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New Directions in Real Algebraic Geometry

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Abstract. This workshop explored the forefront of connections of real algebraic geometry with convex analysis, combinatorics, and computational complexity. Important aspects have been promising interactions with the fields of quantum information theory, discrete geometry, complex and random algebraic geometry.

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Introduction by the Organizers

The workshop *New Directions in Real Algebraic Geometry*, organised by Saugata Basu (West Lafayette), Mario Kummer (Dresden), Tim Netzer (Innsbruck) and Cynthia Vinzant (Seattle) was well attended with 47 in-person and six virtual participants. The 28 talks covered a wide range of topics in real algebraic geometry and its applications. Two of these talks were given by virtual participants, the others by in-person participants. In the following we describe the major lines of investigation emphasized during the workshop.

1. Algebraic geometry and real algebraic geometry

In several talks it has become apparent that prominent concepts and methods of modern algebraic geometry can be profitably applied to classical questions in real algebraic geometry. For instance, Olivier Benoist presented new results on sum of squares representations of nonnegative real-analytic functions where the crucial step is to compute the cohomological dimension of the field of meromorphic functions on certain subsets of normal Stein spaces. Christoph Schulze talked about nonreduced projective schemes that are naturally associated to certain faces of the cone of nonnegative polynomials. Lorenzo Baldi reported on a characterization of the extreme rays of the cone of nonnegative forms on elliptic normal curves in terms of their group law and the real geometry of their 2-torsion points. Matilde Manzaroli gave a bound for the real Betti numbers of a smooth fiber of a real semistable degeneration near a certain special fiber X_0 in terms of the complex geometry of X_0. Rainer Sinn explained the role played by real cubic surfaces and the Schläfli double six in computer vision.

2. Computability and Complexity

Tools from real algebraic geometry are very well suited for studying computability and complexity questions. For example Annie Raymond spoke about the undecidability of the general problem of deciding algebraic inequalities between graph homomorphism densities. Evelyne Hubert spoke about a new approach for optimizing symmetric trigometric functions over $\mathbb{R}^n$ and applications to the exploration of the spectral bound on the chromatic numbers of set avoiding graphs. Peter Bürgisser spoke about the real zeros of random structured polynomial and in particular his proof of a probabilistic version of the real tau conjecture. Hamza Fawzi introduced the notion of k-local Hamiltonian systems and described how to compute their ground energy via a large convex optimization problem based on entropy constraints.

3. Convexity

The connection of real algebraic geometry to convexity and optimization has proven very productive in recent years. The conference also included some talks on this topic. Chiara Meroni explained partial progress on two open problems from convexity theory. One is asking for a classification of zonoids, the other is trying to describe directional convex hulls as semialgebraic sets. Julian Vill spoke about fiber bodies of projected convex sets, and in particular about those of Gram spectrahedra of binary sextics and ternary quartics, in which case the facial structure of the fiber bodies can be determined rather explicitly.

4. Quantum

Recent developments in non-commutative semialgebraic geometry have revealed close connections to quantum information theory and operator algebra. Several talks at the conference covered such topics. Bill Helton explained how non-commutative Positiv- and Nullstellensätze can help finding optimal strategies for quantum non-local games. Ion Nechita spoke about minimal and maximal operator systems, and how they help analyzing compatibility of quantum measurements, in particular the amount of noise one has to add to achieve compatibility. Eli Shamovich reported on the Arveson-Douglas conjecture, related to the compactness of certain commutators of operators. He explained how methods from algebraic geometry can be used to examine the problem. Mirte van der Eyden gave an overview about possible generalizations of abstract operator systems, allowing to apply some of the most important results in the area to problems that did not fit into the framework so far.

5. Geometry of polynomials

Wednesday evening, there were two short talks given remotely. Lior Alon gave a talk introducing the Kurasov-Sarnak construction of Fourier quasicrystals from stable polynomials and the result that all Fourier quasicrystals (with integer weights) can be acheived via this construction. The proof and main objects were closely related with torus actions on varieties and amoebas. This was the subject of the other evening talk by Jan Draisma, who described an algorithm for computing the dimension of the amoeba of a linear space.

On Thursday, Pavel Kurasov gave an overview of the connections between stable polynomials, Fourier quasicrystals, and metric graphs, expanding on the talk of Alon. Greg Knese told us about the possible local structure of singularities of stable hypersurfaces and a connection to the set of rational functions bounded on some domain in $\mathbb{C}^d$. Another topic of discussion was Lorentzian and log-concave polynomials, which generalize stability. Petter Brändén gave an introduction to Lorentzian polynomials, an extension of this notion to cones other than the positive orthant, and several interesting implications for discrete log-concavity in sequences associated to matroids. Nima Anari introduced a different generalization called fractional log-concavity and its implications for sampling algorithms on various delta-matroids.

6. Symmetry

Several talks focused on the study of symmetries in real algebraic geometry. Phillippe Moustrou discussed ideals of polynomials closed under the action of two types of groups: the symmetric group acting by permutations of variables, and the hyperoctahedral group acting by permutation of variables and sign changes. Cordian Riener spoke about the nice properties of the Vandermonde mapping and certain undecidability results arising from the limits of these mappings. Alison Rosenblum spoke about the geometry and topology of Vandermonde variety in the case of Coxeter groups of type B_n. Kevin Shu spoke about hyperbolic polynomials and maps preserving hyperbolicity, and in particular how symmetry in the polynomials allows to connect these two notions.

7. Miscellaneous

Antonio Lerario presented results on the optimal transport problem between algebraic hypersurfaces in complex projective space and explained how to rephrase this as a Riemannian geodesic problem. Dmitrii Pavlov talked about the Zariski closure of Gibbs manifolds, i.e., images of linear spaces of symmetric matrices under the exponential map and explained connections to optimization. Jean-Yves Welschinger spoke about handle decompositions on finite simplicial complexes. These decompositions extend the classical shellings of boundaries of convex polytopes.

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Workshop: New Directions in Real Algebraic Geometry

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Abstracts

Sums of few squares in real-analytic geometry

Olivier Benoist

It was discovered by Hilbert [8] that a positive semidefinite $f \in \mathbb{R}[X_1, \ldots, X_n]$ cannot in general be written as a sum of squares of polynomials. Hilbert's 17th problem, solved by Emil Artin [2] in 1927, shows that it is however always a sum of squares of rational functions.

Theorem 1 (Artin)**.** *Any positive semidefinite $f \in \mathbb{R}[X_1, \ldots, X_n]$ is a sum of squares in $\mathbb{R}(X_1, \ldots, X_n)$.*

Pfister [11] discovered in 1967 a quantitative improvement of Artin's theorem controlling the number of squares required.

Theorem 2 (Pfister)**.** *Any positive semidefinite $f \in \mathbb{R}[X_1, \ldots, X_n]$ is a sum of 2^n squares in $\mathbb{R}(X_1, \ldots, X_n)$.*

In real-analytic geometry, where one considers real-analytic functions (locally given by convergent power series) instead of polynomials, even the analogue of Artin's theorem is still an open problem.

Open question 1. *Are all positive semidefinite real-analytic functions $f : \mathbb{R}^n \to \mathbb{R}$ sums of squares of real-analytic meromorphic functions?*

This question, or variants where one considers real-analytic functions defined on more general normal real-analytic spaces (of dimension n), is known to have a positive answer if $n \leq 2$ [9, 1], or under restrictive hypotheses, such as appropriate compactness hypotheses [10]. Our goal in this talk is to obtain quantitative statements à la Pfister in this context, under such a compactness hypothesis.

Theorem 3. *Let M be a normal real-analytic space of dimension n and let $K \subset M$ be a connected compact set. Then any positive semidefinite $f \in \mathcal{O}(K)$ is a sum of 2^n squares in $\mathcal{M}(K)$.*

In the above statement, we let $\mathcal{O}(K)$ be the ring of real-analytic functions defined in a (non specified) open neighborhood of K, and we define $\mathcal{M}(K)$ to be its field of fractions. An element $f \in \mathcal{O}(K)$ is said to be positive semidefinite if it takes nonnegative values in some neighborhood of K. The following immediate corollary is worth spelling out.

Corollary 4. *Let M be a compact real-analytic manifold of dimension n, and let $f : M \to \mathbb{R}$ be a positive semidefinite real-analytic function. Then f is a sum of 2^n squares of real-analytic meromorphic functions.*

Both Theorem 3 and Corollary 4 are entirely new if $n \geq 3$.

In keeping with the motto that real geometry is complex geometry done equivariantly with respect to the action of the group $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ generated by complex conjugation, it is important for the proof of Theorem 3 to work in the complex-analytic context. The natural setting is that of *Stein spaces* which are the complex-analytic analogues of affine varieties and of their Stein compact subsets, which are those admitting a basis of Stein neighborhoods.

Theorem 5. *Let X be a normal Stein space of dimension n. Let G act on X through an antiholomorphic involution. Let $K \subset X$ be a connected G-invariant Stein compact subset. Then any $f \in \mathcal{O}(K)^G$ which takes nonnegative values on a neighborhood of K^G in X^G is a sum of 2^n squares in $\mathcal{M}(K)^G$.*

Here is a typical example of application of Theorem 5. Choose $X = \mathbb{C}^n$ endowed with the involution $z \mapsto \bar{z}$, and $K \subset X$ to be the closed unit ball. Then $\mathcal{O}(K)$ is the set of power series $\sum_I a_I \underline{z}^I$ in $z_1, \ldots, z_n$ with complex coefficients that have radius of convergence > 1 (i.e. that converge in a neighborhood of K), and $\mathcal{O}(K)^G$ is the subring of those that have real coefficients. Theorem 5 then states that any $f \in \mathcal{O}(K)^G$ that takes nonnegative values in a neighborhood of the closed unit ball in $\mathbb{R}^n$ is a sum of 2^n squares in the fraction field $\mathcal{M}(K)^G$ of $\mathcal{O}(K)^G$.

Theorem 3 follows from Theorem 5 thanks to the works of Cartan, Grauert and Tognoli establishing the existence of Stein complexifications of normal real-analytic spaces (see [4]).

Over fields, the Milnor conjectures proven by Voevodsky [12] provide a bridge between quadratic forms and Galois cohomology. In particular, they imply that the scalars represented by certain quadratic forms (the multiplicative forms of Pfister, among which the sums of 2^n squares quadratic form) are completely controlled by the vanishing of associated Galois cohomology classes. As a consequence, vanishing theorems in Galois cohomology imply representations as sums of few squares results. This technique allows us to reduce Theorem 5 to the following statement, which is the main theorem of the talk.

Theorem 6. *Let X be a normal Stein space of dimension n and let $K \subset X$ be a connected compact Stein subset. Then the field $\mathcal{M}(K)$ has cohomological dimension n.*

The conclusion of Theorem 6 means that the cohomology of the absolute Galois group of $\mathcal{M}(K)$ with value in any finite Galois-module vanishes in degree $> n$. The strategy of its proof is to exploit the fact, due to Hamm [6] and based on Morse theory, that a Stein space of dimension n has the homotopy type of a finite simplicial complex of dimension n, and hence that its singular cohomology vanishes in degree $> n$. To conclude, it remains to prove a theorem comparing étale cohomology (which generalizes Galois cohomology) and singular cohomology.

In algebraic geometry, such a comparison theorem is due to Mike Artin (see [5]). Its proofs are based on fibration arguments (to somehow reduce to the case of curves). Unfortunately, we do not know how to implement such fibrations arguments in the setting of Stein geometry, and one has to devise a new strategy of proof. To this effect, we rely in an essential way on Grauert's bump method as developed by Henkin and Leiterer ([7], see also [3]).

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Undecidability of polynomial inequalities in weighted graph homomorphism densities

Annie Raymond

(joint work with Grigoriy Blekherman, Fan Wei)

Given two simple graphs G and H, let $\hom(H,G)$ denote the number of *homomorphisms* from H to G, which is the set of maps from $V(H)$ to $V(G)$ that send edges of H to edges of G. Furthermore, let $t(H,G) := \frac{\hom(H,G)}{|V(G)|^{|V(H)|}}$ denote the *homomorphism density of H in G*, i.e., the probability that a random map from $V(H)$ to $V(G)$ is a homomorphism. One can extend the usual definition of graph homomorphisms to include target graphs G with edge weights $\mathbf{w} : E(G) \to \mathbb{R}$, denoted as $G_{\mathbf{w}}$:

$$\hom(H, G_{\mathbf{w}}) := \sum_{\substack{\varphi: V(H) \to V(G_{\mathbf{w}}): \\ \varphi \text{ is a homomorphism}}} \prod_{\{i,j\} \in E(H)} w_{\varphi(i),\varphi(j)}.$$

We can define $t(H, G_{\mathbf{w}}) := \frac{\hom(H,G_{\mathbf{w}})}{|V(G_{\mathbf{w}})|^{|V(H)|}}$ analogously. If the edge weights $\mathbf{w}$ of $G_{\mathbf{w}}$ only take values in $\{0,1\}$, we recover the usual definitions of homomorphism numbers and densities.

One of the central topics in extremal combinatorics is the study of algebraic inequalities between homomorphism densities. For example, the famous Sidorenko conjecture [Sid93] states that $t(H,G) - t(K_2,G)^{|E(H)|} \geq 0$ for any bipartite graph H and any graph G. Since $t(H_i, G_{\mathbf{w}}) \cdot t(H_j, G_{\mathbf{w}}) = t(H_iH_j, G_{\mathbf{w}})$ where H_iH_j is the disjoint union of H_i, H_j, any polynomial inequality can be seen as a linear inequality. Formally, define a *quantum graph* f to be a finite formal linear combination of graphs $\sum_{1\leq i\leq k} c_iH_i$ where $k \in \mathbb{N}_+$, $c_i \in \mathbb{R}$, and H_i's are finite graphs [FLS07]. Many extremal combinatorics questions can be reformulated as asking whether

$$t(f, G_{\mathbf{w}}) := \sum c_i t(H_i, G_{\mathbf{w}}) \geq 0$$

is valid for all graphs $G_{\mathbf{w}}$ for certain classes of edge weights $\mathbf{w}$, sometimes including negative real weights, for example, in the theory of Ramsey multiplicity or localness of graph inequalities (e.g., [Lov, Lov11, KVW, FW17, KNN+22, Lov12]).

In 2011, Hatami and Norin [HN11] proved a fundamental result that it is undecidable to determine the validity of polynomial inequalities in homomorphism densities for unweighted graphs. We show undecidability for the corresponding problem for weighted homomorphism densities and numbers. This provides negative answers to questions 17 and 21 of Lovász ([Lov]) which asked to find computationally effective certificates for the validity of homomorphism density inequalities in weighted homomorphism densities.

Proof Idea and Challenges: We first sketch the idea of Ioannidis and Ramakrishnan's short proof [IR95] of the undecidability of inequalities between homomorphism numbers $\hom(H_i, G)$ as a motivation for the proof for homomorphism densities. As in [HN11], this proof is also deduced from Matiyasevich's solution [Mat70] to Hilbert's tenth problem: Given a positive integer k and a polynomial $p(x_1, \ldots, x_k)$ with integer coefficients, the problem of determining whether there exist $x_1, \ldots, x_k \in \mathbb{Z}$ such that $p(x_1, \ldots, x_k) < 0$ is undecidable.

By changing x_i to $-x_i$ when necessary, it is therefore also undecidable to determine whether a polynomial with integer coefficients is always nonnegative for x_i's taking values in $\mathbb{N}$. Thus it suffices to show that for any polynomial with integer coefficients $p(x_1, \ldots, x_k)$, there is a quantum graph f such that $p(x_1, \ldots, x_k) \geq 0$ for all $x_i \in \mathbb{N}$ if and only if $\hom(f, G) \geq 0$ for all G. Let $H_1, \ldots, H_k$ be finite connected graphs with no homomorphisms from one to another and such that each H_i has no non-trivial homomorphism to itself. It is not hard to show that such graphs exist. Since $\hom(H_i, G)\hom(H_j, G) = \hom(H_iH_j, G)$, there is a quantum graph f such that for any graph G,

$$p(\hom(H_1, G), \ldots, \hom(H_k, G)) = \hom(f, G).$$

Crucially, since $\hom(H_i, G) \in \mathbb{N}$, we have that $p \geq 0$ for any $x_1, \ldots, x_k \in \mathbb{N}$ implies that $\hom(f, G) \geq 0$ for all G. On the other hand, for each k-tuple of values $a_1, \ldots, a_k \in \mathbb{N}$, there is a graph G such that $t(H_i, G) = a_i$, for example by letting G be the disjoint union of a_i copies of H_i.

One challenge in generalizing this simple proof to show the undecidability of homomorphism density inequalities is that $t(H_i, G)$ is not necessarily an integer. In [HN11], Hatami and Norin used a result of Bollobás [Bol76] that the convex hull of the set of all possible pairs of edge-triangle densities, i.e., pairs $(t(K_2, G), t(K_3, G))$

for unweighted graphs G, is the convex hull of points $(1,1)$ and $\left(\frac{n-1}{n}, \frac{(n-2)(n-1)}{n^2}\right)$ for $n \in \mathbb{N}$. These extreme points thus provide the needed integer points. This *integrality feature* alone does not lead to undecidability, since nonnegativity of univariate polynomials on integers is a decidable problem.

Given a polynomial $p(x_1, \ldots, x_k)$ in k variables, starting from a particular base graph F, Hatami and Norin use a delicate and intricate construction of a quantum graph f based on different variations of blow-ups of F. As in the proof of the undecidability of homomorphism number inequalities, one needs k "independent" copies of the convex hull to "plug in" $x_1, \ldots, x_k$. Hatami and Norin achieve this by measuring some conditional graph densities, conditioned on the set of graph homomorphisms $\phi : V(F) \to V(G)$. They construct the quantum graph f so that for any graph G,

$$t(f, G) = \sum_{\phi} c_\phi p^*(x_1(\phi), y_1(\phi), \ldots, x_k(\phi), y_k(\phi))$$

where c_ϕ's are constants, p^* is a polynomial whose nonnegativity is closely related to that of p, and $x_i(\phi)$ and $y_i(\phi)$'s are closely related to the density of K_2 and K_3 in some subgraphs $G_i(\phi)$ of G depending on ϕ. When $p \geq 0$ for integer-valued variables, they show that $p^* \geq 0$ by the integrality feature of the aforementioned convex hull and the fact that each individual ϕ enables k copies of this convex hull. A crucial fact is that since the weights $\mathbf{w}$ of $G_{\mathbf{w}}$ are nonnegative (in fact, they are in $\{0,1\}$), the constants c_ϕ are always nonnegative. These two facts imply $t(f, G) \geq 0$ for any G.

There are several difficulties in extending this approach to more arbitrary weights $\mathbf{w}$ in $G_{\mathbf{w}}$. First, much less is known about possible tuples of weighted graph densities. There are certainly fewer valid inequalities when negative weights are allowed. For instance, it is still an open question to characterize all graphs H such that $t(H, G_{\mathbf{w}}) \geq 0$ for all weighted graphs $G_{\mathbf{w}}$ [Lov]. If we try to record again all pairs of edge-triangle densities $(t(K_2, G_{\mathbf{w}}), t(K_3, G_{\mathbf{w}}))$, then any point in $\mathbb{R}^2$ is achievable. Second, the construction that Hatami and Norin used heavily relies on c_ϕ being nonnegative, which is not the case in general for weighted graphs. Lastly, the process to obtain multiple copies of the convex hull with the integrality feature relies on individual ϕ's, and cannot be used in our setting by the previous argument.

