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Mini-Workshop: Skew Braces and the Yang–Baxter Equation

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Abstract. The workshop was focused on the interplay between set-theoretic solutions to the Yang–Baxter equation and several algebraic structures used to construct and understand new solutions. In this vein, the YBE and properties of these algebraic structures are used as a source of inspiration to study other mathematical problems not directly related to the YBE.

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Introduction by the Organizers

Constructing and classifying solutions to the Yang–Baxter equation (YBE) is one of the fundamental problems in mathematical physics. The search for solutions to this crucial equation, which first appeared in physics, has attracted numerous studies in pure mathematics as well. A fruitful approach to constructing and classifying solutions is to identify and study their underlying mathematical structures. Among structures specifically introduced for producing solutions to the YBE, quantum groups are probably the most famous example. Nevertheless, connections with many other mathematical structures were found in the last decade. Some of such structures were specifically introduced in relation to solutions to the YBE, e.g. skew braces, cycle sets and trusses, others were already known, e.g. bijective group 1-cocycles, Hopf–Galois structures and self-distributive systems. These shed new lights and brought new powerful tools into the study of the

YBE: group theory, non-commutative geometry, computational and cohomological algebra, number theory, and knot theory.

This mini-workshop brought together experts in the algebraic approach to the set-theoretic YBE and related fields. Discussions spread into several directions: skew braces and Lie theory, the algebraic theory of skew braces, combinatorial properties of solutions to the YBE, computational aspects, applications, and generalizations.

Summarizing, the topics of the mini-workshop were the following:

The algebraic theory of skew braces: Originally, skew braces were introduced to study set-theoretic solutions to the Yang–Baxter equation. During the mini-workshop, skew braces were widely discussed and their connections to other algebraic structures were explored. The talks of Colazzo, Van Antwerpen, Tsang, Ballester-Bolinches, Letourmy, and Byott discussed recent results on the structure theory of skew braces and possible applications to different topics. Puljic and Vendramin highlighted recent advances regarding skew braces of abelian type and pre-Lie algebras. Stefanello discussed some applications of skew braces to Hopf–Galois theory.

Solutions to the YBE: The mini-workshop contained also several talks on various combinatorial aspects of this famous equation. Lebed discussed indecomposable solutions and the recently introduced "cabling" technique. Doikou, Trappeniers, and Rybołowicz discussed applications of skew braces to construct specific class of solutions. Okniński reported on simple solutions. Lechner discussed an equivalence relation of solutions based on the action of the braid group.

Generalizations: The talks of Rump and Stefanelli presented different generalizations of skew braces and discussed their main properties. Brzeziński presented the Hochschild cohomology for trusses. Iyudu talked about a connection between pre-Lie algebras and trusses. Dietzel's talk was about L-algebras.

A particular accent in some talks was made on open questions and conjectures, which were then discussed among participants. This was especially useful for young researchers, well represented in the mini-workshop. All participants agreed that the meeting had been highly productive, allowing them to keep up with the recent developments in the field, initiate collaborations, and discuss new approaches to open problems in the area.

Mini-Workshop: Skew Braces and the Yang–Baxter Equation

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Abstracts

Braces and pre-Lie algebras

Leandro Vendramin

A *skew brace* is a triple $(A, +, \circ)$, where $(A, +)$ and $(A, \circ)$ are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds for all $a, b, c \in A$. We will mainly consider finite skew braces *of abelian type*, that is, finite skew braces where the group $(A, +)$ is abelian.

In [2], Etingof, Schedler and Soloviev proved that the multiplicative group of a finite skew brace of abelian type is always solvable. A natural question arises: Is every solvable finite group the multiplicative group of a skew brace of abelian type? The answer is no; see [1].

Problem 1. *Find a minimal counterexample.*

A *pre-Lie ring* is an abelian group A with a binary operation $A \times A \to A$, $(a, b) \mapsto a \cdot b$, such that

(1) $(a + b) \cdot c = a \cdot c + b \cdot c$,
(2) $a \cdot (b + c) = a \cdot b + a \cdot c$, and
(3) $(a \cdot b) \cdot c - a \cdot (b \cdot c) = (b \cdot a) \cdot c - b \cdot (a \cdot c)$

for all $a, b, c \in A$.

Computer calculations produce (the number of) isomorphism classes of small pre-Lie rings:

size	2	3	4	5	6	7	8
pre-Lie	2	3	17	2	4	2	178
left nilpotent	1	1	4	1	1	2	28
Novikov	2	3	15	2	4	2	85

Following the usual definitions for skew braces one defines left and right nilpotent pre-Lie rings; see [3, §2].

A pre-Lie ring A is said to be *Novikov* if

$$(a \cdot b) \cdot c = (a \cdot c) \cdot b$$

for all $a, b, c \in A$. One proves that for finite left nilpotent pre-Lie rings, being right nilpotent is equivalent to being Novikov.

Theorem 2 (Smoktunowicz)**.** *Let p be a prime number and n be an integer such that $p > (n + 1)^{n+1}$. Let There exists a bijective correspondence between strongly nilpotent pre-Lie rings of size p^n and strongly nilpotent skew braces of size p^n of abelian type.*

A p-group G is said to be *powerful* if G/G^p is abelian if $p > 2$, or G/G^4 is abelian if $p = 2$, where $G^k = \langle g^k : g \in G\rangle$ is a normal subgroup of G.

Conjecture 3 (Shalev–Smoktunowicz). *Let p be a prime number and A be a powerful skew brace of abelian type of size p^n. Then A is right nilpotent.*

Computer calculations support Conjecture 3. These calculations suggest that the conjecture is true also in the case of arbitrary skew braces of prime-power size.

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Skew-braces, q-braces, and the YBE

Wolfgang Rump

A natural concept of covering of non-degenerate solutions to the set-theoretic Yang-Baxter equation allows studying solutions by means of their universal covering and the action of their fundamental group. The concept of *q-brace* arose in that context. Non-involutive solutions to the YBE are characterized by q-cycle sets, a variant of cycle sets with two operations satisfying three equally shaped equations, and a q-brace with adjoint group G is equivalent to a q-cycle set structure on G that induces a matched pair (G, G). Skew-braces are q-braces with a coupling of the two q-cycle set operations. The observation of Childs et al. that some interactions between skew braces give rise to a sequence of skew braces is related to a basic property of q-braces. Connections between q-braces and skew-braces are illustrated in the lecture.

