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Representation Theory of Quivers and Finite-Dimensional Algebras

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Abstract. This workshop was about the representation theory of quivers and finite-dimensional (associative) algebras, and links to other areas of mathematics, including other areas of representation theory, homological algebra, cluster algebras, algebraic geometry and singularity theory. Particularly active topics included τ-tilting theory, algebras arising from surface triangulations and the study of exact categories and their generalizations.

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Introduction by the Organizers

The representation theory of quivers and finite-dimensional algebras is a vibrant part of modern representation theory. The workshop covered some core topics such as tilting theory and homological conjectures for finite-dimensional algebras. The numerous interactions with other mathematical subjects like Lie theory, algebraic geometry, topology and combinatorics played an equally important role and continue to be a source of inspiration. There were 29 lectures given at the meeting, and what follows is a short survey of their main themes.

Exact categories, triangulated categories and generalizations. Many categories studied in representation theory are either exact or triangulated. These two types have been unified as extriangulated categories by Nakaoka and Palu in 2016. During the meeting several talks were dedicated to this notion. In particular both Nakaoka and Palu gave talks on the generalization of classical constructions in this new setup: Palu reported on a joint work with Nakaoka and Gorsky that introduced the notion of mutation in certain extriangulated categories called 0-Auslander. This new definition unifies mutations defined in exact or triangulated context such as cluster-tilting mutation, silting mutation or mutation of co-t-structures. This moreover gives new examples of mutations arising beyond the triangulated and exact context. Another classical construction arising both in exact and triangulated contexts is the notion of localisation by a certain subcategory: In his talk Nakaoka reported on joint work with Ogawa and Sakai giving a general definition of localisation for extriangulated categories that unifies in particular Verdier localisation and Serre quotients. Pauksztello reported on joint work with Coelho Simões on a generalisation of mutation and reduction for orthogonal subcategories of triangulated categories, generalizing different mutations of simple minded collections (in the non Calabi-Yau setting), and of simple minded systems (in the Calabi-Yau setting).

Hanihara and Kalck were both exploring some triangulated equivalences linking commutative and non commutative algebras. Hanihara gave some equivalences between singularity categories of graded commutative rings constructed from Segre products and higher cluster categories of hereditary algebras, while Kalck described the endomorphism algebra of some tilting object in the bounded derived category of coherent sheaves on a (singular) weighted projective space.

Gratz and Krause were both interested in the lattice of thick subcategories of a triangulated category. Gratz reported on joint work with Stevenson showing that such a lattice is distributive if and only if it is isomorphic to the lattice of opens in a topological space. Krause introduced the notion of commutativity for thick subcategories, and focused his study to the subset of thick subcategories commuting with all thick subcategories. This "center" of the lattice of thick subcategories happens to be a sublattice which is distributive, and so relates with the results of Gratz and to the theory of central support.

Representations of Quivers. An exciting recent development in the representation theory of quivers is the connection with dimension expanders, which are an analogue in linear algebra of the notion of an expander graph. In his lecture, Reineke used representations of quivers to obtain new quantitative results about the existence of dimension expanders. Dimension expanders have also appeared in the work of the participant Eckert in connection with representation amenability of algebras.

Bozec and Schiffmann defined a polynomial counting the dimension of the space of cuspidal functions on the moduli stack of representations of a quiver of a given dimension vector, and conjectured that the polynomial is positive and integral, generalizing Kac's (now proved) conjecture. In a tour de force, Hennecart gave a proof of this using BPS Lie algebras associated to cohomological Hall algebras of preprojective algebras.

Geometry arises when one considers the action of the base change group $\mathrm{GL}(\alpha)$ on the space $\mathrm{Rep}(Q,\alpha)$ of representations of a quiver Q of dimension vector α, and even for Dynkin quivers there are long-standing questions; for example are orbit closures always normal varieties? One approach by Cerulli-Irelli, Feigin and Reineke used a geometric construction based on what are now called projective quotient algebras, and they made a conjecture which could have been one step towards a proof of normality. Unfortunately, in his lecture Cerulli-Irelli gave examples showing that the conjecture is false, so new ideas will be needed for normality.

Vector spaces equipped with a nilpotent endomorphism and an invariant subspace can be viewed as certain representations of the quiver of type A_2 over a truncated polynomial ring $k[x]/(x^n)$. Kosakowska spoke about an ongoing project to understand such representations, and in an elegant lecture, Kvamme gave a result about separated monomorphism categories of representations of a quiver over an abelian category, and used it to give a quiver-theoretic generalization of the fact that the number of indecomposable configurations of a vector space over $\mathbb{F}_p$ equipped an endomorphism of cube zero and an invariant subspace is equal to the number of indecomposable configurations of a finite abelian group of exponent p^3 equipped with a subgroup.

Tilting Theory. For a finite dimensional algebra A which is representation-finite, Plamondon introduced an affine scheme $\widetilde{\mathcal{M}}_A$ which encodes information about 2-term complexes of projective A-modules. For type A quivers it describes the configuration space of n points on $\mathbb{P}^1$, and more generally for Dynkin quivers it is related to a cluster $\mathcal{X}$-variety. He explained the face structure of $\widetilde{\mathcal{M}}_A$ as well as a connection with F-polynomials.

The real Grothendieck group $K_0(\mathrm{proj}A)_{\mathbb{R}}$ of a finite dimensional algebra A has a lot of information on the tilting theory of A. It contains a fan (called the g-fan) whose cones correspond to 2-term silting complexes of A. It is complete if and only if A is τ-tilting finite. Iyama showed that a complete fan of rank 2 can be realized as a g-fan of some finite dimensional algebra if and only if it is sign-coherent. Regarding each vector in $K_0(\mathrm{proj}A)_{\mathbb{R}}$ as a stability condition, one obtains the TF equivalence relation. Asai introduced an open neighborhood of a rigid vector in $K_0(\mathrm{proj}A)$, and applied it to study the rigid part of each vector in $K_0(\mathrm{proj}A)_{\mathbb{R}}$.

Angeleri-Hügel and Laking discussed 2-term (large) cosilting complexes over a finite dimensional algebra A, which correspond bijectively with torsion classes in mod$A$. They are also in bijection with pairs $(\mathcal{Z}, \mathcal{I})$, where $\mathcal{Z}$ is a rigid set of indecomposable pure-injective modules and $\mathcal{I}$ is a set of indecomposable injective modules, generalizing support τ^{-1}-tilting pairs. They characterized mutable points of $(\mathcal{Z}, \mathcal{I})$ as the isolated points in the induced topology of the Ziegler topology, and showed that the mutation at such points is uniquely determined. These results are illustrated with examples coming from cluster-tilted algebras of type $\widetilde{A}$.

A d-cluster tilting subcategory of an abelian category is a d-abelian category (Jasso), and a $d\mathbb{Z}$-cluster tilting subcategory of a triangulated category has a canonical structure of a $(d+2)$-angulated category (Geiss-Keller-Oppermann). The notion of d-torsion classes in a d-abelian category was introduced by Jorgensen.

Treffinger explained recent results on d-torsion classes in a d-cluster tilting subcategory $\mathcal{M}$. Among others, d-torsion classes are precisely subcategories closed under d-extensions and d-quotients, and hence d-torsion classes in $\mathcal{M}$ form a complete lattice. Moreover, d-torsion classes give rise to "support τ_d-tilting modules".

Motivated by Auslander correspondence, Jasso characterized Auslander algebras of $(d+2)$-angulated categories in terms of twisted periodic algebras. He also gave an enhanced version by characterizing dg algebras which are $d\mathbb{Z}$-cluster tilting objects in their perfect derived categories in terms of minimal A_∞-algebra structure.

Algebras arising from surfaces. There is a deep link between the combinatorics of triangulations of surfaces, the representation theory of certain classes of finite-dimensional algebras, and Fomin-Zelevinsky cluster algebras (these are combinatorially defined commutative algebras which arose from the desire to get a better understanding of Lusztig's canonical bases of quantum groups).

Fomin-Zelevinsky cluster algebras can be divided into three types: cluster-finite, mutation-finite and mutation-infinite. Work of Fomin-Shapiro-Thurston (2008) combined with Felikson-Shapiro-Tumarkin (2012), shows that mutation-finite cluster algebras arise from triangulations of marked oriented surfaces (up to a short list of exceptions). The associated Jacobian algebras are of tame representation type, and they are related to several important classical classes of algebras (Brauer graph algebras, gentle algebras and quaternion algebras). The modules over these Jacobian algebras can be parametrized by curves on the marked surface. The intersection pattern of curves leads to information on the dimension of homomorphism and extension spaces between the corresponding modules.

Qin presented his ground breaking results on the connection between different bases of cluster algebras arising from surfaces. He showed that the bracelet basis (parametrized by certain curves on the surface) of these clusters algebras coincides with the set of theta functions studied by Gross, Hacking, Keel and Kontsevich.

Schroll talked on the braid group action on full exceptional sequences in the topological Fukaya category of marked surfaces. Bondal and Polishchuk conjectured that this action is always transitive. Schroll presented a class of counterexamples, where the braid group acts with infinitely many orbits.

Baur discussed how the class of string algebras arises from labelled tilings of surfaces. This generalizes the realization of gentle algebras via triangulations of surfaces. Labardini-Fragoso presented a different generalization by relating marked surface with orbitfold points and the class of semilinear clannish algebras. This can be seen as a first step towards the categorification of cluster algebras which are skew-symmetrizable but not skew-symmetric.

Further topics. Given a real Lie group and a maximal compact subgroup of its complexification, one can define a category of Harish-Chandra modules, and in classical work by Bernstein and others, it was shown that the classification of such modules reduces to the representation theory of suitable associative algebras. For example Gelfand and Ponomarev's 1968 classification of Harish-Chandra modules for $\mathrm{SL}_2(\mathbb{C})$ led to the classification of modules for string algebras. The classification for $SL_2(\mathbb{R})$ is more complicated, but known; in his lecture Burban spoke about the modules for the associative algebras corresponding to Harish-Chandra modules arising from automorphic forms.

Marczinzik has made numerous contributions to the study of homological properties of algebras, and in his lecture he presented more examples, including characterizing the lattices whose incidence algebras are Auslander regular as those which are distributive, and a solution to a question by Green characterizing the Koszul duals of Auslander algebras. Sen spoke about his work on Nakayama algebras, and the question of which are higher Auslander algebras. Conde spoke about a theorem of Koenig, Külshammer and Ovsienko showing that up to Morita equivalence every quasi-hereditary algebra has an exact Borel subalgebra, and her work determining which Morita representative one needs to take. Bobiński presented his short proof of a result of Jaworska-Pastuszak and Pastuszak showing that the Krull-Gabriel dimension of a cluster tilted algebra coincides with that of the corresponding hereditary algebra.

Workshop: Representation Theory of Quivers and Finite-Dimensional Algebras

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Abstracts

Expander representations

Markus Reineke

1. Dimension expanders

Consider the following result in linear algebra:
Let k be an algebraically closed field. For every finite-dimensional k-vector space V, there exist linear operators $F, G \in \mathrm{End}(V)$ such that for all nonzero subspaces $U \subset V$ with $\dim U \leq \frac{1}{2}\dim V$, we have

$$\dim(U + F(U) + G(U)) > \frac{1}{2}(5 - \sqrt{5}) \cdot \dim U.$$

This was proved without the explicit constant in [1], and reproved with determination of the (optimal) constant $(5-\sqrt{5})/2$ in [2]. The fact that the free algebra in two generators admits simple representations on arbitrary finite-dimensional vector spaces can be reformulated as existence of operators $F, G \in \mathrm{End}(V)$ such that $\dim(U + F(U) + G(U)) > \dim U$ for all nonzero proper $U \subset V$; the above statement can be seen as a quantitative version of this. In fact, the claimed property is Zariski-open in the matrix entries of F and G, so that almost all pairs of operators have this property. However, the proof in [2] is completely nonconstructive.

2. Expander representations

Let Q be a finite acyclic quiver, and let $\mu = \frac{\Theta(\_)}{\kappa(\_)} : \mathbb{N}Q_0 \setminus 0 \to \mathbb{Q}$ be a slope function on Q. A finite-dimensional representation V of kQ is called *stable* if $\mu(U) < \mu(V)$ for all nonzero proper subrepresentations $U \subset V$. We consider a quantitative version of stability:

For $\delta \in]0,1[$ and $\varepsilon > 0$, the representation V is called a (δ, ε)-*expander* if for all nonzero subrepresentations $U \subset V$ with $\kappa(U) \leq \delta \cdot \kappa(V)$, we have $\mu(U) \leq \mu(V) - \varepsilon$.

The pair (Q, μ) is said to *exhibit uniform expansion* if for all $\delta \in]0,1[$ there exists $\varepsilon > 0$ and an unbounded (with respect to dimension) family of (δ, ε)-expanders.

It is easily observed that V is stable if and only if, for every $\delta \in]0,1[$, V is a (δ, ε)-expander for some $\varepsilon > 0$. It also follows from the known representation theory of extended Dynkin quivers that (Q, μ) exhibiting uniform expansion forces Q to be wild.

We conjecture that, conversely, every wild quiver Q admits a stability μ such that (Q, μ) exhibits uniform expansion. This can currently be proved for a special class $Q_{\mathbf{a},\mathbf{b}}$ of bipartite quivers. It includes the three-arrow Kronecker quiver K_3, and expander representations $(\mathrm{id}, F, G) : V \to V$ of K_3 can be used to prove the above theorem.

3. Methods

For two dimension vectors $\mathbf{e} \le \mathbf{d}$ for Q, we write $\mathbf{e} \hookrightarrow \mathbf{d}$ if every representation of dimension vector $\mathbf{d}$ admits a subrepresentation of dimension vector $\mathbf{e}$. By a theorem of Schofield and Crawley-Boevey, this can be characterized inductively in terms of the Euler form $\langle \_, \_ \rangle$ of Q; namely, $\mathbf{e} \hookrightarrow \mathbf{d}$ holds iff $\langle \mathbf{e}', \mathbf{d} - \mathbf{e} \rangle \ge 0$ for all $\mathbf{e}' \hookrightarrow \mathbf{e}$.

In particular, $\mathbf{e} \hookrightarrow \mathbf{d}$ implies $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \ge 0$. For the quivers $Q_{\mathbf{a},\mathbf{b}}$ and an unbounded class of dimension vectors $\mathbf{d}$, this quadratic condition on $\mathbf{e}$ can be used to establish that, for every $\delta \in]0,1[$, there exists $\varepsilon > 0$ such that $\mu(\mathbf{e}) \le \mu(\mathbf{d}) - \varepsilon$ whenever $\mathbf{e} \hookrightarrow \mathbf{d}$ and $\kappa(\mathbf{e}) \le \delta \cdot \kappa(\mathbf{d})$. Choosing general representations V of these dimension vectors $\mathbf{d}$ then proves uniform expansion.

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Lattices and thick subcategories

Sira Gratz
(joint work with Greg Stevenson)

Given an essentially small triangulated category $\mathcal{T}$, we are interested in studying its *thick subcategories*, that is its triangulated subcategories which are closed under taking direct summands. These are precisely the subcategories that occur as kernels of exact functors between triangulated categories, and are as such crucial to understand when studying localisations. We denote the set of thick subcategories of $\mathcal{T}$ by $\mathrm{Thick}(\mathcal{T})$.

Famously, the poset $\mathrm{Thick}(\mathcal{T})$ forms a *complete lattice* under inclusion, that is, it has arbitrary *meets* and *joins*, where pair-wise meets for $S, T \in \mathrm{Thick}(\mathcal{T})$ are given by

$$S \wedge T = S \cap T$$

and paire-wise joins are given by

$$S \vee T = \mathrm{thick}(S, T),$$

the smallest thick subcategory containing both S and T.

In classical geometric examples, the lattice $\mathrm{Thick}(\mathcal{T})$, or a distinguished sublattice thereof, is controlled by a space, that is, it can be interpreted as the lattice $\mathcal{O}(X)$ of open subsets of a topological space X, i.e. it is a *spatial frame*. The philosophy of studying thick subcategories through the lense of topology and geometry has been explored fruitfully in the field of tensor triangular geometry: In a tensor triangulated category, the lattice of radical thick tensor ideals is a spatial frame.

Distributivity. In many representation theoretic examples, the lattice of thick subcategories behaves very differently to that of a lattice of opens. As a bare minimum, a lattice of open subsets of a topological space is *distributive*, that is, for any open subsets U, V and W we have

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W).$$

Many classical examples from representation theory do not satisfy this distributivity condition. In fact, the existence of an exceptional pair (E_1, E_2) in $\mathcal{T}$, such that there exists a non-trivial map of some degree from E_1 to E_2 prohibits the $\mathrm{Thick}(\mathcal{T})$ from being distributive. On the other hand, distributivity is enough to guarantee an a priori much stronger property.

Theorem ([GS22]). *Let $\mathcal{T}$ be an essentially small triangulated category. Then $\mathrm{Thick}(\mathcal{T})$ is distributive if and only if it is a spatial frame.*

Approximation by spaces. By the above, we know that whenever $\mathrm{Thick}(\mathcal{T})$ is distributive, then $\mathcal{T}$ is, in terms of its thick subcategories, controlled by a topological space, and we can study its structure in geometric and topological terms. However, it does not provide any insight for the non-distributive case. We provide two free constructions of a topological space associated to any essentially small triangulated category.

Denote by tcat the category of essentially small triangulated categories and exact functors, and by Sob the category of sober topological spaces with continuous maps.

Theorem 1.1 ([GS22]). *There is a functor $\Phi\colon \mathsf{tcat}^{\mathrm{op}} \longrightarrow \mathsf{Sob}$ such that for any essentially small triangulated category K, there is a poset map*

$$f_K\colon \mathrm{Thick}(K) \longrightarrow \mathcal{O}(\Phi(K))$$

preserving arbitrary joins, and the pair $(\mathcal{O}(\Phi(K)), f_K)$ is universal, in the sense that any other poset map $g\colon \mathrm{Thick}(K) \longrightarrow F$ preserving arbitrary joins, where F is a spatial frame, factors through f_K.

Concretely, the space $\Phi(K)$ can be defined as the set $\mathsf{CjSLat}(\mathrm{Thick}(K), \mathbf{2})$ of poset maps preserving arbitrary joins from $T(K)$ to the lattice $\mathbf{2} = \mathcal{O}(\{*\})$ under the topology given by the subbasis $U_\ell = \{p \in \mathsf{CjSLat}(\mathrm{Thick}(K), \mathbf{2}) \mid p(\ell) = 1\}$ for all $\ell \in \mathrm{Thick}(K)$.

The space $\Phi(K)$ is a successful spatial approximation of K in many ways: It is a free, universal, and fully functorial construction. Alas, in the case where $\mathrm{Thick}(K)$ is distributive, we do not in general recover the desired space, i.e. in general the sober space controlling $\mathrm{Thick}(K)$ is properly contained in $\Phi(K)$.

Example 1.2. *Let k be a field and consider $K = \mathrm{D}^{\mathrm{b}}(k \times k)$. Its lattice of thick subcategories is the distributive lattice*

$$\begin{tikzcd}
& K \arrow[dl, dash] \arrow[dr, dash] & \\
\mathsf{M} = \langle (k,0) \rangle \arrow[dr, dash] & & \mathsf{N} = \langle (0,k) \rangle \arrow[dl, dash] \\
& 0 &
\end{tikzcd}$$

and $\mathcal{O}(\Phi(K))$ is consequently computed as

$$\begin{tikzcd}
& X \arrow[d, dash] & \\
& U_1 \arrow[dl, dash] \arrow[dr, dash] & \\
U_{\mathsf{M}} \arrow[dr, dash] & & U_{\mathsf{N}} \arrow[dl, dash] \\
& U_{\mathsf{M}} \cap U_{\mathsf{N}} \arrow[d, dash] & \\
& U_0 = \varnothing &
\end{tikzcd}$$

Note that we have, canonically, $\Phi(K) = \mathrm{Thick}(K)$. This is indeed true for all $K \in \mathsf{tcat}$.

We can fix the issue of losing accuracy in case that $\mathrm{Thick}(K)$ is distributive. However, we do so at the cost of full functoriality. Denote by $\mathsf{CLat}(\mathrm{Thick}(K), \mathbf{2})$ the set of poset maps preserving arbitrary joins and finite meets from $\mathrm{Thick}(K)$ to $\mathbf{2}$. We endow this with a topology with basis $U_\ell = \{p \in \mathsf{CLat}(\mathrm{Thick}(K), \mathbf{2}) \mid p(\ell) = 1\}$ for $\ell \in \mathrm{Thick}(K)$.

Theorem 1.3 ([GS22])**.** *For any $K \in \mathsf{tcat}$ the space*

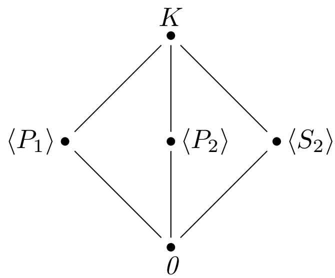
$$\mathrm{Spcnt}(K) = \mathsf{CLat}(\mathrm{Thick}(K), \mathbf{2})$$

is sober and satisfies an appropriate universal property.

If $\mathrm{Thick}(K)$ is distributive, then $\mathcal{O}(\mathrm{Spcnt}(K)) \cong \mathrm{Thick}(K)$ as desired. Furthermore, we have a natural comparison map from $\mathrm{Spcnt}(K)$ to $\Phi(K)$ as well as to several spectral constructions in sufficiently nice settings, such as to the Balmer spectrum [Bal05], the noncommutative spectrum [NVY19] and Matsui's spectrum [Mat19].

However, we note that $\mathrm{Spcnt}(K)$ can be empty.

Example 1.4. *Let k be a field. Let $K = \mathrm{D^b}(\mathsf{mod}\, kA_2)$ be the bounded derived category of finitely generated right kA_2-modules. Then* $\mathrm{Thick}(K)$ *is the lattice*



and $\mathsf{CLat}(\mathrm{Thick}(K), \mathbf{2}) = \varnothing$.

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Derived Endomorphism Algebras in Higher Auslander–Reiten Theory

Gustavo Jasso

(joint work with Fernando Muro)

Let k be a field and A a finite-dimensional algebra over k. Suppose that A is of finite representation type, that is the category $\mathrm{mod}(A)$ of finite-dimensional (right) A-modules admits an additive generator M, say. The algebra $\Gamma := \mathrm{End}_A(M)$ of endomorphisms of M is then an *Auslander algebra*, that is Γ has global dimension at most 2 and dominant dimension at least 2 [Aus71]. The basic paradigm of Auslander–Reiten Theory is that the minimal projective resolutions of simple Γ-modules of projective dimension 2 (the largest possible) correspond to almost-split sequences in $\mathrm{mod}(A)$ [AR75]. More generally, if $d \geq 1$ and M is a d-cluster tilting A-module, then Γ is a $(d+1)$*-dimensional Auslander algebra* in the sense of Iyama [Iya07], that is Γ has global dimension at most $d+1$ and dominant dimension at least $d+1$. We remind the reader that M is a *d-cluster tilting A-module* if the following conditions are equivalent for an indecomposable A-module X:

- X is a direct summand of M.
- For all $0 < i < d$, $\mathrm{Ext}^i_A(X, M) = 0$.
- For all $0 < i < d$, $\mathrm{Ext}^i_A(M, X) = 0$.

Thus, a 1-cluster tilting A-module is simply an additive generator of $\mathrm{mod}(A)$ for the latter two conditions are empty in this case. In this more general context, minimal projective resolutions of simple Γ-modules of projective dimension $d+1$ correspond to *d-almost-split sequences* in $\mathrm{add}(M) \subseteq \mathrm{mod}(A)$, the additive closure of M in $\mathrm{mod}(A)$. Furthermore, up to Morita equivalence, the association $(A, M) \mapsto \mathrm{End}_A(M)$ induces a bijection between:

(1) Pairs (A, M) consisting of a finite-dimensional algebra A and a d-cluster tilting A-module M.
(2) $(d+1)$-Auslander algebras Γ.

The above bijective correspondence is known as the *Auslander–Iyama Correspondence* [Aus71, Iya07].

Suppose now that Λ is a finite-dimensional selfinjective algebra; for simplicity, assume Λ to be basic. We wish to interpret the minimal projective resultions of simple Γ-modules of infinite (!) projective dimension in higher Auslander–Reiten-theoretic terms. For this, it is necessary to enforce a certain periodicity on these resultions. More precisely, we assume that there exists an exact sequence of Λ-bimodules

$$0 \to \Lambda_\sigma \to P_{d+1} \to P_d \to \cdots \to P_2 \to P_1 \to P_0 \to \Lambda \to 0$$

with projective middle terms, where σ is an algebra automorphism of Λ; in this case we say that Λ is *twisted* $(d+2)$*-periodic with respect to* σ. Let S be a simple Λ-module of infinite projective dimension; applying the tensor product functor $S \otimes_\Lambda -$ to the above exact sequence yields the first part of a projective resolution of S that is 'twisted periodic' since the $(d+2)$-syzygy of S is again a simple Λ-module. Thus, the minimal total projective resolution of S is completely determined by the automorphism σ and the truncation

$$Q_{d+1} \to Q_d \to \cdots \to Q_2 \to Q_1 \to Q_0 \to \nu Q_0,$$

where Q_0 is the projective cover of S and νQ_0 is its injective hull. It is natural to wish to interpret the latter complex as an almost split $(d+2)$-angle [IY08, GKO13]. Indeed, a theorem of Amiot [Ami07] in the case $d = 1$ and a generalisation by Lin [Lin19] show that the pair $(\mathrm{proj}(\Lambda), - \otimes_\Lambda \Lambda_{\sigma^{-1}})$ admits a $(d+2)$-angulation, where $\mathrm{proj}(\Lambda)$ is the category of finite-dimensional projective Λ-modules. Conversely, if $\mathrm{proj}(\Lambda)$ admits a $(d+2)$-angulated structure, then Λ must be twisted $(d+2)$-periodic with respect to some algebra automorphism [GSS03, GKO13, Han20]. Furthermore, if Λ arises as the endomorphism algebra of a $d\mathbb{Z}$-cluster tilting object in a triangulated category[1] with finite-dimensional morphism spaces, then $\mathrm{proj}(\Lambda)$ admits a $(d+2)$-angulated structure [GKO13]. The main result in [JM22] refines the above to the following more precise statement (the case $d = 1$ was established in [Mur22]):

[1]That is a (basic) d-cluster tilting object that is isomorphic to its d-fold shift.

Theorem (Derived Auslander–Iyama Correspondence)**.** *Let k be a perfect field. There is a bijective correspondence between the following:*

(1) Quasi-isomorphism classes of DG algebras A such that $H^0(A)$ is a basic finite-dimensional algebra and A is a $d\mathbb{Z}$-cluster tilting object of its perfect derived category $D^c(A)$.

(2) Equivalence classes of pairs (Λ, σ) consisting of a basic finite-dimensional algebra Λ and σ is an algebra automorphism such that Λ is twisted $(d+2)$-periodic with respect to σ.

The correspondence is given by $A \mapsto (H^0(A), \sigma)$, where σ is a choice of algebra automorphism of $H^0(A)$ such that $H^{-d}(A) \cong H^0(A)_\sigma$ as $H^0(A)$-bimodules.

The key ingredient in the proof of the theorem is the *restricted universal Massey product (rUMP) of length* $d+2$ associated to any minimal A_∞-model of A [Kad82, Kel01, LH]. By definition, the rUMP of A is the Hochschild cohomology class

$$u_A \in HH^{d+2,-d}(H^0(A), H^*(A))$$

that is the image of the class $\{m_{d+2}\} \in HH^{d+2,-d}(H^*(A), H^*(A))$ of the higher operation $m_{d+2} \colon H^*(A)^{\otimes d+2} \to H^*(A)[-d]$ under the canonical map

$$HH^{d+2,-d}(H^*(A), H^*(A)) \longrightarrow HH^{d+2,-d}(H^0(A), H^*(A)).$$

Indeed, a further main result in [JM22] is the following variant of the above theorem:

Theorem. *Let k be a perfect field. There is a bijective correspondence between the following:*

(1) Quasi-isomorphism classes of DG algebras A such that $H^0(A)$ is a basic finite-dimensional algebra and A is a $d\mathbb{Z}$-cluster tilting object of its perfect derived category $D^c(A)$.

(2) A_∞-isomorphism classes of minimal A_∞-algebras B with the following properties:
- *The underlying graded algebra of B is concentrated in degrees that are multiples of d, and there exists an invertible element $\varphi \in B^d$.*
- *The rUMP $u_B \in HH^{d+2,-d}(B^0, B)$ is invertible in the Hochschild–Tate cohomology (bigraded) algebra $\underline{HH}^{\bullet,*}(B^0, B)$.*

The correspondence associates to a DG algebra A any of its minimal A_∞-models.

It is interesting to investigate in more detail the existence of additional structures on the DG algebras that arise from the Derived Auslander–Iyama Correspondence.

Conjecture. Let Λ be a basic finite-dimensional algebra that is twisted $(d+2)$-periodic with respect to the Nakayma automorphism ν of Λ. Let A be any DG algebra that corresponds to (Λ, ν) under the Derived Auslander–Iyama Correspondence. Then, A admits a right d-Calabi–Yau structure in the sense of [KS06].