Our proof strategy is to show that the convex hull of ratios of densities of some carefully chosen graphs has the integrality feature even for weighted graphs. We then directly realize multiple copies of the convex hull by using an explicit family of graphs instead of going through a sum depending on ϕ. Remarkably, the convex hull is the same regardless of the weights we use—including unweighted graphs. Another advantage of our proof technique is that we associate each polynomial to an explicit quantum graph.

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Positivity and its Non-Reduced Structures

Christoph Schulze

Non-negativity of polynomials is a key issue in Real Algebraic Geometry. A main objective is the formulation of Positivstellensätze, which are algebraic certificates of positivity. In contrast, though convexity always played a role in this topic, the literature on the convex structure of the set of non-negative polynomials is rather small. The first results towards classifications of extreme rays of cones of non-negative forms just appeared in 2012 (by Blekherman et. al., see [1]) and 2018 (by Kunert and Scheiderer, see [3]) and a complete classification of the faces of the cone of non-negative ternary quartics was given 2014 in Kunert's thesis in [2]. The latter work makes use of cones of locally non-negative polynomials, which were studied in more detail in the affine setting (i.e. without restriction on the degree) in the author's thesis in [4].

Let us introduce some notation for the cones already mentioned. Let $d, n \in \mathbb{N}$. We denote by $\mathcal{P}_{2d}$ the cone of globally non-negative forms inside the forms $\mathbb{R}[\underline{X}]_{2d}$ of degree $2d$ in $n+1$ variables $\underline{X} = (X_0, \ldots, X_n)$. Similarly, given any $P \in \mathbb{P}^n(\mathbb{R})$, we denote by $\mathcal{P}_{2d}^{\mathrm{Loc}}(P)$ the set of forms that are non-negative on some Euclidean neighborhood of P in $\mathbb{P}^n(\mathbb{R})$. Further, there are affine versions: the cone of globally non-negative polynomials (of any degree) $\mathcal{P}$ inside the polynomial ring $\mathbb{R}[\underline{x}]$ in n variables $\underline{x} = (x_1, \dots, x_n)$ and the cones of locally non-negative polynomials $\mathcal{P}^{\mathrm{Loc}}(P)$ with $P \in \mathbb{R}^n$.

Given any (not necessarily exposed) face F of $\mathcal{P}_{2d}$, we consider elements f in the relative interior of F. These are exactly the elements f such that the smallest face of $\mathcal{P}_{2d}$ containing f is F. It is an easy observation that these elements share the same real zero set in $\mathbb{P}^n(\mathbb{R})$. However, all these zeros have to be singularities as we consider non-negative elements. Thus, the question arises if these elements do also share a common non-reduced structure. This is in fact the case as there is an saturated homogeneous ideal $J = \bigoplus_{d' \in \mathbb{N}_0} J_{d'}$ of $\mathbb{R}[\underline{X}]$ that is defined from f "in a geometrical way", independent of the special choice of f and we have $J_{2d} = \mathrm{span}(F)$. Given $d' \in \mathbb{N}_0$, its homogeneous part of degree d' is defined by

$$J_{d'} = \{g \in \mathbb{R}[\underline{X}]_{d'} \mid \exists\, \epsilon > 0\colon f \geq \epsilon \cdot |g| \text{ on } S^n\}.$$

We call ideals of $\mathbb{R}[\underline{X}]$ that arise in this way *positivity ideals*. It is obvious that elements of the form $(X_0^2 + \cdots + X_n^2)^k \cdot f$ with $k \in \mathbb{N}$, and hence also the corresponding faces $J \cap \mathcal{P}_{2(d+k)}$, will give rise to the same ideal J as f. For $d' < d$, $J \cap \mathcal{P}_{2d'}$ is also a face of $\mathcal{P}_{2d'}$, but it may lead to a smaller associated ideal. So there is some $d_0 \in \mathbb{N}_0$ such that just for $d' \geq d_0$, there is a (unique) face of $\mathcal{P}_{2d'}$ that induces J.

In the affine setting, the situation is different. We are working in an infinite-dimensional vector space and there may be faces without any points in the relative algebraic interior. However, it turns out that we obtain similar properties as in the projective setting if we restrict ourselves to the consideration of faces whose linear span is an ideal. We also call them *ideal faces*. Then a proper replacement of the relative interior of an ideal face F are the elements $f \in \mathcal{P}$ such that F is the smallest face containing $f \cdot \mathcal{P}$. The ideal I arising from an ideal face is just its linear span and there is also a similar description as in the projective setting. It is

$$I = \{g \in \mathbb{R}[\underline{x}] \mid \exists\, h \in \mathcal{P}\colon h \cdot f \geq |g| \text{ on } \mathbb{R}^n\}.$$

In the affine setting, we use the same notion of *positivity ideals*. Clearly, we have $I \cap \mathcal{P} = F$, so there is a unique face giving rise to a positivity ideal.

The constructions from above are well-behaved. Given $f \in \mathcal{P}_{2d}$, we may consider the closed subscheme of $\mathbb{P}^n_{\mathbb{R}}$ corresponding to the positivity ideal J that arises from f. Then the restriction of this scheme to an affine chart $\mathbb{A}^n_{\mathbb{R}}$ coincides with the closed subscheme of $\mathbb{A}^n_{\mathbb{R}}$ corresponding to the ideal I that arises from the respective dehomogenization of f. This dehomogenization procedure does also respect the lattice structure of these cones and an additional multiplicative structure that may be defined in a canonical manner. Further, one can also define such ideals for faces of $\mathcal{P}^{\mathrm{Loc}}_{2d}(P)$ and $\mathcal{P}^{\mathrm{Loc}}(P)$. There is a well-defined transfer from a face of $\mathcal{P}_{2d}$ or $\mathcal{P}$ to a corresponding face of $\mathcal{P}^{\mathrm{Loc}}_{2d}(P)$ or $\mathcal{P}^{\mathrm{Loc}}(P)$ and again, this commutes with dehomogenization if $P \in \mathbb{R}^n \subseteq \mathbb{P}^n(\mathbb{R})$.

We will close this abstract with a generalization that goes beyond the study of these cones. In the definition of J and I, it is possible to replace f by any non-negative function on S^n or $\mathbb{R}^n$ and one obtains again a (in the projective case saturated homogeneous) ideal. This leads to a larger class of ideals, but one can show that this class consists exactly of the square roots of positivity ideals. Here, the square root of a subset of the polynomials ring consists of all elements whose square is in the given set. Especially, square roots of positivity ideals are ideals. We call such ideals *absolute ideals*.

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Fiber bodies of spectrahedra

Julian Vill

In [4] Mathis and Meroni study the fiber body of a compact convex set. Informally, this means the following. Choose a convex, compact set $K \subset \mathbb{R}^{n+m}$ and consider the projection $\pi\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ to the first n components. Over each point in the image we have a convex fiber in K. The fiber body of K is the average over all fibers. More precisely, any point y in the fiber body is given by

$$y = \int_{\pi(K)} \gamma(x)\,\mathrm{d}x$$

where $\gamma\colon \pi(K) \to \mathbb{R}^m$ is a measurable section. The fiber body $\Sigma_\pi K$ is itself a compact convex set in $\mathbb{R}^m$. It is the continuous analogue of the Minkowski sum of convex sets. One may also think about a limit object of Minkowski sums when the number of summands tends to infinity. In [4] the fiber body was then studied in several special cases where the support function of the convex body K is known.

Before that the notion of fiber polytopes was introduced and studied by Billera and Sturmfels [1]. These possess beautiful and well-studied combinatorics which led to numerous constructions of polytopes with prescribed combinatorial structure, for example in [10]. Moreover, fiber polytopes are a generalisation of secondary polytopes which are studied in the context of linear programming in [7]. Their combinatoric structure gives rise to certain basis of the optimisation problem.

We study the fiber body of families of spectrahedra. We use a different approach as we do not in general know the support function. The goal is to understand the facial structure of the fiber body given we know the facial structure of the fibers. This can also be seen as a generalization from linear programming to semidefinite programming. However, it is not clear what the meaning of the fiber body is in this situation. It would be interesting to study the fiber body of spectrahedra from this point of view.

We then apply the general setup to study two special families of spectrahedra, namely Gram spectrahedra of binary sextics and of ternary quartics. These are examples in dimensions 3 and 6 respectively.

Gram spectrahedra have a close connection to sum of squares representations of polynomials and have been introduced in [2]. The Gram spectrahedron of a fixed homogeneous polynomial parametrizes its sum of squares representations up to orthogonal equivalence. Afterward, especially the structure of their extreme points was investigated, for example in [3] and [5]. In [6] and [9] the complete facial structure of Gram spectrahedra was studied in the case of ternary quartics and binary forms. As we do not have a description of the support function of Gram spectrahedra, we make extensive use of these results in order to describe the facial structure of the fiber body.

In both cases we were very surprised by the remaining facial structure on the fiber bodies. In the case of binary sextics there is exactly one extreme point on the boundary of the fiber body with a 3-dimensional normal cone ([8, Thm 5.2]). A general positive semidefinite (psd) binary sextic has exactly four such points corresponding to length two sum of squares representations. In the case of ternary quartics the structure of the boundary changes rather drastically. On the one hand, every face has a 1-dimensional normal cone even though the Gram spectrahedron of a general psd ternary quartic has normal cones of dimensions $1, 3, 6$. On the other hand, the fiber body has a 2-dimensional family of 3-dimensional faces whereas a general Gram spectrahedron has no faces of dimension larger than 2 ([8, Thm 6.5]).

Informally, we expect normal cones of the fiber body to be small compared to the normal cones of the fibers, and faces to be larger. This turned out to be true in the cases studied. It would be interesting to know if this also holds for different families of spectrahedra and if these fiber bodies carry information relevant to semidefinite programming.

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Real zeros of random structured polynomials

Peter Bürgisser

Seminal work by Khovanskii [4, 5] showed that the number of nondegenerate real zeros of a fewnomial system in n variables is bounded by n and its number t of monomials. However, Khovanskii's bound is exponential in t, while many people believe that there should be an upper bound, which is polynomial in t, see [7]. We recently proved that [3] "generically" this is indeed the case: more specifically, the expected number of real zeros of random fewnomial systems with prescribed set of exponent vectors can be neatly bounded (similarly as for Kushnirenko's conjecture).

When focusing on structured polynomials, even the univariate case is challenging. Koiran's real tau conjecture [6] for the number of real zeros of a sum of products of sparse polynomials implies the separation of the complexity classes VP and VNP. Proving such separation is the major open question in algebraic complexity theory [2]. Recently, we proved (with I. Briquel) that random univariate polynomials typically have as few real zeros as predicted by the real tau conjecture [1]. The randomness refers here to formulas with fixed combinatorial structure and independent standard Gaussian coefficients.

The proofs rely on tools from the theory of random fields (Kac-Rice formula [8]) and integral geometry.

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Optimal Transpot between algebraic hypersurfaces

Antonio Lerario

(joint work with Paolo Antonini, Fabio Cavalletti)

What is the optimal way to deform a projective hypersurface into another one? In this paper we will answer this question adopting the point of view of measure theory, introducing the optimal transport problem between complex algebraic projective hypersurfaces.

First, a natural topological embedding of the space of hypersurfaces of a given degree into the space of measures on the projective space is constructed. Then, the optimal transport problem between hypersurfaces is defined through a constrained dynamical formulation, minimizing the energy of absolutely continuous curves which lie on the image of this embedding. In this way an inner Wasserstein distance on the projective space of homogeneous polynomials is introduced.

This distance is complete and geodesic: geodesics corresponds to optimal deformations of one algebraic hypersurface into another one. Outside the discriminant this distance is induced by a smooth Riemannian metric, which is the real part of an explicit Hermitian structure. The topology induced by the inner Wasserstein distance is finer than the Fubini–Study one.

To prove these results we develop new techniques, which combine complex and symplectic geometry with optimal transport, and which we expect to be relevant on their own.

We discuss applications on the regularity of the zeroes of a family of multivariate polynomials and on the condition number of polynomial systems solving.

Two convex conjectures for different flavours

Chiara Meroni

(joint work with Fulvio Gesmundo and with Bogdan Raiță and Bernd Sturmfels)

We present two conjectures related to real algebraic geometry, with connections to certain notions of convexity. The first conjecture was originally stated in a joint work with Fulvio Gesmundo [5]. This work is motivated by the *zonoid problem*, which we now introduce. Given $z_1, \dots, z_N \in \mathbb{R}^d$, the Minkowski sum

$$Z = \sum_{i=1}^{N} [-z_i, z_i] \subset \mathbb{R}^d$$

is called *zonotope*, and it is a special type of polytope. A *zonoid* is a limit, in the Hausdorff topology, of zonotopes. According to this definition, zonotopes, and by extension zonoids, are centrally symmetric convex bodies, namely $Z = -Z$. All centrally symmetric convex bodies in $\mathbb{R}^2$ are zonoids; in $\mathbb{R}^d$ for $d > 2$ the set of zonoids is strictly contained in the set of centrally symmetric convex bodies. The zonoid problem, introduced in [3], but already appearing in [2], consists in determining whether a given centrally symmetric convex body is a zonoid. More precisely, assume to have a polynomial $p \in \mathbb{R}[x_1, \dots, x_d]$ such that $K = \{x \in$

$\mathbb{R}^d \,|\, p(x) \geq 0\}$ is a centrally symmetric convex body. Can we decide if K is a zonoid? This point of view on the zonoid problem has been investigated in [9], in the context of o-minimal structures. We turn this question into an algebraic geometry problem by studying the properties of the variety $\partial_a K = \{x \in \mathbb{R}^d \,|\, p(x) = 0\}$, also known as *algebraic boundary* of K. The aim is to gain an understanding of the necessary properties that a polynomial p must satisfy in order for the associated K to be a zonoid. We focus on a particular family of zonoids, called *discotopes* [5, Definition 2.2].

Definition 1. Let $D_1, \ldots, D_N \subset \mathbb{R}^d$ be discs, i.e., linear images of the standard unit ball of any dimension. The associated discotope is their Minkowski sum

$$\mathcal{D} = \sum_{i=1}^{N} D_i.$$

We denote by $\mathcal{E}$ the closure in the Zariski topology of the set of exposed points of $\mathcal{D}$. This is by definition an algebraic variety, contained in the algebraic boundary of $\mathcal{D}$, and we are interested in characterizing its dimension and degree. In particular, since we are interested in exposed points, we might restrict to the case in which $\dim D_i \geq 2$ for all $i = 1, \ldots, N$. We define the quantity $(\star) = \sum_{i=1}^{N}(\dim D_i - 1)$ which captures the sum of the dimensions of the boundaries of the discs. Then the following holds [5, Theorems 4.3, 6.1].

Theorem 2. *Let $\mathcal{D}$ be a generic discotope with $\dim D_i \geq 2$ for all $i = 1, \ldots, N$. If $(\star) \leq d-1$, then $\mathcal{E}$ is an irreducible variety with $\dim \mathcal{E} = (\star)$ and $\deg \mathcal{E} = 2^N$. If $(\star) > d-1$, then $\dim \mathcal{E} = d-1$. In particular, if $\dim D_i = 2$ for all $i = 1, \ldots, N$ then $\mathcal{E}$ is an irreducible variety with $\deg \mathcal{E} \leq 2^N \cdot \binom{N}{d-1}$.*

From the theorem we get no information about irreducibility and degree of $\mathcal{E}$ in the case $(\star) > d-1$ when discs of dimension higher than 2 are involved. The genericity assumption concerns the linear spans of the discs, which we assume to be as transversal as possible. The following is [5, Conjecture 8.2].

Conjecture 1. *Let $\mathcal{D}$ be a generic discotope with $\dim D_i \geq 2$ for all $i = 1, \ldots, N$. Then, $\mathcal{E}$ is irreducible.*

The proof of the first part of the Theorem 2 relies on the construction of geometric joins of varieties. The second part, when $(\star) > d-1$, is based on an interpretation of $\mathcal{E}$ as a determinantal variety. Namely, consider the addition map $\Sigma : \partial D_1 \times \ldots \times \partial D_N \to \mathbb{R}^d \subset \mathbb{C}^d$. It can be showed that $\Sigma^{-1}(\mathcal{E})$ is contained in the critical locus of Σ, and they have the same dimension. Proving that $\operatorname{crit} \Sigma$ is irreducible would imply that our variety $\mathcal{E}$ is irreducible as well. The critical locus can be seen as the determinantal variety of maximal minors of a matrix with linear entries of a specific form. Such a matrix is not generic in the sense of Bertini's theorem, nevertheless in the case of 2-dimensional discs and in all other explicit examples that we computed, it defines an irreducible variety.

We move now to the second conjecture, which is a result of many conversations with Bogdan Raiță and Bernd Sturmfels. The motivation for the problem that we are going to discuss comes from multivariate calculus of variations. The goal is to construct non-constant Lipschitz solutions to the system of PDE's

$$\begin{cases} \mathcal{A}v(x) = 0 & \text{for } x \in \Omega, \\ v(x) \in \mathcal{K} & \text{for almost every } x \in \Omega, \end{cases}$$

where $v : \mathbb{R}^n \to \mathbb{V}$, $\mathcal{A} = \sum_{|\alpha|=\ell} A_\alpha \partial^\alpha$ is an ℓ-homogenenous vectorial linear differential operator with constant coefficients $A_\alpha \in \operatorname{lin}(\mathbb{V}, \mathbb{W})$, and $\mathcal{K} = \{M_1, \ldots, M_N\}$ is finite. Here $\Omega \subset \mathbb{R}^n$ is an open convex set, and $\mathbb{V}, \mathbb{W}$ are finite dimensional real inner product spaces.

Example. Let $\mathcal{A}$ be the curl:

$$\mathcal{A}v = (\partial_k v_{ij} - \partial_j v_{ik}) \quad \text{for} \quad \begin{array}{l} i = 1, \ldots, m, \\ j, k = 1, \ldots n, \end{array}$$

where $v = (v_{\alpha,\beta})_{\alpha,\beta} : \mathbb{R}^n \to \mathbb{R}^{m \times n}$. Then, $\mathcal{A}v = 0$ can be solved for $v = \nabla f$, for some $f \colon \mathbb{R}^n \to \mathbb{R}^m$. Therefore, the question becomes whether there exists such an f satisfying $\nabla f \in \mathcal{K}$, where $\mathcal{K}$ is a given set of finitely-many matrices, and we want the gradient to coincide with all the matrices in $\mathcal{K}$, in distinct regions.

The construction of solutions of the system of PDE's involves computing the Λ-convex hull of $\mathcal{K}$, where Λ is a cone associated to $\mathcal{A}$, known as the *wave cone*. In the example of the curl, the wave cone is the cone of rank-one $(m \times n)$-matrices, and one talks about rank-one convexity. This motivates the study of the rank-one (or, for a more general cone Λ, the directional) convex hull of finitely-many points. A function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is said to be *rank-one convex* if $t \mapsto f(A + tB)$ is a convex function for every matrix $A \in \mathbb{R}^{m \times n}$ and every matrix B of rank one.