Few remarks on filtered trusses and on a geometric interpretation of solutions to CYBE

Natalia Iyudu

First, I will remark that for a truss endowed with appropriate descending filtration, the associated graded structure is a pre-Lie algebra, in case $deg(\alpha(a)) > 2$, for $\alpha(a) = a*(b+c)-a*b-a*c$. Another topic deals with the geometric interpretation of a solution to CYBE coming from the connection between the latter and double Poisson brackets, as well as pre-Calabi-Yau structures.

Derived-indecomposable solutions and skew braces whose elements have a finite number of conjugates

Ilaria Colazzo
(joint work with Maria Ferrara and Marco Trombetti)

A solution to the Yang-Baxter equation (YBE) is a pair (X, r) with X a non-empty set and $r : X \times X \to X \times X$ a bijective map. We assume X to be finite and, if we write $r(x, y) = (\lambda_x(y), \rho_y(x))$ for all $x, y \in X$, we also assume λ_x, ρ_x to be bijective. The problem of classifying and describing such solutions is challenging and usually tackled both using algebraic structures and focusing on subclasses with interesting properties; e.g., skew braces and indecomposable solutions. We refer to [2] for details about the connection between skew braces and solutions to the YBE.

A solution (X, r) is *decomposable* if there exist a non trivial partition $X = X_1 \cup X_2$ such that $X_1 \cap X_2 =$ and $r(X_i \times X_i) \subseteq X_i \times X_i$ $(i = 1, 2)$; it is said *indecomposable* otherwise.

In [1], we focus on a class of indecomposable solutions called *derived-indecomposable*, i.e. solutions, as the name suggests, such that their derived solution is indecomposable. We prove that the structure skew brace $G(X, r)$ of a derived indecomposable solution (X, r) has an intriguing finiteness property: it is a skew brace with the so-called *property* (BS) see [1, Theorem 4.5].

An element x of a skew brace B has the *property* (s) if

$$\text{(s)} \qquad \left|(B, +) : \mathrm{Fix}^r(x) \cap C_x^+\right| \text{ and } \left|(B, \circ) : \mathrm{Fix}^l(x) \cap C_x^\circ\right|$$

are finite, where $\mathrm{Fix}(x) = \{b \in B : b * x = 0\}$, $\mathrm{Fix}^r(x) = \{b \in B : x * b = 0\}$, C_x^+ denotes the centraliser of x in $(B, +)$, and C_x° is the centraliser of x in $(B, \circ)$.

A skew brace B has the *property* (S) if all its elements have the property (s). If there is a positive integer that bounds the cardinality of the sets in (s) for any x we say that B has the *property* (BS).

As explained in [1], we may refer to elements of the type $g * x$, $x * g$, $g \circ xg'$ and $g + x - g$ for some $g \in B$ as *conjugates of* x. And noting that there is a one-to-one correspondence between the set of right cosets of $\mathrm{Fix}^r(x)$ in $(B, +)$ (resp. $\mathrm{Fix}^l(x)$ in $(B, \circ)$) and the set $\{x*b : b \in B\}$ (resp. $\{b*x : b \in B\}$), it is clear that we may refer to skew braces with the property (S) as skew braces whose elements have a finite number of conjugates. We refer to [1], for in-depth results on skew braces with property (S) and (BS). In the following, we will state some open problems.

Question 1. *Let B be a skew brace. Is the set of all elements with property* (s) *an ideal?*

It is not difficult to prove that the answer is affirmative for two-sided skew braces of abelian type.

Now if we focus on periodic skew braces with property (S), i.e. skew braces such that $(B, +)$ coincide with the sets of its torsion elements (note that [1, Theorem 3.13] justifies this definition since for skew braces with property (S) the set of torsion elements of $(B, +)$ coincide with the set of torsion elements of $(B, \circ)$), we would hope to have an analogue of Dietzmann's lemma. Unfortunately, it seems to be not quite clear if a result of this type holds or not.

Question 2 (Question 3.24 in [1]). *Let B be a skew brace with property* (S). *Is it true that any finitely generated ideal of B is finite?*

Finally, an ideal of a skew brace B with property (S) is *good* if every finitely generate ideal J of B with $J \subseteq I$ is finitely generated as a skew brace. It turns out that many natural ideals are good, see [1, Corollary 3.26] and comments afterwards, e.g. $\mathrm{Soc}_n(B)$ is good.

With this terminology, we can reformulate Question 2 as follows.

Question 3. *Is every periodic skew brace with property* (S) *good?*

Actually, as a consequence of Theorems 3.14, 3.25 and 3.27 in [1], it turns out that the previous question is equivalent to the following one: is every skew brace with property (S) good?

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Some problems on L-algebras

Carsten Dietzel

Definition 1. [5] An *L-algebra* is a set X with a distinguished element $e \in X$ and a binary operation $\cdot : X \times X \to X$ such that the following axioms are fulfilled:

$$e \cdot x = x \quad ; \quad x \cdot x = x \cdot e = e$$

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z).$$

$$x \cdot y = e = y \cdot x \Leftrightarrow x = y.$$

An algebraic structure with a binary operation that fulfills the identity in the second line is called a *cycloid*.

The following facts are explained (and proven!) in more detail in [5, 7].

Each L-algebra X is partially ordered via $x \leq y \Leftrightarrow x \cdot y = e$; the greatest element under this order is e.

In a partially ordered set X, let for an element $x \in X$ the *downset* be $x^{\downarrow} = \{y \in X : y \leq x\}$. An L-algebra X is called *self-similar*, if for every $x \in X$, the map $\lambda_x : x^{\downarrow} \to X$, $y \mapsto x \cdot y$ is a bijection. In this case, X has a monoid structure given by $yx = \lambda_x^{-1}(y)$. Furthermore, each self-similar L-algebra has the structure of a $\wedge$-semilattice[1], that can be calculated by $x \wedge y = (x \cdot y)x$.

