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Real Algebraic Geometry with a View toward Koopman Operator Methods

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Abstract. This workshop was dedicated to the newest developments in real algebraic geometry and their interaction with convex optimization and operator theory. A particular effort was invested in exploring the interrelations with the Koopman operator methods in dynamical systems and their applications. The presence of researchers from different scientific communities enabled an interesting dialogue leading to new exciting and promising synergies.

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Introduction by the Organizers

In this workshop we brought together experts, as well as young researchers, working on the following themes: real algebraic geometry, polynomial optimization, moment problems, noncommutative real algebra with connections to quantum physics, and Koopman operator methods.

To stimulate discussions and exchanges during the workshop we scheduled 6 senior and junior speakers giving one hour tutorial and introductory lectures on Monday and Tuesday. These survey-expository talks were (in order of appearence in the schedule):

Rainer Sinn: an introduction to real algebraic geometry, sums of squares (Positivstellensätze), and moment problems.

Alexandre Mauroy: an introduction to Koopman operator methods from numerical and applied perspective.

Etienne de Klerk: an introduction to semidefinite programming and polynomial optimization (moment–sum of squares (SOS) relaxations).
Corbinian Schlosser: on the connection between Koopman operator methods and moment–SOS relaxations.
Bernard Mourrain: on the complexity of Positivstellensätze.
Rainer Nagel: an introduction to Koopman operator methods from "pure" operator theory perspective.

The survey-expository talks were the starting point for the regular research talks of 30 to 40 minutes (20 minutes for the junior speakers). To encourage the dialogue between the various areas we decided to keep a mixed daily thematic structure in the schedule. We also made a concerted effort of giving priority to young speakers. We have profited from the new technical arrangement to have an online session on Wednesday evening, with contributions from 2 online participants, Bruce Reznick and Mehdi Ghasemi, who are both based in North America and could not attend the meeting in person. The talk of Mehdi Ghasemi in the rather late evening hour provided an interesting and unusual counterpoint as he described the applications of moment–SOS based optimization to community safety in the Canadian provinces of Saskatchewan and Alberta.

Some highlights of the workshop:

- In a breakthrough development described in the survey-expository talk of Bernard Mourrain, there emerged a completely new connection between the real algebra side of real algebraic geometry (complexity of Positivstellensätze) and the geometric side (Lojasiewicz inequality and singularities of the boundary of a basic closed semialgebraic set).
- Positive kernels made an important appearence in the survey-expository talk of Bernard Mourrain and played a key role in several talks on Koopman operator methods. This generated a strong interest on part of other operator theorists at the meeting, leading possibly to major new insights in several areas.
- A sum of squares of polynomials is certainly positive. Other ways of certifying positivity have emerged over the years as (far reaching) consequences of the arithmetic mean – geometric mean inequalty, leading to classes such as SAGE / SONC polynomials. This was a topic of several talks (and numerous discussions), bringing these positivity certificates to the level of maturity where they can become new standard tools in real algebraic geometry and convex optimization.

We will give now a summary of the main topics discussed at the workshop by describing the regular research talks (of course, some of the talks were clearly touching more than one area in perfect accordance with the synergetic spirit of this meeting).

Real Algebraic Geometry

The talk of Lorenzo Baldi continued and complemented the survey-expository talk of Bernard Mourrain by discussing lower bounds for the complexity of Postivstellensätze. The talk of Sarah Hess returned to one of the central topics of real algebraic geometry — the gap between the cone of positive polynomials and the cone of sums of squares — by examining a natural sequence of intermediate cones; it both sharpened a classical Hilbert's 1888 theorem and related to recent work of Blekherman and his collaborators. The talk of Konrad Schmuedgen discussed Positivstellensätze for semirings, rather than the more customary quadratic modules or preorderings, of a unital commutative algebra. Alexander Taveira Blomenhofer presented in his talk a semidefinite algorithm for representing given forms as sums of powers of forms (a variant of Waring's problem). The talk of Cynthia Vinzant discussed the principal minors of determinantal polynomials which play a key role in certifying hyperbolicity (much like sums of squares certify positivity). Ngoc Hoang Anh Mai discussed positivity certificates in the context of polynomial optimization on non-compact semialgebraic sets. The talk of David Sawall returned to hyperbolic polynomials (in their non-homogeneous version as real zero polynomials) and gave a counterexample, using matroid techniques pioneered by Branden about a decade ago, to a recent conjecture that two hyperbolic polynomials in overlapping sets of variables can be always viewed as projections of a single hyperbolic polynomial. The concluding talk of the workshop, by Greg Blekherman, discussed polynomials in non-normalized traces of powers of symmetric matrices and showed that their positivity is undecidable; it both related to a large ongoing effort in studying positivity in the presence of group invariance (in this case, the full symmetric group) and contrasted with the results for the normalized trace that were presented in the talks of Jurij Volcic and Igor Klep, see below.

Polynomial Optimization with SAGE / SONC Polynomials

The talk of Thorsten Theobald discussed relative entropy programming and its applications to SAGE (sums of arithmetic geometric exponentials) signomials (exponential sums, which are a generalization of polynomials on the positive orthant), starting with the unconstrained case of global positivity and proceeding to the very recent results in the constrained case of positivity on a polyhedral set; he also described the relation with circuit signomials. This was naturally followed by the talk of Mareike Dressler that described some corresponding Positivstellensätze, their complexity, and open problems. Moritz Schick discussed the cone of SOS cone vs. the SAGE cone and especially the sum of the two. The talk of Bruce Reznick both returned to the original use of the arithmetic mean – geometric mean inequality for proving that certain polynomials such as the Motzkin polynomial are positive, and discussed the SOS-ness of odd powers of positive polynomials that are not SOS.

Moment Problems

Tobias Kuna discussed a (full) moment problem for algebras generated by a nuclear space using the projective limit technology that has been a recent breakthrough in the area of infinite dimensional moment problems; the results are both mathematically compelling and cover a wide range of potential applications. In a related talk, Maria Infusino has shown that the various usual conditions ensuring the existence of a unique representing measure with a compact support: positivity on an Archimedean quadratic module, Carleman condition, continuity with respect to a submultiplicative seminorm, are in fact necessary and sufficient; the resulting description of the support is new even in the finite dimensional case. Philipp di Dio discussed the behavior of positive polynomials and sums of squares, and dually of moment functionals, under the heat semigroup. The talk of Simone Naldi dealt with the computation of certificates for the truncated moment problem.

Noncommutative Real Algebra with Connections to Quantum Physics

Free noncommtative positivity has been a major development at the crossroads of real algebraic geometry and operator algebras that has been featured extensively at the workshops in 2014, 2017, and 2020. Whereas the previous works dealt with positivity of polynomials in free noncommuting variables when evaluated on all matrices (or more general selfadjoint operator algebras), the talk of Jurij Volcic — motivated by rather concrete applications in quantum physics — incorporated a state (a positive linear functional) into the framework. By contrast, the talk of Igor Klep considered polynomial optimization via moment–SOS relaxations and the corresponding positivity certificates for noncommutative polynomials involving a normalized trace rather than a general state.

Koopman Operator Methods

Following the survey-expository talk of Alexandre Mauroy, Benoit Bonnet-Weill presented in his talk a set-valued generalisations of Koopman operators in the spirit of differential inclusions that play a key role in modern control theory. The talk of Oliver Junge discussed an entropic regularization of transfer operators. In a very different setting, regularization was used in the talk of Tobias Sutter for finite dimensional approximation of infinite dimensional linear programs. Positive kernels, which as already mentioned made an important appearence in the survey-expository talk of Bernard Mourrain, and the corresponding reproducing kernel Hilbert spaces, played a key role in the talk of Dimitris Giannakis on the embedding of classical dynamics in a quantum computer. Gary Froyland introduced dynamical systems and transfer operators in elementary settings, including the generalization to the case of time-dependent dynamics. Finally, the talk of Patrick Hermle on a Halmos–von Neumann theorem for action of general groups provided a direct continuation of the survey-expository talk of Rainer Nagel.

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Workshop: Real Algebraic Geometry with a View toward Koopman Operator Methods

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Abstracts

Real Algebraic Geometry and Positivstellensätze

Rainer Sinn

This talk gives an introduction to real algebraic geometry. The focus is on basic notions culminating with various Positivstellensätze. The starting point is a *basic closed semi-algebraic set*

$$K = \{x \in \mathbb{R}^n \colon g_1(x) \geq 0, \ldots, g_r(x) \geq 0\},$$

where $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$ are multivariate polynomials with real coefficients. The goal of a Positivstellensatz is to describe the set

$$\mathcal{P}(K) = \{f \in \mathbb{R}[x_1, \ldots, x_n] \colon f(x) \geq 0 \text{ for all } x \in K\}$$

in terms of obviously nonnegative functions on K. This set $\mathcal{P}(K)$ is known as the *positive cone* or *saturated preorder* of K. It is a convex cone in the real vector space $\mathbb{R}[x_1, \ldots, x_n]$. The main convex cone of obviously nonnegative functions on K for us here is the *preorder* generated by the inequalities g_i defining K:

$$\mathrm{PO}(g_1, \ldots, g_r) = \left\{ \sum_{\alpha \in \{0,1\}^r} \sigma_\alpha g^\alpha \colon \sigma_\alpha \text{ is a sum of squares} \right\}.$$

It is built from the products $g^\alpha = g_1^{\alpha_1} \cdot \ldots \cdot g_r^{\alpha_r}$ of the inequalities, which are clearly nonnegative on K. Those products are then used to weight sums of squares of polynomials σ_α, which are trivially nonnegative on all of $\mathbb{R}^n$. A Positivstellensatz aims to compare such finitely generated preorders to the saturated preorder of K. Here is one central version.

Theorem (Schmüdgen)**.** *Assume that the basic closed semi-algebraic set K is compact. Then every polynomial f that is positive on K (meaning $f(x) > 0$ for all $x \in K$) lies in $PO(g_1, \ldots, g_r)$.*

This theorem uses only geometric/topological assumptions (namely the compactness of K) and holds for any set of inequalities describing K. Other versions, like Putinar's Positivstellensatz, need algebraic assumptions.

Theorem (Putinar)**.** *Suppose that the basic closed semi-algebraic set K is closed and that there exist sums of squares $s_0, s_1, \ldots, s_r$ such that the superlevel set*

$$\{s_0 + s_1 g_1 + \ldots + s_r g_r \geq 0\}$$

is compact. Then for every polynomial f that is positive on K there are sums of squares $\sigma_0, \sigma_1, \ldots, \sigma_r$ such that

$$f = \sigma_0 + \sigma_1 g_1 + \ldots + \sigma_r g_r.$$

The assumption that this superlevel set is compact is called the *archimedean property* of the *quadratic module* generated by the inequalities. This assumption is equivalent to the existence of a natural number N such that there is an identity $N - \sum_{i=1}^{n} x_i^2 = \sigma_0 + \sigma_1 g_1 + \ldots + \sigma_r g_r$ certifying the boundedness of K explicitly.

There are many other Positivstellensätze, in particular for more special sets (like Handelman's Positivstellensatz for polytopes or Pólya's Positivstellensatz for the orthant) or more general rings than the polynomial ring (like coordinate rings of real affine varieties or other real rings).

There is a close connection to *moment problems* in functional analysis. Given a linear functional $L\colon \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}$, the K-moment problem asks if there exists a positive Borel measure μ supported on K such that $L(f) = \int f \mathrm{d}\mu$ for any polynomial f. Clearly, the condition $L(f) \geq 0$ for all nonnegative polynomials $f \in \mathcal{P}(K)$ is necessary for the existence of such a measure. Haviland's Theorem is the converse: this condition is also sufficient.

The K-moment problem is said to be *finitely solvable* if there exists a finitely generated preorder $\mathrm{PO}(g_1, \ldots, g_r)$ that is dense in the positive cone $\mathcal{P}(K)$ (with respect to the finest locally convex topology on $\mathbb{R}[x_1, \ldots, x_n]$). Haviland's Theorem implies that it suffices to check that our candidate L is nonnegative on the finitely generated preorder in this case.

Elementary arguments show that the classical univariate moment problems follow from Haviland's Theorem by showing that the associated finitely generated preorders are in fact saturated (that is equal to the saturated preorder). These examples are *Stieltjes*'s moment problem for $K = [0, \infty)$: a linear functional $L\colon \mathbb{R}[x] \to \mathbb{R}$ comes from a measure if and only if $L(f^2 + xg^2) \geq 0$ for all polynomials $f, g \in \mathbb{R}[x]$. Here $\mathcal{P}(K) = \mathrm{PO}(x)$. For *Hamburger*'s solution of the moment problem for $K = \mathbb{R}$, we need that every nonnegative univariate polynomial is a sum of squares. This follows, for example, from the Fundamental Theorem of Algebra. Finally, also *Hausdorff*'s moment problem for $K = [0, 1]$ is now elementary: $\mathcal{P}(K) = \mathrm{PO}(x, (1 - x))$ follows by induction on the degree of the polynomials.

In the case that K is unbounded, the moment problem tends to not be finitely solvable (at least in dimension at least 2). One notion that plays a role here is stability of preorders. A preorder $\mathrm{PO}(g_1, \ldots, g_r)$ is *stable* if for every $d \in \mathbb{N}$ the intersection $\mathrm{PO}(g_1, \ldots, g_r) \cap \mathbb{R}[x_1, \ldots, x_n]_d$ with the finite-dimensional subspace of all polynomials of degree at most d is contained in the set

$$\left\{ \sum_{\alpha \in \{0,1\}^r} \sigma_\alpha g^\alpha \colon \sigma_\alpha \text{ is a sum of squares}, \deg(\sigma_\alpha) \leq N \right\}$$

for large enough $N \in \mathbb{N}$. So stability is about the existence of *uniform degree bounds*, bounding the degrees of the sums of squares needed to write a polynomial f in it only in terms of the degree d of f.

The simplest example of a stable preorder is the cone of sums of squares $\mathrm{PO}(1)$ itself: If $f = h_1^2 + \ldots + h_s^2$ for real polynomials $h_1, \ldots, h_s$, then $\deg(h_1) \leq \frac{1}{2} \deg(f)$ so that we can choose $N = d$ in the above definition.

Stability is relevant in the context of the moment problem because a simple topological argument implies that stable preorders are closed in the finest locally convex topology on $\mathbb{R}[x_1, \dots, x_n]$. So if the positive cone of K is not finitely generated and any preorder defining K is stable, then the moment problem is not finitely solvable. This indeed tends to happen in the unbounded case. Here is a technically precise version of this.

Theorem (Scheiderer; Kuhlmann/Marshall)**.** *Let $K \subset \mathbb{R}^n$ contain a full-dimensional cone C so that there is a point $x \in K$ and a convex cone C' with non-empty interior such that $x + v \in C \subset K$ for all $v \in C'$. Then any preorder $PO(g_1, \dots, g_r)$ with $K = \{x \in \mathbb{R}^n \colon g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$ is stable. If $\dim(K) \geq 2$, then the positive cone is not finitely generated (as a preorder).*

Let's consider one very explicit example by Stengle: Let $K = [0, 1]$ be the unit interval. The natural preorder could be $\mathrm{PO}(x(1 - x)) = \mathrm{PO}(x, 1 - x)$, which is equal to $\mathcal{P}(K)$. Instead of this natural choice of inequalities defining K, consider $g = x^3(1-x)^3$. Then $[0, 1]$ is still $\{x \in \mathbb{R} \colon g(x) \geq 0\}$. However, $x \in \mathcal{P}(K)$ is not in $\mathrm{PO}(g)$. To see that, write $x = \sigma_0 + \sigma_1 \cdot g$. Since the right hand side is nonnegative, we must have $\sigma_0(0) = 0$. But since σ_0 is a sum of squares, this implies that the derivative of the right hand side also vanishes at 0, which contradicts the evaluation on the left hand side. However, Schmüdgen's Theorem implies that $x + \epsilon \in \mathrm{PO}(g)$ for all $\epsilon > 0$. This shows that $\mathrm{PO}(g)$ is not stable by a limit argument using the fact that the sum-of-squares for $x + \epsilon$ come from a finite-dimensional vector space.

In this workshop there are several talks about lower and upper bounds for the degrees of the sums of squares in certificates of nonnegativity, particularly in the unstable case. The bounds are then not uniform in the degree but depend on additional information on the particular polynomial, mainly its minimum of K. See the contributions of Lorenzo Baldi and Bernard Mourrain for concrete results.

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Introduction to the Koopman Operator

Alexandre Mauroy

In this tutorial talk, we have introduced the so-called semigroup of Koopman operators (or Koopman semigroup) and its basic properties. In particular, we have investigated the interplay between the spectral properties of the Koopman semigroup and the geometric properties of the underlying dynamical system. Finally, focusing on the semigroup defined in the Hardy space of the polydisc, we have applied the framework to global stability analysis.

Definition and Properties. We consider a nonlinear dynamical system described by a flow map $\varphi : \mathbb{R}^+ \times X \to X$, where X is the state space. The semigroup of Koopman operators $(K^t)_{t\geq 0} : \mathcal{F} \to \mathcal{F}$, is defined by the composition $(K^t)f(\cdot) = f \circ \varphi(t, \cdot)$ for all observable $f : X \to \mathbb{C}$, $f \in \mathcal{F}$ [1]. In well-chosen spaces (e.g. $\mathcal{F} = C^0(X)$ if the flow is continuous and X is compact, $\mathcal{F} = L^2(X, \mu)$ if the flow preserves the measure μ), the Koopman semigroup is strongly continuous, i.e. $\lim_{t\downarrow 0} \|K^t f - f\| = 0$ for all $f \in \mathcal{F}$. In this case, the semigroup possesses an infinitesimal generator $A : \mathcal{D}(A) \to \mathcal{F}$, $Af = \lim_{t\downarrow 0} \frac{K^t f - f}{t}$ which is closed and whose domain $\mathcal{D}(A)$ is dense in $\mathcal{F}$ [3]. If the flow is generated by the dynamics $\dot{x} = F(x)$, then we have $Af = F \cdot \nabla f$ for all $f \in \mathcal{D}(A)$.

Spectral Properties. A Koopman eigenfunction $\phi_\lambda \in \mathcal{F}$ satisfies $A\phi_\lambda = \lambda\phi_\lambda$ (which implies $K^t\phi_\lambda = e^{\lambda t}\phi_\lambda$ for all $t > 0$), and $\lambda \in \sigma_p(A)$ is the corresponding Koopman eigenvalue.

Assumption 1. *The flow is holomorphic and generated by the complex dynamics $\dot{z} = F(z)$, $z \in \mathbb{D}^n$, such that $0 = F(0)$ is a globally stable hyperbolic equilibrium and the Jacobian matrix $DF(0)$ has non-resonant eigenvalues λ_j $(j = 1, \dots, n)$.*

If the flow satisfies Assumption 1 and if $\mathcal{F} = H^2(\mathbb{D}^n)$, i.e. $\mathcal{F}$ is the Hardy space of the polydisc $\mathbb{D}^n$, then $\sigma_p(A) = \{\sum_{j=1}^n \alpha_j\lambda_j\}_{\alpha\in\mathbf{N}^n}$ and the set of eigenfunctions $\{\prod_{j=1}^n \phi_{\lambda_j}^{\alpha_j}\}_{\alpha\in\mathbf{N}^n}$ forms a complete basis in $H^2(\mathbb{D}^n)$. Moreover the Koopman semigroup admits the spectral decomposition

$$K^t f = \sum_{\alpha\in\mathbb{N}^n} \langle f, \phi_\alpha\rangle_{H^2_\Phi}\, \phi_{\lambda_1}^{\alpha_1} \cdots \phi_{\lambda_n}^{\alpha_n}\, e^{(\alpha_1\lambda_1+\cdots+\alpha_n\lambda_n)t}$$

in the modulated Hardy space H^2_Φ, with $\|f\|_{H^2_\Phi} = \|g\|_{H^2}$ and $f = g\circ(\phi_{\lambda_1}, \dots, \phi_{\lambda_n})$, $g \in H^2(\mathbb{D}^n)$ [4]. Note that the inner products $\langle f, \phi_\alpha\rangle_{H^2_\Phi}$ are the so-called Koopman modes.

Interplay between Spectral and Geometric Properties. The level sets of the Koopman eigenfunctions capture relevant geometric properties of the associated dynamics. In particular, we have the following results.

Invariant Partition: Let ϕ_0 be a Koopman eigenfunction associated with the eigenvalue $\lambda = 0$. Then, the family of sets

$$\mathcal{S}_0^r = \{x \in X : \phi_0(x) = r \in \mathbb{R}\}$$

is an invariant partition for the flow, i.e. $\varphi(-t, \mathcal{S}_0^r) = \mathcal{S}_0^r$ for all $t > 0$.

Periodic Partition: Let $\phi_{i\omega}$ be a Koopman eigenfunction associated with the eigenvalue $\lambda = i\omega$, with $\omega \in \mathbb{R}_+^*$. Then, the family of sets

$$\mathcal{S}_{i\omega}^{\theta} = \{x \in X : \angle\phi_{i\omega}(x) = \theta \in [0, 2\pi)\}$$

is a T-periodic partition for the flow (with period $T = 2\pi/\omega$), i.e.

$$\varphi(-t, \mathcal{S}_{i\omega}^{\theta}) = \mathcal{S}_{i\omega}^{\theta-\omega t}$$

for all $t > 0$ and $\varphi(-T, \mathcal{S}_{i\omega}^{\theta}) = \mathcal{S}_{i\omega}^{\theta}$. Such a partition is related to the notion of isochrons in the case of limit cycle dynamics [5].

