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Mini-Workshop: New Horizons in Motions in Random Media

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Abstract. The general topic of the mini-workshop *New Horizons in Motions in Random Media* was the study of random walks in random environments, both in their own right and in relation to stochastic homogenization and to models in statistical mechanics, in particular spin system. This is a subject at the intersection of probability, analysis and mathematical physics, and the workshop brought together leading researchers from those areas. While each of these areas has been quite active for decades with many remarkable breakthroughs obtained throughout the years, the workshop provided a unique opportunity to identify principal new objectives and initiate new collaborations.

Mathematics Subject Classification (2020): 60K37, 60G50, 60G60, 35B27, 35R60, 82B31, 82B43, 82B44, 82C41.

Introduction by the Organizers

The mini-workshop *New Horizons in Motions in Random Media*, organised by Sebastian Andres (University of Manchester), Marek Biskup (UCLA), Alessandra Faggionato (La Sapienza Rome) and Martin Slowik (Universität Mannheim) was well attended with 20 participants (16 on-side and 4 digital) from China, Germany, France, Hungary, Israel, Italy, Japan, United Kingdom, and USA. The program consisted of 4 introductory lectures and 12 talks, leaving sufficient time for discussions.

The general topic of the workshop was to study random motions in random media which is a subject at the intersection of probability, analysis and mathematical physics. More specifically the main topics of our prime interest were:

- general random walks in random environment (RWRE),
- random walks in balanced and divergence-free environments,
- stochastic homogenization of reversible random walks,
- reinforced jump processes,
- random walk representations of spin systems and other models in statistical mechanics.

Those areas witnessed remarkable advances over the last years, while there is a significant overlap between these areas and progress there has happened largely in parallel. A number of participants are experts mainly in one of these areas and yet their work transcends to the other areas as well. The workshop provided the unique opportunity to bring some of these experts together and allow them to share their knowledge and technical expertise. In order to achieve those goals, the meeting was run in a format that focused primarily on discussions, problem-solving sessions and networking. Each day was started by one or two long lectures whose purpose was to give overviews of specific areas and thus set the framework for other activities for the rest of the day. Due to the very high level of interaction during those lectures, some of them were continued in the afternoon. The afternoon activities included a number of more specialised talks, and one after-dinner open problem session. The selection of topics and mix of participants stimulated many extensive and fruitful discussions. It also helped initiating new collaborations, and strengthen existing ties between researchers in different fields of mathematics.

The overview lecture to general random walks in random environments by Ofer Zeitouni (Weizmann Institute, Israel) reviewed the classical theory of RWRE, initiated in the early 1970s, with a particular focus on ballisticity conditions, large deviations and perturbative approaches. In the second overview lecture Noam Berger (TU Munich, Germany) summarized the state of the art for random walk in balanced random environment. One subclass of RWREs which is quite well understood is that for which the transition probabilities are reversible. In this case the model is usually prescribed in terms of so called *conductances* which are non-negative numbers attached to the edges of the underlying graph. Major research activity has been devoted to invariance principles for random conductance models, and the most recent state of the art result was presented by Peter Bella (TU Dortmund). Alessandra Faggionato (La Sapienza Rome, Italy) presented recent results and some open problems for reversible random walks on point sets in the Euclidean space derived, for instance, from a Poisson point process by taking its points directly or by considering Voronoi tesselations. Moreover, Paul Dario (U Paris Est, France) explained the classical links between RWRE and the Ginzburg-Landau $\nabla\phi$ interface model in statistical mechanics via the so-called Helffer-Sjöstrand representation.

Random walks in divergence-free environments, a subclass of non-reversible RWREs, are characterized by the fact that the underlying shift-invariant law of the environment remains invariant for the "point of view of the particle". An overview lecture on this topic was delivered by Bálint Tóth (Budapest/Bristol). One example falling into this class are RWRE with a so-called cycle representation, and Weile Weng (TU Berlin, Germany) presented an invariance principle for such RWREs obtained as part of her PhD thesis.

Percolation models were discussed in the overview lectures given by Pierre-Francois Rodriguez (Imperial College London, UK) with a particular focus on the current state of the art for questions concerning phase transitions in random interlacements. David Croydon (RIMS Kyoto, Japan), explained in his talk the Alexander-Orbach conjecture on the heat kernel behaviour of random walks on critical percolation clusters and discussed closely related questions for random walks on self-similar fractals and on certain classes of random trees. In the talk by Artem Sapozhnikov (U Leipzig, Germany) some first results on continuum percolation were discussed which may serve as a basis for future studies of diffusions in degenerate continuum environments.

An introduction to the vertex reinforced jump process (VRJP) and its relation to hyperbolic spin model in statistical mechanics was presented in the overview lecture delivered by Pierre Tarres (NYU Shanghai, China). The talk of Margherita Disertori (U Bonn, Germany) focussed on the random walk representation in terms of the VRJP in the $\mathbb{H}^{2,2}$-model, which is a supersymmetric hyperbolic spin model in statistical mechanics, motivated by the Anderson transition. Further, Silke Rolles (TU Munich, Germany) presented recent results on the VRJP with long-range interactions. In an online presentation, Lorenzo Taggi (La Sapienza Rome, Italy) discussed a general random walk loop soup in relation to several important statistical mechanics models, such as the $O(N)$ model, the dimer model or the Bose Gas.

Mini-Workshop: New Horizons in Motions in Random Media

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Random walk in random environment, I

Ofer Zeitouni

We review some of the history and open problems concerning nearest neighbor random walks in random environment on $\mathbb{Z}^d$; one is given a collection of (typically iid) random vectors $\{\omega(x,e)\}_{x,e\in\mathbb{Z}^d,|e|=1}$ (*the environment*) and a Markov chain X_n with (random) transition kernel $P_\omega\big[X_{n+1} = x+e \,|\, X_n = e\big] = \omega(x,e)$ (*the random walk*). We always assume uniform ellipticity, i.e. the existence of a $\kappa > 0$ so that $\omega(0,e) \geq \kappa$ a.s. for all $|e| = 1$.

Historically, there were several periods of intense activities: the seventies, where much of the results concerning the one dimensional model were laid out, the late nineties and early 00's, where some of the questions concerning $d > 1$ were settled, and the mid '00's till now, where multi-scale ideas came to the fore. Of course, this is only a very rough divisions, and some results (for example, Kalikow's work [12]) are not located within this rough division.

In case $d = 1$, this is a conductance model, with conductances $\prod_{i=1}^{x} \rho_i^{-1}$, where $\rho_x = (1-\omega(x,1))/\omega(x,1)$. The conductances are neither ergodic nor bounded, and this leads to unusual features. In a nutshell, the walk is recurrent iff $\mathbb{E}\big[\log\rho_0\big] = 0$, and is directionally transient otherwise. In case $\mathbb{E}\big[\log\rho_0\big] < 0$, the walks is transient to the right and a critical role is played by the parameter s satisfying $\mathbb{E}\big[\rho^s\big] = 1$: for $s > 1$, the walk is ballistic, for $s \leq 1$ it is sub-ballistic with stable limit laws, for $s \in (1,2)$ there are again stable limit laws (annealed) with a jump-like quenched behavior, and for $s > 2$ one has a CLT, both quenched and annealed. Unusual large deviations, due to *trapping*, are also present. We refer to [16] for a bibliographic account up to 2002, and to [8, 10, 13] for further developments.

In the case of $d \geq 2$, an important role is played by *regeneration times*, which for simplicity we introduce with respect to the preferred direction e_1: those are an increasing sequence of times τ_i so that

$$\tau_{i+1} \;=\; \min\big\{n \geq \tau_i : \max\{X_t \cdot e_1, t < n\} < \min\{X_t \cdot e_i, t \geq n\}, \quad i \geq 0\big\}.$$

Denoting by $\mathbf{P}$ the annealed law $\mathbb{P} \times P_\omega$, it was observed by Sznitman and Zerner that

$$\{\omega(x,\cdot)\}_{x: x\cdot e_1 \in [X_{\tau_i}\cdot e_1, X_{\tau_{i+1}}\cdot e_i)}, \qquad \{X_n\}_{n\in[\tau_i,\tau_{i+1})}$$

forms (under $\mathbf{P}$, for $i \geq 1$) an iid sequence, which moreover has the same law as for $i = 1$ (with $\tau_0 = 0$) under $\mathbf{P}$ conditioned on no-backtracking, i.e. on not visiting the hyperplane $x \cdot e_1 < 0$. From this, Sznitman-Zerner [15] and Zerner [15] were able to deduce the following theorem. Let

$$A_+ \;=\; \big\{X_n \cdot e_1 \to \infty\big\}, \qquad A_- \;=\; \big\{X_n \cdot e_1 \to -\infty\big\}.$$

Theorem 1.

$$\frac{X_n \cdot e_1}{n} \underset{n\to\infty}{\longrightarrow} v_+ \mathbf{1}_{A_+} + v_- \mathbf{1}_{A_-}, \qquad \mathbf{P}\text{-}a.s.$$

Together with the fact that $\mathbf{P}\big[A_- \cup A_+\big] \in \{0,1\}$ (proved in [12]), this puts into focus the following open problem:

Problem 1. *Is* $\mathbf{P}\big[A_+\big] \in \{0,1\}$*?*

As we saw, Problem 1 has a positive answer in $d = 1$, and the case $d = 2$ was settled in [18] using martingale techniques and intersection properties of paths in two dimensions. The case of $d \geq 3$ (which would settle the law of large numbers for X_n) is widely open. We note that the answer to Problem 1 is negative in certain mixing uniformly elliptic environments [6, 11]. We also note that using regeneration times coupled with Green function estimates, one has for $d \geq 5$ that $v_+ \cdot v_- = 0$, see [3].