The conjecture is motivated by the existence of a right d-Calabi–Yau structure on the Amiot–Guo–Keller cluster category [Ami09, Guo11, Kel05a] associated to the derived $(d+1)$-preprojective algebra [Kel11, IO13] of a d-representation finite algebra [IO11], see [KL23] for an announcement of the proof of a much more general theorem on Calabi–Yau structures on Drinfeld quotients.

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Mutation in hereditary extriangulated categories

Yann Palu

(joint work with Mikhail Gorsky, Hiroyuki Nakaoka)

Inspired from categorification of cluster algebras, the notion of mutation arises in many different guises in representation theory. Our aim is to give a common framework for as many of those mutations as possible. This framework takes the form of some "nice" extriangulated structures.

Extriangulated categories [12] axiomatize extension-closed full subcategories of triangulated categories, in a similar way that Quillen exact categories axiomatize extension-closed full subcategories of abelian categories. An extriangulated category is an additive category endowed with some biadditive functor, to be thought of as an Ext^1 bifunctor, and a class of diagrams, called conflations or extriangles, of the form $A \rightarrowtail B \twoheadrightarrow C \overset{\delta}{\dashrightarrow}$, where δ is an element of the abelian group $\mathrm{Ext}^1(C, A)$. One might think of extriangles as conflations where the inflation (resp. deflation) is not necessarily a monomorphism (resp. epimorphism) or alternatively as triangles that cannot be rotated.

The prototypical example of an extriangulated category that carries a theory of mutation is the homotopy category of complexes concentrated in degrees -1 and 0 whose components are finite-dimensional projective modules over a finite-dimensional algebra. It is called the category of two-term complexes and denoted $K^{[-1,0]}(\mathrm{proj}\,\Lambda)$.

Definition: An extriangulated category $\mathcal{C}$ is 0-Auslander if:

(a) for any $X \in \mathcal{C}$, there is an extriangle $P_1 \rightarrowtail P_0 \twoheadrightarrow X \dashrightarrow$ with P_0, P_1 projective;
(b) for any projective P, there is an extriangle $P \rightarrowtail Q \twoheadrightarrow I \dashrightarrow$ with Q projective-injective and I injective.

In other words, an extriangulated category is 0-Auslander if it has enough projectives, global dimension at most one and dominant dimension at least one. An analogous definition using injectives instead of projectives is equivalent to this one. The category of two-term complexes is 0-Auslander, with projectives the complexes concentrated in degree 0 and injectives the complexes concentrated in degree -1. Its only projective-injective is 0.

Fix a Krull–Schmidt, 0-Auslander extriangulated category $\mathcal{C}$ and assume for simplicity that it is Hom-finite with a basic projective generator P.

Definition: Let $R \in \mathcal{C}$ be basic and rigid, i.e. $\mathrm{Ext}^1(R, R) = 0$. Then R is called

(i) *maximal rigid* if, for any X, $R \oplus X$ being rigid implies $X \in \mathrm{add}\, R$;
(ii) *complete rigid* if $|R| = |P|$;
(iii) *tilting* if there is an extriangle $P \rightarrowtail R_0 \twoheadrightarrow R_1 \dashrightarrow$ with $R_0, R_1 \in \mathrm{add}\, R$;
(iv) *silting* if $\mathcal{C}$ is its smallest full subcategory containing R and stable under taking summands, extensions, cones of inflations and fibers of deflations.

Theorem 1: If R is rigid, then $|R| \leq |P|$ (whence the term *complete*), and all four definitions above are equivalent.

Theorem 2: Let R be basic and silting in $\mathcal{C}$. Write $R = \overline{R} \oplus X$ where X is indecomposable and not projective-injective. Then there is a unique, up to isomorphism, Y not isomorphic to X, such that $\overline{R} \oplus Y$ is silting. Moreover, there is an exchange extriangle of the form $X \rightarrowtail E \twoheadrightarrow Y \dashrightarrow$ or $Y \rightarrowtail E' \twoheadrightarrow X \dashrightarrow$, but not both, with $E, E' \in \operatorname{add} \overline{R}$.

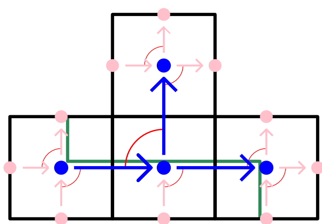
Theorem 2 applies to various settings, recovering many notions of mutations. We give several examples of such applications.

Cluster tilting [7, 3, 9] Let $\mathcal{C}$ be a Hom-finite cluster category, and fix a cluster tilting object $T \in \mathcal{C}$. Let Δ_T be the collection of all those triangles $X \to Y \to Z \xrightarrow{\delta} \Sigma X$, for which δ factors through $\operatorname{add} \Sigma T$. This relative structure appeared in [13] in order to categorify the g-vectors of cluster algebras of finite type, and endow $\mathcal{C}$ with an extriangulated structure by [8]. The extriangulated category $(\mathcal{C}, \Delta_T)$ is 0-Auslander with projectives $\operatorname{add} T$ and injectives $\operatorname{add} \Sigma T$. Its silting objects being precisely the cluster tilting objects, Theorem 2 recovers cluster tilting mutation. More generally, relative tilting mutation can be recovered in a similar way.

Two-term silting [2, 10] Let Λ be a finite-dimensional algebra. The extriangulated category $K^{[-1,0]}(\operatorname{proj} \Lambda)$ is 0-Auslander and its silting objects coincide with the so-called two-term silting complexes, whose theory of mutation is thus recovered by Theorem 2.

Intermediate co-t-structures [6] Let $\mathcal{D}$ be a triangulated category with a co-t-structure whose co-heart we denote by $\mathcal{S}$. Its extended co-heart, the full subcategory $\mathcal{C} = \mathcal{S} * \Sigma\mathcal{S}$, is a 0-Auslander extriangulated category. Under the assumptions of Theorem 2, and combining it with results from [15] and [1], we recover mutation of intermediate co-t-structures.

The non-kissing complex [11, 5, 14] Thomas MCConville generalized the flip of triangulations of a convex polygon to some flip of non-kissing facets. Those are maximal collection of pairwise non-kissing walks that only use NW to SE moves inside a given grid. A kiss is a local configuration where walks meet but do not cross. To each grid, we associate a (blossoming) quiver with relations (Q, I):



The maximal strings of this gentle quiver with relations are in bijection with the walks and we define $\mathcal{W}$ to be the additive full subcategory of the category of representations of (Q, I) generated by the walks. The exact category $\mathcal{W}$ is 0-Auslander with projective-injectives the straight walks (i.e. the string modules corresponding to the maximal paths in (Q, I)). Its indecomposable projectives are the non-simple indecomposable projectives over (Q, I), and similarly for injectives.

Its silting objects are precisely the non-kissing facets and therefore Theorem 2 recovers the flip of non-kissing facets.

New examples? Discussions with Thomas Brüstle and Ralf Schiffler during this workshop showed that the mutation of *maximal almost rigid modules* [4] over a quiver of Dynkin type A_n also follows from Theorem 2.

Question: Are there other examples of mutation theories that are recovered by Theorem 2? Can Theorem 2 be applied to discover new mutations?

Disclaimer: The categorical structures arising from those examples are apparently of very different natures. The category $(\mathcal{C}, \Delta_T)$ is a relative structure of a triangulated category; the category $K^{[-1,0]}(\operatorname{proj}\Lambda)$ is extension-closed in a triangulated category; the category $\mathcal{W}$ is an exact category, and its quotient by the ideal of morphisms factoring through some projective-injective (which is relevant from the combinatorial perspective) is neither exact nor triangulated. Being able to consider all those situations at once shows how extriangulated structures can be useful tools in representation theory.

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Polyharmonic modular forms and representations of the Gelfand quiver

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(joint work with Claudia Alfes-Neumann and Martin Raum)

Recall that the group $G = \mathsf{SL}_2(\mathbb{R})$ acts on complex upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathsf{Im}(\tau) > 0\}$ by linear fractional transformations: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$. For the coordinate $\tau = u + iv$ on $\mathbb{H}$ one considers the *hyperbolic Laplace operator* $\Delta = -v^2 \left(\dfrac{\partial^2}{\partial u^2} + \dfrac{\partial^2}{\partial v^2}\right)$, which acts on the space of smooth functions on $\mathbb{H}$.

Let $\Gamma \subset G = \mathsf{SL}_2(\mathbb{R})$ be a congruence subgroup (e.g. $\Gamma = \mathsf{SL}_2(\mathbb{Z})$) and $\Gamma \xrightarrow{\rho} \mathsf{GL}(V)$ be its finite dimensional representation. The space of polyharmonic modular forms of type (Γ, ρ) is defined as

$$\mathcal{H}\big(\Gamma, (V, \rho)\big) = \left\{ \mathbb{H} \xrightarrow{f} V \;\middle|\; \begin{array}{l} f \text{ is smooth} \\ f(h \cdot \tau) = \rho(\gamma)\big(f(\tau)\big) \text{ for all } h \in \Gamma \\ \Delta^d(f) = 0 \text{ for some } d \in \mathbb{N} \text{ depending on } f \end{array} \right\}.$$

From the analytic point of view of major interest is the case when (V, ρ) is the trivial representation and $d = 1$. To any polyharmonic modular form $f \in \mathcal{H}\big(\Gamma, (V, \rho)\big)$ one can attach a *Harish–Chandra* $(\mathfrak{g}, K)$-module M_f, where $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $K = \mathsf{SO}_2(\mathbb{R})$. Next, the category of Harish–Chandra modules $\mathsf{HC}(\mathfrak{g}, K)$ splits into a union of blocks and M_f automatically belongs to the central block $\mathsf{HC}_\circ(\mathfrak{g}, K)$ (the block containing the trivial $(\mathfrak{g}, K)$-module). It can be shown (see e.g. [1]) that $\mathsf{HC}_\circ(\mathfrak{g}, K)$ is equivalent to the category of finite length nilpotent representations of the Gelfand quiver

$$\begin{tikzcd} - \arrow[r, bend left, "a_-"] & \star \arrow[l, bend left, "b_-"] \arrow[r, bend right, "b_+"'] & + \arrow[l, bend right, "a_+"'] \end{tikzcd} \qquad a_-b_- = a_+b_+.$$

Summing up, we attach to any polyharmonic modular form f a representation M_f of the Gelfand quiver. We prove that M_f is *indecomposable* and provide a full list of representations of the Gelfand quiver having the form M_f for an appropriate $f \in \mathcal{H}\big(\Gamma, (V, \rho)\big)$. This classification generalizes the one of Bringmann and Kudla [2], obtained for *harmonic* Maaß modular forms (i.e. those $f \in \mathcal{H}\big(\Gamma, (V, \rho)\big)$ for which $\Delta(f) = 0$). Our results are also valid for polyharmonic modular forms of arbitrary weight, where other blocks of $\mathsf{HC}(\mathfrak{g}, K)$ do arise.

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Cluster tilting for Segre products

Norihiro Hanihara

1. Background

Let R be a commutative Gorenstein ring, which we assume to be complete local of dimension d, containing a perfect residue field k. One is interested in the category $\mathrm{CM}\, R$ of (maximal) Cohen-Macaulay modules over R, whose stable category $\underline{\mathrm{CM}}\, R$ is a triangulated category, and is moreover $(d-1)$-Calabi-Yau when R is an isolated singularity. On the other hand, given a finite dimensional algebra A and an integer n, one can construct an n-Calabi-Yau triangulated category as the *n-cluster category* $\mathrm{C}_n(A)$, which is by definition the triangulated hull $\mathrm{D^b}(\mathrm{mod}\, A)/-\otimes^{\mathrm{L}}_A DA[-n]$ of the orbit category of the derived category [2, 1, 11]. Recent extensive studies on tilting theory for singularity categories (see [8, 6]) show that there are in fact triangle equivalences

$$\underline{\mathrm{CM}}\, R \simeq \mathrm{C}_{d-1}(A),$$

which gives a deep connection between representation theories of a commutative ring R and of a finite dimensional algebra A.

We intend to study the category $\mathrm{CM}\, R$ in terms of the finite dimensional algebras A, and the most fundamental class of such algebras leads to the following definition of a new representation type.

Definition 1. Let R be a commutative complete Gorenstein local ring. We say that R is of *hereditary representation type* if there exists a finite dimensional hereditary algebra H and a triangle autoequivalence F of $\mathrm{D^b}(\mathrm{mod}\, H)$ such that there is a triangle equivalence

$$\underline{\mathrm{CM}}\, R \simeq \mathrm{D^b}(\mathrm{mod}\, H)/F.$$

We refer to [5] for some variations of the definition. A priori we allow any autoequivalence F, but in fact we shall give examples with the best choice of F.

We would like to give examples of commutative rings of hereditary representation type, which is a non-trivial task; to the best of the author's knowledge only two such examples from quotient singularities are known [10, 12, 13]. To achieve this we give a general construction of cluster tilting objects for some commutative rings R (Theorem 3), and apply Morita-type theorem for Calabi-Yau triangulated categories (Theorem 5) to the stable category $\underline{\mathrm{CM}}\, R$, which yields that R is of hereditary representation type.

2. Cluster tilting for Segre products

Let us start with the following notion which generalizes cluster tilting objects.

Definition 2 ([7, 9])**.** Let R be a commutative Cohen-Macaulay ring. A *CT module* is $M \in \mathrm{CM}\, R$ satisfying the following.

$$\begin{aligned}\mathrm{add}\, M &= \{X \in \mathrm{CM}\, R \mid \mathrm{Hom}_R(M, X) \in \mathrm{CM}\, R\} \\ &= \{X \in \mathrm{CM}\, R \mid \mathrm{Hom}_R(X, M) \in \mathrm{CM}\, R\}\end{aligned}$$

When R is a local isolated singularity of dimension $d+1$, then $\mathrm{Hom}_R(M,N) \in \mathrm{CM}\,R$ if and only if $\mathrm{Ext}^i_R(M,N) = 0$ for $0 < i < d$, thus CT module is exactly the d-cluster tilting object in $\mathrm{CM}\,R$. Also, CT modules are special class of non-commutative creprant resolutions [14] which are one of the important subjects in birational geometry.

Now we state the general existence theorem of CT modules. Let k be a perfect field, $R' = \bigoplus_{i\geq 0} R'_i$ and $R'' = \bigoplus_{i\geq 0} R''_i$ be positively graded commutative Gorenstein normal domains of dimension ≥ 2 such that R'_0 and R''_0 are finite dimensional over k. Let $R = R'\#R''$ be the completion of the Segre product $\bigoplus_{i\geq 0} R'_i \otimes R''_i$. It follows from the computation of local cohomology groups over the Segre product [3] that if R' and R'' has the same negative a-invariant then R is Gorenstein with the same a-invariant.

Theorem 3. Suppose that R' and R'' have the same a-invariant $-p$. If R' (resp. R'') has a CT module M (resp. M'') such that $\mathrm{End}_{R'}(M')$ (resp. $\mathrm{End}_{R''}(M'')$) is positively graded, then $\bigoplus_{l=0}^{p-1} M'(l)\#M''$ is a CT module for R.

Note that the existence of a CT module whose endomorphism ring is positively graded implies the a-invariant is negative, thus R is Gorenstein, and the direct sum makes sense.

Trivially $S \in \mathrm{CM}\,S$ is a CT module when S is regular. As a very special case of Theorem 3, we obtain the following result.

Corollary 4. Let $S_i = k[[x_{i,0},\dots,x_{i,d_i}]]$, $1 \leq i \leq n$ be the power series rings with $\deg x_{i,j} = a_{i,j} > 0$. Suppose that $\sum_{j=0}^{d_i} a_{i,j}$ is common for all $1 \leq i \leq n$. Then the Segre product $S_1\#\cdots\#S_n$ has a CT module.

3. Commutative rings of hereditary representation type

To give examples of commutative Gorenstein rings of hereditary representation type, we need another result than the previous construction of cluster tilting modules in Theorem 3, which is the following Morita-type theorem for Calabi-Yau triangulated categories.

Theorem 5 ([4])**.** Let $\mathcal{T}$ be an algebraic d-Calabi-Yau triangulated category with a d-cluster tilting object T such that $\mathrm{End}_{\mathcal{T}}(\bigoplus_{i=0}^{d-2} T[-i]) =: H$ is hereditary. Then there exists a trianlge equivalence

$$\mathcal{T} \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,H)/\tau^{-1/(d-1)}[1]$$

for a naturally defined $(d-1)$-st root of τ, provided every connected component of H is representation-infinite.

Note that the right-hand-side is a $\mathbb{Z}/(d-1)\mathbb{Z}$-quotient of the usual d-cluster category $\mathrm{C}_d(H) = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,H)/\tau^{-1}[d-1]$, which we therefore denote by $\mathrm{C}_d^{(1/(d-1))}(H)$.

From the construction of Theorem 3, we can find the following examples of commutative rings to which one can apply Theorem 5, hence of hereditary representation type.

Theorem 6. (1) Let $R = k[[x,y]]^{(2)} \# k[[x,y]]^{(2)}$ with $\deg x = \deg y = 1$. Then there exists a triangle equivalence

$$\underline{\mathrm{CM}}\, R \simeq \mathrm{C}_2(kQ), \quad Q = \ \circ \Longrightarrow \circ \Longleftarrow \circ\ .$$

(2) Let $R = k[[x,y,z]] \# k[[u,v]]$ with $\deg x = \deg y = \deg z = 1$ and $\deg u = 1$, $\deg v = 2$. Then there exists a triangle equivalence

$$\underline{\mathrm{CM}}\, R \simeq \mathrm{C}_3^{(1/2)}(kQ), \quad Q = \begin{tikzcd} \circ \arrow[r] \arrow[dr] \arrow[dr, shift left] \arrow[dr, shift right] & \circ \arrow[dl] \arrow[dl, shift left] \arrow[dl, shift right] \\ \circ \arrow[r] & \circ \end{tikzcd} .$$

(3) Let $R = k[[x,y,z]] \# k[[x,y,z]]$ with $\deg x = \deg y = \deg z = 1$. Then there exists a triangle equivalence for the $\widetilde{A}_5$-quiver with triple arrows.

$$\underline{\mathrm{CM}}\, R \simeq \mathrm{C}_4^{(1/3)}(kQ), \quad Q = \begin{tikzcd} \circ \arrow[dr] \arrow[dr, shift left] \arrow[dr, shift right] \arrow[ddr] \arrow[ddr, shift left] \arrow[ddr, shift right] & \circ \arrow[dl] \arrow[dl, shift left] \arrow[dl, shift right] \arrow[ddl] \arrow[ddl, shift left] \arrow[ddl, shift right] \\ \circ \arrow[dr] \arrow[dr, shift left] \arrow[dr, shift right] & \circ \arrow[dl] \arrow[dl, shift left] \arrow[dl, shift right] \\ \circ & \circ \end{tikzcd}$$

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Cluster algebras from surfaces and scattering diagrams

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(joint work with Travis Mandel)

Let S denote a compact oriented surface and M a finite set of marked points in S. $\Sigma = (S, M)$ is called a marked surface. On the one hand, by working with the topology of (curves in) Σ, one can construct a skein algebra $\mathrm{Sk}(\Sigma)$. It has the basis consisting of the bracelets (certain unions of curves). On the other hand, by studying algebraic or geometric structures on Σ, one can construct the cluster algebra A. By [GHKK18], A has the basis consisting of the theta functions (arising from the study of mirror symmetry). It is known that $\mathrm{Sk}(\Sigma) = A$. Roughly speaking, our main result is the following:

Theorem 1 ([MQ23])**.** *The bracelets coincide with the theta functions.*

It has been long expected that the bracelets form the atomic basis for $\mathrm{Sk}(\Sigma)$, i.e., it is the "minimal positive basis". We deduce this conjecture from the "atomicity" properties of theta functions.

Corollary 2. *The bracelets basis is atomic.*

Let us provide more details. We consider curves on Σ such that they either end at M (called arcs) or are closed and contained in $S \setminus M$ (called loops). We impose the mild assumption that Σ has at least one triangulation Δ, by which we mean a maximal collection of non-intersecting non-isotopic curves. A diagram D is union of finitely many curves, which is considered up to homotopy fixing M and the crossings. The skein algebra $\overline{\mathrm{Sk}}(\Sigma)$ associated to Σ is defined as the quotient module of $\oplus_D \mathbb{Z}D$ modulo the Kauffman's skein relations, where D are understood as the homotopic classes. Its multiplication is given by taking the union, i.e., $D \cdot D' := D \cup D'$. The skein algebra $\mathrm{Sk}(\Sigma)$ is defined as the localization of $\overline{\mathrm{Sk}}(\Sigma)$ at the boundary arcs.

For example, let us consider an annulus with one marked point on each boundary component. Its arcs without self-crossings are b_1, b_2, γ_k, see Figure 1. It has a loop L without self-crossing.

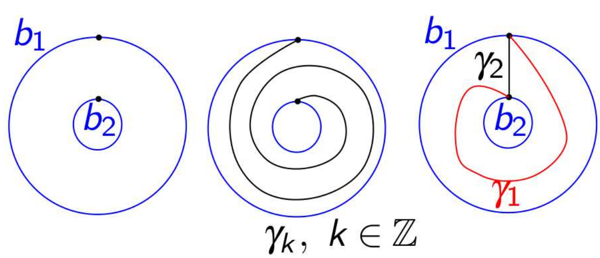

Figure 1. An annulus with two marked points and its curves

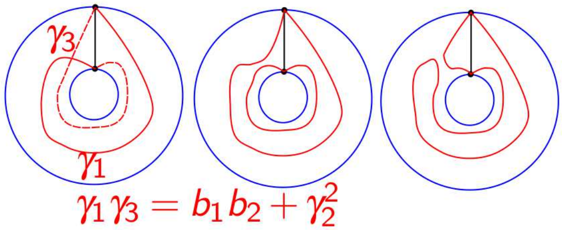

FIGURE 2. A skein relation on the annulus

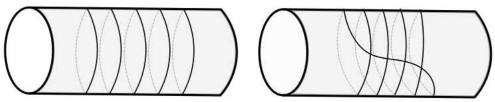

FIGURE 3. Replace multiple loops by a bracelet with self-crossing

By the skein relation, we have $\gamma_k\gamma_{k+2} = b_1b_2 + \gamma_{k+1}^2, \forall k$, see Figure 2. It turns out that the skein algebra is

$$\mathrm{Sk}(\Sigma) =_{\mathbb{Z}[b_1^\pm, b_2^\pm]} \{\gamma_k^{w_k}\gamma_{k+1}^{w_{k+1}}, L^{w_L} | w_i \geq 0\}.$$

The monomials $\gamma_k^{w_k}\gamma_{k+1}^{w_{k+1}}$ are called cluster monomials. For any w-copy of loops isotopic to L, we replace it by a closed loop with $w-1$ self-crossing, called a bracelet $\mathrm{Brac}^w(L)$, see Figure 3. It is known that $\mathrm{Brac}^w(L) = T_w(L)$, where $T_w(\ )$ is the w-th Chebyshev polynomial defined by

$$T_w(z + z^{-1}) = z^w + z^{-w}, \forall w \geq 0.$$

Then the bracelets form a basis of $\mathrm{Sk}(\Sigma)$ [MSW13][FG06].

Notice that $(\gamma_1, \gamma_2, b_1, b_2)$ form a triangulation Δ. The corresponding seed $\mathbf{s}$ consists of a quiver Q (dual graph of Δ) with 4 vertices, and generators a_i on the vertices (we denote $a_1 = \gamma_1, a_2 = \gamma_2, a_3 = b_1, a_4 = b_2$), see [FST08]. Define $x_k = \prod_{i\to k} a_i / \prod_{k\to j} a_j$.

As in [GHKK18], we consider a scattering diagram $\mathfrak{D} = \mathfrak{D}(\mathbf{s})$ in the corresponding Euclidean space. Let us continue this example. Then $\mathfrak{D}$ is a collection of walls in $\mathbb{R}^4$. Let f_i denote the i-th unit vector. A wall is a pair $(\mathfrak{d}, \mathfrak{p}_{\mathfrak{d}})$ such that:

- $\mathfrak{d}$ is a codimension-1 polyhedral cone in $\mathbb{R}^4$ invariant under the translation $\pm f_3$, $\pm f_4$.
- $\mathfrak{p}_{\mathfrak{d}}$ is a formal series in $\mathbb{Z}[\![x_1, x_2]\!]$, called the wall-crossing operator.

A base point $\mathcal{Q}$ is any chosen generic point in $\mathbb{R}^4 \setminus \mathfrak{D}$. For any $m \in \mathbb{Z}^4$, one can construct the theta function$\vartheta_{m,\mathcal{Q}}$ from $\mathfrak{D}$. It takes the form $a^m F_{m,\mathcal{Q}}$, where $F_{m,\mathcal{Q}}$ is a formal series in $\mathbb{Z}[\![x_1, x_2]\!]$ with constant 1. For a different choice $\mathcal{Q}'$, $\vartheta_{m,\mathcal{Q}}$ and $\vartheta_{m,\mathcal{Q}'}$ are related by wall-crossing operators along a path from $\mathcal{Q}$ to $\mathcal{Q}'$.

By [GHKK18], the cluster monomials are theta functions. In this example, it remains to check that $\mathrm{Brac}^w(L)$ are theta functions. Denote $\delta = f_1 - f_2$. We show that, when $\mathcal{Q}$ is chosen close enough to the ray $R_{>0}\delta$,

$$\vartheta_{w\delta,\mathcal{Q}} = a^w + a^{-w} + \text{higher } x\text{-order terms}.$$

Based on this, we can show the Chebyshev recursion:

$$\vartheta_{w\delta,\mathcal{Q}} = T_w(\vartheta_{w,\mathcal{Q}}).$$

We also check that $\vartheta_{\delta,\mathcal{Q}} = L$. Therefore, $\mathrm{Brac}^w(L) = \vartheta_{w\delta,\mathcal{Q}}$ as desired. This example provides a hint for treating general surfaces (more techniques are needed).

Let us discuss the generality of our result. Assume Σ is connected without loss of generality.

Theorem 3. *When Σ is not a once-punctured torus, the quantum (tagged) bracelets coincide with the quantum theta functions.*

For this statement, we need to construct quantum (tagged) bracelets for general surfaces: they are not well-defined in skein algebras if the surface has punctures, but we can construct them in quantum cluster algebras.

Finally, let us remark how the result fails for a once-punctured torus: the tagged arc represents an element in the cluster algebra, which is 4 times the corresponding theta function.

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On braid group actions on full exceptional sequences

Sibylle Schroll

(joint work with Wen Chang and Fabian Haiden)

Full exceptional sequences play an important role as generators of triangulated categories in many different contexts such as bounded derived categories of coherent sheaves and finite dimensional algebras and in the construction of Fukaya Seidel categories. However, for a general triangulated category it is often difficult to decide whether full exceptional sequences exist and if they do exist, how to find all of them. In [9] an action of the braid group on the set of full exceptional sequences in a triangulated category was given and in [2] it was conjectured that, up to shift, this action should be transitive. There are many examples where this has been shown to be true, namely the action has been shown to be transitive for bounded derived categories of projective planes [9], del Pezzo surfaces [8], the Hirzebruch surface of degree two [11] and weighted projective lines [13, 14], the bounded derived categories of hereditary algebras [6, 15] and the topological Fukaya category associated to a graded compact oriented surface with boundary and marked points (as constructed in [10]) in the case that the surface has genus zero [5].

In this talk we report on [3] where we study the braid group action on the set of full exceptional sequences in the topological Fukaya category $\mathcal{F}(S, M, \nu)$ of a graded compact oriented surface with boundary and marked points (S, M) of any genus and where ν is a line field on S, that is an element in $\Gamma(S, \mathbb{P}(TS))$. This category was constructed in [10] where it was shown that it is triangle equivalent to the subcategory of objects with finite total homology of the derived category of a homologically smooth graded gentle algebra. Using results of [5] and [4], we show that there are bijections between the following sets:

- the set of full exceptional sequences in $\mathcal{F}(S, M, \nu)$ up to shift
- marked surfaces (S, M) where $M \subset \partial S$ and M has at least two elements
- (S, M)-framed Hurwitz systems
- simple branched coverings of the complex unit disc with matching paths.

We further show that all four sets have a natural action of the Artin braid group B_n on n strands where n is the rank of the Grothendieck group of $\mathcal{F}(S, M, \nu)$ and that these actions are compatible with the bijections above.

According to Birman-Hilden [1] the braid group B_n is isomorphic to the symmetric mapping class group of a regular branched covering of the disk with n branch points. Recall that the symmetric mapping class group is the subgroup of the mapping class group of the surface S consisting of isotopy classes of orientation preserving diffeomorphisms of S which preserve the fibres of the branched covering. We note that the regular simple branched coverings of the disc are the branched double covers of the disc. Furthermore, the combined actions of the braid group and the mapping class group act transitively on the double covers of the discs with n branch points and with matching paths. Thus for branched double covers of the disc with matching paths the question of the transitivity of the mapping class group action on the set of full exceptional sequences reduces to the question of when the mapping class group is isomorphic to the symmetric mapping class group of the branched covering.

A recent result of Ghaswala and McLeay [7] answers precisely this question. Namely, they show that the symmetric mapping class group of the branched double cover of the disc branched over n points is isomorphic to the mapping class group if and only if $n \leq 3$. They further show that if $n \geq 4$ then the symmetric mapping class group is a subgroup of of infinite index of the mapping class group.

