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Heat Kernels, Stochastic Processes and Functional Inequalities

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Abstract. The workshop provided a forum for recent progress on a wide array of topics at the nexus of Analysis (elliptic, subelliptic and parabolic differential equations), Geometry (Riemannian and sub-Riemannian geometries, metric measure spaces, geometric analysis and curvature), and Probability Theory (Brownian motion, Dirichlet spaces, stochastic calculus and random media). The workshop provides a unique opportunity to encourage and foster interactions between mathematicians who share some common interests but might use different research tools or work in different mathematical settings.

Mathematics Subject Classification (2020): 31C, 60J, 58J.

Introduction by the Organizers

Organized by Masha Gordina (University of Connecticut), Takashi Kumagai (Waseda University, Tokyo), Laurent Saloff-Coste (Cornell University), and Karl-Theodor Sturm (University of Bonn), the workshop was attended by a handful of online participants and 44 in-person participants from Austria, Canada, Finland, France, Germany, Israel, Italy, Japan, Luxembourg, Switzerland, the United Kingdom, and the USA. The program consisted of 28 talks and 6 short contributions, leaving time for informal discussions between participants. The workshop focused on stochastic processes evolving in deterministic or random geometric environments, and analytic and geometric problems in which randomness serves as a useful tool. In this context, the notions of metric measure spaces and Dirichlet spaces play a central role to capture both the key elements of the underlying geometries, the relevant Sobolev spaces, and the interactions with various stochastic processes. In this wide flexible setting, functional inequalities and heat kernel estimates provide essential unifying tools. The problems considered by the participants include discrete problems (e.g., random walks in random environments, discrete random geometries), continuous problems (e.g., partial differential equations, diffusions in Riemannian and sub-Riemannian geometry), and stochastic processes with jumps (e.g., stable processes and their geometry). They range from questions set in classical settings (isoperimetry, extension domains, approximation by Lipschitz functions, homogenization theory) and extensions to non-smooth spaces which often appear naturally via taking limits (e.g., under the Gromov-Hausdorff convergence of metric spaces). One session was devoted to connections between these abstract concepts from metric geometry and stochastic calculus with recent challenges arising in the mathematical analysis of data spaces.

The workshop actively encouraged interactions between established and early career mathematicians from these different areas, providing valuable opportunities for cross fertilization and new developments. One after-dinner session was devoted to short communications. The friendly atmosphere fostered lively questions and discussions around different perspectives in a supportive environment. The list of the talks provided below illustrates the wide variety of the topics treated during the workshop.

Three enlightening talks discussed stochastic processes related to conformal geometry. The talks of Yilin Wang and Eveliina Peltola discussed recent advances in some aspects of Stochastic (or Schramm) Lowner Evolutions (2-d conformal geometry). The closing talk of the workshop, by Lorenzo Dello Schiavo dealt with conformally invariant random fields in higher even dimension in connection with Paneitz operators. Even so the setting was very different, one can add to this group the talk by Naotaka Kajino regarding the quasi-symmetric uniformization problem for symmetric diffusions in the context of metric measure spaces in which a new property of the Sierpiński carpet was established. This solved an open problem that had been discussed during the previous iteration of this meeting three years ago.

Tuesday afternoon session was devoted to the use of metric measure spaces in data science (Thomas Hotz, Thomas Needham), and to problems motivated by such use including the exact computation of the Gromov–Hausdorff distance between round spheres of different dimensions (Facundo Memoli).

Classical geometric analysis and its extension to metric measure spaces was the subject of talks by Emanuel Milman (the multi-bubble problem), Giuseppe Savaré, Nages Shanmugalingam, and Jeremy Tyson (Newton–Sobolev spaces in metric measure spaces, density of Lipschitz functions, modulus of curves), Pekka Koskela (extension operators) and Piotr Hajlasz (Hölder continuous mapings). Patricia Alonzo Ruiz discussed bounded variation functions on metric measure spaces, Moritz Kassmann gave a proof of the divergence theorem, and Theo Sturm presented a sharp integrated form of Lichnerowicz classical spectral gap bound.

Optimal transport was at the center of the talks by Max Fathi (Lipschitz transport via heat flow) and Ronan Herry (transport in the context of point processes).

Several talks were devoted to problems relating geometric notions to stochastic processes and Dirichlet forms. Eva Kopfer presented results concerning Ricci flow and functional inequalities on path space, Nicolas Juillet discussed obstructions to a certain type of coupling on the Heisenberg group; Karen Habermann described the construction of natural processes on hypersurfaces embedded into contact manifolds equipped with a sub-Riemannian structure; Chiara Rigoni presented results based on the notion of distribution-valued Ricci bounds; Tai Melcher discussed a large deviation principle involving sub-Riemannian geometry.

Homogenization theory was the subject of the talks by Zhen-Qing Chen (jump processes) and Jessica Lin (non-divergence form elliptic equations) while heat kernel estimates in different contexts were the subject of the talks by Jean-Dominique Deuschel (gradient estimates in time-dependent random conductance models), Emily Dautenhahn (Riemannian manifolds with ends and Dirichlet boundary condition) and Marvin Weidner (in the setting of non-symmetric non-local operators).

Last but not least, a series of talks were devoted to discrete models and their continuous limits: Marek Biskup discussed the law of the number of visits to the most visited points on regular trees, recent results concerning simple exclusion processes in some random environments were the subject of the talk given by Alessandra Faggionato, and Sebastian Andres discussed first passage percolation for models with long-range correlations. Nina Gantert described new results concerning the speed of biased random walk on dynamical percolation, and David Croydon's talk focused on the sub-diffusive scaling limit for a one-dimensional version of the Mott random walk, which is a hopping in a disordered (random) environment.

Closing remarks. Participants enjoyed the opportunity to interact with mathematicians whose expertise might be in a different but related area of research. New ideas and connections emerged via extensive discussions by participants at different career stages and coming from various mathematical backgrounds. Many commented on the significant progress towards bringing together a more diverse group of participants. Finally, having gone through isolation during the Covid pandemic, the workshop gave us first-hand evidence of the importance of in-person meetings and in-person discussions.

Workshop: Heat Kernels, Stochastic Processes and Functional Inequalities

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Abstracts

SLE, energy duality, and foliations by Weil-Petersson quasicircles

Yilin Wang

(joint work with Morris Ang, Minjae Park, Fredrik Viklund)

Schramm-Loewner evolutions (SLE) are introduced by O. Schramm and are important models of random fractal curves without self-intersection. They describe, e.g., interfaces appearing in the scaling limits of critical 2D statistical mechanics models and are central players in 2D random conformal geometry.

In this talk, we relate two large deviation principles of SLE_κ, one as $\kappa \to 0$ and the other as $\kappa \to \infty$. Since SLE_κ is defined through the Loewner equation driven by $\sqrt{\kappa}$ times the 1-dimensional standard Brownian motion B, their large deviations are closely related to the large deviations of $\sqrt{\kappa}B$. When $\kappa \to 0$, the Schilder's theorem shows that the law of the Brownian paths satisfies a large deviation principle with the rate function which is the Cameron-Martin norm of Brownian motion, namely the Dirichlet energy. When $\kappa \to \infty$, we need to consider the Brownian motion on the circle $\exp(iB_t)$ to obtain a meaningful large deviation principle. Donsker-Varadhan's theorem shows that the average local time of $\exp(i\sqrt{\kappa}B_t)$ satisfies a large deviation principle with rate function

$$L(\rho) = \frac{1}{2}\int_{S^1} (v'(\theta))^2 \, d\theta$$

where $\rho = v(\theta)^2 d\theta$ is a probability measure on S^1 absolutely continuous with respect to the Lebesgue measure $d\theta$. These two large deviation principles naturally give rise to two energies that are the rate functions for SLE_{0+} and SLE_∞. The first one called the Loewner energy, which is associated to a Jordan curve and is finite if and only if the curve is *Weil-Petersson*. Weil-Petersson class is an interesting class of Jordan curves appearing in Teichmüller theory, geometric function theory, and string theory with currently more than 20 equivalent definitions. The second energy is called the Loewner-Kufarev energy [1], which is associated to a *foliation* of the Riemann sphere by Jordan curves [2]. Although the two large deviation principles for Brownian motions have little direct relation between them, plugging them into the Loewner's transform and the SLE duality which relates SLE_κ and $\mathrm{SLE}_{16/\kappa}$ reveals a duality between the Loewner and Loewner-Kufarev energy which is made precise in [2]. More precisely, we show that if the Loewner-Kufarev energy is finite, then all the leaves in the foliation are Weil-Petersson Jordan curves. Moreover, the Loewner energy of Jordan curve γ can be expressed in terms of the minimal Loewner-Kufarev energy of any foliation which contains γ as a leaf.

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Towards a conformal field theory for Schramm-Loewner evolutions?

Eveliina Peltola

Many models in statistical physics can be conveniently described in terms of their *geometric* features, such as *clusters* and *interfaces* (see Fig. 1). Of particular physical interest are so-called critical models, which exhibit fractal properties and (approximate) scale-invariance. For instance, critical Ising model is a spin system on a lattice describing magnetic material whose atoms form a regular crystalline structure. In the planar (2D) case, upon taking the mesh size of the lattice to zero (this is called the *scaling limit*), the model enjoys a strong symmetry, *conformal invariance*. To phrase the conformal invariance mathematically precisely, however, much care is needed. To this end, one possible approach is to describe scaling limits of the clusters and interfaces in terms of random geometric objects.

About 20 years ago, Oded Schramm made a breakthrough pertaining to the description of such interfaces: he introduced "stochastic Loewner evolution", now known as *Schramm-Loewner evolution* (SLE) curves. For a number of models, it has now been proven using (quite specific but very celebrated) discrete complex analysis methods (cf. [1, 2, 3] and references therein) that scaling limits of critical interfaces are indeed described by Schramm's SLEs. Alas, it seems beyond reach to prove by these techniques alone conformal invariance results for more general models, such as Potts, $O(n)$, or random-cluster models (see [4] for recent progress establishing rotational invariance, using still model-specific but quite general tools, but falling short of obtaining scale-invariance and thus conformal invariance).

Schramm argued that SLEs were the only possible random curves that could describe scaling limits of critical interfaces [3]. Indeed, upon requiring *conformal invariance in law* together with a *Markov property* for the growth of the curve, he observed that there is only a one-parameter family of possible random curves in the plane, SLE_κ. The parameter $\kappa > 0$ describes, in particular, fractal properties of SLE_κ curves. Soon after Schramm's breakthrough, John Cardy predicted a relationship between SLE curves and certain "boundary condition changing operators" in critical models [5]. Intuitively, the emergence of an interface from a boundary point should be governed by a "conformal field" $\Phi_{1,2}$, that is, an element in a *conformal field theory* (CFT) describing the scaling limit of the critical model.

Concretely, SLE_κ curves can be generated as *random Loewner evolutions* [7]: the time-evolution of the curve is encoded in a solution of Loewner's differential equation, which in the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ reads

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z, \qquad z \in \overline{\mathbb{H}},$$

where $t \mapsto W_t$ is a real-valued continuous function, *driving function.* The curve is obtained as $\gamma(t) := \lim_{\varepsilon \downarrow 0} g_t^{-1}(W_t + \mathrm{i}\varepsilon)$. For the simplest SLE_κ variant, *chordal* SLE, one takes $W = \sqrt{\kappa} B$, one-dimensional Brownian motion of speed κ. In general, interfaces in critical models with boundary conditions alternating at $2N$ marked boundary points (see Fig. 1) correspond to a collection $(\gamma_1, \ldots, \gamma_N)$ of N interacting random curves (e.g. [8, 9, 10, 11, 12, 13, 14, 15]). For $\kappa \leq 4$, one can

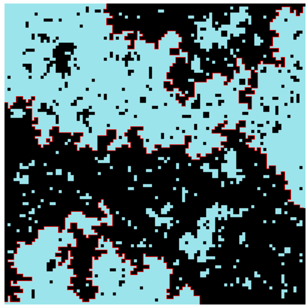

FIGURE 1. Configuration of critical Ising model with alternating boundary conditions (i.e., given boundary segments carry spins $+1$ and the other boundary segments spins -1). The macroscopic random interfaces connecting those boundary points where the boundary condition changes are highlighted. (Figure from [6].)

uniquely characterize these curve families in terms of their conditional laws [11]: by a Markov chain coupling argument, there exists a unique probability measure on N interacting random curves $(\gamma_1, \ldots, \gamma_N)$ such that, for each j, the conditional law of γ_j given $\{\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_N\}$ is chordal SLE_κ. Such a measure can also be constructed by weighting the product measure of N independent SLE_κ curves by a given Radon-Nikodym derivative (cf. [16, 9]). In light of Girsanov's theorem, the marginal law of γ_j in a multiple SLE_κ process in $(\mathbb{H}; x_1 < \cdots < x_{2N})$ is obtained by taking the driving function $W^{(j)}$ of γ_j to be the solution to SDE

$$\begin{aligned} \mathrm{d}W_t^{(j)} &= \sqrt{\kappa}\,\mathrm{d}B_t \\ &\quad + \kappa\,\partial_j \log \mathcal{Z}\big(g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), \ldots, g_t(x_{2N})\big)\,\mathrm{d}t, \end{aligned}$$

with initial value $W_0^{(j)} = x_j$, where $\mathcal{Z}$ is a so-called *partition function* and $g_t(x_i)$ time-evolutions of the other marked points [17]. In the physics parlance [5, 18], the partition functions $\mathcal{Z}$ are examples of CFT correlation functions. While the concept of a "conformal field" is not well-defined in general, one can consider the correlation functions as encoding physical information. In some cases, it is known that a conformal field with given correlation functions can also be constructed as a random generalized function [19, 20]. One might then ask: *Is it possible to construct the appropriate SLE-CFT fields as random distributions?*

To this end, one could try to construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $\Phi_{1,2}$ taking values in a suitable space $S'(\mathbb{R})$ of tempered distributions acting on test functions $f \in S(\mathbb{R})$, such that for all $n \in \mathbb{Z}_{>0}$ the moments are

$$\mathbb{E}\big[(\Phi_{1,2}(f))^n\big] := \int_{\mathbb{R}^n} f(x_1) \cdots f(x_n)\; \mathcal{Z}(x_1, \ldots, x_n)\; \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where the kernels are given by suitable choices of SLE partition functions $\mathcal{Z}$ (here, vanishing when n is odd), would determine the law of $\Phi_{1,2}$ uniquely. One can check that Carleman's condition holds at least when $\kappa \in [2, 6]$ (thus showing uniqueness), while the construction is – to my knowledge – only established in the case where $\kappa = 4$ (using the Gaussian free field; work with Simon Schwarz & in progress).

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Periodic Homogenization of Discontinuous Markov Processes

Zhen-Qing Chen
(joint work with Xin Chen, Takashi Kumagai, Jian Wang)

In this talk, we present some recent results from [4, 7] in the study of periodic homogenization of jump diffusions whose infinitesimal generators are of the following form when acting on $C_b^2(\mathbb{R}^d)$:

$$\begin{aligned}\mathcal{L}f(x) = &\int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\le 1\}}\big)\, k(x,z)\,\Pi(dz)\\ &+ \langle b(x), \nabla f(x)\rangle.\end{aligned} \tag{1}$$

Here $\Pi(dz)$ is a Lévy measure on $\mathbb{R}^d$ that does not charge at the origin 0 and satisfies $\int_{\mathbb{R}^d}(1\wedge |z|^2)\,\Pi(dz) < \infty$; $b(x)$ is a bounded continuous multivariate 1-periodic $\mathbb{R}^d$-valued function, and $k(x,z) : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is a function that is bounded so that $x \mapsto k(x,z)$ is multivariate 1-periodic for each fixed $z \in \mathbb{R}^d$ and

$$\lim_{y\to x} \sup_{z\in\mathbb{R}^d} |k(y,z) - k(x,z)| = 0.$$

Suppose $k(x,z)$ is bounded between two positive constants, and that there is a constant $\beta \in (0,1)$ so that $b(x) = (b_i(x))_{1\le i\le d} \in C_b^\beta(\mathbb{R}^d)$,

$$\sup_{z\in\mathbb{R}^d} |k(x,z) - k(y,z)| \le c_0 |x-y|^\beta, \quad x,y \in \mathbb{R}^d$$

for some $c_0 > 0$, and

$$\Pi(dz) = \mathbb{1}_{\{|r|\le 1\}} \frac{1}{r^{1+\alpha_0}}\, dr d\theta + \mathbb{1}_{\{r>1\}} \frac{1}{r^{1+\alpha}} dr \varrho_0(d\theta) \tag{2}$$

where $\alpha_0 \in (1,2)$, $\alpha \in (0,\infty)$, $\varrho_0(d\theta)$ is a non-negative finite measure on $\mathbb{S}^{d-1}$ and (r,θ) denotes the spherical coordinates of $z \in \mathbb{R}^d$. It follows from [8] that there is a Feller process $X := (X_t)_{t\ge 0}$ having $\mathcal{L}$ as its infinitesimal generator. The Feller process X is irreducible and has the strong Feller property. Since $k(x,z)$ and $b(x)$ are multivariate 1-periodic in x, the process X can be regarded as a Markov process on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Denote by $\mu(dx)$ the stationary probability measure for the quotient process of X on $\mathbb{T}^d$. Define for any $R > 1$,

$$\bar b := \int_{\mathbb{T}^d} b(x)\,\mu(dx), \quad \bar b_R := \int_{\mathbb{T}^d} b_R(x)\,\mu(dx), \quad \bar b_\infty := \int_{\mathbb{T}^d} b_\infty(x)\,\mu(dx).$$

The following assertions are particular cases of the more general results obtained in [4].

(i) Suppose that $k(x,z)$ is a bounded continuous function on $\mathbb{R}^d\times\mathbb{R}^d$ that is multivariate 1-periodic. For any $\varepsilon \in (0,1]$, define $(Y_t^\varepsilon)_{t\ge 0}$ by

$$Y_t^\varepsilon = \begin{cases} \varepsilon X_{t/\varepsilon^\alpha}, & 0<\alpha<1,\\ \varepsilon X_{t/\varepsilon^\alpha} - (\bar b_{1/\varepsilon} + \bar b)t, & \alpha = 1,\\ \varepsilon X_{t/\varepsilon^\alpha} - \varepsilon^{1-\alpha}(\bar b_\infty + \bar b)t, & 1<\alpha<2.\end{cases}$$

Then the process $(Y_t^\varepsilon)_{t\geq 0}$ converges weakly in the Skorohod space, as $\varepsilon\to 0$, to a (possibly non-symmetric) α-stable Lévy process $(\bar X_t)_{t\ge 0}$ having Lévy measure $\frac{\bar k(\theta)}{r^{1+\alpha}}\,dr\,\varrho_0(d\theta)$, where $\bar k:\mathbb{S}^{d-1}\to\mathbb{R}_+$ is defined by

$$\bar k(\theta):=\int_{\mathbb{T}^d}\bar k(x,\theta)\,\mu(dx),\quad \theta\in\mathbb{S}^{d-1},$$

and $\bar k:\mathbb{R}^d\times\mathbb{S}^{d-1}\to\mathbb{R}_+$ satisfies that for all $x\in\mathbb{R}^d$ and $\theta\in\mathbb{S}^{d-1}$,

$$\bar k(x,\theta)=\lim_{T\to\infty}\frac{1}{T}\int_0^T k(x,(r,\theta))\,dr.$$

More precisely, the infinitesimal generator $\bar{\mathcal{L}}f(x)$ of the Lévy process $(\bar X_t)_{t\ge0}$ is given by

$$\begin{cases}\displaystyle\int_{\mathbb{R}^d}(f(x+z)-f(x))\,\bar k(z/|z|)\,\Pi_0(dz) & \alpha\in(0,1),\\ \displaystyle\int_{\mathbb{R}^d}\big(f(x+z)-f(x)-\langle\nabla f(x),z\rangle\mathbb{1}_{\{|z|\le 1\}}\big)\,\bar k(z/|z|)\,\Pi_0(dz) & \alpha=1,\\ \displaystyle\int_{\mathbb{R}^d}(f(x+z)-f(x)-\langle\nabla f(x),z\rangle)\,\bar k(z/|z|)\,\Pi_0(dz) & \alpha\in(1,2),\end{cases}$$

where $\Pi_0(dz):=\mathbb{1}_{\{r>0\}}\frac{1}{r^{1+\alpha}}\,dr\,\varrho_0(d\theta)$. Furthermore, if the finite measure ϱ_0 on $\mathbb{S}^{d-1}$ that does not charge on the set of rationally dependent $\theta\in\mathbb{S}^{d-1}$, then we can take

$$\bar k(\theta)\equiv\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}k(x,z)\,dz\,\mu(dx)\quad\text{for all }\theta\in\mathbb{S}^{d-1},$$

which is a constant, in the statement above. Here we call $\theta=(\theta_1,\cdots,\theta_d)\in\mathbb{S}^{d-1}$ is rationally dependent if there is some non-zero $m=(m_1,\cdots,m_d)\in\mathbb{Z}^d$ so that $\langle m,\theta\rangle=\sum_{i=1}^d m_i\theta_i=0$. Otherwise, we call θ rationally independent. When $d=1$, $\mathbb{S}^0=\{1,-1\}$ so every its member is rationally independent. In particular, if ϱ_0 does not charge on singletons when $d=2$ and does not charge on subsets of $\mathbb{S}^{d-1}$ that are of Hausdorff dimension $d-2$ when $d\ge 3$ (for example, ϱ_0 is γ-dimensional Hausdorff measure with $\gamma\in(d-2,d-1]$), then ϱ_0 does not charge on the set of rationally dependent $\theta\in\mathbb{S}^{d-1}$ and so the result above holds with $\bar k(\theta)\equiv\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}k(x,z)\,dz\,\mu(dx)$.

(ii) When $\alpha=2$, define

$$Y_t^\varepsilon:=\varepsilon X_{\varepsilon^{-2}|\log\varepsilon|^{-1}t}-\varepsilon^{-1}|\log\varepsilon|^{-1}(\bar b_\infty+\bar b)t,\quad t\ge 0.$$

Suppose that

$$k_0:=\lim_{|z|\to\infty}\int_{\mathbb{T}^d}k(x,z)\,\mu(dx)>0.$$

Then $(Y_t^\varepsilon)_{t\ge0}$ converges weakly in $\mathscr{D}([0,\infty);\mathbb{R}^d)$, as $\varepsilon\to0$, to Brownian motion $(\bar X_t)_{t\ge0}$ with the covariance matrix $A=\{a_{ij}\}_{1\le i,j\le d}$ such that

$$a_{ij}:=k_0\int_{\mathbb{S}^{d-1}}\theta_i\theta_j\,\varrho_0(d\theta).$$

(iii) When $\alpha > 2$, define

$$Y_t^\varepsilon := \varepsilon X_{t/\varepsilon^2} - \varepsilon^{-1}(\bar{b}_\infty + \bar{b})t, \quad t \geq 0.$$

Then $(Y_t^\varepsilon)_{t\geq 0}$ converges weakly in $\mathscr{D}([0,\infty);\mathbb{R}^d)$, as $\varepsilon \to 0$, to a d-dimensional Brownian motion $(\bar{X}_t)_{t\geq 0}$ with the covariance matrix

$$A := \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \big(z + \psi(x+z) - \psi(x)\big) \otimes \big(z + \psi(x+z) - \psi(x)\big) k(x,z)\, \Pi(dz)\, \mu(dx).$$

Here $\psi \in \mathscr{D}(\mathcal{L})$ is the unique periodic solution on $\mathbb{R}^d$ to the following equation

$$\mathcal{L}\psi(x) = -b_\infty(x) - b(x) + \bar{b}_\infty + \bar{b}, \quad x \in \mathbb{T}^d \tag{3}$$

with $\mu(\psi) = 0$.

The solution ψ to (3) is called a corrector, whose existence can be deduced from the estimates as well as the gradient estimates for the transition density function of X obtained in [8].

The above homogenization results can be viewed as a counterpart of the celebrated results of Bensoussan, Lions and Papanicolaou [2] and Bhattacharya [3] on periodic homogenizations for diffusion processes. The study of periodic homogenization for discontinuous Markov processes or for non-local operators is relatively recent and is quite limited; see [4, Introduction] for a brief history. We used generator method combined with its connection to martingales in our study [4]. Our results reveal that the limit process depends on the tail of the jumping kernel of the discontinuous Markov process. Though it is used in our approach, unlike the diffusion case, the corrector does not play a role in the homogenized process when $\alpha \leq 2$. We remark that homogenization for non-local operators in random media has been investigated in [5, 6].