Regeneration times are an important tool in proving central limit theorems, both quenched and annealed, and in a multi-scale analysis of RWRE, including effective criteria for ballisticity introduced by Sznitman. They are also important in stating several intriguing open questions, described in details in Berger's lecture in these proceedings. We refer to that talk and to the review [9] for further details.

A variant of regeneration times are *cut times*, which do not require a ballistic behavior but instead work well in high dimension (at least 8) and in environments possessing certain symmetry conditions. In particular, they provide examples of RWREs satisfying that the (annealed) drift at 0 has zero mean but the walk is ballistic, or with annealed expected drift pointing in one direction with the walk transient in another; see [4].

An important direction of research concerns the perturbative regime, i.e. environments in which $|\omega(x,e) - 1/2d| \leq \delta$, $\mathbf{P}$-a.s., for some small δ. Starting with [7], one could ask for central limit theorems in situations where the *law* of the environment is symmetric under rotations or flips that preserve $\mathbb{Z}^d$. The basic result of [7] is that, for $d \geq 3$, an invariance principle indeed holds true for $\delta > 0$ small enough. A different path to an invariance principle, with more explicit induction step was provided (in the context of diffusions in random environments) in [14].

Another approach to the perturbative regime was suggested in [5] and extended in [1, 2]; motivated by the fact that controling exit times from balls adds an extra layer of difficulties to proving an invariance principle, the authors consider $\Pi_{L,\omega}(x,y)$ as the exit measure from a ball of radius L in $\mathbb{Z}^d$. One can compare $\Pi_{L,\omega}$ to the exit measure π_L of simple random walk from the same ball, and prove that for any smooth test function Ψ, and for $d \geq 3$,

$$(1) \qquad \sum_y \big(\Pi_{L,\omega}(0,y) - \pi_L(0,y)\big)\Psi(y/L) \underset{L\to\infty}{\longrightarrow} 0, \qquad \text{in probability,}$$

under a weaker symmetry condition than that of Bricmont-Kupiainen. Work in progress of Bolthausen and the author will prove (1) in the case $d = 2$. This leads to the:

Problem 2. *Does the full invariance principle hold under the Bricmont-Kupiainen conditions, for* $d = 2$*?*

We have omitted from this summary a discussion of large deviations for RWRE and their link with the homogenization of Hamilton-Jacobi-Bellman equations, as well as representation formulae for the quenched and annealed rate functions.

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Random walk in random environment, II

Noam Berger

In continuation with Ofer Zeitouni's talk, here we present two topics: ballisticity criteria for multi-dimensional RWRE and balanced RWRE.

1. Ballisticity condition

As mentioned in Zeitouni's talk, in the one dimensional setting there are simple and verifyable conditions for directional transience (namely that for a given ℓ we have $\lim X_n \cdot \ell = \infty$) and for ballisticity (i.e. that this happens with a positive speed). We write

$$L_\omega(0) \;=\; \sum_x x\,\omega(0,x)$$

for the *local drift*. An environment is called *non-nestling* if

$$0 \;\notin\; \overline{\operatorname{conv}\big(\operatorname{supp}\big(L_\omega(0)\big)\big)},$$

It is called *nestling* if

$$0 \;\in\; \operatorname{conv}\big(\operatorname{supp}\big(L_\omega(0)\big)\big)^o$$

and *marginally nestling* if

$$0 \;\in\; \partial\operatorname{conv}\big(\operatorname{supp}\big(L_\omega(0)\big)\big).$$

It is clear that non-nestling environments are directionally transient and ballistic. Towards the late 70-s one started to ask whether this is a necessary condition. Kalikow [10] found the first counter example to the necessity of the condition: He found nestling examples that are directionally transient. He defined a (then) new condition, now known as Kalikow's condition, which guarantees directional transience, and proved that certain nestling environments satisfy his condition.

Then the topic was quiet for more than 15 years.

The next step happened in the early 2000. In [15] and then in [16] Sznitman defined new ballisticity conditions, named condition (T) and condition (T'). Condition (T) in a direction $\ell \in \mathbb{R}^d \setminus \{0\}$ was defined as one of the following equivalent conditions:

(1) There exists $\alpha > 0$ such that

$$\mathbb{E}\big[\exp\big(\alpha(\tau_2 - \tau_1)\big)\big] \;<\; \infty.$$

(2) for every ℓ' in some open neighbourhood of ℓ,

$$\limsup_{L\to\infty} \frac{1}{L}\log\mathbb{P}\Big[\inf\big\{n : X_n\cdot\ell < -L\big\} < \inf\big\{n : X_n\cdot\ell > L\big\}\Big] \;<\; 0.$$

(3) Let B_L be a cube of side length L one of whose axes is parallel to ℓ, and let $\partial^+ B_L$ be the face in direction ℓ. Then

$$\limsup_{L\to\infty} \frac{1}{L}\log\mathbb{P}\Big[T_{\partial B_L} \neq T_{\partial^+ B_L}\Big] \;<\; 0.$$

In [15] Sznitman proved that condition (T) implies directional transience, ballisticity and an (annealed) CLT. Further, in the same paper it was shown that condition (T) implies Kalikow's condition, but that they are not equivalent.

A year later, in [16] Sznitman introduced a new condition, formally weaker than Conditon (T), and named it condition (T'). The definition of condition (T') is the same as that of condition (T), except that the decay of the various probabilities, instead of exponential, needs to be stretched exponential in every order. similarly to condition (T), condition (T') too implies directional transience, ballisticity and an annealed CLT. In the same paper Sznitman proved that condition (T') is equivalent to an *effective condition*, namely a condition that, at least theoretically, can be verified by observing a finite box rather than the entire space. The effective condition gave rise to a variety of new ballistic examples where the effective criterion could be verified.

The next progress in this direction came in 2014, where Berger, Drewitz and Ramírez [6] showed that condition (T') is equivalent to a power law decay of the return probability, with power $15d + 5$, and showed that this is also equivalent to a (polynomial) effective condition.

Later the power $15d + 5$ was improved to $d - 1$ by Guerra [7] in 2020, and then in a wonderful paper in the same year, Guerra and Ramírez [8] proved that conditions (T) and (T') are equivalent. Thus whenever the return probability decays faster than L^{1-d}, it already decays exponentially with L, and the walk satisfies everything that follows from condition (T), e.g. the results of [17, 4, 14].

2. Balanced RWRE

We now turn to random walks in balanced environments. An environment ω is called *balanced* if for every x,

$$\sum_{y\sim x}\omega(x,y-x)\cdot(y-x) \;=\; 0.$$

Equivalently we may define an environment to be balanced if the random walk on it is a martingale. An important special case is those of environments ω such that

$$\forall_{x,z}\quad \omega(x,z) \;=\; \omega(x,-z).$$

In the nearest neighbour setting the two are equivalent.

In attempt to prove an invariance principle, we hope to use the martingale CLT. Contrary to the case of random walks among random conductances, here the martingale property is trivial whereas the stationarity of the process is often difficult. Counter example shows that there are cases that are ergodic, mixing and elliptic, but satisfy no invariance principle and no stationarity from the point of view of the particle.

The first breakthrough was done in 1982 by Papanicolaou and Varadhan [13], where the problem was solved for the corresponding diffusion problem. There the object of study was diffusion in random environment in *non-divergence form*, which implies that the diffusion is a martingale. Later Lawler [11, 12] transferred the methods to the discrete case and proved an invariance principle, and later a Harnack principle, in the uniform elliptic case under no mixing assumptions.

Later, in 2012, Zeitouni and Guo [9] proved an invariance principle under one of two conditions: the one is an inverse moment assumption on the transition probabilities, and the other is iid combined with ellipticity. This required two different proof, one under each condition.

Then in 2014 Berger and Deuschel [5] removed the requirement of ellipticity in the iid regime, and in 2022, also together with Cohen and Guo [3], also established a Harnack principle in this regime.

At the same time, and completely independently, Armstrong and Smart [2] and then Armstrong and Lin [1] proved, using a completely different set of methods, quantitative homogenization in this regime.

There are a few remaining open problems in this regime, but in my opinion, the most interesting open problems are in connecting this regime to other regimes, particularly the perturbative regime. For example, can we prove an invariance principle for a slightly perturbed balanced environment? can we prove it for a balanced environment which underwent a percolation perturbation?