Definition 3. Let $\mathcal{K} \subset \mathbb{R}^{m \times n}$ be compact. Its rank-one convex hull is

$$\mathcal{K}^{\mathrm{rc}} = \{x \in \mathbb{R}^{m \times n} \mid f(x) \le \min f(\mathcal{K}) \text{ for all rank-one convex } f\}.$$

Alternatively, the rank-one convex hull can be seen as the zero locus of a function, that we are now going to introduce. Let $x \in \mathbb{R}^{m \times n}$. A rank-one *elementary splitting* of the Dirac measure δ_x is the measure $\lambda_1 \delta_{x_1} + \lambda_2 \delta_{x_2}$, with $\lambda_1 + \lambda_2 = 1$, x in the segment $\operatorname{conv}(x_1, x_2)$, and $x_2 - x_1 \in \mathbb{R}^{m \times n}$ of rank-one. Then, a rank-one *laminate of finite order* is defined to be a measure obtained by a finite number of rank-one elementary splittings. It is of the form

$$\nu = \sum_{i=1}^{m} \lambda_i \delta_{x_i}$$

with $\sum \lambda_i = 1$. We call its *center of mass* the point $\overline{\nu} = \sum \lambda_i x_i$. Then, we can rewrite the rank-one convex hull as $\mathcal{K}^{\mathrm{rc}} = \{x \in \mathbb{R}^{m \times n} \mid g(x) = 0\}$, where

$$g(x) = \inf\{\langle \nu, d_{\mathcal{K}}^2(x) \rangle \mid \nu \ \Lambda\text{-laminate of finite order}, \overline{\nu} = x\}.$$

By angle brackets we mean that if $\nu = \sum \lambda_i \delta_{x_i}$ then $\langle \nu, d_{\mathcal{K}}^2(x) \rangle = \sum \lambda_i d_{\mathcal{K}}^2(x_i)$, where $d_{\mathcal{K}}$ is the distance to the set $\mathcal{K}$. This alternative characterization of the rank-one convex hull emphasizes the role of the cone in the construction, since the splittings can be done only along its rays. This leads to our second conjecture.

Conjecture 2. *The rank-one convex hull of a finite set is semialgebraic.*

This result is true in $\mathbb{R}^2$, identified with the space of diagonal 2×2 matrices. In fact, [4, Theorem 1.1] proves that this holds for more general cones in $\mathbb{R}^2$, with a finite number of rays, and provides an algorithm for the computation of the rank-one (or, in general, directional) convex hull. An analogous result [11] describes the case in which the cone Λ consists of d linearly independent vectors in $\mathbb{R}^d$. A very different example of semialgebraic rank-one convex hull in higher dimension appears in [12]. We believe that in higher dimension, where the cones are not necessarily linear but rather a union of algebraic varieties, one should examine the arrangement of the cones placed at each point of $\mathcal{K}$. Heuristically, a positive answer to our conjecture may provide an algorithm to compute rank-one convex hulls; conversely, a negative answer would highlight the complexity of explicitly computing them. For more details on rank-one convexity and directional convexity, we refer to [6, 7, 8, 10, 13]. In addition, we would like to mention that after the talk, Saugata Basu pointed out that a similar geometric construction, known among the analysis community as lamination hull, also appears in the context of cryptography, in relation to the round complexity of randomized functions [1].

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Non-negative forms on cubic curves

LORENZO BALDI
(joint work with G. Blekherman and R. Sinn)

The study and characterization of (global) non-negativity for polynomials (or forms) dates back to D. Hilbert [4], whose classical result characterize the number of variables and the degrees for which the convex cone of non-negative forms on $\mathbb{P}^n$ coincide with the convex cone of sums of squares. In the last decades, the relation between non-negativity and sums of squares received increasing attention: the work of Hilbert has been recently extended by G. Blekherman, R. Sinn, G. Smith and M. Velasco [3], that consider non-negativity and sums of squares on projective varieties. In particular, they characterize the (totally real, non-degenerate, irreducible, projective) varieties where non-negative quadratic forms and sums of squares coincide as those of minimal degree. This result, whose proof requires combining techniques from algebraic and convex geometry, is a major achievement in the recent research area of convex algebraic geometry [2].

In this talk, based on [1], we restrict to the first non-trivial varieties where the cones of non-negative forms and sums of squares are different, i.e. plane cubic curves $C \subset \mathbb{P}^2$. Our goal is to characterize completely the cone of non-negative forms from the point of view of convex geometry.

The simpler case of the projective line $\mathbb{P}^1$ guides our investigation: there, the cone $P_{\mathbb{P}^1,2d}$ of non-negative forms on $\mathbb{P}^1(\mathbb{R})$ of degree $2d$ is full dimensional in $\mathbb{R}[x_0, x_1]_{2d}$, and the all the faces of this convex cone arise as follows. If $\mathcal{F} \subset P_{\mathbb{P}^1,2d}$ is a face, then there exists points $A_i \in \mathbb{P}^1$ and natural numbers k_i such that $\mathcal{F}$ consists of non-negative forms on $\mathbb{P}^1$ that vanish at A_i with multiplicity at least $2k_i$.

We consider the convex cone $P_{C,2d} \subset \mathbb{R}[x_0, x_1, x_2]_{2d}$ of non-negative forms on $C(\mathbb{R})$ of degree $2d$, and we assume that $C \subset \mathbb{P}^2$ is in Weierstarss form. We characterize the extremal rays of the cone $P_{C,2d}$, using the group law $\oplus$ on C: $\mathbb{R}_{\geq 0} \cdot f$ is an extremal ray of $P_{C,2d}$ if and only if $f.C = 2(A_1 + \cdots + A_{3d})$ and either:

- $A_1 \oplus \cdots \oplus A_{3d} = O$, where O the zero of the group law (in this case f is a square); or
- $A_1 \oplus \cdots \oplus A_{3d} = T$, where T is a special 2-torsion point of $(C(\mathbb{R}), \oplus)$.

The analysis is performed for every d, and provides explicit Krivine-Stengle-type certificates of non-negativity for every non-negative form, as of ratios of sums of squares in the function field $\mathbb{R}(C)$ of C. The result obtained is, in a suitable sense, independent from the real topology of the curve, and is connected to the analysis of V. Vinnikov [5] for abstract curves. The special 2-torsion point T is identified using special rational functions in $\mathbb{R}(C)$, given by the embedding of C in $\mathbb{P}^2$.

For higher dimensional faces of $P_{C,2d}$, we retrieve a characterization analogous to the one of $\mathbb{P}^1$. These results are extended to elliptic normal curves, using the Veronese embedding and successively projecting away from points in the Veronese embedding of C.

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S_n and B_n-Specht Ideals

Philippe Moustrou

(joint work with S. Debus, C. Riener, H. Verdure)

An S_n-*invariant ideal* is an ideal I in a polynomial ring $K[x_1, \ldots, x_n]$ such that for every $P \in I$ and $\sigma \in S_n$, $\sigma \cdot P \in I$, where S_n acts by permuting coordinates. We are interested in the points in the corresponding variety $V(I)$, especially their symmetries. In this direction, the degree principle, due to Timofte [1] and revisited by Riener [2] says that if the ideal I is generated by degree d polynomials, then the variety $V(I)$ is non-empty if and only if it contains a point with at most d distinct coordinates.

Up to symmetry, a point x in K^n can be written as

$$x = (\underbrace{x_1, \ldots, x_1}_{\lambda_1}, \ldots, \underbrace{x_k, \ldots, x_k}_{\lambda_k})$$

where $x_i \neq x_j$ whenever $i \neq j$, and $\lambda_i \geq \lambda_{i+1}$ for any i. The sequence $\Lambda(x) = (\lambda_1, \ldots, \lambda_k)$ is then a partition of n: the *orbit-type* of x. In this set-up, the degree principle says that it is enough to look for points in $V(I)$ with an orbit-type of length at most k. In order to have more information on the possible orbit-types, we study *Specht ideals*, which provide a connection between the partitions of n and the irreducible representations of S_n.

For a given partition λ of n, the Specht ideal I_λ is the ideal of $K[x_1, \ldots, x_n]$ generated by all *Specht polynomials* of shape λ. We denote by V_λ the corresponding *Specht variety*. In [3], we study the correspondence between the poset of partitions for the dominance order, the poset of Specht ideals, and the poset of Specht varieties for inclusion relations. More precisely, for two partitions λ and μ of n, we have

$$\mu \text{ dominates } \lambda \Leftrightarrow I_\lambda \subset I_\mu \Leftrightarrow V_\mu \subset V_\lambda.$$

This in turns provides a complete understanding of Specht varieties in terms of orbit types: the Specht variety V_μ is made of all the points x in K^n such that $\Lambda(x)$ is not dominated by μ. This decomposition helps proving results about the algebraic properties of Specht ideals [4, 5]. Moreoever, our results also give information on general S_n-invariant ideals. Depending on the monomials appearing in the polynomials of an S_n-invariant ideal I, we are able to find partitions λ of n such that I contains the Specht ideal I_λ. This then implies that $V_I \subset V_\lambda$, and gives information on the possible orbit types of the points in V. This gives somehow a generalization of the degree principle, giving a better understanding of invariant ideals with respect to representation theory or for algorithmic purposes.

In [6], we adapt our approach to B_n*-invariant ideals*, where B_n acts on $K[x_1, \ldots, x_n]$ by permutation of coordinates and sign changes. There, irreducible representations are in bijection with *bipartitions*, and B_n-Specht ideals and varieties can be defined in a similar way. However, even if several orders on bipartitions were studied, non of them gave the equivalence with the inclusion of Specht ideals. We introduce another order on bipartitions, and show the correspondence with the poset of ideals. This requires a precise study of the poset of bipartitions including the covering cases. Then, we also provide a notion of orbit-type in this situation, in order to get analogues of our decomposition results for S_n-Specht varieties, and similar consequences for B_n-invariant ideals.

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Symmetries at the limit

Cordian Riener

(joint work with Jose Acevedo, Grigoriy Blekherman, Sebastian Debus)

For $n \in \mathbb{N}$ the group S_n of permutations of an n-element set is a well studied object and the natural inclusion $S_n \subset S_{n+1}$ opens the possibility to study phenomena appearing in the limit, when n tends to infinity. In this context it is an interesting question to consider the equivariant algebraic and semi-algebraic geometry of S_∞ invariant sets. The main setup of the talk was considering the limit of the so called *Vandermonde varieties* and *Vandermonde cells*. These are defined via the power sum polynomials $p_a(x) = x_1^a + \ldots + x_n^a$, where $a \in \mathbb{R}_{>0}$. More concretely:

Definition 1. Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d_{>0}$ be a sequence of strictly increasing positive real numbers.

(1) The α-*Vandermonde map* in n variables is defined to be the map

$$\begin{array}{cccc} \nu_{n,\alpha}: & \mathbb{R}^n_{\geq 0} & \longrightarrow & \mathbb{R}^d \\ & x & \longmapsto & (p_{\alpha_1}(x), p_{\alpha_2}(x), \ldots, p_{\alpha_d}(x)) \end{array}.$$

(2) Let

$$\Delta_{n-1} := \left\{ x \in \mathbb{R}^n_{\geq 0} \,:\, x_1 + \ldots + x_n = 1 \right\}$$

and α be a strictly increasing sequence of real numbers larger than 1. The (n,α)-*Vandermonde cell* $\Pi_{n,\alpha}$ is the set $\nu_{n,\alpha}(\Delta_{n-1})$. In the case when $\alpha = (2,\ldots,d)$ we write $\Pi_{n,d}$.

(3) For $2 \leq k \leq d$ and $c \in \{1\} \times \mathbb{R}^{k-1}_{\geq 0}$ we define the associated *generalized positive* α-*Vandermonde variety* to be the fiber over c of the corresponding Vandermonde map, i.e.,

$$V^\alpha_k(c) := \nu^{-1}_{n,(1,\alpha_1,\ldots,\alpha_{k-1})}(c) \cap \mathbb{R}^n_{\geq 0}.$$

Since for $\alpha = (2,\ldots,n)$ the power sum polynomials generate the S_n invariant polynomials, the Vandermonde cells and the Vandermonde varieties in this setup naturally connect to the S_n orbit space of Δ_{n-1} (or generally $\mathbb{R}^n$). They have already been studied by Arnold, Kostov and Ursell [2, 8, 11] in the context of hyperbolic polynomials and symmetric inequalities and can also be used to design algorithms to compute Betti numbers of symmetric semi-algebraic sets which reduce the symmetry in the problem (see [5]). The main question we examined in the talk connected the Vandermonde cells to the following question from computational complexity:

Suppose that we are given a polynomial expression in traces of powers of symmetric matrices, such as, for example

$$2\operatorname{tr}(A^2)\operatorname{tr}(B^6) - \operatorname{tr}(A^4)\operatorname{tr}(B^4) + \operatorname{tr}(A^2)\operatorname{tr}(C^6).$$

Can we decide algorithmically if such a *trace polynomial* is non-negative for all symmetric matrices A, B, C of all possible sizes ? Since for a symmetric $A \in \mathbb{R}^{n\times n}$ we have $\operatorname{tr}(A^d) = \lambda_1^d + \cdots + \lambda_n^d$ this question is indeed naturally connected to the power sum polynomials. Indeed,

$$2\operatorname{tr}(A^2)\operatorname{tr}(B^6) - \operatorname{tr}(A^4)\operatorname{tr}(B^4) + \operatorname{tr}(A^2)\operatorname{tr}(C^6) \geq 0 \;\forall \text{ symmetric } A, B, C \in \mathbb{R}^{n\times n}$$

holds if and only if

$$2p_2(x)p_6(y) - p_4(x)p_4(y) + p_2(x)p_6(z) \geq 0 \;\forall x, y, z \in \mathbb{R}^n.$$

Since we want to ensure that the condition holds for all matrix sizes, one naturally is brought to consider power sum polynomials in an arbitrary number of variables and studying of the Vandermonde cells and their limit when n tends to infinity.

As it turns out, the Vandermonde cells poses as rich combinatorial structure, namely they behave combinatorially similar to cyclic polytopes, in the following sense:

Definition 2. Let $S \subset \mathbb{R}^d$. Then S has the *combinatorial structure* of the cyclic polytope $C(n,d)$ if there exists a homeomorphism

$$\Phi : \operatorname{bd} C(n,d) \to \operatorname{bd} S$$

which is a diffeomorphism when restricted to the relative interior of any face of bd $C(n, d)$. The *vertices* of S are the images of the vertices of $C(n, d)$

The connection to cyclic polytopes was observed experimentally by Melánová, Sturmfels, and Winter [10] and [1] confirms this experimental observation. Building on this combinatorial understanding, one can study the limit. For fixed $d \in \mathbb{N}$ we consider the limiting set, Π_d, i.e., the union over all $\Pi_{n,d}$. Whereas the sets $\Pi_{n,d}$ are by definition semi-algebraic, the following striking result is deduced in [1]

Theorem 3. *The limiting set Π_d is not semi-algebraic for $d \geq 3$.*

The proof of this statement is based on understanding of the singularities that aries and are connected to the combinatorial structure: The number of isolated singularities is growing with n and tends to infinity. Therefore, the limit cannot be semi-algebraic. This result is a first indication that the question of testing non-negativity of symmetric polynomials in the limit is not directly obvious via the Vandermonde map. By further integrating works of Hatami and Norine [7] we arrive at the following stronger statement which appears in [1, Theorem 6.2].

Theorem 4. *The following decision problem is undecidable.*

Instance: *A positive integer k and a trace polynomial $f(X_1, \ldots, X_k)$.*
Question: *Is $f(M_1, \ldots, M_k)$ nonnegative for all real symmetric matrices $M_1, \ldots, M_k$ of all sizes for all $1 \leq i \leq k$?*

It is interesting to remark that it follows from the arguments in [6] that when we replace the usual trace by the normalized trace, i.e. $\frac{1}{n}\operatorname{Tr}(A)$ for a symmetric matrix A of size $n \times n$, the problem becomes decidable.

The computational problem we present in the talk suggests that the geometry of the limit poses interesting questions. Indeed, as mentioned, the sets $\Pi_{n,d}$ are linked to the orbit spaces $\mathbb{R}^n//S_n$. The work on bounding the equivariant Betti numbers of symmetric semi-algebraic sets (for example [4]) brought up the following question.

Question 1. *There exists a polynomial p such that the sum of the equivariant Betti numbers of $S^{(n)}$ is given by $p(n)$?*

The works in [3, 4] have already established polynomial bounds for these equivariant Betti numbers, but a positive answer to the above question would provide a stronger understanding of the behaviour of these quantities when n tends to infinity. Furthermore, in [9] we consider the ring $\mathbb{C}[x_{ij} : i \in \mathbb{N}, j \in [n]]$ and its spectrum $\mathbb{A}^n_\infty$. There is a natural action of the infinite symmetric group S_∞ on the first index in the polynomial ring and on the specturm. Various authors have been able to show that many algebraic properties from finite dimensional rings carry over *up to symmetry*. However, we show in [9] that the quotient space $\mathbb{A}^n_{K,\infty}/G$ is rather badly behaved. However, to understand this better an answer to the following question might shade more light on the symmetry in the limit here:

Question 2. *Is the Kolmogorov quotient $\mathrm{KQ}(\mathbb{A}^n_{K,\infty}/G)$ a spectral space?*

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Betti numbers of real semi-stable degenerations via logarithmic geometry

Matilde Manzaroli

(joint work with Emiliano Ambrosi)

In [AM22], we study the real topology of totally real semi-stable degenerations, with certain technical conditions on the special fiber X_0, and we give a bound for the individual real Betti numbers of a smooth fiber near 0 in terms of the complex geometry of X_0. The subject of interest in [AM22] has its root in the search of refinements of the Smith-Thom inequality; see (1). In the context of toric and tropical degenerations, there have been conjectures (e.g. [Vi80],[Ite17]) and results (e.g. [Ite93], [RS22]) concerning bounds of individual real Betti numbers. In [AM22], thanks to the use of (real) logarithmic geometry (see e.g. [Kat89], [KN89], ([Arg21])), we push ourselves a little further into a purely algebraic geometry context and study degenerations which are not necessarily toric.

Let X be a smooth projective real algebraic variety over $\mathbb{C}$, let $X(\mathbb{C})$ be the set of its complex points and $X(\mathbb{R})$ the set of its real points. For $K = \mathbb{R}, \mathbb{C}$, set $b_i(X(K)) := \dim_{\mathbb{F}_2}(H^i(X(K), \mathbb{Z}/2\mathbb{Z}))$, the i^{th} Betti number of $X(K)$.

1. Individual Betti numbers

The Smith-Thom inequality,

$$\sum_i b_i(X(\mathbb{R})) \leq \sum_i b_i(X(\mathbb{C})) \tag{1}$$

bounds the total Betti number of the real points of X with the total Betti number of its complexification. The problem of finding (non trivial) bounds for the individual Betti numbers $b_i(X(\mathbb{R}))$ in terms of the geometry of $X(\mathbb{C})$ is a central topic in real algebraic geometry. For example, it was conjectured by Viro in [Vi80] that if X is a smooth projective real surface such that $X(\mathbb{C})$ is simply connected then $b_1(X(\mathbb{R})) \leq \dim(H^1(X, \Omega^1_X))$. Even if the conjecture has been disproved by Itenberg in [Ite93], many smooth projective real algebraic varieties constructed with the tools at our disposal (for example, using Viro's patchworking method [Vi83]) verify the inequality

$$b_i(X(\mathbb{R})) \leq \sum_j h^{i,j}(X), \tag{2}$$

where $h^{i,j}(X) := \dim(H^i(X, \Omega^j_X))$.

The main goal of [AM22] is try to understand why it is easier to construct real algebraic varieties satisfying (2) and we look into a purely algebraic geometric setting in which an inequality close to (2) holds. Roughly speaking, the general principle is that, often, the constructions of real varieties with prescribed topology are achieved first by constructing a degenerated version of the pair $(X(\mathbb{C}), X(\mathbb{R}))$, whose irreducible components are simpler to deal with and then by deforming them back in non-trivial ways. Our main results (Theorem 1, Corollary 2), show that if these components are simple, from a cohomological point of view, then the variety obtained by gluing these components satisfies the inequality (2), up to the dimension of the 2-torsion in some cohomology group.