Combining the above results, we obtain a counter example to the conjecture by Bondal and Polishchuk. Namely the braid group action on the set of full exceptional sequences in $\mathcal{F}(S, M, \nu)$ is not transitive if S is a surface with two marked points and either $g = 1$ and two boundary components or $g \geq 2$ and one or two boundary components. Moreover, in these cases there are infinitely many orbits.

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On projective quotient algebras

Giovanni Cerulli Irelli

(joint work with Markus Reineke, Marco Trevisiol and Grzegorz Zwara)

Let k be an algebraically closed field of characteristic zero and let A be a finite-dimensional k-algebra of finite representation type. In [3] it was defined an algebra B_A called the projective quotient algebra associated with A. The algebra B_A is of global dimension at most two and it is endowed with two functors: the restriction functor $res : B_A\text{-mod} \to A\text{-mod}$ and the intermidiate extension functor $c : A\text{-mod} \to B_A\text{-mod}$. Given an A-module M we denote by $\hat{M} = c(M)$. The definition of the algebra B_A generalizes a previous definition given in [1] in the case when $A = kQ$ is the path algebra of a Dynkin quiver. In this case we use the shorthand B_Q. We denote by $\hat{\mathbf{d}} := \mathbf{dim}\hat{M}$ and by $R_B(\hat{\mathbf{d}})$ the corresponding representation variety containing the point $\hat{M}$. We denote by $\mathcal{O}_M$ and $\mathcal{O}_{\hat{M}}$ the orbits of M and $\hat{M}$ under the structure groups, respectively. The importance of the projective quotient algebra for us is given by the following theorem proved in [1] and [3]: the restriction functor realizes the orbit closure of M as a geomtric quotient of both $R_B(\hat{\mathbf{d}})$ and of $\overline{\mathcal{O}_{\hat{M}}}$. Thus, to prove that the orbit closure of M is normal it is enough to prove that either $R_B(\hat{\mathbf{d}})$ or $\overline{\mathcal{O}_{\hat{M}}}$ is normal. Let us restrict our attention to the case of Dynkin quivers considered in [1]. In this case the algebra B_Q is presented by a quiver $\hat{Q}$ with relations $\hat{I}$ [2]. It is straightforward to check that $\overline{\mathcal{O}_{\hat{M}}}$ is an irreducible component of $R_B(\hat{\mathbf{d}})$, and that the number of equations defining $R_B(\hat{\mathbf{d}})$ equals the codimension of $\mathcal{O}_{\hat{M}}$. As a working hypothesis in [2] we formulated the following conjecture: $\overline{\mathcal{O}_{\hat{M}}} = R_B(\hat{\mathbf{d}})$. If this is the case then $\overline{\mathcal{O}_{\hat{M}}}$ is complete intersection and hence Cohen-Macauley. In this project we find examples in type A_n-equioriented, D_5 and A_6 with alternating orientation where the conjecture does not hold. On the positive side we find conditions on a representation of A_n-equioriented, so that the conjecture holds. It is worth noticing that $\hat{M}$ is the minimal element of $R_B(\hat{\mathbf{d}})$ with respect to the Hom-order, and thus our (counter-)examples extend the class of representation varieties where the degeneration order is not equivalent to the Hom-order.

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Quasi-hereditary algebras with exact Borel subalgebras

Teresa Conde

Quasi-hereditary algebras are modeled after the category $\mathcal{O}$ of a complex semi-simple Lie algebra and their exact Borel subalgebras are the counterpart of a Borel subalgebra.

Definition 1. A finite-dimensional algebra[1] A is *quasi-hereditary* with respect to a poset $(Q_0, \leq)$ labelling the isoclasses of simple A-modules if there exist quotients Δ_i of the projective indecomposable modules P_i satisfying the following conditions for every $i \in Q_0$:

(1) $\ker(P_i \twoheadrightarrow \Delta_i)$ is filtered by modules Δ_j with $j > i$;
(2) $\ker(\Delta_i \twoheadrightarrow L_i)$ is filtered by simple modules of the form L_j with $j < i$.

[1] We fix an underlying field and assume throughout that it is algebraically closed.

The quotients Δ_i are called *standard modules*. Quasi-hereditary algebras may be alternatively defined using a class of modules whose role is somewhat dual to that of the standard modules. Such modules are called *costandard* and are denoted by ∇_i. Quasi-hereditary algebras are quite abundant in mathematics and examples include not only blocks of the category $\mathcal{O}$, but also Schur algebras and all finite-dimensional algebras of global dimension at most 2.

For the purpose of understanding the representation theory of quasi-hereditary algebras, one may identify the algebras having the same structure. We call two quasi-hereditary algebras *equivalent* if the corresponding categories of modules filtered by standard modules are equivalent as exact categories, and denote by $[(A, Q_0, \leq)]$ the class of all quasi-hereditary algebras equivalent to a given quasi-hereditary algebra $(A, Q_0, \leq)$. By a result of Dlab and Ringel (see [4]), any two equivalent quasi-hereditary algebras are Morita equivalent through an equivalence of module categories that preserves the "quasi-hereditary structure" (i.e. maps standard modules to standard modules).

Exact Borel subalgebras of quasi-hereditary algebras emulate the role of Borel subalgebras of complex semi-simple Lie algebras.

Definition 2. A subalgebra B of a quasi-hereditary algebra $(A, Q_0, \leq)$ is an *exact Borel subalgebra* if:

(1) the induction functor $A \otimes_B -$ is exact;
(2) the isoclasses of simple B-modules can be labelled by the poset $(Q_0, \leq)$ in such a way that $(B, Q_0, \leq)$ becomes a quasi-hereditary algebra with simple standard modules;
(3) $A \otimes_B L_i^B \cong \Delta_i^A$ for every $i \in Q_0$.

The subalgebra B is *regular* if the induction functor $A \otimes_B -$ induces isomorphisms $\mathrm{Ext}_B^n(L_i^B, L_j^B) \to \mathrm{Ext}_A^n(A \otimes_B L_i^B, A \otimes_B L_j^B)$ for every $i, j \in Q_0$ and every $n \geq 1$.

Not every quasi-hereditary algebra has an exact Borel subalgebra. However, a result of Koenig, Külshammer and Ovsienko establishes the existence of exact Borel subalgebras up to equivalence of quasi-hereditary algebras.

Theorem 3 ([5]). *Let $(A, Q_0, \trianglelefteq)$ be a quasi-hereditary algebra. There exists at least one algebra in $[(A, Q_0, \trianglelefteq)]$ that contains a basic regular exact Borel subalgebra.*

Theorem 3 can be seen as an analogue of the following weak variant of the Wedderburn–Malcev Theorem: in the class of finite-dimensional algebras Morita equivalent to a given algebra A, there exists at least one that contains a basic maximal semi-simple subalgebra. In this latter situation, any two algebras Morita equivalent to A that contain a basic maximal semi-simple subalgebra must be basic and therefore isomorphic, and there exists an isomorphism between them which restricts to an isomorphism between the maximal semi-simple subalgebras. Theorem 3 raises the following questions with respect to uniqueness, in the spirit of Wedderburn–Malcev Theorem:

(1) Consider two equivalent quasi-hereditary algebras, both containing a basic regular exact Borel subalgebra. Do they need to be isomorphic? In

other words, is a quasi-hereditary algebra in $[(A, Q_0, \leq)]$ containing a basic regular exact Borel subalgebra unique up to isomorphism?

(2) Let B be a basic regular exact Borel of some quasi-hereditary algebra in $[(A, Q_0, \leq)]$. Is B unique up to isomorphism?

(3) Let $(A, Q_0, \leq)$ and $(A', Q_0', \leq)$ be two equivalent quasi-hereditary algebras, both containing a basic regular exact Borel subalgebra, say B and B', respectively. Does there exist an isomorphism $f : A \to A'$ that restricts to an isomorphism between B and B'?

Naturally, an affirmative answer to (3) implies an affirmative answer to both (1) and (2). In [2], a positive answer to (1) is provided.

Theorem 4 ([2]). *Let $(A, Q_0, \leq)$ be a quasi-hereditary algebra and let $(l_i)_{i \in Q_0}$ be the sequence of integers defined recursively by the formula*

$$l_i = 1 + \sum_{\substack{j,k \in Q_0 \\ k \leq j < i}} l_k [\nabla_j : L_k] \dim \left(\mathrm{Hom}_A(\Delta_j, \Delta_i)\right) - \sum_{\substack{j \in Q_0 \\ j < i}} l_j [\Delta_i : L_j],$$

where $[X : L]$ denotes the multiplicity of simple A-module L as a composition factor of X. Up to isomorphism, $\mathrm{End}_A(\bigoplus_{i \in Q_0} P_i^{\oplus l_i})^{op}$ is the unique algebra in $[(A, Q_0, \leq)]$ containing a basic regular exact Borel subalgebra.

A considerably stronger uniqueness result, providing an affirmative answer to (3), is proved in [6]. A positive aspect of the approach in [2] is that it gives explicit numerical answers using elementary methods and does not rely on calculations with A_∞-algebras, in contrast with results in [5, 1, 6]. In fact, the recursive formula in Theorem 4 can be slightly modified to define a special matrix $V_{[(A,Q_0,\leq)]}$ which turns out to be quite useful to derive information about regular exact Borel subalgebras.

Theorem 5 ([2]). *Let $(A, Q_0, \leq)$ be a quasi-hereditary algebra. The following hold:*

(1) $(A, Q_0, \leq)$ has a regular exact Borel subalgebra if and only if the linear system of equations $V_{[(A,Q_0,\leq)]} x = (\dim L_i^A)_{i \in Q_0}$ has a solution whose entries are positive integers;

(2) all algebras in $[(A, Q_0, \leq)]$ have a regular exact Borel subalgebra if and only if the radical of every standard A-module is filtered by costandard modules;

(3) The Cartan matrix of any regular exact Borel subalgebra B of a quasi-hereditary algebra in $[(A, Q_0, \leq)]$ is given by the transpose of the product $([\nabla_i^A : L_j^A])_{i,j \in Q_0} \times V_{[(A,Q_0,\leq)]}$.

An interesting aspect of the matrix $V_{[(A,Q_0,\leq)]}$ (and of exact Borel subalgebras in general) is its compatibility with the inductive structure of the algebra $(A, Q_0, \leq)$. The next result is part of joint work in progress with Julian Külshammer.

Theorem 6 ([3]). *Let $(A, Q_0, \leq)$ be a quasi-hereditary algebra and let $e \in A$ be an idempotent supported in a subset Q_0' of Q_0. Assume that A-module A/AeA*

is filtered by standard modules and that its dual $D(A/AeA)$ (regarded as an A-module) is filtered by costandard modules. Then

$$V_{[(A,Q_0,\leq)]} = \begin{pmatrix} V_{[(A/AeA,Q_0\setminus Q_0',\leq)]} & 0 \\ * & V_{[(eAe,Q_0',\leq)]} \end{pmatrix}.$$

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Functorially finite d-torsion classes and τ_d-rigid modules

Hipolito Treffinger

(joint work with Jenny August, Johanne Haugland, Karin Jacobsen, Sondre Kvamme and Yann Palu)

In this report d is a positive natural number, A is a finite dimensional algebra over a field K, $\operatorname{mod} A$ is the category of finitely generated right A-modules and all subcategories are supposed to be full. A subcategory $\mathcal{X}$ of $\operatorname{mod} A$ is said to be generating if for every A-module M there exists an epimorphism $p : X \to M$ where $X \in \mathcal{X}$. Dually, $\mathcal{X}$ is said to be cogenerating if for every A-module M there exists a monomorphism $i : M \to X'$ where $X' \in \mathcal{X}$. A morphism $f\colon X \to Y$ is called *left minimal* if any endomorphism g of Y satisfying $g \circ f = f$ is an isomorphism. The definition of a right minimal morphism is dual.

Given an object $M \in \operatorname{mod} A$ we denote by $|M|$ the number of isomorphism classes of indecomposable direct summands of M. By $\operatorname{add} M$ we denote the category of direct summands of direct sums of U_A. Moreover, $\operatorname{Fac} M$ is the category

$$\operatorname{Fac} M = \{X \in \operatorname{mod} A \colon M' \to X \to 0 \text{ where } M' \in \operatorname{add} M\}.$$

Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} A$ and let M be an object of $\operatorname{mod} A$. A *right $\mathcal{X}$-approximation* of M is a map $f_M \colon X_M \to M$ where $X_M \in \mathcal{X}$ such that every map $g\colon X \to M$ with $X \in \mathcal{X}$ factors through f_M. If every object $M \in \operatorname{mod} A$ has a right $\mathcal{X}$-approximation we say that $\mathcal{X}$ is a contravariantly finite. The definitions of left approximations and covariantly finite subcategories are dual. A subcategory is said to be *functorially finite* if it is both covariantly and contravariantly finite. We start by recalling the definition of d-cluster tilting subcategories [I1, I2].

Definition 1. A functorially finite generating-cogenerating subcategory $\mathcal{M}$ of $\operatorname{mod} \mathrm{A}$ is *d-cluster tilting* if

$$\begin{aligned}\mathcal{M} &= \{X \in \operatorname{mod} \mathrm{A} \mid \operatorname{Ext}^i_A(X, M) = 0 \text{ for all } M \in \mathcal{M} \text{ and all } 1 \leq i \leq d-1\} \\ &= \{Y \in \operatorname{mod} \mathrm{A} \mid \operatorname{Ext}^i_A(M, Y) = 0 \text{ for all } M \in \mathcal{M} \text{ and all } 1 \leq i \leq d-1\}.\end{aligned}$$

Note that if $d = 1$, the conditions of Definition 1 are empty and we obtain that for every algebra A, $\operatorname{mod} \mathrm{A}$ is a 1-cluster tilting subcategory.

The study of the homological aspects of d-cluster tilting subcategories in module categories is usually known as *higher homological algebra.* This is due to the fact that they behave like a higher analogue of abelian categories. This was axiomatised in [Ja] where the notion of d-abelian categories was introduced. Moreover, it was also shown in [Ja] that every d-cluster tilting subcategory of an abelian category is d-abelian. In particular, in any d-cluster tilting subcategory $\mathcal{M}$ the shortest non-split exact sequences known as d-extensions, have $(d+2)$ terms and every morphism in $\mathcal{M}$ admits a d-kernel and a d-cokernel. The reader is referred to [JK] for more on higher homological algebra.

The main object of study in this report is that of functorially finite d-torsion classes. The notion of a torsion classes was introduced in [D] for abelian categories and adapted to d-cluster tilting subcategories of abelian categories in [Jø]. Instead of giving the original definition of d-torsion classes we give as definition the characterisation of d-torsion classes inside a d-cluster tilting subcategory $\mathcal{M} \subset \operatorname{mod} \mathrm{A}$ shown in [D], for the case $d = 1$, and [AHJKPT1], for the general case.

Definition 2. Let A be an algebra and let $\mathcal{U}$ be a subcategory of a d-cluster-tilting $\mathcal{M} \subset \operatorname{mod} \mathrm{A}$. Then $\mathcal{U}$ is a d-torsion class if and only if $\mathcal{U}$ the following conditions are verified:

(1) $\mathcal{U}$ is closed under d-quotients, i.e., for every map $f \colon M \to U$ where $U \in \mathcal{U}$ there exists a d-cokernel

$$M \xrightarrow{f} U \xrightarrow{f_1} U_1 \xrightarrow{f_2} \cdots \xrightarrow{f_d} U_d \to 0$$

such that $U_i \in \mathcal{U}$ for every $i \in \{1, \dots, d\}$.

(2) $\mathcal{U}$ is closed under d-extensions, i.e., for every d-extension

$$0 \to U \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{d-1}} X_d \xrightarrow{f_d} U' \to 0$$

where $U, U' \in \mathcal{U}$ there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "f_1"] \arrow[d] & \cdots \arrow[r, "f_{d-1}"] & X_d \arrow[r, "f_d"] \arrow[d] & U' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & U \arrow[r, "f'_0"] & U'_1 \arrow[r, "f'_1"] & \cdots \arrow[r, "f'_{d-1}"] & U'_d \arrow[r, "f'_d"] & U' \arrow[r] & 0 \end{tikzcd}$$

such that $U_i \in \mathcal{U}$ for every $i \in \{1, \dots, d\}$.

Remark 3. As a consequence of Definition 2 one can show that given a d-torsion class $\mathcal{U} \subset \mathcal{M}$, for every object $M \in \mathcal{M}$ there exists a subobject $U^M \in \mathcal{U}$, known as the torsion subobject of M with respect to $\mathcal{U}$, such that the natural monomorphism $0 \to U^M \to M$ is a minimal right $\mathcal{U}$-approximation. In particular this implies that every d-torsion class is contravariantly finite.

Fix a functorially finite d-torsion class $\mathcal{U}$ in $\mathcal{M} \subset \operatorname{mod} \mathrm{A}$ and let $f_0\colon A \to U_0^A$ be the minimal $\mathcal{U}$-left approximation. Then we can construct an exact sequence

$$A \xrightarrow{f_0} U_0^A \xrightarrow{f_1} U_1^A \xrightarrow{f_2} \dots \xrightarrow{f_{d-1}} U_{d-1}^A \xrightarrow{f_d} U_d^A \to 0$$

where U_i^A is the minimal $\mathcal{U}$-approximation of $\operatorname{coker} f_{i-1}$ for every $i \in \{1, \dots, d\}$. One can show that this sequence is actually a d-cokernel of f_0 and, hence, that f_d is an epimorphism. Moreover, by construction, this is a minimal d-cokernel of f_0, which implies that given any other d-cokernel

$$A \xrightarrow{f_0} U_0^A \xrightarrow{g_1} \tilde{U}_1^A \xrightarrow{g_2} \dots \xrightarrow{g_{d-1}} \tilde{U}_{d-1}^A \xrightarrow{g_d} \tilde{U}_d^A \to 0$$

of f_0 we have that U_i^A is a direct summand of $\tilde{U}_i^A$ for every $i \in \{1, \dots, d\}$. In particular, it follows from Definition 2 that $U_i^A \in \mathcal{U}$ for every $i \in \{1, \dots, d\}$. Set $U_A = \bigoplus_{i=0}^d U_i^A$.

Denote $\tau_d = \tau \circ \Omega^{d-1}$, where Ω is the syzygy functor and τ is the Auslander-Reiten translation in $\operatorname{mod} \mathrm{A}$. If $M \in \operatorname{mod} \mathrm{A}$ is such that $\operatorname{Hom}_A(M, \tau_d M) = 0$ we say that M is τ_d-rigid. In the case $d = 1$ we simply say that M is τ-rigid.

In the following result, due to [AS], for $d = 1$, and [AHJKPT2], for $d > 2$, we compile several properties of U_A and $\mathcal{U}$.

Theorem 4. *Let $\mathcal{U}$ be a functorially finite d-torsion class in $\mathcal{M} \subset \operatorname{mod} \mathrm{A}$. With the notation above the following holds.*

- *(1) U_A is a τ_d-rigid module.*
- *(2) U_A is Ext^d-projective in $\mathcal{U}$, i.e., $\operatorname{Ext}_A^d(U_A, U) = 0$ for every $U \in \mathcal{U}$. Moreover, if $\tilde{U} \in \mathcal{U}$ is Ext^d-projective in $\mathcal{U}$ then $\tilde{U} \in \operatorname{add}(U_A)$.*
- *(3) $\mathcal{U} = \mathcal{M} \cap \operatorname{Fac} U_A$.*
- *(4) Let $P_{\mathcal{U}}$ be the maximal basic projective module such that $\operatorname{Hom}_A(P_{\mathcal{U}}, \mathcal{U}) = 0$. Then $|U_A| + |P_{\mathcal{U}}| = |A|$.*

In what follows, we say that a 1-torsion class is a simply a torsion class and we denote it by $\mathcal{T}$. Moreover the torsion subobject of an object M with respect to a torsion class $\mathcal{T}$ is denoted by tM. The following result of [AJST] shows an intricate relationship between d-torsion classes inside a d-cluster tilting subcategory $\mathcal{M}$ and the torsion classes inside the ambient module category $\operatorname{mod} \mathrm{A}$.

Theorem 5. *Let $\mathcal{U}$ be a d-torsion class in $\mathcal{M} \subset \operatorname{mod} \mathrm{A}$. Then there exists a torsion class $\mathcal{T} \subset \operatorname{mod} \mathrm{A}$ such that $\mathcal{U} = \mathcal{M} \cap \mathcal{T}$ and $U^M = tM$ for every $M \in \mathcal{M}$.*

By fixing $d = 1$ in Theorem 4, for every functorially finite torsion class $\mathcal{T}$ we obtain a τ-rigid module $T = T_0^A \oplus T_1^A$ such that $\mathcal{T} = \operatorname{Fac} T$. Then we can combine this with Theorem 5 to obtain a new way to get functorially finite d-torsion classes, as shown by the following result.

Proposition 6. Let A be an algebra having a d-cluster tilting subcategory $\mathcal{M}$ and let $\mathcal{T} = \operatorname{Fac} T$ a functorially finite torsion class in $\operatorname{mod} A$, where T is as above. Suppose moreover that $\mathcal{U} = \mathcal{M} \cap \mathcal{T}$ is a d-torsion class $U^M = tM$ for every $M \in \mathcal{M}$. Then $\mathcal{U}$ is a functorially finite d-torsion class in $\mathcal{M} \subset \operatorname{mod} \mathrm{A}$.

Suppose that we have a functorially finite torsion class $\mathcal{T}$ as in the previous proposition. Since the d-torsion class $\mathcal{U} = \mathcal{M} \cap \mathcal{T}$ is functorially finite in $\operatorname{mod} \mathrm{A}$ we can play the same game as before and consider the exact sequence

$$T \xrightarrow{g_0} U_0^T \xrightarrow{g_1} U_1^T \xrightarrow{g_2} \dots \xrightarrow{g_{d-1}} U_{d-1}^T \to 0$$

where U_0^T is the minimal left $\mathcal{U}$-approximation of $T = T_0^A \oplus T_1^A$ and U_i^T is the minimal $\mathcal{U}$-approximation of $\operatorname{coker} g_{i-1}$ for every $i \in \{0, \dots, d-1\}$. If we denote by $U_T = \bigoplus_{i=0}^{d-1} U_i^T$, the following result of [AHJKPT2] shows a deep connection between U_T and U_A.

Theorem 7. *Let $\mathcal{M}$ be a d-cluster tilting subcategory of* $\operatorname{mod} \mathrm{A}$ *and let $\mathcal{T}$ be a functorially finite torsion class such that $\mathcal{U} = \mathcal{M} \cap \mathcal{T}$ is a d-torsion class in $\mathcal{M}$ with $U^M = tM$ for every $M \in \mathcal{M}$. Then, with the notation above, the module U_T is an* Ext^d*-projective in $\mathcal{U} = \mathcal{M} \cap \operatorname{Fac} T$. Moreover* $\operatorname{add} U_T = \operatorname{add} U_A$.

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Derived categories of singular varieties and finite dimensional algebras

Martin Kalck
(joint work with Yujiro Kawamata, Carlo Klapproth, Nebojsa Pavic)

1. Introduction and motivation

Throughout this text, X denotes a projective variety over $\mathbb{C}$.

Aim. *Describe the bounded derived category $D^b(X)$ of coherent sheaves on X using derived categories $D^b(R) := D^b(\text{mod } R)$ of finite dimensional algebras R.*

This has been achieved for projective spaces as a first example of tilting theory.

Example 1.1 (Beilinson 1978)**.** *Let $X = \mathbb{P}^n$. There are triangle equivalences*

$$D^b(X) \cong D^b\left(\operatorname{End}_X(\bigoplus_{i=0}^{n} \mathcal{O}(i))\right) \cong \langle \mathcal{O}, \dots, \mathcal{O}(n)\rangle \cong \langle D^b(\mathbb{C}), \dots, D^b(\mathbb{C})\rangle, \tag{1}$$

describing $D^b(X)$ as $D^b(R)$ for a finite dimensional algebra R (tilting), using a full exceptional sequence and as a semiorthogonal decomposition (S.O.D), respectively.

Building on this, full exceptional sequences have been constructed for many smooth varieties, e.g. by Hille & Perling, Kapranov, Kawamata, Kuznetsov.

However, singular projective (Gorenstein) varieties do not admit full exceptional sequences, cf. [5] and also [8]. This motivates the following definition, which generalizes both tilting ($l = 1$) and full exceptional sequences (all $\mathcal{C}_i \cong D^b(\mathbb{C})$).

Definition 1.2 ([5])**.** A *Kawamata semiorthogonal decomposition (KSOD)* is an (admissible) semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{C}_1, \dots, \mathcal{C}_l\rangle, \tag{KSOD}$$

where $\mathcal{C}_i \subseteq \mathsf{Perf}(X)$ or $\mathcal{C}_i \cong D^b(R_i)$ for *finite dimensional* algebras R_i.

Remark 1.3. *Kawamata initiated the study of KSODs for threefolds* [9]*, which was the starting point for our investigations in* [5]*. If X is singular, at least one of the algebras R_i has infinite global dimension. The derived categories $D^b(R_i)$ of these algebras capture the singular information of X and fit into Kuznetsov & Shinder's framework of 'categorical absorption of singularities'* [7]*.*

In the sequel, we describe known constructions of KSODs and also discuss obstructions to the existence of KSODs.

2. Constructions of Kawamata S.O.Ds and tilting

2.1. Curves.

Theorem 2.1 (Burban [1], cf. also [5])**.** *Let X be a connected nodal curve with all irreducible components isomorphic to $\mathbb{P}^1$. Then $D^b(X)$ has a tilting object if and only if the dual intersection graph of X is a tree.*

2.2. **Surfaces.**

Theorem 2.2 (Karmazyn-Kuznetsov-Shinder [6]). *Let X be a projective toric surface. Then X has a Kawamata decomposition*[1]

$$D^b(X) \cong \langle D^b(R_1), \ldots D^b(R_n)\rangle, \tag{2}$$

if and only if $K_0(D^b(X)) \cong \mathbb{Z}^t$. Moreover, in this case, (2) holds for finite dimensional local $\mathbb{C}$-algebras $R_i \cong K_{r_i,s_i}$ described explicitly in (4), see [2].

Definition 2.3. Tuples $\bar{a} = (a_0, \ldots, a_d) \in \mathbb{Z}_{>0}$ define *weighted projective varieties*

$$\mathbb{P}(\bar{a}) := \mathsf{Proj}\ \mathbb{C}[x_0, \ldots, x_d], \qquad \text{where } \deg x_i = a_i. \tag{3}$$

The toric varieties $\mathbb{P}(\bar{a})$ have dimension d and possibly cyclic quotient singularities.

Applying Theorem 2.2 to two-dimensional weighted projective varieties yields:

Example 2.4. *$D^b(\mathbb{P}(1,a,b)) = \langle D^b(K_{b,a}), D^b(K_{a,b'}), D^b(\mathbb{C})\rangle$, for $1 \leq a < b$ coprime, $b'b \equiv 1 \bmod a$ and finite dimensional algebras given by*

$$K_{r,s} \cong \frac{\mathbb{C}\langle z_1 \ldots, z_l\rangle}{\left(\begin{array}{cr} z_i^{\beta_i} & \textit{for all } i \\ z_i z_j & \textit{for } i<j \\ z_i\left(z_i^{\beta_i-2}\right)\left(z_{i-1}^{\beta_{i-1}-2}\right)\cdots\left(z_{j+1}^{\beta_{j+1}-2}\right)\left(z_j^{\beta_j-2}\right)z_j & \textit{for } i>j \end{array}\right)}, \tag{4}$$

where $r/(r-s) = [\beta_1, \ldots, \beta_l]$ is the Hirzebruch–Jung continued fraction, see [2].

2.3. **Higher-dimensional varieties.** The following generalizations of Beilinson's Example 1.1 to singular weighted projective varieties (3) are a part of [3].

Theorem 2.5. *There is an explicit tilting object on $X = \mathbb{P}(1^d, m)$ for all $m, d \geq 1$.*

Theorem 2.6. *Let $X_d = \mathbb{P}(1^d, d)$. There is a Kawamata S.O.D*

$$D^b(X_d) = \langle D^b(R_{X_d}), D^b(\mathbb{C}), D^b(\mathbb{C})\rangle, \tag{5}$$

where R_{X_d} is given by the following quiver Q_d with relations

$$\begin{tikzcd}[column sep=large]
1 \arrow[r, shift left=1.2ex, "x_{11}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=1.2ex, "x_{1d}"'] & 2 \arrow[r, shift left=1.2ex, "x_{21}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=1.2ex, "x_{2d}"'] & \cdots \arrow[r, shift left=1.2ex, "x_{(d-2)1}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=1.2ex, "x_{(d-2)d}"'] & d-1 \arrow[lll, bend left=30, "z_{12}"] \arrow[lll, bend left=45, phantom, "\vdots\ z_{ij,\ i<j}"] \arrow[lll, bend left=60, "z_{(d-1)d}"]
\end{tikzcd}$$

$$\begin{aligned}
x_{(i+1)k}x_{ij} - x_{(i+1)j}x_{ik} &= 0 \\
x_{1j}z_{kl} + x_{1l}z_{jk} - x_{1k}z_{jl} &= 0 \\
z_{kl}x_{(d-2)j} + z_{jk}x_{(d-2)l} - z_{jl}x_{(d-2)k} &= 0 \\
(\mathbb{C}Q_d\{z_{ab} \mid 1 \leq a < b \leq d\}\mathbb{C}Q_d)^2 &= 0 \\
\textit{for all } 1 \leq j < k < l \leq d \textit{ and } 1 \leq i \leq d-3 &
\end{aligned}$$

Remark 2.7. (*a*) *There is a more general version of Theorem 2.6 for certain varieties (including some non-rational varieties!) with a cyclic quotient singularity $\frac{1}{d}(1^d)$ as in $\mathbb{P}(1^d, d)$, cf.* [3].
(*b*) *Theorems 2.5 & 2.6 yield equivalences between singularity categories $D^{sg}(X) := D^b(X)/\mathsf{Perf}(X)$ and $D^{sg}(R) := D^b(R)/K^b(\mathrm{proj}\ R)$, for finite dimensional algebras R. In the special case of Gorenstein singularities, these singular equivalences have also been obtained by Hanihara using a different approach.*

[1] It is admissible if X is Gorenstein.