Answering those questions may involve the development of new methods and is therefore very interesting.

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Localization and delocalization for a class of degenerate convex $\nabla\phi$-interface model

Paul Dario

1. Overview

In this talk, we discuss the properties of a model of stochastic interfaces known as the $\nabla\phi$-*model* or the *Ginzburg-Landau interface model.* The model is formally defined as follows. We consider the discrete d-dimensional torus $\mathbb{T}_L := (\mathbb{Z}/L\mathbb{Z})^d$ of side length $L > 0$ with nearest-neighbor edges and introduce the probability measure on the set $\Omega_L^\circ := \left\{\phi : \mathbb{T}_L \to \mathbb{R} \,:\, \sum_{x\in\mathbb{T}_L} \phi(x) = 0\right\}$ defined by the formula

$$\mu_{\mathbb{T}_L}(d\phi) \;=\; \frac{1}{Z_{\mathbb{T}_L}} \exp\left(-\sum_{x\sim y} V(\phi(x)-\phi(y))\right) \mathrm{d}\phi$$

where $V\colon \mathbb{R}\to\mathbb{R}$ is a symmetric function growing sufficiently fast at infinity, and $\mathrm{d}\phi$ is the Lebesgue measure on the space Ω_L°. The study of the $\nabla\phi$-model was initiated by Brascamp, Lieb and Lebowitz [2] who investigated the question of the localization and delocalization of the interface, i.e., the growth of the variance $\mathrm{var}_{\mathbb{T}_L}[\phi(0)]$ as $L\to\infty$. Specifically, they proved that under one of the following conditions on the potential V:

(1) $V(x) = ax^2 + f(x)$ with $a > 0$ and $f:\mathbb{R}\to\mathbb{R}$ is convex.
(2) There exist constants $c_\pm$, $A, B, C \in (0,\infty)$ such that $0 < c_- \le V''(x) \le c_+ < \infty$ for $|x| \ge A$ and $\left|V(x) - Bx^2\right| \le C$,

the following upper bounds hold

$$\mathrm{var}_{\mu_{\mathbb{T}_L}}[\phi(0)] \;\le\; \begin{cases} C\ln L \text{ in } d=2, \\ \quad C \text{ in } d\ge 3. \end{cases} \tag{1}$$

These bounds are expected to be sharp, and matching lower bounds can be established under some specific assumptions (see [2, 11]).

Since then, the model has been an intense subject of research, and we refer to the pioneering works of [5, 9, 3, 6] and to [4, 11] for an overview of the literature.

In this talk, we are interested in a generalization of the result of [2] to functions $V \in C^2(\mathbb{R})$ satisfying

$$0 < c_- \leq \liminf_{|x| \to \infty} V''(x) \quad \text{and} \quad V''(x) \leq 1. \tag{2}$$

The key tool for the study of the $\nabla\phi$-model is the so-called *Helffer-Sjöstrand representation formula* originally introduced in Helffer-Sjöstrand [7] and used on the model by Naddaf-Spencer [9], Deuschel-Giacomin-Ioffe [3] and Giacomin-Olla-Spohn [6]. The identity, stated below in its probabilistic version and applied to the specific observable $\phi(0)$, reads as follows

$$\operatorname{var}_{\mu_{\Lambda_{\mathbb{T}_L}}} [\phi(0)] = \int_0^\infty \mathbb{E}\left[P_{\mathbf{a}}(t,0)\right] \mathrm{d}t, \tag{3}$$

where the function $P_{\mathbf{a}} : (0,\infty) \times \mathbb{Z}^d \to \mathbb{R}$ is defined to be the solution of the discrete parabolic equation

$$\begin{cases} \partial_t P_{\mathbf{a}} - \nabla \cdot \mathbf{a} \nabla P_{\mathbf{a}} = 0 & \text{in } (0,\infty] \times \mathbb{Z}^d, \\ P_{\mathbf{a}}(0,\cdot) = \delta_0 - \dfrac{1}{|\mathbb{T}_L|} & \text{in } \mathbb{Z}^d, \end{cases}$$

where the environment $\mathbf{a}$ is random, depends on the space and time variables and is formally defined by $\mathbf{a}(t,(x,y)) := V''(\phi_t(y) - \phi_t(x))$, where $(\phi_t)_{t \geq 0}$ is the solution of the *Langevin dynamics*

$$\begin{cases} d\phi_t(x) := \displaystyle\sum_{y \sim x} V'(\phi_t(y) - \phi_t(x))\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t(x) & \text{for } (t,x) \in (0,\infty) \times \mathbb{T}_L, \\ \phi_0(x) = \phi(x) & \text{for } x \in \mathbb{T}_L, \end{cases} \tag{4}$$

where $\left\{B_t(x) : t \geq 0,\, x \in \mathbb{Z}^d\right\}$ is a collection of independent Brownian motions and the initial condition ϕ is distributed according to $\mu_{\mathbb{T}_L}$ independently of the Brownian motions. The symbol $\mathbb{E}$ in (3) then refers to the expectation with respect to the dynamics $(\phi_t)_{t \geq 0}$.

In the case where $V'' \geq c_- > 0$ (which implies $\mathbf{a} \geq c_- > 0$), the Nash inequality [10] and the Poincaré inequality imply that

$$P_{\mathbf{a}} \leq \frac{C}{(1+t)^{\frac{d}{2}}} \exp\left(-\frac{t}{CL^2}\right), \tag{5}$$

which combined with (3) implies the bound (1). In order to extend the result to the functions of the class described in (2), we rely on the works of Mourrat and Otto [8] and Biskup and Rodriguez [1] which (essentially) assert that the bound (5) can be extended to random stationnary coefficient fields $\mathbf{a}$ which may vanish but satisfy an assumption of the form

$$\mathbb{P}\left[\forall t \in [0,T],\, \mathbf{a}(t,(x,y)) > 0\right] \underset{T \to \infty}{\longrightarrow} 0, \tag{6}$$

with a sufficiently fast rate of convergence. Using the assumption (2), this result can be established if we can prove that, for any constant $C > 0$,

$$\mathbb{P}\big[\forall t \in [0,T],\, |\phi(t,x) - \phi(t,y)| \geq C\big] \underset{T\to\infty}{\longrightarrow} 0. \tag{7}$$

In order to prove the bound (4), we decompose the Brownian motions $B_t(x)$ into independent increments and Brownian bridges by writing, for $t \in [0,1]$,

$$X(x) := B_1(x),\ W_t(x) = B_t(x) - tB_1(x) \implies \mathrm{d}B_t(x) = X(x)\mathrm{d}t + \mathrm{d}W_t(x). \tag{8}$$

We may then see the dynamics $(\phi_t)_{t\geq 0}$ as a function of the increments and the Brownian bridges, and we may in particular compute the derivative of the dynamics with respect to the increment $X(x)$. To this end, we use the formula (8), and differentiate both sides of the definition (4) with respect to the increment $X(x)$. This operation gives an equation satisfied by the mapping $\partial\phi/\partial X(x)$ which can be solved using Duhamel's principle and yields, for any $t \geq 1$,

$$\frac{\partial \phi(t,y)}{\partial X(x)} = \sqrt{2}\int_0^1 P_{\mathbf{a}}(t,y;s,x)\,\mathrm{d}s,$$

where $P_{\mathbf{a}}(\cdot,\cdot;s,0)$ is the heat kernel started from the vertex x at time s (i.e., the solution (4) started at time $s \in [0,1]$ instead of 0 and at the vertex $x \in \mathbb{T}_L$ instead of 0). Using this identity and the properties of the heat kernel, one may then prove that

$$\frac{\partial \nabla\phi(1,(x,y))}{\partial X(x)} \geq c_1 > 0, \tag{9}$$

for some constant c_1 (depending only on the dimension). Using that the law of $X(x)$ is Gaussian of variance 1, we may use the property (9) to prove that

$$\mathbb{P}\big[|\nabla\phi(1,(x,y))| \leq C\big] \leq (1-\varepsilon),$$

where $\varepsilon > 0$ depends only on the constant C and the dimension d. This inequality can then be iterates so as to show that, for any $N \in \mathbb{N}$,

$$\mathbb{P}\big[\forall t \in [0,N],\, |\nabla\phi(t,(x,y))| \leq C\big] \leq (1-\varepsilon)^N \underset{N\to\infty}{\longrightarrow} 0,$$

verifying (7) and thus (6).