Aperiodic Partition: Let $\phi_{\sigma+i\omega}$ be a Koopman eigenfunction associated with the eigenvalue $\lambda = \sigma + i\omega$, with $\sigma \in \mathbb{R}_-^*$ and $\omega \in \mathbb{R}_+^*$. Then, the family of sets

$$\mathcal{S}_{\sigma+i\omega}^{\tau} = \{x \in X : |\phi_{\sigma+i\omega}(x)| = e^{\sigma\tau} \in \mathbb{R}_+\}$$

is an aperiodic partition for the flow, i.e. $\varphi(-t, \mathcal{S}_{\sigma+i\omega}^{\tau}) = \mathcal{S}_{\sigma+i\omega}^{\tau-t}$ for all $t > 0$. Such a partition is related to the notion of isostables in the case of dynamics with a hyperbolic equilibrium, when $\lambda = \sigma + i\omega \neq 0$ is the dominant eigenvalue [6].

Stability Properties. There exist direct connections between the stability properties of the Koopman semigroup and the stability properties of the underlying dynamical systems [7]. Moreover, global stability of the dynamics can be directly inferred from the spectral properties of the Koopman semigroup. For instance, it can easily be shown that the intersection of zero level sets of Koopman eigenfunctions associated with eigenvalues λ_i satisfying $\Re\{\lambda_i\} < 0$, i.e.

$$M = \bigcap_{i=1}^{m} \{x \in X | \phi_{\lambda_i}(x) = 0\,, \Re\{\lambda_i\} < 0\},$$

is globally uniformly stable in X. Moreover, a hyperbolic equilibrium $x^* \in X$ is globally uniformly stable in X if and only if there exist n Koopman eigenfunctions $\phi_{\lambda_i} \in C^1(X)$, $i = 1, \ldots, n$, such that $\Re\{\lambda_i\} < 0$ and $\nabla\phi_{\lambda_i}(x^*) \neq 0$ [2].

A Lyapunov function for the flow can also be obtained directly from the Koopman operator framework. Suppose that Assumption 1 holds and let $e_\alpha = z^\alpha$, $\alpha \in \mathbb{N}^n$, be monomials. Then, there exists a sequence $(\epsilon_\alpha)_{\alpha\in\mathbb{N}^n}$ such that the functional

$$\mathcal{V} : \mathcal{D}(\mathcal{V}) \subseteq H^2(\mathbb{D}^n) \to \mathbb{R}_+, \qquad \mathcal{V}(f) = \sum_{\alpha\in\mathbb{N}^n} \epsilon_\alpha |\langle f, e_\alpha\rangle_{H^2}|^2$$

satisfies

$$\frac{d}{dt}\mathcal{V}((K^t)^* f)|_{t=0} < 0 \quad \forall f \in \mathcal{D}(\mathcal{V}) \setminus \mathrm{span}\{\mathbf{1}\},$$

where $(K^t)^*$ is the adjoint of $K(t)$ in $H^2(\mathbb{D}^n)$. Next, consider the evaluation functional $k_z \in H^2(\mathbb{D}^n)$ such that $\langle k_z, f\rangle_{H^2} = f(z)$ and define the domain

$$D = \{z \in \mathbb{D}^n \,|\, k_z \in \mathcal{D}(\mathcal{V})\}.$$

We have that the function

$$V : D \subseteq \mathbb{D}^n \to \mathbb{R}^+, \qquad V(z) = \mathcal{V}(k_z) = \sum_{\alpha\in\mathbb{N}^n} \epsilon_\alpha \left|z^{2\alpha}\right| \quad \forall z \in D$$

is a Lyapunov function for the flow, i.e.

$$V(\varphi^t(z)) < V(z) \qquad \forall z \neq 0.$$

Note that the Lyapunov function can be constructed explicitly. See [8] for more details.

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Relative Entropy Methods in Real Algebraic Geometry

Thorsten Theobald

Relative entropy programs are convex optimization problems which optimize linear functions over affine sections of the relative entropy cone

$$K^n_{\text{rel}} \;=\; \text{cl}\Big\{(x, y, \tau) \in \mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0} \times \mathbb{R}^n \;:\; x_i \ln \frac{x_i}{y_i} \leq \tau_i \;\text{ for } 1 \leq i \leq n\Big\},$$

where cl denotes the topological closure. This class of optimization problems contains as a subclass geometric programming. Building upon the arithmetic-geometric mean inequality (AM/GM inequality), relative entropy programs facilitate to compute nonnegativity certificates of a large class of nonnegative polynomials and of nonnegative signomials. These techniques enrich existing optimization techniques for certifying nonnegativity of polynomials and for the optimization of polynomials, such as semidefinite programming techniques for sum-of-squares certificates in real algebraic geometry.

A *signomial* (also known as *exponential sum* or *exponential polynomial*) is a sum of the form

$$f(x) \;=\; \sum_{\alpha\in\mathcal{T}} c_\alpha \exp(\langle\alpha, x\rangle)$$

with real coefficients c_α and a finite ground support set $\mathcal{T} \subset \mathbb{R}^n$. Here, $\langle \cdot, \cdot \rangle$ is the usual scalar product. When $\mathcal{T} \subset \mathbb{N}^n$, the transformation $x_i = \ln y_i$ gives polynomial functions $y \mapsto \sum_{\alpha \in \mathcal{T}} c_\alpha y^\alpha$ on $\mathbb{R}^n_{>0}$. A signomial f is nonnegative on $\mathbb{R}^n$ if and only if its associated polynomial p is nonnegative on $\mathbb{R}^n_+$.

The following setup connects global nonnegativity certificates of polynomials and of signomials to the AM/GM inequality. An *AGE signomial* ("*arithmetic-geometric exponential*") is a nonnegative signomial with at most one negative coefficient. Finite sums of AGE signomials are nonnegative as well and, for a given ground support $\mathcal{T}$, form the *SAGE cone* $C(\mathcal{T})$ ("sums of arithmetic-geometric exponentials"). For example, the Motzkin signomial is contained in the SAGE cone of its support. SAGE signomials can be expressed in terms of the more elementary *circuit signomials.* Circuit signomials are, in connection with positive exponential monomials, a generating system for the SAGE cone.

While the initial focus of the AM/GM-based optimization of signomials was mostly on unconstrained certificates and unconstrained optimization, the conditional SAGE cone introduced in [2] provides a natural extension to the case of convex constraint sets $X \subset \mathbb{R}^n$. To describe this approach, denote by $\sigma_X(y) = \sup\{y^T x : x \in X\}$ the *support function* of X. Murray, Chandrasekaran and Shah [2] showed that the nonnegativity of a signomial f on X with at most one negative coefficient can be formulated in terms of a relative entropy program involving also the support function σ_X.

A *conditional AGE signomial* is a signomial which has at most one negative coefficient and which is nonnegative on X. For a given ground support $\mathcal{T}$, finite sums of conditional AGE signomials form the *conditional SAGE cone* $C_X(\mathcal{T})$. In joint work with R. Murray and H. Naumann [3], a concept of sublinear circuits has been developed to provide a decomposition of $C_X(\mathcal{T})$ as a finite Minkowski sum of conditional AGE cones $C_X(\mathcal{T}, \lambda)$ induced by the set $\Lambda_X(\mathcal{T})$ of normalized sublinear circuits λ. For an introduction to these techniques from a relative entropy perspective see [4].

Theorem. *For polyhedral X, the conditional SAGE cone decomposes as*

$$C_X(\mathcal{T}) \;=\; \sum_{\lambda \in \Lambda_X(\mathcal{T})} C_X(\mathcal{T}, \lambda) + \sum_{\alpha \in \mathcal{T}} \mathbb{R}_+ \cdot \exp(\langle \alpha, x \rangle).$$

Moreover, the concepts of reduced circuits and reduced sublinear circuits allow to give irredundant decompositions in the unconstrained and the constrained case. In the constrained case, reduced sublinear circuits give the following irredundant decomposition, where $\Lambda^\star_X(\mathcal{T})$ denotes the set of normalized reduced sublinear circuits.

Theorem. *For polyhedral X, we have*

$$C_X(\mathcal{T}) \;=\; \sum_{\lambda \in \Lambda^\star_X(\mathcal{T})} C_X(\mathcal{T}, \lambda) + \sum_{\alpha \in \mathcal{T}} \mathbb{R}_+ \cdot \exp(\langle \alpha, x \rangle)$$

and there does not exist a strict subset Λ *of* $\Lambda_X^\star(\mathcal{T})$ *with*

$$C_X(\mathcal{T}) = \sum_{\lambda\in\Lambda} C_X(\mathcal{T},\lambda) + \sum_{\alpha\in\mathcal{T}} \mathbb{R}_+ \cdot \exp(\langle\alpha, x\rangle).$$

The conditional SAGE cone and the sublinear circuits can also be combined with techniques to exploit symmetries [1].

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Semidefinite and Polynomial Optimization: a Game of Cones

Etienne de Klerk

In this tutorial talk we gave an introduction to the Lasserre hierarchy for polynomial optimization and related problems, from the perspective of conic (linear) optimization. The general form of a conic optimization problem involves two vector spaces over the reals, say E and F, and a nondegenerate bilinear form $\langle\cdot,\cdot\rangle : E\times F \to \mathbb{R}$ (called a *duality* or *pairing*). We equip the spaces E and F with the (locally convex) weak topology of the duality.

The data of the conic optimization problem is a closed, pointed, convex cone $K \subset F$, vectors $a_0, a_1, \dots, a_m \in E$ and a vector $b \in \mathbb{R}^m$, and the problem then takes the form:

$$val = \inf_{x\in K} \left\{ \langle a_0, x\rangle \mid \langle a_i, x\rangle = b_i \quad \forall i \in [m] \right\},$$

where $[m] := \{1, \dots, m\}$.

One obtains the special case of semidefinite programming (SDP) if K is the cone of $n\times n$ symmetric positive semidefinite matrices $\mathbb{S}_+^{n\times n}$ in the vector space $E = F = \mathbb{S}^{n\times n}$, where the duality is now the Euclidean inner product. Similarly one may obtain the general moment problem (GMP) as a special case by setting:

- E the space of continuous functions on a compact set $\mathcal{S} \subset \mathbb{R}^n$
- F the space of signed Radon measures supported on $\mathcal{S}$
- The duality $\langle f, \mu\rangle = \int_{\mathcal{S}} f d\mu$
- $K \subset F$ the cone of nonnegative Radon measures.

The dual cone of $K \subset F$ is defined as

$$K^* := \{ s \in E \mid \langle s, x\rangle \geq 0 \; \forall x \in K \},$$

and the dual conic optimization problem as

$$val^* = \sup_{y \in \mathbb{R}^m} \left\{ \sum_{i=1}^m b_i y_i \mid a_0 - \sum_{i=1}^m y_i a_i \in K^* \right\}.$$

One always has the weak duality relation $val \geq val^*$, and the strong duality theorem gives a sufficient condition for equality.

Theorem. *Assume the cone*

$$\{(\langle a_0, x\rangle, \ldots, \langle a_m, x\rangle) : x \in K\}$$

is closed in $\mathbb{R}^{m+1}$ *and that there is a primal feasible solution. Then* $val = val^*$ *and, if* $val > -\infty$, *there is a primal optimal solution.*

In summary, for continuous functions $f_0, \ldots, f_m$ defined on a compact $\mathcal{S} \subset \mathbb{R}^n$, the GMP takes the form:

$$val := \inf_{\mu \in \mathcal{M}(\mathcal{S})_+} \left\{ \int_{\mathcal{S}} f_0 d\mu \ : \ \int_{\mathcal{S}} f_i d\mu = b_i \quad \forall i \in [m] \right\},$$

with dual problem

$$val^* = \sup_{\mathbf{y} \in \mathbb{R}^m} \left\{ \sum_{i \in [m]} b_i y_i \ : \ f_0(\mathbf{x}) - \sum_{i \in [m]} y_i f_i(\mathbf{x}) \geq 0 \ \forall \mathbf{x} \in \mathcal{S} \right\}.$$

In the dual problem formulation, we have used that the dual cone of $K = \mathcal{M}(\mathcal{S})_+$ is the cone of continuous functions that are nonnegative on $\mathcal{S}$. A special case of the GMP is the constrained optimization problem $\min_{\mathbf{x} \in \mathbb{R}^n} \{f(\mathbf{x}) \mid \mathbf{x} \in \mathcal{S}\}$, obtained by renaming f_0 by f, setting $m = 1$ and $f_1 \equiv 1$, and $b_1 = 1$. In particular we obtain the primal and dual formulations:

$$val := \inf_{\mu \in \mathcal{M}(\mathcal{S})_+} \left\{ \int_{\mathcal{S}} f d\mu \ : \ \int_{\mathcal{S}} 1 d\mu = 1 \right\} = \sup_{y \in \mathbb{R}} \{f(\mathbf{x}) - 1y \geq 0 \ \forall \mathbf{x} \in \mathcal{S}\}.$$

If the case where $f_0, \ldots, f_m$ are *polynomials*, and $\mathcal{S}$ is a semi-algebraic set, one may describe the interior of the dual cone using Positivstellensätze from real algebraic geometry. Now let $\mathcal{S} = \mathcal{S}(\mathbf{g}) \subseteq \mathbb{R}^n$ be a (basic, closed) semi-algebraic set, defined by polynomials $\mathbf{g} = (g_1, g_2, \ldots, g_s)$:

$$\mathcal{S}(\mathbf{g}) = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \ldots, g_s(\mathbf{x}) \geq 0\}.$$

We consider two fundamental cones of polynomials nonnegative on $\mathcal{S}(\mathbf{g})$. The first is the quadratic module generated by $\mathbf{g}$, defined by

$$\mathcal{Q}(\mathbf{g}) = \left\{ \sum_{i=0}^s \sigma_i g_i : \sigma_i \in \Sigma[\mathbf{x}], \quad i = 0, 1, \ldots, s \right\}$$

where $\Sigma[\mathbf{x}]$ is the cone of sums of squared polynomials, and $g_0 \equiv 1$. The second cone is the pre-ordering generated by $\mathbf{g}$,

$$\mathcal{T}(\mathbf{g}) = \left\{ \sum_{I \subseteq [s]} \sigma_I g_I : \sigma_I \in \Sigma[\mathbf{x}], \quad I \subseteq [s] \right\},$$

where $g_I := \prod_{i\in I} g_i$ for $I \subseteq [s]$, and $g_\emptyset \equiv 1$. If $\mathcal{S}(\mathbf{g})$ is compact, $\mathcal{T}(\mathbf{g})$ coincides with the cone of positive polynomials on $\mathcal{S}(\mathbf{g})$.

Theorem (Schmüdgen's Positivstellensatz (1991) [10]). *Assume that $\mathcal{S}(\mathbf{g}) \subseteq \mathbb{R}^n$ is compact. If a polynomial p is positive on $\mathcal{S}(\mathbf{g})$, then $p \in \mathcal{T}(\mathbf{g})$.*

Under an additional assumption, the same holds true for $\mathcal{Q}(\mathbf{g})$.

Theorem (Putinar's Positivstellensatz (1993) [9]). *Assume that $\mathcal{Q}(\mathbf{g})$ is Archimedean, i.e, $\exists f \in \mathcal{Q}(\mathbf{g})$ such that $\{x \in \mathbb{R}^n \mid f(x) \geq 0\}$ is compact. If a polynomial p is positive on $\mathcal{S}(\mathbf{g})$, then $p \in \mathcal{Q}(\mathbf{g})$.*

For polynomial-time computation, we need finite dimensional conic approximations of the quadratic module or pre-ordering, that involve only polynomials of degree at most $r \in \mathbb{N}$. One possibility is to consider the *truncated quadratic module* of order r:

$$\mathcal{Q}(\mathbf{g})_r = \left\{ \sum_{i=0}^{s} \sigma_i g_i : \sigma_i \in \Sigma[\mathbf{x}],\ \deg(\sigma_i g_i) \leq r, \quad i = 0, 1, \ldots, s \right\}.$$

The Lasserre hierarchy for the GMP is obtained by replacing the cone of nonnegative polynomials on $\mathcal{S}(\mathbf{g})$ by $\mathcal{Q}(\mathbf{g})_r$:

$$f_{(r)} := \sup_{\mathbf{y}\in\mathbb{R}^m} \left\{ \sum_{i\in[m]} b_i y_i \ : \ f_0(\mathbf{x}) - \sum_{i\in[m]} y_i f_i(\mathbf{x}) \in \mathcal{Q}(\mathbf{g})_r \ \forall \mathbf{x} \in \mathcal{S} \right\},$$

for $r \in \mathbb{N}$. This problem may be formulated as an SDP [6]. By the Putinar positivstellensatz, one immediately has the following result.

Theorem (Lasserre (2008) [7]). *Assuming primal and dual feasibility and strong duality for the GMP, one has $f_{(r)} \leq f_{(r+1)} \leq val$ for $r = 1, 2, \ldots$, and*

$$\lim_{r\to\infty} f_{(r)} = val,$$

under the Archimedean assumption.

In the talk, we also surveyed recent results on the rate of convergence of the Lasserre hierarchy, based on effective versions of the Putinar positivstellensatz in special cases, e.g. [8, 1, 2]. For example:

- A $val - f_{(r)} = O(1/r^2)$ asymptotic convergence rate result for the Lasserre hierarchy for polynomial optimization on the unit sphere, due to Fang and Fawzi [4]. This result may be extended to the general GMP when $\mathcal{S}(\mathbf{g})$ is the sphere, and the dual GMP problem has an optimal solution [5].
- A recent $val - f_{(r)} = O(1/r)$ asymptotic convergence rate result for the Lasserre hierarchy for polynomial optimization on the hypercube given by $\mathcal{S}(\mathbf{g}) = [-1, 1]^n$, due to Baldi and Slot [3]. In the same paper, there is also a negative result showing $val - f_{(r)} = \Omega(1/r^8)$.

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Algebraic Perspectives on Signomial Optimization

Mareike Dressler

(joint work with R. Murray)

Signomials are obtained by generalizing polynomials to allow for arbitrary real exponents. We consider *signomial programming*, i.e., the minimization of a signomial subject to finitely many signomial inequality constraints. This nonconvex problem has many applications but is in general computationally intractable (NP-hard). An equivalent question addresses the nonnegativity of signomials on certain sets. Commonly one approaches this by finding inner/outer approximations of cones of signomials that are nonnegative on a prescribed set for which membership can be checked efficiently (usually based on Positivstellensätze). We construct arbitrarily strong inner and outer approximations via the concept of conditional sums of arithmetic-geometric exponentials (conditional SAGE or X-SAGE) and a Positivstellensatz for conditional SAGE.

Signomial Rings. Let $\mathcal{A} \subset \mathbb{R}^n$ be a distinguished finite ground set that contains the origin. To every $\boldsymbol{\alpha} \in \mathcal{A}$ we associate a "monomial" basis function $e^{\boldsymbol{\alpha}} \colon \mathbb{R}^n \to \mathbb{R}$ that takes values $e^{\boldsymbol{\alpha}}(\mathbf{x}) = \langle \boldsymbol{\alpha}, \mathbf{x} \rangle$. A *signomial* supported on a finite set $A \subset \mathbb{R}^n$ is a real-linear combination $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in A} c_{\boldsymbol{\alpha}} e^{\boldsymbol{\alpha}}(\mathbf{x})$, where the *support*, denoted by $\operatorname{supp}(f)$, is formally defined as the smallest set $A \subset \mathbb{R}^n$ for which $f \in \operatorname{span}\{e^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in A}$. A *posynomial* is a signomial with only nonnegative terms. The *signomial ring* $\mathbb{R}[\mathcal{A}]$ is the $\mathbb{R}$-algebra generated by basis functions $\{e^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in A}$.

X-SAGE. A signomial is called *X-SAGE* if it can be written as a sum of *X-AGE* functions which are signomials that are nonnegative on a convex set X and have at most one negative term. Checking whether a signomial is X-SAGE can be done via convex *relative entropy programming* (REP). An important property of conditional SAGE is that they preserve sparsity, i.e., if a signomial f is supported on A and has $k \geq 1$ negative coefficients then f is X-SAGE if and only if f is a sum of k X-AGE functions, each supported on A, see [3, 4].

Positivstellensatz for X-SAGE. Let $X \subset \mathbb{R}^n$ be compact convex and $G \subset \mathbb{R}[\mathcal{A}]$ finite. First, we present a characterization of signomials that are positive on sets $K = \{\mathbf{x} \in X : g(\mathbf{x}) \geq 0 \text{ for all } g \in G\}$. Such sets are in general nonconvex.

Theorem 1. *If $f \in \mathbb{R}[\mathcal{A}]$ is positive on K, then there exists an $r \in \mathbb{N}$ for which*

$$\left(\sum_{\boldsymbol{\alpha} \in \mathcal{A}} e^{\boldsymbol{\alpha}}\right)^r f = \lambda_f + \sum_{g \in G} \lambda_g \cdot g,$$

where $\lambda_f \in \mathbb{R}[\mathcal{A}]$ is X-SAGE and each $\lambda_g \in \mathbb{R}[\mathcal{A}]$ is a posynomial.