[1] A $\wedge$-semilattice is a set X with a binary operation $\wedge$ that is idempotent, associative and commutative

It has been shown by Rump, that every L-algebra X is embedded as a sub-L-algebra into a self-similar L-algebra $S(X)$ - its *self-similar closure* - such that $S(X)$ is generated as a monoid by X. The self-similar closure can be shown to be unique up to isomorphism. It can furthermore be shown that the self-similar closure $S(X)$ always has a group of left fractions $G(X) = \{a^{-1}b : a, b \in S(X)\}$, the *structure group* of X.

1. Special classes of L-algebras

Definition 2. [6] A $\wedge$-*semibrace* is a set X with two binary operations $\cdot$, $\wedge$ and a distinguished element $e \in X$, such that $(X, \wedge)$ is a $\wedge$-semilattice and the following conditions hold

$$x \cdot (y \wedge z) = x \cdot y \wedge x \cdot z; \quad e \cdot x = x;$$
$$(x \wedge y) \cdot z = (x \cdot y) \cdot (x \cdot z); \quad x \cdot e = e.$$

As $\wedge$ is a commutative operation, it follows from the lower left equation that $(X, \cdot)$ is a cycloid.

Replacing the $\wedge$-operation by the operation of an abelian group, one gets a brace. It is easy to see that every abelian group A can arise as the additive group of a brace - a canonical choice is the trivial brace associated with A. As there is no (obvious) notion of trivial $\wedge$-semibrace, the following question is far less trivial:

Problem 3. *Which semilattices are the underlying semilattices of a $\wedge$-semibrace? Which semilattices are the underlying semilattices of a $\wedge$-semibraces that is also an L-algebra?*

The next problem is concerned with *right bricks*:

Definition 4. [7] A right brick is a set X, together with binary operations $\cdot, \circ$ and distinguished elements $e, 0 \in X$ such that $(X, \cdot, e)$ and $(X, \circ, e)$ are L-algebras, and setting $\overline{x} = x \cdot 0$ and $\tilde{x} = x \circ 0$, we have

- $\overline{\tilde{x}} = x$,
- $x \cdot \overline{x \circ y} = y \cdot \overline{y \circ x}$ and $x \circ \widetilde{x \cdot y} = y \circ \widetilde{y \cdot x}$,
- $\widetilde{\overline{x} \cdot \overline{y}} = \overline{\tilde{x} \cdot \tilde{y}}$.

On a right brick, the canonical orders determined by $\circ, \cdot$ coincide and make the underlying set a lattice.

For a (possibly noncommutative) field K and an integer $n \geq 1$, let $L(K, n)$ be the lattice of K-linear subspaces of the left vector space K^n. A lattice isomorphism to one of the form $L(K, n)$ is called *desarguesian*.

Problem 5. *Which desarguesian lattices $L(K, n)$ are the underlying lattices of a brick?*

Problem 6. *Can the subspace lattice of a non-desarguesian plane be the underlying lattice of a brick?*

Only positive answers are known - a brick structure on $L(K, n)$ is known to exist in the following cases:

(1) $n \in \{1; 2\}$ (trivial)
(2) There is an anisotropic, hermitean bilinear form on K^n [3]
(3) K is a finite field [2]

By the last result, a desarguesian lattice without a brick structure must necessarily be infinite.

2. Representations of L-algebras

Let K be a field and $b(v, w) = v^\top Aw$ an *anisotropic* K-bilinear form on K^n, where A is a symmetric $n \times n$-matrix. The desarguesian lattice $L(K, n)$ can then be equipped with the structure of an L-algebra $X(b)$ by means of the *Sasaki operation* $U \cdot V = U^\perp + (U \cap V)$ where $U^\perp = \{v \in V : \forall u \in U : b(u, v) = 0\}$. The structure group of $X(b)$ can be described as a *pure paraunitary group*:

Theorem 7. [1] $G(X(b)) \cong \mathrm{PPU}(b) := \{\phi(t) \in K[t, t^{-1}]^{n\times n} \ : \ \phi(t^{-1})^\top A\phi(t) = A \ \wedge \ \varphi(1) = 1\}$.

A need not be symmetric in order to define the Sasaki operation that turns $L(K, n)$ into an L-algebra [3].

Problem 8. *Let $b(v, w) = v^\top Aw$ be an anisotropic, K-bilinear form on K^n that is* not necessarily symmetric. *Find a representation of $G(X(b))$ that is similar to the one described above.*

Let $(X, \circ)$ be a non-degenerate cycle set. Then X admits a *trivialization* X_{tr} on the same set X, given by $x \circ_{\mathrm{tr}} y = y$, which has the structure monoid $M(X_{\mathrm{tr}}) = \mathbb{Z}_{\geq 0}^{|X|}$ and structure group $G(X_{\mathrm{tr}}) = \mathbb{Z}^{|X|}$. The bijective identifications $M(X) \xrightarrow{\sim} M(X_{\mathrm{tr}})$ resp. $G(X) \xrightarrow{\sim} G(X_{\mathrm{tr}})$ result in a brace structure.

Problem 9. *Develop a* trivialization theory *for classes of L-algebras, that relates an L-algebra X to a* trivialization X_{tr}.

Find relations between $S(X)$ and $S(X_{\mathrm{tr}})$ resp. $G(X)$ and $G(X_{\mathrm{tr}})$. Can these relations be described in terms of a brace-like theory?

Problem 10. *Study the trivialization that associates with an L-algebra X the L-algebra X_{tr} with the binary operation*

$$x \cdot_{\mathrm{tr}} y = \begin{cases} e & x \leq y \\ y & x \not\leq y. \end{cases}$$

3. Generalizations and deformations of L-algebras

Non-degenerate solutions to the set-theoretic YBE are parametrized by non-degenerate *q-cycle sets* [8]. In contrast, non-degenerate involutive solutions are parametrized by non-degenerate *cycle sets*. Rump associated with each non-degenerate cycle set an L-algebra, which leads to an approach to Garside structures on structure monoids and -groups of cycle sets. We pose the following question:

Problem 11. *Develop a theory of qL*-algebras *and their monoid- resp. group-valued invariants. Do these invariants carry Garside-like structures?*

If the problem can be solved satisfyingly, we expect the following two consequences:

- handy normal forms for non-involutive structure invariants,
- a better understanding of elementary combinatorial properties, such as growth in structure invariants

Non-degenerate cycle sets can be deformed into L-algebras; both structures share the property of being a cycloid and can be extended to different semigroup structures on $S(X)$ given by $x \oplus y = (x \cdot y)x$.

