases}$$

This system is supplemented with the boundary conditions

$$(BC)\qquad \begin{cases}\mathcal{I}_\nu(x,y,0,\mu) = \mu Q_\nu^+\,,\quad \mathcal{I}_\nu(x,y,Z,-\mu) = \mu Q_\nu^-\,,\quad 0<\mu<1\,,\\ \vec{u}\Big|_{\partial\Omega} = 0\,,\qquad \dfrac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0\,.\end{cases}$$

One can solve the radiative transfer equation for $\mathcal{I}_\nu$ with given temperature field, and average in μ to obtain J_ν, which is then inserted in the heat convection equation. This leads to the following iterative scheme, for $n\ge 1$:

$$J_\nu^n(\vec{x}) = S_\nu(\vec{x}) + \int_0^Z \tfrac{\kappa_\nu}{2}E_1(\kappa_\nu|z-\zeta|)(\kappa_\nu a_\nu J_\nu^{n-1} + \kappa_\nu(1-a_\nu)B_\nu(T^{n-1}))(x,y,\zeta)d\zeta\,,$$

$$\begin{cases}(\vec{u}(\vec{x})\cdot\nabla_{\vec{x}} - \tfrac{c_P}{c_V}\kappa_T\Delta_{\vec{x}})T^n(\vec{x}) = \tfrac{4\pi}{\rho c_V}\displaystyle\int_0^\infty \kappa_\nu(1-a_\nu)(J_\nu^n(\vec{x})-B_\nu(T^n(\vec{x})))d\nu\,,\\ \dfrac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0\,,\end{cases}$$

where $E_1(\tau) := \int_\tau^\infty \frac{e^{-s}}{s}ds$ for all $\tau > 0$, assuming that

$$T^0 \equiv 0\,,\quad J_\nu^0(\vec{x}) = S_\nu(\vec{x}) := \tfrac12\int_0^1 (e^{-\kappa_\nu z/\mu}Q_\nu^+(x,y) + e^{-\kappa_\nu(Z-z)/\mu}Q_\nu^-(x,y))\mu d\mu\,.$$

Theorem A [1]**.** Assume that $0 \le Q_\nu^\pm \le B_\nu(T_M)$, while $0 \le a_\nu \le a_M < 1$ and $0 < \kappa_m \le \kappa_\nu \le \kappa_M$. Assume moreover that

$$\gamma := \sup_{\nu>0}\left((1-a_\nu)\int_0^{\kappa_\nu Z/2} E_1(\theta)d\theta\right) + \sup_{\nu>0}\left(a_\nu \int_0^{\kappa_\nu Z/2} E_1(\theta)d\theta\right) < 1\,.$$

(1) Then one has

$$0 \le S_\nu = J_\nu^0 \le \ldots \le J_\nu^n \le J_\nu^{n+1} \le B_\nu(T_M)\,, \quad 0 = T^0 \le \ldots \le T^n \le T^{n+1} \le T_M\,.$$

(2) The sequence (J_ν^n, T^n) converges exponentially fast to a solution (J_ν, T) of (SSRT) with (BC):

$$\int_\Omega \int_0^\infty (|J_\nu - J_\nu^n| + \kappa_\nu(1-a_\nu)|B_\nu(T) - B_\nu(T_\nu^n)|)(\vec{x})d\nu d\vec{x}$$
$$\le \tfrac{\gamma^n|\Omega|}{1-\gamma}(1 + \tfrac{1}{\kappa_m(1-a_M)})\int_0^\infty \kappa_\nu(1-a_\nu)B_\nu(T_M)d\nu\,.$$

(3) There exists at most one (weak) solution of (SSRT) with (BC) such that $\mathcal{I}_\nu \ge 0$ a.e. on $\Omega \times (-1,1) \times (0,+\infty)$ and $0 \le T \in L^\infty(\Omega)$.

Several numerical applications of Theorem A are presented in [1].

2. Coupling Radiative Transfer with the Boussinesq System

Since $1/c \ll 1$, we may consider a quasi-static coupling of RT with the Boussinesq system, with gravity field $\vec{g}$, and thermal expansion coefficient α:
(RTB)

$$\begin{cases} \vec{\omega}\cdot\nabla_{\vec{x}}I_\nu + \bar{\rho}\bar{\kappa}_\nu I_\nu = \bar{\rho}\bar{\kappa}_\nu a_\nu J_\nu + \bar{\rho}\bar{\kappa}_\nu(1-a_\nu)B_\nu(T)\,, \\ (\partial_t + \vec{u}\cdot\nabla_{\vec{x}})\vec{u} = -\nabla_{\vec{x}}(P/\bar{\rho}) + \mu_F\Delta_{\vec{x}}\vec{u} + (1-\alpha(T-\bar{T}))\vec{g}\,, \quad \mathrm{div}_{\vec{x}}\vec{u} = 0\,, \\ (\partial_t + \vec{u}\cdot\nabla_{\vec{x}})T = \kappa_T\Delta_{\vec{x}}T + \frac{4\pi}{\rho c_P}\int_0^\infty \kappa_\nu(1-a_\nu)(J_\nu - B_\nu(T))d\nu\,. \end{cases}$$