We next consider quantitative homogenization for the special case that the Lévy measure Π in (1) is given by

$$\Pi(dz) = \mathbb{1}_{\{|z|\leq 1\}} \frac{1}{|z|^{d+\alpha_0}}\, dz + \mathbb{1}_{\{|z|>1\}} \frac{1}{|z|^{1+\alpha}}\, dz. \tag{4}$$

Let μ be the stationary probability measure for the quotient process of X on the torus $\mathbb{T}^d$. Assume that $\int_{\mathbb{T}^d} b(x)\mu(dx) = 0$. Denote by $\bar{X}$ the limit process of Y^ε as above. For any $g \in C_c^2(\mathbb{R}^d)$, let $u_\varepsilon(t,x) = \mathbb{E}_x g(Y_t^\varepsilon)$ and $\bar{u}(t,x) = \mathbb{E}_x g(\bar{X}_t)$. They are solutions to the parabolic heat equations with initial value g for the infinitesimal generators of Y^ε and $\bar{X}$, respectively. As a special case of the quantitative homogenization results obtained in [7], where the time-dependent case is also considered, we have for each fixed $T > 0$, there is a positive constant $C = C(g,T)$

depending on g and T so that for $\varepsilon \in (0,1)$,

$$\|u_\varepsilon - \bar{u}\|_{L^2([0,T]\times\mathbb{T}^d;dt\times dx)} \le C(g,T)\begin{cases} \varepsilon^\alpha \wedge \varepsilon^{1-\alpha}, & \alpha \in (0,1), \\ \varepsilon|\log\varepsilon|, & \alpha = 1, \\ \varepsilon^{2-\alpha}, & \alpha \in (1,2), \\ |\log\varepsilon|^{-1}, & \alpha = 2, \\ \varepsilon \vee \varepsilon^{\alpha-2}, & \alpha \in (2,\infty]. \end{cases}$$

The rate for the case of $\alpha \in (1,2)$ is consistent with the convergence rate in the central limit theorem for stable distributions [1].

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Isoperimetric Multi-Bubble Problems - Old and New

Emanuel Milman

(joint work with Joe Neeman)

The classical isoperimetric inequality in Euclidean space $\mathbb{R}^n$ states that among all sets ("bubbles") of prescribed volume, the Euclidean ball minimizes surface area. One may similarly consider isoperimetric problems for more general metric-measure spaces, such as on the n-sphere $\mathbb{S}^n$ and on n-dimensional Gaussian space $\mathbb{G}^n$ (i.e. $\mathbb{R}^n$ endowed with the standard Gaussian measure). Furthermore, one may consider the "multi-bubble" partitioning problem, where one partitions the space into $q-1 \ge 1$ (possibly disconnected) bubbles, so that their total common surface-area is minimal. The classical case, referred to as the single-bubble isoperimetric problem, corresponds to $q = 2$; the case $q = 3$ is called the double-bubble problem, and so on.

In 2000, Hutchings, Morgan, Ritoré and Ros resolved the Double-Bubble conjecture in Euclidean space $\mathbb{R}^3$ [1] (and this was subsequently resolved in $\mathbb{R}^n$ as well) – the optimal partition into two bubbles of prescribed finite volumes (and an exterior unbounded third bubble) which minimizes the total surface-area is given by three spherical caps, meeting at 120°-degree angles. A more general conjecture of J. Sullivan from the 1990's asserts that when $q-1 \leq n+1$, the optimal multi-bubble partition of $\mathbb{R}^n$ (as well as $\mathbb{S}^n$) is obtained by taking the Voronoi cells of q equidistant points in $\mathbb{S}^n$ and applying appropriate stereographic projections to $\mathbb{R}^n$ (and backwards).

In 2018, together with Joe Neeman, we resolved the analogous multi-bubble conjecture on the optimal partition of $\mathbb{G}^n$ into $q-1 \leq n$ bubbles – the unique optimal partition is given by the Voronoi cells of (appropriately translated) q equidistant points [2]. In the talk, we describe our approach in that work, as well as recent progress [3] on the multi-bubble problem on $\mathbb{R}^n$ and $\mathbb{S}^n$. In particular, we show that minimizing partitions are always spherical when $q-1 \leq n$, and we resolve the latter conjectures when in addition $q-1 \leq 5$ (e.g. the triple-bubble conjecture in $\mathbb{R}^3$ and $\mathbb{S}^3$, and the quadruple-bubble conjecture in $\mathbb{R}^4$ and $\mathbb{S}^4$).

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Sobolev spaces on metric measure spaces

Jeremy Tyson

This talks surveys tools commonly used in the theory of Sobolev spaces on metric measure spaces. It serves as a broad review of the recent development, as well as an introduction to Giuseppe Savaré's talk.

Capacitary modulus and Newtonian-Sobolev capacity in metric measure spaces

Giuseppe Savaré

Let $\mathbb{X} := (X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, characterized by a complete and separable metric space (X, d) endowed with a finite positive Borel measure $\mathfrak{m}$. We denote by $\mathrm{RA}(X)$ the collection of all the rectifiable arcs, which we may identify with their arc-length (thus Lipschitz) parametrizations $\gamma : [0,1] \to X$.

A Borel map $g : X \to [0, +\infty]$ is an upper gradient of $f : X \to \mathbb{R}$ [9, 6] if

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_\gamma g \quad \text{for every } \gamma \in \mathrm{RA}(X).$$

We fix an exponent $p \in (1, +\infty)$ and we denote by $\mathrm{L}^p(X, \mathfrak{m})$ (resp. $L^p(X, \mathfrak{m})$) the space of Borel maps $f : X \to \mathbb{R}$ which are p-summable (resp. the usual quotient Lebesgue space where functions which coincide $\mathfrak{m}$-a.e. are identified). $\mathrm{N}^{1,p}(\mathbb{X})$ is the Newtonian-Sobolev class [11, 13] of functions $f \in \mathrm{L}^p(X, \mathfrak{m})$ with an upper gradient g in $\mathrm{L}^p(X, \mathfrak{m})$. It is possible to show [5, 10] that there exists a unique minimal p-*weak* upper gradient $|\mathrm{D}f|_N \in L^p(X, \mathfrak{m})$ such that

$$|\mathrm{D}f|_N \le g \quad \mathfrak{m}\text{-a.e. in } X \quad \text{for every } \mathrm{L}^p\text{-upper gradient } g \text{ of } f.$$

The energy form

$$\mathsf{NE}_p(f) := \int_X |\mathrm{D}f|^p_N(x)\,\mathrm{d}\mathfrak{m}(x), \quad f \in \mathrm{N}^{1,p}(\mathbb{X}),$$

defines a convex and p-homogeneous functional, which is lower semicontinuous w.r.t. the L^p topology. If $f \in \mathrm{Lip}_b(X)$ (the space of bounded Lipschitz real functions on X) it is easy to check that the asymptotic Lipschitz constant

$$|\mathrm{D}f|_a(x) := \lim_{r\downarrow 0} \mathrm{Lip}(f, \mathrm{B}(x,r)), \quad x \in X,$$

is an upper gradient of f, so that $|\mathrm{D}f|_a \ge |\mathrm{D}f|_N$ $\mathfrak{m}$-a.e. in X. Lipschitz functions are dense in NE_p-energy [2, 3]: for every $f \in \mathrm{N}^{1,p}(\mathbb{X})$ there exists a sequence $(f_n) \subset \mathrm{Lip}_b(X)$ converging to f in L^p such that $|\mathrm{D}f_n|_a \to |\mathrm{D}f|_N$ strongly in L^p.

The original proof relies on the construction of the gradient flow of the Cheeger energy, the duality with sub-solutions to Hamilton-Jacobi equations, Kuwada's Lemma, the superposition principle, and a Wasserstein Entropy-Dissipation formula. A more recent and direct proof [12, Sect. 5] relies on a duality argument based on the L^1-Wasserstein metric induced by the intrinsic extended distances

$$\mathsf{d}_g(x_0, x_1) := \inf\Big\{ \int_\gamma g : \gamma \in \mathrm{RA}(X),\ \gamma_0 = x_0,\ \gamma_1 = x_1 \Big\}, \quad x_0, x_1 \in X,$$

where $g : X \to (0, +\infty]$ is a Borel function with $\inf_X g > 0$ (as usual, $\inf \emptyset := +\infty$) and their lower approximations by Lipschitz functions [12, Sect. 2], an argument which is also exploited by the direct construction of [7]. Duality tools have been then further developed in [4].

Notice that g is an upper gradient of f if and only if f is 1-Lipschitz with respect to the extended distance d_g. For every function $f : X \to \mathbb{R}$ bounded from below, we can we thus define the largest 1-Lipschitz function w.r.t. d_g dominated by f, which we denote by $\mathrm{L}_g f$: it is defined by

$$\mathrm{L}_g f(x) := \inf_{y\in X} f(y) + \mathsf{d}_g(x, y).$$

When f, g are coercive (i.e. have compact sublevels) we also have [12, Sect. 2]

$$\mathrm{L}_g f(x) = \sup\Big\{ u(x) : u \in \mathrm{Lip}_b(X),\ u \le f,\ |\mathrm{D}u|_a \le g \Big\}.$$

We can obtain the density result by the composition of the following two monotone approximation techniques.

Theorem 1. *If $f \in \mathrm{L}^p(X, \mathfrak{m})$ has an L^p-upper gradient g, there exist pointwise decreasing sequences of coercive p-summable functions f_n, g_n such that*

$$g_n \text{ is an upper gradient of } f_n \text{ and} \quad \lim_{n\to\infty} \int_X \Big(|f_n - f|^p + |g_n - g|^p \Big)\,\mathrm{d}\mathfrak{m} = 0.$$

If $f : X \to \mathbb{R} \cup \{+\infty\}$ is coercive and has a coercive L^p*-upper gradient g, then there exists a pointwise increasing sequence $f_n \in \mathrm{Lip}_b(X)$ such that*

$$|\mathrm{D}f_n|_a \le g, \quad f_n \uparrow f \quad \textit{in } L^p(X, \mathfrak{m}).$$

We can apply the above result to obtain refined properties (known for PI doubling spaces [5, 10]) of the Newtonian capacity

$$\mathrm{Cap}_p(A) := \inf\Big\{ \int_X \Big(|f|^p + |\mathrm{D}f|^p_{N^{1,p}}\Big)\,\mathrm{d}\mathfrak{m} : f \in N^{1,p}(\mathbb{X}),\ f \ge 1 \text{ on } A\Big\}.$$

Theorem 2. *The Newtonian capacity satisfies the following properties:*

(1) Coercive competitors:

$$\mathrm{Cap}_p(A) = \inf\Big\{ \int_X \Big(|f|^p + g^p\Big)\,\mathrm{d}\mathfrak{m} : f, g \textit{ coercive, } g \textit{ u.g. of } f,\ f \ge 1 \textit{ on } A\Big\}.$$

(2) Tightness: there exists an increasing sequence of compact sets $K_n \subset X$ such that $\lim_{n\to\infty} \mathrm{Cap}(X \setminus K_n) = 0$.

(3) Outer regularity: $\mathrm{Cap}_p(A) = \inf\Big\{\mathrm{Cap}_p(G) : G \textit{ open},\ G \supset A\Big\}$.

(4) Capacity of a compact set K in terms of Lipschitz functions:

$$\mathrm{Cap}_p(K) = \inf\Big\{ \int_X \Big(|f|^p + |\mathrm{D}f|^p_a\Big)\,\mathrm{d}\mathfrak{m} : f \in \mathrm{Lip}_b(X),\ f \ge 1 \textit{ on } K\Big\}.$$

(5) Cap_p *is a Choquet capacity. In particular for every Borel (or Souslin) set $B \subset X$*

$$\mathrm{Cap}_p(B) = \sup\Big\{\mathrm{Cap}_p(K) : K \textit{ compact},\ K \subset B\Big\}.$$

The notion of Modulus [8] of a family of arcs $\Gamma \subset \mathrm{RA}(X)$

$$\mathrm{Mod}_p(\Gamma) := \inf\Big\{ \int_X g^p\,\mathrm{d}\mathfrak{m} : g : X \to [0, +\infty] \text{ Borel}, \int_\gamma g \ge 1, \forall \gamma \in \Gamma\Big\}.$$

is strongly related to the capacity. Setting

$$\Gamma_A := \Big\{\gamma \in \mathrm{RA}(X) : \gamma^{-1}(A) \neq \emptyset\Big\},$$

for $A \subset X$, it is possible to show that [5, 10]

$$\mathrm{Cap}_p(A) = 0 \quad \iff \quad \mathfrak{m}(A) = 0, \quad \mathrm{Mod}_p(\Gamma_A) = 0.$$

Taking inspiration from [1, 12, 4], we propose a new notion of *Capacitary Modulus*, which provides a precise characterization of the capacity: it is defined as

$$\mathrm{CMod}_p(\Gamma) := \inf\Big\{ \int_X \Big(f^p + g^p\Big)\,\mathrm{d}\mathfrak{m} : f(\gamma_1) + \int_\gamma g \ge 1 \text{ for every } \gamma \in \Gamma\Big\},$$

where the infimum is restricted to nonnegative and Borel maps f, g.

Theorem 3. *For every $A \subset X$*

$$\mathrm{Cap}_p(A) = \mathrm{CMod}_p(\Gamma^0_A), \quad \Gamma^0_A := \Big\{\gamma \in \mathrm{RA}(X) : \gamma_0 \in A\Big\}.$$

This identification plays a key role to extend to general metric-measure spaces the well known link between Sobolev functions, quasi-continuity, and quasi-uniform convergence:

Theorem 4. *If $f_n \in \mathrm{N}^{1,p}(\mathbb{X})$, $g_n \in \mathrm{L}^p(X, \mathfrak{m})$ are L^p-Cauchy sequences and g_n is an upper gradient of f_n, then there exists an increasing subsequence $k \mapsto n(k)$ such that $(f_{n(k)})_k$ converges quasi-uniformly to a map $f \in \mathrm{N}^{1,p}(\mathbb{X})$.*

Every function $f \in \mathrm{N}^{1,p}(\mathbb{X})$ is quasi-continuous. Moreover, if two maps in $\mathrm{N}^{1,p}(\mathbb{X})$ coincide $\mathfrak{m}$-a.e., then they coincide quasi-everywhere. In particular, for every open set $G \subset X$

$$\mathrm{Cap}_p(G) = \inf\Big\{ \int_X \Big(|f|^p + |\mathrm{D}f|_N^p\Big)\,\mathrm{d}\mathfrak{m} : f \in N^{1,p}(\mathbb{X}),\ f \geq 1\ \mathfrak{m}\text{-}a.e.\ on\ A\Big\}.$$

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Using the theory of Newton-Sobolev spaces to construct nonlinear fractional Laplacian in compact doubling metric measure spaces

Nageswari Shanmugalingam

Recent progress on analysis in metric measure spaces has resulted in the development of tools needed to study analogs of elliptic PDEs via energy minimization technique for strongly local Sobolev-type function spaces called Newton-Sobolev spaces. Such tools are, however, available only when the measure is doubling and supports a Poincaré-type inequality. When the metric measure space is doubling but does not support a Poincaré inequality, a more viable family of function spaces that encode the geometry of the metric measure space may be non-local. One such possible family of function-spaces are Besov spaces $B^{\theta}_{p,p}(X)$ for $1 \le p < \infty$ and $0 < \theta < 1$.

In the case that X is doubling and supports a p-Poincaré inequality, in collaboration with Sylvester Eriksson-Bique, Gianmarco Giovannardi, Riikka Korte, and Gareth Speight [2], we constructed a notion of fractional Laplacian Δ^{α} on X based on the infinitessimal generator Δ associated with a choice of Cheeger differential structure on X using spectral theory, and we then adapted a technique of Caffarelli and Silvestre to this setting to study regularity properties of solutions to the equation $\Delta^{\alpha} u = 0$ on a domain in X. However, when X does not support any Poincaré inequality and hence is not known to have an operator Δ associated with the corresponding Sobolev class, such a construction and study of solutions is not viable.

In this talk we describe some connections between Newton-Sobolev spaces of functions on uniform domains in metric measure spaces of controlled geometry and Besov classes of functions on the boundary of the domain; we will use this connection to construct a non-local "Dirichlet" form on compact doubling metric measure spaces, and the well-posedness of problems related to associated fractional p-Laplacians for $1 < p < \infty$.

Every compact doubling metric space X can be realized as the boundary (up to biLipschitz change in the metric) of a uniform domain Ω that is equipped with a doubling measure supporting the strongest of all Poincaré inequalities, the 1-Poincaré inequality. This was recently shown in [1]. Moreover, we also showed there that for each $\alpha > 0$ we can construct a measure on this uniform domain that satisfies the doubling and Poincare inequality conditions, such that there is a co-dimension α relationship between the measure on the uniform domain and the measure on the compact metric measure space that forms the boundary of this domain. In [1] we also showed that the trace class of the Newton-Sobolev space $N^{1,p}(\Omega)$ is the Besov class $B^{1-\alpha/p}(X)$. Thus the Besov classes on a compact doubling metric measure space are traces of (weighted) Sobolev classes of functions on a uniform domain Ω that is also doubling and supports a Poincaré inequality, with the additional connection being that the compact metric measure space of interest is the boundary of the uniform domain so that the measure μ_θ on the uniform domain is in a co-dimension θ relationship with the doubling measure ν on the boundary for a fixed $\theta \in (0,1)$.

This connection is useful in constructing a semi-linear form on the Besov class $B^{1-\theta/p}_{p,p}(X)$ as follows: $\mathcal{E}(u,v) = \int_\Omega |\nabla H_p u|^{p-2} \nabla H_p u \cdot \nabla H_p v \, d\mu_\theta$ for $u, v \in B^{1-\theta/p}_{p,p}(X)$, where $H_p u$ is the p-harmonic extension of u to Ω, and ∇ is a fixed choice of a Cheeger differential structure on the Newton-Sobolev class $N^{1,p}(\Omega)$. We show that for each $f \in L^{p'}(X)$ with $\int_X f \, d\nu = 0$ there is a function $u_f \in B^{1-\theta/p}_{p,p}(X)$ such that $\mathcal{E}(u_f, v) = \int_X v \, f \, d\nu$ for each $v \in B^{1-\theta/p}_{p,p}(X)$; thus we solve the problem on X related to the fractional Laplacian associated with the form $\mathcal{E}$. We also show uniqueness up to constants, stability, and Hölder continuity of such solutions, and complete the discussion by describing a related Harnack inequality.

The talk is based on Joint work with Luca Capogna, Ryan Gibara, Josh Kline, and Riikka Korte [3].

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Quasisymmetric Gaussian uniformization is impossible for Brownian motion on the Sierpiński carpet

Naotaka Kajino

(joint work with Mathav Murugan)

This[1] is a continuation of Mathav Murugan's talk at MFO on 18 November 2019 on [4], which concerns the following set $\mathcal{G}_\beta(\mathcal{D})$ defined for $\beta \in (1,\infty)$ and a *metric measure Dirichlet (MMD) space* $\mathcal{D} = (K, d, m, \mathcal{E}, \mathcal{F})$, i.e., a strongly local regular symmetric Dirichlet space $(K, m, \mathcal{E}, \mathcal{F})$ over a locally compact separable metric space (K, d) such that $B_d(x, r) := \{y \in K \mid d(x, y) < r\}$ has compact closure in K for any $(x, r) \in K \times (0,\infty)$:

$$(\mathcal{G}_\beta) \qquad \mathcal{G}_\beta(\mathcal{D}) := \left\{ (\theta,\mu) \;\middle|\; \begin{array}{l} \theta \text{ is a metric on } K \text{ quasisymmetric to } d,\ \mu \\ \text{is a Radon measure on } K \text{ charging no set of} \\ \text{zero } \mathcal{E}\text{-capacity and with full } \mathcal{E}\text{-quasi-support,} \\ (K,\theta,\mu,\mathcal{E},\mathcal{F}^\mu) \text{ satisfies VD and HKE}(\beta) \end{array} \right\}.$$

Here we say that $\mathcal{D}$ satisfies VD if and only if $m(B_d(x, 2r)) \le c_{\mathrm{v}} m(B_d(x, r))$ for any $(x, r) \in K \times (0,\infty)$ for some $c_{\mathrm{v}} \in (0,\infty)$, and that $\mathcal{D}$ satisfies HKE(β) if and only if $(K, m, \mathcal{E}, \mathcal{F})$ has a continuous heat kernel $p = p_t(x, y) : (0,\infty) \times K \times K \to [0,\infty)$ and there exist $c_1, c_2, c_3, c_4 \in (0,\infty)$ such that for any $(t, x, y) \in (0,\infty) \times K \times K$,

$$\text{HKE}(\beta) \qquad \frac{c_1 \mathbf{1}_{[0,c_2]}\bigl(d(x,y)^\beta/t\bigr)}{m(B_d(x,t^{1/\beta}))} \le p_t(x,y) \le \frac{c_3 \exp\bigl(-c_4 \bigl(d(x,y)^\beta/t\bigr)^{\frac{1}{\beta-1}}\bigr)}{m(B_d(x,t^{1/\beta}))}.$$

A metric θ on K is said to be *quasisymmetric* to d ($\theta \overset{\text{qs}}{\sim} d$) if and only if $\theta(x, y)/\theta(x, z) \le \eta\bigl(d(x, y)/d(x, z)\bigr)$ for any $x, y, z \in K$ with $x \ne z$, or equivalently, for any $x \in K$ and any $r, A \in (0,\infty)$ there exists $s \in (0,\infty)$ such that $B_\theta(x, s) \subset B_d(x, r)$ and $B_d(x, Ar) \subset B_\theta(x, \eta(A)s)$, for some homeomorphism

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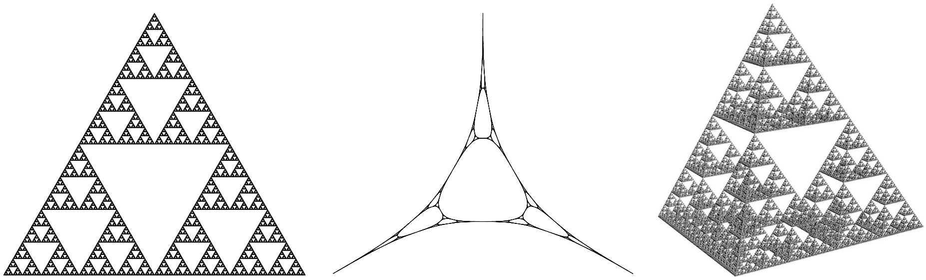

FIGURE 1. Sierpiński gaskets (2-dimensional, harmonic, 3-dimensional)

$\eta : [0,\infty) \to [0,\infty)$. Each μ as in $(\mathcal{G}_\beta)$ is such that *"$\mathcal{E}$ becomes a regular symmetric Dirichlet form on $L^2(K,\mu)$ with core $\mathcal{F} \cap C_{\mathrm{c}}(K)$"*, whose domain is then denoted by $\mathcal{F}^\mu$; see [1, Corollary 5.2.10, (5.2.17) and Theorem 5.2.11] (here $C_{\mathrm{c}}(K)$ is the set of $u\colon K \to \mathbb{R}$ such that u is continuous, and $K \setminus u^{-1}(0)$ has compact closure in K).

It is relatively well known that $\mathcal{G}_\beta(\mathcal{D}) = \emptyset$ for any $\beta \in (1,2)$ (unless K is a singleton); see [4, (1.5) and Lemma 4.7]. Our concern is whether $\mathcal{G}_\beta(\mathcal{D}) \neq \emptyset$ for $\beta = 2$, or at least for $\beta \in (2,\infty)$ arbitrarily close to 2, which is motivated by the following theorem.

Theorem 1 ([6]; see also [4, Theorem 6.33])**.** *Let $\mathcal{D}$ be the MMD space of the Brownian motion on the* 2-dimensional standard Sierpiński gasket *(Figure* 1*, left). Then $\mathcal{G}_2(\mathcal{D}) \neq \emptyset$.*

More precisely, [6] constructed a concrete element of $\mathcal{G}_2(\mathcal{D})$ on the basis of the geometry of the *harmonic Sierpiński gasket* (Figure 1, center). As an answer to the question of whether $\mathcal{G}_\beta(\mathcal{D}) \neq \emptyset$ for a general MMD space $\mathcal{D}$, in [4] we have proved the following theorem.

Theorem 2 ([4, Theorem 2.10])**.** *Let $\mathcal{D} = (K, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space with K having at least two elements. Then $\{\beta \in (1,\infty) \mid \mathcal{G}_\beta(\mathcal{D}) \neq \emptyset\} \in \{[2,\infty), (2,\infty), \emptyset\}$.*

Theorem 2 further raises the questions of *what $\mathcal{D}$ satisfies $\mathcal{G}_2(\mathcal{D}) \neq \emptyset$* and *what $\mathcal{G}_2(\mathcal{D})$ looks like when $\mathcal{G}_2(\mathcal{D}) \neq \emptyset$*. In these regards, in [4] we have proved the following theorem.