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Quenched invariance principle for random conductance model with degenerate ergodic conductances

Peter Bella
(joint work with Mathias Schäffner)

Consider graph $\mathbb{Z}^d$, $d \geq 2$, and to each nearest-neighbor edge xy with $x, y \in \mathbb{Z}^d, |x-y| = 1$ (later denoted $x \sim y$), associate a random conductance $\omega_{xy} \in (0, \infty)$. We consider a random walk $(X_t)_{t \geq 0}$, which moves from a point $x \in \mathbb{Z}^d$ after an exponential clock with mean $(\sum_{x \sim y} \omega_{xy})^{-1}$ rings, to the vertex y chosen among the $2d$ neighbors at random with probability $\omega_{xy}/\sum_{x \sim y} \omega_{xy}$. Equivalently, this reversible Markov process is described by its infinitesimal generator

$$\mathcal{L}f(x) = \sum_{y \in \mathbb{Z}^d, x \sim y} \omega_{xy}(f(y) - f(x)).$$

Starting from the origin, we want to know under which conditions on the conductances the long-time large-scale behavior of the random walk can be described by a (rescaled) Brownian Motion. More precisely, denoting $X^{(n)}(t) := \frac{1}{n} X_{n^2 t}$ for $n \geq 1$ the diffusive rescaling of the walk, we want to show that the law of $X^{(n)}(t)$ converges on any finite time interval to the one of the $\Sigma \cdot W$ - the Brownian Motion with Covariance $\Sigma^T \Sigma$.

Assuming the law of ω_{xy} is stationary and ergodic with respect to shifts $(\tau_z)_{z \in \mathbb{Z}^d}$, one is asking what moments on the conductances and their reciprocals are needed for the above invariance principle to hold for *almost every* realization ω (i.e. in the quenched sense). Assuming $\mathbb{E}\left[\omega_{xy}^p\right] + \mathbb{E}\left[\omega_{xy}^{-q}\right] < \infty$ for every edge xy and some $p, q > 0$, Barlow, Burdzy, and Tumar showed that $p < 1$ and $q < 1$ [2] are not sufficient to yield the quenched invariance principle - while the annealed one (i.e. considering the convergence of $X^{(n)}$ in law instead of almost surely) holds.

On the positive side, using arguments restricted to the plane Biskup [5] (building on the work of Berger and Biskup [4]) showed that the quenched invariance principle (QIP) holds under the condition $p = q = 1$ in the case $\mathbb{Z}^2$. Later Andres, Deuschel, and Slowik [1] showed that QIP holds provided $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ for any $d \geq 2$. Using harmonic coordinates, they decomposed the position of the random walk into a martingale part, and the remainder. While showing the CLT for the martingale part is classical, the challenge is to deal with the remainder. More precisely, the remainder is of the form $\frac{1}{n}\varphi(X_{tn^2})$, where φ denotes the *corrector*. After showing that X_{tn^2} is rarely larger than n, it is enough to show that

$$\lim_{n\to\infty} \frac{1}{n} \sup_{|x|\le n} |\varphi(x)| \ = \ 0,$$

which will follow from the ergodicity in the form

$$\lim_{n\to\infty} \frac{1}{n} \frac{1}{(2n+1)^d} \sum_{x\in[-n,n]^d} |\varphi(x)| \ = \ 0.$$

While up to this point all the arguments work under the assumption $p = q = 1$ (in any dimension), it is this very last step in form of $l^\infty - l^1$ estimate for the corrector which required the stronger assumption $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$.

Building on a previous work with Mathias Schäffner [3] on the local boundedness of solution to degenerate elliptic equations, we improved the above condition to $\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$. Similarly to [1], it is based on the Moser iteration, with two additional ingredients: in the Cacciopolli inequality we choose the optimal cut-off function among the radial symmetric functions, which then allows to use the Sobolev inequality on spheres instead of in the bulk - hence decreasing the exponent d to $d-1$. Surprisingly, Biskup and coauthors [6] showed that for this strategy to work the condition is sharp (modulo the equality case) by showing that in the case $\frac{1}{p} + \frac{1}{q} > \frac{2}{d-1}$ one can construct ergodic environment so that the corrector does not converge to 0 as described above.

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Random Walks in Divergence-Free Random Environments

Bálint Tóth

Let $(\Omega, \mathcal{F}, \pi, (\tau_z : z \in \mathbb{Z}^d))$ be a probability space with an ergodic $\mathbb{Z}^d$-action. Denote by $\mathcal{U} := \{k \in \mathbb{Z}^d : |k| = 1\}$ the set of elements of $\mathbb{Z}^d$ neighbouring the origin which will be the set of possible elementary steps of a nearest neighbour random walk on $\mathbb{Z}^d$. Let $p_k : \Omega \to [0,\infty)$, $k \in \mathcal{U}$, satisfy (π-a.s.) the following *bi-stochasticity* condition

$$\sum_{k\in\mathcal{U}} p_k(\omega) \;=\; \sum_{k\in\mathcal{U}} p_{-k}(\tau_k\omega), \tag{1}$$

and define the continuous-time random walk in random environment (RWRE), $t \mapsto X(t) \in \mathbb{Z}^d$ as the Markovian nearest neighbour random walk with jump rates

$$\mathbf{P}_\omega\big[X(t+dt) = x+k \,|\, X(t) = x\big] \;=\; p_k(\omega, x)\mathrm{d}t,$$

and initial position $X(0) = 0$.

The *environment process* (as seen from the position of the random walker) is, $t \mapsto \eta_t \in \Omega$ defined as

$$\eta_t \;:=\; \tau_{X_t}\omega.$$

This is a pure jump Markov process on the state space Ω. It is well known (and easy to check, see e.g., [8]) that bi-stochasticity (1) of the jump rates p_k is equivalent to stationarity (in time) of the a priori distribution π of the environment process $t \mapsto \eta_t \in \Omega$. Moreover, under the condition (1), spatial ergodicity of $(\Omega, \mathcal{F}, \pi, (\tau_z : z \in \mathbb{Z}^d))$ also implies time-ergodicity of the environment process process $t \mapsto \eta_t \in (\Omega, \mathcal{F}, \pi)$. Hence it follows that the random walk $t \mapsto X(t)$ will have stationary and ergodic annealed increments. (Though, in the annealed setting the walk is not Markovian.)

It is convenient to separate the symmetric and antisymmetric parts of the jump rates, that is, $p_k(\omega) = s_k(\omega) + v_k(\omega)$, where

$$s_k(\omega) \;:=\; \frac{p_k(\omega) + p_{-k}(\tau_k\omega)}{2} \qquad \text{and} \qquad v_k(\omega) \;:=\; \frac{p_k(\omega) - p_{-k}(\tau_k\omega)}{2}.$$

Note that

$$s_k(\tau_x\omega) \;=\; s_{-k}(\tau_{x+k}\omega) \geq 0 \qquad\qquad v_k(\tau_x\omega) \;=\; -v_{-k}(\tau_{x+k}\omega),$$

define a collection nonnegative random conductances on the unoriented edges $\{x, x+k\}$, respectively, a random flow on the oriented edges $(x, x+k)$ of $\mathbb{Z}^d$. Also, obviously, the former dominate the latter:

$$|v_k(\omega)| \;\leq\; s_k(\omega).$$

In terms of these variables the bi-stochasticity (1) reads as *sourcelessness* of the flow

$$\sum_{k\in\mathcal{U}} v_k(\omega) \;=\; 0$$

We assume strong ellipticity and boundedness of the conductances:

$$0 \;<\; s_* \;\leq\; s_k(\omega) \;\leq\; s^* \;<\; \infty, \qquad\qquad \pi-\text{a.s.} \tag{2}$$

and zero mean of the flows

$$\int_\Omega v_k(\omega)\,\mathrm{d}\pi(\omega) \;=\; 0. \tag{3}$$

From ergodicity under π of the environment process $t \mapsto \eta_t$, and from (2) and (3) the strong law of large numbers for the displacement of the random walker readily follows:

$$\lim_{t\to\infty} t^{-1}X(t) \;=\; 0, \qquad \text{a.s..}$$

The notorious H_{-1}-condition for the antisymmetric flows $v_k(\omega)$ imply *diffusive bounds* for the displacement of the random walk. Namely, let for $k, l, \in \mathcal{U}$,

$$C_{k,l}(x) \;:=\; \int_\Omega v_k(\omega) v_l(\tau_x\omega)\,\mathrm{d}\pi(\omega), \qquad \hat{C}_{k,l}(p) \;:=\; \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}x\cdot p}\, C_{k,l}(x),$$

That is: $C_{k,l}(x)$ is the covariance matrix of the drift field, and $\hat{C}_{k,l}(p)$ is its Fourier-transform. The H_{-1}-condition reads

$$\text{(4)} \qquad \int_{[-\pi,\pi)^d} \frac{\sum_{l\in\mathcal{U}} \hat{C}_{l,l}(p)}{\sum_{j=1}^d (1-\cos p_j)}\,\mathrm{d}p \;<\; \infty.$$

This is obviously an infrared bound on the correlation decay of the field $x \mapsto v_k(\tau_x\omega)$. In particular, it also implies (3) which we don't have to assume thus separately. It is equivalent to the existence of a stream tensor $h_{k,l} \in \mathcal{L}^2(\Omega,\pi)$, $k, l \in \mathcal{U}$,

$$h_{k,l}(\omega) \;=\; -h_{-k,l}(\tau_k\omega) \;=\; -h_{k,-l}(\tau_l\omega) \;=\; -h_{l,k}(\omega) \quad v_k(\omega) \;=\; \sum_{l\in\mathcal{U}} h_{k,l}(\omega).$$