Problem 12. *Find other* deformations *of cycle sets into cycloids. Determine the commutative semigroup operations on the structure group $M(X)$ determined by these structures.*

One instance of deformation is given by the *cabling technique* developed by Lebed, Ramirez, and Vendramin [4], which transforms a non-degenerate cycle set into another one.

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Finiteness conditions, nilpotency and solubility of skew left braces

Adolfo Ballester-Bolinches

Skew left braces arise naturally from the study of non-degenerate set-theoretic solutions of the Yang-Baxter equation (YBE): every skew left brace provides a bijective non-degenerate solution and vice versa. The more we know about skew left braces the more we know about their associated bijective non-degenerate solutions of the YBE. Furthermore, as skew left braces are an interaction of two group structures on the same set, it is natural to approach them with group theoretical methods.

We present a sort of analogue of an important strengthened form of the Jordan-Hölder theorem in the context of skew left braces and introduce the notion of finite chief length. Although this result is interesting on their own, it can be used to characterise noetherian and artinian skew left braces introduced in [7]. The Jordan-Hölder theorem also allows us to study right nilpotency of skew left braces by means of their chief factors and relate the chief length of a skew left brace with multipermutation level of the associated solution of the Yang-Baxter equation. Some results in [6] on right nilpotency are improved for skew left braces with chief series ([3]).

Results about central nilpotency of skew left braces introduced by Bonatto and Jedlička in [4] are also presented in this paper. This important class of skew left braces turns out to be the true analog to the class of nilpotent groups and can be characterised by upper and lower central series. We showed that nilpotency of a skew left brace of finite chief length can be also characterised by central chief series. A brace theoretic analog of the well-know Fitting theorem on groups is showed: the product of two nilpotent ideals is nilpotent. As a consequence the ideal generated by all nilpotent ideals of a skew left brace is nilpotent provided that the brace satisfies the maximal condition on ideals: it is called the Fitting ideal and for skew left braces of finite chief length is just the intersection of the centralisers of the chief factors of a given chief series.

The commutator ideal introduced by [5] is used to introduce and study a brace-theoretic property of solubility. We showed that brace nilpotency is to brace solubility what group nilpotency is to group solubility ([1, 2]).

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Nilpotency of skew braces: an overview

Arne Van Antwerpen

Skew braces are particularly effective algebraic structures to both describe and generate bijective non-degenerate set-theoretic solutions of the Yang-Baxter equation. In this talk, we will give an overview of the notions of nilpotency that exist for skew braces and how these interlink. A skew brace $(B, +, \circ)$ is called left (resp. right) nilpotent if the chain of left ideals $B^{n+1} = B * B^n$ (resp. chain of ideals

$B^{(n+1)} = B^{(n)} * B$) ends in $\{0\}$. These notions seem dual to each other, but their behavior is entirely different. Note that for skew braces of nilpotent type, it may be of interest to capture all available nilpotency structures simultaneously to modify the above series to $B^{n+1} = B * B^n + [B, B^n]_+$ and $B^{(n+1)} = B^{(n)} * B + [B^{(n)}, B]_+$.

It was shown in [1] that finite left nilpotent skew braces of nilpotent type can be characterized as skew braces of nilpotent type with nilpotent multiplicative group. Moreover, this is equivalent to the skew brace being a direct product of skew braces of prime power order. This motivates the first problem proposed in this talk.

Problem 1. *Construct skew braces of prime power order.*

Currently, left braces of order dividing p^4 are constructed. Moreover, if $p = 2$, then the classification is finished up to size 64. A first order of attack could be the extension of the classification by Puljic [3] to skew braces. Secondly, one may try to relate nilpotency classes of the additive and multiplicative group with the left nilpotency class. Moreover, the above problem is relevant for all finite skew braces of nilpotent type, as these are iterated factorizations of skew braces of prime power order by the additive Sylow subgroups, which could aid the classification of skew braces of small order. However, in general, this factorization does not hold. As the importance of Sylow subgroups in the theory of finite groups can not be overestimated, the following open problem is crucial. This is a specialization of the same question for solvable groups.

Problem 2. *Which p-groups are multiplicative groups of braces?*

Due to the classification, this has been answered positively for all groups of order dividing p^4, but known to be false. Bachiller showed that not all p-groups are multiplicative groups of braces. His counterexample is based on a classification of pre-Lie algebras, which have recently been connected to braces [4]. As his counterexample is of order p^{10} and $p > 12$, several subproblems arise. Do counterexamples exist for any p? What is the minimal counterexample? Can one give a characterization or necessary condition for a p-group to be the multiplicative group of a brace?

Right nilpotency of skew braces is deeply linked with the behavior of their associated solutions. Indeed, a skew brace of nilpotent type is right nilpotent if and only if its solution is of finite multipermutation level. Moreover, for every solution of finite multipermutation level, the brace structure on $G(X, r)$ is right nilpotent of nilpotent type. Right nilpotency is more well-behaved regarding the ideal structure of the skew brace as it is controlled by both a lower (as above) and upper central series, the so-called s-series. It is known that a skew brace that can be factorized into two right nilpotent skew braces need not be right nilpotent, however, the question remains open if both are assumed to be ideals of the larger brace, which may be a first avenue of investigation for the following problem.

Problem 3. *Explore which factorizations of skew braces (of nilpotent type) produce right nilpotent skew braces.*

This problem is also of interest from the solution point of view. Indeed, factorizations of skew braces are linked with decompositions of set-theoretic solutions.

An exciting class of skew braces to explore the above problems are the centrally nilpotent skew braces. A skew brace $(B, +, \circ)$ is said to be centrally nilpotent, if the chain of ideals $\Gamma_{n+1}(B) = B * \Gamma_n(B) + \Gamma_n(B) + [B, \Gamma_n(B)]_+$ ends in $\{0\}$. This notion is, for finite skew braces, equivalent to being both left and right nilpotent, which makes them the ideal candidates to study the above problems as test cases. Moreover, it was shown [2] that their behavior closely mimics that of nilpotent groups. Hence, their independent study can be started using inspiration coming from finite p-groups. In particular, the following problem comes to mind.