The most general result previously known in this setting is [RS22, Theorem 1.4], which proves a conjecture of Itenberg ([Ite17]). In [RS22], Renaudineau and Shaw show that for a real compact hypersurface X near the $\mathbb{Q}$-regular smooth tropical limit inside a smooth toric variety (see [IKMZ19] for the definitions involved), the inequality (2) holds. The strategy involved to prove such result consists in separately computing complex and real information into combinatorial data ([IKMZ19], [RS22]) and, in a second moment, showing that they are related ([RS22], [ARS21]).

In [AM22] we generalize this result to more general families of varieties, avoiding the use of combinatorial arguments and constructing, via real logarithmic geometry, a space which allows to simultaneously relate real and complex data.

2. Main result

While the methods in [RS22] are mainly of combinatorial nature, tropical geometry allows to construct degenerations of a given subvariety in a smooth toric variety. This is the point of view that we tackle in [AM22], trying to generalize this kind of individual-real-Betti-bounds to the general fiber of an abstract semi-stable degeneration, where combinatorial techniques from tropical and toric geometry are no longer available.

Assume that C is a smooth real curve and $f : X \to C$ a real projective morphism which is smooth outside a real point $0 \in C(\mathbb{R})$ and strictly-semistable around 0, in the sense that the irreducible components of X_0 are smooth and, locally analytically around 0, the family $f : X(\mathbb{C}) \to C(\mathbb{C})$ is isomorphic to the standard semistable degeneration $\mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n, T]/(x_1 \dots x_n - T)) \to \mathrm{Spec}(\mathbb{C}[T])$. Assume furthermore that $f : X \to C$ is totally real, i.e. that the irreducible components of $X_0(\mathbb{C})$ are real. Write $X_0 = \bigcup_{i \in I} X_i$ for the decomposition of X_0 in irreducible components and for every subset $J \subseteq I$ set

$$X_J := \bigcap_{i \in J} X_i \quad \text{and} \quad X_J^0 := X_J \setminus \bigcup_{i \notin J} X_i.$$

Then $X_0 = \coprod_{J \subseteq I} X_J^0$ is a stratification $\mathfrak{I} := \{X_J^0\}_{J \subseteq I}$ of X_0 by smooth real algebraic subvarieties. Fix a refinement $\mathfrak{Z} := \{X_\Delta^0\}$ of $\mathfrak{I}$, made of smooth real algebraic varieties.

In [AM22], we construct, for every ring A and every $q \geq 0$ a canonical cochain complex $C^\bullet_{q,\mathfrak{Z},A}$ of A-modules depending only on the complex geometry of the stratification $\mathfrak{Z}$. ftsRecall that a variety is said to be *maximal*, if (1) is an equality. Inspired by the geometric properties of degenerations constructed from tropical geometry, we consider the following conditions on the members of $\mathfrak{Z}$.

(a) $H^i(X_\Delta^0(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = 0$, for all $i \geq 1$ and $X_\Delta^0 \in \mathfrak{Z}$;
(b) X_Δ^0 is maximal, for all $X_\Delta^0 \in \mathfrak{Z}$;
(c) the mixed Hodge structure on $H^i(X_\Delta^0(\mathbb{C}), \mathbb{Q})$ is pure of type (i,i) and $H^i(X_\Delta^0(\mathbb{C}), \mathbb{Z})$ is torsion free, for all $i \geq 1$ and $X_\Delta^0 \in \mathfrak{Z}$.

Our main result is then the following.

Theorem 1. *Assume that (a),(b) and (c) hold. Then for every $t \in C(\mathbb{R})$ close to 0 one has :*

(1) $b_p(X_t(\mathbb{R})) \leq \sum_q dim(H^p(C^\bullet_{q,\mathfrak{Z},\mathbb{Z}/2\mathbb{Z}}))$.
(2) $dim(H^p(C^\bullet_{q,\mathfrak{Z},\mathbb{Z}} \otimes \mathbb{Q})) = h^{p,q}(X_t)$
(3) $C^\bullet_{q,\mathfrak{Z},\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z} \simeq C^\bullet_{q,\mathfrak{Z},\mathbb{Z}/2\mathbb{Z}}$.

Theorem 1 directly implies the following corollary, which was the main motivation for [AM22].

Corollary 2. *Assume that (a), (b), (c) hold and that $H^p(C^\bullet_{q,\mathfrak{Z},A})$ is torsion free for every $p, q \in \mathbb{N}$. Then for every $t \in C(\mathbb{R})$ close to 0 and every $p \in \mathbb{N}$ one has*

$$b_p(X_t(\mathbb{R})) \leq \sum_q h^{p,q}(X_t).$$

Inspired by the approach used in [Bru22], our basic strategy is to relate the real Betti and the Hodge numbers via the geometry of a common ambient space. The main innovation of [AM22] is the use of (real) logarithmic geometry (see e.g. [Kat89], [KN89], ([Arg21])) to construct and study this common ambient space, which allows to use a more sophisticated and less combinatorial machinery. After the construction of such a space, the cohomology of the general real fiber can be computed by a filtered complex. The idea of the use of filtered complexes is inspired by [RS22], where it was constructed via combinatorial techniques. Since these combinatorial tools are not available in our general setting, we use a different approach based on equivariant cohomology.

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Pinched handle decomposition of finite simplicial complexes

Jean-Yves Welschinger

I will introduce a notion of pinched handle decompositions on finite simplicial complexes and prove their existence after finitely many stellar subdivisions at facets. These decompositions extend the classical shellings of boundaries of convex polytopes for example. They encode a class of compatible discrete Morse functions while conversely every such function on a finite simplicial complex induces a (weak) pinched handle decomposition on its second barycentric subdivision.

Definition 1. A basic (resp. Morse) tile of dimension n and order $k \in \{0, \dots, n+1\}$ is an n-simplex deprived of k codimension one faces (resp. together with possibly a unique face of higher codimension, called its Morse face).

If T is a basic tile of order k, then every non-empty face μ of T has to contain the $(k-1)$-dimensional face $r(T)$, called its *restriction set*, whose missing vertices are opposite to the missing ridges, that is codimension one faces, of T, compare

[8]. The tile $T \setminus \mu$ is said to be *critical* of index k when $\mu = r(T)$ and it is said to be *regular* otherwise. The closed simplex is thus critical of vanishing index and the open simplex critical of index $\dim(\sigma)$.

From the topological viewpoint, an n-dimensional critical tile of index $k \in \{0, \dots, n\}$ is a (simplicial) *pinched handle* of dimension n and index k. It is obtained by pinching onto μ the missing face $\mu \times \theta$ of a piecewise linear handle $\overset{\circ}{\sigma} \times \theta$, where $\overset{\circ}{\sigma}$ is an open k-simplex, θ an $(n-k)$-simplex and μ a codimension one missing face of σ.

Definition 2. A pinched handle decomposition of a finite (relative) simplicial complex S is a filtration $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_N = S$ by (relative) subcomplexes such that for every $p \in \{1, \dots, N\}$, $S_p \setminus S_{p-1}$ consists of a single basic or critical tile supported by a facet, that is a maximal face, of S.

A weaker form of pinched handle decomposition is given by allowing all Morse tiles instead of basic and critical tiles only in this definition. Both notions generalize the classical notion of shellings [2] and I call the latter a Morse shelling, a notion that we introduced in [7] together with Nermin Salepci. Recall that a shellable triangulated manifold has to be PL-homeomorphic to a ball or a sphere, that triangulated spheres need not be shellable [6, 5], though PL-spheres becomes polytopal, hence shellable [2], after sufficiently many barycentric subdivisions [1]. Nevertheless, this number of subdivision can be arbitrarily large. From the algorithmic complexity point of view, deciding collapsibility [9] or shellability [4] is NP-complete, while contractibility is undecidable [10, 9]. A shelling is a pinched handle decomposition without handles of intermediate index.

Theorem 3 (Theorem 1.3 of [11])**.** *Every finite (relative) simplicial complex carries a pinched handle decomposition after finitely many stellar subdivisions at maximal faces. Moreover, the same holds true using stellar subdivisions at codimension one faces instead, or also using mixed ones. Finally, in bounded dimension, both the sequence of subdivisions and the shelling are given by some quadratic time algorithm.*

These pinched handle decompositions, even in their weaker form, recover the homology and cohomology of the complexes via spectral sequences.

Theorem 4 (Theorem 1.2 of [12])**.** *Any Morse shelling on a finite (relative) simplicial complex induces two spectral sequences which converge to its relative homology and cohomology respectively and whose first pages are free graded modules over the critical tiles.*

These weak versions are closely related to the Morse theory of R. Forman [3]. Indeed, every Morse shelling encodes a class of compatible discrete Morse functions whose critical faces are in one-to-one correspondence with the critical tiles of the shelling, preserving the index [7, 12]. And conversely, the following holds.

Theorem 5 (Theorem 1.1 of [13])**.** *Let f be a discrete Morse function on a finite simplicial complex K. Then, the second barycentric subdivision of K carries Morse* *shellings whose critical tiles are in one-to-one correspondence with the critical faces of f, preserving the index.*

This theorem has the following consequence in the smooth category.

Theorem 6 (Corollary 1.2 of [13]). *Let f be a smooth Morse function on a smooth closed manifold M and let $h : K \to M$ be any PL-triangulation on M. Then, as soon as d is large enough, the d-th barycentric subdivision of the simplicial complex K carries Morse shellings whose critical tiles are in one-to-one correspondence with the critical points of f, preserving the index.*

However, discrete Morse functions or Morse shellings do not distinguish between a collapsible complex and an actual triangulated ball, while the critical tiles (or pinched handles) of a pinched handle decomposition on a triangulated manifold provide an obstruction from being shellable, hence being a PL-triangulated ball or sphere, see [13].

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Gibbs manifolds

Dmitrii Pavlov

(joint work with Bernd Sturmfels, Simon Telen)

Given a linear (or, more generally, affine) subspace $\mathcal{L}$ of the space of symmetric $n\times n$ matrices $\mathbb{S}^n \simeq \mathbb{R}^{\binom{n+1}{2}}$, one defines its *Gibbs manifold* $\mathrm{GM}(\mathcal{L})$ to be the image of $\mathcal{L}$ under the exponential map. This map is defined on the space of $n\times n$ matrices by the converging power series $\exp(X) = \sum_{i=0}^{\infty} \frac{X^i}{i!}$. It sends symmetric matrices to positive definite matrices, so $\mathrm{GM}(\mathcal{L})$ is a subset of the positive definite cone.

Gibbs manifolds arise naturally in many areas of science, such as quantum physics [1], statistics [6] and optimization [2]. For instance, in entropic regularization for semidefinite programming [2, Section 5] the optimal point for the regularized program is the unique intersection of the spectrahedron with a Gibbs manifold. More precisely, this is the Gibbs manifold $\mathrm{GM}(\mathcal{L})$ of the linear space $\mathcal{L}$ spanned by the matrices defining the linear constraints.

In some special cases, Gibbs manifolds are semialgebraic. This happens, for instance, if the matrices in $\mathcal{L}$ are pairwise commuting. Though this is not true in general, it is interesting to ask which polynomial equations vanish on the Gibbs manifold of $\mathcal{L}$. This motivates the definition of the Gibbs variety $\mathrm{GV}(\mathcal{L})$, which is the Zariski closure of the Gibbs manifold $\mathrm{GM}(\mathcal{L})$ in the complex space $\mathbb{C}^{\binom{n+1}{2}}$.

Gibbs varieties are a natural noncommutative generalization of toric varieties, since any affine toric variety arises as the Gibbs variety of a linear space spanned by diagonal matrices. In the same spirit, the role of Gibbs manifolds in entropic regularization for semidefinite programming extends the role of positive toric varieties in the linear programming setup of [5].

A peculiar feature of Gibbs varieties is that they are low-dimensional. This is made precise in the following theorem.

Theorem 1 ([2])**.** *Let $\mathcal{L} \subset \mathbb{S}^n$ be an affine space of symmetric matrices of dimension d. The dimension of the Gibbs variety $\mathrm{GV}(\mathcal{L})$ is at most $n+d$. If $\mathcal{L}$ is a linear space, then $\dim \mathrm{GV}(\mathcal{L})$ is at most $n+d-1$.*

The following result gives a formula for the dimension of $\mathrm{GV}(\mathcal{L})$ in terms of the *$\mathcal{L}$-centralizer* of a generic matrix A in $\mathcal{L}$. This is defined as the set of all matrices in $\mathcal{L}$ that commute with A.

Theorem 2 ([3])**.** *Let $\mathcal{L}$ be a linear space of $n \times n$ symmetric matrices of dimension d. Let k be the dimension of the $\mathcal{L}$-centralizer of a generic element in $\mathcal{L}$ and m the dimension of the $\mathbb{Q}$-linear space spanned by the eigenvalues of $\mathcal{L}$. Then $\dim \mathrm{GV}(\mathcal{L}) = m + d - k$.*

Note that by Theorem 2, the upper bound for linear spaces in Theorem 1 is attained when the eigenvalues of $\mathcal{L}$ are $\mathbb{Q}$-linearly independent and the dimension of a generic $\mathcal{L}$-centralizer is one.

Apart from being low-dimensional, Gibbs varieties are also irreducible and unirational under a mild genericity assumption. Moreover, it is possible to find an explicit rational parametrization of a given Gibbs variety. A key ingredient is Sylvester's formula [4]. After such a parametrization is obtained, one can algorithmically implicitize the variety, that is, find its defining equations. This can be done both symbolically and numerically. Implicitization algorithms are implemented and available at `https://mathrepo.mis.mpg.de/GibbsManifolds`.

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Matrix convex sets with polytope base, quantum Latin squares, and compatibility

Ion Nechita

(joint work with Andreas Bluhm, Simon Schmidt)

A polytope $\mathcal{P}$ containing the origin can be characterized in two different but equivalent ways:

- by its facets, as an intersection of half-spaces:

$$\mathcal{P} = \bigcap_{i=1}^{f} \{x \in \mathbb{R}^g \,:\, \langle x, h_i \rangle \leq 1\}, \tag{1}$$

- by its extremal points, as a convex hull:

$$\mathcal{P} = \operatorname{conv}\{v_i\}_{i=1}^k. \tag{2}$$

If we want to allow the elements of the polytope to be tuples of matrices instead of tuples of scalars, these two conditions give rise to two different and *inequivalent* (in general) such matricization, which are both special cases of so-called *matrix convex sets*

- the facet description from Eq. (1) generalizes to the set

$$\mathcal{P}_{\max}(d) := \{(A_1, \ldots, A_g) \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^g \,:\, \langle A, h_i \otimes \rho \rangle \leq 1 \quad \forall i \in [f], \forall \rho \in \mathcal{M}_d^{1,+}(\mathbb{C})\},$$

- and the extremal points description from Eq. (2) generalizes to the set

$$\mathcal{P}_{\min}(d) := \{(A_1, \ldots, A_g) \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^g : \exists\, \mathrm{POVM}\, C \text{ s.t. } A_x = \sum_{i=1}^{k} v_i(x) C_i,\ \forall x \in [g]\}.$$

As our notation suggests, $\mathcal{P}_{\min}$ is the smallest matrix convex set arising from $\mathcal{P}$ and $\mathcal{P}_{\max}$ is the largest.

The appearance of density matrices and POVMs in the definition of the sets $\mathcal{P}_{\min}$, $\mathcal{P}_{\max}$ suggest that there might be a link between such matrix convex sets and quantum information theory. Indeed, in the articles [1, 2, 3], some of the present authors realized that if one takes $\mathcal{P}$ to be the hypercube $[-1,1]^g$, then the following correspondence holds:

$$(2E_1 - I, \ldots, 2E_g - I) \in ([-1,1]^g)_{\max} \iff \{E_i, I - E_i\} \text{ POVMs } \forall i\,.$$

What about $([-1,1]^g)_{\min}$? One of the defining properties that distinguish quantum mechanics from our everyday experience based on classical mechanics is the existence of *incompatible measurements*, i.e., measurements that cannot be performed at the same time [4]. Such measurements are indispensable for detecting quantum non-locality and can therefore be seen as a resource for many quantum information theoretic tasks similar to entanglement. For the measurements that are compatible, a joint measurement exists such that their outcomes can be recovered post-processing the outcomes of the joint measurement. It turns out that membership in $([-1,1]^g)_{\min}$ is related to measurement compatibility

$$(2E_1 - I, \ldots, 2E_g - I) \in ([-1,1]^g)_{\min} \iff \{E_i, I - E_i\} \text{ comptatible POVMs } \forall i\,.$$

The reformulation of measurement compatibility as minimal and matrix convex sets has been instrumental in finding new bounds on the maximal amount of incompatibility available in different situations.

The success of the study of minimal and maximal matrix convex sets for the hypercube suggests the natural question: *What tasks in quantum information theory can be formulated as membership in $\mathcal{P}_{\min}$, $\mathcal{P}_{\max}$ for polytopes $\mathcal{P}$?*. This is the task this paper sets out to solve.

Motivated by the example of measurement compatibility, we define a notion of *polytope operators* and *polytope compatibility.* A tuple of matrices is a $\mathcal{P}$-operator if it is in $\mathcal{P}_{\max}$ and it is $\mathcal{P}$-compatible if it is in $\mathcal{P}_{\min}$. We study equivalent formulations and implications of polytope compatibility and we characterize the elements which are $\mathcal{P}$-compatible if and only if they are $\mathcal{P}$-operators. An informal version of the latter result is the following:

Theorem. Let A be a g-tuple of self-adjoint operators. Then, A is $\mathcal{P}$-compatible for *all* polytopes $\mathcal{P}$ such that they are $\mathcal{P}$-operators if and only if the operators A admit a pairwise commuting dilation N with essentially the same numerical range.

We show that another well-known problem from quantum information theory can be formulated as polytope compatibility, namely the study of *quantum magic squares.* An $N \times N$ block matrix $(A_{ij})_{i,j\in[N]}$ with d-dimensional matrices A_{ij} is a quantum magic square if both its rows $\{A_{ij}\}_{j\in[N]}$ and columns $\{A_{ij}\}_{i\in[N]}$ form POVMs. This can be expressed with the help of the Birkhoff polytope (the set of bistochastic matrices), projected onto its supporting affine subspace. Calling this polytope $\mathcal{B}_N$, we arrive at the following equivalence:

$$A \in (\mathcal{B}_N)_{\max} \iff A \text{ is a quantum magic square.}$$

A quantum magic square is especially simple if it has a hidden structure in terms of a tensor product of permutation matrices and a POVM. In [5], such a quantum magic square is called *semiclassical*, whereas [6] calls such quantum magic squares doubly normalised tensor of positive semi-definite operators. The interest in such objects come from the Birkhoff-von Neumann theorem, which states that the bistochastic matrices are the convex hull of the permutation matrices. The semiclassical magic squares can be seen as a matricization of this idea. However, not all quantum magic squares are semiclassical, but we can characterize the ones that are:

$$A \in (\mathcal{B}_N)_{\min} \iff A \text{ is a semiclassical quantum magic square.}$$

One might be tempted to conjecture that semiclassical quantum magic squares are the ones in which the row and column POVMs are compatible. However, we give an explicit example of a quantum magic square with compatible POVMs which is *not* semiclassical. Using one of our reformulations of $\mathcal{P}$-compatibility as factorization of an associated map through a simplex, we recover the fact that being a semiclassical quantum magic square does not only require the POVMs to be compatible, but also that the post-processing via which they arise from a joint measurement is symmetric (previously observed in [6]).