For computational purposes it is important to note that the representation involves an explicit identity which is affine in f and the "unknown" signomials $\{\lambda_f\} \cup \{\lambda_g\}_{g \in G}$. This result is the first signomial Positivstellensatz to leverage conditional SAGE in the presence of nonconvex constraints and the first to permit irrational exponents. We emphasize that neither X nor K (nor their images under exponential maps) need be semialgebraic. The proof idea is to represent the signomial data via homogeneous polynomials, applying a polynomial Positivstellensatz (Dickinson-Povh [1]), and then mapping back to signomials.

The absence of additional constraining signomials, i.e., $G = \emptyset$, yields:

Corollary 2. *If $f > 0$ on X, then $\left(\sum_{\boldsymbol{\alpha} \in \mathcal{A}} e^{\boldsymbol{\alpha}}\right)^r f$ is X-SAGE for a large enough natural number r.*

Grading Certificates. Once we have decided $r \in \mathbb{N}$ and permissible supports $S_g \supset \operatorname{supp}(\lambda_g)$ for the posynomials, we can use REP to search for an identity given in the Positivstellensatz. Note, by sparsity preservation, we do not need to explicitly bound $\operatorname{supp}(\lambda_f)$, so we have to decide S_g for $g \in G$. Since signomials have no concept of "degree" that is central to polynomial optimization theory we reclaim it now via artificially imposing an $\mathcal{A}$-degree on signomials.

For that let $\mathcal{A}_d$ be the set of sums of at most d vectors from $\mathcal{A}$, then we define the *$\mathcal{A}$-degree* of a signomial f as $\deg_{\mathcal{A}}(f) = \inf\{d : \operatorname{supp}(f) \subset \mathcal{A}_d\}$. We point out that this concept is not intrinsic to signomials. Setting $\mathcal{A} = \operatorname{supp}(f)$, we have $\deg_{\mathcal{A}}(f) = 1$. Let $\mathbb{R}[\mathcal{A}]_d$ denote the space of signomials of $\mathcal{A}$-degree at most d. Since for signomials f, g the inequality $\deg_{\mathcal{A}}(fg) \leq \deg_{\mathcal{A}}(f) + \deg_{\mathcal{A}}(g)$ may be strict, we introduce the so-called *inverse support* of $f \in \mathbb{R}[\mathcal{A}]_d$ as the largest $\mathcal{B} \subset \mathcal{A}_d$ that satisfies $\deg_{\mathcal{A}}(e^{\boldsymbol{\beta}} f) \leq d$ for all $\boldsymbol{\beta} \in \mathcal{B}$ and denote it by $\operatorname{invsupp}_d(f)$.

Hierarchy of Lower Bounds. Given a finite set of signomials $\{f\} \cup G$ and a closed convex set X, we want to compute

$$f_K^\star = \inf_{\mathbf{x} \in K} f(\mathbf{x}), \text{ where } K = \{\mathbf{x} \in X : g(\mathbf{x}) \geq \text{ for all } g \in G\}.$$

Grading the certificates from the Positivstellensatz according to the largest $\mathcal{A}$-degree of the constituent signomials leads us to bounds for this problem.

If $r := d - \deg_{\mathcal{A}}(f) \geq 0$, the *$\mathcal{A}$-degree d SAGE bound* is defined as $f_K^{(d)} := \sup \gamma$ such that $\left(\sum_{\boldsymbol{\alpha}\in\mathcal{A}} e^{\boldsymbol{\alpha}}\right)^r (f - \gamma) = \lambda_f + \sum_{g\in G} \lambda_g \cdot g$, where $\gamma \in \mathbb{R}$ and λ_f and λ_g are X-SAGE signomials supported on $\mathcal{A}_d$ and $\mathrm{invsupp}_d(g)$ for each $g \in G$, respectively. If otherwise $d < \deg_{\mathcal{A}}(f)$, we set $f_K^{(d)} = -\infty$. Note that compared to the representation in the Positivstellensatz where $\{\lambda_g\}_{g\in G}$ are merely posynomials, we take them here to be X-SAGE signomials to increase our chances of finding such an identity. The bounds $f_K^{(d)}$ can be computed via REP.

Corollary 3. *The sequence $f_K^{(1)}, f_K^{(2)}, \ldots$ is nondecreasing and bounded above by $f_K^\star$. If the signomials $\{f\} \cup G$ belong to $\mathbb{R}[\mathcal{A}]$ and X is compact, then*

$$\lim_{d\to\infty} f_K^{(d)} = f_K^\star.$$

This is the first completeness result for minimizing an arbitrary signomial subject to constraints given by a compact convex set and a conjunction of arbitrary (but finitely many) signomial inequalities. It is also the first completeness result for a hierarchy that uses conditional SAGE in the presence of nonconvex constraints.

Through worked examples we illustrate the practicality of this hierarchy in areas such as chemical reaction network theory and chemical engineering. These examples include comparisons to direct global solvers (e.g., BARON and ANTIGONE) and the Lasserre hierarchy (where appropriate).

Outlook: Upper Bounds. Inspired by [2], we additionally develop arbitrarily strong outer approximations of cones of nonnegative signomials and define hierarchies of convex relaxations for approaching the minimum of a signomial from above. Proving the former result requires establishing basic facts on the existence and uniqueness of solutions to signomial moment problems. Interestingly, the completeness of our hierarchy of upper bounds follows from a generic construction and any (hierarchical) inner approximation of the signomial nonnegativity cone. Hence, any signomial Positivstellensatz yields upper bounds for signomial optimization.

Open Questions. Some specific suggestions for lines of future work are:

- How to choose "best" signomial ring $\mathbb{R}[\mathcal{A}]$?
- Detailed study of signomial moment theory and upper bounds.
- Investigate the convergence rates for upper bounds.

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Set-Valued Generalisations of Koopman Operators

Benoît Bonnet-Weill
(joint work with M. Korda)

The last two decades have seen a tremendous success of the Koopman operator theory for analysis of autonomous dynamical systems, see for instance the survey [5]. More recently, this theory was extended to systems with external inputs and to control design problems in several different ways. In [10], the authors defined the Koopman operator for controlled systems by fixing the control to a pre-specified value, whereas in [9] a family of Koopman operators indexed by the control input was considered, and used to carrry out switching control design. Finally, the works [7, 8] investigated the Koopman operator associated with the extended dynamical system evolving on the product space of the original state-space and the space of all control sequences, and used the latter within the model predictive control framework. While the aforelisted methods did produce sound and efficient methods to investigate e.g. the controllability and stabilisability properties of controlled systems, or to design model-based predictors, none of them managed to propose a meaningful generalisation of Koopman operators to dynamics exhibiting a dependence with respect to some input parameter.

Motivated by this observation, we formulate a transposition of the Koopman framework to controlled systems using the tried concepts and grammar of set-valued analysis, for which we refer to [1]. Indeed, it has been known for decades that a controlled system of the form

$$\dot{x}(t) = f(x(t), u(t)), \qquad u(\cdot) \in \mathcal{U}, \tag{1}$$

could be equivalently recast as the differential inclusion

$$\dot{x}(t) \in F(x(t)), \tag{2}$$

wherein the multifunction $F : x \in \mathbb{R}^d \rightrightarrows \{f(x,u) \text{ s.t. } u \in U\} \subset \mathbb{R}^d$ encodes the admissible velocities of the controlled dynamics. This change in perspective has proven to be pivotal in the development of control theory, as evidenced by the reference treatises [6, 11], and is still used to this day to investigate fine properties of controlled systems in very diverse contexts (see e.g. the recent works [2, 3, 4] of the first author in the context of stochastic and mean-field control).

Following this path, we define the *set-valued Koopman operators* $\mathcal{K}_t : \mathcal{X} \rightrightarrows \mathcal{X}$ over some observable space $\mathcal{X}$ as

$$\mathcal{K}_t(\varphi) := \Big\{ \varphi \circ \Phi^u_{(0,t)} \text{ s.t. } u(\cdot) \in \mathcal{U} \Big\} \subset \mathcal{X},$$

for all times $t \in [0,T]$, where $(\Phi^u_{(0,t)})_{t\in[0,T]}$ is the semigroup of flows associated with (1) and generated by some $u(\cdot) \in \mathcal{U}$. We then study the main properties of this object – identifying those amongst the amenable concepts of the classical theory that do persist in this new framework –, propose natural generalisations of the Liouville and Perron-Frobenius operators, and establish a set-valued version of the spectral mapping theorem.

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Operator Theory and Optimization in Dynamical Systems

Milan Korda

In this talk, we present a relation between real algebraic geometry and Perron-Frobenius operators via invariant measures.

We consider time-discrete dynamical systems. That is, for a compact set X and a continuous map $f : X \to X$, let

$$x_{k+1} := f(x_k), \quad x_0 \in X. \tag{1}$$

The main objects in this talk are the Perron-Frobenius operator and invariant measures. The Perron-Frobenius operator embeds the dynamical system (1) into a linear setting [1]. This operator acts on the space of bounded Borel measures $\mathcal{M}(X)$ via

$$P_f : \mathcal{M}(X) \to \mathcal{M}(X), \qquad P_f\mu := f_{\#}\mu,$$

where for Borel sets $A \subset X$ the measure $f_{\#}\mu$ is defined by

$$f_{\#}\mu(A) := \mu(f^{-1}(A)).$$

The operator P_f enjoys many intriguing properties. Among these are that P_f is well defined, linear, bounded, preserves non-negativity of Borel measures, and is dual to the Koopman operator [1]. The second central object for this talk is invariant measures. An invariant measure is a probability measure $\mu \in \mathcal{M}(X)$ with

$$f_{\#}\mu = \mu. \tag{2}$$

Invariant measures play an important and interesting role in the study of the longterm behavior of dynamical systems. We motivate the analysis of invariant measures via the example of the logistic map on $X = [-1, 1]$

$$f : [-1, 1] \to [-1, 1], \qquad f(x) := 2x^2 - 1. \tag{3}$$

We illustrate the chaotic behavior of the logistic map and the probability density of the so-called physical (invariant) measure. We also mention the existence of periodic orbits of arbitrary lengths and a simple relation to the invariant measures supported on these orbits.

Through the lens of the Perron-Frobenius operator, invariant measures can be viewed as spectral objects of P_f. Namely, a probability measure $\mu \in \mathcal{M}(X)$ is invariant if and only if it is a fixed point of P_f, i.e.

$$P_f\mu = \mu. \tag{4}$$

The linear nature of (4) gives rise to several techniques for computing invariant measures. In this talk, we shortly illustrate two such well-established methods – Ulam's method and ergodic averaging – before we focus on a recent optimization approach [2]. The latter method allows us to search for extremal invariant measures, that is, invariant measures that minimize certain cost functions. In its simplest form, the resulting optimization problem reads

$$\begin{array}{ll} \min\limits_{\mu} & \int\limits_X g \, d\mu \\ \text{s.t.} & \mu \in \mathcal{M}(X) \\ & \mu \ \text{ non-negative} \\ & \mu(X) = 1 \\ & P_f\mu = \mu \end{array} \tag{5}$$

where $g : X \to \mathbb{R}$ is an arbitrary but fixed continuous function. The optimization problem (5) is an infinite dimensional conic program on the space of Borel measures. Assuming that $X \subset \mathbb{R}^n$ is a compact semialgebraic set and that f and g are polynomial, the optimization problem (5) can be tackled via the moment-sum-of-squares hierarchy from polynomial optimization [3, 2]. We motivate this observation via a simple connection to the moment formulation of polynomial optimization problems. If, for the moment, we forget the constraint $P_f\mu = \mu$ in (5) we recover the moment formulation of the polynomial optimization

$$\min_{x \in X} g(x).$$

In order to formulate the missing constraint $P_f\mu = \mu$ as a linear moment-constraint we recall the following equivalent formulation of μ being invariant: If $X \subset \mathbb{R}^n$ is compact, then, $\mu \in \mathcal{M}(X)$ is invariant if and only if

$$\int_X p(f(x))\, d\mu(x) = \int_X p(x)\, d\mu(x) \qquad \text{for all } p \in \mathbb{R}[x_1, \ldots, x_n].$$

Following the established line of reasoning for the moment-sum-of-squares hierarchy we obtain a sequence of finite dimensional semidefinite programs whose optimal values converge to the optimal value of (5). Furthermore, the sequence of minimizers in this hierarchy gives rise to a subsequence of pseudo-moments converging to the moment-sequence of a minimizer of (5).

In the last part of the talk, we return to the example of the logistic map (3). We show that the physical measure can be computed via an optimization problem in the form of (5) with a nonlinear but convex cost function. Further, we state how each periodic orbit can be characterized as the support of certain minimizers of (5) for well-chosen polynomials g, we refer to [4] for details. We conclude the talk by mentioning the Christoffel-Darboux kernel as one means for approximating the support of such (invariant) measures and we give an outlook to computing other asymptotic objects such as global attractors via similar techniques based on polynomial optimization.

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On the Effective PositivStellensatz

Bernard Mourrain

Polynomial optimization on a (compact) semi-algebraic set

$$S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_s(x) \geq 0\}$$

where $g_1, \ldots, g_s \in \mathbb{R}[x_1, \ldots, x_n]$ can be reformulated into a convex optimization program, which involves the convex cone $\mathrm{Pos}(S)$ of positive polynomials on S. Its dual involves the cone of moments of measures $\mathcal{M}(S)$ supported on S. As the cone of positive polynomials $\mathrm{Pos}(S)$ and its dual $\mathcal{M}(S)$ are difficult to describe effectively, hierarchies of tractables cones such as sum-of-squares cones also known as Lasserre's hierarchies have been proposed to approximate them. The efficiency of the approach depends on the properties of representation of strictly positive polynomials in these cones, known as the PositivStellensatz.

We review recent results about the Effective PositivStellensatz, on the degree of representation of strictly positive polynomials f on S in terms of algebraic and geometric characteristics of f and the polynomial constraints g_i defining S. We first present degree bounds on the representation of f when S is a simple domain, like the simplex, the unit sphere, the unit ball and the hypercube [6, 3, 4, 7],

Then we describe a transfer method from a general basic semi-algebraic set S to a simple domain D, which involves Łojasiewicz inequalities controlling the behavior of f in terms of the algebraic distance functions [1, 2], and improves the previous known bounds [5].

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Degree Bounds for Putinar's Positivstellensatz on the Hypercube

Lorenzo Baldi

(joint work with L. Slot)

The Positivstellensätze of Schmüdgen and Putinar show that any polynomial f positive on a compact basic semialgebraic set $\mathcal{S}(\mathbf{g})$ can be represented using sums of squares as an element of the preordering $\mathcal{T}(\mathbf{g})$ (Schmüdgen-type representation) or of the quadratic module $\mathcal{Q}(\mathbf{g})$ (Putinar-type representation, under the Archimedean assumption). Recently, there has been large interest in proving effective versions of these results, namely to show bounds on the required degree of the sums of squares in such representations. These *effective* Positivstellensätze have direct implications for the convergence rate of the celebrated moment-SOS hierarchy in polynomial optimization.

In this talk, we restrict to the fundamental case of the hypercube $[-1,1]^n$, defined as a semialgebraic set by the inequalities $1-x_1^2, \ldots, 1-x_n^2$, that was recently analyzed by Baldi and Slot [3]. We show an upper degree bound for Putinar-type representations on $[-1,1]^n$ of the order $O(f_{\max}/f_{\min})$, where $f_{\max}$, $f_{\min}$ are the maximum and minimum of f on $[-1,1]^n$, respectively. Previously, specialized results of this kind were available only for Schmüdgen-type representations, given by Laurent and Slot [4]. On the other hand, for Putinar-type representatons the best available bound for the special case of $[-1,1]^n$ was still the general bound by Baldi, Mourrain and Parusiński [1, 2].

Our specialized analysis, giving the bound of the order $O(f_{\max}/f_{\min})$, improves the exponent of the general bound by a factor of 10. This result is obtained using an outer approximation of $[-1,1]^n$ with a semialgebraic sets defined by a single inequality, and exploiting the previous bounds for Schmüdgen-type representations on hypercubes [4].

Complementing this upper degree bound, we show a lower degree bound of $\Omega(\sqrt[8]{f_{\max}/f_{\min}})$. The bound is obtained performing a local analysis at the vertices of $[-1,1]^n$. This is the first lower bound for Putinar-type representations on a semialgebraic set with nonempty interior described by a standard set of inequalities. Indeed, previously the only known lower bound was due Stengle [6], that exploited the non-standard description of the interval $[0,1]$ with the polynomial $x^3(1-x)^3$. Our approach is different, since we consider the standard set of defining inequalities $1-x_1^2,\dots,1-x_n^2$, and since we exploit the different properties of the quadratic module $\mathcal{Q}(1-x_1^2,\dots,1-x_n^2)$ with respect to the preordering $\mathcal{T}(1-x_1^2,\dots,1-x_n^2)$.

The existence of lower degree bounds can be seen a quantitative version of the *non*-stability property of the quadratic module. The non-stability property for Archimedean quadratic modules, for dimension at least 2, was studied by Scheiderer in [5]. Remarkably, the proof in [5] uses an interior point of the associated semialgebraic, while our proof focuses on a boundary point of the semialgebrac set, as the one of Stengle [6].

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Pure Koopmanism

Rainer Nagel

We start from a topological dynamical system (K,φ), where K is a compact space and $\varphi\colon K\to K$ is continuous. To this object we associate its "Koopman linearization" $(\mathrm{C}(K),T_\varphi)$, where $\mathrm{C}(K)$ is the Banach algebra of all $\mathbb{C}$-valued continuous functions on K and

$$T_\varphi \colon \mathrm{C}(K) \to \mathrm{C}(K), \quad f \mapsto f \circ \varphi$$

is the so called Koopman operator. By Gelfand's theorem (K, φ) and $(\mathrm{C}(K), T_\varphi)$ are (anti-)isomorphic in a categorical sense. As a first exercise we state isomorphic properties in two languages.

$$\begin{aligned} (K, \varphi) &\longleftrightarrow (\mathrm{C}(K), T_\varphi) \\ K \text{ metrizable} &\longleftrightarrow \mathrm{C}(K) \text{ separable} \\ \varphi - \text{invariant closed subset} &\longleftrightarrow T_\varphi - \text{invariant closed ideal} \\ \text{injective/surjective} &\longleftrightarrow \text{surjective/injective} \\ \text{open and closed set} &\longleftrightarrow \text{idempotent } e \in \mathrm{C}(K) \\ K \text{ extremally disconnected} &\longleftrightarrow \mathrm{C}(K) \text{ order complete} \end{aligned}$$

Standart examples of Banach algebras which can be represented as spaces $\mathrm{C}(K)$ are ℓ^∞ and $\mathrm{L}^\infty(X, \Sigma, \mu)$.

In order to obtain a "structured" quotient of (K, φ) we take the Banach algebra with discrete spectrum

$$\mathcal{B} := \overline{\mathrm{lin}}\{f \in \mathrm{C}(K) \mid Tf = \lambda f, |\lambda| = 1\}.$$

If (K, φ) is minimal, we show that $\mathcal{B}$ yields a quotient (L, ψ) of (K, φ) which is homeomorphic to

$$(G, \mathrm{rot}_{g_0})$$

where G is a compact, abelian group having a dense subgroup generated by $g_0 \in G$. We then indicate how the concept of discrete spectrum can be relativized in order to obtain what is called a "Furstenberg tower". The Lyapunovalgebra is briefly presented as another application of "pure Koopmanism".

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Moment Problem for Algebras Generated by a Nuclear Space

Tobias Kuna

(joint work with M. Infusino, S. Kuhlmann, P. Michalski)

In this talk, we considered linear functionals L defined on a unital commutative real algebra A, not necessarily finitely generated, and consider whether they can be represented by a Radon measure on the character space $X(A)$. We assume that the algebra is generated by a vector space V, that means that V is spanned by a set of generators of A. Furthermore, we assume that V is equipped with a topology τ_V which makes V into a locally convex topological vector space. Mainly, we are interested in the case, where τ_V is generated by Hilbertian seminorms, that is seminorms which stem from a symmetric positive semidefinite bilinear form, in particular, we consider topologies which turns (V, τ_V) into a nuclear space. We endow the character space $X(A)$ with the weakest (Hausdorff) topology which makes $\hat{a} : X(A) \to \mathbb{R}$ given by $\alpha \mapsto \alpha(a)$ continuous and consider Radon measures on $X(A)$ with respect to the associated Borel σ-algebra. Given a linear function $L : A \to \mathbb{R}$ with $L(1) = 1$, the moment problem means, that one wants to obtain conditions on L so that there exists a Radon measure ν supported on the set of characters $\{\alpha \in X(A) : \alpha|_V \text{ is } \tau_V\text{-continuous}\}$, assumed to be non-empty, representing L on all of A, that is, for all $a \in A$ it holds

$$L(a) = \int_{X(A)} \hat{a}(\alpha) d\nu(\alpha). \tag{1}$$

This type of moment problem has a wide range of applications in physics and has been studied extensively before which we cannot cover exhaustively here, but see for example [1] for references and more details. This problem reduces to the classical moment problem when A is the algebra of polynomials and V the vector space spanned by the generators of the algebra of polynomials.