Theorem 3 ([4, Proposition 2.11]; [3, Section 4])**.** *Let $\mathcal{D} = (K, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space with K having at least two elements, and let $\mu_{\langle u \rangle}$ be the $\mathcal{E}$-energy measure of $u \in \mathcal{F}$ as defined in* [2, (3.2.14)]. *Then for any $(\theta, \mu) \in \mathcal{G}_2(\mathcal{D})$, the following hold:*

(1) Define for $x, y \in K$

$$d_\mu(x,y) := \sup\{u(x) - u(y) \mid u \in \mathcal{F} \cap C_{\mathrm{c}}(K),\ \mu_{\langle u \rangle} \leq \mu\}.$$

Then for any $x, y \in K$ for some $c_{\theta,\mu} \in [1,\infty)$:

$$c_{\theta,\mu}^{-1} d_\mu(x,y) \leq \theta(x,y) \leq c_{\theta,\mu} d_\mu(x,y).$$

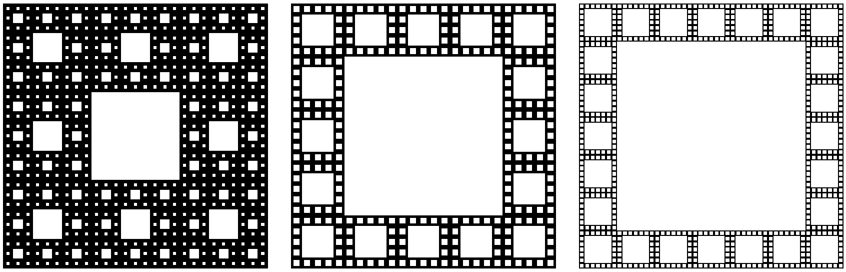

FIGURE 2. Sierpiński carpets SC_ℓ ($\ell = 3, 5, 7$)

(2) Let A be a Borel subset of K. Then $\mu(A) = 0$ if and only if

$$\sup_{u \in \mathcal{F}} \mu_{\langle u \rangle}(A) = 0.$$

Theorem 4 ([4, Theorem 6.35])**.** *Let $N \in \mathbb{N}$ satisfy $N \geq 3$, and let $\mathcal{D}$ be the MMD space of the Brownian motion on the N-*dimensional standard Sierpiński gasket *(see Figure 1, right, for a picture for $N = 3$). Then $\mathcal{G}_2(\mathcal{D}) = \emptyset$.*

It was left open in [4] whether $\mathcal{G}_2(\mathcal{D}) \neq \emptyset$ for the MMD space $\mathcal{D}$ of the Brownian motion on *generalized Sierpiński carpets* (see, e.g., [4, Subsection 6.4] and the references therein for its basics). As our main result, we answer this for those in Figure 2 as follows.

Theorem 5 ([5])**.** *Let $\ell \in \mathbb{N} \setminus \{1\}$ be odd, and let SC_ℓ be the unique non-empty compact subset of $\mathbb{R}^2$ such that $\mathrm{SC}_\ell = \bigcup_{i \in S_\ell} f_{\ell,i}(\mathrm{SC}_\ell)$ (Figure 2), where $f_{\ell,i} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f_{\ell,i}(x) := \ell^{-1} i + \ell^{-1} x$ for $i \in \mathbb{Z}^2$ and $S_\ell := \{i \in \mathbb{Z}^2 \mid f_{\ell,i}([0,1]^2) \subset [0,1]^2 \setminus (\ell^{-1}, 1 - \ell^{-1})^2\}$. Then the MMD space $\mathcal{D} = (K, d, m, \mathcal{E}, \mathcal{F})$ of the Brownian motion on $K := \mathrm{SC}_\ell$, where d is the Euclidean metric and m is the uniform distribution on K, satisfies $\mathcal{G}_2(\mathcal{D}) = \emptyset$.*

Note that SC_3 (Figure 2, left) is nothing but the 2-*dimensional standard Sierpiński carpet.*

We fix the setting of Theorem 5 in the rest of this article. The first step of the proof of Theorem 5 is to note the following theorem and proposition, which we proved in [4] for any generalized Sierpiński carpet in $\mathbb{R}^N$ with arbitrary $N \in \mathbb{N} \setminus \{1\}$. We set $V_0 := K \setminus (0,1)^2$ and, recalling that $h \in \mathcal{F}$ is said to be $\mathcal{E}$-*harmonic* on $K \setminus V_0$ if and only if $\mathcal{E}(h, v) = 0$ for any $v \in \mathcal{F} \cap C_{\mathrm{c}}(K)$ with $v|_{V_0} = 0$, we define $\mathcal{H}_0 := \{h \in \mathcal{F} \mid h \text{ is } \mathcal{E}\text{-harmonic on } K \setminus V_0\}$ and $\mathcal{H}_{\mathcal{G}_2} := \{h + \mathbb{R}\mathbf{1}_K \mid h \in \mathcal{H}_0, (d_{\mu_{\langle h \rangle}}, \mu_{\langle h \rangle}) \in \mathcal{G}_2(\mathcal{D})\}$.

Theorem 6 (a special case of [4, (6.76) and Theorem 6.54])**.** *$\mathcal{G}_2(\mathcal{D}) \neq \emptyset$ (if and) only if $\mathcal{H}_{\mathcal{G}_2} \neq \emptyset$, i.e., there exists $h \in \mathcal{H}_0$ such that $(d_{\mu_{\langle h \rangle}}, \mu_{\langle h \rangle}) \in \mathcal{G}_2(\mathcal{D})$.*

Proposition 1 (a special case of [4, Theorem 6.55])**.** *Let $h \in \mathcal{H}_{\mathcal{G}_2}$. Then the closure of $\{\mathcal{E}(h \circ F_{\ell,w}, h \circ F_{\ell,w})^{-1/2} h \circ F_{\ell,w}\}_{w \in \bigcup_{n=0}^{\infty} S_\ell^n}$ in norm in $(\mathcal{F}/\mathbb{R}\mathbf{1}_K, \mathcal{E})$ is a compact subset of $\mathcal{H}_{\mathcal{G}_2}$, where $F_{\ell,w} := f_{\ell,w_1} \circ \cdots \circ f_{\ell,w_n}|_K$ for $n \in \mathbb{N} \cup \{0\}$ and $w = w_1 \dots w_n \in S_\ell^n$.*

Theorem 5 is obtained by combining Theorem 6, Proposition 1 and the following proposition.

Proposition 2 ([5]). *Let $h_0 \in \mathcal{F}$ be $\mathcal{E}$-harmonic on $K \setminus ([0,1] \times \{0,1\})$ and satisfy $h_0|_{[0,1]\times\{j\}} = j$ for $j \in \{0,1\}$. Then $\max_{K\cap([0,1]\times[0,\ell^{-1}])} h_0 < \ell^{-1}$. Moreover, if $h \in \mathcal{F}$ is $\mathcal{E}$-harmonic on $K \setminus V_0$ and $h|_{[0,1]\times\{0\}} = 0$, then $d_{\mu_{\langle h\rangle}}(x,y) = 0$ for any $x, y \in [0,1] \times \{0\}$.*

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A limit law for the frequent points of a random walk on a regular tree

Marek Biskup

(joint work with Oren Louidor)

Given a random-walk path of a finite length, a frequent point is that the path visits most often. Writing $X = (X_0, X_1, \dots)$ for the path of the simple random walk on $\mathbb{Z}^d$ and using $\ell_n(x) := \sum_{k=0}^{n} 1_{\{X_k = x\}}$ to denote the number of times the path $(X_0, \dots, X_n)$ is at x, we are particularly interested in the random variable $M_n := \max_{x\in\mathbb{Z}^d} \ell_n(x)$ marking the time spent at (any) frequent point.

The study of frequent points of simple random walks on $\mathbb{Z}^d$ was initiated in 1960 by Erdős and Taylor [7]. In $d \geq 3$ they showed that $M_n/\log n$ tends to a d-dependent constant but in $d = 2$ they were only able to bound M_n between two multiples of $(\log n)^2$, conjecturing their upper bound to be sharp. A proof of this conjecture arrived full 40 years later in the celebrated work of Dembo, Peres, Rosen and Zeitouni [6] thanks to which we now know that, in two spatial dimensions, $M_n \sim \frac{4}{\pi}(\log n)^2$ in probability as $n \to \infty$.

Natural follow up questions arise concerning subleading orders and possibly even a (non-degenerate) limit law for suitably centered and scaled M_n or the location of the frequent point(s) and the points that are close runner-ups of these. These are expected to fall under the umbrella of results already known for other logarithmically correlated process (of which the random walk local time ℓ_n is an example): $\sqrt{M_n} - [(2/\sqrt{\pi}) \log n - (1/\sqrt{\pi}) \log\log n]$ should have asymptotically a randomly shifted Gumbel law with the random shift being a total mass of a random measure that governs the distribution of the position of the frequent point.

While the purported random measure has been constructed by Jego [8], a proof of the conjectured behavior for the actual random walk remains elusive. In the work reported here we take up a closely related problem of the random walk on a regular tree that shares the basic features of the two-dimensional simple random walk while being more amenable to analysis due to a self-similar setting.

Given a natural $b \geq 2$, we write $\mathbb{T}_n$ for a rooted b-ary tree of depth n and $\mathbb{L}_n$ for its leaves. Let $X = \{X_t \colon t \geq 0\}$ be a continuous time random walk with a unit jump rate across each edge. Let $\ell_t(x) := \int_0^t 1_{\{X_s = x\}} \mathrm{d}s$ be the time spent at x by the walk up to its actual time t and, writing ϱ for the root of $\mathbb{T}_n$, let $\tau_\varrho := \{t \geq 0 \colon X_t = \varrho\}$ be the actual time of first visit to the root. Motivated by the observation that the leaves of a 4-ary tree of depth n can be identified with a box of side 2^n inside $\mathbb{Z}^2$, and the first exit time of the walk from this box is analogous to the walk on the tree hitting the root, our first result is then:

Theorem 1. *For any $x_n \in \mathbb{L}_n$ and all $u \in \mathbb{R}$,*

$$(1) \quad P^{x_n}\left(\max_{x \in \mathbb{L}_n} \sqrt{\ell_{\tau_\varrho}(x)} \leq \sqrt{\log b}\, n - \frac{1}{\sqrt{\log b}} \log n + u\right) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\big(\mathrm{e}^{-\mathcal{Z} e^{-2u\sqrt{\log b}}}\big),$$

where $\mathcal{Z}$ is an a.s.-positive and finite random variable. In particular,

$$(2) \qquad \frac{1}{n}\left(\max_{x \in \mathbb{L}_n} \ell_{\tau_\varrho}(x) - \big(n^2 \log b - 2n \log n\big)\right) \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}} \log \mathcal{Z} + G,$$

where G is a normalized Gumbel random variable independent of $\mathcal{Z}$.

Most of the proof of this theorem is carried out in a somewhat different setting; namely, studying the maximal time spent at any leaf vertex for the random walk started, and parametrized by the time spent, at the root. The result is then quite analogous except that $\mathcal{Z}$ above gets replaced by a random variable $Z(t)$, where t is the time spent at the root. This $Z(t)$ is allowed to be degenerate to 0; the atom at zero represents the asymptotic probability that the walk never reaches the leaves. The proofs work for t that may even grow with n — and, for t in excess of a constant times $n \log n$, the conclusions subsume earlier results of Abe [1].

The connection between the two settings is supplied by the observation that, in the limit $t \downarrow 0$, conditioning on the walk started at ϱ on hitting the leaves is tantamount to there being just one excursion from the root before hitting the leaves. This initial excursion from ϱ to $\mathbb{L}_n$ deposits no local time on the leaves and, thanks to the symmetries of $\mathbb{T}_n$, the (uniformly random) point where the walk first hits $\mathbb{L}_n$ is immaterial for the maximal time spent at $\mathbb{L}_n$ thereafter. In short, in the limit $t \downarrow 0$ the problem for the walk started at the root reduces to that discussed in the theorem above.

Using these arguments rigorously requires uniform control of the tightness under the conditioning on hitting the leaves in the limit of small t. We then obtain:

Theorem 2. *There exists a constant $c_\star \in (0, \infty)$ depending only on b such that the random variable $\mathcal{Z}$ from Theorem 1 satisfies*

$$(3) \qquad c_\star\, b^{-2n} \sum_{x \in \mathbb{L}_n} \Big(n\sqrt{\log b} - \sqrt{\ell_{\tau_\varrho}(x)}\Big)_+ \ell_{\tau_\varrho}(x)^{1/4}\, \mathrm{e}^{2\sqrt{\log b}\sqrt{\ell_{\tau_\varrho}(x)}} \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}} \mathcal{Z}$$

The object on the left is analogous to so called derivative martingale (although this name is not really justified in the present context) that appeared in the study of other log-correlated processes such as the two-dimensional Gaussian Free Field; see Bramson, Ding and Zeitouni [5] and the present authors [2, 3]. The underlying random measure then just weighs $x \in \mathbb{L}_n$ by the term under the sum.

Further results on the random walk problem include a cross-over of the random walk behavior to that of the Gaussian Free Field on $\mathbb{T}_n$ in the limit $t \to \infty$. This, along with all detailed proofs, can be found in [4].

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Object data analysis – statistics on metric spaces

Thomas Hotz

Complex data arise in many situations, for instance as shapes of objects, as functional data, or as phylogenetic trees. The data are then points on a manifold or an orbifold, or, more generally, on a metric measure space. To analyse them statistics has to be reinvented: What is the mean in a metric measure space? Does it fulfill a law of large number? What about the central limit theorem? We give partial answers to these questions.

Gromov-Wasserstein Distances, Distance Distributions and Inverse Problems for Metric Measure Spaces

Tom Needham

(joint work with Facundo Mémoli)

Classical optimal transport provides a way to compare probability measures defined over the same metric space via the Wasserstein distance. Applications in computer vision, shape analysis and machine learning frequently require one to compare probability measures defined on different metric spaces. To make this precise, we define a *metric measure space (mm-space)* to be a triple $\mathcal{X} = (X, d_X, \mu_X)$ with (X, d_X) a compact metric space and μ_X a fully supported Borel probability measure on X. We then measure the distance between mm-spaces via the *Gromov-Wasserstein (GW) p-distance* [7, 8],

$$d_{\mathrm{GW},p}(\mathcal{X},\mathcal{Y}) :=$$
$$\inf_{\pi \in \mathcal{C}(\mu_X,\mu_Y)} \left(\iint_{X\times Y\times X\times Y} |d_X(x,x') - d_Y(y,y')|^p \, \pi(dxdy)\pi(dx'dy') \right)^{\frac{1}{p}}$$

for $p \in [1,\infty)$, where $\mathcal{C}(\mu_X,\mu_Y)$ is the set of *couplings* of μ_X and μ_Y; that is, probability measures π on $X \times Y$ whose marginals are μ_X and μ_Y, respectively. A coupling π realizing the infimum can be understood intuitively as a "soft-matching" between structurally similar points of the mm-spaces. The first part of the talk summarized applications of the GW framework to machine learning applications, including to computing barycenters of collections of mm-spaces [2], clustering networks [3], transferring segmentations between objects [4] and hypergraph simplification [5].

For finite mm-spaces, computation of $d_{\mathrm{GW},p}$ is a non-convex quadratic programming problem, and such problems are generally intractable. One can estimate $d_{\mathrm{GW},p}$ in applied settings (for finite mm-spaces) via gradient descent—this is the approach taken for the applications described above. However, there is little theory in the literature on the quality of such approximations. It is therefore desirable to have efficiently computable lower bounds on the distance. Examples of such lower bounds can be formulated in terms of distance distributions. To a mm-space $\mathcal{X}$, one associates the *local distance distribution*

$$h_{\mathcal{X}} : X \times \mathbb{R} \to \mathbb{R} : (x,r) \mapsto \mu_X\left(\overline{B_X(x,r)}\right),$$

where $B_X(x,r)$ is the metric ball in X, and the *global distance distribution*

$$H_{\mathcal{X}} : \mathbb{R} \to \mathbb{R} : r \mapsto \int_X h_{\mathcal{X}}(x,r)\mu_X(dx).$$

We then define associated distances L_h and L_H on the space of mm-spaces by the formulas

$$L_h(\mathcal{X},\mathcal{Y}) = \inf_{\pi \in \mathcal{C}(\mu_X,\mu_Y)} \int_{X\times Y} \left(\int_{\mathbb{R}} |h_{\mathcal{X}}(x,r) - h_{\mathcal{Y}}(y,r)| \, dr \right) \pi(dx \times dy)$$

and

$$L_H(\mathcal{X},\mathcal{Y}) = \int_{\mathbb{R}} |H_{\mathcal{X}}(r) - H_{\mathcal{Y}}(r)| \, dr.$$

These pseudometrics are polynomial-time computable for finite spaces and give the desired lower bounds, according to the following result (see [6, Proposition 3] and [8, Section 6]).

Proposition 1. *For mm-spaces $\mathcal{X}$ and $\mathcal{Y}$, we have*

$$L_H(\mathcal{X},\mathcal{Y}) \le L_h(\mathcal{X},\mathcal{Y}) \le d_{\mathrm{GW},1}(\mathcal{X},\mathcal{Y}).$$

A natural question is whether these pseudometrics are true metrics on some subclass of isomorphism classes of mm-spaces. That is, given a class $\mathbf{C}$ of mm-spaces, one would like to know whether, for all $\mathcal{X}, \mathcal{Y} \in \mathbf{C}$, $L_H(\mathcal{X}, \mathcal{Y}) = 0$ or (more weakly) $L_h(\mathcal{X}, \mathcal{Y}) = 0$ imply that $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic as mm-spaces. The second part of the talk described some results in this direction for a few natural classes of mm-spaces. We now describe a few of these results, coming from [6], and pose some related open questions.

The first result gives a counterexample to the *Curve Histogram Conjecture* from [1]. In the following, we consider a plane curve X as a mm-space $\mathcal{X}$ by endowing it with extrinsic Euclidean distance and normalized arclength measure.

Theorem 1. *There exist smooth, convex plane curves $\mathcal{X}$ and $\mathcal{Y}$ such that $L_H(\mathcal{X}, \mathcal{Y}) = 0$ but X and Y do not differ by a rigid motion.*

The theorem is proved by an example, where a curve Y is constructed from a curve X via a certain cut-and-paste procedure. These resulting curves are quite different in Gromov–Wasserstein distance, leading one to ask whether curves can be distinguished *locally* by L_H.

Question 1. *If $\mathcal{X}_t$ is a one-parameter family of smooth plane curves such that $L_H(\mathcal{X}_s, \mathcal{X}_t) = 0$ for all s, t, must it be the case that each pair of curves in the family differs by a rigid motion?*

We also described a sphere rigidity result.

Theorem 2.

- *Let $\mathcal{X}$ be a closed hypersurface in $\mathbb{R}^{d+1}$ endowed with extrinsic Euclidean distance and d-dimensional Hausdorff measure. Assume that X has all principle curvatures bounded above by 1. If $L_H(\mathcal{X}, \mathbb{S}^d) = 0$, where $\mathbb{S}^d$ is the unit sphere, considered as a hypersurface, then $\mathcal{X}$ is isomorphic to $\mathbb{S}^d$.*
- *Let $\mathcal{X}$ be a closed Riemannian d-manifold endowed with geodesic distance and normalized Riemannian measure. Assume that X has all Ricci curvatures bounded below by $d-1$. If $L_H(\mathcal{X}, \mathbb{S}^d) = 0$, where $\mathbb{S}^d$ is the unit sphere, considered as a Riemannian manifold, then $\mathcal{X}$ is isomorphic to $\mathbb{S}^d$.*

In either case, the curvature assumptions can be removed when $d \leq 2$. On the other hand, the curvature assumptions in the theorem are artifacts of the proof strategies and we do not have counterexamples in higher dimensions.

Question 2. *Can the curvature restrictions in Theorem 2 be removed?*

Finally, we considered the case of metric graphs. Roughly, a *metric graph* is a 1-dimensional stratified space X obtained by gluing together finitely many closed intervals along their endpoints. We endow X with shortest path distance and normalized uniform measure to consider it as a mm-space. We have the following local result.

Theorem 3. *For every contractible metric graph $\mathcal{X}$, there exists $\epsilon_{\mathcal{X}} > 0$ such that if another contractible metric graph $\mathcal{Y}$ is $\epsilon_{\mathcal{X}}$-close to $\mathcal{X}$ in Gromov-Hausdorff distance and $L_h(\mathcal{X}, \mathcal{Y}) = 0$, then $\mathcal{Y}$ is isomorphic to $\mathcal{X}$.*

Contractibility is used in the proof of the theorem, but we do not have a counterexample in the general case.

Question 3. *Can the contractibility assumption be removed from Theorem 3?*

We end with one more open question.

Question 4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric graphs with $L_H(\mathcal{X},\mathcal{Y}) = 0$. Must it be the case that $\mathcal{X}$ and $\mathcal{Y}$ are homotopy equivalent?*

We conjecture that the answer is "yes", based on numerical and heuristic arguments. On the other hand, a solution is nontrivial, as it is possible to find homotopy equivalent, but *non*-isomorphic metric graphs $\mathcal{X}$ and $\mathcal{Y}$ such that $L_H(\mathcal{X},\mathcal{Y}) = 0$.

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The Gromov-Hausdorff distance between spheres

Facundo Mémoli

(joint work with Sunhyuk Lim, Zane Smith)

The Gromov-Hausdorff distance is a fundamental tool in Riemannian geometry, due the topology it generates, and also in applied geometry and topological data analysis, as a metric for expressing the stability of the persistent homology of geometric data (e.g. via the Vietoris-Rips filtration). Whereas it is often easy to estimate the value of the distance between two given metric spaces, its precise value is rarely easy to determine. Some of the best (computable) estimates follow from considerations actually related to both the stability of persistent homology and to Gromov's filling radius [3]. However, these turn out to be non-sharp.

Some results which permit calculating the precise value of the Gromov-Hausdorff between certain pairs of spheres $(\mathbb{S}^n, d_n)$ and $(\mathbb{S}^m, d_m)$ (endowed with their geodesic distance) have recently been obtained. These results involve lower bounds which arise from a certain version of the Borsuk-Ulam theorem that is applicable to discontinuous maps, and also matching upper bounds which are induced from specialized constructions of "correspondences" between spheres [2].

The variant of the Borsuk-Ulam theorem mentioned above is due to Dubins and Schwarz [1]. Their result states that, whenever $n > m$, for any odd map $f : \mathbb{S}^n \to \mathbb{S}^m$, the distortion of this (necessarily discontinuous) map satisfies

$$\operatorname{dis}(f) := \sup_{x,x' \in \mathbb{S}^n} |d_n(x, x') - d_m(f(x), f(x'))| \geq \arccos\left(\frac{-1}{m+1}\right) =: \zeta_m.$$

The number ζ_m appearing above is the (geodesic) diameter of the vertex set of the regular simplex inscribed in $\mathbb{S}^m$. For example $\zeta_1 = \frac{2\pi}{3}$, the (angle) between any two vertices of an equilateral triangle inscribed in the unit circle.

In [2] it is proved that given any correspondence R between $\mathbb{S}^n$ and $\mathbb{S}^m$ one can obtain an odd function $f_R : \mathbb{S}^n \to \mathbb{S}^m$ such that $\operatorname{graph}(f_R) \subseteq R$ and its distortion satisfies $\operatorname{dis}(f_R) \leq \operatorname{dis}(R)$. This then leads to

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{\zeta_m}{2} \text{ for all } n \geq m.$$

This in particular means that $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{\pi}{4}$ for all $n > m$.

For $m, n \in \{1, 2, 3\}$ with $m < n$ we construct (an posteriori optimal) correspondence $R_{m,n}$ such that $\operatorname{dis}(R_{m,n}) = \zeta_m$ thus establishing:

$$d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^2) = d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^3) = \frac{\pi}{3} \text{ and } d_{\mathrm{GH}}(\mathbb{S}^2, \mathbb{S}^3) = \frac{\zeta_2}{2}.$$

The construction of $R_{1,3}$ uses ideas related to the Hopf map.

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Quantitative Homogenization of the Invariant Measure for Nondivergence Form Elliptic Equations

Jessica Lin

(joint work with Scott Armstrong, Benjamin Fehrman)

We study the stochastic homogenization of nondivergence form, uniformly elliptic equations of the form

$$-\operatorname{tr}(\mathbf{A}(x)D^2u) = 0, \tag{1}$$

and its parabolic counterpart,

$$\partial_t u - \mathrm{tr}(\mathbf{A}(x)D^2u) = 0, \tag{2}$$

where $\mathrm{tr}(M)$ denotes the trace of a $d\times d$ matrix M, and D^2u is the spatial Hessian of u. The coefficient field $\mathbf{A}(x)$ is a $\mathbb{Z}^d$–stationary random field taking values in the symmetric matrices satisfying, for given constants $0 < \lambda \le \Lambda < \infty$, the uniform ellipticity condition

$$\lambda I_d \le \mathbf{A}(x) \le \Lambda I_d, \quad x \in \mathbb{R}^d\,,$$

and, for some exponent $\alpha_0 \in (0,1]$ and constant $K_0 \in [1,\infty)$, the uniform Hölder continuity condition

$$\big[\mathbf{A}\big]_{C^{0,\alpha_0}(\mathbb{R}^d)} := \sup_{x,y\in\mathbb{R}^d,\, x\neq y} \frac{|\mathbf{A}(x)-\mathbf{A}(y)|}{|x-y|^{\alpha_0}} \le K_0.$$

The assumptions on the underlying probability measure $\mathbb{P}$ are twofold: it is stationary with respect to $\mathbb{Z}^d$ translations, and has a unit range of dependence.

The equations (1) and (2), under the above assumptions, arise in the study of diffusion processes in $\mathbb{R}^d$ in a random environment. Consider the Markov diffusion

$$dX_t = \sigma(X_t)\, dW_t,$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ is a given Holder continuous function satisfying a non-degeneracy condition $\lambda \le \frac{1}{2}\sigma\sigma^T \le \Lambda$. For $\mathbf{A}(x) := \frac{1}{2}\sigma\sigma^T$, it turns out that the infinitesimal generator of the Markov process $\{X_t\}$ is precisely the operator $\varphi \mapsto \mathrm{tr}(\mathbf{A}(x)D^2\varphi)$ in (1), and the parabolic equation (2) is the backward Kolmogorov equation. If σ is itself a stationary random field, then the study of its large-scale behavior is essentially equivalent to that of the large-scale behavior of solutions of (2), which lies in the realm of stochastic homogenization.