The conditions (2) and (4) jointly imply the annealed diffusive bounds

$$0 \;<\; \liminf_{t\to\infty} t^{-1}\mathbf{E}\big[|X(t)|^2\big] \;\le\; \limsup_{t\to\infty} t^{-1}\mathbf{E}\big[|X(t)|^2\big] \;<\; \infty$$

Theorem 1. *Conditions* (1), (2), (4) *are assumed.*

(i) (Source: [9]) *The asymptotic annealed covariance matrix*

$$(\sigma^2)_{ij} \;:=\; \lim_{t\to\infty} T^{-1}\mathbf{E}\big[X_i(T)X_j(T)\big]$$

exists, and it is finite and non-degenerate. For any bounded and continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{T\to\infty} \int_\Omega \Big|\mathbf{E}_\omega\big[f(T^{-1/2}X(T))\big] - \mathbf{E}\big[\mathcal{N}(0,\sigma^2)\big]\Big|\,\mathrm{d}\pi(\omega) \;=\; 0.$$

(ii) (Source: [6], [15]) *Assume the slightly stronger integrability condition* $h_{k,l} \in \mathcal{L}^{2+\varepsilon}(\Omega,\pi)$. *For any bounded continuous function* $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{T\to\infty} \mathbf{E}_\omega\big[f(T^{-1/2}X(T))|\big] \;=\; \mathbf{E}\big[\mathcal{N}(0,\sigma^2)\big], \qquad \pi\text{-}a.s.$$

Physical and mathematical motivations of the problem setting have been shown.

The results were put into the context of [13], [12], [8], [11], [5], [4], [7], [1], [2] (in historical order).

If the H_{-1}-condition (4) fails to hold then typically superdiffusive behaviour of the RWRE is expected and in some cases proved. This was also illustrated with examples from [17], [10], and [3]

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Linear response of random motion in random environment under time-periodic perturbations

Alessandra Faggionato

During the workshop I have outlined some open questions and research subjects on which I am currently working. In one contribution I have explained the connection between the hydrodynamic limit of particles with site-exclusion interaction, jumping on random graphs in $\mathbb{R}^d$ with random conductances, and a particular weak form of CLT for the random walk performed by a single particle (which is an example of random conductance model). In the above setting, the graphs are built on a simple point process and are microscopically disordered but macroscopically homogeneous from a statistical viewpoint. We refer to [1, 2, 3] for details. In another contribution I have indicated some open problems concerning the linear response of random processes in random environment when the weak external field is time-periodic. In this report, I detail this second contribution.

In the last years there has been an intensive investigation of the linear response for random motion in random environment, aiming in particular to derive Einstein's relation rigorously. This fundamental relation states that, modulo a temperature-dependent factor, the mobility matrix of the perturbed particle equals the diffusion matrix of the unperturbed particle (see e.g. [5, 6, 7, 8, 10] and references therein). The perturbation comes from a spatial- and time-homogeneneous weak external field. Regeneration times are in this setting an effective tool since the perturbed particle moves under an asymptotic drift, thus leading to a progressive regeneration of the environment viewed from the particle.

What has not been properly studied yet is the linear response of random motion in random environment under the effect of a weak spatially-homogeneous but time-periodic external field. A case particularly relevant in Physics (cf. [9]) is a field of cosine-type. For example, if $r_{x,y}(\xi)$ represents the probability rate for a jump from x to y for the unperturbed random walk on $\mathbb{Z}^d$ in the random environment ξ, the perturbed jump probability rate at time t would be

$$r^{(\lambda)}_{x,y}(\xi,t) \;:=\; \exp\bigl(\lambda\cos(\omega t)(y-x)\cdot v\bigr)\, r_{x,y}(\xi)\,.$$

In the above formula $0<\lambda\ll 1$ is the tuning parameter, v is a fixed non-zero vector of $\mathbb{R}^d$, $(y-x)\cdot v$ is the scalar product between $y-x$ and v and $\omega = 2\pi/T$, T being the time-period of the external field.

We outline the program to which we are particularly interested. Let us suppose to start e.g. from the random conductance model as unperturbed process. We suppose that $r_{x,y}(\xi)$ equals $r_{0,y-x}(\tau_x\xi)$, where $\tau_x\xi$ denotes the environment translated by $-x$. A first target would be to prove the equilibration of the process "environment viewed from the perturbed particle" towards an oscillatory steady state (OSS), i.e. a law on the path space of the above process left invariant by $T\mathbb{Z}$-time translations. Let us call $\mathbb{P}_\pi$ this OSS, π being its initial distribution on the environment space. Let us call $V_\lambda(t)$ the expected instantaneous velocity of the particle in the OSS at time t, i.e. $V_\lambda(t) := \int \mathbb{P}_\pi(d\xi)\sum_y r^{(\lambda)}_{0,y}(\xi,t)y$. According to Statistical Physics [9], we expect that there exists a frequency–dependent deterministic $d\times d$–matrix $\sigma(\omega)$ such that

$$\partial_{\lambda=0}V_\lambda(t) \;=\; \Re\bigl(\mathrm{e}^{\mathrm{i}\omega t}\sigma(\omega)v\bigr).$$

The above matrix $\sigma(\omega)$ would be the so called *complex mobility matrix.* We expect that $\sigma(\omega)$ converges to the diffusion matrix of the unperturbed random walk as $\omega\to 0$ and we also expect an implicit representation for $\sigma(\omega)$ in terms of $\mathbb{P}_\pi$ similar to the one in [4][Theorem 5.1]. This characterization would correspond to a generalization of the Einstein's relation, as it would express a linear response matrix of the OSS in terms of suitable expectations w.r.t. the unperturbed equilibrium state.

When considering the simplified setting giving by a random walk on $\mathbb{Z}^d$ in a periodized environment, the above program has been implemented and the corresponding results are detailed in [4, Section 5]. As a natural continuation of this work, we are now focusing on random walks in random environments.

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Quenched functional CLT for random walks in bounded cyclic and divergence-free random environments

Weile Weng

(joint work with Jean-Dominique Deuschel, Martin Slowik)

We first show quenched functional CLT (or quenched invariance principle) for random walks in random environments that admit bounded cycle representation with non-negative random cycle weights ($\mathrm{BCR}_{\geq 0}$). In other words, the random environments are generated by random cycle weights associated to a collection of bounded prototype cycles shifting over $\mathbb{Z}^d$ ($d \geq 2$). Precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space for the random environments $\omega \equiv (\omega(x,y))_{(x,y)\in\vec{E}_d}$, where $\vec{E}_d$ denotes the set of oriented edges on $\mathbb{Z}^d$. Then there exists a collection of prototype cycles $\mathcal{C}$ of bounded length $N \in 2\mathbb{N}$ and a collection of non-negative random variables $(W_C)_{C\in\mathcal{C}}$, such that for $\mathbb{P}$-a.e. ω, $(x,y) \in \vec{E}_d$,

(A0) $\omega(x,y) = \sum_{C\in\mathcal{C}} \sum_{z\in\mathbb{Z}^d} W_C(\tau_z\omega) 1_{(x,y)\in C+z}$.

Further we assume

(A1) $\mathbb{P}$ is stationary and ergodic with respect to the space shifts $(\tau_x)_{x\in\mathbb{Z}^d}$, (A2) the symmetric part of ω, i.e., $\forall(x,y)\in\vec{E}_d, \omega^S(x,y):=\frac{1}{2}(\omega(x,y)+\omega(y,x))$ satisfies the following $p-q$ moment condition

$$\mathbb{E}\big[(\omega^S(x,y))^p\big]+\mathbb{E}\big[(\omega^S(x,y))^{-q}\big]\ <\ \infty,$$

where $p,q\in[1,\infty]$, and $\frac{1}{p}+\frac{1}{q}<\frac{2}{d}$.

Now for a given random environment $\omega\in\Omega$, a particle starts from the origin and jumps at rate ω to the nearest-neighbor sites. Let $(X_t)_{t\geq 0}$ be the continuous time random walk of the particle, and define $X_t^{(n)}:=\frac{1}{n}X_{n^2t}, \forall t\geq 0$.

Theorem 1 (Quenched functional CLT (QFCLT) Deuschel, Slowik, Weng 2023$^+$)**.** *Suppose $d\geq 2$, and (A0), (A1), (A2) are satisfied. Then for $\mathbb{P}$-a.e. ω, $(X_t^{(n)})_{t\geq 0}$ converges weakly under the quenched law P_0^ω to a Brownian motion on $\mathbb{R}^d$ with a deterministic covariance matrix Σ that is non-degenerate.*

Random conductance model (RCM) (see [6]) is a special case of $\mathrm{BCR}_{\geq 0}$, with $N=2$. QFCLT has been studied by [12][3][1][5][11] for i.i.d. setting (stricter than (A1)). QFCLT for RCM under (A1)(A2) has been proven by [2] with PDE techniques. Recently, [4] improves [2]'s result by relaxing (A2) to $\frac{1}{p}+\frac{1}{q}<\frac{2}{d-1}, d\geq 3$ (while for $d=2$, QFCLT has been addressed by [6] for the minimal moment requirement $p=q=1$). The $\mathrm{BCR}_{\geq 0}$ model was first appeared in [7], and they show QFCLT under a weak version of uniform ellipticity condition. The model was also included in [9].