The take on this problem, we presented, studies the following question: assume for any finite dimensional subspace W of V we can solve the moment problem on $\langle W \rangle$ the algebra generated by W, more precisely, we can find a measure ν_W which represents the restriction of L to $\langle W \rangle$. When the topology on V is generated by one Hilbertian seminorm q we show that one can obtain a representing measure ν supported on characters whose restrictions to V are q-continuous if and only if there exists another Hilbertian seminorm p on V such that the trace of p with respect to q is finite and the collection $(\nu_W)_W$ is p-concentrated. A sufficient condition for the latter is that $L(a^2) \leq Cp(a)^2$ for some $C > 0$ and all $a \in V$. For a given L there is a canonical choice for p, namely $a \mapsto \sqrt{L(a^2)}$, for which the above bound and hence p-concentration holds automatically. Therefore, one may consider p as given and wonder whether one can find q fulfilling the aforementioned characterization. This can always be done when V is actually equipped with a nuclear topology. We also present a way how to construct q for a given Hilbertian seminorm p, for example when (V, p) is Hausdorff with a dense countable set. Unfortunately, in general q will not be finite on V and then one cannot obtain the representation for all of A but just for a dense sub-algebra of A. Quite easily constructed concrete examples testify that the latter effect appears naturally and that a representation on all of A is actually unachievable in general. Finally, we demonstrated how one can apply the above techniques to the case where $A = S(V)$ the symmetric algebra associated to a nuclear space V. We gave conditions just in terms of a total subset E of V which nevertheless allowed us to get a representation on all of A.

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Amalgamation of Real Zero Polynomials

David Sawall

(joint work with M. Schweighofer)

Consider two polynomials $p \in \mathbb{R}[x, y]$ and $q \in \mathbb{R}[x, z]$ in three blocks of variables $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_m)$ and $z = (z_1, \dots, z_n)$. We call a polynomial $r \in \mathbb{R}[x, y, z]$ an *amalgam* of p and q if $p = r(x, y, 0)$ and $q = r(x, 0, z)$. Clearly, an amalgam of p and q exists if and only if $p(x, 0) = q(x, 0)$.

We want to investigate when an amalgam of two polynomials exists which is real zero [HV].

Definition 1. *A polynomial $p \in \mathbb{R}[x]$ is called* real zero *if for all $a \in \mathbb{R}^\ell$ the univariate polynomial*

$$p(ta) \in \mathbb{R}[t]$$

has only real roots.

Note that in particular $p(0) \neq 0$. A univariate polynomial p with $p(0) \neq 0$ is real zero if and only if it is real-rooted. Being real zero means that the polynomial restricted to any line through the origin splits in $\mathbb{R}[t]$.

Clearly, if $p \in \mathbb{R}[x, y]$ is real zero then so is $p(x, 0)$. Thus it only makes sense to ask for real zero amalgams of real zero polynomials. Schweighofer conjectured the following in his preprint [Sch] from March 2020:

Conjecture 2. *Let $p \in \mathbb{R}[x, y]$ and $q \in \mathbb{R}[x, z]$ be real zero polynomials such that $p(x, 0) = q(x, 0)$. Then there exists a real zero amalgam $r \in \mathbb{R}[x, y, z]$.*

In this talk I gave a counterexample to this conjecture. It is based on a counterexample of amalgamation of matroids and we will use some of the theory of stable polynomials.

Definition 3. *Let $p \in \mathbb{R}[x]$ be a real zero polynomial. The set*

$$\{a \in \mathbb{R}^\ell \mid \forall t \in (0, 1) : p(ta) \neq 0\}$$

is called the rigidly conex set *of p.*

Helton and Vinnikov introduced rigidly convex sets in [HV] and showed that rigidly convex sets are indeed convex (the homogeneous analogues of rigidly convex sets (hyperbolicity cones) were introduced much earlier by Gårding [Går]).

Under the assumption that Conjecture 2 was true Schweighofer showed that for every rigidly convex set C and any finite union of two-dimensional subspaces of $\mathbb{R}^\ell$ there exists a spectrahedron S containing C and agreeing with it on the finite union of two-dimensional subspaces. This is a very weak version of the *Generalized Lax Conjecture* which conjectures that every rigidly convex set is a spectrahedron. For this application, Schweighofer only needed to amalgamate real zero polynomials with $\ell = 2$ shared variables. Our counterexample has $\ell = 6$ shared variables and there is a theorem due to Poljak and Turzík [PT1] which says that there cannot be a counterexample for $\ell = 2$ of the type we are considering here. They showed that any two matroids on sets S and T can be amalgamated if $S \cap T$ has at most two elements (which corresponds to the case of two shared variables).

Definition 4. *Let S be a finite set and $\mathcal{I}$ be a non-empty set of subsets of S. Then $M = (S, \mathcal{I})$ is called a* matroid *on S if the following two hold:*

- *if $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$,*
- *if $I, J \in \mathcal{I}$ with $|J| > |I|$ then there exists $a \in J \setminus I$ such that $I \cup \{a\} \in \mathcal{I}$.*

The elements of $\mathcal{I}$ are called independent sets *of M and the maximal independent sets* bases *of M.*

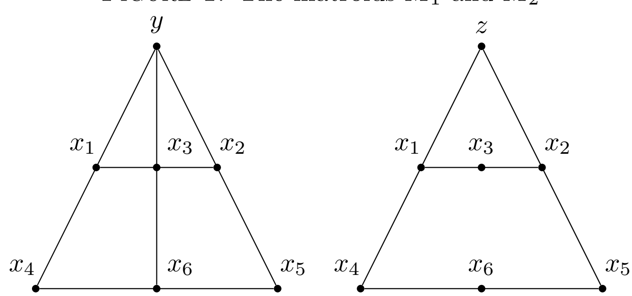
Instead of specifying a matroid by its independent sets it is sufficient to specify its bases (and of course its underlying set). One should think about S as a finite set of vectors in some vector space and about $\mathcal{I}$ as the subsets of S which are linearly independent.

Definition 5. *To a matroid M on the set $\{1, \dots, \ell\}$ with set of bases $\mathcal{B}$ we associated the multi-affine* basis generating polynomial

$$p_M := \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i.$$

Poljak and Turzík [PT2] showed that the following two matroids M_1 and M_2 do not have a matroid amalgam:

FIGURE 1. The matroids M_1 and M_2



Every 3-element subset of $\{x_1, \dots, x_6, y\}$ respectively $\{x_1, \dots, x_6, z\}$ is a basis of M_1 respectively M_2 if it does not lie on a line in the picture above. Let $e_3 \in \mathbb{R}[x, y]$ be the third elementary symmetric polynomial in the seven variables $x_1, \dots, x_6, y$.

We consider the real stable basis generating polynomials

$$p_{M_1} = e_3 - x_1x_2x_3 - x_4x_5x_6 - yx_1x_4 - yx_2x_5 - yx_3x_6$$

and

$$p_{M_2} = e_3(x, z) - x_1x_2x_3 - x_4x_5x_6 - zx_1x_4 - zx_2x_5.$$

Note that not every basis generating polynomial is stable. Using (delta-)matroids and some more theory of stable polynomials, we showed that there is no stable amalgam of p_{M_1} and p_{M_2}.

Furthermore, we showed that the polynomials $p_{M_1}(x_1 + 1, \dots, x_6 + 1, y)$ and $p_{M_2}(x_1+1, \dots, x_6+1, z)$ are real zero and that any real zero amalgam $r \in \mathbb{R}[x, y, z]$ would give a stable amalgam $r(x_1 - 1, \dots, x_6 - 1, y, z)$ of p_{M_1} and p_{M_2}. This concludes the counterexample for Conjecture 2.

We hope that the application by Schweighofer mentioned above can still be saved and conjecture the following.

Conjecture 6 (weak real zero amalgamation conjecture)**.** *Let $p \in \mathbb{R}[x_1, x_2, y]$ and $q \in \mathbb{R}[x_1, x_2, z]$ be real zero polynomials such that $p(x, 0) = q(x, 0)$. Then there exists a real zero amalgam of p and q, i.e., a real zero polynomial $r \in \mathbb{R}[x, y, z]$ such that*

$$p = r(x, y, 0) \quad \textit{and} \quad q = r(x, 0, z).$$

More generally, we want to investigate when there exist real zero amalgams of real zero polynomials. Schweighofer showed in [Sch] that for $\ell = m = n = 1$, for quadratic p and q and for $\ell = 0$ there always exists a real zero amalagam of p and q (even of degree at most $\max\{\deg p, \deg q\}$).

Conjecture 7 (strong real zero amalgamation conjecture)**.** *Let $p \in \mathbb{R}[x_1, x_2, y]$ and $q \in \mathbb{R}[x_1, x_2, z]$ be real zero polynomials both of degree at most d such that $p(x, 0) = q(x, 0)$. Then there exists a real zero amalgam of p and q of degree at most d.*

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State Polynomials: Positivity and Applications

Jurij Volčič
(joint work with I. Klep, V. Magron, J. Wang)

This talk introduces (noncommutative) state polynomials, which are polynomial expressions in noncommuting variables $x_1, \dots, x_n$ and formal state ς of their products. For example, $p = \varsigma(x_1x_2x_1)\varsigma(x_1)^2 + 2\varsigma(x_2^2x_1) - 3$ is a state polynomial, and $f = \varsigma(x_1x_2)\varsigma(x_1)x_2x_1 - \varsigma(x_2^2x_1) - x_2x_1x_2 + 4$ is a noncommutative state polynomial. They can be naturally evaluated on a n-tuple of bounded self-adjoint operators on a real Hilbert space $\mathcal{H}$, and a state on $B(\mathcal{H})$ (i.e., a unital positive continuous functional $B(\mathcal{H}) \to \mathbb{R}$). Concretely, if $X = (X_1, X_2) \in B(\mathcal{H})$ and λ is a state on $B(\mathcal{H})$, then $p(\lambda, X) = \lambda(X_1X_2X_1)\lambda(X_1)^2 + 2\lambda(X_2^2X_1) - 3 \in \mathbb{R}$ and $f(\lambda, X) = \lambda(X_1X_2)\lambda(X_1)X_2X_1 - \lambda(X_2^2X_1) - X_2X_1X_2 + 4I \in B(\mathcal{H})$.

Positivity and optimization of state polynomials naturally arise in functional analysis and quantum information theory. More precisely, one is interested in certifying positivity of a state polynomial over all finite-dimensional Hilbert spaces, or separable Hilbert spaces, subject to polynomial constraints on the operator variables and the state. The derived theory has both noncommutative (due to formal operator variables generating the free algebra) and commutative (due to formal state symbols generating the infinitely generated polynomial ring) aspects, and builds upon real algebra [9], free real algebraic geometry [2], and invariant/representation theory connecting the previous two [12, 8]. The talk presents algebraic certificates of positivity (Positivstellensätze) and optimization in this dimension-independent setting, and demonstrates their applications to polynomial Bell inequalities in quantum networks.

The first main result is the resolution of the state polynomial analog of Hilbert's 17th problem for positivity on matrix tuples and matricial states.

Theorem 1. *Let p be a state polynomial. The following are equivalent:*

(1) $p(\lambda, X) \geq 0$ for all states λ and tuples X of self-adjoint operators on separable Hilbert spaces;
(2) $p(\lambda, X) \geq 0$ for all matricial states λ and matrix tuples X;
(3) p is a quotient of sums of products of elements of the form $\varsigma(hh^*)$ for a noncommutative state polynomial h.

For example, the Cauchy-Schwarz inequality admits an algebraic certificate of global positivity

$$\varsigma(x_1^2)\varsigma(x_2^2) - \varsigma(x_1x_2)^2 = \frac{\varsigma\Big(\big(\varsigma(x_1^2)x_2 - \varsigma(x_1x_2)x_1\big)^2\Big)}{\varsigma(x_1)^2}.$$

Theorem 1 guarantees that every state polynomial inequality is an algebraic consequence of states of hermitian squares. Moreover, for such inequalities, dimension-independent matrix positivity implies operator positivity, which contrasts the failure of analogous implication for trace polynomials (where only tracial states are considered) by the refutation of Connes' embedding conjecture [4]. Since Theorem 1 characterizes only positive state polynomials but not noncommutative state polynomials, one is left with the following open problem.

Problem 2. *Find a state-of-hermitian-square certificate for unconstrained positivity of noncommutative state polynomials.*

A very special case of Problem 2 for state polynomials in one matrix variable is given in [5].

The second main result is an archimedean Positivstellensatz for positivity subject to noncommutative state polynomial constraints. To a set C of noncommutative state polynomials we associate the analog of a basic semialgebraic set,

$$K_C = \big\{(\lambda, X)\colon X_j = X_j^* \in B(\mathcal{H}),\ \lambda \text{ a state on } B(\mathcal{H}),\ c(\lambda, X) \succeq 0 \text{ for all } c \in C\big\}$$

where $\mathcal{H}$ is an infinite-dimensional separable real Hilbert space. Furthermore, the role of a quadratic module is taken by

$$Q_C = \left\{ \sum_i \varsigma(h_i c_i h_i^*)\colon c_i \in \{1\} \cup C,\ h_i \text{ nc state polynomials} \right\}.$$

We say that C is algebraically bounded if $r - x_1^2 - \cdots x_n^2 = \sum_i f_i c_i f_i^*$ for some $r > 0$, $c_i \in \{1\} \cup C$ and noncommutative polynomials f_i. Algebraically bounded sets lead to quadratic modules that are archimedean in a suitable sense, and admit the following Positivstellensatz.

Theorem 3. *Let p be a state polynomial, and C an algebraically bounded set of noncommutative state polynomials. The following are equivalent:*

(1) $p \geq 0$ on K_C;
(2) $p + \varepsilon \in Q_C$ for all $\varepsilon > 0$.

Note that in Theorem 3, checking positivity over operator variables is necessary in general, as matrices of all sizes as in Theorem 1 are no longer sufficient. This observation holds already for freely noncommutative polynomials (without state symbols), where matrix evaluations are ample enough only for convex K_C. Furthermore, while state polynomial positivity admits analogs of Artin's solution to Hilbert's 17th problem and Putinar's Positivstellensatz, one can somewhat surprisingly show that there are no straightforward analogs of Krivine-Stengle and Schmüdgen Positivstellensätze for the preordering generated by Q_C. These remarks lead to the following questions.

Problem 4. *Are matrix evaluations sufficient for checking positivity on a convex K_C? Is there an analog of Krivine-Stengle Positivstellensatz for K_C if one considers evaluations on unbounded self-adjoint operators?*

Notwithstanding these open problems, Theorem 3 leads to an efficient state polynomial optimization procedure. Namely, given a state polynomial p and an algebraically bounded C, there is a hierarchy of semidefinite programs whose solutions form a convergent decreasing sequence with limit $\sup_{K_C} p$. This construction is a variant of Lasserre's SDP hierarchy, which has been successfully applied in commutative polynomial optimization [3], noncommutative polynomial eigenvalue optimization [10], and trace polynomial optimization [6]. While the size of SDPs in our hierarchy grows very quickly, this can be mitigated by using sparsity, symmetry and conditional expectation reductions. Furthermore, we obtain a stopping criterion based on flatness in the dual SDP, which yields (through a version of the Gelfand-Naimark-Segal construction) an explicit finite-dimensional optimizer for $\sup_{K_C} p$.

Finally, the above state polynomial optimization was developed with a view toward polynomial Bell inequalities in quantum networks. Let us conclude by giving two concrete applications in this vein. Firstly, abbreviating

$$\operatorname{cov}(x,y) = \varsigma(xy) - \varsigma(x)\varsigma(y)$$

and then maximizing

$$\begin{aligned} &\operatorname{cov}(x_1,y_1)+\operatorname{cov}(x_1,y_2)+\operatorname{cov}(x_1,y_3)+\operatorname{cov}(x_2,y_1)\\ +&\operatorname{cov}(x_2,y_2)-\operatorname{cov}(x_2,y_3)+\operatorname{cov}(x_3,y_1)-\operatorname{cov}(x_3,y_2)\end{aligned}$$

subject to

$$x_i^2 = y_j^2 = 1, \quad [x_i,y_j]=0$$

gives the maximal quantum violation 5 of a covariance Bell inequality with the classical value 2, which answers a question of [11]. Secondly, maximizing

$$\begin{aligned} -\frac{1}{8}\Big(\varsigma((x_1+x_2)y_1(z_1+z_2)) - \varsigma((x_1-x_2)y_2(z_1-z_2))\Big)^2&\\ +\varsigma((x_1+x_2)y_1(z_1+z_2)) + \varsigma((x_1-x_2)y_2(z_1-z_2))&\end{aligned}$$

subject to

$$\begin{aligned} x_i^2 = y_j^2 = z_k^2 = 1, \quad [x_i,y_j]=[y_j,z_k]=[z_k,x_i]=0,&\\ \varsigma(uv)=\varsigma(u)\varsigma(v) \quad \text{for products } u \text{ in } x_i \text{ and products } v \text{ in } z_k&\end{aligned}$$

gives the maximal quantum violation $2\sqrt{2}$ of a Bell inequality in a bilocal scenario with the classical value 2, which answers a question of [1]. More generally, state polynomial optimization can be applied to any network of parties and entanglement sources, to produce convergent upper bounds on quantum violations of polynomial Bell inequalities.

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Entropic Transfer Operators

Oliver Junge

We propose a novel approach to the following problem from dynamical systems: *Given a deterministic map $F : X \to X$ on a large (continuous and possibly high-dimensional) state space X, find a stochastic (i.e. distribution-valued) map F^N on a small (i.e. finite) state space $X^N \subset X$ that "captures the most relevant features" of F.* Here, $N \in \mathbb{N}$ denotes a discretization scale and we will often be interested in the limit $N \to \infty$. Our idea for the construction of F^N is to smooth the action of the deterministic map F on X^N by means of an entropically regularized optimal transport plan.

Concerning the stochastic formulation, we shall assume that F possesses some interesting invariant probability measure μ (i.e. a natural resp. SRB-measure) and consider the stochastic formulation of F's deterministic dynamics via the transfer operator $T : L^2(\mu) \to L^2(\mu)$ given by $Th := dF_\#(h\mu)/d\mu$ (where $F_\#\nu$ denotes the push-forward of the measure ν under F). Intuitively, if points are distributed according to $h\mu$ on X, then their images are distributed according to $Th\mu$. We then define F^N via a transfer operator $T^{N,\epsilon} : L^2(\mu^N) \to L^2(\mu^N)$, where μ^N is an approximate invariant probability measure for the stochastic dynamics, and ϵ denotes the magnitude of regularization.

A similar regularization can be applied to T itself, yielding the operator

$$T^\epsilon : L^2(\mu) \to L^2(\mu).$$

Informally, compared to T, the regularized operator T^ϵ smooths spatial structures below the length scale $\sqrt{\epsilon}$. We show that (a suitable extension of) $T^{N,\epsilon}$ converges to T^ϵ in operator norm when μ^N converges to μ as $N \to \infty$ for fixed $\epsilon > 0$. That is, for sufficiently high N, the analysis of $T^{N,\epsilon}$ can reveal structural properties of the dynamics of F above the length scale $\sqrt{\epsilon}$.

Concerning the "most relevant features", we shall adopt the point of view that these are determined by the peripheral spectrum of T^ϵ. For example, if F exhibits an (almost) n-cycle, then this will be represented in the spectrum of T^ϵ by an n-tuple of eigenvalues close to $\lambda_k = e^{2\pi i k/n}$. Similarly, if F shows metastable behaviour, i.e. the state space X decomposes into k almost invariant sets, the spectrum of T^ϵ contains k real eigenvalues close to 1.

In numerical experiments, our method successfully recovers known metastable behaviour in a 30 dimensional system from molecular dynamics.

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From Infinite to Finite Programs: Explicit Error Bounds with an Application to Approximate Dynamic Programming

Tobias Sutter

(joint work with P. M. Esfahani, D. Kuhn, J. Lygeros)

Infinite dimensional linear programming offers remarkable modeling power and emerges in different fields including engineering, economics, operations research. This class of programs subsumes prominent problems such as finite dimensional optimization problems, the generalized moment problem, and dynamic programming as special cases. The exact solution to these programs is in general far beyond the reach of analytical methods or classical numerical techniques, motivating the study of tractable approximations schemes.

While there are various approximation techniques for infinite linear programs, the literature still lacks a scheme that enjoys provably explicit performance guarantees. In this talk we will propose a novel approach via finite (random) tractable convex programs. The idea uniquely builds on tools from two growing areas of randomized optimization and structural convex optimization. The proposed scheme involves a "regularization" idea that is essential to establish a-priori as well as a-posterior performance guarantees. To this end, we generalize several existing results in the randomized as well as structural convex optimization algorithms to an infinite dimensional setting. The theoretical results of this method, as a special case, offers an effective approximate dynamic programming scheme along with explicit performance guarantees for different classes of Markov decision processes (discounted or long-time average costs).

The importance of these results is evidenced by their broad potential applicability where decisions and data are two prominent features, opening new vistas towards data-driven decision-making. As reinforcement learning and "Big Data" gain ever-increasing significance, we foresee that these results can further open the door to other fields such as machine learning, operations research, stochastic model predictive control, finance, business analytics, and many more where randomized algorithms and convex optimization algorithms have already seen success.