The classical qualitative result of stochastic homogenization in the nondivergence form setting was proved more than forty years ago [6, 7]. It roughly states that, on an event of $\mathbb{P}$–probability one, solutions of (1) and (2) converge in the large-scale limit to solutions of the homogenized equations, in which the coefficient $\mathbf{A}(x)$ is replaced by a constant, deterministic matrix $\bar{\mathbf{A}}$. This result can be equivalently formulated in terms of the corresponding Markov process $\{X_t\}$ as the statement that, on an event of $\mathbb{P}$–probability one, the rescaled process $X_t^\varepsilon := \varepsilon X_{t/\varepsilon^2}$ converges in law as $\varepsilon \to 0$ to a Brownian motion with covariance $(2\bar{\mathbf{A}})^{\frac{1}{2}}$.

Quantitative homogenization has been studied from a variety of perspectives, in both continuum and discrete models, using PDE techniques [3, 1, 2, 7, 8], probability techniques [4, 5], and some combination thereof. We present here a new approach, showing that quantitative homogenization can be proved by considering the unique stationary *invariant measure* μ. The density m of $\mu = m(x)\,dx$ is the unique $\mathbb{Z}^d$–stationary random field which takes values in $(0,\infty)$, has mean $\langle m\rangle = 1$, and solves the forward Kolmogorov equation (independent of the time variable), which formally is the "doubly divergence form" equation

$$-\sum_{i,j=1}^{d} \partial_{x_i}\partial_{x_j}\big(\mathbf{A}_{ij}(x)m(x)\big) = 0.$$

Under a qualitative ergodicity assumption, the effective matrix $\bar{\mathbf{A}}$ can be identified as the ensemble average of $m\mathbf{A}$, and the ergodic theorem then implies that

$$(3) \qquad m\big(\tfrac{\cdot}{\varepsilon}\big) \rightharpoonup 1 \quad \text{and} \quad m\big(\tfrac{\cdot}{\varepsilon}\big)\mathbf{A}\big(\tfrac{\cdot}{\varepsilon}\big) \rightharpoonup \bar{\mathbf{A}} \quad \text{as } \varepsilon \to 0, \text{ weakly in } L^1_{\mathrm{loc}}, \ \mathbb{P}\text{–a.s.}$$

This pair of weak limits can then be shown to imply homogenization by purely deterministic, PDE arguments. Alternatively, one can use (3) to prove the quenched CLT for the process X_t^ε directly by applying the martingale CLT and then observing that, thanks to (3), the quadratic variation of this process is averaging to $\bar{\mathbf{A}}$.

In order to analyze m, we provide a *quenched construction* of m via the parabolic Green function $P : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, solving

$$\begin{cases} \partial_t P(\cdot,\cdot,y) - \mathrm{tr}(\mathbf{A}D^2 P(\cdot,\cdot,y)) = 0 & \text{in } (0,\infty)\times\mathbb{R}^d, \\ P(0,\cdot,y) = \delta(\cdot - y) & \text{on } \mathbb{R}^d, \end{cases}$$

for every $y \in \mathbb{R}^d$. Moreover, we consider the homogenized Parabolic Green function $\overline{P}$ solving

$$\begin{cases} \partial_t \overline{P}(\cdot,\cdot - y) - \mathrm{tr}(\bar{\mathbf{A}}D^2 \overline{P}(\cdot,\cdot - y)) = 0 & \text{in } (0,\infty)\times\mathbb{R}^d, \\ \overline{P}(0,\cdot,y) = \delta(\cdot - y) & \text{on } \mathbb{R}^d. \end{cases}$$

The "classical" constructions of m, see for example [6, 7, 9], rely on analyzing the *environment from the point of view of the particle*; instead of performing an analysis on the "physical space" $\mathbb{R}^d$, one studies the trajectories of the diffusion $\tau_{X_t}\mathbf{A}$ lying in the probability space Ω.

In this work, we aim to analyze m in a quenched ($\mathbb{P}$-a.e.) fashion, and thus we first construct m via a quenched approach. We prove that for each $y \in \mathbb{R}^d$, the limit

$$m(y) := \lim_{t\to\infty} \int P(t,x,y)\,dx$$

exists $\mathbb{P}$-almost surely, and we prove a quantitative homogenization result which quantifies the convergence (or homogenization) of the parabolic Green function

$$(4) \qquad P(t,x,y) \xrightarrow{t\to\infty} m(y)\overline{P}(t,x-y).$$

It is easy to see that $m(y)$ defined above is in fact the same unique, ergodic invariant measure constructed in [6, 7, 9]. By quantifying the convergence in (4), we are then able to obtain quantitative estimates on the convergence in (3).

Our results have several other consequences for the model at hand. The first is that our quantification of (4) implies a heat kernel (or Nash-Aronson) bound on the parabolic Green function P on large scales. Indeed, we show that for t sufficiently large (in a very precise quantitative sense),

$$cm(y)t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{ct}\right) \leq P(t,x,y) \leq Cm(y)t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{Ct}\right),$$

with $m \in L^\infty_{loc}$. In general, bounds of this type are *not true* for equations in nondivergence form.

Finally, we also obtain a quantitative ergodicity result for the environmental process. We let $\mathbf{P}$ denote the probability measure associated to the Markov diffusion $\{X_t\}$. It is known that

$$\frac{1}{t}\int_0^t \mathbf{A}(X_s)\,ds \to \bar{\mathbf{A}}, \quad \text{as } t \to \infty, \quad \mathbb{P} \otimes \mathbf{P}\text{–a.s.},$$

and our results imply a quantitative estimate on the above convergence.

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Globally Lipschitz transport maps

Max Fathi

(joint work with Dan Mikulincer, Yair Shenfeld)

There are many concentration and functional inequalities for probability measures that can be proved using Caffarelli's contraction theorem [1], which states that quadratic optimal transport maps from a Gaussian measure onto a 1-uniformly log-concave measure is 1-Lipschitz, in any dimension. A natural question is to find other settings where the map is globally Lipschitz, and some further results have been obtained, for example [2].

One setting where it is reasonable to expect lipschitz maps is the following:

Conjecture 1. *Let γ be the standard Gaussian measure on $\mathbb{R}^n$, and μ be a probability measure of the form* $\exp(f)d\gamma$*, with f Lipschitz. Is the quadratic optimal transport map from γ onto μ globally Lipschitz, with a Lipschitz norm that depends only on $||f||_{lip}$, and not on n?*

This conjecture is motivated by the fact that if we linearize around small perturbations of γ the Monge-Ampère equation that the optimal transport map solves, then we obtain a PDE whose solution does satisfy the analogous bound.

Kim and Milman [4] gave an alternative construction of transport maps, that are different from OT maps, but which nonetheless satisfy the same Lipschitz property as the OT map in the setting of Caffarelli's theorem. Using this construction, we obtain the following:

Theorem 1. *In the setting of the above conjecture, there exists a (not necessarily optimal) transport map that is* $\exp(c||f||_{lip}^2)$*-Lipschitz, with c a fixed numerical constant.*

In particular, the estimate is dimension-free. The square-exponential dependence on $||f||_{lip}$ can be checked to be optimal on some examples.

The result extends to non-Gaussian source measures, with density of the form $\exp(-V)$ with V uniformly convex with bounded third derivatives. It also extends to transport maps on manifolds, under conditions on the Ricci and Riemann curvature tensors, as well as their derivatives. For example, for the sphere, we obtain the following:

Theorem 2. *Let σ be the normalized volume measure on $\sqrt{n}\mathbb{S}^{n-1}$ (that is a sphere of radius $\sqrt{n}$ in $\mathbb{R}^n$). Let μ be a probability measure of the form* $\exp(f)d\sigma$*, with f L-Lipschitz. Then there is a transport map form σ onto μ that is $C(L)$-Lipschitz, for a constant that only depends on L, and not on the dimension.*

The proof studies the heat flow transport map of [4] in a Riemannian setting, combined with regularity estimates for the heat flow proved in [3].

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Generalized Ricci flow and functional inequalities

Eva Kopfer

(joint work with Jeffrey Streets)

In the analysis of Ricci flow, the classic Bochner formula for gradients plays a key role. This basic formula underlies gradient estimates for solutions to the heat equation along Ricci flow, and yields functional inequalities such as Wasserstein distance monotonicity [6], and universal Poincaré and log-Sobolev inequalities [4]. Furthermore, these functional inequalities can be used to characterize supersolutions to Ricci flow [3, 6]. Later, through a broad extension of the Bochner formula to functions on path space, Haslhofer-Naber gave a characterization of solutions to Ricci flow [3] in terms of universal functional inequalities. Here, we extend this circle of ideas to the setting of *generalized Ricci flow* [5]. A one-parameter family of metrics and two-forms (g_t, b_t) is a solution of generalized Ricci flow [8] if

$$\partial_t g = -2\mathrm{Rc} + \tfrac{1}{2}H^2, \qquad \partial_t b = -d_g^* H, \qquad H = H_0 + db,$$

where $dH_0 = 0$ and

$$H^2(X,Y) = \langle X \lrcorner H, Y \lrcorner H \rangle.$$

It is natural to express this equation using the curvature of the unique metric connection with torsion H, referred to as a *Bismut connection.* If we let D denote the Levi-Civita connection, the relevant Bismut connection is then

$$\nabla := D + \tfrac{1}{2}g^{-1}H, \qquad \mathrm{Rc}^{\nabla} = \mathrm{Rc} - \tfrac{1}{4}H^2 - \tfrac{1}{2}d_g^* H.$$

It follows that the generalized Ricci flow can be expressed as

$$\partial_t (g - b) = -2\mathrm{Rc}^{\nabla},$$

where Rc^{∇} is the Ricci tensor of the Bismut connection. The flow equation arises in the two-dimensional σ-model [7].

We will show a generalization of the infinite-dimensional Bochner formula for the Malliavin gradient on path space along Ricci flow as in [1, 3]. The starting point of these constructions is to define a connection on the frame bundle of the spacetime associated to a time-dependent Riemannian manifold, originally employed in Hamilton's proof of the Harnack inequality for Ricci flow [2]. It turns out that it is possible to incorporate the two-form potential b_t into this construction in a way that fits very naturally with the generalized Ricci flow equation. For a family (M^n, g_t, b_t) defined for $t \in I$, we define a connection ∇ on $\pi^* TM \to M \times I$ which extends the given action of ∇ via

$$\nabla_t Y = \partial_t Y + \tfrac{1}{2}\partial_t (g_t - b_t)(Y, \cdot)^{\sharp_{g_t}}.$$

This operator admits a key Bochner formula, which is central to our constructions. In particular, given $(g_t, H_t = H_0 + db_t)$ a general one-parameter family, and u a solution of the time-dependent heat equation, one has that

$$\nabla_t \mathrm{grad}_{g_t} u = \Delta \mathrm{grad}_{g_t} u - \left(\mathrm{Rc}^{\nabla} + \tfrac{1}{2}\partial_t (g_t - b_t)\right)\left(\mathrm{grad}_{g_t} u, \cdot\right)^{\sharp_{g_t}}.$$

Thus, along a solution to generalized Ricci flow, the gradient of a solution to the heat equation itself satisfies a pure heat equation using the adapted derivative ∇. The main goal is to give an extension of the Bochner formula above to path space. We use the connection ∇ on spacetime defined above together with the antidevelopment map to give the Eels-Elworthy-Malliavin construction of Brownian motion in this setting. This in turn gives a notion of parallel gradient for martingales. We then prove a formula on the evolution of parallel gradients of martingales which generalizes the Bochner identity above:

$$d(\nabla_\sigma^\perp F_\tau) = \langle \nabla_\tau^\perp \nabla_\sigma^\perp F_\tau,\, dW_\tau \rangle + (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_\tau(\nabla_\tau^\perp F_\tau)\, d\tau.$$

This is a generalization of the Bochner formula described above, which occurs as the case where F is a one-point cylinder function.

The path-space Bochner formula above can be used to give many equivalent characterizations of generalized Ricci flow, e.g. gradient estimates:

$$|\nabla_x \mathbb{E}[F]|^2 \leq \mathbb{E}[|\nabla_0^\perp F|^2].$$

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Optimal Transport for Point Processes

Ronan Herry

(joint work with Lorenzo Dello Schiavo, Kohei Suzuki)

On a metric measure space, (X, m, d), a Poisson random measure with intensity m is a random measure characterized by the two following facts:

(i) for all Borel sets $B \subset X$, $\eta(B)$ has a Poisson distribution with mean $\eta(B)$,

(ii) for all disjoint Borel sets $B_1, \ldots, B_l \subset X$, $\eta(B_1), \ldots, \eta(B_l)$ are independent.

As for Gaussian processes, Poisson processes are ubiquitous in probability theory. Among other common properties, they share the existence of an orthogonal systems of "chaoses". Consequently, they admit a "differential calculus", known as the Malliavin calculus, completely characterised by their probabilistic properties. In particular, it is possible to define a gradient operator D, a divergence operator $\mathsf{D}^\star$, and a Ornstein–Uhlenbeck semi-group (P_t), that plays the role of the heat semi-group. Due to the probabilistic nature of these objects, we expect the geometric and functional analytic results one can deduce from this differential calculus to be independent of properties of the underlying space. However, due to the non-local nature of Poisson processes making rigorous this analogy is particularly intricate. For instance, both the Gaussian measures and the Poisson measures satisfy an exact commutation relation à la Bakry–Emery:

$$\mathsf{D}\mathsf{P}_t = \mathrm{e}^{-t}\mathsf{P}_t\mathsf{D}.$$

In a diffusive setting, this commutation is known to characterise synthetic Ricci curvature bound from below, and entail numerous consequence from the point of view of geometrico-analytic inequalities on the underlying space. Since Poisson processes are not diffusive, drawing functional analytic consequences of this commutation is much more difficult in the Poisson setting than in the Gaussian setting (see [2] for results in that direction).

On a suitable class of metric measure spaces, the theory of Ricci curvature bounds à la Bakry–Emery is equivalent to a convexity property of the relative entropy at the level of the Wasserstein transport distance. In this work, we construct a variational distance on the space of discrete measures, inspired by a non-local formulation of the quadratic optimal transport à la Benamou–Brenier [1]. Such ideas go back to [3, 4, 5] in the setting of Markov chains on finite state space and jump processes on $\mathbb{R}^d$. We stress that due to the lack of canonical distance in our setting no direct formulation of optimal transport is available. We show that this transport distance shares the same properties as the usual transport distance over a Riemannian manifold with Ricci curvature bounded from below. For instance, we obtain that

- the Ornstein–Uhlenbeck semi-group is the gradient flow of the relative entropy;
- the Poisson space has a Ricci curvature, in the entropic sense, bounded below by 1;
- our distance satisfies an HWI inequality.

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Oscillations of BV measures on unbounded nested fractals

Patricia Alonso Ruiz
(joint work with Fabrice Baudoin)

Bounded variation (BV) measures play an important role in the characterization of the perimeter of sets: In the Euclidean setting, the BV space is usually defined as

$$\mathrm{BV}(\mathbb{R}^n)=\{f\in L^1(\mathbb{R}^n)\colon \|Df\|(\mathbb{R}^n)<\infty\},$$

where

$$\|Df\|(\mathbb{R}^n)=\sup\Big\{\int_{\mathbb{R}^n} f\mathrm{div}\varphi\,dx\colon \varphi\in C_c^\infty(\mathbb{R}^n),\ \|\varphi\|_\infty\le 1\Big\}. \tag{1}$$

The latter is called the *total variation* of f. Replacing $\mathbb{R}^n$ by any Borel subset $B\subset\mathbb{R}^n$, the BV-measure associated with $f\in \mathrm{BV}(X)$ is given by $\|Df\|(\cdot)$. In the 1920s, Caccioppoli observed that the perimeter of a Borel set $E\subset\mathbb{R}^n$ coincides with the total variation of $\mathbf{1}_B$, i.e.

$$\mathrm{Perimeter}(E)=\|\mathbf{1}_E\|(\mathbb{R}^n).$$

Later on, BV functions were characterized by Korevaar and Schoen in [1] and by Miranda, Pallara, Paronetto and Preunkert [2] respectively as

$$\|D_f\|(\mathbb{R}^n)\simeq\lim_{r\to 0^+}\frac{1}{r}\int_X\int_{B(x,r)}\frac{|f(x)-f(y)|}{r^n}dx \tag{2}$$

$$\|D_f\|(\mathbb{R}^n)=\frac{\sqrt{\pi}}{2}\lim_{t\to 0^+}\frac{1}{\sqrt{t}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(x)-f(y)|p_t(x,y)\,dy\,dx, \tag{3}$$

where $p_t(x,y)$ denotes the standard Gaussian heat kernel.

To define BV measures in more general metric measure spaces, several approaches have been developed over the years that mimic (1), (2) or (3) by using the upper gradients of Koskela and Shanmugalingam, Korevaar-Schoen-type functionals, or heat semigroups à la Ledoux. In the context of metric measure spaces satisfying Gaussian heat kernel estimates, both (2) and (3) are known to be equivalent [3]. However, finding an exact characterization of the BV measure associated with a BV function, that is proving the equality in (3) is a challenging task, which in the semigroup approach was achieved for Carnot groups in [4] and only recently certain Riemannian manifolds in [5].

When the underlying space is an unbounded planar nested fractal like the unbounded Sierpinski gasket $\mathrm{SG}^{\langle\infty\rangle}$ one can still write analogue expressions after suitable changes of the exponents involved. In particular, estimates for the corresponding heat kernel were studied by Kumagai in [6].

In the present talk we discover that in the case of $\mathrm{SG}^{\langle\infty\rangle}$ an oscillatory phenomenon will actually prevent BV measures of being uniquely determined. In other words, the limits in (2) and (3) may not exist. More precisely, it is possible to prove that when $E \subset \mathrm{SG}^{\langle\infty\rangle}$ is a finite union of cells as in displayed in Figure 1, there are positive and bounded periodic functions Φ and Ψ independent of E such that

$$\lim_{r\to 0^+} \Psi(-\ln r)\frac{1}{r^{\alpha_1 d_W}} \int_X \int_{B(x,r)} \frac{|\mathbf{1}_E(x)-\mathbf{1}_E(y)|}{r^n} dx = |\partial E| \tag{4}$$

$$\lim_{t\to 0^+} \Phi(-\ln t)\frac{1}{t^{\alpha_1}} |\mathbf{1}_E(x)-\mathbf{1}_E(y)| p_t(x,y)\, d\mu(y), d\mu(x) = |\partial E|. \tag{5}$$

Here, d_H denotes the Hausdorff dimension, d_W the so-called walk dimension and α_1 a suitable critical exponent of $\mathrm{SG}^{\langle\infty\rangle}$, while $|\partial E|$ equals the number of points in the boundary of E.

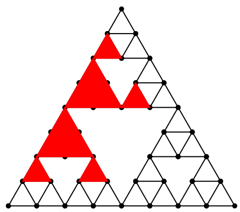

Figure 1. Unions of cells in the unbounded Sierpinski gasket.

While this oscillatory behavior was expected in view of the on-diagonal oscillations of the heat kernel at small scales in these settings, see e.g. [7], the non-uniqueness of the BV measure is not straightforward to obtain. In fact, we can presently prove that the function Ψ in (4) is non-constant by constructing specific sequences whose limits do not coincide. The question is still open for Φ in (5).

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Heat kernel estimates on manifolds with ends with mixed boundary condition

Emily Dautenhahn
(joint work with Laurent Saloff-Coste)

In this short talk, we discussed the results of [1].

The heat kernel is a fundamental object of interest to both probabilists and analysts. In $\mathbb{R}^n$, it is both the transition density of Brownian motion and the fundamental solution of the heat equation. More generally, the abstract theory of Dirichlet spaces enables us to consider heat kernels on a variety of metric measure spaces.

While there is an explicit formula for the heat kernel in $\mathbb{R}^n$, in other spaces the best we can hope for is to obtain "matching" two-sided bounds on the heat kernel, that is, upper and lower bounds of a similar form that differ by constants in certain places. In many cases where such bounds exist, they take the form of two-sided Gaussian bounds: if $p(t,x,y)$ denotes the heat kernel on a metric measure space (X,d,μ), then this space is said to have two-sided Gaussian bounds if there exist constants $c_1, c_2, c_3, c_4 > 0$ such that

$$\frac{c_1}{V(x,\sqrt{t})}\exp\Big(-\frac{d^2(x,y)}{c_2 t}\Big) \le p(t,x,y) \le \frac{c_3}{V(x,\sqrt{t})}\exp\Big(-\frac{d^2(x,y)}{c_4 t}\Big),$$

where $V(x,r)$ denotes the volume of a ball of radius r centered at x (with respect to the measure μ) and $d(x,y)$ denotes the distance between x and y.

In particular, if we fix a point $o \in X$, where the heat kernel on X satisfies bounds of this form, then we have

$$\frac{c_1}{V(o,\sqrt{t})} \le p(t,o,o) \le \frac{c_2}{V(o,\sqrt{t})}.$$

In this talk, we saw an example of a class of spaces where such Gaussian bounds do not hold but we nevertheless obtain two-sided heat kernel estimates. In particular, we considered connected sums of cones in the plane $\mathbb{R}^2$ with Dirichlet (zero) boundary condition. Heat kernel estimates on connected sums of manifolds without boundary condition or with solely Neumann (zero normal derivative) boundary condition can be found in [2]; the contribution of [1] is to allow for Dirichlet condition or a mixture of Dirichlet and Neumann condition. Both of these papers can be considered as partial answers to the following gluing problem:

> Given several "pieces" with heat kernels that satisfy two-sided Gaussian bounds, we glue them together in some fashion. Can we estimate the heat kernel on this new glued object?

An abbreviated version of one of the main theorems of [1] states that given a connected sum of Riemannian manifolds, $M = M_1 \# \cdots \# M_k$, with a subset $\Omega = \Omega_1 \# \cdots \# \Omega_k$ encoding the mixed (or solely Dirichlet) boundary condition, then under the assumptions that (1) the heat kernels of the M_i satisfy two-sided Gaussian bounds, (2) the Ω_i are uniform domains in their closures, and (3) the set of Dirichlet boundary condition is not too wild, then for any $t > 1$ and $o \in \Omega$ fixed in the central gluing set, there exist constants c_1, c_2 such that

$$\frac{c_1}{V_{\min,h}(\sqrt{t})} \leq p(t,o,o) \leq \frac{c_2}{V_{\min,h}(\sqrt{t})}.$$

Here $V_{\min,h}$ is a minimum volume over the ends (evaluated at particular points) that depends upon a global harmonic function h.

In particular, denote by Ω the space formed by gluing three cones in $\mathbb{R}^2$ of apertures $\alpha_1, \alpha_2, \alpha_3$ via compact set K, where we impose Dirichlet condition on all of the boundary. Then for any $o \in K$,

$$c_1 t^{-\left(1+\frac{\pi}{\alpha_{\max}}\right)} \leq p_\Omega(t,o,o) \leq c_2 t^{-\left(1+\frac{\pi}{\alpha_{\max}}\right)},$$

where $\alpha_{\max} := \max\{\alpha_1, \alpha_2, \alpha_3\}$. Hence the heat kernel on Ω is controlled by the *largest* cone; the intuition is that due to the presence of Dirichlet boundary, if we start in the gluing set and want to survive long enough to return to the gluing set, we should stay in the largest cone where the Dirichlet boundary is further away. Note this is faster decay than in $\mathbb{R}^2$ itself, where the heat kernel decays like t^{-1}. Moreover, since the volume of balls in Ω remains roughly the same as the volume of balls in $\mathbb{R}^2$ itself, the above heat kernel estimate is not of the two-sided Gaussian form.

The results of [1] hold for a much larger class of manifolds with ends and give heat kernel estimates for any $t > 0$ and any two points x, y. The proof of these results relies on using ideas for treatment of Dirichlet boundary condition found in [3] and the h-transform technique to generalize the results of [2]. One of the major components of the proof is the construction of a suitable global harmonic function h with certain properties. The construction of h is as in [4], although it is important and requires additional effort to show that h can be estimated by harmonic profiles of the ends.

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Large deviations principles for (some) subRiemannian random walks

Tai Melcher

(joint work with Maria Gordina and Jing Wang)

Among the first large deviations results that one learns is Cramér's theorem, which states that for a random walk on $\mathbb{R}^d$ with iid increments, the probability that the sample average exceeds its mean decays at an exponential rate given by Λ^*, the Fenchel-Legendre transform of the log moment generating function of the increment distribution. In the simplest case that the increments are standard normal random variables, one computes $\Lambda^*(x) = \frac{1}{2}|x|^2$, and finds that the rate is exactly half the square of the distance of x from the expected value of 0.

We prove large deviations results for random walks on stratified nilpotent Lie groups. For such groups there is a collection of vectors generating the Lie algebra, which equips the groups with a natural but degenerate geometry. We consider random walks with increments in only these directions and show that, under certain constraints on the distribution of the increments, a large deviations principle holds with a natural rate function adapted to the subRiemannian geometry of these spaces.

More particularly, we consider a Carnot group G with Lie algebra $\mathfrak{g} = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ generates the Lie algebra. By the Chow–Rashevskii theorem, this bracket generating condition implies that any two points in the group are connected by a "horizontal" path, that is, an absolutely continuous path γ whose Maurer-Cartan form $c_\gamma(t) := (L_{\gamma(t)^{-1}})_*\dot{\gamma}(t)$ lies in $\mathcal{H}$ for a.e. t. We assume that $\mathcal{H}$ comes equipped with an inner product, and thus are able to define the horizontal, or subRiemannian, distance $\rho_{cc}(x, y)$ as the infimum of the length of all the horizontal paths connecting x and y. Note that we assume no other inner product structure on the rest of $\mathfrak{g}$!