Problem 4. *Study centrally nilpotent skew braces of maximal nilpotency class.*

It should be, of course, duly noted which nilpotency class to consider. The one derived from the above series seems a natural choice, an equivalent version of this series as $\Gamma_{[n+1]}(B) = \sum_{i=1}^{n} \Gamma_{[i]} * \Gamma_{[n+1-i]} + [\Gamma_{[i]}, \Gamma_{[n+1-i]}]_+$ has proven useful in calculations. Note that, in contrast to group theory, non-trivial braces of order p^2 exist!

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On solutions of the set-theoretic Yang–Baxter equation with a parameter from a skew brace

Bernard Rybołowicz

On solutions of the set-theoretic Yang-Baxter equation with a parameter from a skew brace. To every skew brace, we can associate a solution of the set-theoretic Yang-Baxter equation. Moreover, it is involutive if a skew brace is a brace. Recently, we extended this result and showed that by considering some natural deformations of the lambda map, we associate with one skew brace many solutions. Moreover, we can assign non-involutive solutions to two-sided braces for every element different from identity. During this talk, I will analyze and present how those deformations of the lambda map naturally appear when one considers quotients of skew braces.

Classification of unitary R-matrices by braid group characters

Gandalf Lechner
(joint work with Tasarla Deadman)

1. The classification programme for unitary R-matrices

The structure of the set of the solutions to the Yang-Baxter Equation (YBE) remains mysterious despite its connections to many fields in mathematics and its manifold applications. Classification results on all solutions, or some large subset of solutions, seem hard to come by unless one considers them up to a suitable equivalence relation.

Since applications in quantum physics often require *unitary* solutions, we consider here pairs (R, V) where V is a finite-dimensional Hilbert space and $R : V \otimes V \to V \otimes V$ a unitary endomorphism satisfying the YBE, namely

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$

We denote the collection of all such "R-matrices" by $\mathcal{R}(V)$ and propose to consider it up to a natural equivalence relation related to braid group characters.

Recall that any $R \in \mathcal{R}(V)$ defines unitary representations $\rho_R^{(n)}$ of the braid groups B_n, $n \in \mathbb{N}$, on the tensor powers $V^{\otimes n}$, by mapping the i-th Artin generator σ_i to $R_i = \mathrm{id}_V^{\otimes(i-1)} \otimes R \otimes \mathrm{id}_V^{\otimes(n-i-1)}$.

Definition 1. [2] Two unitary R-matrices R, S are called equivalent, written $R \sim S$, if for all $n \in \mathbb{N}$ their B_n-representations are unitarily equivalent.

An equivalent way of expressing $R \sim S$ is to say that R and S are defined on Hilbert spaces of the same dimension and define the same normalized character $\chi_R = \chi_S$ of the infinite braid group B_∞ [3].

The above equivalence leads to large equivalence classes and hence to the much more accessible (but still challenging) problem of classifying $\mathcal{R}(V)$ up to $\sim$. In the special case of *involutive* (meaning $R^2 = 1$) unitary solutions, a complete classification has been achieved. The equivalence classes of involutive solutions are in explicit bijection with pairs of Young diagrams with $\dim V$ cells in total [2]. Also all unitary Temperley-Lieb solutions of the YBE have been classified, here the classifying data are the spectrum of R and the value a spectral projection takes in a Markov trace [1].

In general, the classification problem for $\mathcal{R}(V)/\!\sim$ is open. This is partly due to the fact that only few accessible invariants for $\sim$ are known, namely a) the characteristic polynomial of R, capturing all character values of the form $\chi_R(\sigma_1^k)$, $k \in \mathbb{Z}$ and b) the characteristic polynomial of the partial trace[1] of R, capturing all character values of cycle form, namely

$$\chi_R(\sigma_1\sigma_2\cdots\sigma_{n-1}) = (\dim V)^{-n} \cdot \mathrm{Tr}_V((\mathrm{ptr}R)^{n-1}).$$

[1]Recall that the partial trace is the map $\mathrm{ptr} : \mathrm{End}(V \otimes V) \to \mathrm{End}(V)$, $\mathrm{ptr} = \mathrm{Tr}_V \otimes \mathrm{id}_{\mathrm{End}(V)}$.

2. Racks, Quandles and Set-Theoretic Solutions to the YBE

Let X be a finite set and $r : X \times X \to X \times X$ a *set-theoretic* bijective, non-degnerate solution ("set-theoretic solution", for short) to the YBE. Then its linearisation, namely the Hilbert space $V = \mathrm{span}(X)$ spanned by X as an orthonormal basis, and the linear extension R of r, yields a unitary solution to the linear YBE.

The classification of the subset $\mathcal{R}_{\mathrm{set}}(V) \subset \mathcal{R}(V)$ coming from non-degenerate set-theoretic solutions up to $\sim$ is currently being investigated in the PhD project of the first named author. We here report on some observations and results in this project.

Recall that invertible maps $r : X^2 \to X^2$ of the special form

$$r(x, y) = (\lambda_x(y), x), \qquad x, y \in X,$$

with $\lambda_x \in S_X$ bijections, solve the YBE if and only if $x \triangleright y := \lambda_x(y)$ is a rack structure on X. We will refer to such solutions as *rack solutions* for short.

Proposition 2. [4, 5] *Any $R \in \mathcal{R}_{\mathrm{set}}(V)$ is equivalent to a rack solution.*

This result tells us that up to equivalence, we may restrict attention from $\mathcal{R}_{\mathrm{set}}(V)$ to rack solutions. As the only involutive rack solution is the flip $r(x, y) = (y, x)$, we also have:

Corollary 3. *All non-degenerate set-theoretic involutive solutions on X are equivalent.*

Comparing with the results of [2], we see that this is due to the assumption of non-degeneracy – in fact, many of the normal form R-matrices in [2] are set-theoretic (but degenerate).

Isomorphic racks $(X, \triangleright) \cong (X', \triangleright')$ give rise to equivalent R-matrices, since the isomorphism produces a permutation matrix intertwiner for the R-matrices. Given the much larger flexibility of equivalence $\sim$ in comparison to rack isomorphism, it is not expected that equivalence of two rack solutions implies isomorphism of their underlying racks. Nonetheless, examples of non-isomorphic equivalent racks are non-trivial to find. The main question in this context is to understand how much of the structure of a rack $(X, \triangleright)$ is encoded in the B_∞-character of its associated rack solution r.