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Intermediate Cones between the SOS and PSD Cones: Quartics and Sextics

Sarah Hess
(joint work with C. Goel, S. Kuhlmann)

In this talk, we construct a filtration of intermediate cones between the cone $\mathcal{P}_{n+1,2d}$ of forms in $n+1$ variables of degree $2d$ that are positive semidefinite and its subcone $\Sigma_{n+1,2d}$ of forms that admit a sum-of-square representation via a Gram matrix approach [CLR95] and analyze it for strict inclusions. The investigation of the relationship between $\mathcal{P}_{n+1,2d}$ and $\Sigma_{n+1,2d}$ dates back to Hilbert's 1888 seminal paper [Hil88], where it is demonstrated that $\mathcal{P}_{n+1,2d} = \Sigma_{n+1,2d}$ if and only if $n+1=2$ or $2d=2$ or $(n+1,2d)=(3,4)$. Thus, in any inequality case, at least one inclusion in our induced cone filtration has to be strict. However, it is not clear which one and how many there are.

More precisely, let $\mathcal{F}_{n+1,2d}$ be the vector space of all real forms in $n+1$ variables with degree $2d$ and let $(m_0(X),\ldots,m_k(X))$ be the lexicographically ordered monomial basis of half degree d forms (in $n+1$ variables). For any $f \in \mathcal{F}_{n+1,2d}$, denote the set of all Gram matrices associated to f by $\mathcal{G}^{-1}(f)$ and, for any $A \in \mathcal{G}^{-1}(f)$, let q_A be the associated quadratic form. Moreover, let V be the d^{th}-Veronese embedding (of $\mathbb{P}^n$), then

$$\begin{aligned}\mathcal{P}_{n+1,2d} &= \{f \in \mathcal{F}_{n+1,2d} \mid \exists A \in \mathcal{G}^{-1}(f)\colon q_A|_{V(\mathbb{P}^n)(\mathbb{R})} \geq 0\} \text{ and}\\ \Sigma_{n+1,2d} &= \{f \in \mathcal{F}_{n+1,2d} \mid \exists A \in \mathcal{G}^{-1}(f)\colon q_A|_{\mathbb{P}^k(\mathbb{R})} \geq 0\}.\end{aligned}$$

For $i=0,\ldots,k-n$, set

$$H_i := \{[z] \in \mathbb{P}^k \mid \exists x \in \mathbb{C}^{n+1}\colon (z_0,\ldots,z_{n+i}) = (m_0(x),\ldots,m_{n+i}(x))\}$$

and let V_i be the smallest projective variety in $\mathbb{P}^k$ containing H_i. This leads to a filtration $V_{k-n} \subseteq \ldots \subseteq V_0$ of irreducible projective varieties containing $V(\mathbb{P}^n)$ in which each inclusion is strict. In particular, $V_{k-n} = V(\mathbb{P}^n)$, $V_0 = \mathbb{P}^k$ and $\dim(\mathbb{P}^k) - \dim(V(\mathbb{P}^n)) = k-n$. Furthermore, each inclusion in the corresponding filtration of sets of real points $V_{k-n}(\mathbb{R}) \subseteq \ldots \subseteq V_0(\mathbb{R})$ is also strict.

We now construct a cone filtration

$$C_0 \subseteq \ldots \subseteq C_{k-n} \tag{1}$$

by setting $C_i := \{f \in \mathcal{F}_{n+1,2d} \mid \exists A \in \mathcal{G}^{-1}(f)\colon q_A|_{V_i(\mathbb{R})} \geq 0\}$ for $i=0,\ldots,k-n$. Clearly, $C_{k-n} = \mathcal{P}_{n+1,2d}$ and $C_0 = \Sigma_{n+1,2d}$, since $V_{k-n} = V(\mathbb{P}^n)$ and $V_0 = \mathbb{P}^k$, respectively. Hence, (1) is in fact an intermediate cones filtration between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$.

For the equality cases $(2,2d)$, $(n+1,2)$ and $(3,4)$, no inclusions in (1) can be strict by Hilbert's 1888 Theorem. It thus remains to consider the inequality cases $(n+1,2d)_{n\geq 2,d\geq 2} \neq (3,4)$ where at least one strict inclusion in (1) has to occur (again by Hilbert's 1888 Theorem). The below given general query arises:

Problem 1. *For $(n+1,2d)_{n\geq 2,d\geq 2} \neq (3,4)$, which inclusions in (1) are strict?*

In the aim of giving an answer to the above problem, we firstly observe that each V_i is an irreducible, totally-real, nondegenerate projective variety with codimension i. Moreover, by an application of Bézout's Theorem [Har77, Theorem 7.7], we are secondly able to determine the degrees of $V_0, \ldots, V_n$, and V_{n+1} if $n \leq 2$, respectively.

Proposition. *Let $n, d \in \mathbb{N}$ and $i = 0, \ldots, n$, then $\deg(V_i) = i + 1$. Moreover, if $n \leq 2$, then also $\deg(V_{n+1}) = n + 2$.*

This result is crucial, as it allows us to conclude that $V_0, \ldots, V_n$, and V_{n+1} if $n \leq 2$, are projective varieties of minimal degree, respectively. From this, the next result follows:

Theorem. *Let $n, d \in \mathbb{N}$ and $i = 0, \ldots, n-1$, then $C_i = C_{i+1}$. Moreover, if $n \leq 2$, then also $C_n = C_{n+1}$.*

The remaining inclusions $C_n \subseteq \ldots \subseteq C_{k-n}$ in (1) thus have to be investigated for strictness. We conjecture the following:

Conjecture. *For $(n+1, 2d)_{n \geq 2, d \geq 2} \neq (3, 4)$ and $i = n+1, \ldots, k-n-1$, the inclusion $C_i \subseteq C_{i+1}$ is strict. Furthermore, if $n \geq 3$, then also the inclusion $C_n \subseteq C_{n+1}$ is strict.*

In this talk, we discuss a proof for the above conjecture in the cases $(n+1, 4)_{n \geq 3}$ and $(n+1, 6)_{n \geq 2}$. Thus, in combination with the above general theorem, we provide a full answer to Problem 1 for $(n+1)$-ary quartics ($n \geq 3$) and $(n+1)$-ary sextics ($n \geq 2$). Our conjecture consequently reduces to an investigation of the following special problem:

Problem 2. *Is it true that for $(n+1, 2d)_{n \geq 2, d \geq 4}$ and $i = n+1, \ldots, k-n-1$, the inclusion $C_i \subseteq C_{i+1}$ is strict. Moreover, if $n \geq 3$, is it also true that the inclusion $C_n \subseteq C_{n+1}$ is strict?*

We are currently preparing a paper [GHK+] that extends our findings about the strict inclusions in the $(n+1)$-ary quartics ($n \geq 3$) and $(n+1)$-ary sextics ($n \geq 2$) cases given in this talk (based on [GHK]) to arbitrary inequality cases. To this end, we will introduce a *degree-jumping principle* that allows us to maintain the separating property of a given form $f \in C_{i+1} \backslash C_i \subseteq \mathcal{P}_{n+1,2d}$ when going over to higher degrees 2δ for $\delta \geq d$ while the number of variables $n+1$ remains unchanged. We will therefore give an answer to the special Problem 2. Consequently, the above conjecture will be proven and the general Problem 1 will be solved.

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The Odd Powers of the Motzkin Polynomial, etc.

Bruce Reznick

For positive integers n, d, let $F_{n,d}$ denote the real forms of degree d in n variables. A form p is *psd* if $p(a) = p(a_1, \dots, a_n) \geq 0$ for all $a \in \mathbb{R}^n$; p is *sos* if there exist forms f_j so that $p = \sum_{j=1}^{r} f_j^2$. A form p is *positive definite* if $a \neq 0$ implies $p(a) > 0$. Following Choi and Lam [1], let $P_{n,m}$ and $\Sigma_{n,m}$ denote the closed convex cones of psd forms and sos forms in $F_{n,m}$ (for even m). Let $E(P_{n,m})$ denote the set of extremal elements of $P_{n,m}$; if p is positive definite, then $p \notin E(P_{n,m})$. Let $\Delta_{n,m} = P_{n,m} \setminus \Sigma_{n,m}$. Hilbert proved that $\Delta_{n,m} \neq \emptyset$ iff $n \geq 3$ and $m \geq 6$ or $n \geq 4$ and $m \geq 4$, without naming specific examples. His 17th Problem asks whether, for $p \in P_{n,m}$, there exists some q in some $F_{n,d}$ so that $q^2 p \in \Sigma_{n,m+2d}$. Artin answered yes, but unconstructively. Choi and Lam studied M, S, Q in $\Delta_{3,6}$ and $\Delta_{4,4}$; M below was found by Motzkin. Another early example in $\Delta_{3,6}$ was R, found by Robinson. Each of these forms is extremal.

$$M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2,$$
$$S(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2,$$
$$Q(w, x, y, z) = w^4 + x^2y^2 + x^2z^2 + y^2z^2 - 4wxyz,$$
$$R(x, y, z) = x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2.$$

Stengle [7] proved in 1979 that $T(x, y, z) = x^3z^3 + (y^2z - x^3 - z^2x)^2 \in \Delta_{3,6}$, and moreover, $T^{2k+1} \in \Delta_{3,6(2k+1)}$ for every positive odd integer $2k+1$. He said that I had proved that , $S^{2k+1} \in \Delta_{3,6(2k+1)}$ by different arguments. In 1982, Choi, Dai, Lam and I [2] cited Stengle's claimed the same for M instead of S. No proofs for M or S were given at the time.

There are some key old school techniques in the proof of this result for M. First if p is psd and $p(a) = 0$ for $a \neq 0$, then $p = \sum f_j^2$, implies $f_j(a) = 0$. If now also $p^{2k+1} = \sum f_j^2$, then each f_j vanishes at least to $2k$-th order at a. When a is a unit vector, more information is easily derived. Using multinomial notation: $i = (i_1 \dots, i_n)$, $x^i = x_1^{i_1} \cdots x_n^{i_n}$ and $f \in \mathbb{R}[x_1, \dots, x_n]$ with $f(x_1, \dots, x_n) = \sum_i a(f;i)x^i$, define $C(f) = cvx\{i : a(f;i) \neq 0\}$. Then [4]: $p(x) = \sum_{k=1}^{n} q_k^2(x) \implies C(q_k) \subseteq \frac{1}{2}C(p)$.

Second, by generalizing polarization, the following is known [3] for sos even forms which are even ($a(f;i) \neq 0$ implies $i \in (2\mathbb{Z})^n$). Suppose $p = \sum_i h_i^2$ is an even form, then we may write $p = \sum_j q_j^2$, where $q_j(x) = \sum_i c_j(i)x^i$ and all i's in a particular q_j belong to a single congruence class in $(\mathbb{Z}/2\mathbb{Z})^n$.

Third, if $f_j(t) \in \mathbb{R}[t]$ and for $s \geq 1$, $(t^2-1)^{2s} = \sum_{j=1}^r f_j(t)^2$. Then an elementary argument shows that $f_j(t) = \lambda_j(t^2-1)^s$ and $\sum_j \lambda_j^2 = 1$.

Theorem. *For any odd integer $2k+1 \geq 1$, M^{2k+1} is not sos.*

For the proof, note that if $M^{2k+1} = \sum f_j^2$, then

$$C(M^{2k+1}) = cvx\{(8k+4, 4k+2, 0), (4k+2, 8k+4, 0), (0, 0, 12k+6)\},$$
$$C(f_j) \subseteq cvx\{(4k+2, 2k+1, 0), (2k+1, 4k+2, 0), (0, 0, 6k+3)\}.$$

Since $M(0,0,1) = 1$, it follows that $1 = \sum_{j=1}^r f_j(0,0,1)^2$, and the non-zero values can only occur in those f_j which contain $x^0y^0z^{6k+3}$ as a summand; that is, those f_j in which the exponents of x and y are both even. If $x^{2a}y^{2b}z^{6k+3-2a-2b}$ occurs in f_j and $2a = 0$, then $2b = 0$; the largest possible value of $2a$ in any f_j is $4k+2$, but x^{4k+2} must be paired with y^{2k+1}, so the largest exponent of x where the exponent of y is also even is at most $4k$. Thus, $f_j(t,1,1) = c_{j,(0,0,6k+3)} + t^2h_j(t)$ and $deg(f_j(t,1,1)) \leq 4t$.

Now note that $M(t,1,1) = (t^2-1)^2$, and so $(t^2-1)^{4k+2} = \sum_j f_j(t,1,1)^2$, $f_j(t,1,1)$ must be a multiple of $(t^2-1)^{2t+1}$ and so $f_j(t,1,1) \equiv 0$. Thus, for all j, $c_{j,(0,0,6k+3)} = 0$, contradicting that the sum of the squares is 1. $\square$

We remark without proof that the same argument applies to a larger set of forms. Suppose $q(y,z) \in P_{2,m}$ for some *even* form q. Then no odd power of the psd even form $z^mM(x,y,z) + x^2(y^2-z^2)^2q(y,z)$ is sos. (This was proved the morning after the talk, which is how things go at MFO.)

With minor variations, it can be proved in the same way that S^{2k+1} and Q^{2k+1} are never sos, and by looking purely at the zeros, R^3 and S_t^3 are not sos, where S_t is a family in $\Delta_{3,6} \cap E(P_{3,6})$ ([5, §6.5]) which interpolates between S and R.

We foolishly make the following conjecture:

Conjecture. *If p is psd, not sos and extremal, then p^{2k+1} is not sos.*

At this point, I should mention a remarkable 2012 theorem of Claus Scheiderer [6], a special case of which is that if $p \in P_{n,m}$ is positive definite, then for sufficiently large r, $p^r \in \Sigma_{n,rm}$. There is no contradiction between this result and the conjecture, because positive definite forms can't be extremal.

Stengle's example is not extremal: define $T_c(x,y,z) = cx^3z^3 + (y^2z - x^3 - z^2x)^2$. Then it is not hard to show that $T_c \in P_{3,6}$ for $|c| \leq \sqrt{256/27}$, so $T = T_1 \notin E(P_{3,6})$. Stengle's argument shows that T_c^{2k+1} is not sos if $0 < c \leq \sqrt{256/27}$.

Let $M_c(x,y,z) = x^4y^2 + x^2y^4 + z^6 - cx^2y^2z^2$. Then $M_c \in \Delta_{3,6}$ if $c \in (0,3]$ and M_3 is the Motzkin form. Using binomial squares plus one monomial square, we can show that $M_c^3 \in \Sigma_{3,18}$ if $c^3 \leq \frac{15}{13}$. More elaborate variations show that we can increase c to about 1.1336, which is almost certainly not optimal.

Define $\Sigma_{n,m}(2k+1) = \{f \in P_{n,m} \mid f^{2k+1} \in \Sigma_{n,(2k+1)m}\}$, so that we have $\Sigma_{n,m}(1) = \Sigma_{n,m}$. A natural question is whether $\Sigma_{n,m}(2k+1)$ is a closed convex cone. Closed is easy; convex seems difficult. I'd guess the answer is "no," but wouldn't bet any money on it. A positive answer would mean that if p^{2k+1} and q^{2k+1} are sos, then so is $(p+q)^{2k+1}$. I can prove a weaker result.

Theorem. *Suppose p and q are polynomials and p^{2k+1} and q are both sos. Then $(p+q)^{2k+1}$ is also sos.*

As a hint of the proof, $(p+q)^3 = p^3 + q(3p^2 + 3pq + q^2)$ and moreover also $3p^2 + 3pq + q^2 = \frac{3}{4}(2p+q)^2 + \frac{1}{4}q^2$ is sos. (Added: It can also be shown that if p^3 and q^3 are sos, then so is $(p+q)^5$. More generally, I conjecture that if p^{2i+1} and q^{2j+1} are sos, then so is $(p+q)^{2i+2j+1}$.

This result is already enough to prove that there exists $c_k \in [0,3)$ so that

$$\{c \mid (x^4y^2 + x^2y^4 + 1 - cx^2y^2)^{2k+1} \text{ is sos}\} = (-\infty, c_k].$$

Observe that if p is psd and p^{2k+1} is sos and $\ell > k$, then $p^{2\ell+1} = (p^{\ell-k})^2 \cdot p^{2k+1}$ will also be sos, so the sequence (c_k) is non-decreasing. What is $\lim_{k\to\infty} c_k$?

So, to sum up:

Question. *If p is psd and not sos, is p^3 sos?*

Answer. *Maybe, maybe not.*

Question. *If p and p^3 is psd and not sos, can p^5 be sos?*

Answer. *I'd guess "yes", but I don't have an example here.*

In a similar vein, let M be the Motzkin form and for $c \geq 0$, let

$$M_c(x,y,z) = M(x,y,z) + c(x^2+y^2+z^2)^3.$$

When $c > 0$, Scheiderer's theorem implies that there exists $N(c)$ so that M_c^t is sos whenever t is an odd integer $\geq N(c)$. Let $I_{2k+1} = \{c \mid M_c \in \Sigma_{3,6(2k+1)}\}$, and let $\beta_{2k+1} = \inf(I_{2k+1})$; it's not hard to show that $I_{2k+1} = [\beta_{2k+1}, \infty)$. Further, $\lim_{k\to\infty} \beta_{2k+1} = 0$ and β_{2k+1} is a non-increasing sequence of positive numbers, so there are infinitely many k for which $\beta_{2k+1} > \beta_{2k+3}$. For such a k, and $f = M_{\beta_{2k+3}}$, f^{2k+1} is not sos and f^{2k+3} is sos.

Finally, why is this question interesting, except as a curiosity? For one thing, if f^{2k+1} is sos, then f can be written as a sum of squares in which the denominators are f^k. This makes it easier to produce a certificate that f is psd. Just check if f, f^3, f^5, etc. are sos. Stop when you get tired. It also gives the opportunity to produce interesting intermediate sets of forms between $\Sigma_{n,m}$ and $P_{n,m}$, even if they might not be cones. This question was of interest to Stengle and Scheiderer in the context of Positivstellensatzen, which is beyond the scope of this talk. We definitely need more quantitative information in this direction!

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An Intrinsic Characterization of Moment Functionals in the Compact Case

Maria Infusino
(joint work with S. Kuhlmann, T. Kuna, P. Michalski)

We deal with the following instance of moment problem for compactly supported measures, which is general enough to include infinite dimensional situations such as measures supported in infinite dimensional spaces or linear functionals defined on infinitely generated algebras.

Question 1. *Let A be a unital commutative $\mathbb{R}$–algebra (not necessarily finitely generated) whose character space $X(A)$ is non-empty. Given a linear functional $L\colon A \to \mathbb{R}$ with $L(1) = 1$, does there exist a Radon measure ν on $X(A)$ such that*

$$L(a) = \int_{X(A)} \alpha(a) d\nu(\alpha) \qquad \text{for all } a \in A \tag{1}$$

and the support of ν is compact*?*

Recall that:

- the *character space* $X(A)$ consists of all $\mathbb{R}$–algebras homomorphisms from A to $\mathbb{R}$ and is here endowed with the weakest (Hausdorff) topology such that for each $a \in A$ the function $\hat{a}\colon X(A) \to \mathbb{R}$, $\alpha \mapsto \alpha(a)$ is continuous;
- a *Radon measure* ν on $X(A)$ is a non-negative measure on the Borel σ–algebra w.r.t. $\tau_{X(A)}$ that is locally finite and inner regular w.r.t. compact subsets of $X(A)$;
- the *support of* ν, denoted by $\mathrm{supp}(\nu)$, is the smallest closed subset C of $X(A)$ for which $\nu(X(A) \setminus C) = 0$ holds.

If a Radon measure ν as in (1) does exist, then we call ν a *representing Radon measure for* L and we say that L is a *moment functional.* In fact, if the support of a representing Radon measure is compact, then the representation in (1) is unique (see e.g., [5, Section 3.3] or [6, Chapter 12]).

In [3, Theorem 1.2] we provide the following characterization all moment functionals with compactly supported representing measure solely in terms of a new growth condition intrinsic to the given linear functional (see (2)), which also allows us to exactly identify the compact support of the representing measure.

Theorem 2. *Let $L\colon A \to \mathbb{R}$ be linear with $L(A^2) \subseteq [0, \infty)$ and $L(1) = 1$. Then there exists a unique representing Radon measure ν_L for L with compact support* *if and only if*

$$\sup_{n\in\mathbb{N}} \sqrt[2n]{L(a^{2n})} < \infty \text{ for all } a \in A. \tag{2}$$

Moreover, in this case,

$$\operatorname{supp}(\nu_L) = \left\{\alpha \in X(A) : |\alpha(a)| \leq \sup_{n\in\mathbb{N}} \sqrt[2n]{L(a^{2n})} \text{ for all } a \in A\right\}. \tag{3}$$

To establish the existence of the unique representing measure ν_L, we exploit the recent general version of the classical Nussbaum theorem in [2, Theorem 3.17], whose applicability is guaranteed because our growth condition (2) implies Carleman's condition, i.e. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(a^{2n})}} = \infty$ for all $a \in A$. Also the exact identification of the support of the representing measure in (3) entirely relies on the growth condition(2), but is the most surprising and novel feature of Theorem 2. In fact, to the best of our knowledge, the other results available in the literature on Question 1 characterize the existence of a compactly supported representing measure in terms of the non-negativity of the starting functional on an Archimedean quadratic module (see e.g., [4]) or in terms of its continuity w.r.t. submultiplicative seminorms (see e.g., [1]), but the support is only shown to be contained in a compact set associated with the considered quadratic module resp. submultiplicative seminorm and so not exactly identified (for more references, see [3]).