For $X_1, X_2, \ldots$ iid random variables in $\mathcal{H}$, consider random walks of the form

$$S_n := \exp(X_1) \ldots \exp(X_n).$$

We prove the following large deviations result for $\{(D_{1/n}S_n)\}_{n=1}^\infty$, where, for $\alpha > 0$, we let $D_\alpha : G \to G$ denote the standard dilation on G adapted to its stratification.

Theorem 1. *Suppose that $\{X_k\}_{k=1}^\infty$ are iid mean 0 random variables in $\mathcal{H}$ and either*

(i) G is step 2 and the distribution of the X_k's is sub-Gaussian, or
(ii) G is step ≥ 3 and the distribution of the X_k's bounded.

Then $\{D_{1/n}S_n\}_{n=1}^\infty$ satisfies a large deviations principle with rate function

$$J(x) := \inf\left\{ \int_0^1 \Lambda^*(c_\gamma(t))\,dt : \gamma \text{ horizontal, } \gamma(0) = e, \gamma(1) = x \right\},$$

where Λ^* is the Fenchel-Legendre transform of the log moment generating function of the X_k's.

To demonstrate the naturalness of this rate function, consider the step 2 case with increments having the standard normal distribution on $\mathcal{H}$. In this case, again $\Lambda^*(x) = \frac{1}{2}|x|^2$, and so the rate function is given by

$$\begin{aligned} J(x) &= \inf\left\{\frac{1}{2}\int_0^1 |\dot\gamma(t)|^2_{\gamma(t)}\,dt : \gamma \text{ horizontal}, \gamma(0)=e, \gamma(1)=x\right\} \\ &= \inf\left\{\frac{1}{2}E(\gamma) : \gamma \text{ horizontal}, \gamma(0)=e, \gamma(1)=x\right\}, \end{aligned}$$

the exact minimum energy to reach x from e in time 1. Using that a horizontal curve is a minimizer of the energy if and only if it is a minimizer of the length, one may further show in this case that

$$J(x) = \frac{1}{2}\rho^2_{cc}(e,x),$$

and we note the comparison with the classical Cramér's theorem.

There are related large deviations results for random walks on Lie groups in [1] and [3], with closely related rate functions in a variational form. In [3], the author studies random walks on general Lie groups, requiring more restrictions on the distributions of the increments including boundedness, not allowing for the normal distribution, for example, which we see results in the most natural rate function. In [1], the authors study random walks on nilpotent Lie groups, but their method of proof does not reveal that the infimum appearing in the rate function is achieved over horizontal paths.

Our method of proof uses classical large deviations tools, including the contraction principle and an exponential approximation argument. However, arguments are complicated, for example, by the resistance of horizontal paths to simple approximations. The result should certainly also hold for groups of step 3 or larger with the weaker requirement of sub-Gaussian distributions; however, the exponential approximation argument (the only part of the proof that relies on these assumptions on the distribution) becomes more complicated in the higher step case as the combinatorial arguments used in the step 2 case are difficult to generalize.

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Large scale limit of simple exclusion processes and resistor networks on point processes with random conductances

Alessandra Faggionato

We are interested in transport properties of disordered media. We present here some results concerning the large scale limit for exclusion processes and resistor networks built on point processes in $\mathbb{R}^d$. An important tool in the derivation of these scaling limits has been homogenization theory for random walks on point processes with random conductances [6].

• *Random weighted graphs on point processes.* We first introduce a very large class of random weighted graphs on point processes. To this aim we fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and call environments its elements ω. We then fix a simple point process, i.e. a measurable map $\Omega \ni \omega \mapsto \hat{\omega} \in \{\text{locally finite subsets of } \mathbb{R}^d\}$ (see [3] for the topology of the last space). We also fix a conductance field, i.e. a map $c : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \ni (\omega, x, y) \mapsto c_{x,y}(\omega) \in [0, +\infty)$ such that $c_{x,y}(\omega) = c_{y,x}(\omega)$. As it will be clear later, the relevant values of the conductance field are for $x \neq y$ in $\hat{\omega}$. Finally we introduce the weighted undirected graph $\mathcal{G}(\omega)$ with vertex set $\hat{\omega}$, edge set $\{\, \{x,y\} \,:\, x \neq y \text{ in } \hat{\omega}\,,\ c_{x,y}(\omega) > 0\,\}$ and weight of the edge $\{x,y\}$ given by $c_{x,y}(\omega)$, which is called the conductance of $\{x,y\}$ similarly to [2].

We assume that the abelian group $\mathbb{G} = \mathbb{R}^d$ or $\mathbb{G} = \mathbb{Z}^d$ acts both on the Euclidean space and on the probability space in a covariant way. To simplify the notation, we restrict here to the case $\mathbb{G} = \mathbb{R}^d$ although our results cover also the case $\mathbb{G} = \mathbb{Z}^d$, which is particularly suited when dealing with lattice graphs. The action $(\tau_g)_{g\in\mathbb{G}}$ of $\mathbb{G}$ on the Euclidean space is given by translations, and for simplicity we take here $\tau_g x := x + g$. We denote by $(\theta_g)_{g\in\mathbb{G}}$ the action of $\mathbb{G}$ on the probability space (cf. [3]). Roughly, $\theta_g\omega$ is the new environment when we apply the translation τ_{-g} on the Euclidean space. The two actions are linked by the covariant relations

$$
\begin{aligned}
&\widehat{\theta_g\omega} = \tau_{-g}(\hat{\omega})\,,\\
&c_{x,y}(\theta_g\omega) = c_{\tau_g x, \tau_g y}(\omega)\,,
\end{aligned}
$$

that we assume. We assume also that the law $\mathcal{P}$ of the environment is stationary and ergodic for the action $(\theta_g)_{g\in\mathbb{G}}$ and that the intensity of the point process $m := \int d\mathcal{P}(\omega) \sharp(\hat{\omega} \cap [0,1]^d)$ is finite and positive ($\sharp A$ denotes the number of points in A). We call $\mathcal{P}_0$ the Palm distribution associated to $\mathcal{P}$. Roughly $\mathcal{P}_0 = \mathcal{P}(\cdot|0 \in \hat{\omega})$. Finally we require some basic moment bounds:

$$
\sum_{x\in\hat{\omega}:x\neq 0} c_{0,x}(\omega) \in L^1(\mathcal{P}_0)\,, \qquad \sum_{x\in\hat{\omega}:x\neq 0} c_{0,x}(\omega)|x|^2 \in L^1(\mathcal{P}_0)\,.
$$

Although the above formalism can appear abstract, it allows to describe several relevant models also with different geometrical features as explained in [6, 7, 8]. For examples, supercritical percolation clusters and Delaunay triangulations (thought as dual graph of Voronoi tessellations) are covered by the above formalism.

• *Effective homogenized matrix.* We introduce the effective homogenized matrix as the $d \times d$ nonnegative symmetric matrix D such that, for all $a \in \mathbb{R}^d$,

$$a \cdot Da = \inf_{f \in L^\infty(\mathcal{P}_0)} \frac{1}{2} \int d\mathcal{P}_0(\omega) \sum_{x \in \hat{\omega}} c_{0,x}(\omega) \left(a \cdot x - \nabla f(\omega, x)\right)^2 ,$$

where $\nabla f(\omega, x) := f(\theta_x \omega) - f(\omega)$. In [6] we have proved that the above matrix describes indeed the homogenization of the massive Poisson equation for the random walk on $\hat{\omega}$ with jump rates $c_{x,y}(\omega)$. We point out that D can be degenerate.

• *Symmetric simple exclusion processes.* Fixed the environment ω, we consider the symmetric simple exclusion process (cf. [12]) on the vertex set $\hat{\omega}$ with infinitesimal generator given by

$$\mathcal{L}_\omega f(\eta) = \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) \eta(x) \big(1 - \eta(y)\big) \left[f(\eta^{x,y}) - f(\eta)\right] .$$

We recall that the particle configuration is described by an element $\eta \in \{0,1\}^{\hat{\omega}}$, where $\eta(x)$ is the occupation number at the vertex x. The configuration $\eta^{x,y}$ is obtained from η by exchanging the values at x and y.

In the symmetric simple exclusion process particles perform independent random walks on $\hat{\omega}$ with probability rate $c_{x,y}(\omega)$ for a jump from x to $y \neq x$ with exception of the exclusion rule: when a particle attempts to jump to a vertex occupied by some other particle, the jump is suppressed.

Let $\mathcal{M}$ be the space of Radon measures on $\mathbb{R}^d$ with the vague topology. Let $(P_t)_{t \geq 0}$ be the Markov semigroup of the Brownian motion on $\mathbb{R}^d$ with diffusion matrix $2D$. Given $\mathfrak{m} \in \mathrm{Prob}(\{0,1\}^{\hat{\omega}})$, let $\mathbb{P}_{\mathfrak{m},\omega}$ be the law of the symmetric simple exclusion process on $\hat{\omega}$ with initial distribution $\mathfrak{m}$. Finally, given $\epsilon > 0$ and $\eta \in \{0,1\}^{\hat{\omega}}$, let $\pi_\epsilon(\eta)$ be the empirical measure $\pi_\epsilon(\eta) := \epsilon^d \sum_{x \in \hat{\omega}} \eta(x) \delta_{\epsilon x}$.

Then, under few additional minor technical assumptions, we have:

Theorem 1 (cf. [8, 9]) *For $\mathcal{P}$–a.a. ω the following holds:*
Let $\rho_0 : \mathbb{R}^d \to [0,1]$ be measurable and define $\rho : \mathbb{R}^d \times [0,\infty) \to [0,1]$ as $\rho(x,t) := P_t \rho_0(x)$. For $\epsilon > 0$ fix $\mathfrak{m}_\epsilon \in \mathit{Prob}\big(\{0,1\}^{\hat{\omega}}\big)$ such that, when $\eta \stackrel{\mathcal{L}}{\sim} \mathfrak{m}_\epsilon$, it holds $\pi_\epsilon(\eta) \stackrel{\epsilon \downarrow 0}{\to} \rho_0(x)dx$ in probability in $\mathcal{M}$. Then, for all $T > 0$, when $(\eta_s)_{s \geq 0} \stackrel{\mathcal{L}}{\sim} \mathbb{P}_{\mathfrak{m}_\epsilon,\omega}$ we have

$$\big(\pi_\epsilon(\eta_{\epsilon^{-2}t})\big)_{0 \leq t \leq T} \stackrel{\epsilon \downarrow 0}{\to} \big(\rho(x,t)dx\big)_{0 \leq t \leq T} \text{ in probability}$$

in $D([0,T], \mathcal{M})$.

Above the symbol $A \stackrel{\mathcal{L}}{\sim} B$ means that A and B have the same law. The proof of the above result is based on duality and homogenization and is provided in [8]. In [9] we explain how to remove the so called Assumption (SEP) from [8].

The above theorem can be applied to many specific models as discussed in [8]. We mention the symmetric exclusion process with random jump rates on $\mathbb{Z}^d$ (or another lattice), on the supercritical percolation cluster on $\mathbb{Z}^d$ (or another lattice), on the Delaunay triangulation defined as the dual graph of the Voronoi tessellation of $\hat{\omega}$ (cf. [10]), the symmetric exclusion process obtained by adding the exclusion interaction to the Mott variable range hopping.

For previous related hydrodynamic results for the exclusion process with random symmetric rates we mention [4, 5, 11, 13]

• *Random resistor networks.* Given $\ell > 0$ we consider the box, stripe and half-stripes[1]

$$\begin{cases} \Lambda_\ell := (-\ell/2, \ell/2)^d \,, & S_\ell := \mathbb{R} \times (-\ell/2, \ell/2)^{d-1} \\ S_\ell^- := \{x \in S_\ell \,:\, x_1 \leq -\ell/2\} \,, & S_\ell^+ := \{x \in S_\ell \,:\, x_1 \geq \ell/2\} \,. \end{cases} \tag{1}$$

Fixed the environment ω we consider the ℓ–parametrized resistor network on S_ℓ with node set $\hat{\omega} \cap S_\ell$. To each unordered pair of nodes $\{x, y\}$, such that $\{x, y\} \cap \Lambda_\ell \neq \emptyset$ and $c_{x,y}(\omega) > 0$, we associate an electrical filament of conductance $c_{x,y}(\omega)$. We then consider the electrical current $\sigma_\ell(\omega)$ flowing along the first direction when fixing the electrical potential equal to 0 on S_ℓ^- and equal to 1 on S_ℓ^+.

The scaling limit of $\sigma_\ell(\omega)$ is described by the following theorem, which holds under very weak additional assumptions:

Theorem 2. (cf. [7]) *Suppose that e_1 is an eigenvector of the effective homogenized matrix D. Then for $\mathcal{P}$–a.a. ω it holds $\lim_{\ell \to +\infty} \ell^{2-d} \sigma_\ell(\omega) = m D_{1,1}$, where m is the intensity of the simple point process.*

We refer to [7] for a generalized scaling limit theorem without the assumption that e_1 is an eigenvector of D. The above theorem is an universal result, since it covers a huge class of models as discussed [7]. For their relevance to transport in disordered model, we mention in particular the resistor networks built on the supercritical percolation cluster in $\mathbb{Z}^d$ and the Miller–Abrahams resistor network used to model the electron Mott variable range hopping in doped semiconductors [1, 14].

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Improved Lichnerowicz Inequality in terms of the L^{-p}-Mean of the Ricci Curvature

Karl-Theodor Sturm

We derive a new lower bound for the spectral gap on a closed nonnegatively curved Riemannian manifold or, more generally, on a compact $\mathrm{RCD}(k, N)$ space with a variable lower Ricci bound $k : X \to \mathbb{R}_+$:

$$\lambda_1 \geq \frac{N}{N-1} \left\|k\right\|_{-p} \qquad \text{with } p := \left(\frac{N-1}{N}\right)^2 < 1. \tag{1}$$

Here $\|f\|_{-p} := \left(\int_X f^{-p} d\bar{m}\right)^{-1/p}$ for any $p > 0$ and $f : X \to \mathbb{R}_+$ with $\bar{m} := \frac{1}{m(X)}\, m$.

Our estimate in terms of the L^{-p} norm of k for $p = 1 - 1/N$ improves upon the celebrated Lichnerowicz estimate (1958, case $p = \infty$)

$$\lambda_1 \geq \frac{N}{N-1} \left\|k\right\|_{-\infty} = \frac{N}{N-1} \cdot \inf_x k(x)$$

and the estimate of Veysseire (2010, case $p = 1$)

$$\lambda_1 \geq \left\|k\right\|_{-1} = \left(\int_X \frac{1}{k(x)}\, d\bar{m}(x)\right)^{-1}.$$

It is based on a generalized Carron-Rose estimate

$$\lambda_1 \geq \frac{N}{N-1} \cdot \inf \operatorname{spec}\left(-\frac{N}{N-1}\Delta + k\right). \tag{2}$$

which in turn is derived by means of a self-improved Bakry-Émery estimate

$$\Gamma_2(f) \geq k|\nabla f|^2 + \frac{1}{N}(\Delta f)^2 + \frac{N}{N-1} \left\|\nabla^2 f - \frac{1}{N}\Delta f \cdot \mathbf{1}\right\|^2. \tag{3}$$

Interpolation of Sobolev extension operators

Pekka Koskela
(joint work with Riddhi Mishra)

Given a domain Ω in an n-dimensional Euclidean space, we denote by $W^{1,p}(\Omega)$ the usual first order Sobolev space, consisting of all functions in $L^p(\Omega)$ whose first order distributional derivatives also belong to Ω. When $n = 1$, our domain Ω is necessarily an interval and there is a bounded linear extension operator that extends each function in our Sobolev space to a function in the corresponding space defined on the entire real line.

When $n \geq 2$, the situation is more complicated and validity of the extension property depends on the geometry of the domain in question. For example, for an outward quadratic cusp in the plane, the extension property fails for all $1 \leq p < \infty$. For the exterior of this cusp, the extension property fails for all $1 \leq p \leq \infty$. On the other hand, by results of Calderon and Stein, one can extend Sobolev functions from a Lipschitz domain, boundedly and linearly, for all $1 \leq p \leq \infty$.

Besides of extensions, one could also consider traces, but this leads to the same class of domains when $1 < p \leq \infty$. More precisely, if we know that every function in $W^{1,p}(\Omega)$ coincides with the restriction of a Sobolev function (in the Sobolev space with same indices) defined on the entire Euclidean space, then there is a bounded linear extension operator that extends functions to the global Sobolev space. Moreover, this extension operator can be chosen canonically in the sense that we can use a single operator for all the values of p for which the trace operator is onto. For these results see [2].

Suppose now that Ω has the above extension property for two different values of p, say for p_1, p_2 with $1 < p_1 < p_2$. Then we can choose a single operator that extends functions from our two different Sobolev spaces, boundedly and linearly. In the case of Lipschitz domain, it follows that our extension operator also does the same for $W^{1,p}(\Omega)$ for $p_1 < p < p_2$. It is very tempting to conjecture that this holds in general: by interpolation our operator maps the interpolation spaces of our Sobolev spaces into the interpolation spaces of the global Sobolev spaces. The latter spaces are full Sobolev spaces while the former ones are subspaces of the corresponding Sobolev spaces.

Partial progress towards the conjecture was obtained already quite some time ago [1]: if the dimension of our Euclidean spaces is n and Ω has the extension property for some $p_1 \geq n$, then it has the extension property for all $p > p_1$.

Unfortunately, we have very recently found a counterexample: there exists a planar domain that has the extension property for all the other values of p but 2. We are currently trying to generalise the idea behind our construction to see for which ranges of exponents the interpolation property fails. In the planar case, it seems that, given $1 < q < 2$, we can construct a domain that has the extension property precisely when $1 \leq p < q$ or $p > 2$. There are still pairs of p_1, p_2 for which we do not understand the situation and we have only started to consider the higher dimensional setting.

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Hölder continuous mappings, differential forms and the Heisenberg groups

Piotr Hajłasz

Young [8] defined Stielties integrals $\int_a^b f dg$ of Hölder continuous functions, $f, g \in C^{0,\alpha}$, $\alpha > 1/2$, and this theory plays an important role in the stochastic calculus. More recently, three groups of researchers developed, completely independently, a multidimensional version of the Young integral $\int f dg_1 \wedge \ldots \wedge dg_n$ with different applications in mind: Brezis and Nguyen [1, 2] as an abstract theory; Conti, De Lellis, and Székelyhidi [3], in the context of convex integration and rigidity of isometric immersions; Züst [9] in the setting of currents with the applications to the Heisenberg groups. Since the functions are merely Hölder continuous, the theory can be regarded as the theory of distributional Jacobians of Hölder continuous maps.

In the recent work [5] we used the related theory of distributional pullbacks of differential forms under Hölder maps to study Hölder continuous mappings into the Heisenberg groups. In particular we found a simple proof of a generalization of a well known theorem of Gromov that is discussed below.

The Heisenberg group $\mathbb{H}^n$ equipped with the Carnot-Carathéodory metric d_c is homeomorphic to $\mathbb{R}^{2n+1}$, but its Hausdorff dimension equals $2n+2$. On compact sets the metric d_c satisfies $|p-q| \lesssim d_c(p,q) \lesssim |p-q|^{1/2}$. In fact, in some directions the metric d_c is comparable to the Euclidean metric, while in other directions it is of fractal nature. The Heisenberg group is also isomorphic to the standard contact structure on $\mathbb{R}^{2n+1}$, and the metric d_c gives a natural way of measuring distances between points along curves tangent to the contact distribution. It is well known that every Legendrian (i.e., horizontal) map $f : U \subset \mathbb{R}^m \to \mathbb{H}^n \cong \mathbb{R}^{2n+1}$, that is smooth or Lipschitz, must satisfy $\operatorname{rank} Df \leq n$, and hence there is no Lipschitz embedding of U into $\mathbb{H}^n$, if $m \geq n+1$. On the other hand, any smooth or Lipschitz map $f : U \to \mathbb{R}^{2n+1}$, is $\frac{1}{2}$-Hölder continuous as a map into $\mathbb{H}^n$. In this context Gromov in his seminal work [4], initiated the program of investigating properties of Hölder continuous mappings into the Heisenberg group. He proved the following non-embedding result [4, 3.1.A]:

Theorem 1. *There is no $C^{0,\gamma}$-Hölder continuous embedding $f : \mathbb{R}^m \to \mathbb{H}^n$, if $m \geq n+1$, and $\gamma > \frac{n+1}{n+2}$.*

The original proof is actually quite difficult since it requires the h-principle and microflexibility. In [5] we used distributional pullbacks of differential forms to prove the following result:

Theorem 2. *Suppose that* $m \geq n+1$,

$$\frac{1}{2} \leq \gamma \leq \frac{n+1}{n+2} \quad \textit{and} \quad \theta = \frac{n+1}{n} - \frac{2\gamma}{n}.$$

Then, there does not exist a map

$$f \in C^{0,\gamma+}(\mathbb{B}^m, \mathbb{H}^n) \cap C^{0,\theta}(\mathbb{B}^m, \mathbb{R}^{2n+1})$$

or a map

$$f \in C^{0,\gamma}(\mathbb{B}^m, \mathbb{H}^n) \cap C^{0,\theta+}(\mathbb{B}^m, \mathbb{R}^{2n+1}),$$

such that $f|_{\mathbb{S}^{m-1}}$ *is a topological embedding.*

Here, $C^{0,\alpha}(\mathbb{B}^m, X)$ stands for the class of Hölder continuous mappings into a metric space X. Moreover, $f \in C^{0,\alpha+}(\mathbb{B}^m, X)$ means that

$$\lim_{t \to 0^+} \sup \left\{ \frac{d(f(x), f(y))}{|x-y|^\alpha} : \; x, y \in \mathbb{B}^m, \; 0 < |x-y| \leq t \right\} = 0.$$

When $\gamma = \frac{n+1}{n+2}$, then $\theta = \frac{n+1}{n+2}$, and hence Theorem 1 follows as this special case of Theorem 2.

I a forthcoming work [6] we develop a comprehensive analysis of pullbacks of differential forms under Hölder continuous mappings with particular emphasis on applications to geometry and topology of the Heisenberg groups. In particular we apply this theory to study Hölder continuous homotopy groups of the Heisenberg group, and we obtain far reaching generalizations of the results of [7]. Currently the paper is 92 pages long and we expect it to exceed 100 pages.

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L^p couplings of Heisenberg Brownian motions

Nicolas Juillet
(joint work with Michel Bonnefont)

The Heisenberg group is a kind of non-commutative vector space homeomorphic to $\mathbb{R}^{2d+1}$ for which the natural Brownian motion is composed of d planar Brownian Motions and the sum of their Levy areas on the last coordinate (labeled $2d+1$). Different kinds of couplings of two Heisenberg Brownian motions have been considered in the literature. In [1] we look at the minimal L^p distance between two co-adapted versions. We compare it with the L^p Wasserstein distance between the (Heisenberg) heat kernels as a function of time. Our study is inspired by equivalent characterizations of the Ricci curvature of Riemannian manifolds that involve the rate of control for the L^p distance of co-adapted Brownian motions on the one side, and the Wasserstein distance on the other side, as for instance sumed up in [4].

The talk was organized as follows:

(1) The Heisenberg group $\mathbb{H}$
(2) Results
(3) Sketch of proofs

I spent about half of the time in introducing the first Heisenberg group presented as a non commutative vector space (first part of the talk). My presentation was based on the analogy with the parallelogram rule used for adding two vectors of $\mathbb{R}^2$. The vectors of $\mathbb{R}^2$ can be replaced by the equivalence classes of curves

$$\{\gamma : [0,\ell] \to \mathbb{R}^2 |\ \gamma(0) = (0,0)\}/\sim$$

with $(\gamma \sim \gamma') \Leftrightarrow (\gamma(1) = \gamma'(1))$. Adding vectors is the same as concatenating curves. Here the concatenation is defined by

$$[\gamma\cdot\gamma'](t) = \begin{cases} \gamma(t) & \text{if } t \le \ell, \\ \gamma(\ell) + \gamma'(t-\ell) & \text{if } \ell < t \le \ell + \ell'. \end{cases}$$

The first Heisenberg group is about what happens when one reinforces this equivalence relation as follows:

$$(\gamma \sim \gamma') \Leftrightarrow \begin{cases} \gamma(1) = \gamma'(1) \\ \dfrac{1}{2}\displaystyle\int_0^{\ell} \gamma \wedge \dot\gamma\ dt = \dfrac{1}{2}\displaystyle\int_0^{\ell'} \gamma' \wedge \dot\gamma'\ dt. \end{cases}$$

The second equation is interpreted as the equality of the algebraic areas swept by the curves. In coordinates the equivalence class is isomorphic to $\mathbb{R}^2 \times \mathbb{R}$ through the map

$$\gamma \longmapsto (x(\ell), y(\ell); \frac{1}{2}\int_0^{\ell} (x\dot y - y\dot x)\ dt).$$

The concatenation then corresponds to the product

$$(x,y;z)\cdot(x',y';z') = (x+x', y+y'; z+z'+\frac{1}{2}(xy'-yx'))$$

where $\frac{1}{2}(xy' - yx')) = \frac{1}{2}\gamma(1) \wedge \gamma'(1)$ is the oriented area of the triangle with ends $(0,0)$, $\gamma(\ell)$ and $\gamma(\ell) + \gamma'(\ell')$ in this order, i.e the triangle used for the sum of the vectors $\overrightarrow{0\gamma(\ell)}$ and $\overrightarrow{\gamma(\ell)\gamma'(\ell')}$ in $\mathbb{R}^2$. The order gives the sign for this triangle area and lies at the origin of the fact that $\mathbb{H}$ is a non commutative vector space.