To show QFCLT, the canonical three steps are

(1) *martingale decomposition*
Let $\Pi:\Omega\times\mathbb{Z}^d\to\mathbb{R}^d$ be the random position field, with $\Pi(\omega,x):=x, \forall\omega,x$. We are looking for a decomposition $\Pi=\chi+\Phi$ with
- $\chi,\Phi:\Omega\times\mathbb{Z}^d\to\mathbb{R}^d$ are random fields.
- $L^\omega\Phi^i(\omega,\cdot)=0$ on $\mathbb{Z}^d$, for $\mathbb{P}$-a.e. $\omega,\forall i\in[d]\equiv\{1,\cdots d\}$, where L^ω is the quenched generator, with

$$L^\omega f(x)\ =\ \sum_{y\sim x}\omega(x,y)(f(y)-f(x))$$

for all $f:\mathbb{Z}^d\to\mathbb{R}$ bounded.
- For $\forall i\in[d]$, $(\chi^i(\cdot,z))_{z\sim 0}\in L^2_{\mathrm{pot}}$, which is the closure of horizontal gradients of bounded and measurable functions from Ω to $\mathbb{R}$ in the Hilbert space L^2_{cov}. We refer further details to [6].

(2) *sublinearity of the corrector*
To show the χ vanishes under the diffusive scaling, in the sense that $\frac{1}{n}\chi(\cdot,X_{n^2t})$ converges weakly to 0 with respect to P_0^ω, for $\mathbb{P}$-a.e. ω, it is enough to have for $\mathbb{P}$-a.e. ω,

$$\forall i\in[d]:\lim_{n\to\infty}\frac{1}{n}\sup_{x\in B(n)}|\chi^i(\omega,x)|\ =\ 0,$$

with $B(n) := \{x \in \mathbb{Z}^d : ||x||_\infty \leq n\}, n \in \mathbb{N}$. It is essential to obtain following quenched energy estimate (QEE),

$$\frac{\mathcal{E}^\omega(\eta f)}{|B(n)|} \leq CN^2||\nabla\eta||^2_{\ell^\infty}||\mu^\omega||_{p,B(n)}||f||^2_{p_*,B(n)},$$

where η is the cut-off function on $B(n)$, f is non-negative L^ω-subharmonic, $\mu^\omega(x) := \sum_{y\sim x}\omega(x,y)$, $||\cdot||_{p,B(n)}$ is the space-averaged ℓ^p norm, and finally p_* is the Hölder conjugate of p.

(3) *functional CLT for quenched martingales*
The weak convergence of $\frac{1}{n}\Phi(\omega, X_{n^2t})$ in the quenched probability to non-degenerate d-dimensional Brownian motion is dealt with Helland's martingale functional CLT (see [8]).

We prove the theorem by adapting the PDE techniques used in [2] to show (2). Compared to RCM (the reversible case), the difference for the BCR$_{\geq 0}$ case arises first in the construction of corrector χ, since one cannot obtain χ via the orthogonal decomposition of Π over $L^2_{\text{cov}} = L^2_{\text{pot}} \bigoplus L^2_{\text{sol}}$ (see [6][2]). Instead, to solve the Poisson equation $L^\omega\chi(\omega,x) = L^\omega\Phi(\omega,x)$ for $\mathbb{P}$-a.e. ω, and $x \in \mathbb{Z}^d$, the bounded cyclic structure provides two nice properties to apply the Lax-Milgram theorem, which are *sector condition* and the *bounded operator* property of the local drift $V^i(\omega) := \omega_{e_i} - \omega_{-e_i}$. Precisely, $\forall\phi,\varphi : \Omega \to \mathbb{R}$ suitable, $\mathcal{E}(\phi,\varphi) := \mathbb{E}[\phi(-\mathcal{L}\varphi)]$, and $\mathcal{L}\varphi(\omega) = \sum_{y\sim x}\omega(x,y)(\varphi(\tau_y\omega) - \varphi(\tau_x\omega))$,

$$\mathcal{E}(\phi,\varphi)^2 \leq 4N^2\mathcal{E}(\phi,\phi)\mathcal{E}(\varphi,\varphi)$$

$$\mathbb{E}[V^i\phi]^2 \leq \frac{1}{2}N^3\mathbb{E}[\mu]\mathcal{E}(\phi,\phi)$$

The second difference, or rather challenge, is to obtain (QEE). For which we decompose the quenched generator into symmetric and anti-symmetric part along the cycles and evaluate them separately, where bounded cycle length N plays a role.

The BCR$_{\geq 0}$ is a special case of doubly stochastic random environment (or divergence-free random environment). [13] showed QFCLT for this case under the assumption that ω^S satisfies uniform elliptity and is (L^2-) bounded, and the stream tensor that is associated with the anti-symmetric part ω^A exists and is $L^{2+\varepsilon}$ integrable. Since stream tensor can be characterized as length four cycles with anti-symmetric values when the orientation is reversed, it motivates and aspire to adapt [2]'s PDE scheme to show QFCLT under (A1) and (A2)-*type* moment assumption to the following two cases

- divergence-free random environment admits stream tensor (or stream cycle) representation[1], which requires the existence of the second moment of ω^A,
- random environment admit bounded cycle representation with $\mathbb{R}$-valued cycle weights, as long as $\omega \geq 0$.

[1]This is an ongoing joint work with B. Fehrman and M. Slowik.

We particularly emphasize that the result for divergence-free case does not cover the result for the $BCR_{\geq 0}$ case, since the former needs $p \geq 2$ for ω^A, but the latter does not assume this even for ω^S ($\geq \omega^A$).

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Random Walks and Percolation

Pierre-François Rodriguez

Consider the 'porous medium' obtained by eliminating sites from a d-dimensional torus ($d \geq 3$) of large side length N by running a random walk up to time uN^d, for a parameter $u \in (0, \infty)$. The walk deletes the vertices it visits. This model features in an article by M. Brummelhuis and H.J. Hilhorst [5]; its rigorous study was initiated in work of I. Benjamini and A.-S. Sznitman [1].

Similarly as its short-ranged (SR) 'Bernoulli' analogue, in which vertices are removed *independently* with probability $p = e^{-u}$, the vacant set of the walk has long been conjectured to undergo an abrupt percolation phase transition across a non-degenerate value $u_*(d) \in (0, \infty)$, across which a unique macroscopic connected component in the vacant set disappears, leaving room for tiny clusters only. Due to the long-range (LR) effects induced by the walk, the critical regime $u \approx u_*$

is nonetheless believed to display rather drastically different behavior than its (SR)-cousin, governed by other scaling laws.

The vacant set of the walk is known [13] to have a (local) 'Gibbsian' limit $\mathcal{V}^u \subset \mathbb{Z}^d$ as $N \to \infty$, characterized by the exact formula

$$\mathbb{P}[\mathcal{V}^u \supset K] \;=\; \exp\{-u\mathrm{cap}(K)\},$$

for all finite $K \subset \mathbb{Z}^d$, where $\mathrm{cap}(K)$ refers to the capacity of K. The set $\mathcal{V}^u$ is the vacant set of random interlacements, introduced by Sznitman in [12]. The long-range dependence inherent to the model is articulated by the fact that

$$\text{(LR)} \qquad \mathbb{C}\mathrm{ov}\big[1_{\{x\in\mathcal{V}^u\}}, 1_{\{y\in\mathcal{V}^u\}}\big] \;\sim\; c(d,u)|x-y|^{2-d}, \;\text{ as } |x-y| \to \infty,$$

which renders its analysis delicate. The strong correlations implied by (LR) do however lead to certain rigidity constraints (see for instance [4] for an illustrative example), which, if properly accounted for, can lead to a much improved understanding, notably in the near-critical regime.

The talk will survey existing results regarding the above phase transition, both for the random walk model and its infinite-volume limit $\mathcal{V}^u$. Various findings exhibiting well-behaved sub- and supercritical phases, targeting for instance a quenched functional central limit theorem and Gaussian heat kernel bounds on the infinite cluster [9, 10], thus mirroring by now classical results in the (SR)-case [2, 3, 8, 11], are so far only known in perturbative regimes. The discussion will lead up to several recent developments, one based on ongoing work with H. Duminil-Copin, S. Goswami, F. Severo, A. Teixeira, regarding the long purported equality of various critical parameters naturally associated to this phase transition. We will also get a glimpse of its associated universality class, which, following a likely scenario, presumably corresponds to that recently derived in work with A. Prévost and A. Drewitz [6] for a related model in $d = 3$ exhibiting the same (LR)-dependence.