Analyzing the equivalence of our growth condition (2), the positivity condition in [4], and the continuity condition in [1] independently of the existence of the representing measure ν_L, we also determine explicit descriptions of $\operatorname{supp}(\nu_L)$ in terms of the largest Archimedean quadratic module on which L is non-negative and in terms of the smallest submultiplicative seminorm w.r.t. which L is continuous (see [3, Corollary 3.13]). All the above mentioned equivalent conditions and the characterizations of $\operatorname{supp}(\nu_L)$ are collected in the following result.

Corollary 3. *Let $L\colon A \to \mathbb{R}$ be linear with $L(A^2) \subseteq [0,\infty)$ and $L(1) = 1$. Then the following are equivalent.*

(i) There exists a unique representing Radon measure ν_L for L with compact support.
(ii) $\sup_{n\in\mathbb{N}} \sqrt[2n]{L(a^{2n})} < \infty$ for all $a \in A$.
(iii) L is p–continuous for some submultiplicative seminorm p on A.
(iv) L is Q–positive for some Archimedean quadratic module Q in A.

In this case,

$$\begin{aligned}
\operatorname{supp}(\nu_L) &= \left\{\alpha \in X(A) : |\alpha(a)| \leq \sup_{n\in\mathbb{N}} \sqrt[2n]{L(a^{2n})} \text{ for all } a \in A\right\} \\
&= \{\alpha \in X(A) : \alpha \text{ is } p_L\text{−continuous}\} \\
&= \{\alpha \in X(A) : \alpha(a) \geq 0 \text{ for all } a \in Q_L\}.
\end{aligned}$$

where

- *p_L is the seminorm defined by $p_L(a) := \sup_{n\in\mathbb{N}} \sqrt[2n]{L(a^{2n})}$ for all a in A, which is in fact the smallest submultiplicative seminorm on A w.r.t. which L is continuous.*
- *Q_L is the quadratic module generated by $\sup_{n\in\mathbb{N}} \sqrt[2n]{L(a^{2n})} \pm a$ with $a \in A$, which is in fact the largest Archimedean quadratic module in A on which L is non-negative.*

Thanks to Theorem 2, we also derive an explicit formula for computing the measure of singletons in $\operatorname{supp}(\nu_L)$, which in turn gives a sufficient condition for $\operatorname{supp}(\nu_L)$ to be countable as well as a necessary and sufficient condition for $\operatorname{supp}(\nu_L)$ to be a finite set (for more details, see [3, Section 3.4]).

We also construct and compare two locally convex topologies on A closely related to the growth condition (2) and making A a topological algebra. Moreover, we show that if A is endowed with a locally convex topology belonging to a certain class, then assuming the growth condition (2) only on the generating elements of a dense subalgebra of A is sufficient for the existence of ν_L (see [3, Section 4]).

In future work we would like to investigate whether some of the techniques used for the results here presented could be exploited to solve some instances of the moment problem in the non-compact case as well as in the truncated case. Also the tractability of the growth condition in concrete applications is still to be explored, and might be a valuable tool to ensure the convergence of sequences of outer approximations of the compact support.

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Embedding Classical Dynamics in a Quantum Computer

Dimitrios Giannakis

(joint work with S. Das, A. Ourmazd, P. Pfeffer, J. Schumacher, J. Slawinska)

The simulation of classical dynamical systems by quantum systems is a problem of growing interest in recent years, motivated at least in part by the potential of quantum computation to solve classically intractable computational problems. Mathematically, quantum theory has deep connections with the operator-theoretic formulation of ergodic theory [3], particularly from an algebraic perspective that associates to a classical dynamical system an abelian algebra of observables (functions of the state) and to a quantum system a non-abelian operator algebra of quantum observables acting on Hilbert space. Leveraging these connections provides a natural route for designing quantum algorithms for simulation of classical dynamics.

In this talk, we present a framework [4] for simulating a class of measure-preserving, ergodic dynamics in a quantum computer that is based on a combination of techniques from Koopman operator theory and harmonic analysis. We focus on a class of continuous-time, measure-preserving, ergodic flows with finitely generated pure point spectrum. By the Halmos–von Neumann theorem, such systems are topologically conjugate to irrational rotations on tori, and we can take advantage of the structured nature of these systems to design quantum algorithms that are asymptotically consistent with the Koopman evolution of observables of the classical system.

Our approach is based on an embedding that represents classical states x in $X \equiv \mathbb{T}^d$ by density operators $\rho_x : \mathcal{H} \to \mathcal{H}$ acting on a reproducing kernel Hilbert space, $\mathcal{H}$, of complex-valued functions on the classical state space X. The Hilbert space $\mathcal{H}$ is dense in the space of continuous functions on X and it is chosen such that (i) it is a Banach *-algebra with respect to pointwise function multiplication and complex conjugation; and (ii) the group of Koopman operators (composition operators) $U^t : \mathcal{H} \to \mathcal{H}$ with $U^t f = f \circ \Phi^t$ is well-defined as a strongly continuous unitary evolution group on $\mathcal{H}$ [1]. Here, $\Phi^t : X \to X$ is the classical dynamical flow

The Banach algebra structure of $\mathcal{H}$ provides, through its regular representation, a mapping $\pi : \mathcal{H} \to B(\mathcal{H})$ from classical observables in $\mathcal{H}$ into quantum observables in $B(\mathcal{H})$ that act as multiplication operators, $(\pi f)g := fg$. Moreover, by virtue of the reproducing property of $\mathcal{H}$ we have the identity $f(x) = \operatorname{tr}(\rho_x(\pi f))$ for every $x \in X$ and $f \in \mathcal{H}$. The latter, allows us to consistently represent pointwise evaluation of classical observables by quantum mechanical expectation values.

Quantum mechanical observables $a \in B(\mathcal{H})$ evolve under the action of the Koopman operator as $a \mapsto U^t a U^{t*}$. The dual picture to this is evolution is the evolution of density operators ρ_x as $\rho_x \mapsto U^{t*} \rho U^t$. The latter, can can be thought of as a lifted version of the evolution of classical probability measures under the transfer operator (the dual to the Koopman operator).

To arrive at a finite-dimensional quantum system that is implementable on a quantum computer, we project the states ρ_x of the infinite-dimensional quantum system on $\mathcal{H}$ onto finite-rank density operators on a 2^n-dimensional tensor product Hilbert space associated with n qubits. Importantly, due to the group structure of the spectrum of the Koopman operator for pure point spectrum systems, the unitary evolution operators associated with the projected system admit a tensor-product factorization allowing efficient implementation in a quantum circuit. In particular, a quantum circuit utilizing $O(n^2)$ quantum gates simulates the evolution and measurement of classical observables in a subspace of $\mathcal{H}$ of dimension 2^n.

We illustrate our approach with quantum circuit simulations of low-dimensional dynamical systems, as well as actual experiments on the IBM Quantum System One that show promising results for low qubit numbers ($n \leq 4$).

A pertinent research question stemming from this work is whether similar quantum algorithms can be constructed for measure-preserving ergodic flows that are not of pure-point-spectrum type, particularly weak-mixing or mixing systems where the point spectrum of the Koopman operator on L^2 is trivial, consisting of only a single simple eigenvalue 1 with constant corresponding eigenfunctions. In such cases, it is possible to consistently approximate (in the sense of strong convergence of spectral measures) the generator of the Koopman group on L^2 by operators acting on suitably constructed RKHSs that have discrete spectra [2]. However, the spectra of these regularized operators do not have a group structure analogous to the spectrum of the Koopman generator for a pure-point-spectrum system, posing obstacles to efficient approximation within quantum circuits. We believe that exploring ways of deriving efficient and consistent quantum algorithms for such systems is an interesting topic for future work.

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Trace-Positive Noncommutative Polynomials and the Unbounded Tracial Moment Problem

Igor Klep

(joint work with C. Scheiderer, J. Volčič)

This talk concerns noncommutative polynomials, i.e., elements of a free algebra, with our primary focus being evaluations in matrices and their (normalized) traces.

Consider the following noncommutative lift of the Motzkin polynomial,

$$M = x_2x_1^4x_2 + x_2^2x_1^2x_2^2 - 3x_2x_1^2x_2 + 1.$$

It is well-known (see, e.g., [KS08], that for any pair of real symmetric $n \times n$ matrices X_1, X_2, the evaluation

$$M(X_1, X_2) := X_2X_1^4X_2 + X_2^2X_1^2X_2^2 - 3X_2X_1^2X_2 + I_n$$

has nonnegative trace. Alternately, as observed by Quarez [Qua15], M admits a Hilbert 17th-type sum of squares up to commutators certificate with denominators.

Namely, M is a sum of commutators c and sums of hermitian squares s with denominators for

$$\begin{aligned} s &= (1-x_1^2x_2^2)^*(1+x_1^2)^{-1}(1-x_1^2x_2^2)+x_2(x_1^3-x_1)(1+x_1^2)^{-1}(x_1^3-x_1)x_2 \\ &\quad +(x_2^2-1)x_1(1+x_1^2)^{-1}x_1(x_2^2-1), \\ c &= 2\big[x_2,[(1+x_1^2)^{-1},x_2]\big]. \end{aligned}$$

The main results of the talk present a weak version of Artin's solution to Hilbert's 17th problem for trace positive noncommutative polynomials. More precisely, we establish the following:

Theorem 1. *The following are equivalent for a noncommutative polynomial f:*

(i) $\tau(f(X)) \geq 0$ *for all finite von Neumann algebras $(\mathcal{F},\tau)$ and tuples X of self-adjoint operators in $\mathcal{F}$;*[1]

(ii) f *lies in the closure of $\mathcal{K}$ with respect to the finest locally convex topology on the free algebra;*

(iii) *for every $\varepsilon>0$ there exists $r\in\mathbb{N}$ such that*

$$f+\varepsilon\sum_{j=1}^{n}\sum_{k=0}^{r}\frac{1}{k!}x_j^{2k}$$

is a sum of hermitian squares and commutators in the free algebra.

Here, $\mathcal{K}$ is the convex cone of all noncommutative polynomials that can be written as sums of hermitian squares and commutators of elements in the localized subalgebra

$$\mathbb{R}\langle x_1,\dots,x_n,(1+x_1^2)^{-1},\dots,(1+x_n^2)^{-1}\rangle$$

of the free skew field (=universal skew field of fractions of the free algebra). Thus the equivalence between (i) and (ii) of Theorem 1 can be seen as a weak solution to the tracial Hilbert's 17th problem. The equivalence between (i) and (iii) is the tracial Störungspositivstellensatz and was inspired by the commutative analog due to Lasserre [Las06].

The cone $\mathcal{K}$ is known not to be closed, e.g. the homogenized noncommutative Motzkin polynomial

$$x_2^2x_1^2x_2^2+x_1^2x_2^2x_1^2+x_3^6-3x_1x_2x_3^2x_2x_1$$

is trace positive (thus in $\overline{\mathcal{K}}$ by Theorem 1), but is not in $\mathcal{K}$. Since $\overline{\mathcal{K}}$ seems difficult to work with, it would be interesting to have more malleable descriptions for it. One possible candidate is the so-called dagger closure often considered in real algebraic geometry:

Open Problem. *Is $\overline{\mathcal{K}}=\mathcal{K}^\dagger$, where*

$$\mathcal{K}^\dagger := \{f \mid \textit{there is an nc polynomial } g \textit{ such that } f+\varepsilon g\in\mathcal{K} \textit{ for every } \varepsilon>0\}?$$

The key step to establishing Theorem 1 through convex duality is solving a tracial moment problem. We prove the following:

[1] Routine considerations show that one can restrict to II_1 factors $\mathcal{F}$ here.

Theorem 2. *Let φ be a unital linear functional on the free algebra in $x_1, \ldots, x_n$.*

(a) *There exists a finite von Neumann algebra $(\mathcal{F}, \tau)$ and tuple of power-integrable self-adjoint operators X affiliated with $\mathcal{F}$ s. t. $\varphi(p) = \tau(p(X))$ for all noncommutative polynomials p if and only if $\varphi(\mathcal{K}) = \mathbb{R}_{\geq 0}$.*

(a) *The equivalent conditions in (a) hold if there is $M > 0$ s. t. $\varphi(x_j^r) \leq r!M^r$ for all $j = 1, \ldots, n$ and even $r \in \mathbb{N}$.*

The recent negative answer to Connes' embedding problem [JNVWY] implies that in general, one cannot restrict (i) in Theorem 1 or (a) in Theorem 2 to finite-dimensional von Neumann algebras (i.e., matrix algebras). This brings us to the following question.

Open Problem. *Find a simple and explicit example of a noncommutative polynomial that is trace positive in all matrix algebras but has negative trace in a II_1-factor.*

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The Minkowski Sum of Sums of Squares and Sums of Nonnegative Circuit Forms

Moritz Schick

(joint work with M. Dressler, S. Kuhlmann)

An n-ary form over $\mathbb{R}$ is positive semidefinite, if it takes nonnegative values on $\mathbb{R}^n$. The set of such forms of degree $2d$ is a convex cone denoted by $P_{n,2d}$. In general, checking membership in $P_{n,2d}$ is difficult, therefore one often considers membership in appropriate convex subcones. On the one hand, the sums of squares cone $\Sigma_{n,2d}$ has a long history in Mathematics, with results going back to Hilbert's seminal work in [4]. On the other hand, the sums of nonnegative circuit forms cone $C_{n,2d}$ is a rather newly established cone, first formally defined in [5].

Motivated by [3], we study the convex hull of $\Sigma_{n,2d} \cup C_{n,2d}$, i.e. the Minkowski sum $(\Sigma + C)_{n,2d} := \Sigma_{n,2d} + C_{n,2d}$. Clearly, the following chain of inclusions hold:

$$\Sigma_{n,2d} \cup C_{n,2d} \subseteq (\Sigma + C)_{n,2d} \subseteq P_{n,2d}.$$

Thus, one can ask the following:

Question 1. *For which case of $n, 2d$ is $\Sigma_{n,2d} \cup C_{n,2d} \subsetneq (\Sigma + C)_{n,2d}$?*

Question 2. *For which case of $n, 2d$ is $(\Sigma + C)_{n,2d} = P_{n,2d}$?*

In the course of this talk, we presenting explicit forms separating $(\Sigma + C)_{n,2d}$ from $\Sigma_{n,2d} \cup C_{n,2d}$ and $P_{n,2d}$ from $(\Sigma + C)_{n,2d}$. Regarding Question 1, we show the following.

Theorem 3. *The identity $\Sigma_{n,2d} \cup C_{n,2d} = (\Sigma + C)_{n,2d}$ holds if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.*

The if-part follows by Hilbert's 1888 Theorem [4], since for $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$, we have $\Sigma_{n,2d} = (\Sigma + C)_{n,2d} = P_{n,2d}$. For the only if part, it suffices to show that for the two base cases $(n, 2d) \in \{(3, 6), (4, 4)\}$, it holds $P_{n,2d} \backslash (\Sigma + C)_{n,2d} \neq \emptyset$. Indeed, we show that such forms can be found via an SOS-perturbation of the Motzkin form [6] ($(n, 2d) = (3, 6)$) and the Choi-Lam form [2] ($(n, 2d) = (4, 4)$).

Question 2 on the other hand is fully answered in [1], where the author shows the following.

Theorem 4. *The identity $(\Sigma + C)_{n,2d} = P_{n,2d}$ holds if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.*

The proof in [1] does not provide explicit forms in $P_{n,2d} \backslash (\Sigma + C)_{n,2d} = P_{n,2d}$ for $n \geq 3, 2d \geq 4, (n, 2d) \neq (3, 4)$. However, it can indeed be shown that for the base cases $(n, 2d) \in \{(3, 6), (4, 4)\}$ such examples are given by the Robinson forms [7]. Examples for higher degrees can be achieved with a suitable multiplier. For higher number of variables, the same examples apply.

Furthermore, we show that the cone $(\Sigma + C)_{n,2d} = P_{n,2d}$ has the following properties:

(1) It is a proper cone (i.e. convex, closed, pointed and has nonempty interior) in the finite dimensional vector space $H_{n,2d}$ of n-ary forms of degree $2d$.
(2) It is neither closed under multiplication nor under linear transformation of variables.
(3) Membership in $(\Sigma + C)_{n,2d}$ can be checked via a combination of semidefinite and relative entropy programming.

In our future work, we aim for an exploitation of the cone $(\Sigma + C)_{n,2d}$ in a polynomial optimization setting. The goal is to find hierarchies of lower bounds for unconstrained or constrained polynomial optimization problems that can be computed efficiently using membership conditions in $(\Sigma + C)_{n,2d}$.

An open question regarding the interplay of the cones $\Sigma_{n,2d}$ and $C_{n,2d}$ is a characterization of their intersection $\Sigma_{n,2d} \cap C_{n,2d}$.

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Positivstellensätze for Semirings

Konrad Schmüdgen
(joint work with M. Schötz)

This talk is about Positivstellensätze for semirings of a commutative unital real algebra A. Let C be a convex cone in A such that $1 \in C$. We define

$$C^\dagger := \{a \in A : (a + \varepsilon 1) \in C \textit{ for all } \varepsilon > 0\}.$$

If S is a semiring of A (that is, S is a convex cone with $1 \in S$ which is invariant under multiplication), then C is called an S-module if $ac \in C$ for all $a \in S$ and $c \in C$.

It is shown that if C is an S-module of an Archimedean semiring S, then $C^\dagger$ is an Archimedean quadratic module and $S^\dagger$ is an Archimedean preordering. Also, if C is an Archimedean quadratic module, then $C^\dagger$ is an Archimedean preordering. This implies that the Archimedean Positivstellensätze for semirings and quadratic modules can be derived from each other. In particular it yields a unified operator-theoretic approach to both results.

Further, a number of notions and results on general semirings in A are developed. Another main result is a general Positivstellensatz with denominators for semirings of filtered algebras. As an application of this theorem, a denominator-free Positivstellensatz for the cylindrical extension of an algebra with Archimedean semiring is obtained. A number of applications and illustrating examples of these results are given.

The talk is based on joint work with Matthias Schötz, Arxiv 2207.02748.

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A Halmos-von-Neumann Theorem for Actions of General Groups

Patrick Hermle
(joint work with H. Kreidler)

A classical problem of topological dynamics is to determine whether given topological G-systems, G is a topological group, are isomorphic. Every G-system (G, K, φ), i. g. a continuous group action φ of G on a compact space K, induces a strongly continuous *Koopman representation*

$$T_\varphi \colon G \to \mathcal{L}(\mathrm{C}(K)), \quad g \mapsto T_{\varphi_g^{-1}},$$

where $T_{\varphi_g^{-1}} f = f \circ \varphi_{g^{-1}}$ for every $f \in \mathrm{C}(K)$. A G-system has *discrete spectrum* (see [1]) if

$$\mathrm{C}(K) = \overline{\bigcup \{M \subseteq \mathrm{C}(K) \mid M \text{ finite-dimensional } T_\varphi \text{ invariant subspace}\}}^{\|\cdot\|}.$$

The classical theorem of Halmos and von Neumann (see [2]) solves the isomorphism problem for minimal $\mathbb{Z}$-systems with discrete spectrum completely in the sense by addressing three aspects:

(i) (**Uniqueness**) The point spectrum of the Koopman operator T_{φ_1} is a complete isomorpism invariant.
(ii) (**Representation**) Every system is isomorphic to a rotation system on a compact monothetic group.
(iii) (**Realisation**) For every subgroup Γ of the unit disk there exists a $\mathbb{Z}$-system such that $\Gamma = \sigma(T_{\varphi_1})$.

We generalize this theorem replacing $\mathbb{Z}$ by an arbitary topological group (see [3]). While the proof of the classical Halmos-von Neumann theorem uses the structure of compact abelian groups (e. g. *Pontryagin duality*), the non-commutative situation is more intricate. Instead of character theory, we have to deal in the non-commutative case with finite-dimensional representation of dimension larger than 1. Therefore, we introduce the dual object

$$\hat{G} := \mathrm{Irr}(G)/\sim$$

of a topological group G, where $\mathrm{Irr}(G)$ are the irreducible finite-dimensional representations, and use the non-commutative analogue of Pontryagins duality, the so called *Tannaka-Krein duality* (see [4]).

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A Semidefinite Algorithm for Powers-of-Forms Decomposition

Alexander Taveira Blomenhofer

In powers-of-forms (POF) decomposition, we are given forms f_d of degree $d \cdot k$ in n variables for (various) values $d \in D$, e.g. $D = \{0, 1, 2, 3\}$, and we are asked to find k-forms $q_1, \dots, q_m$ such that $f_d = \sum_{i=1}^m q_i^d$ for all $d \in D$. For fixed d, this amounts to finding a k-Waring decomposition of f of rank m. It turns out that if $d \geq 3$ and m is not too large, general POF decompositions of rank m will be unique for the forms they describe, up to trivialities. The algorithmic task to recover the unique addends of a POF decomposition is poorly understood, but has applications in algebraic statistics, as e.g. the parameter estimation problem for mixtures of centered Gaussians from moments is a POF problem, where $k = 2$. We aim to understand for which ranks m, recovering the unique addends of some typical rank-m POF decomposition is possible with an efficient algorithm. In this talk, I generalize a classical algorithmic uniqueness result for 1-Waring decompositions to the case of all $k \geq 1$. The key ingredient is a condition that the second order power sum $\sum_{i=1}^m q_i^2$ has a particularly simple Gram spectrahedron.