The metric side of $\mathbb{H}$ appears while giving to $(x, y; z)$ a (non homogeneous) norm corresponding to the the minimal length of a curve γ between $(0,0)$ and (x, y) that encloses the algebraic area z. The problem of finding such a curve simply translates as Dido problem –an ancient isoperimetric problem presented in the Eneid. For $(x, y) \neq (0, 0)$ the curve is uniquely determined up to parametrization and it is a piece of planar circle or a line if $z = 0$. Up to Lipschitz multiplicative constants the minimizing length has value $\sqrt{x^2 + y^2} + |z|^{1/2}$. Note that this quantity behaves well with respect to the scaling $\delta_\lambda : (x, y, z) \mapsto (\lambda x, \lambda y, \lambda^2 z)$.

The second part was about three results obtained with Michel Bonnefont in [1]:

- If two Heisenberg Brownian motions B_t and B'_t have planar coordinates evolving in mirror of each other until these planar part meet, the norm of $(B_t)^{-1} \cdot B'_t$ to the power $p \in]0, 1[$ has expectation bounded with respect to time.
- For any power $p \geq 2$ it is not possible to find a co-adapted coupling such that the same quantity (that is $t \mapsto \mathbb{E}(\|(B_t)^{-1} \cdot B'_t\|^p)$) is bounded. Note that for $p \in [1, 2[$, it is not known if a bounded coupling exists.
- There exists a constant $C > 0$ such that when t is fixed there exists a coupling (B_t, B'_t) that corresponds to a translation for the planar part and with $\mathbb{E}[\|(B_t)^{-1} \cdot B'_t\|] \leq CW_1(B_0, B'_0)$. One has to stress that is coupling only at fixed time t and not a co-adapted coupling of processes. With respect to this result a remaining challenge would be to exhibit a coupling such that $\|(B_t)^{-1} \cdot B'_t\| \leq CW_\infty(B_0, B'_0)$. This estimate is in fact indirectly known through of combination of works by Li [5] and Kuwada [3].

In the third part I very briefly gave ideas of our proofs. The setting of the second result is the same as the one of Kendall [2] where it is proved that there exists successful couplings of Heisenberg Brownian motions, in particular such that $\mathbb{P}(B_t = B'_t) \to_{t\to\infty} 1$. The proof of the third result partially relies on the optimal transport on $\mathbb{R}$ with the concave cost $c(z, z') = \sqrt{|z' - z|}$ and the fact that the corresponding Wasserstein distance is a norm on the space of probability measures.

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Radial process on a model hypersurface in a contact sub-Riemannian model space

Karen Habermann
(joint work with Davide Barilari)

Let M be a smooth manifold of dimension $2n+1$ for $n \geq 1$, let $\mathcal{D}$ be a contact structure on M, and let g be a smooth fibre inner product on $\mathcal{D}$. Since this gives rise to a contact manifold $(M, \mathcal{D})$ and as $(\mathcal{D}, g)$ defines a sub-Riemannian structure on the manifold M, the triple $(M, \mathcal{D}, g)$ is called a contact sub-Riemannian manifold.

We assume that there exists a global one-form ω on M such that $\mathcal{D} = \ker \omega$ and $\omega \wedge (\mathrm{d}\omega)^n \neq 0$. Such a global one-form ω is called contact form for the contact structure $\mathcal{D}$. The existence of a contact form ω ensures that the manifold M is orientable as it can then be oriented by the volume form

$$\Omega = \omega \wedge (\mathrm{d}\omega)^n \ .$$

We further impose the normalisation condition that

$$(\mathrm{d}\omega)^n|_{\mathcal{D}} = n! \operatorname{vol}_g \ ,$$

with vol_g denoting the volume form on the distribution $\mathcal{D}$ induced by the fibre inner product g.

Let S be an orientable hypersurface embedded in the contact manifold $(M, \mathcal{D})$. We denote by $C(S)$ the set of characteristic points of S, namely the set of points $x \in S$ such that $T_x S = \mathcal{D}_x$. Observe that $C(S)$ is a closed subset of S, which implies that $S \setminus C(S)$ is a well-defined hypersurface in M. Once the orientations of S and M are fixed, we have a unique unit and normal vector field N compatible with the orientations and defined as follows.

The sub-Riemannian normal vector field N along a hypersurface $S \setminus C(S)$ in a contact sub-Riemannian manifold $(M, \mathcal{D}, g)$ is the unit-length vector field in the distribution $\mathcal{D}$, that is,

$$\omega(N) = 0 \quad \text{and} \quad g(N, N) = 1 \ ,$$

such that, for any vector field Y on $S \setminus C(S)$ and in the distribution $\mathcal{D} \cap TS$,

$$g(N, Y) = 0 \ ,$$

and where, for any positively oriented local orthonormal frame $(Z_1, \ldots, Z_{2n})$ for $S \setminus C(S)$, the frame $(N, Z_1, \ldots, Z_{2n})$ for M is a positively oriented.

Using the volume form Ω on M and the sub-Riemannian normal vector field N along $S \setminus C(S)$, we introduce a volume form μ on $S \setminus C(S)$ by setting

$$\mu = \iota_N \Omega \ ,$$

that is, μ is obtained by contracting the form Ω with the vector field N restricted to $S \setminus C(S)$.

The horizontal gradient $\nabla_S f$ of a smooth function $f \colon S \setminus C(S) \to \mathbb{R}$ is the unique vector field in the distribution $\mathcal{D} \cap TS$ such that, for any vector field Y in $\mathcal{D} \cap TS$,

$$g(\nabla_S f, Y) = \mathrm{d}f(Y) \ .$$

The intrinsic sub-Laplacian Δ on $S \setminus C(S)$ is defined as the divergence with respect to the volume form μ of the horizontal gradient ∇_S, that is, for a smooth function $f\colon S \setminus C(S) \to \mathbb{R}$, we have

$$\Delta f = \operatorname{div}_\mu \left(\nabla_S f \right) \ .$$

Note that due to the imposed normalisation condition which links the contact form ω to the fibre inner product g, the intrinsic sub-Laplacian Δ depends only on the choice of embedded hypersurface S and contact sub-Riemannian manifold $(M, \mathcal{D}, g)$.

1. Model hypersurface in a contact sub-Riemannian model space

The model spaces for contact sub-Riemannian manifolds arise by equipping the Euclidean space $\mathbb{R}^{2n+1}$, the sphere S^{2n+1} and a hyperboloid H^{2n+1}, respectively, with a standard contact structure $\mathcal{D}$ and the following fibre inner product g on $\mathcal{D}$. For $\mathbb{R}^{2n+1}$, we choose g such that $(\mathbb{R}^{2n+1}, \mathcal{D}, g)$ gives rise to the higher-dimensional Heisenberg group $\mathbb{H}^n$. For the sphere S^{2n+1} embedded in $\mathbb{R}^{2n+2}$, we choose $k \in \mathbb{R}$ positive and set, with $\langle \cdot, \cdot \rangle$ denoting the Euclidean inner product on $\mathbb{R}^{2n+2}$,

$$g(\cdot, \cdot) = \frac{1}{k^2} \left. \langle \cdot, \cdot \rangle \right|_{\mathcal{D}} \ .$$

This gives rise to a one-parameter family of model spaces with underlying manifold S^{2n+1} and parameter $k > 0$. Similarly, for the hyperboloid H^{2n+1} embedded in $\mathbb{R}^{2n,2}$, we use the flat Lorentzian metric η on $\mathbb{R}^{2n,2}$ and $k \in \mathbb{R}$ positive to define

$$g(\cdot, \cdot) = \frac{1}{k^2} \left. \eta(\cdot, \cdot) \right|_{\mathcal{D}} \ ,$$

which yields a one-parameter family of model spaces with underlying manifold H^{2n+1} and parameter $k > 0$.

The hypersurface which we consider embedded in $\mathbb{R}^{2n+1}$, in S^{2n+1} and in H^{2n+1}, respectively, serves as a model hypersurface in the corresponding model space and can be identified with the Euclidean space $\mathbb{R}^{2n}$, the sphere S^{2n} and a hyperboloid $\widetilde{H}^{2n}$, respectively, with a unique characteristic point in the first and third case, and with two antipodal characteristic points in the second case.

2. Radial process on a model hypersurface

For a model hypersurface in a contact sub-Riemannian model space, the volume form μ can be determined explicitly, see [2, Proposition 6].

Proposition. *Let $(M, \mathcal{D}, g)$ be a $(2n+1)$-dimensional contact sub-Riemannian model space. Set $I = (0, \frac{\pi}{k})$ if $M = S^{2n+1}$ associated with parameter $k > 0$ and set $I = (0, \infty)$ otherwise. Define $h_k\colon I \to \mathbb{R}$ by, for $r \in I$,*

$$h_k(r) = \begin{cases} r & \text{if } M = \mathbb{R}^{2n+1} \\ k^{-1} \sin(kr) & \text{if } M = S^{2n+1} \\ k^{-1} \sinh(kr) & \text{if } M = H^{2n+1} \end{cases} \ .$$

For the model hypersurface S in the model space $(M,\mathcal{D},g)$ as well as in suitable coordinates $(r,\varphi_1,\ldots,\varphi_{2n-1})$ for $S\setminus C(S)$ with $r\in I$, $\varphi_1,\ldots,\varphi_{2n-2}\in[0,\pi]$ and $\varphi_{2n-1}\in[0,2\pi)$, the volume form μ defined on $S\setminus C(S)$ is given by

$$\mu=\frac{n!}{2}\left(h_k(r)\right)^{2n}\left(\prod_{i=1}^{2n-2}\left(\sin(\varphi_i)\right)^{2n-i-1}\right)\mathrm{d}r\wedge\mathrm{d}\varphi_1\wedge\cdots\wedge\mathrm{d}\varphi_{2n-1}\ .$$

This result then allows us to identify the radial process induced by the intrinsic sub-Laplacian Δ away from characteristic points, see [2, Theorem 7].

Theorem. *Let $(M,\mathcal{D},g)$ be a $(2n+1)$-dimensional contact sub-Riemannian model space. For the model hypersurface S in the model space $(M,\mathcal{D},g)$, the radial part of the stochastic process with generator $\frac{1}{2}\Delta$ on $S\setminus C(S)$ is the Bessel process of order $2n+1$ if $M=\mathbb{R}^{2n+1}$, a Legendre process of order $2n+1$ if $M=S^{2n+1}$, and a hyperbolic Bessel process of order $2n+1$ if $M=H^{2n+1}$.*

For surfaces in three-dimensional contact sub-Riemannian model spaces, the above theorem is implied by [1, Theorem 1.5] because the intrinsic sub-Laplacian Δ arises as the limit of Laplace–Beltrami operators built by means of Riemannian approximations to the sub-Riemannian structure which use the Reeb vector field, see [2, Theorem 5].

Since a Bessel process of order $2n+1$ and a hyperbolic Bessel process of order $2n+1$ for $n\geq 1$ almost surely neither hits the origin nor explodes in finite time, and as a Legendre process of order $2n+1$ for $n\geq 1$ almost surely hits neither endpoint of the interval $(0,\frac{\pi}{k})$, it is an immediate consequence of the theorem that in all model cases considered the intrinsic sub-Laplacian Δ defined on $S\setminus C(S)$ is stochastically complete. On the other hand, the geometry induced on the hypersurface $S\setminus C(S)$ is not geodesically complete. This can be seen by noting that a radial ray, that is, a path along the radial direction emanating from one of the characteristic points, parameterised by arc length is a geodesic which cannot be extended indefinitely towards the characteristic point.

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Heat kernel estimates for (nonsymmetric) nonlocal operators

MARVIN WEIDNER
(joint work with Moritz Kassmann)

The aim of this talk is to discuss heat kernel estimates for nonlocal operators

$$-Lu(x) = 2\,\mathrm{p.v.}\int_{\mathbb{R}^d} (u(x) - u(y))K(x, y)\mathrm{d}y, \tag{L}$$

where $K : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ is a measurable function, called jumping kernel. The prototype example of an operator L is the fractional Laplacian $-(-\Delta)^{\alpha/2}$, which corresponds to the choice $K(x, y) = c(d, \alpha)|x - y|^{-d-\alpha}$ for some $\alpha \in (0, 2)$ and a constant $c(d, \alpha) > 0$ such that

$$\mathcal{F}((-\Delta)^{\alpha/2}u)(\xi) = |\xi|^\alpha \mathcal{F}(u).$$

Consider the Cauchy problem

$$\begin{cases} \partial_t u - Lu &= 0, \quad \text{in } (0, \infty) \times \mathbb{R}^d, \\ u(0) &= u_0, \quad \text{in } \mathbb{R}^d,\ u_0 \in L^2(\mathbb{R}^d). \end{cases} \tag{CP}$$

We say that $p : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ is a fundamental solution to (CP) if

$$u(t, x) = \int_{\mathbb{R}^d} p(t, x, y)u_0(y)\mathrm{d}y$$

for every $u_0 \in L^2(\mathbb{R}^d)$. In case $L = -(-\Delta)^{\alpha/2}$, the fundamental solution has the following polynomial decay:

$$p(t, x, y) = \left(e^{-t|\cdot|^\alpha}\right)^\vee (x - y) \asymp t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x - y|^{d+\alpha}}.$$

A celebrated result due to Chen and Kumagai (see [2]) states that all nonlocal operators whose jumping kernels are comparable to the one of the fractional Laplacian share the same two-sided estimates of their fundamental solutions.

Theorem 1. *Let K be a symmetric jumping kernel satisfying for some $\Lambda \geq 1$:*

$$\Lambda^{-1}|x - y|^{-d-\alpha} \leq K(x, y) \leq \Lambda|x - y|^{-d-\alpha},\ \forall x, y \in \mathbb{R}^d. \tag{$K^{\asymp}$}$$

Then, the fundamental solution p corresponding to the Cauchy problem (CP) *associated with L satisfies, for all $t > 0$ and $x, y \in \mathbb{R}^d$*

$$c_1\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x - y|^{d+\alpha}}\right) \leq p(t, x, y) \leq c_2\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x - y|^{d+\alpha}}\right),$$

for some $c_1, c_2 > 0$ that depend only on d, α, Λ.

This result can be interpreted as a nonlocal counterpart of the famous Aronson bounds for second order operators in divergence form. In his celebrated article [1], Aronson proves that for any bounded, uniformly elliptic, diffusion matrix A and any drift coefficient b, d with $|b|^2, |d|^2 \in L^\theta(\mathbb{R}^d)$, where $\theta \in (\frac{d}{2}, \infty]$, the fundamental solution corresponding to the Cauchy problem (CP) associated with

$$Lu = \mathrm{div}(A\nabla u) - \mathrm{div}(bu) + d\nabla u$$

satisfies two-sided Gaussian bounds. This result can be interpreted as an application of the De Giorgi-Nash-Moser theory, as its proof heavily relies on the parabolic Harnack inequality.

The talk is split into two parts. In the first part, we present a new proof of the upper estimate in Theorem 1, which is based on the original ideas of Aronson's proof for local operators (see [5]). Our proof has the advantage that it is quite explicit. Moreover, as opposed to many existing proofs of heat kernel estimates for nonlocal operators that exist in the literature, it solely relies on analytic arguments.

A key idea of our proof is the consideration of associated truncated operators

$$-L^{\rho}u(t,x) := 2 \text{ p.v.} \int_{B_\rho(x)} (u(t,x)-u(t,y))K(x,y)\mathrm{d}y, \ \ \rho > 0,$$

and proving in a first step an estimate for the heat kernel p_ρ corresponding to the Cauchy problem corresponding to L^ρ, before proving an estimate for p in a second step. The proof relies on two main ingredients. First, we prove that any weak solution to (CP) satisfies a weighted L^2-estimate of the form

$$\sup_{t\in(0,s)} \int_{\mathbb{R}^d} H(t,x)u^2(t,x)\mathrm{d}x \le \int_{\mathbb{R}^d} H(0,x)u_0^2(x)\mathrm{d}x,$$

whenever H satisfies for some $C = C(d,\Lambda) > 0$

$$C\Gamma^{\rho}_{(\alpha)}(H^{1/2},H^{1/2}) \le -\partial_t H, \ \ \text{in } (0,s)\times\mathbb{R}^d, \tag{LY}$$

where $\Gamma^{\rho}_{(\alpha)}(H^{1/2},H^{1/2})(x) = \int_{B_\rho(x)} (H^{1/2}(x)-H^{1/2}(y))^2|x-y|^{-d-\alpha}\mathrm{d}y.$

$\Gamma^{\rho}_{(\alpha)}(u,u)$ can be interpreted as a nonlocal counterpart of $|\nabla u|^2$ and the estimate (LY) can be regarded as a nonlocal Li-Yau type inequality. One can prove that there is $\nu > 0$ such that for every $0 < s < \frac{1}{4\nu}\rho^\alpha$, and $y \in \mathbb{R}^d$

$$H(t,x) = e^{-\log(\frac{\rho^\alpha}{\nu[t]})\left(\frac{|x-y|}{3\rho}\vee 1\right)}, \ \ \text{where } [t] = 2s-t,$$

satisfies (LY). Second, we establish an $L^\infty - L^2$-estimate for weak subsolutions to (CP) with respect to L^ρ:

$$u(s,y)^2 \le cs^{-\frac{d}{\alpha}}\left(\frac{\rho^\alpha}{s}\right)^{\frac{d}{\alpha}} \sup_{t\in(0,s)} \int_{B_{2\rho}(y)} u^2(t,x)\mathrm{d}x. \tag{$L^\infty - L^2$}$$

Such estimate follows by a nonlocal De Giorgi-type iteration. Moreover, we explain how our method can be extended to heat kernel estimates on doubling metric measure spaces under the presence of a possibly time-inhomogeneous jumping kernel satisfying an upper bound of mixed polynomial type and a Faber-Krahn inequality.

In the second part of the talk, we report on recent results on two-sided heat kernel estimates for nonsymmetric nonlocal operators of the form (L). Here, 'nonsymmetric' means that the jumping kernel K does not have to be a symmetric function in x, y. While the regularity theory and heat kernel estimates are well known in the symmetric case, such questions have not yet been systematically studied for nonsymmetric nonlocal operators.

We explain that under the assumption that for some $\theta \in (\frac{d}{\alpha}, \infty]$ and $\Lambda > 0$, the jumping kernel satisfies $(K^{\asymp})$ and

$$\text{(K1)} \qquad \left\| \int_{\mathbb{R}^d} \frac{|K_a(\cdot, y)|^2}{K_s(\cdot, y)} \mathrm{d}y \right\|_{L^\theta(\mathbb{R}^d)} \leq \Lambda,$$

the associated bilinear form

$$\mathcal{E}^K(f, g) = 2 \iint_{\mathbb{R}^d\mathbb{R}^d} (f(x) - f(y))g(x)K(x, y)\mathrm{d}y\mathrm{d}x$$

is a regular lower bounded semi-Dirichlet form. Here, K_s and K_a denote the symmetric and antisymmetric part of K, respectively. As a consequence, one can associate with L a stochastic process and a semigroup (P_t). Due to the lack of symmetry of K, the dual semigroup $(\widehat{P}_t)$ does not satisfy the Markov property. This is a huge obstruction when establishing heat kernel estimates in the nonsymmetric case. Our main result reads as follows:

Theorem 2 ([6])**.** *Let $T > 0$. Let K be such that for some $\Lambda \geq 1$, and $\theta \in (\frac{d}{\alpha}, \infty]$ it holds* (K1) *and* $(K^{\asymp})$*. Then, the fundamental solution p to* (CP) *exists, and there are $c_1, c_2, c_3 > 0$ depending only on $d, \alpha, \theta, \Lambda$ such that*

$$\text{(1)} \qquad p(t, x, y) \geq c_1 \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x - y|^{d+\alpha}} \right), \ \forall t \in (0, T), \ x, y \in \mathbb{R}^d,$$

$$\text{(2)} \qquad p(t, x, y) \leq c_2 \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x - y|^{d+\alpha}} \right), \ \forall t \in (0, T), \ |x - y| \leq c_3 T^{1/\alpha}.$$

To the best of our knowledge, this is the first time, heat kernel estimates are established in the nonsymmetric case. Even on-diagonal upper bounds were not known before. In order to establish the lower bound (1), we rely on a weak parabolic Harnack inequality which holds true for nonnegative weak solutions to

$$\partial_t u - Lu = 0 \text{ in } Q, \quad \text{ or } \quad \partial_t u - \widehat{L}u = 0 \text{ in } Q$$

for some time-space cylinder $Q \subset (0, \infty) \times \mathbb{R}^d$. Such result has recently been established in [3] by adapting the nonlocal Moser iteration technique to nonsymmetric operators. The upper estimate (2) can be established with the help of an $L^\infty - L^2$-estimate similar to $(L^\infty - L^2)$ for nonsymmetric operators, as it was proved in [4], and an extension of the nonlocal Aronson technique, which was addressed in the first part of the talk.

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First passage percolation with long-range correlations

Sebastian Andres
(joint work with Alexis Prévost)

1. Positivity of the time constant in first passage percolation

First passage percolation has been a central topic in probability theory and an active research area since more than 50 years, see [4] for a survey. To define the model, consider the integer lattice $\mathbb{Z}^d$, $d \geq 2$, equipped with the set of nearest-neighbour edges E_d. Let $(t_e^\omega)_{e \in E_d}$ be a family of non-negative random weights. Then, the random pseudo-metric $d^\omega(x,y)_{x,y \in \mathbb{Z}^d}$ associated with first passage percolation (or FPP distance in short) is given by

$$d^\omega(x,y) = \inf_{\pi:\, x \overset{\pi}{\leftrightarrow} y} \sum_{e \in \pi} t_e^\omega,$$

where the infimum is taken over all simple paths π of edges connecting x to y.

From now on let us suppose that $(t_e^\omega)_{e \in E_d}$ stationary and ergodic with $\mathbb{E}[t_e^\omega] < \infty$ for any $e \in E_d$. Then, by Kingman's subadditive ergodic theorem, for all $x \in \mathbb{Z}^d \setminus \{0\}$, there exists a constant $\mu_u(x) \in [0,\infty)$ such that

$$\lim_{n \to \infty} \frac{d^\omega(0,nx)}{n} = \mu_u(x), \quad \mathbb{P}\text{-a.s. and in } L^1.$$

The constant $\mu_u(x)$ is called the time constant and depends on the choice of the direction x and the law of the weights. Our aim is to find conditions under which the time constant is strictly positive so that $d^\omega(0,nx)$ grows linearly. In the case where the weights $(t_e^\omega)_{e \in E_d}$ are i.i.d. a simple criterion [6, Theorem 6.1] is that

$$\mu_u(x) > 0 \text{ if and only if } \mathbb{P}(t_e^\omega = 0) < p_c, \qquad \text{for any } x \in \mathbb{Z}^d \setminus \{0\}, \tag{1}$$

where p_c is the critical parameter for i.i.d. Bernoulli bond percolation on $\mathbb{Z}^d$.

For a general ergodic family $(t_e^\omega)_{e \in E_d}$, it is known that the time constant can be zero even when all the weights are strictly positive, cf. e.g. [2, Theorems 2.4 and 2.5]. For our main result we will suppose that the law of the weights is described by a one-parameter family of probability measures $(\mathbb{P}^u)_{u \in I}$, indexed by a sprinkling parameter u in some open interval $I \subset \mathbb{R}$, satisfying the following conditions:

- **P1**: spatial ergodicity;
- **P2:** stochastic monotonicity in u;
- **P3**: weak decorrelation for monotone events.

Very similar conditions have been introduced initially in [5] in order to study the graph distance in percolation models with long-range correlations. This framework also includes certain models with polynomial correlation decay. As one prime example let us consider the discrete Gaussian free field (DGFF) $(\phi_x)_{x\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$ with $d \geq 3$, i.e. the centred Gaussian field with covariance function given by the Green's function of the simple random walk on $\mathbb{Z}^d$. Let $\mathbb{P}^u$ be the law of $(\phi_x+u)_{x\in\mathbb{Z}^d}$. (In the context of the DGFF we consider FPP on vertices rather than edges.) Then $(\mathbb{P}^u)$ satisfies **P1**–**P3**, see [3, Proposition 4.1] based on arguments in[7]. Further prominent examples include Ginzburg-Landau $\nabla\phi$ interface models or random interlacements, see [9] and [8] for closely related results.

Theorem 1 ([3]). *Suppose that $(\mathbb{P}^u)_{u\in I}$ satisfies* **P1**–**P3** *and $\mathbb{E}^u[t^\omega_e] < \infty$ for any $e \in E_d$ and $u \in I$. Further, assume*

$$\liminf_{L\to\infty} \sup_{x\in\mathbb{Z}^d} \mathbb{P}^u\Big(B(x,L) \longleftrightarrow B(x,2L)^c \ \textit{in}\ \{e\in E_d : t^\omega_e = 0\}\Big) = 0, \quad u \in I.$$

Then $\mu_u(x) > 0$ for all $x \in \mathbb{Z}^d \setminus \{0\}$.