Finally, we find that polytope compatibility corresponds in general the compatibility of POVMs with common elements under restricted post-processing. Any collection of POVMs which share elements can be represented as a hypergraph, where each POVM element is a vertex and which POVM elements belong to the same POVM is represented by hyperedges. Such hypergraphs are in one-to-one correspondence with polytopes $\mathcal{P}$ having vertices with rational coordinates. Being a $\mathcal{P}$-operator then corresponds to being a POVM with the desired common elements. These POVMs are $\mathcal{P}$-compatible if and only if the POVMs are compatible and have a joint measurement from which they arise under restricted post-processing.

We illustrate this in an example where we consider two POVMs with a common element A, such that the POVMs become

$$(A, B, I - A - B) \qquad (A, C, I - A - C). \tag{3}$$

The polytope to which this compatibility structure corresponds is a pyramid with square basis. (A, B, C) is in the minimal matrix convex set corresponding to the pyramid if and only if the two POVMs above are compatible and have a joint POVM Q with five elements from which they arise as

$$A = Q_1 \qquad B = Q_4 + Q_5 \qquad C = Q_3 + Q_5.$$

We conclude the section with an explicit counterexample that not all compatible POVMs as in (3) have a joint POVM of the form above, which shows that the restricted post-processing is indeed necessary.

Thus, in summary, this work generalizes the correspondence between measurement incompatibility and the minimal and maximal matrix convex sets of the hypercube. We find that polytope compatibility is in one-to-one correspondence with measurement compatibility with common elements and restricted post-processing.

As an example, we find that being a semiclassical magic square corresponds to being Birkhoff-polytope compatible.

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Arveson-Douglas essential normality conjecture - a bridge between operator algebras and algebraic geometry

Eli Shamovich

Let us fix a number $d \in \mathbb{N}$ and consider the polynomial ring $\mathbb{C}[z_1, \ldots, z_d]$. For a multi-index $\alpha \in (\mathbb{N} \cup \{0\})^d$, we define the length $|\alpha| = \sum_{j=1}^d \alpha_j$ and the corresponding monomial $z^\alpha = \prod_{j=1}^d z_j^{\alpha_j}$. One can endow the polynomial ring with various inner products. One, in particular, stands out due to its importance in many areas of mathematics and physics. This inner product, which is known in algebraic geometry as the Bombieri inner product (see, for example, [5]), is given by

$$\langle z^\alpha, z^\beta \rangle = \delta_{\alpha\beta} \frac{\prod_{j=1}^d \alpha_j!}{|\alpha|!} = \frac{\alpha!}{|\alpha|!}.$$

Completing the polynomial ring with respect to the corresponding norm yields a Hilbert space that we will denote H^2_d. This Hilbert space is known in physics as the symmetric (or bosonic) Fock space and as the Drury-Arveson space in function theory and operator algebras community. The Drury-Arveson space is a space of analytic functions on the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ with certain convergence conditions on the Taylor coefficients at the origin. By the Bombieri inequality, for every two homogenous polynomials $p, q \in \mathbb{C}[z_1, \ldots, z_d]$ of degrees n and m, respectively, we have

$$\frac{n!m!}{(n+m)!} \|p\|\|q\| \leq \|pq\| \leq \|p\|\|q\|$$

In particular, it implies that multiplication by a polynomial extends to a bounded operator on H^2_d. We will denote these multiplication operators by M_p. In particular, we will focus on the operators of multiplication by coordinates M_{z_j}, $1 \leq j \leq d$. The multiplier algebra $\mathcal{M}_d$ of H^2_d is the weak operator topology closed algebra generated by the M_{z_j} in $B(H^2_d)$. The space H^2_d and its multiplier algebra enjoy several universality properties discovered by Drury [10], Arveson [2], and Agler and McCarthy [1]. Arveson proved that for every $1 \leq i, j \leq d$, the operator $M_{z_j}M^*_{z_i} - M^*_{z_i}M_{z_j}$ is compact. Recall, that an operator on a Hilbert space is called compact if it maps the unit ball into a precompact subset. Intuitively, compact operators are small, since one can approximate a compact operator by finite-rank ones in norm. Therefore, we say that the operators M_{z_j} are essentially normal. Moreover, we get an exact sequence

$$0 \to K \to C^*(1, M_{z_1}, \ldots, M_{z_d}) \to C(\partial\mathbb{B}_d) \to 0.$$

Here, K stands for the ideal of compact operators, $C^*(1, M_{z_1}, \ldots, M_{z_d})$ is the unital C^*-algebra generated by our multiplication operators, and $C(\partial\mathbb{B}_d)$ is the algebra of continuous functions on the boundary of the ball.

Given a radical homogenous ideal $I \subset \mathbb{C}[z_1, \ldots, z_d]$, we can associate to I two Hilbert spaces $[I]$, that is the closure of I in H^2_d and $\mathcal{H}_I$ that is the orthogonal complement of $[I]$. It is rather immediate that $[I]$ is invariant for M_{z_j} and, therefore, $\mathcal{H}_I$ is coinvariant (i.e., invariant under the $M^*_{z_j}$). Let us denote by P_I the orthogonal projection onto $\mathcal{H}_I$ and for $1 \leq j \leq d$, $S_j = P_I M_{z_j}|_{\mathcal{H}_I}$. Arveson [3] and Douglas [8] have conjectured that the operators S_j are also essentially normal. Nowadays, this question is the first part of the Arveson-Douglas essential normality conjecture. If true, this implies that we have an exact sequence:

$$0 \to K \to C^*(1, S_1, \ldots, S_d) \to C(\partial\mathbb{B}_d \cap V(I)) \to 0.$$

Here, $V(I)$ is the cone cut out by the homogenous ideal I in $\mathbb{C}^d$ and $C(\partial\mathbb{B}_d \cap V(I))$ is the algebra of continuous functions on the intersection of $V(I)$ with the boundary of the ball. The exact sequence gives rise to a K-homology class via the Brown-Douglas-Fillmore theory [6] on the space $\partial\mathbb{B}_d \cap V(I)$. This class, in turn, can be translated (at least in the smooth case) to a K-theory class on the homogeneous variety corresponding to I (see [9]). Whether the corresponding K-homology class is non-trivial, even if the operators S_j are essentially normal, is the second part of the Arveson-Douglas essential normality conjecture.

The essential normality conjecture was established in many cases. For example, the case of monomial ideals was proved by Arveson [4] and Douglas [7]. The case of principal ideals and the cases for $d = 2$ and $d = 3$ was established by Guo and Wang [12]. The case of smooth projective varieties was settled by Engliš and Eschmeier [11] (see also [9]).

Shalit [13] proposed an interesting approach to the conjecture. We say that the ideal I has the stable division property, if there exists a constant $C > 0$ and a homogeneous generating set $f_1, \ldots, f_k \in I$, such that for every homogeneous $h \in I$, there exists $g_1, \ldots, g_k \in \mathbb{C}[z_1, \ldots, z_d]$, such that $h = \sum_{j=1}^k f_k g_k$ and $\sum_{j=1}^k \|f_k g_k\|^2 \leq C\|h\|^2$. It was shown by Shalit that if $I \subset \mathbb{C}[z_1, \ldots, z_d]$ is a homogenous ideal with the stable division property, then the corresponding operators $S_1, \ldots, S_d$ are essentially normal. It is an open question whether the converse holds.

Lastly, one can obtain a condition on angles between subspaces. Let $\mathcal{H}$ be a Hilbert space an $\mathcal{K}, \mathcal{L} \subset \mathcal{H}$ be closed subspaces. Let $\mathcal{K}' = \mathcal{K} \cap (\mathcal{K} \cap \mathcal{L})^{\perp}$ and $\mathcal{L}' = \mathcal{L} \cap (\mathcal{K} \cap \mathcal{L})^{\perp}$. The Friedrichs angle between $\mathcal{K}$ and $\mathcal{L}$ is the acute angle θ, such that

$$\cos\theta = \sup\left\{|\langle x, y\rangle| \mid x \in \mathcal{K}',\, y \in \mathcal{L}', \text{ and } \|x\| = \|y\| = 1\right\}.$$

Consider the last differential in the Koszul complex of the variables. Namely,

$$D\colon \mathbb{C}[z_1, \ldots, z_d] \otimes \mathbb{C}^d \to \mathbb{C}[z_1, \ldots, z_d]$$

given by $D\left(\sum_{j=1}^d p_j \otimes e_j\right) = \sum_{j=1}^d z_j f_j$. Here, the e_j are the vectors of the standard basis in $\mathbb{C}^d$. Let $(\ker D)_n$ be the n-th graded component of the kernel of D. Let I_n stand for the n-th graded component of I. In an ongoing joint work with Kennedy, we have shown that the first part of the essential normality conjecture is equivalent to the fact that the cosine of the Friedrichs' angle between $(\ker D)_n$ and $I_n \otimes \mathbb{C}^d$ tends to 0 with n.

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Abstract cone systems

Mirte van der Eyden
(joint work with Tim Netzer, Gemma De les Coves)

In free semialgebraic geometry one studies families of semialgebraic sets that live in matrix spaces of growing dimension. These sets have some common description; for example in case of a free spectrahedron, one can define every level by the same tuple of hermitian matrices $(B_1, \dots, B_d)$:

$$\mathcal{S}_s(B_1, \dots B_d) = \{(A_1, \dots, A_d) \in \mathrm{Her}_s^d | A_1 \otimes B_1 + \dots + A_d \otimes B_d \geq 0\},$$

where Her_s are the s by s hermitian matrices and by ≥ 0 we mean positive semidefinite (psd). By studying the entire family $\{\mathcal{S}_s\}_{s \geq 0}$ at once, one can prove stronger results then by just looking at the levels separately [8].

Abstract operator systems are a closely related concept that originated from operator theory, with the same spirit of having a free dimension [5, 7]. An abstract operator system (AOS) consists of a vector space V with involution, together with a proper cone D_s (i.e. convex, closed, salient and with non-empty interior) inside the hermitian part of $\mathrm{Mat}_s(V)$, for every $s \in \mathbb{N}$. The cones have to be chosen such that one can move between the levels by matrix contractions:

$$A \in D_s \Rightarrow \forall X \in \mathrm{Mat}_{s,t}(\mathbb{C}), \quad X^* A X \in D_t.$$

In the higher levels, the interaction between the cones and the tensor product becomes particularly insightful. Starting with a cone $D_1 \subseteq V$ at the groud level, there are multiple ways to define the cone of positive elements in the higher levels. Well-studied examples of operator systems over the cone of positive semidefinite matrices are the operator system of separable matrices (arising from the *minimal* tensor product), psd matrices, and block-positive matrices (from the *maximal* tensor product) [3]. These examples have clear connections to quantum information theory, where one studies quantum states and measurements, modelled by psd matrices, and is interested for example in determining if a quantum state is separable (i.e. part of the minimal AOS over the psd cone) or otherwise entangled. Some results in this direction can be found here [3, 6, 4].

There are multiple recent works on similar notions of entanglement between general cones in finite dimensional vector spaces, not necessarily matrix spaces [1, 2]. Related to that, there has been a lot of interest in general probabilistic theories, where one tries to single out quantum theory from more general probabilistic theories where states are no longer psd matrices but elements from cones in a vector space (see [9] and references therein). Abstract operator systems are no longer capable of capturing these results, because they are always based on matrix spaces.

Motivated by this, we propose a generalized version of AOS, called the Abstract cone system (ACS). We no longer restrict to matrix spaces (and their positive semidefinite cones), but instead look at tensor products of V with arbitrary finite dimensional vector spaces with involution. We reprove the most important theorems about AOS in this more general case: the equivalent of the Choi-Effros theorem that every abstract operator system has a concrete realization, the existence of minimal and maximal ACS over a given cone at the ground level and the duality between cone systems with a finite dimensional realization (which are free spectrahedra) and finitely generated ones.

More specifically, ACS are based on the following structure. The role that is usually played by the matrix spaces $\mathrm{Mat}_s(\mathbb{C})$ and the contraction maps that guarantee the compatibility between the cones, now has to be played by other vector spaces and other maps. We formalize this by choosing a subcategory of the category ${}^*\mathsf{FVec}$, the category with finite dimensional vector spaces with involution $*$ as objects, and as morphisms all $*$-linear maps. This subcategory $\mathcal{S}$ has to be monoidal, meaning that $\mathbb{C}$ is in there, and that whenever S and T are objects in $\mathcal{S}$, then $S \otimes T$ is also in there. Moreover, it has to respect duals, so for $S \in \mathcal{S}$, also its dual space $S' \in \mathcal{S}$. Together with a few more technical details, this provides the structure on which we can define an abstract cones system:

Definition 1 (Abstract cone system)**.** An *abstract cone system* consists of an object $V \in {}^*\mathsf{FVec}$, together with a family of proper cones $D_S \subseteq V \otimes S$ for all $S \in \mathcal{S}$, such that:

$$A \in D_S \Rightarrow \forall \phi \in \mathrm{Hom}_{\mathcal{S}}(S,T), \quad (\mathrm{id} \otimes \phi)A \in D_T,$$

where by $\mathrm{Hom}_{\mathcal{S}}(S,T)$ we denote the set of morphisms between $S, T \in \mathcal{S}$ that are part of the subcategory.

It is now interesting to explore the different subcategories that we can choose, and what type of examples can be fit into this generalized framework. Some examples are the following:

- The minimal subcategory consists only of $\mathbb{C}$, with $\mathrm{Hom}_{\mathcal{S}}(\mathbb{C},\mathbb{C}) = \mathbb{R}_{\geq 0}$ as morphisms. An abstract operator system over this subcategory will be a vector space V with any proper cone $D \subseteq V_h$.
- Adding one other vector space S to this minimal subcategory, immediately requires us to include an infinite number of objects of the form

$$S_{n,m} = S^{\otimes n} \otimes S'^{\otimes m}$$

  in the subcategory. The morphisms between all these vector spaces can be determined by choosing a proper cone $C_s \subseteq S_h$ and defining $\mathrm{Hom}_{\mathcal{S}}(S,\mathbb{C})$ to be all the maps that are positive on this cone. There are many ways to extend this to a choice for all morphisms in the subcategory, but two distinct choices are the maps that are completely positive on the minimal and maximal tensor products of the cone C_s and its dual cone $C_s^\vee$:

$$C_{n,m} = C_s^{\otimes_{\mathrm{min/max}} n} \otimes_{\mathrm{min/max}} (C_s^\vee)^{\otimes_{\mathrm{min/max}} m}$$

- Finally, when choosing as elements all vector spaces of bounded operators on finite dimensional Hilbert spaces, $\mathcal{B}(H)$, and as morphisms the set

$$\{\phi : \mathcal{B}(H_1) \to \mathcal{B}(H_2) | \phi(T) = f^\dagger T f \quad \forall f \in \mathrm{Hom}(H_1,H_2)\},$$

  we are exactly back to the case of abstract operator systems, but in a basis independent way.

This talk was about ongoing work in this direction. In the near future, we will focus on expanding the list of examples and connecting to recent work and open problems in quantum information theory and general probabilistic theories.

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A classical quantum game: Perfection vs. The Nullstellensatz

John William Helton

(joint work with dam Bene Watts, Jared Hughes, Daniel Kane, Igor Klep and Zehong Zhang)

The talk concerned a rigid class of equations which make sense both for matrix unknowns or for binary unknowns. The ultimate issue is to compare the satisfiability (solvability) of these equations. The problem comes from quantum games and amounts to understanding the advantage of allowing a 'perfect' quantum strategy over restricting players to using a classical strategy.

Satisfiability (SAT) problems are heavily studied in computer science for $m\times n$-systems of linear equations which one must solve mod 2. They generate a class of such systems randomly and find that asymptotically there is a critical threshold c, meaning that with high probability

$$\begin{aligned} m/n < c &\quad \text{implies there is a solution;} \\ m/n > c &\quad \text{implies there is no solution.} \end{aligned}$$

For example, for 3XOR SAT c exists and is about 0.92; see [1].

There is a k player XOR game and quantum game analog of this and historically these have been an influential guide to quantum vs classical behavior. That quantum entanglement exists was shown by experiments on a particular 2XOR game. 2XOR games were fully understood by Tsirelson in the 1980s, who showed they reduce to a SDP. About 15 years ago it was shown that quantum strategies could have unbounded advantage over classical strategies in the context of 3XOR games. Open remained: given a 3XOR game determine if it does or does not have 'perfect' quantum strategy. Is this problem decidable?

Work with collaborators settles this by reducing quantum strategy production to a new class of linear SAT like problems. The reduction starts at a high level with a noncommutative directional real nullstellensatz applied to a special class of toric ideals [2]. Next comes a tricky calculation [3].

Having an algorithim allows us to run many experiments which confront us with questions about the critical threshold [4]. So now we are developing theory for that. Our speculation is that

$$cSAT = cGAME = cQUANTUMGAME$$

holds for the thresholds. This is a long slog with progress: bounding the difference above within 4 percent and we are near a proof that the first equality is true.

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Amoeba dimensions

Jan Draisma

(joint work with Eggleston, Pendavingh, Rau, and Yuen)

1. Amoeba dimensions for general varieties

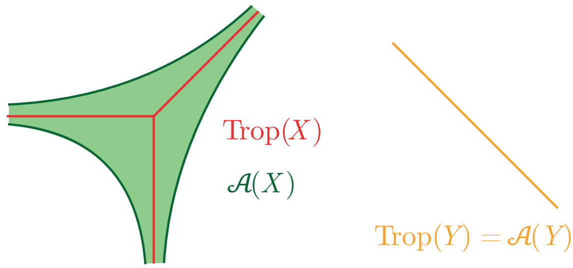
Let $X \subseteq (\mathbb{C}^*)^n$ be an irreducible closed subvariety. The *amoeba* of X, denoted $\mathcal{A}(X)$, is the image of X under the map

$$\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n, (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|).$$

Note that $\mathcal{A}(X)$ is the image of a semialgebraic set in $(\mathbb{R}_{>0})^n$ under the diffeomorphism $\log : \mathbb{R}_{>0} \to \mathbb{R}$ applied coordinate-wise, so $\dim_{\mathbb{R}} \mathcal{A}(X)$ has a well-defined meaning. The *tropicalisation* of X, denoted $\mathrm{Trop}(X)$, equals $\lim_{t\to\infty} \frac{1}{t}\mathcal{A}(X)$ in a suitable version of the Hausdorff metric.

Much is known about the structure of $\mathrm{Trop}(X)$; e.g., it is the support of a pure $\dim_{\mathbb{C}} X$-dimensional polyhedral complex, which is connected through codimension one. Much less is known, in this generality, about $\mathcal{A}(X)$. Indeed, even its dimension behaves mysteriously. For instance, $X := \{(x, y) \mid x + y = 1\}$ and

$Y := \{(x,y) \mid x \cdot y = 1\}$ are both curves but have amoebas of dimensions and 2, respectively:



An obvious upper bound to $\dim_{\mathbb{R}} \mathcal{A}(X)$ is $2\dim_{\mathbb{C}} X$. On the other extreme, if $T \subseteq (\mathbb{C}^*)^n$ is a subtorus of dimension k, given as the image of a monomial map $(\mathbb{C}^*)^k \to (\mathbb{C}^*)^n$ corresponding to a matrix $B \in \mathbb{Z}^{n \times k}$, then $\mathcal{A}(T) = B \cdot \mathbb{R}^k$ is a k-dimensional, rather than $2k$-dimensional, linear subspace of $\mathbb{R}^n$.