As a first observation, note that if someone was to give us a basis $u_1, \dots, u_m$ of the space $q_1, \dots, q_m$, this would allow to consider the following reduction to the better-understood case $k = 1$: Consider the evaluation map

$$\varphi\colon \mathbb{R}[Y_1, \dots, Y_m] \to \mathbb{R}[q_1, \dots, q_m], Y_i \mapsto u_i$$

which is a homomorphism of graded algebras from the polynomial ring in m variables $Y_1, \dots, Y_m$ to the algebra generated by $\mathbb{R}[q_1, \dots, q_m]$. To make the previous statement true, we endow $\mathbb{R}[Y_1, \dots, Y_m]$ with the grading by the degree and we grade $\mathbb{R}[q_1, \dots, q_m]$ by $\frac{1}{2}$ the degree. The kernel of φ is the ideal of relations of $q_1, \dots, q_m$. The restriction $\varphi|_{\leq 3}$ to the graded components of degree at most 3 is thus an isomorphism of $\mathbb{R}$-vector spaces if and only if there are no algebraic relations of $q_1, \dots, q_m$ of degree at most 3. Given access to the basis $u_1, \dots, u_m$, the inverse map $\varphi_{\leq 3}^{-1}$ can be evaluated algorithmically by solving a linear system. In particular, it is then possible to compute the preimages g_1, g_2, g_3 of f_1, f_2, f_3 under the map φ. It is then straightforward that g_1, g_2, g_3 admit a joint POF decomposition

$$g_d = \sum_{i=1}^m \ell_i^d$$

where ℓ_i is the preimage of q_i. This reduced to a degree-3 POF decomposition where the number of addends $\ell_1, \dots, \ell_m$ equals the number of variables $Y_1, \dots, Y_m$. Algorithms for this special case are classically known. In other words, if $q_1, \dots, q_m$ do not satisfy any algebraic relations of degree ≤ 3, then recovering the space of $\langle q_1, \dots, q_m \rangle$ from the powers sums solves the problem.

Note that generic $q_1, \dots, q_m$ will not satisfy any relations of degree at most 3 if m is not too large. Indeed, for $m \leq n$, they will not satisfy any algebraic relation at all, but even for m slightly larger than n, the degree of algebraic relations will usually be higher than 3. E.g. from Bézout's theorem it follows that $n+1$ general quadratics will have a principal ideal of relations generated by a single polynomial of degree $2^m \geq 4$ (for $m \geq 2$).

The question remains how to recover the space $\langle q_1, \ldots, q_m \rangle$. This is where I make use of the second order power sum. Decompositions $f_2 = \sum_{i=1}^N p_i^2$ come with an associated subspace $\langle p_1, \ldots, p_N \rangle$ which is the image of the Gram tensor $\sum_{i=1}^N p_i^{\otimes 2} \in \mathrm{Gram}(f_2)$ associated with the decomposition. It turns out that this subspace is equal for all Sum-of-Squares decompositions $G \in \mathrm{Gram}(f_2)$ that share the same supporting face F. Let us write U_F for the subspace associated with each face F of the Gram spectrahedron of f_2.

Now, a Gram spectrahedron might have infinitely many faces, but sometimes the Gram spectrahedron of f_2 has a particularly simple facial structure. E.g. the most simple case is when $\mathrm{Gram}(f_2)$ is a singleton set, i.e. up to orthogonal transformations there is a unique Sum-of-Squares decomposition. It is then possible to compute the unique Sum-of-Squares decomposition $\sum_{i=1}^m q_i^{\otimes 2}$ of f_2 and recover the space from it.

With geometrical arguments on the real variety $V_{\mathbb{R}}(q_1, \ldots, q_m)$, it is easy to show that singleton Gram spectrahedra occur for typical choices of m quadratics at least as long as $m \leq n-1$. However, a numerical study I conduct with the SDP solver `MOSEK` shows that singleton Gram spectrahedra still typically occur also for pairs of values (m, n) where m is larger than n.

The aforementioned algorithm suggests that a natural problem in real algebra: What is the maximum number $m(n)$ for which there exists a Euclidean open subset $\mathcal{U}$ of the space of m-tuples of quadratics in n variables such that for all $q = (q_1, \ldots, q_m) \in \mathcal{U}$, $\sum_{i=1}^m q_i^2$ has a singleton Gram spectrahedron? So far, it is only known to me that $n - 1 \leq m(n) < \binom{n+1}{2}$.

Time-Dependent Moments from PDEs: The Heat Equation as an Example

Philipp J. di Dio

Partial differential equations (PDEs), (real) algebraic geometry (non-negative polynomials), and the theory of moments are intensively studied fields in mathematics. Much studied is the interaction between real algebraic geometry and the moment problem. Less studied is the interaction between these and partial differential equations. We present here recent results. We explicitly treat the heat equation. It is joint work with R. Curto, M. Korda, and V. Magron. The work is financed by the DFG project DI-2780/2-1 and the research fellowship of Ph. J. di Dio at the Zukunfskolleg of the University of Konstanz, funded as part of the Excellence Strategy of the German Federal and State Government.

Introduction. The heat equation is

$$\partial_t u = \Delta u$$
$$u(\,\cdot\,, 0) = u_0 \quad \text{on } \mathbb{R}^n$$

with $\Delta := \partial_1^2 + \dots + \partial_n^2$ and u_0 a Schwartz function. Its unique solution is $u(\,\cdot\,,t) = \Theta_t * u_0$, the convolution of the heat kernel Θ_t with the initial data u_0. By duality, the heat equation also acts on polynomials $u_0 = p_0 \in \mathbb{R}[x_1,\dots,x_n]$. The unique solution is then given by

$$p(\,\cdot\,,t) = \Theta_t * p_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot \Delta^k p_0 = \sum_{k=0}^{\lfloor (\deg p_0)/2 \rfloor} \frac{t^k}{k!} \cdot \Delta^k p_0 \in \mathbb{R}[x_1,\dots,x_n,t].$$

The convolution with the non-negative heat kernel shows that non-negativity is preserved while the second formulation is valid since the sum is finite (polynomials are entire vectors with respect to Δ). Hence,

$$e^{t\cdot\Delta} : \mathbb{R}[x_1,\dots,x_n] \to \mathbb{R}[x_1,\dots,x_n]$$

for all $t \in \mathbb{R}$ (not just $t \geq 0$) and the set of non-negative polynomials is invariant for all $t \geq 0$. Additionally, applying the Richter Theorem it is clear that from the convolution also sums of squares are mapped to sums of squares. In each case we have

$$\deg_x\, p(\,\cdot\,,t) = \deg_x\, p_0$$

for all $t \in \mathbb{R}$, i.e., in summary

$$e^{t\cdot\Delta} : \mathrm{Pos}(n,d) \to \mathrm{Pos}(n,d) \quad \text{and} \quad e^{t\cdot\Delta} : \mathrm{SOS}(n,d) \to \mathrm{SOS}(n,d)$$

for all $t \geq 0$. Again, by duality we have the group structure

$$e^{t\cdot\Delta} : \mathbb{R}[x_1,\dots,x_n]^* \to \mathbb{R}[x_1,\dots,x_n]^*$$

on the linear functionals on $\mathbb{R}[x_1,\dots,x_n]$. For $t \geq 0$ moment functionals are mapped to moment functionals, since non-negative polynomials are mapped to non-negative polynomials.

Results. The following (surprising) observations are the starting point of this investigation:

(1) the Motzkin polynomial $f_{\mathrm{Motz}}(x,y) = 1 - 3x^2y^2 + x^4y^2 + x^2y^4$ becomes a sum of squares $e^{\Delta} f_{\mathrm{Motz}} \in \mathrm{SOS}(2,6)$,

(2) the Robinson polynomial

$$f_{\mathrm{Rob}}(x,y) = 1 - x^2 - y^2 - x^4 + 3x^2y^2 - y^4 + x^6 - x^4y^2 - x^2y^4 + y^6$$

becomes a sum of squares $e^{t\Delta} f_{\mathrm{Rob}} \in \mathrm{SOS}(2,6)$ for all

$$t \geq \tau_{\mathrm{Rob}} \in \left(\frac{20\,946}{100\,000}, \frac{20\,947}{100\,000} \right),$$

(3) the Schmüdgen polynomial

$$\begin{aligned} f_{\mathrm{Schm}}(x,y) \quad &= \quad (y^2 - x^2)x(x+2)[x(x-2) + 2(y^2-4)] \\ &\qquad +200[(x^3-4x)^2 + (y^3-4y)^2] \end{aligned}$$

becomes a sum of squares for all $t \geq 2 \cdot 10^{-4}$,

(4) the Berg–Christensen–Jensen polynomial

$$f_{\mathrm{BCJ}}(x,y) = 1 - x^2y^2 + x^4y^2 + x^2y^4$$

becomes a sum of squares for all $t \geq \frac{1}{6}$.

For the Choi–Lam polynomial we can even determine the exact time, when it becomes a sum of squares.

Theorem 1. *Let*

$$f_{\mathrm{CL}}(x,y,z) := 1 - 4xyz + x^2y^2 + x^2z^2 + y^2z^2 \quad \in \mathrm{Pos}(3,4) \setminus \mathrm{SOS}(3,4)$$

be the Choi–Lam polynomial. Then

$$e^{t\Delta} f_{\mathrm{CL}} \in \begin{cases} \mathrm{Pos}(3,4) \setminus \mathrm{SOS}(3,4) & \text{for } t \in [0, 1/9), \text{ and} \\ \mathrm{SOS}(3,4) & \text{for } t \in [1/9, \infty). \end{cases}$$

To prove this result we have to look at all Gram matrix representations of $e^{t\Delta} f_{\mathrm{CL}}$. This Gram matrix proof enables us to prove the following.

Theorem 2. *There exists a $\tau_{3,4} \geq 1/9$ such that*

$$e^{t\cdot\Delta}\mathrm{Pos}(3,4) \subseteq \mathrm{SOS}(3,4)$$

for all $t \geq \tau_{3,4}$.

Connecting this to the moment problem we get the following.

Corollary. *Let $\tau_{3,4}$ be minimal from Theorem 1 and $L : \mathbb{R}[x,y,z]_{\leq 4} \to \mathbb{R}[x,y,z]_{\leq 4}$ be such that $e^{-\tau_{3,4}\cdot\Delta} L$ is strictly square positive. Then L is a moment functional.*

All given examples have in common that the highest degree part is a sum of squares. If this is not the case, the time-evolution with respect to the heat equation never becomes a sum of squares.

Theorem 3. *Let $p_0 \in \mathbb{R}[x_1, \ldots, x_n]$ be non-negative on $\mathbb{R}^n$ such that the highest degree part is not a sum of squares. Then $e^{t\cdot\Delta} p_0$ is not a sum of squares for all $t \in \mathbb{R}$.*

The case of $(n, 2d) = (n, 2)$ is trivial, since then $\mathrm{Pos}(n,2) = \mathrm{SOS}(n,2)$ by Hilbert's Theorem. The second case $(3,4)$ is solved by Theorem 2. Only the cases $(2, 2d)$ are by Theorem 3 the only other possibilities when non-negative polynomials can become sum of squares. This is still open.

Open Problem 1. *For which $d \in \mathbb{N}$ does there exist a $\tau_{2,2d} \geq 0$ such that*

$$e^{\tau_{2,2d}\cdot\Delta}\mathrm{Pos}(2,2d) \subseteq \mathrm{SOS}(2,2d).$$

We only looked at the operator $A = \Delta$. Hence, we have the following second open problem.

Open Problem 2. *For $(n,d) \in \mathbb{N} \times 2\mathbb{N}$ is there an (linear) operator A such that there exists a $\tau_{n,d}(A) \geq 0$ such that*

$$e^{t\cdot A}\mathrm{Pos}(n,d) \subseteq \mathrm{SOS}(n,d)$$

for all $t \geq \tau_{n,d}(A)$?

We thank the organizers for the conference at Oberwolfach and the possibility to present these research results.

An Elementary View to Transfer Operators

Gary Froyland

Transfer operators provide a powerful spectral approach to solving many problems in dynamics. In this tutorial-style presentation I will introduce dynamical systems and transfer operators in elementary settings. Dynamical systems that expand all directions in phase space, or more generally that have well-defined directions of uniform expansion and contraction are well-suited to transfer operator techniques. For such dynamics, there are suitable Banach spaces upon which the transfer operator is quasi-compact. Quasi-compactness provides an elegant approach (the Nagaev–Guivarc'h approach) [1]–[3] to proving statistical limit laws such as the central limit theorem, local central limit theorem, and large deviation principle; see also the survey [8] A spectral approach to extreme value theory is developed by Keller in [5]. Quasi-compactness of the transfer operator enables rigorous computation of fixed points, which correspond to physical invariant measures of the dynamics, and of the variance in the central limit theorem, and the rate function in a large deviation principle [10]–[11].

Real-world dynamics requires time-dependent dynamics and we introduce transfer operator cocycles. One must now define analogues of eigenvalues (Lyapunov exponents), eigenfunctions (Oseledets spaces), quasi-compactness, and isolated spectrum. With the help of multiplicative ergodic theory the necessary extensions of these concepts to the time-dependent or random dynamical systems can be carried out [6]–[7]. One may then try to construct a Nagaev–Guivarc'h-style approach to statistical limit laws for random dynamical systems. The fundamentals of this program have been carried out in [9], [12]. In the case of extreme value theory, one requires an extension of differentiable perturbation theory from deterministic dynamics [4] to the random situation [13]. Such an extension enables a very general extreme value theory [13] for random dynamics.

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Determinantal Polynomials and the Principal Minor Map

Cynthia Vinzant
(joint work with A. Al Ahmadieh)

The principal minor map takes an $n \times n$ matrix to the vector of its 2^n principal minors.The quest for an algebraic description of the image of this map dates back to 19th century classical algebraic geometry. This talk described a connection between this problem and certain classes of determinantal representations.

The image of an $n \times n$ matrix A under the principal minor map is given by $\varphi(A) = (A_S)_{S \subseteq [n]}$ where A_S is the determinant of the $|S| \times |S|$ submatrix of A whose rows and columns are indexed by S. By convention we take $A_\emptyset = 1$. Understanding the image of this map is a question in classical algebraic that has reemerged due to its connections with determinantal point processes, which are discrete probabilistic models with useful properties for sampling and computation.

There are nontrivial equations vanishing on the image of $n \times n$ symmetric matrices under this map starting at $n = 3$ and general $n \times n$ matrices starting at $n = 4$. In 1897, Nanson [7] gave some of the algebraic relations on $\varphi(\mathbb{C}^{4\times 4})$, a characterization that was completed by Lin and Sturmfels in 2009 [6]. A characterization for general n remains open. For symmetric matrices, Holtz and Sturmfels [4] show invariant under an action of the group $\mathrm{SL}_2(\mathbb{C})^n \rtimes S_n$ and conjectured that the vanishing of polynomials in the orbit of the *hyperdeterminant* under this group cuts out the image of the principal minor map over $\mathbb{C}$. This conjecture was resolved by Oeding [8]. We use a connection with determinantal representations to extend the results of Oeding to arbitrary fields and show that no such finite uniform description is possible in the general case.

The connection to determinantal polynomials is that the principal minors of an $n \times n$ matrix A are exactly the coefficients of the polynomial

$$f_A = \det\left(\operatorname{diag}(x_1, \ldots, x_n) + A\right) = \sum_{S \subseteq [n]} A_S \prod_{i \in [n] \setminus S} x_i. \tag{1}$$

We can there translate the problem of characterizing the image of the principal minor map to that of characterizing which multiaffine polynomials have such a representation. When the matrix A is Hermitian, then the polynomial f_A is real stable, meaning that it has no zeroes whose imaginary parts are all positive. Brändén [3] showed that a multiaffine polynomial $f = \sum_{S \subseteq [n]} a_S \prod_{i \in [n] \setminus S} \in \mathbb{R}[x_1, \ldots, x_n]$ is stable if and only if for all $i, j \in [n]$, the polynomial

$$\Delta_{ij}(f) = \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} - f\frac{\partial^2 f}{\partial x_i \partial x_j}$$

is nonnegative on $\mathbb{R}^n$. Kummer, Plaumann, and Vinzant [5] showed that f has a determinantal representation (1) with a real symmetric matrix A if and only if each of the polynomials $\Delta_{ij}(f)$ are squares in $\mathbb{R}[x_1, \ldots, x_n]$. The forward direction of this relies on Dodgson condensation. In [1], show that this extends to arbitrary fields k. Moreover, for $n = 3$, the quadratic discriminant of the polynomial $\Delta_{12}(f) \in k[x_3]$ is exactly Cayley's $2 \times 2 \times 2$ hyperdeterminant, which shows that Oeding's results extends to arbitrary fields.

In [2], we show that f has a determinantal representation (1) with a Hermitian matrix A if and only if each of the polynomials $\Delta_{ij}(f)$ are a sum of two squares $g^2 + h^2$ in $\mathbb{R}[x_1, \ldots, x_n]$ (alternatively a Hermitian square $(g + ih)(g - ih)$). Using this characterization, we show that for every $n \geq 4$, the image of $n \times n$ Hermitian matrices under the principal minor map is cut out by the image of the equations and inequalities defining the image for $n = 4$ under the group $\mathrm{SL}_2(\mathbb{C})^n \rtimes S_n$.

One might wonder if the image of general $n \times n$ matrices under the principal minor map can be described by the orbit of some finite set of equations and inequalities under $\mathrm{SL}_2(\mathbb{C})^n \rtimes S_n$. This turns out to be impossible. As in the symmetric and Hermitian case, one can show that for any i, j, the polynomial $\Delta_{ij}(f_A)$ must factor as a product of two multiaffine polynomials. The converse does not hold [2, Example 3.2]. Moreover, one can construct a family of multiaffine polynomials f_{2n+1} for $n \geq 2$ with the property that f_{2n+1} does not have a determinantal representation (1) but that after specializing any variable, it does. For example,

$$f_5 = x_1(x_3x_4 + 1)(x_2x_5 + 1) + (x_2x_3 + 1)(x_4x_5 + 1).$$

The polynomial $\Delta_{12}(f) = (x_3 - x_5)(x_3x_4 + 1)(x_4x_5 + 1)$ does not factor as the product of *two* multiaffine polynomials but that its specialization in any variable does. We show that the coefficient vector of f_{2n+1} satisfies conditions inherited from the image of $2n \times 2n$ matrices under the principal minor map but does not belong to the image of the principal minor map itself. This precludes the existence of a finite description of the image of general $n \times n$ using only the group action of $\mathrm{SL}_2(\mathbb{C})^n \rtimes S_n$. We end with two questions.

Question 1. *In [5], the authors show that if some power f^m of a polynomial f can be written as $\det(A_0 + x_1A_1 + \ldots + x_nA_n)$ for some $m \in \mathbb{Z}_+$ and real symmetric matrices $A_0, \ldots, A_n$ with $A_j \succeq 0$ for $j \geq 1$, then $\Delta_{ij}(f)$ is a sum of squares in $\mathbb{R}[x_1, \ldots, x_n]$ for all $i, j \in [n]$. Does the converse hold?*

Question 2. *What are equations describing the image of $\mathbb{C}^{5\times 5}$ under the principal minor map? Can the image of $\mathbb{C}^{n\times n}$ be cut out by equations obtained from the action of $\mathrm{SL}_2(\mathbb{C})^n \rtimes S_n$ and some increasing combinatorial structures?*

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ML-Degree of Statistical Models and the Beta-Invariant of a Matroid

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(joint work with M. Michałek)

We start with a common question from the field of statistics: Given a family $(f_\Sigma)_{\Sigma\in S}$ of probability distributions on $\mathbb{R}^m$ and some observed data vectors $d_1, \ldots, d_k \in \mathbb{R}^m$, which of the f_Σ describes the observed data best? To answer this question we usually want to find $\Sigma \in S$ such that

$$\log\left(\prod_{i=1}^{k} f_\Sigma(d_i)\right)$$

is maximized.

We will focus on the special case of linear concentration models, where some linear subspace $\mathcal{L} \subset \mathbb{R}^{m\times m}$ is fixed and the the family of probability distributions is given by

$$f_\Sigma : \mathbb{R}^m \to \mathbb{R}, x \mapsto \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{1}{2}x^{\mathrm{t}}\Sigma^{-1}x\right)$$

where $\Sigma^{-1} \in \mathcal{L}$ is positive semi-definite and invertible.