To illustrate this results, let us consider the special case of the level sets of the Gaussian free field $(\phi_x)_{x\in\mathbb{Z}^d}$ in $d \geq 3$. We denote by h_* the critical parameter for the percolation of the associated level sets $\{x \in \mathbb{Z}^d : \phi_x \geq h\}$, $h \in \mathbb{R}$.

Theorem 2 ([3]). *Assume that the weights $(t^\omega_x)_{x\in\mathbb{Z}^d}$ have the same law under $\mathbb{P}^h$ as $(\mathbb{1}_{\{\phi_x<h\}})_{x\in\mathbb{Z}^d}$. Then, for all $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$\mu_h(x) > 0 \ \textit{if}\ h > h_* \ \textit{and}\ \mu_h(x) = 0 \ \textit{if}\ h < h_*.$$

Moreover, for all $h > h_*$ and $\delta > 4$, there exist positive constants c_1 and c_2 such that for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\mathbb{P}^h\left(d^\omega(0,nx) \leq c_1 n\right) \leq \exp\left(-\frac{c_2 n}{\log(n)^{\delta \mathbb{1}_{d=3}}}\right).$$

Theorem 2 can thus be seen as the equivalent of (1) but for the level sets of the Gaussian free field, except for the unknown case $h = h_*$.

2. Applications to random Schrödinger operators

Next we will discuss two applications to the *random conductance model (RCM)*, which is a well established model for a random walk in random environment. We endow $(\mathbb{Z}^d, E_d)$ with positive random weights $\{a^\omega(e), e \in E_d\}$, where we refer to $a^\omega(e)$ as the *conductance* of an edge $e \in E_d$. Write $\mu^\omega(x) := \sum_{y\sim x} a^\omega(x,y)$ and $\nu^\omega(x) := \sum_{y\sim x} 1/a^\omega(x,y)$, $x \in \mathbb{Z}^d$. In a first application of Theorem 1 we can turn the upper heat kernel bounds in [2] for RCMs with general speed measures into genuine Gaussian estimates (see [3, Section 3.1]). In a second application we study the Green's function $g^\omega(x,y)$ associated with the random Schrödinger operator of the form

$$\big(\mathcal{L}^\omega f\big)(x) \;=\; \sum_{y\sim x} a^\omega(x,y)\,\big(f(y)-f(x)\big) - h\,\kappa^\omega(x)\,f(x),$$

where $\kappa^\omega : \mathbb{Z}^d \to [0,\infty)$ is a positive ergodic killing measure (or potential) and $h \in [0,1]$ is a scalar. Then, the associated random walk X is the continuous-time Markov process on $\mathbb{Z}^d$, which jumps from y to z at rate $a^\omega(x,y)$ and is killed when visiting x at rate $h\kappa^\omega(x)$.

Theorem 3 ([3])**.** *Let $d \geq 3$. Assume there exist $p, q \in (1,\infty]$ with $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ such that $\mathbb{E}^u\big[\mu^\omega(0)^p\big] + \mathbb{E}^u\left[\nu^\omega(0)^q\right] + \mathbb{E}^u\left[\kappa^\omega(0)^p\right] + \mathbb{E}^u\left[\kappa^\omega(0)^{-1}\right] < \infty$. Then, for $\mathbb{P}^u$-a.e. ω and any $x \in \mathbb{Z}^d$, there exists $N(\omega,x)$ such that for all $y \in \mathbb{Z}^d$ with $|x-y| \geq N(\omega,x)$ and all $h \in [0,1]$,*

$$g^\omega(x,y) \leq c_3\, F_\gamma\big(h\,|x-y|^2\big)\,|x-y|^{2-d} \max_{z \in B(x,n)^c} \Big(e^{-c_4\sqrt{h} d^\omega_\kappa(x,z)}\Big),$$

where $n = |x-y|/4$, $F_\gamma(r) := (1+r)^\gamma(1+1/r)^{1/2}$ for some $\gamma > 0$ and

$$d^\omega_\kappa(x,y) := \inf_{\pi:\, x \overset{\pi}{\leftrightarrow} y} \sum_{e=(e^-,e^+)\in\pi} \left(1 \wedge \frac{\kappa^\omega(e^-) \wedge \kappa^\omega(e^+)}{a^\omega(e)}\right)^{1/2}.$$

This upper bound on $g^\omega(x,y)$ exhibits an artificial polynomial factor $F_\gamma\big(h\,|x-y|^2\big)$ in terms of the graph distance, and an exponential decay governed by an FPP-distance d^ω_κ. Notice that d^ω_κ is dominated by the graph distance, but, as discussed above, it can become much smaller. However, under the additional assumption that the weights in the FPP-distance d^ω_κ satisfy conditions **P1**–**P3**, we can exploit Theorem 1 in order to derive an exponential decay with respect to the graph distance.

Theorem 4 ([3])**.** *In addition to the assumptions in Theorem 3, assume that $\omega \mapsto t^\omega_e := \big(1 \wedge \frac{\kappa^\omega(e^-)\wedge\kappa^\omega(e^+)}{a^\omega(e)}\big)^{1/2}$ is monotone, and the family of measures $(\mathbb{P}^u)_{u\in I}$ satisfies conditions* **P1**–**P3**. *Then, for $\mathbb{P}^u$-a.e. ω and any $x \in \mathbb{Z}^d$, there exists $N(\omega,x)$ such that for all $y \in \mathbb{Z}^d$ with $|x-y| > N(\omega,x)$ and all $h \in [0,1]$,*

$$g^\omega(x,y) \;\leq c_5\,|x-y|^{2-d} \exp\big(-c_6\sqrt{h}|x-y|\big).$$

For the proof of the Green kernel bound in Theorem 3 we follow the strategy established by Agmon in [1] to show exponential decay bounds on eigenfunctions of Schrödinger operators in $\mathbb{R}^d$.

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Biased random walk on dynamical percolation

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(joint work with Sebastian Andres, Dominik Schmid, Perla Sousi)

Biased random walks in random environment have been studied intensively over the last years, the most prominent examples being biased random walk on percolation clusters, see [3, 18, 6, 11] and biased random walks on Galton-Watson trees, see [15, 17, 1, 5]. We focus on the existence and monotonicity properties of the linear speed of the walk and we refer to [2, 9, 10, 12, 13, 14, 16, 8] for a non-exhaustive list of some recent works and to [4] for a survey.

We consider biased random walk in dynamical percolation and study the linear speed of the walk as a function of the parameters. The model is defined as follows. Consider $\mathbb{Z}^d$ and an initial state $\eta \in \{0,1\}^E$ of the edges, E = edges of $\mathbb{Z}^d$. An edge e is **open** at time t if $\eta_t(e) = 1$, and **closed** otherwise.

Fix $\mu \geq 0$ and $p \in [0,1]$. $(\eta_t)_{t\geq 0}$ with $\eta_0 = \eta$ defined as follows: each edge $e \in E$ has an independent Poisson process of rate μ. If there is a point of the Poisson process at time t, we refresh the state of e in η_t, i.e. we declare e open with probability p and closed with probability $1-p$, independently of all other edges and previous states of e. Define a continuous-time random walk $(X_t)_{t\geq 0}$ in the environment $(\eta_t)_{t\geq 0}$ with bias parameter $\lambda > 0$: set $X_0 = 0$ and assign a rate 1 Poisson clock to the particle. When the clock rings at time t and the random walker is currently at a site x, choose one of the neighbours y of x with probability

$$
\begin{aligned}
p(x, x+\mathrm{e}_1) &= \frac{\mathrm{e}^{\lambda}}{Z(\lambda)},\\
p(x, x-\mathrm{e}_1) &= \frac{\mathrm{e}^{-\lambda}}{Z(\lambda)},\\
p(x, x\pm\mathrm{e}_i) &= \frac{1}{Z(\lambda)} \text{ for } i \in \{2,\dots,d\}
\end{aligned}
$$

where $Z(\lambda) = \mathrm{e}^{\lambda} + e^{-\lambda} + 2d - 2$ is a normalizing factor.
If $\eta_t(\{x,y\}) = 1$, the random walker moves from x to y, and it stays at x, otherwise. $(X_t, \eta_t)_{t\geq 0}$ is a λ-biased random walk on dynamical percolation with parameters μ and p. Note that in contrast to the static case, i.e. to the biased random walk on a percolation cluster, we can also take values of p for which the percolation is subcritical. Recall the following invariance principle.

Theorem 1 (Yuval Peres, Alexandre Stauffer, Jeffrey Steif)**.** *For $d \geq 1$, $\mu > 0$, $p \in (0,1)$ and $\lambda = 0$, there exists $\sigma \in (0,\infty)$ so that*

$$\left(\frac{X_{kt}}{\sqrt{k}}\right)_{t\in[0,1]} \overset{(d)}{\to} (\sigma B_t)_{t\in[0,1]}$$

in $D[0,1]$ as $k \to \infty$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

We show the following results.

Theorem 2. *Let $d \geq 1$ and let $(X_t, \eta_t)_{t\geq 0}$ be a λ-biased random walk on dynamical percolation on $\mathbb{Z}^d$ with parameters $\mu > 0$ and $p \in (0,1)$. Then for all $\lambda > 0$, there exists $v_1(\lambda) = v_1(\lambda,\mu,p)$ such that almost surely*

$$\lim_{t\to\infty} \frac{X_t}{t} = (v_1(\lambda), 0, \dots, 0)\,.$$

Further, the function $\lambda \mapsto v_1(\lambda)$ is strictly positive for all $\lambda > 0$, continuously differentiable and satisfies with σ from the previous theorem

$$\lim_{\lambda\to 0} v_1'(\lambda) = \sigma^2. \tag{1}$$

Moreover, we have, for all choices of $p \in (0,1)$ and $\mu > 0$,

$$\lim_{\lambda\to\infty} v_1(\lambda,\mu,p) = \frac{\mu p}{1+\mu-p}\,. \tag{2}$$

(1) is known as “Einstein relation”.

We now address the following question: is the speed increasing as a function of the bias for λ large enough? This is true for biased random walks on Galton-Watson trees without leaves, see [5] and for biased random walk among i.i.d., uniformly elliptic conductances, see [7].

Theorem 3 (Eventual monotonicity of the speed for $d \geq 2$)**.** *Consider the biased random walk on dynamical percolation on $\mathbb{Z}^2$ for $d \geq 2$. For all $p \in (0,1)$ and $\mu > 0$ there exists some $\lambda_0 = \lambda_0(p,\mu)$ such that the following holds.*

1. *The speed $v_1(\lambda)$ is strictly increasing for all $\lambda \geq \lambda_0$ provided that $\mu^2 > p(1-p)$.*
2. *The speed $v_1(\lambda)$ is strictly decreasing for all $\lambda \geq \lambda_0$ provided that $\mu^2 < p(1-p)$.*

We know that the speed is monotone in the bias for the following choices of the parameters μ and p.Theorem

Proposition 1. *Fix $p \in (0,1)$. There exists some constant $\tilde{\mu} = \tilde{\mu}(p) > 0$ such that for all $\mu > \tilde{\mu}$, we have that $\lambda \mapsto v_1(\lambda,\mu,p)$ is increasing in $\lambda > 0$.*

Proposition 2. *Fix $\mu > 0$. There exists some constant $\tilde{p} = \tilde{p}(\mu) > 0$ such that for all $p \in (\tilde{p},1)$, we have that $\lambda \mapsto v_1(\lambda,\mu,p)$ is increasing in $\lambda > 0$.*

Ingredients of our proofs are:

- A clever definition of regeneration times (whose law does not depend on λ!), due to Jonathan Hermon and Perla Sousi, see also Yuval Peres, Alexandre Stauffer, Jeffrey Steif.

- Comparison with the $\lambda = 0$ case via Radon-Nikodym derivatives
- Coupling arguments.

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Sub-diffusive scaling regimes for one-dimensional Mott variable-range hopping

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(joint work with Ryoki Fukushima, Stefan Junk)

Across [2] and [3], anomalous, sub-diffusive scaling limits for a one-dimensional version of the Mott random walk are derived. In the first article, the setting we consider nonetheless results in polynomial space-time scaling. In this case, the limiting process can be viewed heuristically as a one-dimensional diffusion with an absolutely continuous speed measure and a discontinuous scale function, as given by a two-sided stable subordinator. Corresponding to intervals of low conductance in the discrete model, the discontinuities in the scale function act as barriers off which the limiting process reflects for some time before crossing. Our proof relies on a recently developed theory that relates the convergence of processes to that of associated resistance metric measure spaces. In the second article, we consider a regime that exhibits even more severe blocking (and sub-polynomial scaling). We establish that, for any fixed time, the appropriately-rescaled Mott random walk is situated between two environment-measurable barriers, the locations of which have an extremal scaling limit. Moreover, we give an asymptotic description of the distribution of the Mott random walk between the barriers that contain it.

To provide more detail, we now introduce the model of interest. Let $\dots < \omega_{-2} < \omega_{-1} < \omega_0 = 0 < \omega_1 < \omega_2 < \cdots$ be the atoms of a homogeneous Poisson process on $\mathbb{R}$ with intensity $\rho \in (0,\infty)$, conditioned to have an atom at zero (i.e. sampled according to the relevant Palm distribution). The points $\omega = (\omega_i)_{i\in\mathbb{Z}}$ represent electron localisation sites, and to capture corresponding energy marks, we suppose $E = (E_i)_{i\in\mathbb{Z}}$ is an independent and identically distributed (i.i.d.) family of random variables on $\mathbb{R}$, independent of $(\omega_i)_{i\in\mathbb{Z}}$. For a given realisation of the environment variables (ω, E), we define conductances $(c^{\alpha,\beta,\lambda}(x,y))_{x,y\in\omega}$ by setting

$$c^{\alpha,\beta,\lambda}(\omega_i,\omega_j) := \exp\left(-|\omega_i-\omega_j|^\alpha - \beta U(E_i,E_j) + \lambda(\omega_i+\omega_j)\right),$$

where $U\colon \mathbb{R}\times\mathbb{R} \to [0,1]$ is a symmetric function and $\alpha > 0$, $\beta \geq 0$, $\lambda \in \mathbb{R}$ are parameters. Note that, in addition to terms depending on the spatial separation and energy marks (i.e. $|\omega_i - \omega_j|^\alpha$ and $\beta U(E_i,E_j)$), we model the effect of an external field with $\lambda(\omega_i+\omega_j)$. The version of the Mott random walk we study is the continuous-time Markov chain $X^{\alpha,\beta,\lambda} = (X_t^{\alpha,\beta,\lambda})_{t\geq 0}$ on ω with generator

$$(L^{\alpha,\beta,\lambda} f)(\omega_i) := \sum_{j\in\mathbb{Z}} \frac{c^{\alpha,\beta,\lambda}(\omega_i,\omega_j)}{c^{\alpha,\beta,\lambda}(\omega_i)} \left(f(\omega_j) - f(\omega_i)\right),$$

where $c^{\alpha,\beta,\lambda}(\omega_i) := \sum_{j\in\mathbb{Z}} c^{\alpha,\beta,\lambda}(\omega_i,\omega_j)$ is the invariant measure. (Note that this is not well-defined for all α,β,λ.) We write $P^{\alpha,\beta,\lambda}$ for the law of $X^{\alpha,\beta,\lambda}$ started from 0, conditional on (ω,E); this is the so-called quenched law of $X^{\alpha,\beta,\lambda}$. The corresponding annealed law is obtained by integrating out the randomness of the environment, i.e. $\mathbb{P}^{\alpha,\beta,\lambda} := \int P^{\alpha,\beta,\lambda}(\cdot)\mathbf{P}(\mathrm{d}\omega \mathrm{d}E)$, where $\mathbf{P}$ is the probability measure on the probability space upon which the pair (ω,E) is built. Specifically, we assume $P^{\alpha,\beta,\lambda}$ and $\mathbb{P}^{\alpha,\beta,\lambda}$ are probability measures on the space of càdlàg functions $D([0,\infty),\mathbb{R})$, equipped with an appropriate Skorohod topology.

It is known that when $\alpha < 1$ or when $\alpha = 1$ and the density of localisation sites is suitably high, that is, when $\rho > 1$, the symmetric Mott random walk undergoes homogenisation. Indeed, in these cases, one has that, for any value of $\beta \geq 0$ and $\mathbf{P}$-a.e. realisation of (ω, E), under the quenched law,

$$\left(n^{-1}X^{\alpha,\beta,0}_{n^2t}\right)_{t\geq 0} \xrightarrow[n\to\infty]{} (B_{\sigma^2 t})_{t\geq 0}, \tag{1}$$

in distribution, where $(B_t)_{t\geq 0}$ is a standard Brownian motion, and $\sigma^2 \in (0,\infty)$ is a deterministic constant [1]. On the other hand, it was also established in [1] that when $\alpha = 1$ and $\rho \leq 1$, the limit at (1) is valid with respect to the annealed law, but with a limiting diffusion constant $\sigma^2 = 0$. The principal goal of [2] was to describe the appropriate scaling for the Mott random walk in this sub-diffusive regime in the symmetric case ($\lambda = 0$). Note that the following statement only applies to the case $\rho < 1$ for technical convenience. The boundary case $\rho = 1$ is also discussed in [2]. Illustrating our results, in Figures 1 and 2 we present some simulations of the Mott random walk (in which time runs upwards.)

Theorem 1. *For $\alpha = 1$ and every $\rho < 1$ and $\beta, \lambda \geq 0$, it holds that as $n \to \infty$,*

$$\mathbb{P}^{\alpha,\beta,\lambda/n}\left((n^{-1}X_{n^{1+1/\rho}t})_{t\geq 0} \in \cdot\right)$$

converge weakly as probability measures on $D([0,\infty),\mathbb{R})$ to the law of a continuous Markov process $Z^{\beta,\lambda}$, which can be described as a generalized one-dimensional diffusion with scale function given by a certain 'λ-tilted' version of a ρ-stable Lévy process and speed measure $e^{2\lambda r/\rho}\mathrm{d}r$.

Our second paper, [3], deals with the case when $\alpha > 1$. For simplicity, we present here a result for a single time point. In the paper, we also provide a functional version, and discuss the distribution of the Mott random walk between its infimum and supremum.

Theorem 2. *Fix $\alpha > 1$, $\lambda \in \mathbb{R}$, $\beta = 0$ and $\rho = 1$. For each fixed $t > 0$, as $n \to \infty$,*

$$\mathbb{P}^{\alpha,\beta,\lambda/n}\left(\left(n^{-1}\underline{X}_{nL^{-1}(nt)}, n^{-1}\overline{X}_{nL^{-1}(nt)}\right) = \left(m^{-1}_{n,-}(t), m^{-1}_{n,+}(t)\right)\right) \to 1.$$

where $\underline{X}$ and $\overline{X}$ are the running infimum and supremum of X, $L(u)$ is given by slowly-varying function $e^{\log^{1/\alpha}(u)}$ for $u \geq 1$, and $m^{-1}_{n,-}$ and $m^{-1}_{n,-}$ are environment-measurable processes that converge in distribution to certain extremal processes.

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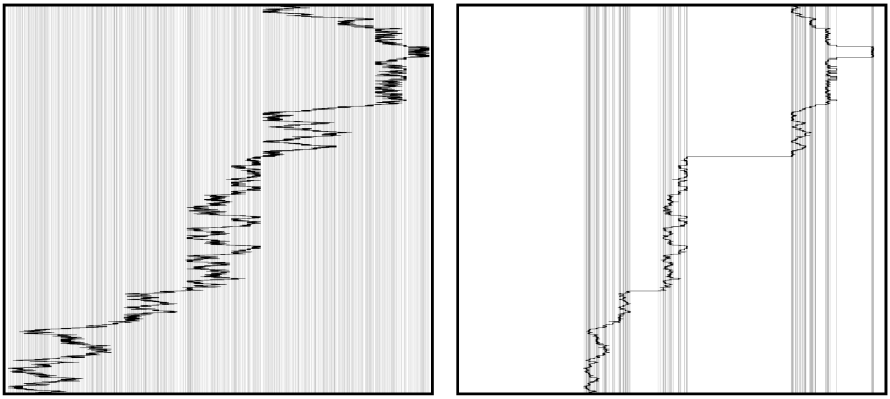

FIGURE 1. Simulation of $(X_t)_{t\geq 0}$ in the case $\alpha = 1$, $\rho = 0.75$, $\beta = \lambda = 0$, and $3 \cdot 10^6$ steps. The left graph shows the process in physical space, with vertical lines indicating the environment $\{\omega_i : i \in \mathbb{Z}\}$. The vertical lines in the right graph denote the coordinates in 'resistance space'. In resistance space, the process behaves like the trace of Brownian motion, meaning it cannot easily cross large gaps. In physical space, the gaps in the environment ω disappear asymptotically, but their effect is still visible.

Gradient estimates of the heat kernel in the Random Conductance Model

JEAN-DOMINIQUE DEUSCHEL
(joint work with Takashi Kumagai, Martin Slowik)

It is well known that the heat kernel associated with a uniform elliptic operator in divergence form has nice regularity properties, in particular Hölder continuity in both time and space variables, cf. Nash-Moser-De Giorgi [3, 10, 8]. These results can be extended to the discrete setting for the heat kernel of a random walks with symmetric jump rates, uniformly in the diffusive scaling parameter, cf. [4]. However, unless the parameters are smooth enough, the picture is quite different for the gradient of the heat kernel. In particular in the random media setting, one cannot expect good estimates in the quenched regime i.e. for almost all realisation of the environment, but only in the annealed i.e. averaged over the law of the environment, cf. [5, 6, 9, 2]. The analytical method used in [5] and [6, 9, 2] relies heavily on the uniform ellipticity assumption, so that it is not well suited for degenerate situations such as unbounded coefficients or random walks on random percolation clusters. Recently [1] proposed a different approach based on a more robust entropy method for a discrete time random walk on the supercritical percolation cluster. They obtained very sharp annealed results for the discrete gradient of the corresponding heat kernel, which they used in order to prove a Liouville principle for sublinear harmonic functions. Our objective in the present talk is

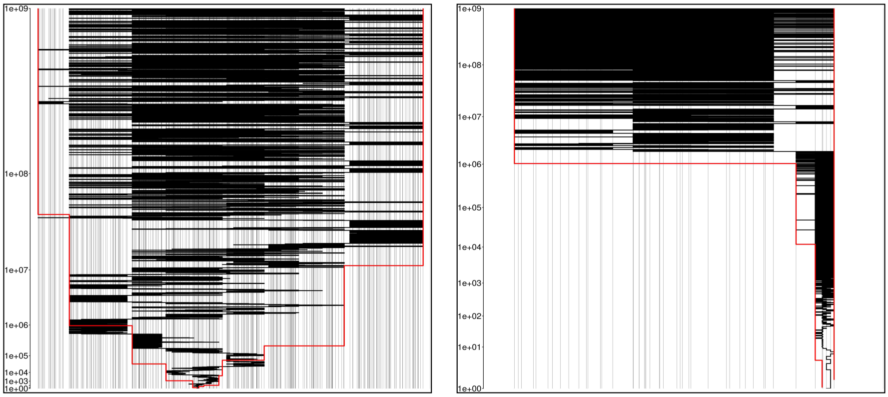

FIGURE 2. Simulation of the Mott random walk X for 10^9 steps with $\alpha = 1.5$ (left) and $\alpha = 2.5$ (right), and, in each case, $\lambda = \beta = 0$ and $\rho = 1$. Space runs along the horizontal axis, and time runs upwards, with the vertical axes being rescaled as $t \mapsto L^{-1}(t)$ (note that L depends on α). The vertical lines indicate the atoms of ω, and the red curves indicate the ω-measurable processes $m_{n,+}$ and $m_{n,-}$, which describe the space-time region that the Mott random walk is highly likely to both be contained within and fully explore.

to adapt the entropy method to the variable speed random walk with dynamical random conductances, bounded from below but unbounded from above. Under the minimal assumption of space-time stationarity of the environment and finite first moment of the conductance we then obtain sharp, scaling invariant annealed on-diagonal estimate for the first and second discrete derivative of the heat kernel, which we then use in order to prove local limit theorem for the annealed heat kernel and discrete first derivative. Moreover assuming further off diagonal estimates, we can extend these annealed results to the Green function and corresponding first and second discrete derivatives.

We consider a random walk among time-dependent random conductances. In recent years the long-time behaviour of this model under diffusive rescaling has been intensively studied, and – depending on the assumptions on the law of the environment – is fairly well understood. In this talk, we discuss how to obtain first and second space derivatives of the annealed transition density using entropy estimates that has been developed in the time-independent setting in a paper by Benjamini, Duminil-Copin, Kozma, and Yadin [1].

Consider the d-dimensional Euclidean lattice, $(\mathbb{Z}^d, E^d)$, for $d \geq 1$, where E^d denotes the set of all oriented nearest neighbor bonds, i.e. $E^d := \{(x, y) : x, y \in \mathbb{Z}^d, |x-y| = 1\}$. We also write $x \sim y$ if $(x, y) \in E^d$. The graph $(\mathbb{Z}^d, E^d)$ is endowed with a family $\omega = \{\omega_t(e) : e \in E^d,\, t \in \mathbb{R}\} \in \omega := (0, \infty)^{\mathbb{R} \times E^d}$ of time-dependent, positive weights.