3. Obstructions to Kawamata semiorthogonal decompositions

We state a special case of the main result in [4]. It indicates that odd-dimensional varieties with hypersurface singularities typically do not admit KSODs.

Theorem 3.1. *Let X be a projective Gorenstein variety over $\mathbb{C}$. Assume that*

(1) the dimension of X is odd.

(2) the bounded derived category of coherent sheaves $D^b(X)$ admits a KSOD.

The following implications hold for every isolated hypersurface *singularity s of X:*

(a) if s is an ADE-hypersurface singularity*, then s is an A_1-singularity.*

(b) if $D^{sg}(\widehat{\mathcal{O}}_s) \cong D^{sg}(S)$ for a 3-fold $\mathsf{Spec}(S)$ admitting a small *resolution of singularities*[2]*, then s is an A_1-singularity.*

Remark 3.2. (1) *A special case of Theorem 3.1 shows: if an odd-dimensional projective variety X with only* ADE-hypersurface singularities *admits a tilting object, then X is nodal, i.e. all its singularities are of type A_1.*
(2) *The key idea to prove Theorem 3.1 is to compare the singularity categories of X and of finite dimensional Gorenstein algebras R, respectively. More precisely, we look at the quivers of the endomorphism algebras of special generators called cluster-tilting objects. In $D^{sg}(R)$ these quivers cannot have loops or 2-cycles, whereas, in our setting, if the* corresponding *quivers in $D^{sg}(X)$ contain arrows, then they also have loops or 2-cycles*[3]*. Hence, these quivers contain no arrows, which implies $D^{sg}(\widehat{\mathcal{O}}_s) \cong D^{sg}(A_1)$. This shows that $\widehat{\mathcal{O}}_s$ is an A_1-singularity.*

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[2] That is, a resolution of singularities such that all fibres are at most one-dimensional.

[3] Here, we use that the cluster-tilting object in $D^{sg}(S) \cong D^{sg}(\widehat{\mathcal{O}}_s) \subseteq D^{sg}(X)$, corresponds to a small resolution of singularities. During my talk, I only assumed that some cluster-tilting object in $D^{sg}(\widehat{\mathcal{O}}_s)$ exists and I do not know whether this is sufficient for the implication (b).

Localization of extriangulated categories

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(joint work with Yasuaki Ogawa, Arashi Sakai)

1. Introduction

Extriangulated category can be regarded as a common generalization of exact categories and triangulated categories. It is defined as an additive category, equipped with some extra structure ([7]). One of the advantages to use extriangulated categories is that this class is closed by some basic operations such as taking extension-closed subcategories, ideal quotients, and relative theories. As *localization* is another basic operation for additive categories, it will be natural to look for a unified formulation of localizations of extriangulated categories.

In a collaboration with Yasuaki Ogawa and Arashi Sakai [6], we have shown that the localization of an extriangulated category by a class of morphisms satisfying some conditions can be equipped with a natural, universal structure of an extriangulated category. This construction unifies the Serre quotient of abelian categories and the Verdier quotient of triangulated categories. In fact, it unifies the following types of localizations involving abelian/exact/triangulated categories known in the literature.

(i) Verdier quotient of a triangulated category [9].
(ii) Serre quotient of an abelian category.
(iii) Localization of an exact category by a percolating subcategory given by Henrard, Kvamme and van Roosmalen [2].
(iv) Rump's localization of an exact category [8] by a biresolving subcategory.
(v) Localization of an extriangulated category with respect to a Hovey twin cotorsion pair [7]. This contains localizations of abelian/exact categories with respect to nice model structures (abelian model structures by Hovey [3],[4], exact model structures by Gillespie [1], whose counterpart in triangulated categories is given by Yang [10]) as typical cases.

2. Main theorem

In the following let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category ([7, Definition 2.12]), which we assume to be small for simplicity. If $\delta\in\mathbb{E}(C,A)$ satisfies $\mathfrak{s}(\delta)=[A\xrightarrow{x}B\xrightarrow{y}C]$, we write as $A\overset{x}{\rightarrowtail}B\overset{y}{\twoheadrightarrow}C\overset{\delta}{\dashrightarrow}$ and call this sequence an $\mathfrak{s}$-*triangle* or *extriangle*. The sequence $A\overset{x}{\rightarrowtail}B\overset{y}{\twoheadrightarrow}C$ is called a *conflation*, morphisms x,y are called *inflation* and *deflation*, respectively. Let $\mathscr{S}$ be a class of morphisms in $\mathscr{C}$ which contains all isomorphisms, closed by compositions and finite direct sums. Take an ideal quotient $p\colon\mathscr{C}\to\overline{\mathscr{C}}=\mathscr{C}/[\mathcal{N}_{\mathscr{S}}]$, where we define $\mathcal{N}_{\mathscr{S}}$ to be the full subcategory of $\mathscr{C}$ given by

$$\mathcal{N}_{\mathscr{S}}=\{N\in\mathscr{C}\mid(N\to 0),(0\to N)\in\mathscr{S}\}.$$

Let $L\colon \overline{\mathscr{C}} \to \widetilde{\mathscr{C}}$ be the localization of $\overline{\mathscr{C}}$ by $\overline{\mathscr{S}} = p(\mathscr{S})$, and put $Q = L \circ p\colon \mathscr{C} \to \widetilde{\mathscr{C}}$. Our main theorem is as follows.

Theorem 1 ([6, Theorem 3.5])**.** *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ and $\mathscr{S}$ be as above. Assume that $\mathscr{S}$ satisfies $\mathscr{S} = p^{-1}(\overline{\mathscr{S}})$ and the following conditions.*

(MR1) *$\overline{\mathscr{S}}$ satisfies 2-out-of-3 condition with respect to compositions.*

(MR2) *$\overline{\mathscr{S}}$ is a multiplicative system.*

(MR3) *Let $A \overset{x}{\rightarrowtail} B \overset{y}{\twoheadrightarrow} C \overset{\delta}{\dashrightarrow}$ and $A' \overset{x'}{\rightarrowtail} B' \overset{y'}{\twoheadrightarrow} C' \overset{\delta'}{\dashrightarrow}$ be any pair of extriangles, and suppose that $a \in \mathscr{C}(A, A'), c \in \mathscr{C}(C, C')$ satisfies $a\delta = \delta' c$. If $\overline{a}, \overline{c} \in \overline{\mathscr{S}}$, then there exists $\overline{b} \in \overline{\mathscr{S}}$ which makes*

$$\begin{tikzcd}
A \arrow[r, "\overline{x}"] \arrow[d, "\overline{a}"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "\overline{y}"] \arrow[d, "\overline{b}"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "\overline{c}"] \\
A' \arrow[r, "\overline{x}'"'] & B' \arrow[r, "\overline{y}'"'] & C'
\end{tikzcd}$$

commutative in $\overline{\mathscr{C}}$.

(MR4) *$\{\overline{v} \circ \overline{x} \circ \overline{u} \mid x$ is an inflation, $\overline{u}, \overline{v} \in \overline{\mathscr{S}}\}$ is closed by compositions. Dually for deflations.*

Then $\widetilde{\mathscr{C}}$ has a natural structure of an extriangulated category $(\widetilde{\mathscr{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}})$ with a universal exact functor $(Q, \mu)\colon (\mathscr{C}, \mathbb{E}, \mathfrak{s}) \to (\widetilde{\mathscr{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}})$.

As a brief sketch of the construction, for any pair of objects A, C, the abelian group $\widetilde{\mathbb{E}}(C, A)$ is given by the quotient set of

$$\{(C \xleftarrow{\overline{t}} Z \overset{\delta}{\dashrightarrow} X \xleftarrow{\overline{s}} A) \mid \overline{s}, \overline{t} \in \overline{\mathscr{S}}, \delta \in \mathbb{E}(Z, X)\}$$

by some equivalence relation. For each equivalence class $[\overline{t} \backslash \delta / \overline{s}] \in \widetilde{\mathbb{E}}(C, A)$ to which $(C \xleftarrow{\overline{t}} Z \overset{\delta}{\dashrightarrow} X \xleftarrow{\overline{s}} A)$ belongs, the associated conflation $A \xrightarrow{Q(x \circ s)} Y \xrightarrow{Q(t \circ y)} C$ in $\widetilde{\mathscr{C}}$ is given by using an extriangle $X \overset{x}{\rightarrowtail} Y \overset{y}{\twoheadrightarrow} Z \overset{\delta}{\dashrightarrow}$ in $\mathscr{C}$. See [6] for details.

3. Relation to known constructions

Let $\mathscr{N} \subseteq \mathscr{C}$ be a full additive subcategory closed by isomorphisms.

Definition 2. $\mathscr{N} \subseteq \mathscr{C}$ is called a *thick* subcategory if it is moreover closed by direct summands, and satisfies 2-out-of-3 condition for conflations.

We assume $\mathscr{N} \subseteq \mathscr{C}$ is thick in the rest. Define classes of morphisms $\mathcal{L}_{\mathscr{N}}, \mathcal{R}_{\mathscr{N}}$ in $\mathscr{C}$ by

$$\begin{aligned}
\mathcal{L}_{\mathscr{N}} &= \{f \mid \text{there is a conflation } X \overset{f}{\rightarrowtail} Y \twoheadrightarrow N \text{ with } N \in \mathscr{N}\}, \\
\mathcal{R}_{\mathscr{N}} &= \{f \mid \text{there is a conflation } N \rightarrowtail X \overset{f}{\twoheadrightarrow} Y \text{ with } N \in \mathscr{N}\},
\end{aligned}$$

and let $\mathscr{S}_{\mathscr{N}}$ be the class of all finite compositions of morphisms in $\mathcal{L}_{\mathscr{N}}$ and $\mathcal{R}_{\mathscr{N}}$. We have $\mathscr{N}_{\mathscr{S}_{\mathscr{N}}} = \mathscr{N}$. In [6], we have shown that $\mathscr{S}_{\mathscr{N}}$ satisfies the assumption of Theorem 1 in each of the following two cases.

Case A: $\mathcal{N} \subseteq \mathscr{C}$ is *biresolving*, namely any object C in $\mathscr{C}$ admits a deflation $N \twoheadrightarrow C$ and an inflation $C \rightarrowtail N'$ for some $N, N' \in \mathcal{N}$.

Case B: $\mathcal{N} \subseteq \mathscr{C}$ is *percolating*, namely any morphism $f \in \mathscr{C}(X,Y)$ admits a factorization $f = i \circ d$ through some $N \in \mathcal{N}$ where $X \overset{d}{\twoheadrightarrow} N$ is a deflation and $N \overset{i}{\rightarrowtail} Y$ is an inflation. Moreover we assume that $\mathcal{N}$ satisfies the following technical conditions (a),(b).

(a) If $f \in \mathscr{C}(A,B)$ is a split monomorphism such that $\overline{f}$ is an isomorphism in $\overline{\mathscr{C}}$, then there exist $N \in \mathcal{N}$ and $j \in \mathscr{C}(N,B)$ such that $[f\ j] \colon A \oplus N \to B$ is an isomorphism in $\mathscr{C}$.

(b) $\operatorname{Ker}\big(\mathscr{C}(X,A) \xrightarrow{l\circ -} \mathscr{C}(X,B)\big) \subseteq [\mathcal{N}](X,A)$ holds for any $X \in \mathscr{C}$ and any $l \in \mathcal{L}_{\mathcal{N}}(A,B)$. Dually, $\operatorname{Ker}\big(\mathscr{C}(C,X) \xrightarrow{-\circ r} \mathscr{C}(B,X)\big) \subseteq [\mathcal{N}](C,X)$ holds for any $X \in \mathscr{C}$ and any $r \in \mathcal{R}_{\mathcal{N}}(B,C)$.

Remark 3. If $\mathscr{C}$ is an exact category, then the above definition of a percolating subcategory agrees with the original definition in [2, Definition 2.8], and the technical conditions are always satisfied.

If $\mathcal{N} \subseteq \mathscr{C}$ is a biresolving thick subcategory as in the above Case A, then we have the following ([6, Section 4.3]).

- $\mathscr{S}_{\mathcal{N}} = \mathcal{R}_{\mathcal{N}} \circ \mathcal{L}_{\mathcal{N}}$ holds, and $\mathscr{S}_{\mathcal{N}}$ satisfies the assumption of Theorem 1.
- $\overline{\mathscr{S}_{\mathcal{N}}}$ agrees with the set of monomorphic-epimorphic morphisms in $\overline{\mathscr{C}}$.
- $\widetilde{\mathscr{C}}$ becomes a triangulated category.

Example 4. Case A recovers the following.

(1) If $\mathscr{C}$ is triangulated, then any thick subcategory is biresolving and $\widetilde{\mathscr{C}}$ is nothing but the Verdier quotient of $\mathscr{C}$ by $\mathcal{N}$.

(2) If $\mathscr{C}$ is exact, then it recovers the localization of an exact category by a biresolving subcategory given by Rump in [8].

(3) If $\mathscr{C}$ is a weakly idempotent complete extriangulated category equipped with a Hovey twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$, then we can associate a biresolving thick subcategory $\mathcal{N}$ consisting of cones of morphisms in $\{V \xrightarrow{f} S \mid V \in \mathcal{V}, S \in \mathcal{S}\}$. As above the resulting localization $\widetilde{\mathscr{C}}$ is triangulated, which was also shown in [7, Theorem 6.20]. (We remark that equivalence between condition (WIC) in [7, Condition 5.8] and weak idempotent completeness is shown by Klapproth in [5, Proposition 2.7].)

If $\mathcal{N} \subseteq \mathscr{C}$ is a percolating thick subcategory satisfying technical conditions as in the above Case B, then we have the following ([6, Section 4.4]).

- $\mathscr{S}_{\mathcal{N}} = \mathcal{L}_{\mathcal{N}} \circ \mathcal{R}_{\mathcal{N}}$ holds, and $\mathscr{S}_{\mathcal{N}}$ satisfies the assumption of Theorem 1.
- In general $\widetilde{\mathscr{C}}$ is extriangulated by Theorem 1. Moreover $\widetilde{\mathscr{C}}$ becomes exact if $\mathcal{N}$ satisfies the following condition which is a bit stronger than (b).

(b)′ $\operatorname{Ker}\big(\mathscr{C}(X,A) \xrightarrow{x\circ -} \mathscr{C}(X,B)\big) \subseteq [\mathcal{N}](X,A)$ holds for any $X \in \mathscr{C}$ and any inflation $x \in \mathscr{C}(A,B)$. Dually, $\operatorname{Ker}\big(\mathscr{C}(C,X) \xrightarrow{-\circ y} \mathscr{C}(B,X)\big) \subseteq [\mathcal{N}](C,X)$ holds for any $X \in \mathscr{C}$ and any deflation $y \in \mathscr{C}(B,C)$.

For example if $\mathscr{C}$ is exact, this condition is trivially satisfied.

Example 5. Case B recovers the following. See [6, Section 4.4] for the detail and other typical instances.

(1) If $\mathscr{C}$ is exact, then it recovers the localization of an exact category by a percolating subcategory given by Henrard, Kvamme and van Roosmalen in [2].
(2) In particular if $\mathscr{C}$ is abelian, it recovers the Serre quotient of $\mathscr{C}$ by $\mathcal{N}$.

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Indecomposables in the separated monomorphism category

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(joint work with Nan Gao, Julian Külshammer, Chrysostomos Psaroudakis)

1. Introduction

In this talk we consider a classical question in representation theory, namely to determine the indecomposable objects in a given category. We are interested in the separated monomorphic representations of a quiver, defined below.

Let $Q = (Q_0, Q_1)$ be a finite acyclic quiver with vertices Q_0 and arrow Q_1, and let $\mathcal{A}$ be an abelian category. For an arrow $\alpha \in Q_1$ we let $s(\alpha)$ and $t(\alpha)$ denote its source and target, respectively. Consider the category $\operatorname{rep}(Q, \mathcal{A})$ of representations of Q over $\mathcal{A}$. Explicitly, its objects are tuples

$$((A_{\mathtt{i}})_{\mathtt{i}\in Q_0}, (A_\alpha)_{\alpha\in Q_1})$$

where $A_{\mathtt{i}}$ is an object in $\mathcal{A}$ for all $\mathtt{i} \in Q_0$, and where $A_\alpha \colon A_{s(\alpha)} \to A_{t(\alpha)}$ is a morphism in $\mathcal{A}$ for all $\alpha \in Q_1$. For simplicity we skip the index sets when writing representations. A morphism

$$(A_{\mathtt{i}}, A_\alpha) \to (B_{\mathtt{i}}, B_\alpha)$$

in $\operatorname{rep}(Q, \operatorname{mod}\Lambda)$ is a collection of morphism $(\varphi_{\mathtt{i}} \colon A_{\mathtt{i}} \to B_{\mathtt{i}})_{\mathtt{i} \in Q_0}$ such that the following diagram

$$\begin{tikzcd} A_{\mathtt{i}} \arrow[r, "A_\alpha"] \arrow[d, "\varphi_{\mathtt{i}}"] & A_{\mathtt{j}} \arrow[d, "\varphi_{\mathtt{j}}"] \\ B_{\mathtt{i}} \arrow[r, "B_\alpha"] & B_{\mathtt{j}} \end{tikzcd}$$

commutes for every $\alpha \in Q_1$. We are interested in the subcategory of separated monomorphisms.

Definition 1. A representation $(A_{\mathtt{i}}, A_\alpha) \in \operatorname{rep}(Q, \mathcal{A})$ is called a *separated monomorphism* if the canonical morphism

$$\bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha) = \mathtt{i}}} A_{s(\alpha)} \xrightarrow{(A_\alpha)} A_{\mathtt{i}}$$

is injective for all $\mathtt{i} \in Q_0$. The subcategory of separated monomorphisms is denoted by $\operatorname{mono}(Q, \mathcal{A})$.

If $Q = \mathtt{1} \to \mathtt{2}$ and $\mathcal{A} = \operatorname{mod}\Lambda$ is the category of finitely generated right modules over a ring Λ, then we are just studying submodules of finitely generated Λ-modules. Such situations have been studied since the beginning of the 20th century, see for example [4, 2]. There is now a large body of work on such questions. We mention some recent results below. Here k denotes a field.

- In [6] it was shown that $\operatorname{mono}(\mathtt{1} \to \mathtt{2}, \operatorname{mod}\Lambda)$ is exact and has Auslander–Reiten sequences if Λ is an Artin algebra. This was generalized in [3] to $\operatorname{mono}(Q, \operatorname{mod}\Lambda)$ for an arbitrary finite acyclic quiver Q.
- The Auslander–Reiten quiver of $\operatorname{mono}(\mathtt{1} \to \mathtt{2}, \operatorname{mod} k[x]/(x^n))$ was determined in [7] for $n \leq 6$. It turns out that it is representation-finite when $n \leq 5$, and is tame when $n = 6$.
- The Auslander–Reiten quiver of $\operatorname{mono}(\mathtt{1} \to \mathtt{2} \to \mathtt{3}, \operatorname{mod} k[x]/(x^n))$ was determined in [5] for $n \leq 4$. It turns out that it is representation-finite when $n \leq 3$, and is tame when $n = 4$.
- The Auslander–Reiten quiver of $\operatorname{mono}(\mathtt{1} \to \mathtt{2} \to \mathtt{3} \to \mathtt{4}, \operatorname{mod} k[x]/(x^3))$ and $\operatorname{mono}(\mathtt{1} \to \mathtt{2} \to \cdots \to \mathtt{m}, \operatorname{mod} k[x]/(x^2))$ was determined in [9]. In these cases it is representation-finite.
- Less is known about the separated monomorphism category over $\mathbb{Z}/p^n\mathbb{Z}$ (for p a prime) than over $k[x]/(x^n)$, even though both are artinian uniseral rings of Loewy length n. This is due to $\mathbb{Z}/p^n\mathbb{Z}$ not being a finite-dimensional algebra, and therefore certain techniques in representation-theory cannot be applied to it. An old open question is the "Birkhoff problem", on describing the category $\operatorname{mono}(\mathtt{1} \to \mathtt{2}, \operatorname{mod}\mathbb{Z}/p^6\mathbb{Z}))$. The

question is attributed to the paper [1] by Birkhoff. Note that the indecomposables in $\mathrm{mono}(\mathtt{1} \to \mathtt{2}, \mathrm{mod}\, \mathbb{Z}/p^n\mathbb{Z})$ have been computed in [8] for $n \leq 5$. In [7] the authors ask whether the structure of $\mathrm{mono}(\mathtt{1} \to \mathtt{2}, \mathrm{mod}\, \mathbb{Z}/p^6\mathbb{Z}))$ is similar to the structure of $\mathrm{mono}(\mathtt{1} \to \mathtt{2}, \mathrm{mod}\, k[x]/(x^6))$.

2. A (representation) equivalence

Now assume $\mathcal{A}$ is an abelian category with enough injectives. Then the category $\mathrm{rep}(Q, \mathcal{A})$ is also an abelian category with enough injectives. One can show that $\mathrm{mono}(Q, \mathcal{A})$ is an extension-closed subcategory of $\mathrm{rep}(Q, \mathcal{A})$, and is therefore an exact category in the sense of Quillen. Furthermore, $\mathrm{mono}(Q, \mathcal{A})$ has enough injective objects as an exact category, but they are different from the injectives in $\mathrm{rep}(Q, \mathcal{A})$. Let $\overline{\mathrm{mono}}(Q, \mathcal{A})$ and $\overline{\mathcal{A}}$ denote the injectively stable categories. The following definition is due to Auslander.

Definition 2. A *representation equivalence* is a full and dense functor F which reflects isomorphisms (i.e. if $F(f)$ is an isomorphism then f is an isomorphism).

Theorem 3. *The composite*

$$\mathrm{mono}(Q, \mathcal{A}) \to \mathrm{rep}(Q, \mathcal{A}) \to \mathrm{rep}(Q, \overline{\mathcal{A}})$$

induces a representation equivalence

$$\Phi \colon \overline{\mathrm{mono}}(Q, \mathcal{A}) \to \mathrm{rep}(Q, \overline{\mathcal{A}}).$$

Furthermore, if Q has at least one arrow, then Φ is an equivalence if and only if $\mathcal{A}$ is hereditary.

Since Φ is a representation equivalence, it induces a bijection between isomorphism classes of indecomposable objects in $\overline{\mathrm{mono}}(Q, \mathcal{A})$ and in $\mathrm{rep}(Q, \overline{\mathcal{A}})$. If $\mathcal{A}$ is nice, e.g. the category of finitely generated modules of an Artin algebra, then this is also in bijection with isomorphism classes of non-injective objects in $\mathrm{mono}(Q, \mathcal{A})$. This is very useful if $\overline{\mathcal{A}}$ is abelian, since we reduce the study of indecomposables in the exact category $\mathrm{mono}(Q, \mathcal{A})$ to the study of indecomposables in the abelian category $\mathrm{rep}(Q, \overline{\mathcal{A}})$.

3. Artinian uniserial rings of Loewy length 3

Let $\mathbb{F}_p$ be the finite field with p elements, where p is a prime number. Then there is an equivalence

$$\overline{\mathrm{mod}}\, \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{\cong} \overline{\mathrm{mod}}\, \mathbb{F}_p[x]/(x^3).$$

By Theorem 3 we get a bijection between isomorphism classes of indecomposable non-injective objects in $\mathrm{mono}(Q, \mathrm{mod}\, \mathbb{Z}/p^3\mathbb{Z})$ and in $\mathrm{mono}(Q, \mathrm{mod}\, \mathbb{F}_p[x]/(x^3))$. In fact, this can be extended to a bijection between all indecomposable objects, which preserves the underlying partition vector. We explain what this means below.

Assume Λ is a commutative artinian uniserial ring of Loewy length n, e.g. $\mathbb{Z}/p^n\mathbb{Z}$ or $k[x]/(x^n)$ for k a field. Then there is a bijection between isomorphism classes of finitely generated Λ-modules, and partitions, i.e. sequences $\overline{\alpha} = (\alpha_1 \geq$

$\alpha \geq \ldots \geq \alpha_m$) with $\alpha_1 \leq n$. Explicitly the bijection is given by sending $\overline{\alpha}$ to the Λ-module

$$M(\overline{\alpha}) \coloneqq \bigoplus_{i=1}^{m} M(\alpha_i)$$

where $M(\alpha_i)$ is the unique (up to isomorphism) indecomposable Λ-module with length α_i. Given a representation $(M_{\mathtt{i}}, M_\alpha) \in \operatorname{rep}(Q, \operatorname{mod}\Lambda)$, we have an associated partition $\overline{\alpha}^{\mathtt{i}}$ for each $\mathtt{i} \in Q_0$, defined by $M(\overline{\alpha}^{\mathtt{i}}) \cong M_{\mathtt{i}}$. The tuple $(\overline{\alpha}^{\mathtt{i}})_{\mathtt{i}\in Q_0}$ is called the *partition vector* of $(M_{\mathtt{i}}, M_\alpha)$.

Theorem 4. *There is a bijection between isomorphism classes of indecomposable representations in* $\operatorname{mono}(Q, \operatorname{mod}\mathbb{Z}/p^3\mathbb{Z})$ *and in* $\operatorname{mono}(Q, \operatorname{mod}\mathbb{F}_p[x]/(x^3))$ *which preserves the partition vector.*

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Tiled surfaces, string algebras and laminations

Karin Baur

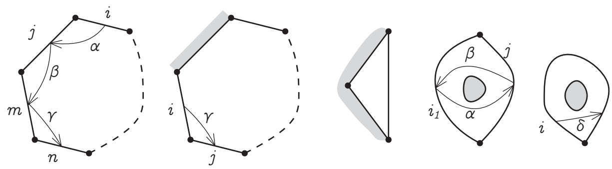
(joint work with Raquel Coelho Simoes, Bethany R. Marsh)

A geometric module for string algebras, [2]. A *string algebra* is a finite dimensional algebra $A = kQ/I$ where k is an algebraically closed field, $Q = (Q_0, Q_1)$ a finite quiver and I an admissible ideal which is generated by paths in Q of length at least two. Furthermore, for every $v \in Q_0$ there are at most two incoming arrows and at most two outgoing arrows and the arrows of Q satisfy:
(S1) For every $a \in Q_1$ there is at most one $b \in Q_1$ such that $ba \notin I$ and at most one arrow c with $ac \notin I$.

String algebras are a large class of algebras which in particular contain the *gentle algebras*. The latter satisfy in addition:
(G1) for every arrow a there is at most one arrow b' such that $b'a \in I$ and at most one arrow c' such that $ac' \in I$.
(G2) The ideal I is generated by paths of length two.

If we do not require A to be finite dimensional but still assume the above conditions, including (G1) and (G2), A is called *locally gentle*.

All these algebras are amenable to geometric descriptions by tiled surfaces. By this we mean a subdivision of an oriented surface with marked points on boundary components and in the interior (called punctures) such that the tiles under this are of the following shapes. The vertices of these shapes are boundary vertices or punctures.



The first two tiles are polygonal of size at least 3, with no boundary segment or a single boundary segment (shaded). We use this to define the quiver Q: The arcs in the tiling of the surface correspond to the vertices of Q and arrows correspond to angles in the tiles, as indicated in the figure: there is an arrow $i \to j$ in Q if and only the arcs for j follows the arc of i clockwise within a tile. The relations of I arise as follows: any path of length two inside a tile is an element of I. For example, the first tile comes with the relations $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$.

Geometric modules for gentle algebras (and their module categories) have been given in [1], [6], for locally gentle algebras in [7].

Since for string algebras, we may have longer relations, we need an additional geometric feature on tiled surfaces. We introduce *labels* on tilings: a label is a sequence of $t \geq 3$ arcs $(i_1, i_2, \ldots, i_t)$ which share a common vertex, such that there are arrows $\alpha_j : i_j \to i_{j+1}$ in Q. Then the path given by a label is a generator for I. This allows us to obtain the extra monomial relations needed for string algebras. If $A = kQ/I$ arises from a tiling with labels, we say that A is a labelled tiling algebra. Two examples of labelled tilings and the associated quivers with relations are in Figure 1. The labels are indicated in red.