The two invariants mentioned in the first section translate as follows. The characteristic polynomial of the linearisation R of r exactly encodes the conjugacy class of r in S_{X^2}. It is currently not clear to us which conjugacy classes in S_{X^2} are realized by rack structures on X.

To translate the partial trace of R, we first recall that the *square map* of a rack $(X, \triangleright)$ is the map $\mathrm{Sq} : X \to X$, $\mathrm{Sq}(x) := x \triangleright x$. The square map of a rack is always a bijection of X. Any $\pi \in S_X$ arises as the square map of some rack, for instance from the permutation solution example $x \triangleright y = \pi(y)$, $x, y \in X$.

Lemma 4. *The partial trace of the linearisation of a rack solution r is the linearisation of the square map of the underlying rack.*

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$N(d)$	1	0	1	1	3	2	5	3	8	1	9	10	11	0
$E_{23}(d)$	–	–	–	–	1	1	1	0	4	–	6	4	2	–

FIGURE 1. $d = |X|$, $N(d)$: number of isomorphism classes of indecomposable quandles of size d, $E_{23}(d)$: number of pairs of classes with equivalent B_2- and B_3-actions on X^2 and X^3.

Hence the characteristic polynomial of the partial trace of the linearisation of a rack solution exactly encodes the conjugacy class of the square map in S_X, and all conjugacy classes are realized.

However, these two conjugacy classes (of r in S_{X^2} and of Sq in S_X) appear to be insufficient to determine the equivalence class of r up to $\sim$. For instance, when restricting to quandles (i.e. racks with $x \triangleright x = x$ for all $x \in X$) the square map is always trivial, but many non-equivalent quandles ought to exist.

To shed some light on this question, we did a numerical investigation of small indecomposable quandles provided in the gap package rig [6]. Searching for quandles with equivalent B_2- and B_3-representations (with set-theoretic intertwiners), we found only few pairs of candidates for equivalent solutions. For example, there exists a single pair out of the 10 possible ones for $|X| = 7$, and no pair at all out of the three possible ones for $|X| = 8$ (see Figure 1). It is currently not yet decided whether these pairs have equivalent B_n-representations for arbitrary numbers of strands n.

We expect to find examples of non-isomorphic but equivalent quandles within these data, for instance by considering Alexander quandles.

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Pre-Lie algebras and classification of braces of cardinality p^4

Dora Puljic

The connection between pre-Lie algebras over R and left nilpotent R-braces was described by W. Rump, along with a method for constructing a brace from a pre-Lie algebra. This method can be applied to pre-Lie algebras over finite fields - it has been shown that nilpotent pre-Lie rings of cardinality p^n correspond to strongly nilpotent braces of the same cardinality, for sufficiently large primes p. These braces are explicitly obtained from the corresponding pre-Lie rings by the construction of the group of flows. Hence this connection can be leveraged for the classification of strongly nilpotent braces of cardinality p^4, thereby finishing the classification of braces of cardinality p^4 as not right nilpotent braces of cardinality p^4 have been classified. In this talk, we will describe the method for constructing braces from pre-Lie algebras and present classification results.

Non-abelian simple groups which can occur as the additive group of a skew brace with solvable multiplicative group

Cindy (Sin Yi) Tsang

Let $B = (B, +, \circ)$ be a finite skew brace. It is natural to ask how its multiplicative group B° and additive group B^+, in terms of their group-theoretic properties say, are related. For example, it is known that:

(1) If B° is cyclic, then B^+ has to be supersolvable. [2]
(2) If B° is abelian, then B^+ has to be metabelian. [1]
(3) If B° is nilpotent, then B^+ has to be solvable. [2]

There is also a conjecture, due to Byott, that:

(4) If B° is insolvable, then B^+ has to be insolvable.

This seems extremely difficult to prove. Instead, let us consider its converse, which is known to be false by [1]. In other words, there exist finite skew braces $B = (B, +, \circ)$ with B° solvable but B^+ insolvable. In this talk, we discuss the following theorem from [3], which completely characterizes the non-abelian simple groups that can occur as B^+ for such a skew brace.

Theorem 1. *Let A be a finite non-abelian simple group. There exists a skew brace $B = (B, +, \circ)$ with B° solvable and $B^+ \simeq A$ if and only if A is isomorphic to one of the following:*

(a) $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(8)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$, M_{11}*;*
(b) $\mathrm{PSL}_2(q)$ *with $q \neq 2, 3$ a prime power.*

The proof uses the theory of factorization of groups and a method to construct regular subgroups in the holomorph via fixed-point free pairs of homomorphisms.

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Skew brace Schur covers with a view towards representations

Thomas Letourmy
(joint work with Leandro Vendramin)

In 1911, Issai Schur introduced Schur covers of groups as a tool to classify projective representations. A Schur cover of a group is a certain universal central extension. This object's strength is based on the *lifting property*, i.e. given a cover E of a group G, one can "lift" each *projective representation* of G to a *representation* of E. Hence it reduces the problem of classifying projective representations to classifying representations of a cover. We recently introduced notions of Schur covers and projective representations for skew braces. We showed that skew braces Schur covers have a lifting property. In general, skew braces can have multiple Schur covers. However, they are unique up to a weaker equivalence relation: isoclinism. After briefly recalling the classical theory, I will present our main results on skew braces, Schur covers, and isoclinisms.

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Hochchild cohomology for trusses

Tomasz Brzeziński
(joint work with James Papworth)

In an attempt to quatise bi-Hamiltonian systems, Cariñena, Grabowski and Marmo [1] introduced *Nijenhuis operators* and *Nijenhuis products* on associative algebras. Given an algebra A over a field k, and a linear operator N on A a *Nijenhuis product* on A is defined, for all $a, b \in A$,

$$a \circ_N b = N(a)b - N(ab) + aN(b).$$

This product is associative if and only if the *Nijenhuis torsion*

$$T_N : A \to A, \qquad a \mapsto N(a \circ_N b) - N(a)N(b),$$

is a 2-cocycle in the Hochschild cohomology of A (with coefficients in A). The operator N whose torsion is zero is called a *Nijenhuis operator.*

An extension of the Nijenhuis products to the case of affine spaces with bi-affine associative multiplication or associative *afgebras* indicates a natural 2-cocycle condition for the Nijenhuis torsion (now defined relatively to a point in an affine space). Since afgebras are in particular trusses (in the same way that associative algebras are rings), this gives the general form of a Hochschild 2-cocycle for a truss. Taking this as a starting point we describe the full Hochschild cochain complex for trusses. This is now defined relatively to a chosen element of a truss, but the associated cohomology heaps are independent of the choice of this element up to isomorphism. We illustrate this definition with calculation of the cohomology heaps for trusses built on an abelian group A and with products $ab = a$ and $ab = a + b$. In the latter case the cohomology is calculated up to degree 4, and the general form is conjectured.