Further, consider a probability measure, $\mathbb{P}$, on the measurable space $(\Omega, \mathcal{F})$, and we write $\mathbb{E}$ to denote the corresponding expectation with respect to $\mathbb{P}$.

Assumption 1. *Assume that $\mathbb{P}$ satisfies the following conditions:*

(i) $\mathbb{P}$ is stationary and ergodic with respect to time-space shifts.

(ii) The time dependent random conductances are $\mathbb{P}$-a.s. symmetric, that is,

$$\mathbb{P}\big[\omega_t(x,y) = \omega_t(y,x) \quad \forall\, t \in \mathbb{R}\big] \;=\; 1.$$

(iii) $\mathbb{E}\big[\omega_t(e)\big] < \infty$ for all $e \in E^d$ and $t \in \mathbb{R}$.

For any fixed realization $\omega \in \Omega$, we consider the time-inhomogeneous Markov process, $X \equiv \big((X_t : t \geq s), \mathbb{P}^{\omega}_{s,x} : (s,x) \in \mathbb{R} \times \mathbb{Z}^d\big)$ in the random environment ω, where $\mathbb{P}^{\omega}_{s,x}$ denotes the law of X on $\mathcal{D}(\mathbb{R}, \mathbb{Z}^d)$, the space of $\mathbb{Z}^d$-valued càdlàg functions on $\mathbb{R}$, starting at time s in the vertex x, i.e.

$$\mathbb{P}^{\omega}_{s,x}\big[X_s = x\big] \;=\; 1 \qquad \mathbb{P}\text{-a.s.}$$

The time-dependent generator, $\mathcal{L}^{\omega}_t$, acts on bounded functions $f \colon \mathbb{Z}^d \to \mathbb{R}$ as

$$(\mathcal{L}^{\omega}_t f)(x) \;\coloneqq\; \sum_{y:(x,y)\in E^d} \omega_t(x,y)\,\big(f(y) \,-\, f(x)\big).$$

For any $s \in \mathbb{R}$, we denote by $(P^{\omega}_{s,t} : t \geq s)$ the Markov semigroup associated to the Markov process X, i.e. $(P^{\omega}_{s,t} f)(x) = \mathbb{E}^{\omega}_{s,x}[f(X_t)]$ for any bounded function $f \colon \mathbb{Z}^d \to \mathbb{R}$, $s \leq t$ and $x \in \mathbb{Z}^d$. Moreover, for any $x, y \in \mathbb{Z}^d$ and $s \leq t$, we denote by

$$p^{\omega}_{s,t}(x,y) \;\coloneqq\; \mathbb{P}^{\omega}_{s,x}\big[X_t = y\big] \qquad \text{and} \qquad \bar{p}_{s,t}(x,y) \;\coloneqq\; \mathbb{E}\big[p^{\omega}_{s,t}(x,y)\big].$$

the (quenched) transition density with respect to the counting measure (heat kernel of the so-called VSRW) and the annealed (averaged) transition density, respectively. As a consequence of the stationarity with respect to time-space shifts we have

$$p^{\tau_{h,z}\omega}_{s,t}(x,y) \;=\; p^{\omega}_{s+h,t+h}(x+z, y+z) \qquad \forall\, h \in \mathbb{R},\, z \in \mathbb{Z}^d.$$

In particular, in view of Assumption 1-(i), it holds

$$\bar{p}_{s,t}(x,y) \;=\; \bar{p}_{0,t-s}(0, y-x) \;=\; \bar{p}_{s-t,0}(x-y, 0).$$

As a further consequence of Assumption 1 the averaged mean displacement of the stochastic process, $(X_t : t \geq 0)$, behaves diffusively.

Proposition 1. *Suppose that Assumption 1 holds. Then, there exists $C_1 < \infty$ such that the following holds for all $T > 0$,*

$$\mathbb{E}\Big[\mathbb{E}^{\omega}_{0,0}\Big[\sup_{0 \leq t \leq T} |X_t|\Big]\Big] \;\leq\; C_1 \sqrt{T}.$$

Our further results rely on the following additional assumption.

Assumption 2. *There exists a non-random constant $C_2 > 0$ such that*

$$\mathbb{P}[\omega_t(x,y) \geq C_2] \;=\; 1 \qquad \textit{for any } (x,y) \in E^d.$$

Our main objective is to prove a spatial derivative estimate of the annealed heat kernel. For a given function $f\colon \mathbb{Z}^d \to \mathbb{R}$ the discrete spatial derivative in direction $(0,x) \in E^d$ is defined by

$$\nabla_x f(y) \;:=\; f(x+y) - f(y), \qquad \forall\, y \in \mathbb{Z}^d.$$

Theorem 1. *Suppose that Assumption 1 and 2 hold true.*

(i) There exists $C_3 < \infty$ such that for any $(0,x) \in E^d$, $y, y' \in \mathbb{Z}^d$ and $t > 0$,

$$\mathbb{E}\Big[\big|\nabla_x^1\, p_{0,t}^{\omega}(y,y')\big|^2\Big]^{1/2} \;\leq\; C_3\,(1 \vee t)^{-(d+1)/2}.$$

Moreover, the same estimate holds if $\nabla_x^1\, p_{0,t}^{\omega}(y,y')$ is replaced by $\nabla_x^2\, p_{0,t}^{\omega}(y',y)$.

(ii) There exists $C_4 < \infty$ such that, for any $p \in [1,2]$, $(0,x) \in E^d$, $y, y' \in \mathbb{Z}^d$ and $t > 0$

$$\bigg(\sum_{y' \in \mathbb{Z}^d} \mathbb{E}\Big[\big|\nabla_x^1\, p_{0,t}^{\omega}(y,y')\big|^p\Big]\bigg)^{1/p} \;\leq\; C_4\,(1 \vee t)^{-(1+d(1-1/p))/2}.$$

Moreover, the same estimate holds if $\nabla_x^1\, p_{0,t}^{\omega}(y,y')$ is replaced by $\nabla_x^2\, p_{0,t}^{\omega}(y',y)$.

(iii) For any $(0,x), (0,x') \in E^d$, $y, y' \in \mathbb{Z}^d$, and $t > 0$,

$$\begin{aligned}\big|\nabla_x^2 \nabla_{x'}^2\, \bar{p}_{0,t}(y,y')\big| \;&\leq\; C_4\,(1 \vee t)^{-(d+2)/2},\\ \mathbb{E}\Big[\Big|\nabla_{-x}^1 \nabla_{x'}^2\, p_{0,t}^{\omega}(y,y')\Big|\Big] \;&\leq\; C_4\,(1 \vee t)^{-(d+2)/2}.\end{aligned}$$

Off-diagonal upper bound and Green kernel estimates. In this subsection, we consider annealed off-diagonal upper bounds under the additional quenched assumption as follows.

Assumption 3. *For some $\alpha, \beta, \gamma \in (0,\infty)$ assume that there exist non-random constants $c_1, c_2 \in (0,\infty)$ and positive random variables $\{N_y(\omega) : y \in \mathbb{Z}^d\}$ such that the following holds true: for $\mathbb{P}$-a.e. ω and for all $y, y' \in \mathbb{Z}^d$, $s \geq 0$ such that $|y-y'| \vee t^{1/2} \geq N_y(\omega)$*

$$p_{s,s+t}^{\omega}(y,y') \;\leq\; \begin{cases} c_1 |y-y'|^{-\alpha}, & \text{when } t \leq |y-y'| \\ c_1 t^{-d/2}\bigg(1 \vee \dfrac{|y-y'|}{\sqrt{t}}\bigg)^{-\beta}, & \text{when } t \geq |y-y'|. \end{cases}$$

Moreover,

$$\mathbb{P}\big[N_y(\omega) \geq L\big] \;\leq\; c_2\, L^{-\gamma}, \quad \forall\, L > 0.$$

Proposition 2. *Suppose that Assumption 1 and 2 hold. Additionally, let Assumption 3 be satisfied for some $\alpha, \beta, \gamma \in (0,\infty)$. Then, there exist $C_9 < \infty$ such that* *for any* $(0,x) \in E^d$, $y, y' \in \mathbb{Z}^d$ *and* $t \geq 1$,

$$\mathbb{E}\Big[\big|\nabla_x^1 \, p_{0,t}^{\omega}(y,y')\big|^2\Big]^{1/2} \leq C_9 \begin{cases} |y-y'|^{-\alpha} + t^{-d/2}\,|y-y'|^{-\gamma/2}, & \text{if } t \leq |y-y'|, \\ t^{-(d+1)/2}, & \text{if } t \geq |y-y'|^2, \\ t^{-(d+1)/2}\left(\frac{|y-y'|}{\sqrt{t}}\right)^{-\beta/2} + t^{-d/2}\,|y-y'|^{-\gamma/2}, & \text{otherwise.} \end{cases}$$

Moreover, the same estimate holds if $\nabla_x^1 \, p_{0,t}^{\omega}$ *is replaced by* $\nabla_x^2 \, p_{0,t}^{\omega}$.

Our next result deals with the annealed CLT. For this purpose, let

$$p_{s,t}^{\omega,n}(y,y') \;:=\; n^d p_{n^2 s, n^2 t}^{\omega}([ny],[ny']), \qquad s<t, y,y' \in \mathbb{R}^d$$

be the rescaled heat kernel,

$$\nabla_{x'}^{2,n}\, p_{s,t}^{\omega,n}(y,y') \;:=\; n^{d+1}\, \nabla_{x'}^2 \, p_{n^2 s,n^2 t}^{\omega}([ny],[ny'])$$

the rescaled first derivative and

$$\nabla_x^{2,n}\nabla_{x'}^{2,n}\, p_{s,t}^{\omega,n}(y,y') \;:=\; n^{d+2}\, \nabla_x^2\nabla_{x'}^2 \, p_{n^2 s,n^2 t}^{\omega}([ny],[ny'])$$

the rescaled second derivative. We write $\bar{p}_{s,t}^{n}(0,x) = \mathbb{E}[p_{s,t}^{\omega,n}(0,x)]$.

Then annealed estimates we derived in the previous section hold uniformly in the scaling parameter n, that is under Assumption 1 and 2, there exists $c_1 < \infty$ such that for any $y,y' \in \mathbb{R}^d$, $(0,x),(0,x') \in E^d$ and $t>0$,

$$\mathbb{E}\big[|\nabla_x^{2,n}\, p_{0,t}^{\omega,n}(y,y')|^2\big]^{1/2} \;\leq\; c_1\, t^{-(d+1)/2}.$$

$$\big|\nabla_x^{2,n}\nabla_{x'}^{2,n}\, \bar{p}_{0,t}^{n}(y,y')\big| \;\leq\; c_1 t^{-(d+2)/2}.$$

Let $p_t^{\Sigma}(0,\cdot)$ be the density of $\mathcal{N}(0,t\Sigma)$, the centered gaussian distribution on $\mathbb{R}^d$ with variance $t\Sigma$.

Proposition 3. *Under Assumption 1 and 2, then for each* $t>0$*, the annealed law of* $n^{-1}X_{n^2 t}$ *converges weakly to* $\mathcal{N}(0,t\Sigma)$*. Moreover for each* $t_0>0, K>0$

$$\lim_{n\to\infty} \sup_{t\geq t_0, |x|\leq K} \big|\bar{p}_t^n(0,x) - p_t^{\Sigma}(0,x)\big| \;=\; 0$$

and for each $e_i \sim 0, i=1,...,d$

$$\lim_{n\to\infty} \sup_{t\geq t_0, |x|\leq K} \big|\nabla_{e_i}^{2,n}\, \bar{p}_t^n(0,x) - \partial_i p_t^{\Sigma}(0,x)\big| \;=\; 0.$$

Using the local CLT we can near diagonal annealed lower bounds for the annealed heat kernel and its first derivative. (Note that the usual arguments to obtain on-diagonal lower bound (see for instance [7, Proposition 4.3.4]) cannot be used for time-dependent case since the proof uses symmetry of the heat kernel.) This shows that our upper bound is correct, at least near the diagonal. Notice that a local CLT and corresponding lower bound for the annealed second derivative is still open.

Proposition 1. *Under Assumptions 1 and 2, the following near-diagonal lower bound holds: For all $c_1 > 0$, there exists $c_2 > 0$ and $T_1 > 0$ such that*

$$\bar{p}_{0,t}(0,y) \;\geq\; c_2 t^{-d/2} \qquad \forall\, |y| \leq c_1 t^{1/2},\; t \geq T_1.$$

and

$$\left|\nabla_z^2\, \bar{p}_{0,t}(0,y)\right| \;\geq\; c_2 t^{-(d+1)/2} \qquad \forall\, z \sim 0,\, |y| \leq c_1 t^{1/2},\; t \geq T_1.$$

$$\left|\nabla_z^1\, \bar{p}_{0,t}(0,y)\right| \;\geq\; c_2 t^{-(d+1)/2} \qquad \forall\, z \sim 0,\, |y| \leq c_1 t^{1/2},\; t \geq T_1.$$

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Tamed spaces – Dirichlet spaces with distribution-valued Ricci bounds

Chiara Rigoni

(joint work with Matthias Erbar, Karl-Theodor Sturm, Luca Tamanini)

Synthetic lower Ricci bounds have proven to be a powerful concept for analyzing the geometry of singular spaces. The most prominent versions of such synthetic Ricci bounds are the Eulerian formulation in the setting of Dirichlet spaces by Bakry–Émery and the Lagrangian formulation in the setting of metric measure spaces by Lott–Villani and Sturm.

Bakry and Émery, in their seminal paper [5], characterized synthetic lower Ricci bounds $K \in \mathbb{R}$ for a given strongly local Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ in terms of the generalized Bochner inequality

$$\Gamma_2(f) \geq K \cdot \Gamma(f).$$

Here Γ denotes the carré du champ associated with $\mathcal{E}$ and Γ_2 the iterated carré du champ. For the canonical Dirichlet space with $\mathrm{X} = \mathsf{M}$, $\mathfrak{m} = \mathrm{vol}_{\mathsf{g}}$, and $\mathcal{E}(f) = \frac{1}{2}\int_{\mathsf{M}} |\nabla f|^2 \, \mathrm{d}\mathfrak{m}$ on a Riemannian manifold (M, g) this reads as

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle \geq K \cdot |\nabla f|^2, \tag{1}$$

which in turn is well known – due to Bochner's equality – to be equivalent to $\mathrm{Ric}_{\mathsf{g}} \geq \mathrm{K} \cdot \mathsf{g}$.

A synthetic notion of lower Ricci curvature bounds in the setting of metric measure spaces based on optimal transport has been developed by Lott, Villani and Sturm in [8, 9, 10], leading to a fruitful study of analytic and geometric properties of these structures. In particular, Ambrosio, Gigli and Savaré in a series of seminal papers [1, 2] developed a powerful first order calculus on such spaces leading to natural notions of (modulus of the) gradient, energy functional (called Cheeger energy), and heat flow. For so-called infinitesimally Hilbertian spaces the Cheeger energy is quadratic and defines a Dirichlet form and (under minimal assumptions) the Eulerian and Langrangian approaches to synthetic Ricci bounds have been shown to be equivalent [3, 6, 4], providing in particular a Bochner inequality for metric measure spaces.

In this talk, we develop a generalization of the concept of synthetic lower Ricci bounds that goes far beyond the framework of uniform bounds. Indeed, many important properties and quantitive estimates which typically are regarded as consequences of uniform lower Ricci bounds also hold true in much more general settings.

Our notion of **tamed spaces** will refer to Dirichlet spaces $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ which admit a distribution-valued lower Ricci bound, formulated as a canonical generalization of (1). Roughly speaking, we are going to replace the constant K in (1) by a distribution κ and to consider the inequality in distributional sense, that is, as

$$\int_{\mathrm{X}} \varphi\, \Gamma_2(f) \, \mathrm{d}\mathfrak{m} \geq \big\langle \kappa, \varphi\, \Gamma(\mathrm{f})\big\rangle$$

for all sufficiently regular f and $\varphi \geq 0$. The distributions κ to be considered will lie in the class $\mathcal{F}^{-1}_{\mathrm{qloc}}$. Here $\mathcal{F}^{-1}$ denotes the dual space of the form domain $\mathcal{F} = \mathsf{D}(\mathcal{E})$ and $\mathcal{F}^{-1}_{\mathrm{qloc}}$ denotes the class of κ's for which there exists an exhaustion of X by quasi-open subsets $G_n \nearrow \mathrm{X}$ such that κ coincides on each G_n with some element in $\mathcal{F}^{-1}_{G_n}$.

Already in the case of Riemannian manifolds, our new setting contains plenty of important examples which are not covered by any of the concepts of "spaces with uniform lower Ricci bounds", among them we have

(i) *"Singularity of Ricci at ∞"*: Smooth Riemannian manifolds with Ricci curvature bounded from below in terms of a continuous – but unbounded – function which globally lies in the Kato class.

(ii) *"Local singularities of Ricci"*: Riemannian manifolds with (synthetic) Ricci curvature bounded from below in terms of a locally unbounded function which lies in L^p for some $p > n/2$. Such "singular" manifolds for instance are obtained from smooth manifolds by ground state transformations (see e.g. [7]), conformal transformations, or time changes with singular weight functions.

(iii) *"Singular Ricci induced by the boundary"*: Riemannian manifolds with boundary for which the second fundamental form is bounded from below in terms of a (possibly unbounded) function which lies in L^p w.r.t. the boundary measure for some $p > n - 1$. Such manifolds with boundaries in particular appear as closed subsets of manifolds without boundaries.

(iv) *"Singular Ricci at the rim"*: Doubling of a Riemannian surface with boundary leads to a (nonsmooth) Riemannian surface which admits a uniform (synthetic) lower Ricci bound if and only if the initial surface has convex boundary.

Given a Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ and a distribution $\kappa \in \mathcal{F}^{-1}_{\mathrm{qloc}}$, the crucial quantities to formulate our synthetic lower Ricci bound will be the **taming energy** $\mathcal{E}^\kappa$ – a singular zero-order perturbation of $\mathcal{E}$ – and the **taming semigroup** $(P_t^\kappa)_{t\geq 0}$. The latter allows for a straightforward definition via the Feynman-Kac formula as

$$P_t^\kappa f(x) := \mathbb{E}_x\Big[e^{-A_t^\kappa}\, f(B_t)\Big]$$

in terms of the stochastic process $\big(\mathbb{P}_x, B_t\big)_{x\in\mathrm{X}, t\geq 0}$ properly associated with $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ and in terms of the local continuous additive functional $(A_t^\kappa)_{t\geq 0}$ associated with κ. We say that the distribution κ is **moderate** if

$$\sup_{t\in[0,1]}\ \sup_{x\in\mathrm{X}} \mathbb{E}_x\Big[e^{-A_t^\kappa}\Big] < \infty.$$

In this case, $(P_t^\kappa)_{t\geq 0}$ defines a strongly continuous, exponentially bounded semigroup on $L^2(\mathrm{X}, \mathfrak{m})$ and thus it generates a lower bounded, closed quadratic form $\big(\mathcal{E}^\kappa, \mathsf{D}(\mathcal{E}^\kappa)\big)$. The latter indeed can be identified with the relaxation of the quadratic form

$$\dot{\mathcal{E}}^\kappa(f) := \mathcal{E}(f) + \mathcal{E}_1(\psi_n, f^2)$$

defined on a suitable subset of $\cup_n \mathcal{F}_{G_n}$ where $(G_n)_n$ denotes an exhaustions of X by quasi-open sets G_n such that $\kappa \in \mathcal{F}^1_{G_n}$, and where we set $\psi_n := (-\mathsf{L}_{\mathsf{G_n}} + 1)^{-1}\kappa$. We also provide a condition on κ which guarantees that $\dot{\mathcal{E}}^\kappa$ is closable, in which case $\mathcal{E}^\kappa$ is its closure.

At this point, we say that a Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ is **tamed** if there exists a moderate distribution $\kappa \in \mathcal{F}^{-1}_{\mathrm{qloc}}$ such that the following *Bochner inequality* holds true:

$$\mathcal{E}^{\kappa/2}\big(\varphi, \Gamma(f)^{1/2}\big) + \int \varphi \, \frac{1}{\Gamma(f)^{1/2}} \Gamma\big(f, \mathsf{Lf}\big)\, \mathrm{d}\mathfrak{m} \le 0 \tag{2}$$

for all f and $\varphi \geq 0$ in appropriate functions spaces. In this case, κ will be called **distribution-valued lower Ricci bound** or **taming distribution** for the Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$.

Theorem 1. *A moderate distribution $\kappa \in \mathcal{F}^{-1}_{\mathrm{qloc}}$ is taming for the Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ if and only if the following* gradient estimate*, briefly $\mathsf{GE}_1(\kappa, \infty)$, holds true:*

$$\Gamma(P_t f)^{1/2} \le P_t^{\kappa/2}\big(\Gamma(f)^{1/2}\big) \tag{3}$$

for all $f \in \mathcal{F}$.

In the case of a constant κ, (3) reads as $\Gamma(P_t f)^{1/2} \le e^{-\kappa t/2} P_t\big(\Gamma(f)^{1/2}\big)$ which is the well-known, “improved version” of the gradient estimate in the Bakry-Émery theory.

Besides the fundamental gradient estimates, tamed spaces share many important properties with spaces which admit uniform lower Ricci bounds. A selection of important *quantitative* properties is listed below.

Theorem 2. *Assume that the Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ is tamed by a 2-moderate distribution $\kappa \in \mathcal{F}^{-1}_{\mathrm{qloc}}$. Then the following functional inequalities hold true, say for $t \le 1$,*

(i) Local Poincaré inequality:

$$P_t(f^2) - (P_t f)^2 \le Ct\, P_t(\Gamma f);$$

(ii) Reverse local Poincaré inequality:

$$P_t(f^2) - (P_t f)^2 \ge t/C\, \Gamma(P_t f);$$

(iii) Local log-Sobolev inequality:

$$P_t(f \log f) - P_t f \log(P_t f) \le \int_0^t P_s P^{\kappa}_{t-s}\left(\frac{\Gamma f}{f}\right) \mathrm{ds};$$

(iv) Reverse local log-Sobolev inequality:

$$P_t(f \log f) - P_t f \log(P_t f) \ge \int_0^t \frac{\Gamma(P_t f)}{P_s^{\kappa/2} P_{t-s} f}\, \mathrm{dsA}$$

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Conformally Invariant Random Fields, Quantum Liouville Measures, and Random Paneitz Operators on Riemannian Manifolds of Even Dimension

Lorenzo Dello Schiavo
(joint work with Ronan Herry, Eva Kopfer, and Karl-Theodor Sturm)

We construct and study conformally invariant random fields of distributions on large classes of *even-dimensional* Riemannian manifolds (M, g). These *co-polyharmonic Gaussian fields* $h = h_g$ are centered Gaussian fields characterized by their covariance kernels k, exhibiting a precise logarithmic divergence:

$$\left| k(x, y) - \log \frac{1}{d(x, y)} \right| \le C\,.$$

They share the fundamental quasi-invariance property under conformal transformations: if $g' = e^{2\varphi} g$, then

$$h_{g'} \overset{(\mathrm{d})}{=} e^{n\varphi} h_g - C \cdot \mathrm{vol}_{g'}$$

for an appropriate random variable $C = C_\varphi$.

In terms of the co-polyharmonic Gaussian field h, we define the *Liouville Quantum Gravity measure*, a random measure on M, heuristically given as

$$d\mu_g^h(x) := e^{\gamma h(x) - \frac{\gamma^2}{2} k(x,x)}\, d\mathrm{vol}_g(x)\,,$$

and rigorously obtained as almost sure weak limit of the right-hand side with h replaced by suitable regular approximations h_ℓ, $\ell \in \mathbb{N}$. Again, these measures share a crucial quasi-invariance property under conformal transformations: if $g' = e^{2\varphi} g$, then

$$d\mu_{g'}^{h'}(x) \overset{(\mathrm{d})}{=} e^{F^h(x)}\, d\mu_g^h(x)$$

for an explicitly given random variable $F^h(x)$.

In terms on the Liouville Quantum Gravity measure, we define the *Liouville Brownian motion* on M and the *random critical Graham–Jenne–Mason–Sparling* *operators*. Finally, we present an approach to a conformal field theory in arbitrary even dimensions with an ansatz based on Branson's Q-curvature: we give a rigorous meaning to the *Polyakov–Liouville measure*

$$d\boldsymbol{\nu}_g(h) = \frac{1}{Z_g} \exp\left(- \int \Theta\, Q_g h + m e^{\gamma h} d\mathrm{vol}_g \right) \exp\left(-\frac{a_n}{2} \mathfrak{p}_g(h,h) \right) dh$$

for suitable positive constants Θ, m, γ and a_n, and we derive the corresponding *conformal anomaly*.

The set of *admissible* manifolds is conformally invariant. It includes all compact 2-dimensional Riemannian manifolds, all compact non-negatively curved Einstein manifolds of even dimension, and large classes of compact hyperbolic manifolds of even dimension. However, not every compact even-dimensional Riemannian manifold is admissible.

Our results concerning the logarithmic divergence of the kernel k — defined as the Green kernel for the Graham–Jenne–Mason–Sparling operators on (M, g) — rely on new sharp estimates for heat kernels and higher order Green kernels on arbitrary compact manifolds.

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