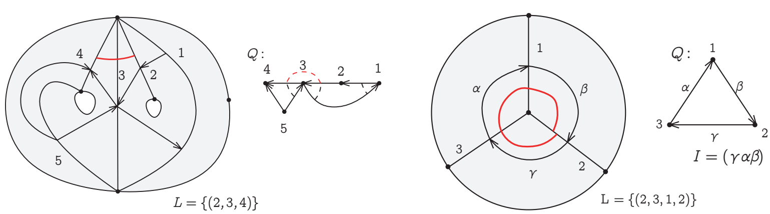


FIGURE 1. Labelled tiled surfaces, with associated quivers with relations

Our main result is the following:

Theorem ([2]). *The following are equivalent:*

(1) A is a string algebra.
(2) $A = B/R$ with B locally gentle and R is an ideal generated by paths of length ≥ 2 such that every oriented cycle in A has a relation.
(3) A is a labelled tiling algebra.

A key ingredient is the fact that we can view a string algebra as the quotient of a locally gentle algebra, see point (2) above. The surface corresponding to a string algebra $A = kQ/I$ arises from an associated locally gentle algebra B and the surface associated to it. This surface is not unique as in order to find B, we can choose which relations to remove from the ideal I. We conjecture that if we choose B to be finite dimensional, the surface is unique, up to rotating certain collections of tiles.

Laminations for tiled surfaces, [3]. In the second part of the talk we introduce the notion of laminations on a surface tiled by polygon: every tile is a polygon of size ≥ 3 and with at most one boundary edge. The laminations in [4], see also [5], can be viewed as the case of a triangulated surface.

In [3], we aim to determine the size of a maximal lamination and to define a mutation operation on curves in a maximal lamination.

Laminations are collections of non-crossing curves such that whenever a curve successively crosses two edges of a tile, these two edges have to be adjacent. Let $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$ be the arcs in the tiling where $e_1, \dots, e_n$ are the arcs whose endpoints are both punctures and $e_{n+1}, \dots, e_m$ are arcs with endpoints a puncture and a vertex on the boundary, $m > n$. Any curve c crossing the tiling gives a coordinate vector with entries in $\mathbb{Z}^m$ as follows: We say that c crosses e_i positively, if the arc e_i together with the immediate predecessor and successor of arcs crossed by c form an S-shape. We say that c crosses e_i negatively, if they form a Z-shape. And c crosses e_i trivially if it crosses these three arcs in a fan (or does not meet e_i). We associate $+1$ to any positive crossing, -1 to any negative crossing. Then any lamination gives a coordinate vector in $\mathbb{Z}^m$. We say that c is the *ith elementary lamination* if its coordinate vector has entry $+1$ in position i and 0 else. We write c_i for this curve. A lamination is *maximal* if no curve distinct from all its components can be added without introducing crossings. The *size* of a lamination is the number of different connected components it contains (up to isotopy). The collection $\{c_1, \dots, c_n\}$ forms a maximal lamination of size n.

We aim to determine the size of maximal laminations which cross all edges $e_{n+1}, \dots, e_m$ trivially, i.e. laminations with support on $e_1, \dots, e_n$ only. Any curve in such a lamination, either spirals at both ends into (different) punctures or is a closed curve. For simplicity, we now restrict to coordinate vectors in $\mathbb{Z}^n$ (as the remaining entries are all 0). We show in [3] that if a coordinate vector of a lamination contains a 0 entry, it is not maximal.

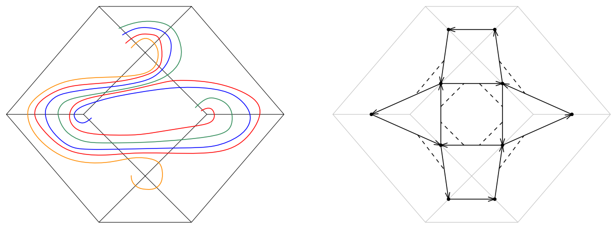


Figure 2. Lamination on tiled disk and the quiver of the tiling

Conjecture. *Let L be a maximal lamination on a tiled surface with polygonal tiles which is supported on n internal edges. Then L has at most size n. If the lamination contains no closed loops, the size is equal to n.*

Figure 2 shows a maximal lamination on a hexagon on the left. Denote the internal edges by e_1, e_2, e_3, e_4 clockwise around the inner quadrilateral, starting with its north west edge. The coordinate vector is $(6, -5, -5, 4)$ in that order.

Tilings by polygons as above give rise to locally gentle algebras, as in the first part of this abstract. The vertices of the quiver are the arcs of the tiling (the non-boundary edges). Every internal vertex has degree 4. In any internal tile of size r we have a clockwise r-cycle. Any r-tile on the boundary gives rise to a linearly oriented quiver on $r-1$ vertices. An example is on the right in Figure 2. Every clockwise cycle is full of relations, i.e. any path of length two in a clockwise cycle is 0. The connected components of any lamination correspond to a (non-crossing) collection of non-zero walks in the quiver which are infinite or closed. So they correspond to infinite dimensional string modules and to band modules. We expect that the walks of a laminations with no loops give rise to a rigid infinite dimensional string module.

We have a proof of the above conjecture for certain tilings, using a reduction technique on internal edges and showing that the size of a lamination strictly decreases if the reduction is not at a closed loop.

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Simple-mindedness: tilting, reduction, mutation

David Pauksztello
(joint work with N. Broomhead, R. Coelho Simões, D. Ploog, J. Woolf, A. Zvonareva)

1. Setting

Throughout, D will be a Hom-finite, Krull-Schmidt, $\mathbf{k}$-linear triangulated category with shift functor $\Sigma\colon \mathsf{D} \to \mathsf{D}$. For simplicity, we will assume that $\mathbf{k}$ is an algebraically closed field. When D has a Serre functor, it will be denoted $\mathbb{S}\colon \mathsf{D} \to \mathsf{D}$. If X is a subcategory or collection of objects of D then ${}^{\perp}\mathsf{X} = \{d \in D \mid \mathrm{Hom}_{\mathsf{D}}(d,x) = 0\, \forall x \in \mathsf{X}\}$ and $\mathsf{X}^{\perp} = \{d \in D \mid \mathrm{Hom}_{\mathsf{D}}(x,d) = 0\, \forall x \in \mathsf{X}\}$.

2. Tilting and (co-)t-structures

A *torsion pair* in D is a pair of full subcategories (X,Y), each closed under direct summands, such that

(1) $\mathrm{Hom}_{\mathsf{D}}(x,y) = 0$ for each $x \in \mathsf{X}$ and $y \in \mathsf{Y}$;
(2) $\mathsf{D} = \mathsf{X} * \mathsf{Y} = \{d \in \mathsf{D} \mid$ there exists a triangle $x \to d \to y \to \Sigma x$ with $x \in \mathsf{X}$ and $y \in \mathsf{Y}\}$.

A torsion pair (X,Y) is called a *t-structure* if $\Sigma\mathsf{X} \subset \mathsf{X}$ and $\Sigma^{-1}\mathsf{Y} \subset \mathsf{Y}$, and is called a *co-t-structure* if $\Sigma^{-1}\mathsf{X} \subset \mathsf{X}$ and $\Sigma\mathsf{Y} \subset \mathsf{Y}$. The subcategory X is called the *aisle* of the torsion pair and the subcategory Y is called the *co-aisle*. If (X,Y) then its *heart*, $\mathsf{H} = \mathsf{X} \cap \Sigma\mathsf{Y}$ is an abelian subcategory of D. A t-structure (X,Y) is called *bounded* if

$$\mathsf{D} = \bigcup_{i \geq j} \Sigma^i \mathsf{H} * \Sigma^{i-1}\mathsf{H} * \cdots * \Sigma^j \mathsf{H}.$$

Since $\mathsf{H} \subset \mathsf{X}$ and $\mathsf{H} \subset \Sigma\mathsf{Y}$ we have $\mathrm{Hom}_{\mathsf{D}}(\Sigma^i h_1, h_2) = 0$ for each $h_1, h_2 \in \mathsf{H}$ and $i > 0$.

A *torsion pair* in an abelian category H is a pair of full subcategories $(\mathcal{T},\mathcal{F})$ such that $\mathrm{Hom}_{\mathsf{H}}(t,f) = 0$ for each $t \in \mathcal{T}$ and $f \in \mathcal{F}$, and $\mathsf{H} = \mathcal{T} * \mathcal{F} = \{h \in \mathsf{H} \mid$ there exists a short exact sequence $0 \to t \to h \to f \to 0$ with $t \in \mathcal{T}$ and $f \in \mathcal{F}\}$.

Theorem 2.1 ([6, Proposition 2.1])**.** *Let (X,Y) be a t-structure in D with heart H. Suppose $(\mathcal{T},\mathcal{F})$ is a torsion pair in H. Then $\big(\mathsf{X} * \Sigma^{-1}\mathcal{T}, \Sigma^{-1}(\mathcal{F} * \mathsf{Y})\big)$ is a t-structure in D with heart $\mathsf{K} = \mathcal{F} * \Sigma^{-1}\mathcal{T}$; see Figure 1.*

A subcategory X of D is *contravariantly finite* if each object $d \in \mathsf{D}$ admits a morphism $f\colon x_d \to d$ such that $\mathrm{Hom}_{\mathsf{D}}(x,f)\colon \mathrm{Hom}_{\mathsf{D}}(x,x_d) \twoheadrightarrow \mathrm{Hom}_{\mathsf{D}}(x,d)$ is surjective for each $x \in \mathsf{X}$. *Covariantly finite* subcategories are defined dually and a subcategory is *functorially finite* if it is contravariantly and covariantly finite.

Theorem 2.2 ([4, Corollary 2.8])**.** *Suppose D is a Hom-finite, Krull-Schmidt, saturated triangulated category. Let (X,Y) be a bounded t-structure in D with heart H. The following are equivalent:*

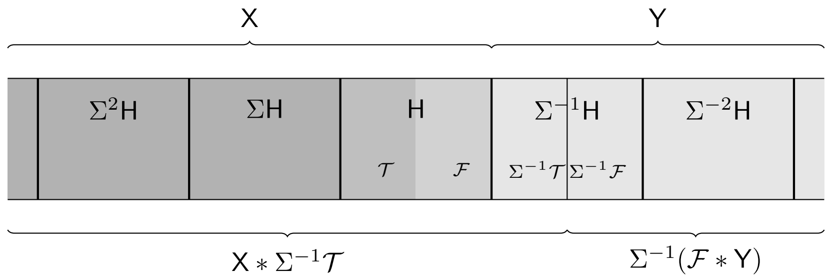

Figure 1. Schematic showing the t-structure (X,Y) and the Happel-Reiten-Smalø tilted t-structure $\big(\mathsf{X} * \Sigma^{-1}\mathcal{T}, \Sigma^{-1}(\mathcal{F} * \mathcal{Y})\big)$ at the torsion pair $(\mathcal{T},\mathcal{F})$ in the heart $\mathsf{H} = \mathsf{X} \cap \Sigma\mathsf{Y}$.

(1) H *is contravariantly finite (resp. covariantly finite) in* D*.*
(2) H *has enough injectives (resp. projectives).*
(3) (X,Y) *has a right (resp. left) adjacent co-t-structure, i.e. there is a co-t-structure* $(\mathsf{Y},\mathsf{Y}^{\perp})$ *(resp.* $({}^{\perp}\mathsf{X},\mathsf{X})$*).*

Prototypical examples of saturated triangulated categories are $\mathsf{D}^b(\mathsf{mod}\, A)$ and $\mathsf{D}^b(\mathsf{coh}\, X)$, where A is a finite-dimensional algebra of finite global dimension and X is a smooth projective variety. A more technical version of Theorem 2.2 is true without the restriction that D is saturated in [4, Theorem 2.4].

Corollary 2.3. Suppose D is a Hom-finite, Krull-Schmidt, saturated triangulated category. If H is functorially finite in D and $(\mathcal{T},\mathcal{F})$ is a torsion pair in which $\mathcal{T}$ and $\mathcal{F}$ are functorially finite, then the HRS-tilted heart $\mathsf{K} = \mathcal{F} * \Sigma^{-1}\mathcal{T}$ is also functorially finite in D.

Proof. If (X,Y) is a t-structure, then $({}^{\perp}\mathsf{X},\mathsf{X})$ is a co-t-structure if and only if X is functorially finite in D. If H is functorially finite in D then so is the torsion class $\mathcal{T}$. In particular, by [13, Lemma 5.3], $\mathsf{X} * \Sigma^{-1}\mathcal{T}$ is functorially finite in D. Hence, $\big({}^{\perp}(\mathsf{X} * \Sigma^{-1}\mathcal{T}), \mathsf{X} * \Sigma^{-1}\mathcal{T}\big)$ is a co-t-structure in D. One argues similarly with the torsionfree class. □

3. Simple-minded objects

Definition. A collection of objects S of D is an *orthogonal collection*, if for each $s_1, s_2 \in \mathsf{S}$ Schur's lemma holds, i.e.

$$\mathrm{Hom}_{\mathsf{D}}(s,t) = \begin{cases} \mathbf{k} & \text{if } s_1 \simeq s_2, \\ 0 & \text{otherwise.} \end{cases}$$

An orthogonal collection S is a *simple-minded collection* (*SMC*) [12] if

(1) it is an ∞-*orthogonal collection*, i.e. $\mathrm{Hom}_{\mathsf{D}}(\Sigma^i s_1, s_2) = 0$ for each $i > 0$ and $s_1, s_2 \in \mathsf{S}$, and,

(2) $\mathsf{D} = \bigcup_{i \geq j} \Sigma^i \langle \mathsf{S} \rangle * \Sigma^{i-1} \langle \mathsf{S} \rangle * \cdots * \Sigma^j \langle \mathsf{S} \rangle$, i.e. $\langle \mathsf{S} \rangle$ is the heart of a bounded t-structure in D.

For $w \geq 1$, S is a *w-simple-minded system* (*w-SMS*) [2, 11] if

(1) it is a *w-orthogonal collection*, i.e. $\mathrm{Hom}_{\mathsf{D}}(\Sigma^i s_1, s_2) = 0$ for each $1 \leq i \leq w-1$ and $s_1, s_2 \in \mathsf{S}$, and,
(2) $\mathsf{D} = \Sigma^{w-1} \langle \mathsf{S} \rangle * \cdots * \Sigma \langle \mathsf{S} \rangle * \langle \mathsf{S} \rangle$

Theorem 3.1 ([5, Theorem 3.3]). *Let S be an orthogonal collection of D and suppose $\mathsf{T} \subseteq \mathsf{S}$. Then $\langle \mathsf{T} \rangle$ is functorially finite in $\langle \mathsf{S} \rangle$.*

Remark 3.2. *If S is a w-SMS in D, then, as a consequence of condition* (2) *in the definition, $\langle \mathsf{S} \rangle$ is functorially finite in D, see* [3, Corollary 2.9]. *By Theorem 3.1 it follows that if $\mathsf{T} \subseteq \mathsf{S}$ then $\langle \mathsf{T} \rangle$ is also functorially finite in D. In particular, functorial finiteness of the extension closure of a w-orthogonal collection is a necessary condition for that collection to occur as a subcollection of a w-SMS.*

If S is an orthogonal collection and $\mathsf{T} \subseteq \mathsf{S}$ then $\big(\langle \mathsf{T} \rangle, \mathsf{T}^{\perp} \cap \langle \mathsf{S} \rangle\big)$ and $\big({}^{\perp}\mathsf{T} \cap \langle \mathsf{S} \rangle, \langle \mathsf{T} \rangle\big)$ are "torsion pairs" in $\langle \mathsf{S} \rangle$ with functorially finite "torsion class" and functorially finite "torsionfree class", respectively. In the case that S is an SMC, then the two "torsion pairs" above are genuine torsion pairs in the abelian sense.

4. Reduction and mutation

Let T be an orthogonal collection and U be a collection of objects of D. Provided that $\langle \mathsf{T} \rangle$ is functorially finite in D, we can define two mutation operations on U with respect to T. The *right mutation* of U at T is obtained by taking for each object $u \in \mathsf{U}$ a minimal right $\langle \mathsf{T} \rangle$-approximation $u_t \to \Sigma u$ and extending it a distinguished triangle,

$$t_u \to \Sigma u \to \mathsf{R}_{\mathsf{T}}(u) \to \Sigma t_u,$$

and setting $\mathsf{R}_{\mathsf{T}}(\mathsf{U}) = \{\mathsf{R}_{\mathsf{T}}(u) \mid u \in \mathsf{U}\}$. *Left mutation* is defined analogously, see [3] for precise details.

In analogy with [8] for cluster-tilting/silting mutation, in [3] a pair of collections of objects (U, V) is called a T-*mutation pair* if $\mathsf{U} = \mathsf{L}_{\mathsf{T}}(\mathsf{V})$ and $\mathsf{V} = \mathsf{R}_{\mathsf{T}}(\mathsf{U})$.

When T is a subcollection of a w-SMS, the extension closure $\langle \mathsf{T} \rangle$ is automatically functorially finite by Theorem 3.1 and as such mutation is always defined. However, if T is a subcollection of an SMC this is not automatic. This motivates the following definition, which permits us to discuss mutation of SMCs.

Definition. An SMC S in D is called *strong* if $\langle \mathsf{S} \rangle$ is functorially finite in D.

Theorem 4.1 ([3, Theorems 4.1 & 5.1]). *Suppose T is an orthogonal collection such that*

(1) $\langle \mathsf{T} \rangle$ is functorially finite in D; and,
(2) $\mathbb{S}\Sigma\mathsf{T} = \mathsf{T}$ or $\mathrm{Hom}_{\mathsf{D}}(\Sigma t_1, t_2) = 0$ for each $t_1, t_2 \in \mathsf{T}$.

Let Z be a subcategory of D such that (Z, Z) is an T-mutation pair satisfying,

(**Z1**) Z *is closed under extensions and direct summands;*
(**Z2**) *the cones in* D *of maps in* Z *lie in* $\langle\mathsf{T}\rangle * \mathsf{Z}$*; and*
(**Z3**) *the cocones in* D *of maps in* Z *lie in* $\mathsf{Z} * \langle\mathsf{T}\rangle$.

Then there is a functor $\langle 1\rangle \colon \mathsf{Z} \to \mathsf{Z}$ *and for each morphism* $f \colon x \to y$ *in* Z *there is a diagram* $x \xrightarrow{f} y \longrightarrow z_f \longrightarrow x\langle 1\rangle$ *giving rise to a class of triangles* Δ *which makes* D *into a triangulated category.*

The key point is that the shift functor $\langle 1\rangle \colon \mathsf{Z} \to \mathsf{Z}$ is defined via the right mutation formula with respect to T. In particular, if $\mathsf{T} = \{0\}$ then $\langle 1\rangle = \Sigma$.

This result allows one to obtain a reduction result for w-SMSs and SMCs analogous to the reduction results for w-cluster-tilting subcategories and silting subcategories obtained in [1, 7, 8]. We state the result fo w-SMSs and SMCs together. The result for w-SMSs is due to [3, Theorem 6.6] and the result for SMCs is due to [9, Theorem 3.1]. An alternative proof in the SMC case in the same spirit as the SMS case is given in [4, Theorem A.2] of the appendix to that article.

Theorem 4.2. *Let* T *be a* w*-orthogonal (resp.* ∞*-orthogonal) collection and*

$$\mathsf{Z} = \begin{cases} \{d \in \mathsf{D} \mid \mathrm{Hom}_{\mathsf{D}}(\Sigma^i t, d) = 0\, \forall t \in \mathsf{T} \text{ and } 0 \leq i \leq w\} & \text{if } \mathsf{T} \text{ is } w\text{-orthogonal;} \\ \mathsf{Z} = {}^{\perp}(\Sigma^{\leq 0}\mathsf{T}) \cap (\Sigma^{\geq 0}\mathsf{T})^{\perp}) & \text{if } \mathsf{T} \text{ is } \infty\text{-orthogonal.} \end{cases}$$

Then, (Z, Z) *is a* T*-mutation pair satisfying the hypotheses of Theorem 4.1. Moreover, there is bijection,*

$$\{w\text{-SMSs (resp. SMCs) in } \mathsf{D} \text{ containing } \mathsf{T}\} \xleftrightarrow{1-1} \{w\text{-SMSs (resp. SMCs) in } \mathsf{Z}\}.$$

The key observation in this theorem is that a right mutation on the left-hand side of the bijection corresponds to a shift on the right-hand side of the bijection. Therefore, the question of whether the mutation of a w-SMS or an SMC is again a w-SMS or an SMC boils down to asking whether the shift of a w-SMS or an SMC is again a w-SMS or an SMC, which is tautologous. The following theorem recovers [10, Theorem 6.3] in the case of w-SMSs and generalises the SMC mutation theory for derived categories of finite-dimensional algebras of [12].

Theorem 4.3. *Let* T *be a* w*-orthogonal (resp.* ∞*-orthogonal) collection such that* $\langle\mathsf{T}\rangle$ *is functorially finite in* D*. Suppose* (U, V) *is a* T*-mutation pair. Then* $\mathsf{U} \cup \mathsf{T}$ *is a* w*-SMS (resp. strong SMC) if and only if* $\mathsf{V} \cup \mathsf{T}$ *is a* w*-SMS (resp. strong SMC).*

Remark 4.4. *Corollary 2.3 says that tilting a functorially finite aisle (resp. coaisle) at a functorially finite torsion (resp. torsionfree) class produces another functorially finite aisle (resp. coaisle). That is, the property of having an adjacent co-t-structure is preserved by tilting at functorially finite torsion pairs providing a conceptual homological explanation behind the Koenig-Yang correspondences and their compatibility with mutation.*

Tilting at torsion pairs whose torsion (resp. torsionfree) class is generated by a subset of simple objects is simple-minded mutation. In particular, simple tilts of length hearts with enough projectives and enough injectives produce length hearts with enough projectives and enough injectives. This means that "algebraic" hearts are well behaved within the space of Bridgeland stability conditions.

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Mutation of closed sets in the Ziegler spectrum - Part I

Lidia Angeleri Hügel

(joint work with Rosanna Laking, Francesco Sentieri)

For a finite dimensional algebra A, the lattice $\operatorname{tors} A$ of torsion pairs in the category $\operatorname{mod} A$ of finite dimensional modules is controlled by mutation. Minimal inclusions of functorially finite torsion classes are encoded by mutation of associated compact 2-term silting complexes, as shown in [1]. The non-functorially finite case is captured by large 2-term cosilting complexes and their mutations inside the derived category $\mathcal{D}(\operatorname{Mod} A)$.

Definition [9]**.** Let $\mathrm{K}^{\mathrm{b}}(\mathrm{Inj} A)$ denote the category of bounded complexes of injective A-modules. A complex $\sigma \in \mathrm{K}^{\mathrm{b}}(\mathrm{Inj} A)$ which is concentrated in degrees 0 and 1 is said to be a *2-term cosilting complex* if it satisfies

(i) $\mathrm{Hom}_{\mathcal{D}(\operatorname{Mod} A)}(\sigma^I, \sigma[1]) = 0$ for all sets I, and

(ii) $\mathrm{K}^{\mathrm{b}}(\mathrm{Inj} A)$ is generated as a thick subcategory by the class $\operatorname{Prod} \sigma$ of all arbitrary products of copies of σ and their direct summands.

Two cosilting complexes σ and σ' are said to be *equivalent* if $\operatorname{Prod} \sigma = \operatorname{Prod} \sigma'$.

There is a bijection between equivalence classes of 2-term cosilting complexes and torsion pairs in $\operatorname{mod} A$: given a cosilting complex σ, we take its zero-th cohomology $C = H^0(\sigma)$, consider the torsion pair $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}, \operatorname{Cogen} C)$ in $\operatorname{Mod} A$ cogenerated by C, and assign to σ its restriction $(\mathbf{t}, \mathbf{f}) = (\mathcal{T} \cap \operatorname{mod} A, \mathcal{F} \cap \operatorname{mod} A)$ to $\operatorname{mod} A$. It is proved in [4] that under this bijection irreducible cosilting mutation corresponds to minimal inclusion of torsion classes and thus determines the Hasse quiver of $\operatorname{tors} A$.

Here irreducible mutation is defined as in [2, 1] by picking an indecomposable summand of σ, approximating it by "the rest" and replacing it by the cone (or co-cone) of this approximation. When working with large cosilting complexes, however, this approximation by "the rest" requires some care: our objects in general don't have indecomposable decompositions and we are forced to work with infinite direct products rather than finite direct sums.

The aim of this talk is to show that, despite these difficulties, large mutation amounts to an operation on sets of indecomposable modules and shares important features with classical silting mutation.

To this end, we consider the Ziegler spectrum $\operatorname{Zg}(A)$ of A, a topological space whose points are given by the isomorphism classes of indecomposable *pure-injective* modules. Recall that the pure-injective modules over a finite-dimensional algebra A are precisely the modules which are direct summands of direct products of finite-dimensional modules. By [5, 6] we know that the module C defined above is pure-injective, and it follows from [7] that $\mathcal{Z} = \operatorname{Prod} C \cap \operatorname{Zg}(A)$ is a closed subset of $\operatorname{Zg}(A)$. It turns out that the cosilting complex σ is determined by $\mathcal{Z}$ together with the set $\mathcal{I}$ of representatives of the indecomposable injective A-modules in $\operatorname{Ker}(\operatorname{Hom}_A(C, -))$.

Definition. A pair $(\mathcal{Z}, \mathcal{I})$ is a *rigid pair* if it satisfies

(i) $\mathcal{Z}$ is a subset of $\operatorname{Zg}(A)$, and $\mathcal{I}$ is a set of indecomposable injective modules;
(ii) $\mathcal{Z}$ is *rigid*, i.e. if M, N are in $\mathcal{Z}$ and μ_M, μ_N are their minimal injective copresentations, then $\operatorname{Hom}_{\mathcal{D}(\operatorname{Mod} A)}(\mu_M, \mu_N[1]) = 0$;
(iii) $\operatorname{Hom}_{\mathcal{D}(\operatorname{Mod} A)}(\mathcal{Z}, \mathcal{I}) = 0$.

The rigid pairs which are maximal among all rigid pairs are called *cosilting pairs*.

Proposition. The assignment $\sigma \mapsto (\mathcal{Z}, \mathcal{I})$ defines a bijection between equivalence classes of 2-term cosilting complexes and cosilting pairs.

Via this bijection we can define a notion of *mutation of cosilting pairs*. It amounts to exchanging elements of $\mathcal{Z} \cup \mathcal{I}$ with suitable indecomposable pure-injective or injective modules. However, unlike classical mutation, in general not every point in a cosilting set is *mutable*, i.e. can be replaced. But an important property is preserved.

Theorem. [3] Let $(\mathcal{Z}, \mathcal{I})$ be a cosilting pair.

(1) If X is a mutable point of $(\mathcal{Z}, \mathcal{I})$, then there is exactly one way to replace X to obtain a new cosilting pair $(\mathcal{Z}', \mathcal{I}')$.

(2) **(IC)** If X is not a mutable point of $(\mathcal{Z}, \mathcal{I})$, then X lies in $\mathcal{Z}$ and $(\mathcal{Z}, \mathcal{I})$ is the only cosilting pair which extends $(\mathcal{Z} \setminus \{X\}, \mathcal{I})$.

Our proof of statement (2) requires to assume the validity of the Isolation Condition **(IC)**. We refer to Part II and to [8, §5.3.2] for details on this condition.

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Mutation of closed sets in the Ziegler spectrum - Part II

Rosanna Laking

(joint work with Lidia Angeleri Hügel, Francesco Sentieri)

Let A be a finite-dimensional algebra over a field K. This talk follows on from Part I in which Lidia Angeleri Hügel explained how cosilting pairs $(\mathcal{Z}, \mathcal{I})$ in $\operatorname{Mod} A$ can be used to parametrise 2-term cosilting complexes in $\mathcal{D}(\operatorname{Mod} A)$ up to equivalence. In Part I it was explained that the irreducible mutations of 2-term cosilting complexes can be seen on the level of the corresponding cosilting pair $(\mathcal{Z}, \mathcal{I})$ as replacing a *mutable* element of $\mathcal{Z} \cup \mathcal{I}$ by a (uniquely determined) indecomposable module to obtain a new cosilting pair $(\mathcal{Z}', \mathcal{I}')$.

Let $(\mathcal{Z}, \mathcal{I})$ be a cosilting pair and let σ be the corresponding 2-term cosilting complex. The elements of $X \in \mathcal{Z} \cup \mathcal{I}$ correspond bijectively to the indecomposable complexes α_X obtained as direct summands of arbitrary products of σ (i.e. contained in $\operatorname{Prod}(\sigma)$). We say that a module $X \in \mathcal{Z} \cup \mathcal{I}$ is **mutable** if there is a mutation of σ at the corresponding complex $\alpha_X \in \operatorname{Prod}(\sigma)$. In the first part of this talk we address the following question: when is $X \in \mathcal{Z} \cup \mathcal{I}$ mutable?

The Ziegler spectrum and mutablity. In Part I it was explained that the sets $(\mathcal{Z}, \mathcal{I})$ consist of indecomposable pure-injective modules, that is, indecomposable modules that arise as direct summands of products of finite-dimensional modules. The isomorphism classes of these modules form the points of a topological space called the Ziegler spectrum; we will often identify an isomorphism class with a representative of the class. It turns out that, in some important cases, the question of whether a module in $\mathcal{Z}\cup\mathcal{I}$ is mutable can be answered in terms of this topology.