In [3], Nisse and Sottile study the structure of $\mathcal{A}(X)$. In particular, they observe that a torus action always causes a dimension drop for $\mathcal{A}(X)$. The argument is as simple as it is elegant: if X is stable under a k-dimensional subtorus $T \subseteq (\mathbb{C}^*)^n$, then the quotient map $X \to X/T \subseteq (\mathbb{C}^*)^n/T \cong (\mathbb{C}^*)^{n-k}$ has a corresponding natural map $\mathcal{A}(X) \to \mathcal{A}(X/T) \subseteq \mathbb{R}^{n-k}$ whose fibres are translates of the k-dimensional space $\mathcal{A}(T)$. In particular,

$$\dim_{\mathbb{R}} \mathcal{A}(X) = \dim_{\mathbb{R}} \mathcal{A}(X/T) + k \leq 2\dim_{\mathbb{C}}(X/T) + k \leq 2\dim_{\mathbb{C}} X - k. \tag{1}$$

Furthermore, Nisse and Sottile prove that the maximal dimension drop is caused by a maximal-dimensional torus action:

Theorem 1 ([3]). *The amoeba dimension* $\dim_{\mathbb{R}} \mathcal{A}(X)$ *of an irreducible closed subvariety* $X \subseteq (\mathbb{C}^*)^n$ *is at least* $\dim_{\mathbb{C}} X$*, with equality if and only if* X *is a torus orbit.*

It is natural to wonder, conversely, whether a dimension drop is *always* caused by a torus action. This is not the case: e.g., if X is any variety with $\frac{n}{2} < \dim_{\mathbb{C}} X < n$, then $\dim_{\mathbb{R}} \mathcal{A}(X) \leq n < 2\dim_{\mathbb{C}} X$, whereas most varieties of this dimension do not admit any torus action. This caused Nisse-Sottile to consider a weaker version of torus actions. Slightly modifying their definition, for the purpose of this abstract, we say that a torus $T \subseteq (\mathbb{C}^*)^n$ *nearly acts* on X if $\mathcal{A}(T) \subseteq T_p\mathcal{A}(X)$ for most $p \in \mathcal{A}(X)$. For instance, the torus $(\mathbb{C}^*)^2$ nearly acts on X in the first example above. This example shows that a near torus action does not always cause a dimension drop in the amoeba dimension. However, in [2] we proved that the converse *is* true:

Proposition 2 ([2]). *If* $\dim_{\mathbb{R}} \mathcal{A}(X) < 2\dim_{\mathbb{C}} X$*, then some positive-dimensional torus* T *nearly acts on* X*.*

Now if T nearly acts on X, then $\dim_{\mathbb{R}} \mathcal{A}(X) = \dim_{\mathbb{R}} \mathcal{A}(\overline{T \cdot X})$. Furthermore, the variety $Y := \overline{T \cdot X}$ is stable under T, so that, to compute its amoeba dimension, one can use the argument in (1). Since Y/T lives in a lower-dimensional torus, one can use this argument inductively. The conclusion is then the following formula for the amoeba dimension:

Theorem 3 ([2])**.** *For any irreducible closed subvariety $X \subseteq (\mathbb{C}^*)^n$, we have*

$$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{2 \dim_{\mathbb{C}} \overline{T \cdot X} - \dim_{\mathbb{C}} T \mid T \text{ a subtorus of } (\mathbb{C}^*)^n\}.$$

This theorem has the interesting consequence that $\operatorname{Trop}(X)$ determines the amoeba dimension of X:

Corollary 4 ([2])**.** *For any irreducible closed subvariety $X \subseteq (\mathbb{C}^*)^n$, we have*

$$\begin{aligned}\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{&2 \dim_{\mathbb{R}}(U + \operatorname{Trop}(X)) - \dim_{\mathbb{R}} U \mid \\ &U \text{ subspace of } (\mathbb{R})^n \text{ spanned by rational vectors}\},\end{aligned}$$

where $U + \operatorname{Trop}(X)$ is the Minkowski sum.

Open question 1. *Is there an algorithm that takes as input a d-dimensional rational polyhedral fan with support F and computes the expression in the corollary with* $\operatorname{Trop}(X)$ *replaced by F?*

2. Amoeba dimensions of linear spaces

Now assume that $V \subseteq \mathbb{C}^n$ is a linear subspace for which $X := V \cap (\mathbb{C}^*)^n$ is nonempty. Then $\operatorname{Trop}(X)$ is the matroid fan of the loopless matroid M on $[n]$ in which a subset I is independent if the projection $V \to \mathbb{C}^I$ is surjective. In particular, by the corollary, $\dim_{\mathbb{R}} \mathcal{A}(X)$ is uniquely determined by M. This raises the question for an explicit matroidal formula. We have found such a formula:

Theorem 5 ([1])**.** *In the setting above, $\dim_{\mathbb{R}} \mathcal{A}(X)$ equals the minimum, over all partitions of $[n]$ into nonempty subsets $P_1, \ldots, P_k$, of*

$$\sum_{i=1}^{k} (2 \operatorname{rk}(P_i) - 1).$$

Furthermore, this dimension can be computed deterministically in polynomial time in the bit size of a matrix with row space V.

A few remarks are in order:

(1) The inequality $\leq$ is easy and follows from the fact that, for any such partition $P_1, \ldots, P_k$ of $[n]$, we have $V \subseteq \prod_i V_i$ where V_i is the projection of V in $\mathbb{C}^{P_i}$; the -1 is caused by the fact that each vector space V_i is stable under a 1-dimensional torus.
(2) The proof of $\geq$ is independent of our earlier paper [2] and is specific for linear spaces; we do not get a new formula for the amoeba dimension of a general variety.
(3) We actually show that for a general loopless matroid M on $[n]$, the function that maps $E \subseteq [n]$ to the minimum over all partitions of E into nonempty subsets $P_1, \ldots, P_k$ of the expression in the theorem is the rank function

of a new matroid M' on $[n]$, and we give an algorithm for computing the rank of E that needs only a polynomial number of rank evaluations in M.

To conclude, we now have two formulas for the amoeba dimension of X when $X = V \cap (\mathbb{C}^*)^n$, with V a linear subspace of $\mathbb{C}^n$. Both of these depend only on the matroid defined by V, and must of course give the same value. This raises the following question.

Open question 2. *Let M be an arbitrary loopless matroid on $[n]$ and let $F \subseteq \mathbb{R}^n$ be the support of the matroid fan of M. Is it true that*

$$\min\{2\dim_{\mathbb{R}}(U+F) - \dim_{\mathbb{R}} U \mid U \textit{ subspace of } (\mathbb{R})^n \textit{ spanned by rational vectors}\}$$

equals

$$\min\{\sum_{i=1}^{k}(2\,\mathrm{rk}_M(P_i) - 1) \mid \bigsqcup_i P_i = [n] \textit{ and all } P_i \neq \emptyset\}?$$

It is not hard to see $\leq$ by taking for U the space spanned by the characteristic vectors of the P_i. For the converse we have no idea!

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Quasicrystals and Lee-Yang (stable) polynomials

Lior Alon

(joint work with Alex Cohen, Cynthia Vinzant)

This talk presents our recent results along with a line of works, initiated by Kurasov and Sarnak [2] and followed by Olevskii and Ulanovskii [3], that relates three, a-priori different, mathematical objects:

(I) Fourier quasicrystals (crystalline measures) with $\mathbb{N}$-valued coefficients.
(II) Real-rooted exponential polynomials, and
(III) Lee-Yang (stable) polynomials.

Following Ruelle [1], a multivariate polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ is called *Lee-Yang polynomial* if it has no zeros in $\mathbb{D}^n \cup (\mathbb{C} \setminus \overline{\mathbb{D}})^n$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc. One fundamental example is a determinant,

$$p(z_1, z_2, \ldots, z_n) = \det(\mathrm{diag}(z_1, \ldots, z_n) + U),$$

where U is any fixed $n \times n$ unitary matrix. The name Lee-Yang polynomials refers to the elegant proof of the Lee-Yang Circle Theorem by Brändén and Borcea. Lee-Yang polynomials relate to real-rooted exponential polynomials as follows.

If $p(z_1,\dots,z_n)=\sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}}\mathbf{z}^{\boldsymbol{\alpha}}$ is Lee-Yang and $\ell=(\ell_1,\dots,\ell_n)\in\mathbb{R}_+^n$, then the univariate exponential polynomial

$$f(x)=p(\exp(ix\ell))=p(e^{ix\ell_1},\dots,e^{ix\ell_n})=\sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}}e^{ix\langle\ell,\boldsymbol{\alpha}\rangle}$$

is real-rooted, since $(e^{ix\ell_1},\dots,e^{ix\ell_n})\in\mathbb{D}^n\cup(\mathbb{C}\setminus\overline{\mathbb{D}})^n$ whenever $\mathrm{Im}(x)\neq 0$.

A recent work of Kurasov and Sarnak [2] related the above observation to a special class of one-dimensional *crystalline measures* called *Fourier quasicrystals*. Let $\mathcal{S}(\mathbb{R})$ denote the space of *Schwartz functions*: smooth functions on $\mathbb{R}$ with $\lim_{|x|\to\infty}|x^n\frac{d^m}{d^m x}f(x)|=0$ for all $n,m\in\mathbb{Z}_{\geq 0}$. If $f\in\mathcal{S}(\mathbb{R})$ then so does its Fourier transform $\hat{f}\in\mathcal{S}(\mathbb{R})$. If there exist discrete (locally finite) sets $\Lambda,S\subset\mathbb{R}$ and complex coefficients $(a_x)_{x\in\Lambda},(c_k)_{k\in S}$, such that for any $f\in\mathcal{S}(\mathbb{R})$,

$$\sum_{x\in\Lambda}a_xf(x)=\sum_{k\in S}c_k\hat{f}(k),$$

in the sense that the right and left sums converge to the same finite limit, then $\mu=\sum_{x\in\Lambda}a_x\delta_x$ is called a *crystalline measure*. It is called a *Fourier Quasicrystal* (FQ) if in addition to the above, the sums on both sides converge absolutely.

Theorem 1. [2] *Let $p(z_1,\dots,z_n)$ be a Lee-Yang polynomial, let $\boldsymbol{\ell}\in\mathbb{R}_+^n$, let Λ be the zero set of $f(x)=p(\exp(ix\boldsymbol{\ell}))$, and for $x\in\Lambda$ let $a_x\in\mathbb{N}$ be the multiplicity of x as a zero of f. Then $\mu_{p,\boldsymbol{\ell}}:=\sum_{x\in\Lambda}a_x\delta_x$ is an FQ.*

Consequently, Olevskii and Ulanovskii [3] showed that for any FQ with $a_x\in\mathbb{N}$ for all $x\in\Lambda$, there is some real-rooted exponential polynomial f whose zero set and multiplicities are Λ and $(a_x)_{x\in\Lambda}$. This motivated us to ask whether there is also an associated pair of Lee-Yang p and $\boldsymbol{\ell}\in\mathbb{R}_+^n$.

Theorem 2. [4] *Let $f(x)=\sum_{j=0}^{s}c_je^{\lambda_jx}$ with $c_j,\lambda_j\in\mathbb{C}$. If f is real-rooted, then there is a Lee-Yang polynomial $p\in\mathbb{C}[z_1,\dots,z_n]$ and $\boldsymbol{\ell}\in\mathbb{R}_+^n$ such that*

$$f(x)=e^{\lambda_0x}p(\exp(ix\boldsymbol{\ell})),$$

and the entries of $\boldsymbol{\ell}$ are $\mathbb{Q}$-linearly independent.

As a result, the Kurasov-Sarnak construction comprises every possible FQ with $\mathbb{N}$-valued coefficients.

Theorem 3. [4, 3] *If $\mu=\sum_{x\in\Lambda}a_x\delta_x$ is an FQ with $a_x\in\mathbb{N}$ for all $x\in\Lambda$, then $\mu=\mu_{p,\boldsymbol{\ell}}$ as in the Kurasov-Sarnak construction, for some Lee-Yang $p(z_1,\dots,z_n)$ and $\boldsymbol{\ell}\in\mathbb{R}_+^n$ with $\mathbb{Q}$-linearly independent entries.*

After establishing this equivalence between FQ's with $\mathbb{N}$-valued coefficients and $(p,\boldsymbol{\ell})$ pairs, and after providing some motivation and background for non-periodic FQ's, I will present new (unpublished) results on the relation between properties of $\mu_{p,\boldsymbol{\ell}}$ and properties of p. In particular, showing that $\mu_{p,\boldsymbol{\ell}}$ is periodic if and only if p is binomial (or a power of such), and that the irreducible decomposition of p provides a decomposition of $\mu_{p,\boldsymbol{\ell}}$ into sum of periodic and non-periodic measures, where the support of the non periodic parts have uniform bound on their intersection with any periodic set. Furthermore, if p has multi-degree $\mathbf{d}=(d_1,\dots,d_n)$, so that $p(z_1,\dots,z_n)$ has degree d_j in z_j, and if $(x_j)_{j\in\mathbb{Z}}$ are the zeros of $p(\exp(ix\boldsymbol{\ell}))$, counted increasingly with multiplicity, then $x_j = \frac{2\pi}{\langle \mathbf{d},\boldsymbol{\ell}\rangle} j + O(1)$, where the $O(1)$ error term is uniformly bounded, and the gaps $x_{j+1} - x_j$ have a well defined gap distribution $\rho_{p,\boldsymbol{\ell}}$. That is, for any h continuous on $\mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(x_{j+1} - x_j) = \int h d\rho_{p,\boldsymbol{\ell}},$$

where $\rho_{p,\boldsymbol{\ell}}$ is given implicitly in terms of the torus zero set of p.

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Hereditary Lorentzian polynomials

Petter Brändén

(joint work with Jonathan Leake)

We introduce the class of hereditary polynomials. These polynomials share fundamental properties with volume polynomials of Chow rings of simplicial fans.

We characterize hereditary polynomials that are hereditary Lorentzian. This gives a characterization of Chow rings that satisfy Hodge-Riemann relations of degree 0 and 1.

Local theory of stable polynomials and bounded rational functions

Greg Knese

(joint work with K. Bickel, J.E. Pascoe, A. Sola)

The admissible numerator problem amounts to describing the bounded rational functions on a domain in $\mathbb{C}^d$. More precisely, given a domain $D \subset \mathbb{C}^d$ and a polynomial $p \in \mathbb{C}[x_1,\dots,x_d]$ with $p(x) \neq 0$ for $x \in D$, is it possible to give a simple description of the set

$$\{Q \in \mathbb{C}[x_1,\dots,x_d] : \exists C > 0 \text{ such that } |Q/p| \le C \text{ on } D\}?$$

In one dimension this is simply a matter of cancelling boundary poles, so the question is only interesting in several variables where it is likely very challenging in complete generality. In this talk we focus on a particular domain, namely the unit bidisk

$$\mathbb{D}^2 = \{(x_1,x_2) \in \mathbb{C}^2 : |x_1|, |x_2| < 1\}.$$

The admissible numerator question can be approached by focusing on single boundary zero of p and converting to the bi-upper half plane

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{C}^2 : \Im x, \Im y > 0\}.$$

In this setting we then focus on a polynomial p with no zeros in $\mathbb{H}^2$ and $p(0, 0) = 0$. We present a complete local description of the zero sets of such stable polynomials in terms of constrained Puiseux series expansions. Specifically, the branches of $p(x, y)$ break up into two types:

- The *real stable* type of the form
$$0 = y + \phi(x)$$
where ϕ is analytic at 0, $\phi(0) = 0$, $\phi'(0) > 0$, and ϕ has all real power series coefficients.
- The *pure stable* type of the form
$$0 = y + q(x) + x^{2L}\psi(x^{1/k})$$
where $q \in \mathbb{R}[x]$ has degree less than $2L$, $q(0) = 0$, $q'(0) > 0$, and ψ is analytic at 0 with $\Im\psi(0) > 0$.

It turns out that the pure stable case is the only case of interest in the admissible numerator problem. The following polynomial

$$[p](x, y) = \prod(y + q(x) + ix^{2L})$$

where the product is taken over all of the branches of p has the property that

$$|p/[p]|$$

is bounded above and below for $(x, y) \in \mathbb{H}^2$ close to $(0, 0)$.

In particular, in the local admissible numerator problem we can always replace p with the simpler $[p]$.

Our solution to the admissible numerator problem is then described as follows. Let $\mathcal{I}_p$ be the set of polynomials Q such that Q/p is bounded in $\mathbb{H}^2$ for (x, y) close to $(0, 0)$ and let $\mathcal{I}$ be the product ideal

$$\mathcal{I} = \prod(y + q(x), x^{2L})$$

where again the product is taken over all local branches of p.

Our main theorem is

$$\mathcal{I}_p = \mathcal{I}.$$

One half of this was proven in our paper and the other half was proven by J. Kollár.

Metric graphs, stable polynomials and Fourier quasicrystals

Pavel Kurasov

(joint work with Peter Sarnak)

Spectra of Laplacians on metric graphs, also known as *quantum graphs*, have been intensively studies in recent year due to possible applications to nano-physics. Let us restrict our studies to finite metric graphs formed from compact intervals $e_n,\ n = 1, 2, \dots, N$, of lengths ℓ_n and Laplace operators $-\frac{d^2}{dx^2}$ with standard vertex conditions:

- the function is continuous at the vertex (continuity condition);
- the sum of outgoing first derivates at the vertex is equal to zero (Kirchhoff condition).

One of the most interesting results is the trace formula connecting the spectrum $\lambda_j = k_j^2$ to geometric and topologic properties of the metric graph [9, 2, 4, 3]:

$$\underbrace{\sum_{k_n \neq 0} \left(\delta(k-k_n) + \delta(k+k_n)\right)}_{\text{spectral information}} = \underbrace{\underbrace{1-\beta_1}_{=\chi} \delta(k) + \frac{\mathcal{L}}{\pi} + \frac{1}{\pi} \sum_{p \in \mathcal{P}} l(\text{prim}\,(p)) S_{\text{v}}(p) \cos k\ell(p)}_{\text{geometric/topologic information}}$$

where

- $\mathcal{L} = \sum_{n=1}^{N} \ell_n$ - the total length of the graph;
- χ – Euler characteristic of Γ;
- β_1 – number of independent cycles in Γ;
- $\mathcal{P}$ – the set of closed oriented paths p on Γ;
- $\ell(p)$ – length of the closed path p;
- $S_{\text{v}}(p)$ – product of all vertex scattering coefficients along the path p.

This formula is a direct generalisation of the classical Poisson summation formula and coincides with it if the graph is just one interval with Neumann conditions at the end points. In contrast to similar formulas for Riemannian manifolds the obtained trace formula is exact.