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Random walks on random graphs in critical regimes

David A. Croydon

Interest in understanding random walks on percolation clusters goes back at least to 1976, when the Nobel prize-winning physicist de Gennes raised questions about how the mean-square displacement of what he called the 'ant in the labyrinth' behaved near the critical threshold of percolation [15]. Describing the expected anomalous behaviour of random walks on percolation clusters at criticality itself soon thereafter became a question of interest. In this direction, Alexander and Orbach famously conjectured that the spectral dimension of a critical percolation cluster on $\mathbb{Z}^d$ (i.e. the exponent $d_s := \lim_{t\to\infty} -2\log p_t(x,x)/\log t$ capturing the on-diagonal decay of the corresponding heat kernel) is given by 4/3, independent of the dimension of the underlying space [1]. (This statement is now only expected to be true in suitably high dimensions.) Mathematically, a remarkable contribution was made by Kesten in 1986 [17, 18]. In these papers, he not only constructed what he called an 'incipient infinite cluster' for critical percolation in two dimensions, which can be viewed as a critical percolation cluster conditioned to be infinite, but also showed that the associated random walk is sub-diffusive, in that it moves a distance smaller than $n^{\frac{1}{2}-\varepsilon}$ in time n. (See [14] for a related estimate.) Moreover, in [18], Kesten undertook a similar study for the random walk on a critical Galton-Watson tree, and proved more detailed results in that case, including one that supported the Alexander-Orbach conjecture. In my overview talk, I discussed progress towards understanding random walks on random graphs in critical regimes since Kesten's seminal work.

The two main focuses of my talk were as follows. (I highlight that the citations to the literature are not meant to provide an exhaustive survey of the relevant research areas.)

Heat kernel estimates: It is known that sub-Gaussian heat kernel (i.e. transition density) estimates for a random walk on an infinite, locally finite, connected graph are closely connected to the volume growth and effective resistance of the graph in question. For instance, if the volume of a ball of radius r, with respect to the graph distance d, is given up to multiplicative constants by r^α and the effective resistance between vertices x and y is given, again up to multiplicative constants, by $d(x,y)^{\beta-\alpha}$, then (under certain technical conditions) the transition density of the associated random walk satisfies a bound of the form

$$p_t(x,y) \;\leq\; Ct^{-\alpha/\beta}\exp\left(-\left(\frac{d(x,y)^\beta}{Ct}\right)^{\frac{1}{\beta-1}}\right),$$

where C is a constant, and a similar lower bound holds, see [4, Theorem 1.3]. Significant progress has been made in deriving related heat kernel estimates under weaker conditions that are suitable for random graphs; see [22] for some generally applicable bounds. Notable examples where this has been done (with $\alpha = 2$ and $\beta = 3$) include critical Galton-Watson trees [6], critical oriented percolation clusters in high dimensions [5] and critical (unoriented) percolation clusters in high dimensions [21]. In particular, in each of these cases, the Alexander-Orbach conjecture has been shown to hold. Evidence that it does not hold in lower dimensions is provided in [16].

Scaling limits: Using the framework of Kigami's resistance forms (see [19, 20]), in [12], it is shown that if one can rescale the effective resistance metrics and invariant measures of the random walks on a sequence of graphs in such a way that they converge to a non-trivial limit, then so do the corresponding stochastic processes. See [13] for an earlier version of this result, and [3] for a related result that applies in the case of tree-like metric spaces. This viewpoint has proved particularly useful for understanding the random walks on random graphs in critical regimes, and can be applied to deduce the scaling limits of random walks on critical Galton-Watson trees from [9, 10], on critical Erdős-Rényi random graphs from [11] and on stable looptrees, which can be related to the boundary of critical percolation clusters on random planar maps, from [2]. Moreover, the resistance-convergence criterion is closely related to the 'four conditions' given for deriving scaling limits of ants in high-dimensional labyrinths of [8], which have been applied to obtain a scaling limit for the random walk on a critical branching walk in high dimensions, a structure expected to be extremely closely related to a critical percolation cluster [7].

Many of the approaches followed in the above areas grew out of the study of self-similar fractals. One of the original motivations for studying the stochastic processes on such spaces was to give insight into how stochastic processes behaved on media with disorder at all scales, as was expected in the case of critical percolation clusters. (Suggesting that analysis for the two kinds of model should be similar, the early physics work of Alexander and Orbach, for example, also included predictions concerning the Sierpinski gasket.) By now, the techniques have developed to the point where they are capable of handling models of random graphs with random fractal scaling limits. And, questions relating to random walks on critical percolation clusters on $\mathbb{Z}^d$ are starting to be answered. It is exciting to envisage how this work will continue to progress over the coming years...

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Continuum percolation models

Artem Sapozhnikov
(joint work with Yingxin Mu)

In the first part of this talk, we discuss several examples of continuum percolation models in $\mathbb{R}^d$, such as the Boolean model, the Brownian interlacements and the Poisson cylinder model. Recently, some progress has been made in understanding of the percolation phase transition for the vacant set of these models, see e.g. [1, 6, 2, 4, 10, 3]. The goal of this exposition is to stimulate research on motions in continuum random media.

In the second part of the talk, we present our resent results about the uniqueness of the infinite connected component in the vacant set of Brownian interlacements. Brownian interlacements, introduced by Sznitman in [8], is the continuous counterpart of random interlacements on $\mathbb{Z}^d$ (see [7]); while random interlacements is a Poisson cloud of doubly-infinite random walks on $\mathbb{Z}^d$, Brownian interlacements is a Poisson cloud of doubly-infinite Wiener sausages of fixed positive radius, whose density is controlled by a parameter $\alpha > 0$. Due to similarities in the constructions as well as large-scale properties of random walks and Wiener sausages, Brownian interlacements share basic properties with random interlacements, such as slow algebraic decay of correlations and absence of the so-called finite energy property.

In [5],we prove that in any dimension $d \geq 3$ and for any density parameter $\alpha > 0$,

the vacant set of Brownian interlacements contains at most one infinite connected component almost surely.

The uniqueness for the vacant set of random interlacements on $\mathbb{Z}^d$ was proved by Teixeira in [9]. The situation in continuum is much more delicate because of complicated microstructure. Our method is different from the one of Teixeira and, in fact, gives a new robuster proof of his result.

One of the key ingredients in our proof is a result about microscopic uniqueness for ensembles of independent Wiener sausages, which may be of independent interest: Let $W^{(1)}, \ldots, W^{(K)}$ be independent Brownian motions in $\mathbb{R}^d$ ($d \geq 3$) started on the boundary of the balll $B(0,2)$ and let

$$\mathcal{V}_K \;=\; \mathbb{R}^d \setminus \Big(\bigcup_{k=1}^{K} \bigcup_{t_k=0}^{\infty} B\big(W^{(k)}_{t_k}, 1\big) \Big)$$

be the vacant set of the ensemble of K respective Wiener sausages of radius 1. Then for any $K \geq 1$ and $\varepsilon \in (0,1)$,

$$\mathbb{P}\left[\begin{array}{c} \mathcal{V}_K \cap B(0,\varepsilon) \text{ contains at least 2} \\ \text{connected components} \end{array} \right] \;\leq\; C \log^m\big(\tfrac{1}{\varepsilon}\big) \varepsilon^{d+1},$$

for some dimension dependent constants C and m. The exponent $d+1$ in the bound is sharp for $K \in \{1, 2\}$.

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Self-interacting random walks and statistical physics

Pierre Tarrès

(joint work with C. Sabot, M. Disertori, F. Merkl, S. Rolles)

Let $G = (V, E)$ be a non-oriented locally finite graph and, for all $e \in E(G)$, let $W_e : \mathbb{N} \longrightarrow (0, \infty)$ be a weight function. Then $(X_n)_{n\in\mathbb{N}}$ is called an *Edge Self-Interacting Random Walk (ESIRW)* if, for all $n \geq 0$, $X_n = i$, then

$$\mathbb{P}\big[X_{n+1} = j \mid X_k,\, k \leq n\big] \;=\; 1_{i\sim j}\frac{W_{\{i,j\}}(Z_n(\{i,j\}))}{\sum_{k\sim i} W_{\{i,k\}}(Z_n(\{i,k\}))}.$$

with

$$Z_n(\{i,j\}) \;:=\; \sum_{k=1}^{n} 1_{\{X_{k-1},X_k\}=\{i,j\}}$$

number of crossings of the nonoriented edge $\{i, j\}$.

Similarly, a *Vertex Self-Interacting Random Walk (VSIRW)* has the same definition, replacing the numbers of crossings of edges $Z_n(\{i, j\})$ by the numbers of visits to vertices

$$Z_n(j) \;=\; \sum_{k=0}^{n} 1_{\{X_k=j\}}.$$

The *Edge (resp. Vertex) Reinforced Random Walk (ERRW, resp. VRRW)* correspond to the affine case $W_e(n) = a_e + n$, and were first introduced by Coppersmith and Diaconis [3].