Definition ([10]): Let $\mathrm{Zg}(A)$ denote the set of equivalence classes of indecomposable pure-injective modules. For each $f\colon M \to N$ in mod A, define the subset $(f) := \{L \in \mathrm{Zg}(A) \mid \mathrm{Hom}_A(f, L) \text{ is not surjective }\} \subseteq \mathrm{Zg}(A)$. Then the sets $\{(f) \mid f \in \text{mod } A\}$ form a basis of open sets of the **Ziegler topology** on $\mathrm{Zg}(A)$. The set $\mathrm{Zg}(A)$ with this topology is known as the **Ziegler spectrum**.

The setting where a topological interpretation of mutability is available is when the isolation condition (IC) holds for A. This condition can be found, for example, in [7]; due to its technical nature, we will not define it here. We note, however, that it is known to hold for many important classes of algebras, such as tame hereditary algebras, many algebras of domestic representation type (e.g. domestic gentle, cycle-finite, strongly simply-connected and multicoil algebras) or any finite dimensional algebra over countable fields. In fact, no examples are known where this condition fails. The result where (IC) is assumed is labelled accordingly.

Theorem ([2]): Let $(\mathcal{Z},\mathcal{I})$ be a cosilting pair in Mod A and let $M \in \mathcal{Z}$. Then the following statements hold.

1. Every module $I \in \mathcal{I}$ is mutable.
2. The set $\mathcal{Z}$ is a closed set of $\mathrm{Zg}(A)$.
3. **(IC)** The module M is mutable if and only if M is isolated in $\mathcal{Z}$ with the subspace topology.

The (isoclasses of) indecomposable finite-dimensional A-modules are isolated in $\mathrm{Zg}(A)$ and so it follows from the theorem that finite-dimensional modules in $\mathcal{Z}$ are always mutable. A cosilting pair $(\mathcal{Z},\mathcal{I})$ consists of finite-dimensional modules if and only if $(\oplus_{M\in\mathcal{Z}}M, \oplus_{I\in\mathcal{I}}I)$ is a support τ^{-1}-tilting pair. Thus our result recovers the well-known result that mutation is possible at any summand of a support τ^{-1}-tilting pair [1].

In our joint work [2] we are also able to identify the mutable elements of $\mathcal{Z}$ without using (IC). Indeed, it was explained in Part I that there is a torsion pair $(\mathcal{T},\mathcal{F})$ in Mod A associated to $(\mathcal{Z},\mathcal{I})$. We show that a module $M \in \mathcal{Z}$ is mutable if and only if there exists a left almost split morphism $M \to N$ in $\mathcal{F}$ such that the kernel and cokernel are finitely presented. Indeed, it is also straightforward to see from this perspective that finite-dimensional modules in $\mathcal{Z}$ are always mutable.

Example: cluster-tilted algebras of type $\widetilde{A}$. We end the talk with an example illustrating mutation of cosilting pairs for a family of finite-dimensional algebras called cluster-tilted algebras of type $\widetilde{A}$ [4]. These algebras have been shown to coincide with the surface algebras $A(\Gamma)$ corresponding to a triangulation Γ of an annulus S with marked points M in the boundary [3]. Recall that the arcs in Γ yield the vertices of a quiver with relations for $A(\Gamma)$; later we will use the notation I_α for the injective indexed by the vertex $\alpha \in \Gamma$.

The Ziegler spectrum of $A(\Gamma)$ is known [8]: it consists of string modules $M(\alpha)$ (parametrised by finite or asymptotic curves α in the surface) and band modules

$B(\lambda, n)$ (parametrised by (λ, n) in the set $K^* \times \mathbb{N} \cup \{\infty, -\infty\}$) plus one additional band module called the generic module G.

The cosilting pairs have also been classified [6]: the finite-dimensional cosilting pairs are parametrised by triangulations Δ of (S, M) and the infinite-dimensional cosilting pairs are parametrised by pairs (Ω, P) where Ω is an asymptotic triangulation of (S, M) (in the sense of Baur and Dupont [5]) and $P \subseteq K^*$. Indeed they are given as follows:

- Given a triangulation Δ as above, we have a cosilting pair $(\mathcal{Z}, \mathcal{I})$ in $\operatorname{Mod} A(\Gamma)$ with $\mathcal{Z} = \{M(\alpha) \mid \alpha \in \Delta \setminus \Gamma\}$ and $\mathcal{I} = \{I_\alpha \mid \alpha \in \Delta \cap \Gamma\}$.
- Given a pair (Ω, P) as above, we have a cosilting pair $(\mathcal{Z}, \mathcal{I})$ in $\operatorname{Mod} A$ with $\mathcal{Z} = \{M(\alpha) \mid \alpha \in \Delta \setminus \Gamma\} \cup \{B(\lambda, \infty) \mid \lambda \in P\} \cup \{B(\lambda, -\infty) \mid \lambda \notin P\} \cup \{G\}$ and $\mathcal{I} = \{I_\alpha \mid \alpha \in \Delta \cap \Gamma\}$.

The topology on $\operatorname{Zg}(A(\Gamma))$ has been described explicitly and the Krull-Gabriel dimension of $A(\Gamma)$ is shown to be equal to two [9]; this implies that (IC) holds for $A(\Gamma)$. We can therefore apply the results of Parts I and II of this talk and conclude that every module in such an $\mathcal{Z}$ is mutable except the generic module G. Moreover, the mutation of string modules is given by a flip operation (described for asymptotic arcs in [5]) and the mutation of band modules is given by exchanging the module $B(\lambda, \infty)$ for the module $B(\lambda, -\infty)$.

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BPS Lie algebra of 2-Calabi–Yau categories and positivity of cuspidal polynomials of quivers

Lucien Hennecart

(joint work with Ben Davison and Sebastian Schlegel Mejia)

2-Calabi–Yau categories are ubiquitous in representation theory and algebraic geometry. They arise as the categories of

(1) Representations of the (deformed or not, additive or multiplicative) preprojective algebra Π_Q of a quiver Q, or more generally of 2-Calabi–Yau algebras,
(2) Representations of the (twisted or not) fundamental group algebra of a compact Riemann surface S,
(3) Semistable sheaves on (non-necessarily compact) symplectic surfaces.

This is a report on the preprints [4] and [5].

Setup. We let $\mathcal{A}$ be one of the categories defined above. We are here especially interested in the category $\mathcal{A} = \mathrm{Rep}(\Pi_Q)$ of finite dimensional representations of the preprojective algebra of a quiver Q. We refer to [4] for the general case. We let $(M,N)_{\mathcal{A}} \coloneqq \sum_{j\in\mathbf{Z}}(-1)^j\mathrm{ext}^j(M,N)$ be the Euler form of $\mathcal{A}$.

Throughout, Q denotes a finite quiver, i.e. a pair of a set of vertices Q_0 and a set of arrows Q_1, both finite, along with two maps $s,t\colon Q_1\to Q_0$ assigning to an arrow its *source* and *target*. We form the *doubled quiver* $\overline{Q}=(Q_0,\overline{Q_1})$ by adding an arrow α^* to each arrow $\alpha\in Q_1$, with α^* given in the opposite orientation of α. The preprojective algebra is the quotient

$$\Pi_Q \coloneqq \mathbf{C}\overline{Q}/\langle \sum_{\alpha\in Q_1}[\alpha,\alpha^*]\rangle.$$

Generalised Kac–Moody Lie algebra for a monoid with bilinear form. For a pair $\overline{M}=(M,(-,-))$ of a monoid with a bilinear form $(-,-)\colon M\times M\to\mathbf{Z}$, we define

$$\Sigma_{\overline{M}} \coloneqq \Big\{m\in R^+_{\overline{M}} \mid \text{ for any nontrivial decomposition } m=\sum_{j=1}^r m_j, m_j\in M, \text{ one has } 2-(m,m) > \sum_{j=1}^r(2-(m_j,m_j))\Big\}$$

the set of primitive positive roots

and

$$\Phi^+_{\overline{M}} \coloneqq \Sigma_{\overline{M}}\cup\{lm\colon l\geq 2, m\in\Sigma_{\overline{M}} \text{ with } (m,m)=0\}$$

the set of simple positive roots.

The Cartan matrix is $A_{\overline{M}} \coloneqq ((m,n))_{m,n\in\Phi^+_{\overline{M}}}$. We assume that positive diagonal coefficients are equal to 2 and off-diagonal coefficients are nonpositive. For a

$\Phi^+_{\overline{M}}\times\mathbf{Z}$-vector space V, we define the Lie algebra $\mathfrak{n}_{\overline{M},V}$ as the Lie algebra generated by V with the relations

$$\begin{aligned}[][v,w]&=0 &\quad& \text{if } (\deg(v),\deg(w))=0\\ \mathrm{ad}(v)^{1-(\deg(v),\deg(w))}(w)&=0 && \text{if } (\deg(v),\deg(v))=2\end{aligned}$$

for homogeneous $v,w\in V$, where $\deg\colon V\to\Phi^+_{\overline{M}}$.

The associative algebra generated by V with the same relations is canonically isomorphic to the enveloping algebra $\mathbf{U}(\mathfrak{n}_{\overline{M},V})$.

The BPS Lie algebra of 2 Calabi–Yau categories. We let $\mathfrak{M}_{\mathcal{A}}$ be the stack of objects of $\mathcal{A}$, $\mathcal{M}_{\mathcal{A}}$ be the moduli space of semisimple objects in $\mathcal{A}$ and $\mathtt{JH}\colon\mathfrak{M}_{\mathcal{A}}\to\mathcal{M}_{\mathcal{A}}$ be the Jordan–Hölder map, sending an object of $\mathcal{A}$ to its semisimplification with respect to some Jordan–Hölder filtration. We let $\mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$ be the (Abelian) category of perverse sheaves on $\mathcal{M}_{\mathcal{A}}$. Using the monoid structure $\oplus\colon\mathcal{M}_{\mathcal{A}}^{\times 2}\to\mathcal{M}_{\mathcal{A}}$ given by the direct sum, we make $\mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$ a tensor category by defining the tensor product $\mathscr{F}\boxdot\mathscr{G}=\oplus_*(\mathscr{F}\boxtimes\mathscr{G})$. We let $M_{\mathcal{A}}:=\pi_0(\mathcal{M}_{\mathcal{A}})$ be the monoid of connected components of $\mathcal{M}_{\mathcal{A}}$. We let $\mathcal{M}_{\mathcal{A},0}$ be the connected component of the zero object of $\mathcal{A}$. An algebra object in $\mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$ is a triple $(\mathscr{F}\in\mathrm{Perv}(\mathcal{M}_{\mathcal{A}}),m\colon\mathscr{F}\boxdot\mathscr{F}\to\mathscr{F},\eta\colon\underline{\mathbf{Q}}_{\mathcal{M}_{\mathcal{A},0}}\to\mathscr{F})$ satisfying the usual axioms. Algebra objects in the category of bounded below constructible complex $\mathcal{D}^+_{\mathrm{c}}(\mathcal{M}_{\mathcal{A}})$ are defined in the same way.

Theorem 1 (Davison–H–Schlegel Mejia, 2022, [4, 5])**.**

(1) There is a cohomological Hall algebra product on the complex of constructible sheaves $\mathscr{A}_{\mathcal{A}}:=\mathtt{JH}_*\mathbb{D}\underline{\mathbf{Q}}^{\mathrm{vir}}_{\mathfrak{M}_{\mathcal{A}}}$, making it an algebra object in $\mathcal{D}^+_{\mathrm{c}}(\mathcal{M}_{\mathcal{A}})$,

(2) The constructible complex $\mathscr{A}_{\mathcal{A}}$ is semisimple and concentrated in nonnegative perverse degrees,

(3) The degree 0 perverse cohomology ${}^{\mathfrak{p}}\mathcal{H}^0(\mathscr{A}_{\mathcal{A}})$ has an induced algebra structure in $\mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$.

The relative BPS algebra of $\mathcal{A}$ is defined as $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}:={}^{\mathfrak{p}}\mathcal{H}^0(\mathscr{A}_{\mathcal{A}})$. The *absolute* BPS algebra is obtained by taking the derived global sections: $\mathrm{BPS}_{\mathcal{A},\mathrm{Alg}}:=\mathrm{H}^*(\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}})$.

For $\mathcal{A}=\mathrm{Rep}(\Pi_Q)$, these results were proven in [2]. The proof in the generality exposed here relies on the neighbourhood theorem for 2-Calabi–Yau categories in [3].

Theorem 2 (Davison–H–Schlegel Mejia, 2023, [4, 5])**.** *The BPS algebra* $\mathrm{BPS}_{\mathcal{A},\mathrm{Alg}}$ *is isomorphic to the enveloping algebra of the generalised Kac–Moody Lie algebra associated to the pair $(M_{\mathcal{A}},(-,-)_{\mathcal{A}})$ generated by*

$$\mathrm{IC}(\mathcal{M}_{\Phi^+_{\mathcal{A}}}):=\bigoplus_{a\in\Sigma_{\mathcal{A}}}\mathrm{IC}(\mathcal{M}_{\mathcal{A},a})\oplus\bigoplus_{\substack{a\in\Sigma_{\mathcal{A}},(a,a)_{\mathcal{A}}=0\\ l\geq 2}}\mathrm{IC}(\mathcal{M}_{\mathcal{A},a}),$$

the intersection cohomology of some connected components of the moduli space of semisimple objects in $\mathcal{A}$ (note the specificity for isotropic roots).

Idea of the proof. This theorem is proven for the relative BPS algebra $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}$. First, using the neighbourhood theorem of [3], we show that it suffices to prove this theorem for $\mathcal{A} = \mathrm{Rep}(\Pi_Q)$ for all quivers Q. By the neighbourhood theorem again, we prove the result for preprojective algebras by induction on the set of pairs $(Q, \mathbf{d})$ of a quiver Q and a dimension vector $\mathbf{d} \in \mathbf{N}^{Q_0}$ supported on the whole of Q. We take advantage of the fact that $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}$ is a semisimple perverse sheaf on $\mathcal{M}_{\mathcal{A}}$. We then rely on one of the main theorems of [6] which gives an explicit and combinatorial description of the top CoHA of the strictly seminilpotent stack. □

At this point, one may define the relative BPS Lie algebra of $\mathcal{A}$ as the sub-Lie algebra of $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}$ generated by $\mathcal{IC}(\mathcal{M}_{\Phi^+_{\mathcal{A}}})$. The absolute BPS Lie algebra is $\mathrm{BPS}_{\mathcal{A},\mathrm{Lie}} \coloneqq \mathrm{H}^*(\mathcal{BPS}_{\mathcal{A},\mathrm{Lie}})$.

When $\mathcal{A}$ is the category of representations of a 2-Calabi–Yau algebra A, there is an other approach for defining the BPS Lie algebra using the critical cohomological Hall algebra associated to the 3-Calabi–Yau completion of A. In [4], we prove that both definitions lead to canonically isomorphic Lie algebras.

Corollary 3. *The BPS Lie algebra is isomorphic to the generalised Kac–Moody Lie algebra associated to the pair $(\pi_0(\mathcal{M}_{\mathcal{A}}), (-,-)_{\mathcal{A}})$ generated by $\mathrm{IC}(\mathcal{M}_{\Phi^+_{\mathcal{A}}})$.*

Theorem 4 (Davison, [2])**.** *For $\mathcal{A} = \mathrm{Rep}(\Pi_Q)$, the character of the BPS Lie algebra is given by*

$$\mathrm{ch}(\mathrm{BPS}_{\Pi_Q,\mathrm{Lie}}) = \sum_{\mathbf{d}\in\mathbf{N}^{Q_0}} A_{Q,\mathbf{d}}(q^{-2})z^{\mathbf{d}}.$$

Constructible Hall algebra and cuspidal polynomials. We let $\mathrm{Rep}(Q, \mathbf{F}_q)$ be category of representations of Q over the finite field with q elements $\mathbf{F}_q$. The *constructible* Hall algebra of Q is the space

$$H_{Q,\mathbf{F}_q} \coloneqq \mathrm{Fun_c}(\mathrm{Rep}(Q, \mathbf{F}_q)/\sim, \mathbf{C})$$

of finitely supported functions on the set of isomorphism classes of representations of Q over $\mathbf{F}_q$. The algebra structure comes from the extension structure of the category $\mathrm{Rep}(Q, \mathbf{F}_q)$ and is given by some convolution product:

$$(f \star g)([R]) \coloneqq \sum_{N\subset R} q^{\frac{1}{2}\langle [R/N],[N]\rangle_Q} f([R/N])g([N]),$$

Dually, a twisted coproduct Δ can be defined:

$$\Delta(f)([M],[N]) = \frac{q^{-\frac{1}{2}\langle M,N\rangle_Q}}{|\mathrm{Ext}^1_Q(M,N)|} \sum_{\xi\in\mathrm{Ext}^1(M,N)} f([X_\xi])$$

where X_ξ is the middle term of the short exact sequence determined by ξ.

The character of $H_{Q,\mathbf{F}_q}$ is given by the formulas with plethystic exponentials

$$\begin{aligned}\mathrm{ch}(H_{Q,\mathbf{F}_q}) \coloneqq \sum_{\mathbf{d}\in\mathbf{N}^{Q_0}} M_{Q,\mathbf{d}}(q)z^{\mathbf{d}} &= \mathrm{Exp}_z\left(\sum_{\mathbf{d}\in\mathbf{N}^{Q_0}} I_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\right)\\ &= \mathrm{Exp}_{z,q}\left(\sum_{\mathbf{d}\in\mathbf{N}^{Q_0}} A_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\right)\end{aligned}$$

where the polynomials $M_{Q,\mathbf{d}}(q)$ (resp. $I_{Q,\mathbf{d}}(q)$, resp. $A_{Q,\mathbf{d}}(q)$) count all (resp. indecomposable, resp. absolutely indecomposable) $\mathbf{d}$-dimensional representations of Q over $\mathbf{F}_q$.

The space of *cuspidal functions* is the space of primitive elements for the coproduct Δ: $H^{\mathrm{cusp}}_{Q,\mathbf{F}_q} = \bigoplus_{\mathbf{d}\in\mathbf{N}^{Q_0}} H^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$, $H^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}] \coloneqq \{f \in H_{Q,\mathbf{F}_q}[\mathbf{d}] \mid \Delta(f) = f\otimes 1 + 1\otimes f\}$. Bozec and Schiffmann proved ([1]) that the functions $C_{Q,\mathbf{d}}(q) \coloneqq \dim_{\mathbf{C}} H^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$ are polynomials in q. They conjectured that these polynomials have nonnegative coefficients for $\mathbf{d}\in\Sigma_{\Pi_Q}$.

Theorem 5 (Davison–H–Schlegel Mejia, 2023, [4, 5])**.** *For $\mathbf{d}\in\Sigma_{\Pi_Q}$, $C_{Q,\mathbf{d}}(q)\in\mathbf{N}[q]$. Furthermore, $C_{Q,\mathbf{d}}(q) = \mathrm{IP}(\mathcal{M}_{\Pi_Q,\mathbf{d}})(q^{-\frac{1}{2}})$ (intersection Poincaré polynomial).*

The proof of Theorem 5 relies on the interpretation of *absolutely* cuspidal polynomials as the $\mathbf{N}^{Q_0}\times\mathbf{Z}$-graded multiplicity of the space of simple positive roots of a $\mathbf{N}^{Q_0}\times\mathbf{Z}$-graded generalised Kac–Moody algebra having the generating series of Kac polynomials as character ([1]). Theorem 5 is then deduced from Corollary 3 and Theorem 4. Theorem 5 also provides qualitative informations on the cuspidal polynomials: $C_{Q,\mathbf{d}}(q)$ is monic and of degree $1-\langle\mathbf{d},\mathbf{d}\rangle_Q$ for $\mathbf{d}\in\Sigma_{\Pi_Q}$.

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Nakayama Algebras and Wide Subcategories

Emre Sen

(joint work with Gordana Todorov, Shijie Zhu)

1. Introduction

Among the artin algebras of finite representation type, Nakayama algebras are special since all indecomposable modules are uniserial. Although module categories are well-understood, many classification problems are still open. Here we report results of [STZ1] and [STZ2] in which we give complete classifications of cyclic Nakayama algebras which are (dominant) Auslander-Gorenstein and linear Nakayama algebras which are higher Auslander respectively.

There is an extensive literature on the homological properties of Nakayama algebras by the works of Ringel, Gustafson, Madsen, Marczinzik to name a few. One recurring theme of all these works is the Gustafson's function which takes a Kupish series of a Nakayama algebra and interprets socles of projectives modules combinatorially. In [S1], we developed a functorial approach to handle homological dimensions, briefly the core idea of the syzygy filtration method is constructing the algebra $\boldsymbol{\varepsilon}(\Lambda)$ called *syzygy filtered algebra* whose category of modules are equivalent to the category of modules filtered by the *second* syzygies of the cyclic Nakayama algebra Λ over the algebraically closed field $\mathbb{K}$.

We present the construction shortly. Let $\mathcal{S}(\Lambda)$ be the complete set of representatives of socles of projective modules over Λ. The syzygy filtered algebra $\boldsymbol{\varepsilon}(\Lambda)$ is the endomorphism algebra of projective covers of Auslander-Reiten translates of elements of $\mathcal{S}(\Lambda)$, i.e.

$$\boldsymbol{\varepsilon}(\Lambda) := \operatorname{End}_\Lambda \mathcal{P} \quad \text{where} \quad \mathcal{P} = \bigoplus_{S\in\mathcal{S}(\Lambda)} P(\tau S).$$

Let $\mathcal{B}(\Lambda)$ be the complete set of isomorphism classes of the second syzygies of simple modules with projective dimension greater than one and $Filt(\mathcal{B}(\Lambda))$ be the category of $\mathcal{B}(\Lambda)$-filtered Λ-modules. Then, $Filt(\mathcal{B}(\Lambda))$ is equivalent to mod-$\boldsymbol{\varepsilon}(\Lambda)$. It turns out that $\boldsymbol{\varepsilon}(\Lambda)$ is again a Nakayama algebra in most cases (i.e. $\operatorname{gldim}\Lambda \neq 2$), hence enables us to use mathematical induction on certain homological dimensions. For details we refer to [S1] and [R1].

A finite dimensional artin algebra A is called higher Auslander if global and dominant dimensions of A are equal [I]. In [S2], based on the syzygy filtered algebra construction we gave the complete classification of cyclic Nakayama algebras which are higher Auslander algebras. We define the defect of algebra Λ as the number of indecomposable injective but non-projective modules and denote it by $\operatorname{def}\Lambda$.

Theorem 1.1. [S2] (*i*) *If Λ is a cyclic Nakayama algebra of rank n which is a higher Auslander algebra of global dimension k, then there exists a unique cyclic Nakayama algebra Λ' of rank $n + \operatorname{def}\Lambda$ which is a higher Auslander algebra of global dimension $k + 2$ and $\boldsymbol{\varepsilon}(\Lambda') \cong \Lambda$.*
(*ii*) *Let (Λ, τ) denote the algebra $\Lambda = \Lambda_1 \times ... \times \Lambda_t$ with connected linear Nakayama*

algebras $\Lambda_1, \dots, \Lambda_t$ and the cyclic permutation τ of the simple Λ-modules, such that the restriction to the simple Λ_i-modules is the Auslander-Reiten translation for the simple Λ_i-modules, and τ maps the simple projective Λ_i-module to the simple injective Λ_{i-1}-module[1] *(with $\Lambda_0 = \Lambda_t$). If (Λ, τ) is a higher Auslander algebra of global dimension k and rank n, then there exists a unique cyclic connected Nakayama algebra Λ' of rank $n + \operatorname{def}\Lambda$ which is a higher Auslander algebra of global dimension $k+2$ and $\boldsymbol{\varepsilon}(\Lambda') \cong \Lambda$.*

This means that any cyclic Nakayama algebra which is a higher Auslander algebra can be uniquely constructed from Nakayama algebras of smaller ranks by reversing the syzygy filtration process. Therefore, the classification of all cyclic Nakayama algebras which are higher Auslander algebras reduces to the classification of linear ones.

We report the result from the upcoming joint work with G. Todorov and S. Zhu. Relying on Theorem 1.1, it is natural to ask under which conditions one can reverse the syzygy filtration process uniquely. The key observation is the invariance of the defects.

Theorem 1.2. [STZ1] *If Λ is a cyclic Nakayama algebra, then there exists unique cyclic Nakayama algebra Λ' such that $\boldsymbol{\varepsilon}(\Lambda') \cong \Lambda$ and* $\operatorname{def}(\Lambda') = \operatorname{def}(\Lambda)$.

Recall that an algebra is called Auslander-Gorenstein if injective dimension is bounded by dominant dimension of the algebra [IS]. Recently dominant Auslander-Gorenstein algebras introduced in [CIM], namely for each projective module P, $\operatorname{in.dim} P \leq \operatorname{dom.dim} P$. We apply Theorem 1.2 to give a complete classification of Nakayama algebras which are (dominant) Auslander-Gorenstein algebras. In other words, we prove that any cyclic Nakayama algebra which is an Auslander-Gorenstein algebra can be uniquely constructed from Nakayama algebras of smaller ranks by reversing the syzygy filtration process. Therefore, it is enough to classify 2-Auslander-Gorenstein algebras which we state below.

Theorem 1.3. [STZ1] *Cyclic Nakayama algebra is 2-Auslander-Gorenstein of infinite global dimension iff resolution quiver $(S_i \mapsto \tau \operatorname{soc} P(S_i))$ has one cycle and some length one branches where nodes in the cycle correspond to minimal projectives and leaves correspond to projective-injective modules with nonzero defect.*

2. Wide Subcategories Cogenerated by Projective-Injective Modules

The syzygy filtration method was developed for cyclic Nakayama algebras and provided a unified approach to handle some homological dimensions. It is natural to ask whether we can extend it for other classes of algebras. It seems that the correct tool might be wide subcategories.

Let $\hat{P}$ be the additive generator of projective-injective A-modules where A is an artin algebra with $\operatorname{dom.dim} A \geq 1$. Then $\mathcal{F} := \{M \in \text{mod-}A \mid M \text{ is submodule of } \hat{P}\}$ is a torsion-free class. The wide subcategory associated to $\mathcal{F}$ is

[1]The term Nakayama cycle was coined by Ringel for (Λ, τ)

$$(1) \qquad \mathcal{W}(\mathrm{cogen}\,\hat{P}) := \{X \in \mathcal{F} \,|\, \forall (g : X \to Y) \in \mathcal{F}, \text{then } \mathrm{coker}(g) \in \mathcal{F}\}\,.$$

Proposition 2.1. [S1, Prop 2.36] *If Λ is a cyclic Nakayama algebra, then the category $Filt(\mathcal{B}(\Lambda))$ is equivalent to the wide subcategory cogenerated by projective-injective Λ-module $\hat{P}$, i.e. $Filt(\mathcal{B}(\Lambda)) \cong \mathcal{W}(\mathrm{cogen}\,\hat{P})$.*

Proposition above suggests that the syzygy filtered algebra construction might be carried into some other classes of algebras via the wide subcategories. First, we use it in the case of **linear** Nakayama algebras. Since linear Nakayama algebras are representation directed, the wide subcategory has always a simple object coming from projective cover of simple injective module which we call trivial.

Theorem 2.2. [STZ2] *(i) If Λ is a connected linear Nakayama algebra which is higher Auslander, then there exists a unique connected linear Nakayama algebra Λ' which is higher Auslander such that non-trivial component of $\mathcal{W}(\Lambda')$ is equivalent to* mod-Λ.

(ii) Let (Λ, τ) denote the product $\Lambda = \Lambda_1 \times ... \times \Lambda_t$ of connected linear Nakayama algebras $\Lambda_1, \dots, \Lambda_t$ such that the restriction to the simple Λ_i-modules is the Auslander-Reiten translation for the simple Λ_i-modules, and τ maps the simple projective Λ_i-module to the simple injective Λ_{i-1}-module. If (Λ, τ) is a higher Auslander algebra, then there exists a unique connected linear Nakayama algebra Λ' which is higher Auslander such that non-trivial component of $\mathcal{W}(\Lambda')$ is equivalent to mod-(Λ, τ).

In conclusion, the classification problem reduces to the detection of small rank linear Nakayama algebras in which we extend results of the recent work of Ringel [R2] where he considers linear Nakayama algebras with monotone Kupish series which are higher Auslander.

Example 2.3. *For instance 3-Auslander algebras constructed from $\bigoplus_{i=1}^{m} \mathbb{A}_n$ are given by Kupisch series $(n^n, (2n, 2n-1, \dots, n, n^{n-1})^{m-1}, n, n-1, \dots, 1)$. Similarly, any 4-Auslander algebra can be constructed from Auslander algebras which are of the form $((2,3)^d, 2, 2, 1)$. For instance 4-Auslander algebras constructed from $\bigoplus_{i=1}^{m}(2,3,2,2,1)$ is $(2, (3,3,3,3,2,3,2)^{m-1}, 3,3,3,3,2,2,1)$.*

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The Krull–Gabriel dimension of cluster-titled algebras

Grzegorz Bobiński

We present a short proof of the following result due to Jaworska-Pastuszak and Pastuszak [7].