It appears that this formula is extremely interesting for Fourier analysis since it provides explicit examples of crystalline measures, which can be defined following Y. Meyer as [7]:

A discrete measure μ is **crystalline** *if it is a tempered distribution and if the measure itself and its Fourier transform $\hat{\mu}$ are sums of delta functions with discrete supports:*

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \qquad \hat{\mu} = \sum_{s \in S} b_s \delta_s.$$

Collecting all delta functions on the left hand side the trace formula can be written as:

$$\sum_{k_n \neq 0} \left(\delta(k-k_n) + \delta(k+k_n)\right) - \chi\delta(k) = \frac{\mathcal{L}}{\pi} + \frac{1}{\pi} \sum_{p \in \mathcal{P}} l(\text{prim}\,(p)) S_{\text{v}}(p) \cos k\ell(p).$$

One may obtain similar summation formulas starting from multivariate stable polynomials [5]. The supports of the corresponding measures are described as zeroes of trigonometric polynomials. If the multivariate polynomials in addition are symmetric, *i.e.* invariant under involution $z_j \mapsto 1/z_j$, then the trigonometric polynomials have only real zeroes and the corresponding measures are crystalline measures. It appears that all one-dimensional crystalline measures are given by real-rooted trigonometric polynomials [8]. It was proven recently that all such polynomials can be obtained using our construction via multivariate stable polynomials [1]. Spectral properties of Laplacians on metric graphs are further described in [6, 3].

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Entropy constraints for ground state optimization

Hamza Fawzi
(joint work with Omar Fawzi, Samuel O. Scalet)

A fundamental computational problem in quantum many-body theory is to compute the ground energy of local Hamiltonians. Consider a multipartite Hilbert space $\mathcal{H} = \otimes_{v\in V}\mathbb{C}^d$ with local dimension d, on a finite set of sites V. We consider here 2-local Hamiltonians, where the interaction can be modeled by a graph $G = (V, E)$ on the set of sites V, and where a Hamiltonian term h_{ij} is attached to each edge $ij \in E$

$$H = \sum_{ij\in E} h_{ij}, \tag{1}$$

where each h_{ij} acts nontrivially only on sites i and j.

The *ground energy* of H is defined as its smallest eigenvalue. Due to the special structure of H, its matrix representation is generally sparse and thus one can apply standard methods such as Lanczos iterations to compute its minimal eigenvalue.

However, since the dimension of H grows exponentially with $|V|$, this is only feasible for moderate values of $|V|$. It is of considerable theoretical and practical interest to find efficient algorithms, that scale polynomially in $|V|$, to approximate the ground energy of local Hamiltonians.

The smallest eigenvalue of H admits the following variational formulation:

$$\lambda_{\min}(H) = \min_{\psi\in\mathcal{H}} \frac{\langle\psi, H\psi\rangle}{\langle\psi,\psi\rangle}. \tag{2}$$

Variational methods posit a certain form for the state $\psi = \psi_\theta$, and find the value of the parameters θ that minimize the objective function of (2). As such, these methods provide upper bounds on $\lambda_{\min}(H)$. A prominent example are tensor network states, which have been extremely successful and in particular give provably efficient algorithms for gapped systems in one dimension.

Another class of methods that have been studied in the literature are based on convex relaxations and provide lower bounds on $\lambda_{\min}(H)$. For 2-local Hamiltonians H of the form (1), computing the energy $\langle\psi, H\psi\rangle$ only requires knowledge of the two-body marginals ρ_{ij} of $|\psi\rangle\langle\psi|$ for $ij \in E$. If we denote by $\mathcal{C}$ the set of two-body marginals that are consistent with a global state on V, i.e.,

$$\mathcal{C} = \mathcal{C}_{d,V,E} = \left\{(\rho_{ij})_{ij\in E} : \exists\rho \in \mathsf{D}(\otimes_{v\in V}\mathbb{C}^d),\ \text{s.t. } \rho_{ij} = \operatorname{tr}_{V\setminus\{i,j\}}\rho\right\} \tag{3}$$

where $\mathsf{D}(\mathcal{H})$ denotes the set of density operators on a Hilbert space $\mathcal{H}$, then one can write the ground energy problem for a 2-local Hamiltonian (1) as a linear optimization problem over $\mathcal{C}$:

$$\lambda_{\min}(H) = \min_{(\rho_{ij})\in\mathcal{C}} \sum_{ij\in E} \operatorname{tr}[h_{ij}\rho_{ij}]. \tag{4}$$

To make this approach tractable, it is required to have a computationally efficient representation of the convex set $\mathcal{C}$. Unfortunately, it is highly likely that $\mathcal{C}$ does not have any simple representation, e.g., it is known that the problem of checking membership in $\mathcal{C}_{2,[n],\binom{[n]}{2}}$ is QMA-hard.

Rather than aiming to describe $\mathcal{C}$, we are interested in constructing efficient outer relaxations of $\mathcal{C}$, i.e., tractable convex sets $\widehat{\mathcal{C}}$ such that $\mathcal{C} \subset \widehat{\mathcal{C}}$. Replacing $\mathcal{C}$ by $\widehat{\mathcal{C}}$ in (4) would then yield a lower bound on $\lambda_{\min}(H)$. Such relaxations $\widehat{\mathcal{C}}$ can be constructed by identifying *necessary conditions* that any set of marginals (ρ_{ij}) which are globally consistent must satisfy. Most relaxations that have been constructed in the literature are based on semidefinite programming. We describe here the most popular approaches:

• A simple relaxation can be obtained by simply imposing that the two-body marginals are consistent on the intersection of their supports, i.e., one can take

$$\widehat{\mathcal{C}}_E^{\mathrm{loc}} = \left\{(\rho_{ij})_{ij\in E} : \rho_{ij} \succeq 0, \operatorname{tr}\rho_{ij} = 1\ \ \forall ij \in E,\ \ \operatorname{tr}_j\rho_{ij} = \operatorname{tr}_{j'}\rho_{ij'}\ \ \forall ij, ij' \in E\right\}. \tag{5}$$

This relaxation can be made tighter by introducing higher-order marginals of ρ. This gives a hierarchy of the form $\mathcal{C} = \widehat{\mathcal{C}}_N^{\mathrm{loc}} \subset \widehat{\mathcal{C}}_{N-1}^{\mathrm{loc}} \subset \cdots \subset \widehat{\mathcal{C}}_2^{\mathrm{loc}} \subset \widehat{\mathcal{C}}_E^{\mathrm{loc}}$, where $N = |V|$.

• The Lasserre/sum-of-squares relaxation stems from the observation that if $|\psi\rangle$ is a global state on $\mathcal{H}$, then $\langle\psi, O^\dagger O\psi\rangle \geq 0$ for any observable O acting on $\mathcal{H}$. In particular if O is a l-local operator, then $O^\dagger O$ is at most $2l$-local, and $\langle\psi, O^\dagger O\psi\rangle$ is linear in the expectation values $m_F = \langle\psi, F\psi\rangle$ of $2l$-local observables F. It turns out that the infinite family of constraints

$$\langle\psi, O^\dagger O\psi\rangle \geq 0 \;\; \forall O \; l\text{-local observable on } \mathcal{H} \tag{6}$$

can be encoded as a single positive semidefinite (psd) constraint of a matrix whose entries are linear in the expectation values (m_F).

In this contribution, we are interested in relaxations for the convex set $\mathcal{C}$ that go beyond semidefinite programming. In particular, we are interested in relaxations that use *entropies* of the local marginals. Recall that the von Neumann entropy of the system A for the state ρ is defined by

$$S(A)_\rho = -\mathrm{tr}\,\rho_A \log \rho_A,$$

where ρ_A denoted the reduced state of ρ on the system A. In addition, the conditional entropy is defined by

$$S(A|B)_\rho = S(AB)_\rho - S(B)_\rho.$$

An important property about the latter is that it is concave in ρ_{AB}; this follows from the identity

$$S(A|B)_\rho = -D(\rho_{AB}\|\mathrm{id}_A \otimes \rho_B)$$

where $D(\rho\|\sigma) = \mathrm{tr}\,[\rho(\log\rho - \log\sigma)]$ is the relative entropy function, which is jointly convex in (ρ, σ).

Our main contribution is to study two families of entropy constraints that yield new strengthened relaxations for the ground energy problem. These relaxations are obtained by imposing inequalities on the conditional entropies of local marginals of the global state, and can be combined with any of the existing semidefinite relaxations. By virtue of the concavity of the conditional entropy function, these relaxations are all convex and can be solved efficiently using tools from convex optimization [FS22].

The first family of entropy constraints come from weak monotonicity, and the second family of constraints, that we call Markov Entropy Decomposition (MED) constraints, are motivated by the work of Poulin and Hastings [PH11] and are based on combining the chain rule together with strong subadditivity. Though weak monotonocity constraints are in many cases stronger than MED constraints, we show that in general the two families are not comparable. Our main message is that for many natural Hamiltonians, imposing entropy constraints can lead to significantly tighter bounds compared to simple consistency conditions captured by $\widehat{\mathcal{C}}_l^{\mathrm{loc}}$. We also show limitations on the gains that can be obtained using weak monotonicity constraints (and also MED in many settings): entropy constraints involving l sites are implied by consistency constraints on $2l-1$ sites. As a result, as the size of the matrix variables involved in $\widehat{\mathcal{C}}_l^{\mathrm{loc}}$ is exponential in l, entropy constraints can at most lead to a quadratic improvement in terms of the size of the matrix variables.

An intriguing open question arising from our work is to construct other entropy constraints that could lead to tighter relaxations. This question is related to obtaining inequalities for the so-called quantum entropy cone (see e.g., [LW05]) though it differs in several respects: in our case, the dimension of the subsystems is fixed, the number of systems involved is bounded by l, and in order to obtain convex relaxations, we look for expressions that are concave in the state ρ, e.g., conical linear combinations of conditional entropies.

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Symmetrically Hyperbolic Polynomials

Kevin Shu

(joint work with Greg Blekherman, Julia Lindberg)

A polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ is said to be hyperbolic in a direction $v \in \mathbb{R}^n$ if p is homogeneous, $p(v) \neq 0$, and for any $x \in \mathbb{R}^n$, the univariate polynomial $p(x + tv)$ has only real roots. These polynomials turn out to be surprisingly pervasive in real algebraic geometry, optimization, and probability.

These polynomials are natural generalizations of the notion stable polynomials in the homogeneous case. The key example of such polynomials is the determinant of a symmetric matrix. The fact that a symmetric matrix has real eigenvalues is equivalent to the fact that $\det(X)$ is hyperbolic with respect to $I \in \mathrm{Sym}_2(\mathbb{R}^n)$.

Definition 1. We say that a polynomial is *symmetrically hyperbolic* if it is invariant under permutations of variables and it is hyperbolic with respect to $\vec{1}$, the all 1's vector.

Definition 2. We say that a homogeneous symmetric polynomial of degree d is *hook-shaped* if it is of the form

$$p(x) = \sum_{i=1}^{d} a_i e_1^{d-i}(x) e_i(x),$$

where e_i is the elementary symmetric polynomial of degree i, and $a_i \in \mathbb{R}$ for $i = 1, \dots, d$.

Definition 3. A linear map $T : \mathbb{R}[t]_{n,0} \to \mathbb{R}[t]_{d,0}$ is a 0-sum hyperbolicity preserver if $T(\mathcal{H}_{n,0}) \subseteq \mathcal{H}_{d,0}$. T is diagonal if $T(t^{n-k}) = \gamma_k t^{d-k}$ for all $k = 0 \dots d$, and $T(t^{n-k}) = 0$ for $k > d$

Definition 4. If p is a hook-shaped polynomial, then the *associated operator* to p is the function $T : \mathbb{R}[t]_{n,0} \to \mathbb{R}[t]_{d,0}$ defined as follows. If $g(t) \in \mathbb{R}[t]_{n,0}$ is a polynomial so that

$$g(t) = a(t - r_1)(t - r_2)\dots(t - r_n),$$

for $r_1, \dots, r_n \in \mathbb{C}$, then we let

$$T(g)(t) = ap(\vec{r} - \vec{1}t),$$

where $\vec{r}$ denotes the vector whose entries are the roots r_i. We then extend this definition to all $g \in \mathbb{R}[t]_{n,0}$ by continuity (note that this is possible because the coefficients of $T(g)$ are polynomials in the coefficients of g).

We might ask how the notion of a 0-sum hyperbolicity preserver relates the usual notion of hyperbolicity preserver considered in [1, 2]. Clearly, if $\hat{T} : \mathbb{R}[t]_n \to \mathbb{R}[t]_d$ is a diagonal linear map that preserves hyperbolicity for all polynomials in $\mathbb{R}[t]_n$, then the restriction of $\hat{T}$ to $\mathbb{R}[t]_{n,0}$ is a 0-sum hyperbolicity preserver. We might ask if all such 0-sum hyperbolicity preservers occur in this way.

Definition 5. Let $T : \mathbb{R}[t]_{n,0} \to \mathbb{R}[t]_{d,0}$ be a 0-sum hyperbolicity preserver. We say T is *extendable* if there exists a diagonal linear map $\hat{T} : \mathbb{R}[t]_n \to \mathbb{R}[t]_d$ so that $\hat{T}$ is a hyperbolicity preserver, and T is the restriction of $\hat{T}$ to $\mathbb{R}[t]_{n,0}$.

Theorem 6. *Let $T : \mathbb{R}[t]_{n,0} \to \mathbb{R}[t]_{d,0}$ be a 0-sum hyperbolicity preserver. If $d \leq 4$, then T is extendable, and moreover, this is the case if and only if*

$$T((x - 1)^{n-1}(x + n - 1))$$

has real roots, where $d - 1$ of them have the same sign.

On the other hand, if $d \geq 5$, then there exists a 0-sum hyperbolicity preserver which is not extendable.

Using this theorem, we can obtain characterizations of hook-shaped symmetrically hyperbolic polynomials of low degree. We can say the most for cubics:

Theorem 7. *Let p be a cubic symmetric polynomial. Then the following are equivalent:*

(1) *p is symmetrically hyperbolic.*
(2) *The operator associated to p is a 0-sum hyperbolicity preserver.*
(3) *$p(b + \vec{1}t)$ is real rooted, where b is any coordinate vector.*
(4) *$\Delta_{\vec{1}\vec{1}}p$ is sum of squares (SOS).*

We can also characterize quartic hook-shaped polynomials:

Theorem 8. *Let p be a quartic hook-shaped symmetric polynomial. Then the following are equivalent:*

(1) *p is symmetrically hyperbolic.*
(2) *The operator associated to p is a 0-sum hyperbolicity preserver.*
(3) *Let $q(t) = p(b + \vec{1}t)$, where b is a coordinate vector. Then $q(t - \frac{1}{n})$ is real rooted, with at least 3 roots having the same sign.*

Finally, we will say more about a quintic example.

Theorem 9. *Let*

$$p = 4500e_5 - 220e_1e_4 + 7e_1^2e_3,$$

which is in 5 *variables. Then,*

- *p is symmetrically hyperbolic.*
- *p's associated operator is not extendable*
- *$\Delta_{\vec{1}\vec{1}}p$ is SOS*
- *p is not SOS hyperbolic.*

Here are some comments on the proofs of this theorem. We are able to show that p is symmetrically hyperbolic by using a computer to show that $\Delta_{\vec{1}\vec{1}}p$ is SOS. Similarly, we show that for some $u \in \mathbb{R}^n$, $\Delta_{\vec{1}u}p$ is not SOS using a computer. Let T be the associated operator to p. We show that T is not extendable by noting that

$$T((x-1)^4(x+4)) = 750(x-1)^2(x-2)^2(x+6),$$

and that in general, if

$$T((x-1)^4(x+4)) = (x-a)^2(x-b)^2(x+2a+2b),$$

for $a, b \in \mathbb{R}_+$, then T is not extendable.

Open Problems

Our results leave open several interesting questions. Firstly, we might wonder whether or not the natural extension of the Branden-Borcea characterization to 0-sum hyperbolicity preservers might hold:

Conjecture 1. *Let $T : \mathbb{R}[t]_{n,0} \to \mathbb{R}[t]_{d,0}$ be diagonal. Then T is a 0-sum hyperbolicity preserver if and only if*

$$T((x-1)^{n-1}(x+n-1))$$

has real roots with $d-1$ having the same sign.

This conjecture would also immediately characterize hook-shaped symmetric polynomials.

We have shown that this holds for $d \leq 4$, and also that the sign condition on the roots of $T((x-1)^{n-1}(x+n-1))$ is necessary for all d. Furthermore, we have extensive computational evidence that this conjecture holds when $d \leq 6$, using the following procedure: we chose a real rooted polynomial $q \in \mathbb{R}[t]_{d,0}$ with $d-1$ roots of the same sign. For this polynomial q, there is a unique diagonal map T satisfying $T((x-1)^{n-1}(x+n-1)) = q$, and a unique hook-shaped symmetric polynomial p whose associated operator is T. We then verified that $\Delta_{\vec{1},\vec{1}}p$ is a sum of squares. This leads us to an additional conjecture, which in a sense generalizes the Newton inequalities:

Conjecture 2. *If p is a hook-shaped symmetric polynomial, then p is symmetrically hyperbolic if and only if $\Delta_{\vec{1},\vec{1}}p$ is SOS.*

One reason this may be difficult is that there are examples of hook-shaped symmetrically hyperbolic polynomials p where p is not SOS hyperbolic.

A more speculative conjecture is as follows:

Conjecture 3. *If p is hook-shaped and symmetrically hyperbolic, then the associated operator of p is extendable if and only if p is weakly SOS-hyperbolic.*

We have a large amount of computational evidence that for cubics and quartics that this holds in the sense that we cannot find any examples of such polynomials which are not SOS-hyperbolic. Our evidence in the case of quintic polynomials is more limited, though we have not disproven this conjecture yet.

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lg-concavity and sector-stability of polynomials

Nima Anari

In this talk, I explain two properties of polynomials with nonnegative coefficients that are connected to fast mixing of random walks. A fractionally lg-concave polynomial is one that becomes lg-concave as a function over $\mathbb{R}^n_{>0}$ after variables are raised to a constant fractional power. I will show that the generating polynomial of a distribution being fractionally lg-concave implies the existence of fast sampling distributions for the distribution. Then I explain examples of combinatorial distributions coming from matroid theory and the theory of delta-matroids that satisfy fractional lg-concavity. The proof uses sector-stability of distributions associated to these delta-matroids.

Two View Chiral Reconstructions and Real Cubic Surfaces

Rainer Sinn

(joint work with Andrew Pryhuber, Rekha R. Thomas)

We discuss projective reconstructions in computer vision for pinhole cameras from the point of view of real algebraic geometry and *chirality*. A *pinhole camera* is modelled as a 3×4 matrix A with real entries and maximal rank 3. This defines a rational map $A\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ that is not defined in the 1-dimensional nullspace of A. The codomain of this map is a picture taken by the pinhole camera A. A *scene* is a collection of cameras and world points, i.e., points in $\mathbb{R}P^3$. The goal of image reconstruction is to find such a scene given the pictures that the cameras have taken. We will only look at the case of 2 cameras here and assume that the input is a set of *point pairs*, which means a set $\mathcal{P} = \{(x_i, y_i) \in \mathbb{R}P^2 \times \mathbb{R}P^2 \colon i = 1, 2, \ldots, k\}$ of matched points in both images. The goal is to find cameras A_1 and A_2 and points $X_i \in \mathbb{R}P^3$ such that $A_1 X_i = x_i$ and $A_2 X_i = y_i$. A projective transformation of $\mathbb{R}P^3$ of the scene (so cameras and world points) preserves these linear equations so that projective reconstructions come in orbits under the projective linear group.

The goal of this talk is to discuss the underlying projective algebraic geometry which goes back to the 19th century. The connection is particularly intriguing in the case $k = 5$ of five point pairs and two cameras. We will see real cubic surfaces with 27 real lines as well as a Schläfli double six consisting of twelve real lines.