Assume that this problem is posed on a smooth domain Ω of $\mathbf{R}^3$, with unit outward normal $\vec{n}_{\vec{x}}$ at $\vec{x} \in \partial\Omega$, with boundary conditions

$$(BC') \quad I_\nu(t,\vec{x},\vec{\omega}) = I_\nu^b(\vec{x},\vec{\omega})\,, \quad \vec{\omega}\cdot n_{\vec{x}} < 0\,, \quad \vec{u}\Big|_{\partial\Omega} = 0\,, \quad T\Big|_{\partial\Omega} = T^b \equiv T^b(\vec{x})\,,$$

and initial conditions

$$(IC) \qquad \vec{u}\Big|_{t=0} = \vec{u}^{in}\,, \qquad T\Big|_{t=0} = T^{in}\,.$$

The buoyancy term in the Boussinesq equation couples the heat and the motion equations, and precludes using the monotonicity argument of the preceding section. The following result is based on a priori "energy" estimates.

Theorem B [2]**.** Let $T^{in} \in H^1(\Omega)$ satisfy the boundary condition $T^{in}\Big|_{\partial\Omega} = T^b$, and let $\vec{u}^{in} \in H_0^1(\Omega)$ satisfy $\mathrm{div}_{\vec{x}}\vec{u}^{in} = 0$. Assume that there exist $T_m, T_M \in \mathbf{R}$ s.t.

$$0 < T_m \le T^b \le T_M \quad \text{and} \quad B_\nu(T_m) \le I_\nu^b \le B_\nu(T_M)\,.$$

Then there exists some finite time $\tau > 0$ and a unique strong solution of the system (RTB) with boundary conditions (BC') and initial conditions (IC) on the time interval $[0, \tau)$. This solution satisfies $\vec{u} \in L^\infty(0, \tau; H_0^1(\Omega)) \cap L^2(0, \tau; H^2(\Omega))$,

$$0 < T_m \le T^b \le T_M\,, \quad \text{and} \quad B_\nu(T_m) \le I_\nu^b \le B_\nu(T_M)\,.$$

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Primitive equations coupled with moisture dynamics and justification of hydrostatic balance

Jinkai Li

(joint work with S. Hittmeir, R. Klein, E. Titi, G. Yuan)

In this talk we will present some recent results on the well-posedness of the coupled system of the primitive equations with the moisture system for the warm cloud. Multi-phases and phase changes are taken into consideration and both the simplified case and the thermodynamically refined case will be considered. For the simplified case, we assume that the dry air and water vaper have the same gas constants and heat capacities and ignore the heat capacity of the liquid water. Some results rigorously justifing the hydrostatic approximatoin from the Navier-Stokes equations to the primitive equations in both the frameworks of strong solutions and z-weak solutions will also be presented.

On the way to the limit: time-parallel algorithms for oscillatory, multiscale PDEs

Beth A Wingate

(joint work with J. Rosemeier, T. Haut, T. Andrews, C. Cotter, H. Yamazaki)

This work focuses on phase averaging techniques, inspired by the method of fast singular limits, for taking large time steps in time-parallel time stepping methods. We briefly introduce time-parallel time stepping. We discuss a proof of 'parareal' convergence [1] whose ingredients include bounds from fast singular limits in the case when epsilon -¿ 0 and arguments developed by Gander and Hairer [3]. Second, we discuss the more general case, when epsilon finite (and not necessarily small) where we have a new proof of how the combination of mapping and phase averaging works as a coarse propagator for parareal [2]. We give a few examples. Finally, we end with a brief outline of 3 new lines of inquiry: a proof of convergence for multi-level parareal [4] (with Juliane Rosemeier), alternate mapping to capture more phase averaging (with Tim Andrews). Finally, we show a first explorations of this type of mapping and averaging on the sphere [5] with Hiroe Yamazaki and Colin Cotter.

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On singular limits for the Rayleigh-Benard problems

Eduard Feireisl

(joint work with P. Bella, F. Oschmann)

We consider a general compressible viscous and heat conducting fluid confined between two parallel plates and heated from the bottom. The time evolution of the fluid is described by the Navier-Stokes-Fourier system considered in the regime of low Mach and Froude numbers suitably interrelated. The asymptotic limit is identified as the Oberbeck-Boussinesq system supplemented with non-local boundary conditions for the temperature deviation.

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