Theorem 1. *Let H be a hereditary algebra over a field k. If C is a cluster tilted algebra of type H, then*

$$\operatorname{KG-dim} C = \operatorname{KG-dim} H.$$

In the above theorem $\operatorname{KG-dim} A$ stands for the Krull–Gabriel dimension of an algebra A. We remark that the Krull-Gabriel dimension of hereditary algebras is well-known. Namely, we have the following compilation of classical results due to Auslander [1], Geigle [5] and Baer [2].

Theorem 2. *Let H be a hereditary algebra over a field k. Then*

$$\operatorname{KG-dim} H = \begin{cases} 0 & \textit{if } H \textit{ is of Dynkin type}, \\ 2 & \textit{if } H \textit{ is of Euclidean type}, \\ \infty & \textit{otherwise.} \end{cases}$$

For completeness we present the definition of the Krull-Gabriel dimension. For an abelian category $\mathcal{A}$, we define a so-called Krull–Gabriel filtration $(\mathcal{A}_\alpha)$ of $\mathcal{A}$ by full subcategories $\mathcal{A}_\alpha$, where α is either an ordinal or -1, as follows:

$$\mathcal{A}_\alpha := \begin{cases} 0 & \text{if } \alpha = -1, \\ \{A \in \mathcal{A} : A \text{ is of finite length in } \mathcal{A}/\mathcal{A}_\beta\} & \text{if } \alpha = \beta + 1, \\ \bigcup_{\beta<\alpha} \mathcal{A}_\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

If there exists α such that $\mathcal{A}_\alpha = \mathcal{A}$, then we call the minimal such α the Krull–Gabriel dimension of $\mathcal{A}$ and write $\operatorname{KG-dim} \mathcal{A} = \alpha$. Otherwise, the Krull–Gabriel dimension of $\mathcal{A}$ is undefined and we write $\operatorname{KG-dim} \mathcal{A} = \infty$.

Now, for a k-category $\mathcal{B}$, let $\mathcal{F}(\mathcal{B})$ be the category of finitely presented contravariant functors from $\mathcal{B}$ to the category of k-vector spaces. We call $\mathcal{B}$ right coherent if $\mathcal{F}(\mathcal{B})$ is an abelian category. This is the case if either $\mathcal{B}$ is abelian (for example, the category of finite dimensional modules over a finite dimension algebra) or triangulated (for example, the cluster category $\mathcal{C}_H$ for a hereditary algebra H).

Finally, if A a finite dimensional algebra, then we put

$$\operatorname{KG-dim} A := \operatorname{KG-dim} \mathcal{F}(\operatorname{mod} A).$$

There are two results we use in order to prove Theorem 1. In order to formulate the first one, which is due to Geigle [6], we need an additional notion. Let X be an object of a k-category $\mathcal{B}$. A morphism $f\colon Y \to X$ is called right almost split if f is not a split epimorphism and any morphism $Z \to X$ which is not a split epimorphism factors through f.

Theorem 3. *Let $\mathcal{B}$ be a Krull–Schmidt Hom-finite right coherent k-category. Assume $\mathcal{N}$ is a finite class of indecomposable objects in $\mathcal{B}$ such that for each $X \in \mathcal{N}$ there exists a right almost split morphism $Y \to X$. If $\operatorname{add}\mathcal{N} \neq \mathcal{B}$, then*

$$\text{KG-dim}\,\mathcal{F}(\mathcal{B}) = \text{KG-dim}\,\mathcal{F}(\mathcal{B}/[\mathcal{N}]),$$

where $[\mathcal{N}]$ denotes the ideal of maps in $\mathcal{B}$, which factor through direct sums of objects in $\mathcal{N}$.

The second result is the following theorem due to Buan, Marsch and Reiten [4].

Theorem 4. *Let T be a cluster tilting object in the cluster category $\mathcal{C}_H$, for a hereditary algebra H. If $C := \operatorname{End}_{\mathcal{C}_H}(T)^{\mathrm{op}}$, then we have an equivalence of categories*

$$\operatorname{mod} C \simeq \mathcal{C}_H/[\Sigma T].$$

Since there exist almost split triangles in $\mathcal{C}_H$ [3], $\mathcal{B} := \mathcal{C}_H$ satisfies the assumptions of Theorem 3. Thus if $C := \operatorname{End}_{\mathcal{C}_H}(T)^{\mathrm{op}}$ is a cluster tilted algebra of type H, then by using Theorems 3 and 4 (for $T := T$ and $T := H$), we get

$$\text{KG-dim}\,C = \text{KG-dim}\,\mathcal{F}(\mathcal{C}_H) = \text{KG-dim}\,H.$$

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Auslander regular algebras and Koszul duality

René Marczinzik
(joint work with Aaron Chan and Osamu Iyama)

A noetherian ring A is called *Auslander regular* if it has finite global dimension and there exists a minimal injective coresolution

$$0 \to A \to I^0 \to I^1 \to \ldots \to I^n \to 0$$

such that the flat dimension of each term I^i is at most i. This is a non-commutative generalisation of the well studied regular local commutative algebras and famous examples of non-commutative Auslander regular algebras include Weyl algebras and universal enveloping algebras of finite dimensional Lie algebras. Assume for the rest of this note that all algebras are finite dimensional algebras over a field. For finite dimensional algebras, the most important class of Auslander regular algebras are higher Auslander algebras, blocks of category $\mathcal{O}$ ([KMM]) and incidence algebras of distributive lattices ([IM]). We introduce the new class of *dominant Auslander regular* algebras as algebras of finite global dimension such that every indecomposable projective module P has the property that the injective dimension of P is at most the dominant dimension of P. We use this new class of algebras to generalise the classical Auslander algebras and the higher Auslander correspondence with cluster tilting modules. Dominant Auslander regular algebras have several advantages compared to the classical higher Auslander algebras. For example they are invariant under glueing and under Koszul duality. We use this last property to answer an old question by Green ([Gre]) on the characterisation of the Koszul dual of Auslander algebras that were studied in [GM1] and [GM2]. Namely, we show that a finite dimensional Koszul algebra A is the Koszul dual of a non-semisimple Auslander algebra if and only if A has the following three properties:

(1) The Loewy length of A is equal to 3.
(2) An indecomposable projective A-module P is injective if and only if it has Loewy length 3.
(3) A is dominant Auslander regular.

This leads to the following open question: For which representation-finite algebra B is the Koszul dual of the Auslander algebra of B higher Auslander? We show that this is the case when B is hereditary or selfinjective and use this to describe a rich new class of higher Auslander algebras and higher cluster tilting modules. For more details and proofs we refer to [CIM].

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Semilinear clannish algebras associated to triangulations of surfaces with orbifold points

Daniel Labardini-Fragoso

(joint work with Raphael Bennett-Tennenhaus)

The main aim of this talk is to present [2]. We construct semilinear clannish algebras for the colored triangulations of a surface with marked points and orbifold points, and prove that they are Morita-equivalent to the Jacobian algebras of the species with potential constructed by Geuenich and myself a few years ago in [3, 4].

Surfaces with marked points and orbifold points. A *surface with marked points and orbifold points* is a triple $\mathbf{\Sigma} = (\Sigma, \mathbb{M}, \mathbb{O})$ consisting of

- A compact, connected, oriented, two-dimensional real manifold Σ with (possibly empty) boundary $\partial\Sigma$;
- a finite set of *marked points* $\emptyset \neq \mathbb{M} \subseteq \Sigma$ with at least one point from each connected component of $\partial\Sigma$; points in $\mathbb{P} := \mathbb{M} \setminus \partial\Sigma$ are called *punctures*;
- a (possibly empty) finite set of *orbifold points* $\mathbb{O} \subseteq \Sigma \setminus \partial\Sigma$.

An *arc on* $\mathbf{\Sigma}$ is a curve k on Σ that connects either a pair of points in $\mathbb{M}$, or a point in $\mathbb{M}$ and a point in $\mathbb{O}$, and satisfies the following conditions:

- except for its endpoints, k is disjoint from $\partial\Sigma \cup \mathbb{M} \cup \mathbb{O}$;
- except possibly for its endpoints, k does not cross itself;
- k is not homotopically trivial in $\Sigma \setminus (\mathbb{M} \cup \mathbb{O})$ rel $\mathbb{M} \cup \mathbb{O}$;
- k is not homotopic in Σ rel $\mathbb{M} \cup \mathbb{O}$ to a boundary segment of Σ;
- k is not a loop closely enclosing a single orbifold point.

There are two types of arcs: those connecting points in $\mathbb{M}$, called *non-pending arcs*, and those connecting a point in $\mathbb{M}$ to a point in $\mathbb{O}$, called *pending arcs*.

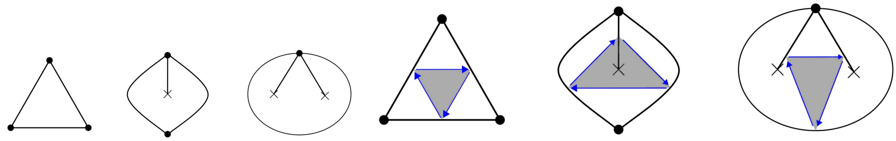
Arcs are considered up homotopy rel $\mathbb{M} \cup \mathbb{O}$. Two arcs are *compatible* if there are representatives in their homotopy classes rel $\mathbb{M} \cup \mathbb{O}$ that do not intersect in $\Sigma \setminus \partial\Sigma$. A *triangulation* of $\mathbf{\Sigma}$ is a maximal collection of pairwise compatible arcs. Each triangulation τ of $\mathbf{\Sigma}$ splits Σ into finitely many triangles.

From now on, we shall assume that $\mathbf{\Sigma}$ satisfies one of the following conditions:

$$(1) \qquad \partial\Sigma \neq \emptyset \text{ and } \mathbb{M} \subseteq \partial\Sigma \text{ (so } \mathbb{P} = \emptyset), \quad \text{or} \quad \partial\Sigma = \emptyset \text{ and } |\mathbb{M}| = 1 \text{ (so } \mathbb{P} = \mathbb{M}).$$

Then there are three possible types of triangles for any given triangulation τ of $\mathbf{\Sigma}$, namely, those shown in Figure 1.

FIGURE 1. Left: The tree types of triangles of a triangulation. Right: The cells of the CW-complex $X(\tau) = (X_n(\tau))_{n=0,1,2}$.



Colored triangulations. Define a CW-complex $X(\tau) = (X_n(\tau))_{n=0,1,2}$ by:

0-cells: $X_0(\tau) := \tau$ (i.e., each arc in τ becomes a 0-cell)

1-cells: $X_1(\tau) := \{$arrows connecting 0-cells clockwisely inside triangles$\}$

2-cells: $X_2(\tau) := \{$3-cycles on $X_1(\tau)$ arising from triangles of τ, up to rotation$\}$.

Let $C_n(\tau) := \mathbb{F}_2 X_n(\tau)$ be the $\mathbb{F}_2$-vector space with basis $X_n(\tau)$, where $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$, and define a chain complex $C_\bullet(\tau)$ and a cochain complex $C^\bullet(\tau)$ as follows

$$\begin{aligned} C_\bullet(\tau): &\qquad 0 \longrightarrow C_2(\tau) \xrightarrow{\partial_2} C_1(\tau) \xrightarrow{\partial_1} C_0(\tau) \longrightarrow 0 \\ &\qquad \partial_2(\alpha\beta\gamma) := \alpha+\beta+\gamma, \quad \partial_1(\alpha) := h(\alpha) - t(\alpha) \\ C^\bullet(\tau) := \mathrm{Hom}_{\mathbb{F}_2}(C_\bullet(\tau), \mathbb{F}_2): &\qquad 0 \longleftarrow C^2(\tau) \xleftarrow{\partial_2^\vee} C^1(\tau) \xleftarrow{\partial_1^\vee} C^0(\tau) \longleftarrow 0 \end{aligned}$$

Following [4], we define a *colored triangulation* of $\boldsymbol{\Sigma} = (\Sigma, \mathbb{M}, \mathbb{O})$ to be a pair (τ, ξ) consisting of a triangulation τ of $\boldsymbol{\Sigma}$ and a 1-cocycle $\xi \in \ker(\partial_2^\vee) \subseteq C^1(\tau)$, i.e. a choice $(\xi_a)_{a \in X_1(\tau)}$ of elements of $\mathbb{F}_2 = \{0,1\}$, subject to the condition that for every 2-cell $\alpha\beta\gamma \in X_2(\tau)$ one must have $\xi_\alpha + \xi_\beta + \xi_\gamma = 0 \bmod 2$.

Semilinear clannish algebras. Let K be any field, and suppose we are given:

- a finite quiver $\widehat{Q}$, not necessarily loop-free;
- a set $\mathbb{S} \subseteq \widehat{Q}$ of *special loops*;
- a field automorphism $\sigma_a \in \mathrm{Aut}(K)$ for each arrow $a \in \widehat{Q}_1$;
- a set Z of paths on $\widehat{Q}$ of length at least 2;
- a degree-2 polynomial $q_s \in K[s;\sigma_s]$ for each $s \in \mathbb{S}$, where $K[s;\sigma_s]$ is the *skew-polynomial ring* in s with coefficients in K, skewed by σ_s.

Suppose further that the following conditions are met:

- for each vertex $k \in \widehat{Q}_0$, at most two arrows of $\widehat{Q}$ end (resp. start) at k;
- for each arrow $a : k \to j$ not in $\mathbb{S}$, at most one arrow b of $\widehat{Q}$ ends (resp. starts) at k and satisfies $ab \notin Z$ (resp. $ba \notin Z$);
- no path belonging to Z has a special loop as its first or last arrow.

Consider the ring $S := \times_{k \in \widehat{Q}_0} K$ and the S-S-bimodule $\bigoplus_{a \in \widehat{Q}_1} K^{\sigma_a} \otimes_K K$, where for any field automophism $\sigma \in \mathrm{Aut}(K)$, we define K^σ to be the twisted K-K-bimodule having K as underlying abelian group, with left K-action $z * m := zm$ and right K-action $m * z := m\sigma(z)$ for $z \in K$ and $m \in K$. Following [1], we denote the tensor algebra $K_{\boldsymbol{\sigma}}\widehat{Q} := T_S(\bigoplus_{a\in\widehat{Q}_1} K^{\sigma_a} \otimes_K K)$, and say that the quotient

$$K_{\boldsymbol{\sigma}}\widehat{Q}/\langle Z \cup \{q_s \mid s \in \mathbb{S}\}\rangle$$

is a *semilinear clannish algebra.*

As in [2], we associate a semilinear clannish algebra to each colored triangulation (τ, ξ) as follows. Fix one of the following two choices of field K:

$$K := \mathbb{C} \quad \text{or} \quad K := \mathbb{R}$$

Set $\widehat{Q}(\tau)$ to be the quiver obtained from $(X_0(\tau), X_1(\tau))$ by adding a loop s_j at each pending arc j of τ. Further, let $\theta : \mathbb{C} \to \mathbb{C}$ be complex conjugation and define

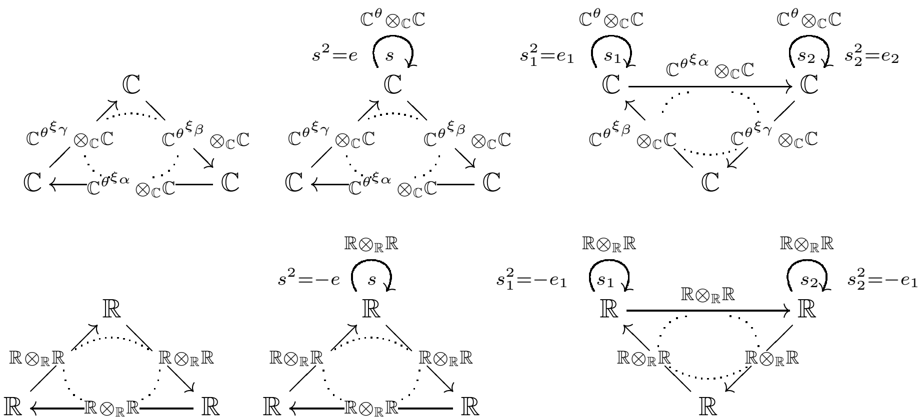
$$\mathbb{S}(\tau) := \{s_j \mid j \in \tau \text{ is pending}\} \qquad \text{i.e., all loops are special;}$$

$$\sigma_a := \begin{cases} \theta^{\xi_a}|_K & \text{if } a \in X_1(\tau) \\ \theta|_K & \text{if } a \in \mathbb{S}(\tau) \end{cases} \qquad \text{for each arrow } a \text{ of } \widehat{Q}(\tau);$$

$$Z(\tau) := \{\alpha\beta \mid \exists\gamma \in X_1(\tau) \text{ such that } \alpha\beta\gamma \in X_2(\tau) \text{ up to rotation of cycles}\},$$

$$q_{s_j} := \begin{cases} s_j^2 - 1 \in \mathbb{C}[s_j;\theta] & \text{if } K = \mathbb{C} \\ s_j^2 + 1 \in \mathbb{R}[s_j;\mathbb{1}_{\mathbb{R}}] & \text{if } K = \mathbb{R} \end{cases} \qquad \text{for all pending arcs } j \text{ of } \tau.$$

These definitions can be mnemotechnically visualized for the triangles from Figure 1 as follows ($K = \mathbb{C}$ first, $K = \mathbb{R}$ afterwards):



We show in [2] that $K_{\boldsymbol{\sigma}}\widehat{Q}(\tau)/I(\tau,\xi)$, where $I(\tau,\xi) := \langle Z \cup \{q_{s_j} \mid s_j \in \mathbb{S}(\tau)\}\rangle$, is a semilinear clannish algebra.

Jacobian algebras. Let (τ, ξ) be a colored triangulation of $\boldsymbol{\Sigma}$. Pick one of the following two assignments $\mathbf{F} = (F_k)_{k\in\tau}$ of fields F_k for $k \in \tau$:

$$F_k := \begin{cases} \mathbb{R} & \text{for all } k \text{ pending;} \\ \mathbb{C} & \text{for all } k \text{ non-pending;} \end{cases} \quad \text{or } F_k := \begin{cases} \mathbb{C} & \text{for all } k \text{ pending;} \\ \mathbb{R} & \text{for all } k \text{ non-pending.} \end{cases}$$

We shall say that the assignment on the left *is B-like*, and to that the one on the right *is C-like*. For each arrow $a \in X_1(\tau)$, $a : k \to j$, set $g(\tau,\xi)_a := \theta^{\xi_a}|_{F_j \cap F_k}$ and

$$A(\tau,\xi)_a := \begin{cases} (F_j \otimes_{\mathbb{R}} F_k)^2 = (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R})^2 & \text{if } k, j \text{ are pending and } \mathbf{F} \text{ is } B\text{-like;} \\ F_j \otimes_{\mathbb{R}} F_k = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} & \text{if } k, j \text{ are pending and } \mathbf{F} \text{ is } C\text{-like;} \\ F_j^{g(\tau,\xi)_a} \otimes_{F_j \cap F_k} F_k & \text{if at least one of } k, j \text{ is non-pending.} \end{cases}$$

These assignments of fields and bimodules can be mnemotechnically visualized for the triangles from Figure 1 as follows (B-like assignment first, C-like afterwards)

$$\begin{tikzcd} & \mathbb{C} \arrow[dr, "\mathbb{C}^{\theta^{\xi_\beta}}\otimes_{\mathbb{C}}\mathbb{C}"] & \\ \mathbb{C} \arrow[ur, "\mathbb{C}^{\theta^{\xi_\gamma}}\otimes_{\mathbb{C}}\mathbb{C}"] & & \mathbb{C} \arrow[ll, "\mathbb{C}^{\theta^{\xi_\alpha}}\otimes_{\mathbb{C}}\mathbb{C}"] \end{tikzcd} \quad \begin{tikzcd} & \mathbb{R} \arrow[dr, "\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}"] & \\ \mathbb{C} \arrow[ur, "\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}"] & & \mathbb{C} \arrow[ll, "\mathbb{C}^{\theta^{\xi_\alpha}}\otimes_{\mathbb{C}}\mathbb{C}"] \end{tikzcd} \quad \begin{tikzcd} \mathbb{R} \arrow[rr, "(\mathbb{R}\otimes_{\mathbb{R}}\mathbb{R})^2"] & & \mathbb{R} \arrow[dl, "\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}"] \\ & \mathbb{C} \arrow[ul, "\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}"] & \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{R} \arrow[dr, "\mathbb{R}\otimes_{\mathbb{R}}\mathbb{R}"] & \\ \mathbb{R} \arrow[ur, "\mathbb{R}\otimes_{\mathbb{R}}\mathbb{R}"] & & \mathbb{R} \arrow[ll, "\mathbb{R}\otimes_{\mathbb{R}}\mathbb{R}"] \end{tikzcd} \quad \begin{tikzcd} & \mathbb{C} \arrow[dr, "\mathbb{R}\otimes_{\mathbb{C}}\mathbb{C}"] & \\ \mathbb{R} \arrow[ur, "\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}"] & & \mathbb{R} \arrow[ll, "\mathbb{R}\otimes_{\mathbb{R}}\mathbb{R}"] \end{tikzcd} \quad \begin{tikzcd} \mathbb{C} \arrow[rr, "\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}"] & & \mathbb{C} \arrow[dl, "\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}"] \\ & \mathbb{R} \arrow[ul, "\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}"] & \end{tikzcd}$$

Let $R := \times_{k \in \tau} F_k$, $A(\tau,\xi) := \bigoplus_{a \in X_1(\tau)} A(\tau,\xi)_a$. Thus, $A(\tau,\xi)$ is an R-R-bimodule, so one can form the (complete) tensor ring of $A(\tau,\xi)$ over R. Following [3, 4], we can define an "obvious" (super-)potential $W(\tau,\xi) \in T_R(A(\tau,\xi))$ as the sum of "obvious" degree-3 cycles in $T_R(A(\tau,\xi))$, and take the *cyclic derivatives* of $W(\tau,\xi)$ to define the *Jacobian algebra* $\mathcal{P}(A(\tau,\xi), W(\tau,\xi))$. Roughly, for each arrow a and cycle c, the cyclic derivative $\partial_a(c)$ is defined to be the $g(\tau,\xi)_a^{-1}$-linear part of the usual sum of paths obtained by deleting each occurrence of a in c (with the reordering yx if $c = xay$), see [3]. Since the F_k are not necessarily all the same field, the notion of path has to be enhanced, both a and ia (resp. ai) may be paths whenever a is an arrow with $F_{h(a)} = \mathbb{C}$ (resp. $F_{t(a)} = \mathbb{C}$). This way, e.g., ab and aib are distinct paths if $t(a) = h(b)$, $F_{h(a)} = \mathbb{R}$, $F_{t(a)} = \mathbb{C}$, $F_{t(b)} = \mathbb{R}$. On the other hand, if $t(a) = h(b)$, $F_{h(a)} = \mathbb{C}$, $F_{t(a)} = \mathbb{R}$, $F_{t(b)} = \mathbb{C}$, then

$$ab = \frac{1}{2}(ab - iabi) + \frac{1}{2}(ab + iabi),$$
$$(ab - iabi)i = i(ab - iabi), \qquad (ab + iabi)i = -i(ab + iabi),$$

i.e., $\frac{1}{2}(ab - iabi)$ and $\frac{1}{2}(ab + iabi)$ are the $\mathbb{1}$-linear part and the θ-linear part of ab.

Main result. Essential to the proof of our main result are the simple observations that $\mathbb{C} \cong \mathbb{R}[s]/\langle s^2+1\rangle = \mathbb{R}[s, \mathbb{1}_{\mathbb{R}}]/\langle s^2+1\rangle$ and $\mathbb{R} \simeq_{\text{Morita}} \mathbb{C}[s;\theta]/\langle s^2-1\rangle$.

Theorem 1. [2] *If $\mathbf{\Sigma}$ is a surface with marked points and orbifold points satisfying* (1), *then for any colored triangulation (τ,ξ) of $\mathbf{\Sigma}$, the Jacobian algebra $\mathcal{P}(A(\tau,\xi), W(\tau,\xi))$ and the semilinear clannish algebra $K_{\boldsymbol{\sigma}}\widehat{Q}(\tau,\xi)/I(\tau,\xi)$ are Morita-equivalent ($K = \mathbb{C}$ in the B-like situation, $K = \mathbb{R}$ in the C-like situation).*

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Configuration spaces and "curvy" associahedra from associative algebras

Pierre-Guy Plamondon

(joint work with Nima Arkani-Hamed, Hadleigh Frost, Giulio Salvatori and Hugh Thomas)

Let k be a field and fix a finite-dimensional k-algebra Λ. We will define an affine variety $\widetilde{\mathcal{M}}_\Lambda$ which reflects the τ-tilting theory of Λ and recovers varieties arising in geometry and in the computation of scattering amplitudes.

1. Algebraic setting: 2-term complexes of projectives

We denote by $K^{[-1,0]}(\operatorname{proj}\Lambda)$ the category of *2-term complexes* of finitely generated projective Λ-modules. It is the full subcategory of the homotopy category $K^b(\operatorname{proj}\Lambda)$ whose objects are the complexes supported in cohomological degrees -1 and 0. This category enjoys the following properties:

(1) Since it is an extension-closed full subcategory of a triangulated category, it is *extriangulated* in the sense of [11].

(2) As an extriangulated category, it has enough projectives and enough injectives. The projective objects are the complexes supported in degree 0 and the injective ones are those supported in degree -1.

(3) It has homological dimension at most 1 and dominant dimension at least 1, and so is a 0-Auslander category in the sense of [7].

(4) The cohomology functor H^0 induces an equivalence of k-linear categories
$$K^{[-1,0]}(\operatorname{proj}\Lambda)/(\Lambda[1]) \xrightarrow{\sim} \operatorname{mod}\Lambda,$$
where $(\Lambda[1])$ is the ideal of morphisms factoring through an injective object.

(5) The category has Auslander–Reiten–Serre duality in the sense of [8], and in particular has almost-split conflations.

2. The varieties

From now on, we assume that Λ has finite representation type. For any two objects X and Y of $K^{[-1,0]}(\operatorname{proj}\Lambda)$, we define a "compatibility degree" $c(X,Y)$ as
$$c(X,Y) := \dim \operatorname{Ext}^1(X,Y) + \dim \operatorname{Ext}^1(Y,X).$$

Definition 1. For each indecomposable object X of $K^{[-1,0]}(\operatorname{proj}\Lambda)$, let u_X be a variable.

(1) The *u-equation* associated to X is the polynomial equation

$$u_X + \prod_{Y \text{ indecomposable}} u_Y^{c(X,Y)} = 1.$$

(2) Define

$$\widetilde{\mathcal{M}}_\Lambda := \operatorname{Spec}\left(\mathbb{C}[u_X \mid X \text{ indec.}]/\langle u\text{-equations}\rangle\right)$$

and

$$\mathcal{M}_\Lambda := \operatorname{Spec}\left(\mathbb{C}[u_X^{\pm 1} \mid X \text{ indec.}]/\langle u\text{-equations}\rangle\right).$$

Example 2. If $\Lambda = k$, then there are only two indecomposable objects in the category $K^{[-1,0]}(\operatorname{proj}\Lambda)$, namely Λ and its shift $\Sigma\Lambda$. The two u-equations are the same: $u_\Lambda + u_{\Sigma\Lambda} = 1$. Thus

$$\widetilde{\mathcal{M}}_\Lambda := \operatorname{Spec}\left(\mathbb{C}[u_\Lambda, u_{\Sigma\Lambda}]/\langle u_X + u_{\Sigma X} - 1\rangle\right).$$

Example 3. If $\Lambda = kQ$, where Q is the quiver of type A_n given by $1 \to \cdots \to n$, then the u-equations can be described combinatorially as follows. Take a disk with $n+3$ marked points on its boundary numbered clockwise from 1 to $n+3$, and consider its diagonals $[i,j]$ joining points i and j (with $1 \le i < j \le n+3, |i-j| \ge 2$). For two diagonals $[i,j]$ and $[k,\ell]$, let $I([i,j],[k,\ell])$ be 1 if the two diagonals intersect and 0 if they don't.

There is a bijection between the diagonals of the disk and the indecomposable objects of $K^{[-1,0]}(\operatorname{proj}\Lambda)$ such that the u-equations become

$$u_{[i,j]} + \prod_{[k,\ell]} u_{[k,\ell]}^{I([i,j],[k,\ell])} = 1.$$

These equations appear in work Koba and Nielsen [10] on scattering amplitudes, and in work of Brown [2] on configuration spaces of points in a projective line.

Example 4. If $\Lambda = kQ$, with Q any orientation of a Dynkin diagram of type ADE, then we recover the u-equations of [1].

Example 5. If Λ is a gentle algebra arising from a grid as in [3, 12], then the u-equations are related to a conjecture of Early [6].

3. Face structure

Theorem 6 (Arkani-Hamed, Frost, P., Salvatori, Thomas)**.** *Let R be an indecomposable rigid object of $K^{[-1,0]}(\operatorname{proj}\Lambda)$. Then the Zariski-closed subset of $\widetilde{\mathcal{M}}_\Lambda$ defined by putting $u_R = 0$ is isomorphic to $\widetilde{\mathcal{M}}_{\Lambda'}$, where Λ' is the τ-tilting reduction of Λ with respect to R (see* [9]*).*

We view $\widetilde{\mathcal{M}}_{\Lambda'}$ as the "face" of $\widetilde{\mathcal{M}}_\Lambda$ at $u_R = 0$. In type A_n, this confers a "face" structure to $\widetilde{\mathcal{M}}_\Lambda$ which is the same as that of the associahedron. We thus view $\widetilde{\mathcal{M}}_\Lambda$ as a "curvy" associahedron.