The connection is based on an important tool in the computer vision literature called a *fundamental matrix*. A fundamental matrix F is a 3×3 real matrix of rank 2. It encodes the relative location of two cameras up to projective transformation: if $A_1 = [I_3|0]$ and $A_2 = [G|t]$, where I_3 is the 3×3 identity matrix and G is an invertible 3×3 matrix, then the associated fundamental matrix is $[t]_\times G$, where $[t]_\times$ is the matrix representing the linear map $v \mapsto t \times v$ given by the cross product $\times$ on $\mathbb{R}^3$. The main result here says that there exists a projective reconstruction of the point correspondence $\mathcal{P}$ as defined above if there is a fundamental matrix F such that $y_i^\top F x_i = 0$ for $i = 1, 2, \ldots, k$ (and such that F is $\mathcal{P}$-regular). These k linear equations in F are known as the *epipolar equations*.

However, we are interested in reconstructions where the points X_i lie in front of the cameras A_1 and A_2. This can be modelled by inequality constraints in $\mathbb{R}P^3$ and translated to the setup of fundamental matrices. The *chiral epipolar inequalities* are $g_i(F)g_j(F) \geq 0$ for $1 \leq i < j \leq k$, where $g_i(F) = y_i^\top [t]_\times F x_i$ for a generator t of the left kernel of F. Since such a vector t can be written in terms of quadratic functions in the entries of F by Cramer's rule, the polynomial $g_i(F)$ is cubic in the entries of F. So the product $g_i g_j$ has even degree in those variables. In particular, its sign is invariant under scaling of F by real numbers.

To connect this general setup with cubic surfaces in $\mathbb{P}^3$ and the geometry of the 27 lines on it, we assume from now on that $k = 5$ and that the point pairs $\mathcal{P}$ are generic. Then the epipolar equations define a 3-dimensional subspace of $\mathbb{P}^8 = \mathbb{P}(\mathrm{Mat}_{3\times 3})$. Since a fundamental matrix is supposed to have rank 2, we are interested in the intersection of the determinantal hypersurface in $\mathbb{P}^8$ with this $\mathbb{P}^3$ defined by the epipolar equations. This gives our cubic surface $S \subset \mathbb{P}^3$. By genericity, this surface is smooth and therefore contains exactly 27 real lines. In fact, since the rank 1 matrices $x_i y_i^\top$ defining the surface (via $0 = y_i^\top F x_i = \mathrm{trace}\left(F(x_i y_i^\top)\right)$) are real, it follows from a dimension count in linear algebra, that S contains 10 real lines, namely $W_{x_i} = \{M \in S\colon M x_i = 0\}$ and symmetrically $W^{y_i} = \{M \in S\colon y_i^\top M = 0\}$ for $i = 1, 2, \ldots, 5$. Moreover, the 5 lines W_{x_i} given in terms of right kernels are pairwise skew because every point on S is a matrix of rank 2. The same holds for the 5 lines W^{y_i} given in terms of left kernels. Two lines W_{x_i} and W^{y_j} intersect if and only if $i \neq j$. In fact, these lines are contained in a Schläfli double six, meaning that there are points x_6 and y_6 in $\mathbb{R}P^2$ such that the 6 lines W_{x_i} are pairwise skew, W^{y_j} are pairwise skew, and W_{x_i} intersects W^{y_j} if and only if $i \neq j$. By the classification of the real forms of cubic surfaces, this is enough to conclude that the cubic surfaces S arising in the context of chiral reconstruction always have 27 real lines. Moreover, S is isomorphic to the blow up of $\mathbb{P}^2$ in $x_1, x_2, \ldots, x_6$ as well as the blow up of $\mathbb{P}^2$ in $y_1, y_2, \ldots, y_6$.

To interpret the chiral epipolar inequalities from this perspective, it suffices to show that the hypersurface defined by g_i intersects S in the two lines W_{x_i} and W^{y_i} on S so that the inequality $g_i g_j$ can only change sign along the four lines $W_{x_i} \cup W^{y_i} \cup W_{x_j} \cup W^{y_j}$. Since these lines arise as the exceptional divisors or strict transforms of conics in $\mathbb{P}^2$ under the isomorphism with the blow up of $\mathbb{P}^2$ in $x_1, \ldots, x_6$, we can see the semi-algebraic subset of S defined by these inequalities in the corresponding image plane of the camera A_1 that we are looking for.

More details, further references, and illustrating pictures can be found in our paper [1].

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Optimization of trigonometric polynomials with symmetry

Evelyne Hubert

(joint work with Tobias Metzlaff, Philippe Moustrou, Cordian Riener)

Given a n–dimensional lattice $\Omega \subseteq \mathbb{R}^n$, a trigonometric polynomial is a function
$$f : \mathbb{R}^n \to \mathbb{R},\, u \mapsto f(u) := \sum_{\omega \in \Omega} c_\omega \, e^{2\pi \mathrm{i} \langle \omega, u \rangle},$$
where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product and the finitely many nonzero coefficients $c_\omega \in \mathbb{C}$ satisfy $c_{-\omega} = \overline{c_\omega}$. Such functions are good L^2–aproximations for Λ–periodic functions, where Λ is the dual lattice. This paper offers a new approach to optimizing such a trigometric function, over $\mathbb{R}^n$, when this latter is invariant under a crystalographic reflection group. We show how the problem can then be reduced to polynomial optimization on a semi–algebraic set and handled with a variation on Lasserre hierarchy. The resulting algorithm is applied to the exploration of the spectral bound on the chromatic numbers of set avoiding graphs.

In the literature of trigonometric optimization, one often regards the lattice simply as a free $\mathbb{Z}$–module, that is, $\Omega = \mathbb{Z}^n$, ignoring the geometry and only taking central symmetry into account. For the purpose of optimization, a hierarchy of Hermitian sums of squares reinforcements provides a numerical solution [13, 3]. Alternatively, one can apply Lasserre's hierarchy with complex variables [25], where one has to restrict to the compact torus.

In this article, Ω is the weight lattice of a crystallographic root system in $\mathbb{R}^n$. Root and weight lattices provide optimal configurations for a variety of problems in geometry and information theory, with incidence in physics and chemistry. The $\mathcal{A}_2$ root lattice (the hexagonal lattice) is classically known to be optimal for sampling, packing, covering, and quantization in the plane [11, 26], but also proved, or conjectured, to be optimal for energy minimization problems [34, 9]. From an approximation point of view, weight lattices of root systems describe Gaussian cubature [30, 31], a rare occurence on multidimensional domains.

The distinguishing feature of the lattices associated to crystallographic root system is their intrisic symmetry. This latter is given by the so called Weyl group $\mathcal{W}$, a finite group generated by orthogonal reflections w.r.t. $\langle \cdot, \cdot \rangle$. It is this feature that we emphasize and offer to exploit in an optimization context. We present a new approach to numerically solve the trigonometric optimization problem

$$f^* := \min_{u\in\mathbb{R}^n} f(u)$$

under the assumption of crystallographic symmetry, that is, for $A \in \mathcal{W}$, we have $f(A\,u) = f(u)$, or equivalently $c_{A\,\omega} = c_\omega$. The first step of our approach is a symmetry reduction that translates the trigonometric optimization above to the problem of optimizing a polynomial over a semi–algebraic set, a subject that ripened in the last two decades [27, 32, 33, 28, 29, 10, 18]. The second step of our approach is thus an adaptation of Lasserre's hierarchy of moment relaxations and sums of squares reinforcements. We indeed modify the hierarchy introduced in [22, 23] to work directly in the basis of generalized Chebyshev polynomials. These are not homogeneous but naturally filtered by a weighted degree, different from the usual degree.

The simplest case of this symmetry reduction scheme, the univariate case, is obvious but maybe worth reviewing to get the initial idea. The group is then $\mathcal{W} = \{1, -1\}$ and the invariance condition is thus $f(-u) = f(u)$ for all $u \in \mathbb{R}$. That implies that one can write $f(u) = \sum_{k\in\mathbb{N}} \frac{c_k}{2}\,(\exp(2\pi \mathrm{i}\, ku) + \exp(-2\pi\mathrm{i}\, ku)) = \sum_{k\in\mathbb{N}} c_k\, \cos(2\pi\, ku) = \sum_{k\in\mathbb{N}} c_k\, T_k(\cos(2\pi\, u))$, where $\{T_k\}_{k\in\mathbb{N}}$ are the Chebyshev polynomials of the first kind. As a consequence we have $f^* := \min_{u\in\mathbb{R}^n} f(u) = \min_{z^2\leq 1} \sum_{k\in\mathbb{N}} c_k T_k(z)$, a polynomial optimization problem with semi–algebraic constraints.

With $\Omega = \mathbb{Z}^n$ and $\mathcal{W} = \{1, -1\}^n$, one can use products of univariate Chebyshev polynomials to operate a similar symmetry reduction. This is the $\mathcal{A}_1\times\ldots\times\mathcal{A}_1$ case. We look at all the lattices associated to crystallographic root systems, offering a wider range of domains of periodicity (hexagon, rhombic dodecahedron, icositetrachoron, ...) and simplices of any dimension, or cartesian products of these, as fundamental domains. The key to the symmetry reduction then is the existence and properties of generalized Chebyshev polynomials. They allow to rewrite any invariant trigonometric polynomials as polynomials of the fundamental generalized cosines. These generalized Chebyshev polynomials arose in different contexts, in particular in the search of multivariate orthogonal polynomials [14, 16, 21, 7]. A more recent development is the description of their domain of orthogonality, the image of the generalized cosines, as a compact semi–algebraic set given by a unified and explicit polynomial matrix inequality [24]. Such a description is necessary to proceed algorithmically with the obtained polynomial optimization problem.

In the algorithmic approach, we solve a primal/dual semi–definite program (SDP) that models a moment–relaxation/sums of squares reinforcement in terms of generalized Chebyshev polynomials. Our Maple package GeneralizedChebyshev[1] allows to compute the parameters of the SDP, specifically the matrices which impose the semi–definite constraints. Beyond that, the package offers a large variety of tools, including the matrices from [24], a function to rewrite invariants in terms of generalized Chebyshev polynomials and an implemented recurrence formula for their computation. We can thus compare our method with the one in [13] in practice. We observe in several examples that the quality of the approximation is improved, while the computational complexity is reduced.

As a second set of contributions, we apply our method to the computation of spectral bounds for chromatic numbers of set avoiding graphs. The first such graph considered was the Euclidean distance graph [35, 5, 6, 12], where the vertices are the points of $\mathbb{R}^n$ and the set to be avoided is the sphere. As set of vertices we consider either $\mathbb{R}^n$, or a lattice thereof. As for the set to be avoided we mostly consider the boundary of a polytope with crystallographic symmetry. Choosing appropriate discrete measures on the polytope, the spectral bound from [5] made specific to the chromatic number can be expressed as the solution of a max–min optimization problem on a trigonometric polynomial. Our symmetry reduction technique then allows us to retrieve, with simple proofs, the chromatic number of the $\mathcal{C}_n$ lattice, of the graph avoiding the crosspolytope of radius 2 in $\mathbb{Z}^n$, and of the graph avoiding the cube in $\mathbb{R}^n$. In other cases, we apply our optimization algorithm to compute lower bounds numerically. We improve on [17] by +2 for the chromatic number of $\mathbb{Z}^4$ avoiding the crosspolytope of radius 4. We also give further bounds for the rhombic dodecahedron as well as the icositetrachroron.

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Vandermonde Varieties in Type B

Alison Rosenblum
(joint work with Saugata Basu)

This talk concerned the topology of Vandermonde varieties in the setting of type B symmetry. Recent results of Basu and Riener leverage symmetry relative to the action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{R}^n$ in the study of the cohomology of semialgebraic sets. We hope to extend these principles to the next major class of symmetry, and begin by studying a key class of symmetric sets known as Vandermonde varieties in the type B setting.

Let S be a set defined by the polynomials $\{f_1, \ldots, f_s\}$ with each f_j contained in the set $\mathbb{R}[X_1, \ldots, X_n]_{\leq d}^{\mathfrak{S}_n}$ of symmetric polynomials of degree at most d. The action of $\mathfrak{S}_n$ on $\mathbb{R}^n$ (by permuting variables) induces an action on the cohomology spaces $H^i(S)$, where throughout we will assume coefficients in $\mathbb{Q}$. This action allows us to apply the isotypic decomposition from representation theory to each $H^i(S)$:

$$H^i(S) \cong_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} m_{i,\lambda}(S)\, \mathbb{S}^\lambda$$

where the decomposition is indexed over partitions λ of the number n. $\mathbb{S}^\lambda$ denotes the Specht module associated to λ, and $m_{i,\lambda}(S)$ is the multiplicity of S^λ in $H^i(S)$. The dimensions of the Specht modules are well known and may be computed using the "hook length formula," so in order to compute the ith Betti number, i.e., the dimension of $H^i(s)$, one need only compute each multiplicity. A priori, the number of partitons grows exponentially with n. However, Basu and Riener proved that partitions which are too long relative to d cannot appear with positive multiplicity in the decomposition.

Theorem 1 (Restriction Theorem for Partitions, Basu and Riener, [3])**.** *For* $d \geq 2$, $m_{i,\lambda}(S) = 0$ *when* $\operatorname{length}(\lambda) \geq 2d + i - 1$.

As a consequence, Basu and Riener in [3] develop an algorithm for computing the first l Betti numbers of such a set S, with complexity bounded by $(snd)^{2^{O(d+l)}}$.

One key tool in the restriction theorem above is a class of sets known as *Vandermonde varieties*. These were classically studied in a trio of papers by Arnold [1], Givental [4], and Kostov [5]. In the case of the symmetric group acting on $\mathbb{R}^n$ (refered to as type A), we will define Vandermonde varieties using the Newton power sums $p_{A,m}^{(n)} = X_1^m + \cdots + X_n^m$. For a positive integer $d \leq n$ and a point $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we define the type A Vandermonde variety

$$V_{A,d,\mathbf{y}} = \{\mathbf{x} \in \mathbb{R}^n \mid p_{A,1}^{(n)}(\mathbf{x}) = y_1, \ldots, p_{A,d}^{(n)}(\mathbf{x}) = y_d\}$$

We will be most interested in studying the portion of $V_{A,d,\mathbf{y}}$ that is contained in a fundamental region relative to the group's action. In the present case, we will take the *Weyl chamber*, $\mathcal{W}_A^{(n)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \leq \cdots \leq x_n\}$. We will denote the restriction of a Vandermonde variety to this set by $Z_{A,d,\mathbf{y}}^{(n)} = V_{A,d,\mathbf{y}}^{(n)} \cap \mathcal{W}_A^{(n)}$. Basu and Riener showed in [3] that each non-empty $Z^{(n)}_{A,d,\mathbf{y}}$ together with its intersections with the walls of the Weyl chamber forms a regular cell complex (an open subset X of $\mathbb{R}$ is said to be *regular* if X and its closure are homeomorphic as a pair to the open ball of some dimension and its closure). We have now done the same in the next major class of symmetry.

Type B refers to the action of the group $W_B(n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ of signed permutations. Elements act on $\mathbb{R}^n$ by both permuting coordinates and changing their sign. To define Vandermonde varieties in type B, we draw upon their charicterization as the intersection of level sets of the first d generators of the ring of polynomials invariant relative to the symmetry at hand. In the case of type B, a convenient choice is the Newton power sums of even degree, $p^{(n)}_{B,m} = X_1^{2m} + \cdots + X_n^{2m}$. For $1 \le d \le n$ and $\mathbf{y} \in \mathbb{R}^d$, the type B Vandermonde variety is then

$$V_{B,d,\mathbf{y}} = \{\mathbf{x} \in \mathbb{R}^n \mid p^{(n)}_{B,1}(\mathbf{x}) = y_1, \ldots, p^{(n)}_{B,d}(\mathbf{x}) = y_d\}$$

The Weyl chamber in type B is defined by $\mathcal{W}^{(n)}_B = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_1 \le \cdots \le x_n\}$. We have shown that regularity continues to hold for the type B restricted Vandermonde variety.

Theorem 2. *For every $1 \le d \le n$ and $\mathbf{y} \in \mathbb{R}^n$, the interior of $Z^{(n)}_{B,d,\mathbf{y}}$ is either empty, a single point, or a regular cell of dimension $n - d$. It follows that this set together with the intersection of $Z^{(n)}_{B,d,\mathbf{y}}$ with the walls of $\mathcal{W}^{(n)}_B$ is a regular cell complex.*

The proof of Basu and Riener in [3] involves a concept known as *monotonicity*, as develped by Basu, Gabrielov, and Vorobjov in [2]. Monotonicity in type A allows for an interesting shortcut in the type B regularity proof. The regularity of the Vandermonde varieties in type B is expected to lay the fondation for proving a type B analogue of Basu and Riener's restriction theorem. Their paper first establishes the restriction theorem (or something very similar) for Vandermonde varieties. We have put in place many of the ingredients for a type B restriction theorem for Vandermonde varieties.

To understand the (co)homology of a symmetric set by studying it only on a fundamental region, we must pay specific attention to the walls. The walls of the fundamental region correspond to Coxeter generators $\mathrm{Cox}_A(n)$ (resp. $\mathrm{Cox}_B\, n)$) of the given group. In the type A case, we may use the adjacent transpositions $s_j = (j\ j+1)$ as generators and focus on the geometric interpretation of s_j as the wall obtained by intersecting $\mathcal{W}^{(n)}$ with $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j = x_{j+1}\}$. In type B, we may use the generators in $\mathrm{Cox}_A(n)$ and add one generator s_0 which corresponds to changing the sign on the first coordinate. The associated wall would be the intersection of $\mathcal{W}^n_B$ with $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 = x_1\}$. We have used the type B regularity theorem above in conjunction with type B analogues of several of Arnold, Givental, and Kostov's theorems to establish the following statement about relative cohomologies. Let $\mathcal{W}^{(n)}_{B,s_j}$ denote the wall corresponding to s_j.

Theorem 3. *Let* $d \geq 2$ *and* $\mathbf{y} \in \mathbb{R}^d$. *If* $T \subset \mathrm{Cox}_B(n)$, *then*

$$H^i\left(Z^{(n)}_{B,d,\mathbf{y}}, Z^{(n)}_{B,d,\mathbf{y}} \cap \left(\bigcup_{s \in T} \mathcal{W}^{(n)}_{B,s}\right)\right) = 0$$

when $\mathrm{card}(T) \geq 2d + i$

This provides the vanishing of direct summands in a lesser-known decomposition of the homology spaces, given below. We assume that d is at least 2.

$$H_*(V^{(n)}_{B,d,\mathbf{y}}) \simeq_{W_B(n)} \bigoplus_{T \subset \mathrm{Cox}_B(n)} H_*\left(Z^{(n)}_{B,d,\mathbf{y}}, Z^{(n)}_{B,d,\mathbf{y}} \cap \left(\bigcup_{s \in T} \mathcal{W}^{(n)}_{B,s}\right)\right) \otimes \Psi^{(n)}_{B,T}$$

where $\Psi^{(n)}_{B,T}$ is what is known as the Solomon module in type B_n indexed by T. Basu and Riener used a very similar result in type A to prove their restriction theorem for Vandermonde varieties, and so we expect the above vanishing to yield a type B restriction theorem first for Vandermonde varieties and then for general symmetric semialgebraic sets.

We would eventually like to extend these results to all finite reflection groups. Sufficiently indecomposible reflection groups can be classified into a small number of types (technically, this would be the classification of Coxeter systems). We have addressed or are addressing types A and B. The above results hold equally well for type C, and so only type D, the dihedral groups, and a few scattered exceptional types remain. While we could establish our general result by checking case by case, it would be nice to also determine the principles underlying our arguments, in order to establish an overarching proof.

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