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Indecomposable set-theoretic solutions of the Yang–Baxter equation of square-free cardinality

Jan Okninski

Indecomposable involutive non-degenerate set-theoretic solutions (X, r) of the Yang–Baxter equation of cardinality $p_1 \cdots p_n$, for different prime numbers $p_1, \ldots, p_n$, are studied. It is proved that they are multipermutation solutions of level $\leq n$. In particular, there is no simple solution of a non-prime square-free cardinality. Indecomposable solutions of cardinality $p_1 \cdots p_n$ that are multipermutation of level n are constructed, for every nonnegative integer n. The talk is based on a recent joint work with Ferran Cedo.

Non-involutive solutions of the set-theoretic YBE and *p*-deformed braidings

Anastasia Doikou

Motivated by recent findings on the derivation of parametric non-involutive solutions of the Yang-Baxter equation we reconstruct the underlying algebraic structures, called near skew braces. Using the notion of the near skew braces we produce new multi-parametric, non-degenerate, non-involutive solutions of the set-theoretic Yang-Baxter equation. These solutions are generalisations of the known ones coming from (skew) braces. Bijective maps associated to the inverse solutions are also constructed. Furthermore, we introduce the generalized notion of p-deformed braidings and we show that every p-braiding is a solution of the braid equation. We also show that certain multi-parametric maps within the near skew braces provide special cases of p-braidings. (Work in collaboration with B. Rybolowicz)

Studying solutions of the Yang–Baxter equation through skew braces

Senne Trappeniers
(joint work with Marco Castelli)

Left braces, and later skew left braces, were introduced as a tool to study and produce solutions of the set-theoretic Yang–Baxter equation. Ever since then, this connection remains one of the driving forces behind research on skew braces and has led to natural notions like the multipermutation level, socle, cycle bases, etc. of a (skew) brace.

In this talk, based on joint work with M. Castelli [1], we introduce new and extend established results on some of the aforementioned notions. Moreover, we show that also the automorphisms and in some cases the endomorphisms of a solution can be studied through its associated skew brace.

It is proved in [1] that indecomposable multipermutation solutions have no proper subsolutions. It would be interesting to study the class of (indecomposable) solutions with no proper subsolutions. In particular, a first question is whether there exist non-multipermutation solutions which have this property. As the image of a homomorphism of solutions is a subsolution, every endomorphism of a finite solution with no proper subsolutions is necessarily an automorphism. In particular, this means that every endomorphism of a finite indecomposable multipermutation solution is an automorphism. Once again the natural question arises whether there exist finite (indecomposable) solutions with this property, that are not multipermutation solutions and if so, how one could characterise such solutions.

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Semi-affine structures on groups as a tool for studying semi-braces

Paola Stefanelli

A *(left) semi-brace* [1, 3] is a triple $(B, +, \circ)$ with B a set, $+$ and $\circ$ binary operations on B such that $(B, +)$ is a semigroup, $(B, \circ)$ is a group, and $a \circ (b + c) = a \circ b + a \circ (a^- + c)$, for all $a, b, c \in B$, where a^- is the inverse of a with respect to $\circ$. Moreover, if $(B, +)$ is a left cancellative semigroup, then B is said to be a *left cancellative semi-brace*. If $(B, +)$ is a group, then B is a *skew brace* [2]; if $(B, +)$ is an abelian group, then B is a *brace* [4].

In [5], Rump proved the existence of a correspondence between affine structures on a group G and braces with G as a multiplicative structure. If $G = (B, \circ)$ is a group, a map $\sigma : B \to \mathrm{Sym}_B$ is an *affine structure on* G if it is an anti-homomorphism from G to the permutation group on B such that, set $\sigma_a = \sigma(a)$, for every $a \in B$, it holds $a \circ \sigma_a(b) = b \circ \sigma_b(a)$, for all $a, b \in B$.

This correspondence can be extended to skew braces and, more in general, to semi-braces, by using *semi-affine structures on groups* introduced in [6]. If $G = (B, \circ)$ is a group, a semigroup anti-homomorphism $\sigma : B \to B^B$ from G to the set of all maps from B into itself is a *semi-affine structure on* G if it holds the identity

$$\forall\ a, b, c \in B \quad \sigma_a\left(b \circ \sigma_b\left(c\right)\right) = \sigma_a\left(b\right) \circ \sigma_{\sigma_a(b)}\sigma_a\left(c\right).$$

If $\sigma\left(B\right) \subseteq \mathrm{Sym}_B$, we say that σ is a *cancellative semi-affine structure*. In such a case, denoted by 0 the identity of G, if in addition $\sigma_a\left(0\right) = 0$, for every $a \in B$, we say that σ is a *groupal semi-affine structure*. We have that cancellative semi-affine structures correspond to cancellative semi-braces and groupal semi-affine structures correspond to skew braces. In particular, Rump's affine structures are special groupal semi-affine structures. Furthermore, there exists a categorical equivalence between semi-affine structures and semi-braces.

Theorem 1. *If $G = (B, \circ)$ is a group and $\varphi, \omega : B \to B^B$ are semi-affine structures on G such that the equalities $\varphi_a\omega_b = \omega_b\varphi_a$ and*

$$(c) \qquad \varphi_{b\circ\omega_a(b)^-} = \omega_{\varphi_a\omega_a(b)\circ\omega_a(b)^-}$$

are satisfied, for all $a, b \in B$, then the map $\sigma : B \to B^B$ given by $\sigma_a := \varphi_a\omega_a$, for every $a \in B$, is a semi-affine structure on G.