Denote by $\mathcal{L}^{-1}$ the complexification of the Zariski-closure of the set

$$\{\Sigma \in \mathcal{R}^{m\times m} \mid \Sigma^{-1} \in \mathcal{L}\}.$$

With the initially described optimization problem in mind, it is natural to define the Maximum-Likelihood-degree (ML-degree) of $\mathcal{L}$ to be the number of complex critical points $\Sigma \in \mathcal{L}^{-1}$ of the function

$$\Sigma \mapsto \log\left(\prod_{i=1}^{k} f_\Sigma(d_i)\right)$$

where $d_1, \ldots, d_k \in \mathbb{R}^m$ are general and k is sufficiently large. It is known that for a generic choice of $\mathcal{L}$ this number agrees with the degree of the variety $\mathcal{L}^{-1} \subseteq \mathbb{C}^{m\times m}$. This also happens in the special case where $\mathcal{L}$ only contains diagonal matrices, where both numbers can be shown to be an invariant of the matroid associated with $\mathcal{L}$. We will therefore now consider $\mathcal{L} \subseteq \mathbb{C}^m$ and think of it as a space of diagonal $m\times m$-matrices. After projectivizing, $\mathcal{L}^{-1}$ is then given as the image of the (rational) cremona map

$$\begin{aligned}
&\mathrm{crem} : \mathbb{P}(\mathbb{C}^m) \dashrightarrow \mathbb{P}(\mathbb{C}^m)\\
&(x_1 : \ldots : x_m) \mapsto (x_1^{-1} : \ldots : x_m^{-1}) = (x_2 \cdot \ldots \cdot x_m : \ldots : x_1 \cdot \ldots \cdot x_{m-1}).
\end{aligned}$$

Let M be the matroid associated to $\mathcal{L}$ in the following way: Consider the intersections of the coordinate hyperplanes with $\mathcal{L}$, i.e. $H_i := \{x_i = 0\} \cap \mathcal{L}, i = 1, \ldots, m$, then M is the matroid on $\{1, \ldots, m\}$, where a k-element subset $\{i_1, \ldots, i_k\}, k \leq m$ is independent if and only if $\mathrm{codim}_{\mathcal{L}}(H_{i_1} \cap \cdots \cap H_{i_k}) = k$. Let $\chi_M(q) \in \mathbb{Z}[q]$ and $\overline{\chi_M}(q) = \chi_M(q)/(q-1)$ be the characteristic and reduced characteristic polynomials of M.

In [3] it is shown that the coefficients of $\overline{\chi_M}(q)$ are (up to sign) given by the multidegree of the graph of the cremona map in $\mathbb{P}(\mathbb{C}^m) \times \mathbb{P}(\mathbb{C}^m)$. In particular the ML-degree of $\mathcal{L}$ and the degree of $\mathcal{L}^{-1}$ are both given as $\overline{\chi_M}(0)(-1)^{\mathrm{rank}(M)}$. This is an invariant of the matroid M but it is not the beta-invariant. However, both the ML-degree and the beta-invariant of a matroid are connected to counting bounded regions of a certain hyperplane arrangement, so it is natural to ask if there is a related matroid $\tilde{M}$ such that its beta-invariant $\beta(\tilde{M}) := \overline{\chi_{\tilde{M}}}(1)(-1)^{\mathrm{rank}(M)}$ equals the degree of $\mathcal{L}^{-1}$. In fact we can construct such a matroid from $\mathcal{L}$ in the following way:

(1) Consider the orthogonal complement (with respect to the standard inner product) $\mathcal{L}^\perp$ of $\mathcal{L}$ and shift it by some generic $u \in \mathbb{C}^m$.
(2) Intersect with the coordinate hyperplanes to obtain an arrangement of affine hyperplanes $H_i := \{x_i = 0\} \cap (\mathcal{L}^\perp + u), i = 1, \ldots, m$.
(3) Describe each H_i as the kernel of an affine linear form $q_i \in \mathbb{C}[x_1, \ldots, x_{m-d}]$, where $d = \dim(\mathcal{L})$ and we identify $\mathcal{L}^\perp$ with $\mathbb{C}^{m-d}$. Homogenize each q_i using a new variable y. The new kernels of the q_i give a hyperplane arrangement in $\mathcal{L}^\perp \oplus \mathbb{C}$. We add one more hyperplane

$$H_0 := \{(x, y) \in \text{Ł}^\perp \oplus \mathbb{C} \mid y = 0\}.$$

(4) From the new hyperplane arrangement we build a matroid $\tilde{M}$ on $m+1$ elements as described above, then $\beta(\tilde{M}) = \deg(\mathcal{L}^{-1})$.

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Positivity Certificates and Polynomial Optimization on Non-Compact Semialgbraic Sets

Ngoc Hoang Anh Mai

(joint work with J.-B. Lasserre, V. Magron)

The work is concerned with polynomial optimization on non-compact semialgebraic sets. Its spirit and main motivation is to voluntarily avoid the *big-ball trick* which reduces the problem to the compact case. The big-ball "trick" is to simply assume that the global minimum is attained in some *a priori known* ball B_M centered at zero of radius $M > 0$ potentially large. Therefore, by adding this additional constraint to the definition of the feasible set, one is back to the compact case.

Why? This "trick" has definitely some merit since in some practical applications such an M can be sometimes determined with *ad-hoc* arguments. However, it is not satisfactory from a mathematical point of view. Indeed after one has found a minimizer $x^\star \in B_M$, one is still left with the question: *Is really $x^\star$ a global minimizer? Was M chosen sufficiently large?* In other words, in doing so one does not obtain an certificate that $x^\star$ is a global minimizer. As we will see, the challenge is to *adapt* some certificates of positivity on non-compact sets already available in the literature, to turn them into a practical algorithm.

Reznick proves in [6] that any positive definite form can be multiplied by a large enough power of $\|x\|_2^2$ to become a sum of powers of linear forms (which is in particular a sums of squares (SOS) of polynomial). For this specific class of nonnegative polynomials, Reznick's result provides a suitable decomposition into SOS of rational functions, which can be practically computed via semidefinite programming (SDP). An interesting related result is the Positivstellensatz [5] of Putinar and Vasilescu.

Denote by $\mathbb{R}[x]$ the ring of real polynomials in vector of variable $x = (x_1, \dots, x_n)$. Let $\Sigma[x] \subset \mathbb{R}[x]$ stand for the set of SOS polynomials. Given $g = \{g_1, \dots, g_m\}$ in $\mathbb{R}[x]$, define

$$\begin{aligned} S(g) &:= \{x \in \mathbb{R}^n : \, g_j(x) \geq 0, \; j = 1, \dots, m\}, \\ Q(g) &:= \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j g_j \, : \, \sigma_j \in \Sigma[x] \right\}. \end{aligned}$$

Theorem 1 (Putinar–Vasilescu [5]). *Let $\theta \in \mathbb{R}[x]$ be the quadratic polynomial $x \mapsto \theta(x) := 1 + \|x\|_2^2$, and denote by $\tilde{p} \in \mathbb{R}[x, x_{n+1}]$ the homogeneous polynomial associated with $p \in \mathbb{R}[x]$, defined by $x \mapsto \tilde{p}(x) := x_{n+1}^{\deg(p)} p(x/x_{n+1})$.*

(1) Let $f \in \mathbb{R}[x]$ such that $\tilde{f} > 0$ on $\mathbb{R}^{n+1}\backslash\{0\}$. Then $\theta^k f \in \Sigma[x]$ for some $k \in \mathbb{N}$.

(2) Let $f, g_1, \ldots, g_m \in \mathbb{R}[x]$ satisfy the following two conditions:
   (a) $f = f_0 + f_1$ such that $\deg(f_0) < \deg(f_1)$ and $\tilde{f}_1 > 0$ on $\mathbb{R}^{n+1}\backslash\{0\}$;
   (b) $f > 0$ on $S(g)$.
   Then $\theta^{2k} f \in Q(g)$ for some $k \in \mathbb{N}$.

As a consequence, they also obtain:

Corollary 2 (Putinar–Vasilescu [5]). *Let $\theta := 1 + \|x\|_2^2$.*

(1) Let f be a polynomial in $\mathbb{R}[x]$ of degree at most $2d$ such that $f \geq 0$ on $\mathbb{R}^n$. Then for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\theta^k(f + \varepsilon\theta^d) \in \Sigma[x]$.

(2) Let $f \in \mathbb{R}[x]$ such that $f \geq 0$ on $S(g)$. Let $d \in \mathbb{N}$ such that $2d > \deg(f)$. Then for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\theta^{2k}(f + \varepsilon\theta^d) \in Q(g)$.

As already mentioned, our approach is to treat the non-compact case frontally and avoid the big-ball trick. Our contribution is threefold:

I. We first provide an alternative proof of Corollary 2, *with an explicit degree bound* on the SOS weights, by relying on Jacobi's technique in the proof of [3, Theorem 7]; this is crucial as it has immediate implications on the algorithmic side. More precisely, the degrees of the SOS weights σ_j are bounded above by $k + d - \lceil \deg(g_j)/2 \rceil$. First, one transforms the initial polynomials to homogeneous forms, then one relies on Putinar's Positivstellensatz for the compact case, and finally one transforms back the obtained forms to dehomogenized polynomials. As a consequence, with $\varepsilon > 0$ fixed, arbitrary, this degree bound allows us to provide hierarchies $(\rho_k^i(\varepsilon))_{k \in \mathbb{N}}$, $i = 1, 2, 3$ for unconstrained polynomial optimization (where $m = 0$ and $i = 1$) as well as for constrained polynomial optimization ($m \geq 1$ and $i = 2, 3$). Computing each $\rho_k^i(\varepsilon)$ boils down to solving a single SDP, with strong duality property. For k sufficiently large, $\rho_k^i(\varepsilon)$ becomes an upper bound for the optimal value $f^\star$ of the corresponding polynomial optimization problem (POP) $\min\limits_{x \in S(g)} f(x)$. If this problem has an optimal solution $x^\star$, the gap between $\rho_k^i(\varepsilon)$ and $f^\star$ is at most $\varepsilon\theta(x^\star)^d$. The related convergence rates are also analyzed in these sections.

II. In the second contribution, we provide a new algorithm to find a feasible solution in the semialgebraic set $S(g)$. The idea is to include appropriate additional spherical equality constraints $\varphi_t := \xi_t - \|x - a_t\|_2^2$, $t = 0, \ldots, n$, in $S(g)$ so that the system $S(g \cup \{\pm\varphi_0, \ldots, \pm\varphi_n\})$ has a unique real solution. The nonnegative reals $(\xi_t)_{t=0}^n$ are computed with an adequate moment-SOS hierarchy. Moreover, this solution might be extracted in certain cases by checking whether some (moment) matrix satisfies a flat extension condition (see [1, 4]).

III. Finally we use this method to approximate a global minimizer of f on $S(g)$. Namely, we fix $\varepsilon > 0$ small and find a point in $S(g \cup \{\rho_k^i(\varepsilon) - f\})$. This procedure works in certain cases, even if the set of minimizers is infinite. This is in deep contrast with the extraction procedure of [2] (via some flat extension condition) which works only for finite solution sets. Assuming that the set of solutions is finite, one may compare our algorithm with the procedure from [2] as follows. On the one hand, the latter extraction procedure provides global optimizers, provided that one has solved an SDP-relaxation with sufficiently large "k" (so as to get an appropriate rank condition). On the other hand, our algorithm that adds spherical equality constraints "divides" the problem into $n+1$ SDP relaxations with additional constraints but with smaller order "k" (which is *the* crucial parameter for the SDP solvers).

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On Algebraic Certificates for the Truncated Moment Problem

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(joint work with D. Henrion, M. Safey El Din)

Let $K = S(\boldsymbol{g}) \subset \mathbb{R}^n$ be a basic semialgebraic set defined by polynomial inequalities $g_1(x) \geq 0, \ldots, g_k(x) \geq 0$, where $\boldsymbol{g} = (g_1, \ldots, g_k) \in \mathbb{R}[x]^k$.

Given $d \in \mathbb{N}$ and a sequence of real numbers $\boldsymbol{y} = (y_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}_d^n}$,

$$\mathbb{N}_d^n = \{\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n : \sum_i \alpha_i \leq d\},$$

the *truncated moment problem* is the question of determining whether there exists a nonnegative Borel measure μ on $\mathbb{R}^n$, with support in K, and such that

$$y_\alpha = \int_K x^{\boldsymbol{\alpha}} \, d\,\mu, \qquad \text{for all } \boldsymbol{\alpha} \in \mathbb{N}_d^n.$$

In this case one says that $\boldsymbol{y}$ is *moment-representable* with respect to $\boldsymbol{g}$. It is the truncated version of the classical *moment problem*, see [5, 10].

The truncated moment problem can be interpreted as a convex conic feasibility program $L \cap C$, where the cone is the set $C = \mathscr{P}(K)_d$ of polynomials nonnegative on K, of degree at most d, and the linear space L is the hyperplane defined by the vanishing of the Riesz functional $\mathscr{L}_{\boldsymbol{y}} : \mathbb{R}[x]_{\leq d} \to \mathbb{R}$ defined by

$$\mathscr{L}_{\boldsymbol{y}}\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}^n_d} p_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}\right) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n_d} p_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}},$$

that is $L = \ker(\mathscr{L}_{\boldsymbol{y}})$.

When K is compact, the classical duality between moments and positive polynomials states that $\boldsymbol{y}$ is moment-representable whenever $\mathscr{L}_{\boldsymbol{y}}$ is nonnegative on $\mathscr{P}(K)_d$, in other words, if the mentioned conic program is *weakly feasible* (L intersects $\mathscr{P}(K)_d$ but does not intersect its interior). In this case there exists an atomic measure $\mu = \sum_{i=1}^s c_i\delta_{x_i}$ whose moment sequence of degree $\leq d$ is $\boldsymbol{y}$: such a measure is a (real) solution of a highly structured polynomial system of type multivariate Vandermonde. An open question from a computational point of view is how to compute efficiently such a measure, that represents a *certificate of representability* for the input vector: the user can compute again the moments of the measure by simply evaluating the integral and check that the input vector coincides with these moments.

On the other side of the coin, $\boldsymbol{y}$ is not moment-representable exactly when the conic program $L\cap\mathscr{P}(K)_d$ is *strongly feasible*: in algebraic terms, when there exists a polynomial $p \in \mathscr{P}(K)_d$, (strictly) positive on K, in L, the kernel of $\mathscr{L}_{\boldsymbol{y}}$. In our contribution we study algorithmic aspects of the computation of *irrepresentability algebraic certificates* when $\boldsymbol{y}$ is not moment-representable. We show the existence of explicitly strictly positive polynomials in the kernel of the Riesz functional. These have the form

$$p = 1 + \sigma_0 + \sigma_1 g_1 + \sigma_2 g_2 + \cdots + \sigma_k g_k \in 1 + Q(\boldsymbol{g})$$

where $Q(\boldsymbol{g})$ is the quadratic module associated with the description $\boldsymbol{g}$, and p satisfies $\mathscr{L}_{\boldsymbol{y}}(p) = 0$. One example which has been discussed during the talk at MFO is the following: fix $n = 2$ and $d = 6$, and let $\boldsymbol{g}$ be the vector $\boldsymbol{y} \in \mathbb{R}^{28}$ given by

$$\begin{aligned} y_{00} &= 32 & y_{22} &= 30 \\ y_{20} = y_{02} &= 34 & y_{60} = y_{06} &= 128 \\ y_{40} = y_{04} &= 43 & y_{42} = y_{24} &= 28. \end{aligned}$$

and all the entries are zeroes. A polynomial certifying that $\boldsymbol{y}$ is not a moment vector in the unit ball $K = \{\boldsymbol{a} \in \mathbb{R}^2 : 1 - a_1^2 - a_2^2 \geq 0\}$ is

$$p = 1 + \frac{8}{9}(1 - x^2 - y^2).$$

Remark that p certifies that the subvector $(y_{00}, y_{10}, y_{01}, y_{20}, y_{11}, y_{02})$ is already not moment-representable. After the talk, Greg Blekherman suggested an easier way of certifying that the previous vector is not moment-representable by means of binomial inequalities [3].

Our contribution shows also that there exist *rational certificates of unrepresentability*: when $\boldsymbol{y}$ is over $\mathbb{Q}$, the polynomial p can be chosen with rational coefficients. However, by the result of C. Scheiderer [9], even if p is in $\mathbb{Q}[\boldsymbol{x}]$, it is possible that any of its positivity certificates for the membership $p \in 1 + Q(\boldsymbol{g})$ are not rational. It is an open question whether such phenomenon can actually occur in this context.

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Matrix Polynomials, Symmetric Polynomials and Undecidability

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(joint work with J. Acevedo, S. Debus, C. Riener)

Our main object of interest is the so-called Vandermonde map which appears naturally in several contexts and provides connections between different mathematical domains. We begin with the following motivating problem: suppose that we are given a polynomial expression in traces of powers of symmetric matrices, such as

$$2\,\mathrm{tr}(A^2)\,\mathrm{tr}(B^6) - \mathrm{tr}(A^4)\,\mathrm{tr}(B^4),$$

is there an algorithm to decide whether this expression is nonnegative for all symmetric matrices A, B of all sizes? What happens if we replace trace by normalized trace $\widetilde{\mathrm{tr}}(A) = \frac{\mathrm{tr}\,A}{n}$, where n is the size of the matrix?

One of our main results is that the first (unnormalized) problem is *undecidable*, while the second one is *decidable*. The key to the hardness of the unnormalized problem is the fascinating geometry of *the image of the probability simplex under the Vandermonde map*. Some geometric properties of this set were observed in different areas of mathematics making it an important and beautiful object to study.

For any $n \times n$ matrix A recall that $\operatorname{tr}(A^d) = \lambda_1^d + \cdots + \lambda_n^d$, where λ_i are the eigenvalues of A. We use p_d to denote the d-th power sum polynomial:

$$p_d(x) = x_1^d + \cdots + x_n^d.$$

We see that testing whether $2\operatorname{tr}(A^2)\operatorname{tr}(B^6) - \operatorname{tr}(A^4)\operatorname{tr}(B^4)$ is nonnegative on all symmetric matrices of all sizes is equivalent to understanding whether $2p_2(x)p_6(y) - p_4(x)p_4(y)$ is nonnegative on all real vectors x and y of any dimension. Define the *d-th Vandermonde map* $\nu_{n,d}$ by sending a point in $\mathbb{R}^n$ to its image under the first d power sums:

$$\nu_{n,d}(x) = (p_1(x), \ldots, p_d(x)).$$

Let Δ_{n-1} be the probability simplex in $\mathbb{R}^n$: Δ_{n-1} consists of all vectors with nonnegative coordinates with the sum of coordinates equal to 1. We call the image $\nu_{n,d}(\Delta_{n-1})$ of the probability simplex under the Vandermonde map *the (n,d)-Vandermonde cell* and denote it by $\Pi_{n,d}$. Observe that the first coordinate of $\Pi_{n,d}$ is identically 1, and so we may project it out, and see $\Pi_{n,d}$ as the subset of $\mathbb{R}^{d-1}$, which is the image of Δ_{n-1} under $(p_2, \ldots, p_d)$.

Since $2p_2(x)p_6(y) - p_4(x)p_4(y)$ is an even homogeneous polynomial, deciding whether it is nonnegative for all $x, y \in \mathbb{R}^n$ is equivalent to deciding whether the polynomial $2a_1b_3 - a_2b_2$ is nonnegative on the product $\Pi_{n,3} \times \Pi_{n,3}$, where $a_i = p_i(x)$ and $b_i = p_i(y)$.

We reach two important conclusions: first, we are interested in nonnegativity of polynomials on (products of) Vandermonde cells $\Pi_{n,d}$, and second, to consider matrices of all sizes we need to take the *limit of the Vandermonde cell* $\Pi_{n,d}$ *as n goes to infinity*.

The Vandermonde cell $\Pi_{n,d}$ is a compact subset of $\mathbb{R}^{d-1}$, and our first main result is that $\Pi_{n,d}$ has *the combinatorial structure of a cyclic polytope*, verifying an experimental observation of [8].

For a fixed d the sets $\Pi_{n,d}$ form an increasing sequence of sets in $\mathbb{R}^{d-1}$. Let Π_d be the closure of the union of $\Pi_{n,d}$. We show that the set Π_d has the combinatorial structure of an *infinite cyclic polytope*, and that Π_d is not semialgebraic for all $d \geq 3$. Reduction needed to show undecidability of the unnormalized trace problem is borrowed from the one used by Hatami and Norine in [6] in the context of homomorphism density inequalities in graph theory. The set used by Hatami and Norine is essentially a linear transformation of the set Π_3, and the reduction is based on the geometry of Π_3. In particular this shows that deciding validity of matrix power trace inequalities is already undecidable if we only use second, fourth and sixth matrix powers, and we need at most 11 matrix variables for the problem to become undecidable. We note that the geometry of Π_3 was also used directly by Blekherman, Raymond and Wei [2] to show undecidability of homomorphism density inequalities with arbitrary edge weights.

We also consider the image of Δ_{n-1} under elementary symmetric polynomials. Our previous results on the boundary structure transfer over by using Newton's identities. We write $E_{n,d} := (e_1, \ldots, e_d)(\Delta_{n-1})$ and denote the limit image by E_d.

We show that the convex hull of $E_{n,d}$ is an actual cyclic polytope. This helps us reprove and slightly generalize the result of Choi, Lam and Reznick [5] on test sets for nonnegativity of even symmetric sextics. We note that the convex hull result can be traced to the work of Bollobás in extremal graph theory [4].

Testing nonnegativity of univariate normalized trace polynomials was considered by Klep, Pascoe and Volčič [7] where the authors proved a Positivstellensatz in the univariate case. Geometrically, such normalized trace polynomials correspond to power means. Nonnegativity of polynomials in power means was investigated by Blekherman and Riener in degree 4 [3] and more generally by Acevedo and Blekherman [1]. We briefly illustrate the connection with the Vandermonde map. Decidability of the normalized trace problem follows quickly from the work in [3]. As before we can consider the image of the normalized Vandermonde map, and fixing d take the (closure of the) limit as n goes to infinity. As explained in [1] the geometry of the limit is drastically different. For instance, the limit of the normalized Vandermonde map of the unit simplex Δ_{n-1} corresponds to the set of the first d moments of a probability measure supported on $\mathbb{R}_{\geq 0}$, and it is well-known that this set can be described by linear matrix inequalities [9]. In particular, the limit is semialgebraic for all d.

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