4. Parametrization

From now on, we assume moreover that $k = \mathbb{C}$. We let $\Lambda \cong kQ/I$ with I an admissible ideal, and we let n be the number of vertices of Q. For any Λ-module M, its *F-polynomial* is

$$F(M) = \sum_{\mathbf{d}\in\mathbb{N}^n} \chi\left(\mathrm{Gr}_{\mathbf{d}}(M)\right) \mathbf{y}^{\mathbf{d}} \in \mathbb{Z}[y_1, \ldots, y_n],$$

where $\mathrm{Gr}_{\mathbf{d}}(M)$ is the submodule Grassmannian of M of dimension vector $\mathbf{d}$ and χ is the Euler characteristic. This definition takes its root in the categorification of cluster algebras [4]. If X is an object of $K^{[-1,0]}(\operatorname{proj}\Lambda)$, we write $F(X)$ instead of $F\left(H^0(X)\right)$.

Theorem 7 (Arkani-Hamed, Frost, P., Salvatori, Thomas)**.** *For any indecomposable non-projective object X of $K^{[-1,0]}(\operatorname{proj}\Lambda)$, let $\tau X \to E_X \to X$ be an almost-split conflation. Then the map*

$$\Phi : \mathbb{C}[\mathcal{M}_\Lambda] \longrightarrow \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}][F(X)^{-1} \mid X \textit{ indec.}]$$

defined by

$$u_X \mapsto \begin{cases} \frac{F_E}{F_{\tau X} F_X} & \textit{if } X \textit{ is neither } P_i \textit{ nor } \Sigma P_i \\ y_i \frac{F_E}{F_{\tau X} F_X} & \textit{if } X = \Sigma P_i \\ \frac{F_{\mathrm{rad}\, P_i}}{F_{P_i}} & \textit{if } X = P_i \end{cases}$$

is surjective.

The proof uses a multiplication formula of [5] for F-polynomials and results on Grothendieck group and g-vectors from [13]. We conjecture that the map Φ is in fact an isomorphism.

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Fans in tilting theory: Rank 2 case

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(joint work with Toshitaka Aoki, Akihiro Higashitani, Ryoichi Kase, Yuya Mizuno)

For each finite dimensional algebra A, 2-term silting complexes of A give rise to a nonsingular fan in the real Grothendieck group of A, which we call the *g-fan* of A. An important problem in tilting theory is to classify complete g-fans. In this note, we give an answer for rank 2 case. More explicitly, we show that complete g-fans of rank 2 are precisely sign-coherent fans of rank 2. As a by-product of our method, we prove that for each positive integer N, there exists a finite dimensional algebra A of rank 2 such that the Hasse quiver of the poset of 2-term silting complexes of A has precisely N connected components.

We start with recalling basic notions from [F]. Let $\mathbb{R}^d$ be an Euclidean space with inner product $\langle\cdot,\cdot\rangle$. A *convex polyhedral cone* σ in $\mathbb{R}^d$ is a set of the form

$$\mathrm{cone}\{v_1,\dots,v_s\} := \{\sum_{i=1}^{s} r_i v_i \mid r_i \geq 0\} \subset \mathbb{R}^d,$$

where $v_1,\dots,v_s \in \mathbb{R}^d$. It is called *rational* if each v_i can be taken from $\mathbb{Q}^d$, and *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. A *supporting hyperplane* of σ is a hyperplane $\mathrm{Ker}\langle u,-\rangle$ in $\mathbb{R}^d$ given by some $0 \neq u \in \mathbb{R}^d$ satisfying $\sigma \subset \{v \in \mathbb{R}^d \mid \langle u,v\rangle \geq 0\}$. A *face* of σ is the intersection of σ with a supporting hyperplane of σ.

A *fan* Σ in $\mathbb{R}^d$ is a collection of cones in $\mathbb{R}^d$ such that

- each face of a cone in Σ is also contained in Σ, and
- the intersection of two cones in Σ is a face of each of those two cones.

A fan Σ in $\mathbb{R}^d$ is called *complete* if $\bigcup_{\sigma\in\Sigma}\sigma = \mathbb{R}^d$, and *nonsingular* if each maximal cone in Σ is generated by a $\mathbb{Z}$-basis for $\mathbb{Z}^d$.

The following class of fans is our main concern.

Definition 1. A *sign-coherent fan* is a pair (Σ,σ_+) of a nonsingular fan Σ in $\mathbb{R}^d$ and a cone $\sigma_+ = \mathrm{cone}\{e_i \mid 1 \leq i \leq d\} \in \Sigma$ of dimension d such that $-\sigma_+ \in \Sigma$ and the following condition is satisfied.

- For each $\sigma \in \Sigma$, there exists $\epsilon \in \{\pm 1\}^d$ such that $\sigma \subseteq \mathrm{cone}\{\epsilon_1 e_1,\dots,\epsilon_d e_d\}$.

Now we introduce the g-fan. Let A be a finite dimensional algebra over a field k, $\mathrm{proj}A$ the category of finitely generated projective A-modules, and $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}A)$ the homotopy category of bounded complexes on $\mathrm{proj}A$. The split Grothendieck group $K_0(\mathrm{proj}A)$ of $\mathrm{proj}A$ is isomorphic to that of the triangulated category $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}A)$, and is a free abelian group with basis consisting of the isomorphism classes of indecomposable projective A-modules. Consider the real Grothendieck group

$$K_0(\mathrm{proj}A)_{\mathbb{R}} := K_0(\mathrm{proj}A) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{|A|}.$$

We call $T = (T^i, d^i) \in \mathrm{K}^{\mathrm{b}}(\mathrm{proj}A)$ *presilting* if $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(\mathrm{proj}A)}(T, T[\ell]) = 0$ for all positive integers ℓ, and *silting* if it is presilting and generates $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}A)$ as a thick subcategory. We call T *2-term* if $T^i = 0$ for all $i \neq 0, -1$.

Definition 2. [H, DIJ] For a 2-term presilting complex $T = T_1 \oplus \cdots \oplus T_\ell$ of A with indecomposable direct summands T_i, let $C(T) := \mathrm{cone}\{[T_1], \ldots, [T_\ell]\} \subset K_0(\mathrm{proj}A)_{\mathbb{R}}$. The *g-fan* of A is

$$\Sigma(A) := \{C(T) \mid T : \text{2-term presilting complex of } A\}.$$

This is in fact a fan. More strongly, the following basic result holds.

Proposition 3. [A, DIJ] *Let A be a finite dimensional algebra over a field k.*

(1) $(\Sigma(A), C(A))$ is a sign-coherent fan in $K_0(\mathrm{proj}A)_{\mathbb{R}}$.
(2) $\Sigma(A)$ is complete if and only if A is τ-tilting finite (that is, A has only finitely many isomorphism classes of basic 2-term silting complexes).

There are many other conditions which are satisfied by g-fans, e.g. idempotent reductions, Jasso reductions, pairwise positivity [AHIKM1]. We pose the following problem.

Problem 4. Characterize complete sign-coherent fans in $\mathbb{R}^d$ which can be realized as g-fans of some finite dimensional algebras.

The following main result gives a simple answer to the case $d = 2$.

Theorem 5. [AHIKM2] *Let k be an arbitrary field. For each complete sign-coherent fan (Σ, σ_+) in $\mathbb{R}^2$, there exists a finite dimensional k-algebra A and an isomorphism $K_0(\mathrm{proj}A)_{\mathbb{R}} \simeq \mathbb{R}^2$ of $\mathbb{R}$-vector spaces such that $(\Sigma(A), C(A)) \simeq (\Sigma, \sigma_+)$.*

To prove this, we give the following three results, which realize certain combinatorial operations for fans in the level of algebras:

- Gluing Theorem,
- Rotation Theorem,
- Subdivision Theorem.

As another application of our method, we prove the following result.

Theorem 6. [AHIKM2] *Let k be an arbitrary field. For each positive integer N, there exists a finite dimensional k-algebra with $|A| = 2$ such that the Hasse quiver of the poset of 2-term silting complexes of A has precisely N connected components.*

This follows from Rotation Theorem and another type of Gluing Theorem.

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The rigid parts of the elements of the real Grothendieck groups

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(joint work with Osamu Iyama)

Let A be a finite dimensional algebra over a field, and $\mathsf{proj}\, A$ be the category of finitely generated projective A-modules.

A complex U in the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is said to be *presilting* if $\mathsf{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(U, U[>0]) = 0$. Since $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is Krull-Schmidt, any complex $X \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ has a unique decomposition $X = \bigoplus_{i=1}^{m} X_i$ into indecomposable direct summands up to isomorphisms and reordering, so X has a maximum presilting direct summand, which we can call the *rigid part* of X.

Based on decompositions of 2-term complexes $X = (X^{-1} \to X^0)$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$, Derksen-Fei [5, Definition 4.3] introduced *canonical decompositions* $\theta = \bigoplus_{i=1}^{m} \theta_i$ in the Grothendieck group $K_0(\mathsf{proj}\, A)$. We say that $\theta \in K_0(\mathsf{proj}\, A)$ is *rigid* if there exists some 2-term presilting complex U such that $[U] = \theta$. Since any element $\theta \in K_0(\mathsf{proj}\, A)$ admits a unique canonical decomposition up to reordering [5, Theorem 4.4], the rigid part of θ can be defined in a quite natural way.

However, in this definition, it is still unclear whether the rigid part of $m\theta$ is always m times of the rigid part of θ for each positive integer $m \in \mathbb{Z}_{\geq 1}$, because there are not enough results on the relationship between the canonical decompositions of θ and $m\theta$.

To avoid this problem partially, we define the rigid part of each element in the real Grothendieck group $K_0(\mathsf{proj}\, A)_{\mathbb{R}} := K_0(\mathsf{proj}\, A) \otimes_{\mathbb{Z}} \mathbb{R}$ without using canonical decompositions. In our definition of rigid parts, the rigid part η of θ satisfies the following properties.

(a) η is rigid.
(b) We can take the "direct sum" of η and $\theta - \eta$.
(c) The element η is the "maximum" element satisfying (a) and (b).
(d) For any positive real number $r \in \mathbb{R}_{>0}$, $r\eta$ is the rigid part of $r\theta$.

Below we explain these properties precisely.

(a). First, to define rigid elements in $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$, we define the *presilting cones* $C^\circ(U), C(U)$ in $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ for each basic 2-term presilting complex $U = \bigoplus_{i=1}^m U_i$ with U_i indecomposable by

$$C^\circ(U) := \sum_{i=1}^m \mathbb{R}_{>0}[U_i], \quad C(U) := \sum_{i=1}^m \mathbb{R}_{\geq 0}[U_i].$$

By [1, Theorem 2.27, Corollary 2.28], the elements $[U_1], \ldots, [U_m]$ are linearly independent, so the presilting cones $C^\circ(U), C(U)$ are m-dimensional. Under this preparation, $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ is said to be *rigid* if there exists some basic 2-term presilting complex U such that $\theta \in C^+(U)$.

(b). We next consider the "direct sum" of η and $\theta - \eta$. For this purpose, we use the two *numerical torsion pairs* $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta), (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ associated to each $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ introduced by Baumann-Kamnitzer-Tingley [3, Proposition 3.1]. These are defined in the category $\mathsf{mod}\,A$ of finitely generated A-modules as follows:

$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{M \in \mathsf{mod}\,A \mid \theta(N) \geq 0 \text{ for any quotient } N \text{ of } M\},\\
\mathcal{F}_\theta &:= \{M \in \mathsf{mod}\,A \mid \theta(L) < 0 \text{ for any submodule } L \neq 0 \text{ of } M\},\\
\mathcal{T}_\theta &:= \{M \in \mathsf{mod}\,A \mid \theta(N) > 0 \text{ for any quotient } N \neq 0 \text{ of } M\},\\
\overline{\mathcal{F}}_\theta &:= \{M \in \mathsf{mod}\,A \mid \theta(L) \leq 0 \text{ for any submodule } L \text{ of } M\}.
\end{aligned}$$

We say that two element $\theta, \eta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ is *TF equivalent* if both $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_\eta, \mathcal{F}_\eta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_\eta, \overline{\mathcal{F}}_\eta)$ hold.

Based on results by Yurikusa [6, Proposition 3.3] and Brüstle-Smith-Treffinger [4, Proposition 3.27], I proved that the presilting cone $C^\circ(U)$ for each basic 2-term presilting complex $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ is a TF equivalence class in [2, Proposition 3.11]. Thus, we write $(\overline{\mathcal{T}}_U, \mathcal{F}_U)$ and $(\mathcal{T}_U, \overline{\mathcal{F}}_U)$ for the corresponding numerical torsion pairs.

For our purpose, it is important to use the open neighborhood N_U of $C^\circ(U)$ for each basic 2-term presilting complex $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ given by

$$N_U := \{\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \mathcal{T}_\theta,\ \mathcal{F}_U \subset \mathcal{F}_\theta\},$$

which firstly appeared in [2, Section 4]. Its closure $\overline{N_U}$ satisfies

$$\overline{N_U} = \{\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \overline{\mathcal{T}}_\theta,\ \mathcal{F}_U \subset \overline{\mathcal{F}}_\theta\}.$$

Therefore, $\overline{N_U}$ is a union of TF equivalence classes, and moreover, we can check that $\overline{N_U}$ is a rational polyhedral cone in $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$. For any basic 2-term presilting complexes U, V, the complex $U \oplus V$ is presilting if and only if $C^\circ(V) \subset \overline{N_U}$, so if $\eta \in C^\circ(U)$, we can think that the direct sum $\eta \oplus (\theta - \eta)$ is "admitted" if and only if $\theta - \eta \in \overline{N_U}$.

(c). Let U be a basic 2-term presilting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$. Then, for each $\theta \in \overline{N_U}$, we would like to show the set

$$H := \{\eta \in C(U) \mid \theta - \eta \in \overline{N_U}\}.$$

has a maximum element.

The key point is the facets (the faces of codimension one) of $\overline{N_U}$. If $U = \bigoplus_{i=1}^m U_i$ with U_i indecomposable, then since $\overline{N_U} = \bigcap_{i=1}^m \overline{N_{U_i}}$, any facet F of $\overline{N_U}$ admits some $i \in \{1, 2, \dots, m\}$ such that F is contained in some facet of $\overline{N_{U_i}}$. We have proved the uniqueness of this property.

Theorem 1. Let $U = \bigoplus_{i=1}^m U_i$ be a basic 2-term presilting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with U_i indecomposable. Then, for any facet F of $\overline{N_U}$, there uniquely exists $i \in \{1, 2, \dots, m\}$ such that F is contained in exactly one facet of $\overline{N_{U_i}}$.

By using this, we have that H has a maximum element.

Proposition 2. Let $U = \bigoplus_{i=1}^m U_i$ be a basic 2-term presilting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with U_i indecomposable, and $\theta \in \overline{N_U}$. Then, the set $H = \{\eta \in C(U) \mid \theta - \eta \in \overline{N_U}\}$ is of the form

$$\left\{ \sum_{i=1}^m x_i [U_i] \mid x_i \in [0, a_i] \ (i \in \{1, 2, \dots, m\}) \right\}$$

with $a_1, a_2, \dots, a_m \in \mathbb{R}_{\geq 0}$.

Therefore, under the setting of the proposition, we set

$$\eta_U(\theta) := \sum_{i=1}^m a_i [U_i] \in C(U).$$

If $\theta \in N_U$, then $\eta_U \in C^+(U)$ holds. Now we can define the rigid part of each $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ as follows.

Definition 3. Let $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$. We can take the maximum basic 2-term presilting complex U in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ such that $\theta \in N_U$. Then, we call $\eta_U(\theta)$ the *rigid part* of θ.

This definition gurantees the desired "maximum" property in the following sense: if a basic 2-term presilting complex V and $\eta \in C^\circ(V)$ satisfy $\theta - \eta \in \overline{N_V}$, then $\eta = \sum_{i=1}^m b_i [U_i] \in C(U)$ with $b_i \in [0, a_i]$ in the notation of Proposition 2.

(d). The definition of $\eta_U(\theta)$ gives a map $\overline{N_U} \to C(U)$. This map is piecewise $\mathbb{R}$-linear, so we have the following.

Corollary 4. Let $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ and $r \in \mathbb{R}_{>0}$. Then, the rigid part of $r\theta$ is r times of θ.

We end this abstract by stating the relationship between canonical decompositions.

Corollary 5. Let K be an algebraically closed field and $\theta \in K_0(\mathsf{proj}\, A)$. Then, there exists some positive interger $l \in \mathbb{Z}_{\geq 1}$ such that, for each $k \in \mathbb{Z}_{\geq 1}$, the following elements coincide.

(a) The rigid part of $kl\theta$ in our definition.
(b) The rigid part of $kl\theta$ defined by canonical decompositions.
(c) k times of the rigid part of $l\theta$ in our definition.
(d) k times of the rigid part of $l\theta$ defined by canonical decompositions.

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Combinatorial invariants of invariant subspaces of nilpotent linear operators

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(joint work with Markus Schmidmeier)

In [2, 3, 4, 5, 6] we developed and introduced several combinatorial tools and invariants (Littlewood-Richardson tableaux, Klein tableaux, arc diagrams, socle tableaux, standard Young tableaux) that control some properties of invariant subspaces of nilpotent linear operators. We want to present how one can apply these tools and invariants to investigate geometric and algebraic properties of invariant subspaces of nilpotent linear operators.

Throughout we assume that k is a field and $\Lambda = k[[T]]$ is the k-algebra of power series. By a nilpotent k-linear operator we mean a Λ-module of the form

$$N_\alpha = \bigoplus_{i=1}^{s} \Lambda/(T^{\alpha_i}),$$

where $\alpha = (\alpha_1, \dots, \alpha_s)$ is a partition.

We are interested in the category $\mathcal{S}$ of all triples (N_α, N_β, f), where

$$f : N_\alpha \to N_\beta$$

is a monomorphism of Λ-modules. Morphisms in this category are defined in the natural way. We call objects of $\mathcal{S}$ invariant subspaces of nilpotent linear operators and identify them with short exact sequences of nilpotent linear operators.

Theorem of Green and Klein ([1, 7]) states that for given partitions α, β, γ, there exists a short exact sequence

$$0 \longrightarrow N_\alpha \longrightarrow N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

of nilpotent linear operators if and only if there is a Littlewood-Richardson (LR-) tableau Γ of shape (α, β, γ). Moreover, we have $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(s)}]$, where for all i:

$$N_{\gamma^{(i)}} \cong N_\beta / T^i f(N_\alpha).$$

We call Γ the LR-tableau of this short exact sequence.

Let α' be the transpose of the partition α. Recall that an LR-tableau of shape (α, β, γ) is a filling of the skew diagram $\beta \setminus \gamma$ with α'_1 boxes with entry 1, α'_2 boxes with entry 2, etc. such that

- the entries are weakly increasing in each row, strictly increasing in each column,
- for each $c \geq 0$, $\ell \geq 2$ there are at least as many entries $\ell - 1$ on the right hand side of the c-th column as there are entries ℓ.

An LR-tableau Γ can be viewed as a sequence of partitions

$$\Gamma = [\gamma^{(0)}, \dots, \gamma^{(s)}]$$

where $\gamma^{(i)}$ denotes the region in the Young diagram β which contains the empty boxes and boxes with entries $1, \dots, i$.

Let $k = \overline{k}$ be an algebraically closed field. Fix partitions α, β, γ. Denote by $\mathbb{H}_\alpha^\beta = \operatorname{Hom}_k(N_\alpha, N_\beta) = \mathbb{M}_{|\alpha|,|\beta|}(k)$ the affine variety of all $|\alpha| \times |\beta|$-matrices, where $|\alpha| = \alpha_1 + \ldots + \alpha_s$. We work with the Zariski topology. Consider the subset $\mathbb{V}_{\alpha,\gamma}^\beta \subset \mathbb{H}_\alpha^\beta$ consisting of all monomorphisms $f : N_\alpha \to N_\beta$ such that $\operatorname{Coker} f \cong N_\gamma$. For an LR-tableau Γ of shape (α, β, γ), denote by $\mathbb{V}_\Gamma \subseteq \mathbb{V}_{\alpha,\gamma}^\beta$ subset consisting of all f such that (N_α, N_β, f) has type Γ.

Note that

$$\mathbb{V}_{\alpha,\gamma}^\beta = \dot{\bigcup} \mathbb{V}_\Gamma,$$

were the union is indexed by the set $\mathcal{T}_{\alpha,\gamma}^\beta$ of all LR-tableaux of shape (α, β, γ). For $\Gamma, \Delta \in \mathcal{T}_{\alpha,\gamma}^\beta$ we define define (reflexive and anti- symmetric) relation

$$\Delta \preceq_{\text{bound}} \Gamma \iff \mathbb{V}_\Gamma \cap \overline{\mathbb{V}}_\Delta \neq \emptyset$$

Denote by $\leq_{\text{bound}}$ the transitive closure of $\preceq_{\text{bound}}$.

In [4, 5, 3], applying combinatorial properties of LR-tableaux, the order $\leq_{\mathsf{bound}}$ is investigated.

We introduce a new combinatorial invariants: socle tableaux. The socle tableau for (N_α, N_β, f) is defined in the following way. Let $\operatorname{soc}^i N_\alpha = \{x \in N_\alpha \; ; \; T^i x = 0\}$. The socle seguence

$$0 \subset f(\operatorname{soc} N_\alpha) \subset \cdots \subset f(\operatorname{soc}^{s-1} N_\alpha) \subset f(\operatorname{soc}^s N_\alpha) = f(N_\alpha),$$

gets a sequence of epimorphisms:

$$N_\beta \to N_\beta / f(\operatorname{soc} N_\alpha) \to \cdots \to N_\beta / f(\operatorname{soc}^{s-1} N_\alpha) \to N_\beta / f(N_\alpha),$$

and hence also a decreasing sequence of partitions:

$$\beta = \sigma^{(0)} \supseteq \sigma^{(1)} \supseteq \cdots \supseteq \sigma^{(s-1)} \supseteq \sigma^{(s)} = \gamma,$$

where $\sigma^{(i)}$ is such that $N_{\delta^{(i)}} \simeq N_\beta / f(\operatorname{soc}^i N_\alpha)$. We call $\Sigma = [\sigma^{(s)}, \sigma^{(s-1)}, \dots, \sigma^{(0)}]$ the socle tableau of (N_α, N_β, f).

An S-tableau of shape (α, β, γ) is a filling of the skew diagram $\beta \setminus \gamma$ with α'_1 boxes with entry 1, α'_2 boxes with entry 2, etc. such that

- in each row, the entries are weakly decreasing, in each column, the entries are strictly decreasing,
- for each vertical line, and each natural number ℓ, there are at most as many entries $\ell + 1$ on the left hand side of the line as there are entries ℓ.

The following theorems hold.

Theorem [2]. *There exists a short exact sequence of Λ-modules*

$$\eta \quad : \quad 0 \longrightarrow N_\alpha \xrightarrow{f} N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

if and only if there exists an S-tableau Σ of shape (α, β, γ).

Moreover, $\Sigma = [\sigma^{(s)}, \dots, \sigma^{(0)}]$, where for all i

$$N_{\sigma^{(i)}} \cong N_\beta / f(\mathrm{soc}^i N_\alpha).$$

Let $\hom_{\mathcal{S}}(X, Y) = \mathrm{len}\mathrm{Hom}_{\mathcal{S}}(X, Y)$, $P_\ell^m = (\Lambda/(p^\ell), \Lambda/(p^m), \iota)$.

Theorem [2]. *For $X = (N_\alpha, N_\beta, N_\gamma)$, the following invariants are equivalent in the sense that each one determines both of the others.*

(1) The socle tableau $\Sigma = \Sigma_X$.
(2) The LR-tableau $\Gamma^* = \Gamma_{X^*}$ of the dual object X^*.
(3) The Hom-matrix $H = (h_\ell^m)_{\ell,m}$ where $h_\ell^m = \hom_{\mathcal{S}}(P_\ell^m, X)$.

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On the syzygy categories over dimer tree algebras and their skew group algebras

Ralf Schiffler
(joint work with Khrystyna Serhiyenko)

This is an overview of the results in [7, 8, 9]. A *dimer tree* algebra A is the Jacobian algebra of a quiver Q (without loops and 2-cycles) with a canonical potential such that

(1) every arrow lies in a chordless oriented cycle;
(2) the dual graph is a tree.

By definition, dimer tree algebras are 2-Calabi Yau tilted algebras [1] and therefore non-commutative Gorenstein algebras of Gorenstein dimension one [5]. The category of non-projective syzygies over A is equivalent to the stable category of (maximal) Cohen Macaulay modules $\underline{\mathrm{CMP}}\,A$ as well as to the singularity category of A. The category $\underline{\mathrm{CMP}}\,A$ is a triangulated category whose shift is given by the inverse syzygy functor Ω^{-1}. Moreover, $\underline{\mathrm{CMP}}\,A$ is 3-Calabi-Yau [5].

In [7], we introduce a derived invariant, the *total weight*, of a dimer tree algebra and show that it is an even integer, which we denote by $2N$. Then we construct a $2N$-gon $\mathcal{S}$ with checkerboard pattern $\rho(i)$, $i \in Q_0$, and show that the category $\underline{\mathrm{CMP}}\,A$ is equivalent to the category $\mathrm{Diag}\,\mathcal{S}$ of 2-diagonals in $\mathcal{S}$. This equivalence maps indecomposable syzygies to 2-diagonals, irreducible morphisms to 2-pivots, and Auslander-Reiten triangles to meshes. Moreover, it commutes with the shift functors and the Auslander-Reiten translations in both categories. In particular, the syzygy functor Ω in $\underline{\mathrm{CMP}}\,A$ corresponds to the clockwise rotation R by π/N in $\mathrm{Diag}\,\mathcal{S}$. The inverse Auslander-Reiten translation in $\underline{\mathrm{CMP}}\,A$ is given by Ω^2 and in $\mathrm{Diag}\,\mathcal{S}$ by R^2. Furthermore, the radical of the indecomposable projective module $P(i)$ is mapped to the line $\rho(i)$ of the checkerboard pattern in $\mathcal{S}$.

It follows that the projective resolutions of A-modules are periodic of period N or $2N$. Using a result of [3], we then obtain that $\underline{\mathrm{CMP}}\,A$ is equivalent to the 2-cluster category $\mathcal{C}^2_{\mathbb{A}_{N-3}} = \mathcal{D}^b(\mathrm{mod}\,H)/\tau^{-1}[2]$ of type $\mathbb{A}_{N-3}$.

In particular, the number of indecomposable Cohen-Macaulay modules over A is $N(N-2)$. We say the dimer tree algebra is of *finite Cohen-Macaulay type* $\mathbb{A}$.

The name dimer tree algebra stems from the fact that the checkerboard pattern on $\mathcal{S}$ can be extended to a dimer model (or Postnikov diagram) on a slightly larger disk.

In [9]. we then consider dimer tree algebras with an admissible group action by a group $G = \{1, \sigma\}$ of order two. The skew group algebra AG has been studied in [6, 2]. Inspired by their results, we show that AG is of finite Cohen-Macaulay type $\mathbb{D}$. In fact, $\underline{\mathrm{CMP}}\,AG$ is equivalent to the category of 2-arcs in a punctured N-gon and therefore to the 2-cluster category of Dynkin type $\mathbb{D}_{\frac{N+1}{2}}$. The action of the non-trivial group element σ induces an action on the checkerboard $2N$-gon given by a rotation by angle π. Therefore, the checkerboard pattern on the $2N$-gon induces a checkerboard pattern on the punctured N-gon, and the equivalence of categories again commutes with the shift functors and the Auslander-Reiten translations in both categories.

We also give examples of 2-Calabi-Yau tilted algebras of finite Cohen-Macaulay types $\mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$.

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Central support for triangulated categories

Henning Krause

The notion of support for objects of an essentially small triangulated category requires a space, and it is therefore an interesting task to construct such a space from the lattice of thick subcategories. Several options are discussed in recent work of Gratz and Stevenson [3]. In fact they point out that a distributive lattice of thick subcategories is automatically a spatial frame, so isomorphic to a lattice of open subsets of a topological space. In my talk I reported on own recent work [4] which has been inspired by this beautiful observation. I turns out that distributivity follows when a sublattice consists of thick subcategories which are pairwise commuting (i.e. any morphism between these subcategories factors through an object in the intersection). In particular, the centre of the lattice of thick subcategories is a distributive lattice. This yields a space which one may use to define central support for objects of a triangulated category. It is interesting to note that for a tensor triangulated category any thick tensor ideal is central (so commuting with any other thick tensor ideal), provided that the tensor category is rigid. In that way we recover Balmer's spectrum of a (rigid) tensor triangulated category [1]. Also, thick subcategories are central when they are defined via the cohomological support given by a central ring action, as in work of Benson, Iyengar, and myself

[2]. Thus central support offers a common generalisation of several existing notions of support. As a bonus one obtains Mayer–Vietoris sequences for any pair of commuting thick subcategories. We refer to [4] for details.

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