Assuming that φ, ω are cancellative, we have that (c) can be replaced by the identity

$$(c') \qquad \forall\ a, b \in B \quad \varphi_{\omega_{a^-}(b)\circ b^-} = \omega_{\varphi_a(b)\circ b^-}.$$

In addition, if conversely, σ is a cancellative semi-affine structure on G, set $\varphi_a := \sigma_a$ and $\omega_a := \mathrm{id}_B$, for every $a \in B$, then φ and ω are semi-affine structures that trivially satisfy the assumptions of the previous theorem.

Problem 2. *Let G be a group. Find all* "trivially decomposable" *cancellative semi-affine structures on G to construct all cancellative semi-affine structures on G itself.*

Problem 3. *Extend the description of cancellative semi-affine structures on groups to the general case.*

Problem 4. *Find systematic constructions of skew braces that are not bi-skew braces through groupal semi-affine structures on groups.*

Problem 5. *Construct two-sided skew braces through groupal semi-affine structures on groups.*

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Hopf–Galois theory via skew braces

Lorenzo Stefanello
(joint work with Senne Trappeniers)

The study of Hopf–Galois structures, valuable tools for dealing with classical problems in arithmetic or field theory in a more general context, has been enriched by an unexpected connection with skew braces, algebraic structures whose role in several areas of mathematics has been shown to be relevant. Although not bijective, this connection has motivated several quantitative results developed in recent years; however, a look at the literature seems to suggest that structural results have been few and mostly partial.

The goal of this talk is to present a new version of this connection, introduced in a joint work with S. Trappeniers [1], as stated in the following theorem.

Theorem 1. *Let L/K be a finite Galois extension with Galois group $(G,\circ)$. Then Hopf–Galois structures on L/K are in bijective correspondence with operations $\cdot$ such that $(G,\cdot,\circ)$ is a skew brace. Explicitly, given such an operation $\cdot$, the corresponding Hopf–Galois structure consists of the K-Hopf algebra $L[G,\cdot]^{(G,\circ)}$, where $(G,\circ)$ acts on L via Galois action and on $(G,\cdot)$ via the map λ of the skew brace $(G,\cdot,\circ)$, with action on L given as follows:*

$$\left(\sum_{\sigma\in G}\ell_\sigma\sigma\right)\star x=\sum_{\sigma\in G}\ell_\sigma\sigma(x).$$

As a consequence of this new perspective, we can not only describe Hopf–Galois structures explicitly and bijectively via skew braces, but also relate structural properties of both settings; several notions in skew brace theory, like basic substructures, (semi)direct products, and nilpotency have their counterpart in Hopf–Galois theory. Accordingly, known and new results can be explained and derived. For example, a long-standing problem of Hopf–Galois theory, the study of the surjectivity of the Hopf–Galois correspondence, can be translated into a very natural problem in skew brace theory, allowing us to construct several new classes of Hopf–Galois structures for which a certain rare but desirable property holds.

The final part of the talk is devoted to questions. A natural research direction to follow is to try to push the new connection between Hopf–Galois structures and skew braces further by finding new results and concepts that translate from one context to another. For example, notions of isomorphisms, extensions, and representations seem to be reasonable starting points for this inquiry. Furthermore, as various studied notions of skew braces have an impact in Hopf–Galois theory due to the new connection, it would be interesting, given a finite group $(G,\circ)$, to develop efficient ways to obtain large classes of skew braces of the form $(G, \cdot, \circ)$ satisfying some prescribed properties.

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Cabling of involutive YBE solutions, with applications to decomposability

Victoria Lebed

(joint work with Leandro Vendramin)

In 2005, Rump gave an unexpected proof of the 1996 conjecture of Gateva-Ivanova, stipulating that an involutive YBE solution X is decomposable whenever it is square-free, that is, its diagonal map T is the identity. In 2021, Camp–More and Sastriques extended this decomposability result to the case when the order of T is coprime to the size of X. In this talk, we will present several new cycle decomposition patterns of T implying the decomposability of X. At the heart of our machinery for producing decomposability tests lie certain cabling operations on solutions, which become surprisingly elementary in the skew brace framework. Several other applications of cabling will be discussed, including structural results on skew braces, and a conceptual interpretation of the Dehornoy class, a combinatorial invariant naturally appearing in the Garside-theoretic approach to involutive solutions.

Constructing some simple skew braces

Nigel Byott

Constructions for finite simple braces have been given by Bachiller [1] and by Cedó, Jespers and Okniński [2] via matched products. Little seems to be known, however, about simple skew braces which are not braces (i.e. which have non-abelian additive group). One can give examples where the multiplicative group is a finite simple group (and the additive group need not be simple). Indeed Tsang [3] has determined all such cases where the additive group is soluble. (See her talk at this conference.) Recently Vendramin showed by a computer search that the smallest simple skew braces $(B, +, \circ)$ which are not braces have order 12. There are two such braces, both with $(B, +) \cong A_4$ and $(B, \circ) \cong C_3 \rtimes C_4$.

In this talk, I will present a fairly explicit construction for a simple skew brace $(B, +, \circ)$ of size $p^p q$, where p, q are any prime numbers such that q divides $(p^p - 1)/(p - 1)$. Both the additive group $(B, +)$ and multiplicative group $(B, \circ)$ of B are soluble but not abelian. Indeed $(B, +) \cong V \rtimes C_q$, where V is an elementary abelian group of order p^p, and $(B, \circ) \cong C_q \rtimes P$ where P is a certain group of order p^p and exponent p^2 with nilpotency class $p - 1$. The group $(B, \circ)$ is constructed explicitly as a regular subgroup of the holomorph of $(B, +)$. The resulting skew brace is indeed simple since the only proper non-trivial normal subgroup of $(B, +)$ has order p^p but $(B, \circ)$ has no normal subgroup of this order. Moreover B is not isomorphic to its opposite skew brace, so these new simple skew braces occur in pairs. The smallest case $p = 2$, $q = 3$ gives simple skew braces of size 12 as found by Vendramin.

Some open questions regarding this construction are:

(1) If B' is a simple skew brace with $(B', +) \cong (B, +)$ and $(B', \circ) \cong (B, \circ)$, then must B' be isomorphic to either B or its opposite?
(2) If B' is any simple skew brace of size $p^p q$ (for primes p and q as above), then must B' be isomorphic to either B or its opposite?

References

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