We start by a short survey on the behaviors of those walks. First, we note that the W-urns, corresponding to the case of star-shaped graphs, display a phase transition in behavior around the linear reinforcement case. We also note that the VRRW and ERRW offer strikingly different behavior on $\mathbb{Z}$: the VRRW a.s. eventually localizes on five consecutive sites [11], whereas it is easy to show that the ERRW visits all sites infinitely often (i.o.). Note that, by a simple Borel-Cantelli argument, one can show that, for any ESIRW with W nondecreasing and $\sum W(n)^{-1} = \infty$, if the walk visits one site i.o., then all sites are a.s. visited i.o. if the graph is connected.

Then we review the results by Tóth [12] on the ESIRW on $\mathbb{Z}$, using Ray-Knight (1963), or equivalently, Kesten-Kozlov-Spitzer (1975) arguments to find the various scalings for the position of the random walk for $W = W_e$ not depending on the edges.

The rest of the talk focuses on ERRW on general graphs. It is partially exchangeable in the sense of Diaconis and Freedman (1980), i.e. the probability of a path only depends on the number of crossings of edges, which implies that it is a mixture of Markov Chains, in other words a Random Walk in Random Environment (RWRE).

Now, we have shown with Sabot [8] that the ERRW can be seen (at jump times) as mixture (for random weights β) of the so-called Vertex-Reinforced Jump Process (VRJP) [4], also partially exchangeable, which jumps from i to j at time t a rate $\beta_{ij} L_j(t)$, where

$$L_j(t) \;=\; 1 + \int_0^t 1_{\{Y_s = j\}} \mathrm{d}s$$

is the local time (plus one) spent by the process at site j.

Next, we explain a new argument that allows to guess the mixing measure, based on a Bayesian approach, see the OOPS minicourse on YouTube (2019) for more details. That measure for the VRJP can be seen as the marginal in horospherical coordinates of the supersymmetric hyperbolic sigma model $\mathbb{H}^{2|2}$ in quantum field theory [8], studied by Disertori, Spencer and Zirnbauer [6]. We also explain the link between the VRJP with a random Schrödinger operator [9, 10]. All those techniques allow one to deduce that there is a unique phase transition between recurrence and transience in dimension $d \geq 3$ [8, 6, 1, 5, 10, 7]. We also discuss the Dynkin isomorphisms obtained by Bauerschmidt, Helmuth and Swan [2], which appear for the VRJP directly in hyperbolic coordinates $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$ coordinates.

Finally, we discuss a few open problems on self-interacting random walks.

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Macroscopic loops in random walk loop soups

Lorenzo Taggi
(joint work with Alexandra Quitmann)

We consider a general system of interacting random loops which includes several models of interest, such as the *Spin O(N) model, random lattice permutations*, a version of the *interacting Bose gas* in discrete space and of the *loop O(N) model.* We consider the system in $\mathbb{Z}^d$, with $d > 2$, and prove the occurrence of macroscopic loops, whose length is proportional to the volume of the system.

1. Definitions

Let $\mathbb{T}_L$ be a torus of side length L in $\mathbb{Z}^d$, whose elements can be identified with the set $\{x = (x_1, \dots, x_d) \in \mathbb{Z}^d \ : \ x_i \in (-\frac{L}{2}, \frac{L}{2}] \text{ for each } i = 1, \dots d\}$. Let $\mathcal{L}$ be the set of rooted oriented loops, i.e., finite ordered sequences of vertices in $\mathbb{T}_L$, $\ell = \big(\ell(0), \ell(1), \dots \ell(k)\big)$, such that $\ell(i)$ is a nearest-neighbour of $\ell(i-1)$ for each $i \in \{1, \dots, k\}$, $\ell(k) = \ell(0)$ and $k > 1$. For any such sequence $\ell = \big(\ell(0), \ell(1), \dots, \ell(k)\big) \in \mathcal{L}$, we denote by $|\ell| := k$ the length of the loop ℓ. We let $\Omega := \cup_{n=0}^{\infty} \mathcal{L}^n$ be the configuration space, whose elements are ordered collections of rooted oriented loops. Given any configuration $\omega \in \Omega$ we denote by $|\omega|$ the *number of loops*, i.e., $|\omega|$ is defined as the integer $n \in \mathbb{N}_0$ such that $\omega \in \mathcal{L}^n$. For any $\omega \in \Omega$, we define by

$$n_x(\omega) \ := \sum_{n=1}^{|\omega|} \sum_{j=0}^{|\ell_n|-1} \mathbf{1}_{\{\ell_n(j)=x\}}$$

the *local time* at $x \in \mathbb{T}_L$.

We first define a *potential*, $v\colon \mathbb{Z}^d \to \mathbb{R}$ satisfying $v(x) = v(-x)$ for each $x \in \mathbb{Z}^d$ and make the interaction periodic under torus translations by introducing the function $v_L\colon \mathbb{T}_L \times \mathbb{T}_L \to \mathbb{R}$, which depends on v and L, and is defined as

$$v_L(x,y) \;:=\; \sum_{z\in\mathbb{Z}^d} v(y+Lz-x).$$

Moreover, we define the *weight function*, $U:\mathbb{N}_0 \to \mathbb{R}_0^+$, which weights the local time at sites and may, for example, suppress configurations with local time above a certain threshold. We also introduce two parameters $\lambda, N \in \mathbb{R}^+$. Our measure assigns to any realisation $\omega \in \Omega$ with $n \in \mathbb{N}_0$ loops, $\omega = \big(\ell_1, \ell_2, \dots \ell_n\big) \in \mathcal{L}^n$, the weight,

(1)

$$\mathcal{P}_{L,U,v,N,\lambda}(\omega) \;:=\; \frac{1}{\mathcal{Z}_{L,U,v,N,\lambda}} \frac{1}{n!} \prod_{i=1}^{n} \frac{\lambda^{|\ell_i|}}{|\ell_i|} \left(\frac{N}{2}\right)^n \prod_{x\in\mathbb{T}_L} U(n_x(\omega)) \exp\big(-\mathcal{V}_L(\omega)\big),$$

where for any $\omega \in \Omega$,

$$\mathcal{V}_L(\omega) \;:=\; \sum_{i=1}^{|\omega|}\sum_{j=1}^{|\omega|}\sum_{m=0}^{|\ell_i|-1}\sum_{n=0}^{|\ell_j|-1} v_L\big(\ell_i(m), \ell_j(n)\big),$$

and $\mathcal{Z}_{L,U,v,N,\lambda}$ is a normalisation constant, to which we refer as *partition function.*

1.1. **Special cases.** *Interacting Bose gas.* When $N = 2$, and $U(n) = 1$ for any $n \in \mathbb{N}_0$, our loop soup is a version of the discrete Bose gas in the grand-canonical ensemble with chemical potential $\log\lambda$ and unit inverse temperature, the only difference with the Bose gas in continuous space is that the particles are located in $\mathbb{Z}^d$ rather than in $\mathbb{R}^d$ and that a single-step random walk trajectory rather than a Brownian bridge of time β (the inverse temperature) connects two consecutive particles.

Spin $O(N)$ model and BFS representation. When $N \in \mathbb{N}$, $v(x) = 0$ for any $x \in \mathbb{Z}^d$, and

$$U(n) \;=\; \frac{\Gamma(\frac{N}{2})}{2^n\Gamma(\frac{N}{2}+n)},$$

our model corresponds to the Brydges, Fröhlich and Spencer representation of the Spin O(N) model with inverse temperature $\lambda \geq 0$ [1].

Lattice permutations, loop $O(N)$ model and other models. When $v(x) = 0$ for any $x \in \mathbb{Z}^d$ and

$$U(n) \;=\; \begin{cases} 1 & \text{if } n \leq R, \\ 0 & \text{otherwise,} \end{cases}$$

our model is such that the local time at each vertex is upper bounded by $R \in \mathbb{N}$. When $R = 1$, our model reduces to random lattice permutations [4], which, in turn, reduce to the double dimer model when $\lambda = \infty$ [4].

2. Main result about the occurrence of macroscopic loops

Our main theorem states that, if λ is large enough, the expected length of any loop is of the order of the volume of the torus. In particular, the probability of existence of a loop connecting any pair of sites having distance proportional to the diameter of the box is uniformly positive. Such a general result requires a further assumption on the potential, to which we refer as *separability*. Here we provide two main examples of potentials fulfilling such an assumption, i.e.,

$$v_1(x) \;=\; \alpha \mathbf{1}_{\{x=0\}} - \beta\, \mathrm{e}^{-\iota |x|_1} \mathbf{1}_{\{x\neq 0\}}, \text{ and } v_2(x) \;=\; \alpha \mathbf{1}_{\{x=0\}} - \beta\, |x|_1^{-s} \mathbf{1}_{\{x\neq 0\}},$$

for parameters $\alpha, \beta, \iota, s \in [0,\infty)$ such that $s > d$, where $|\cdot|_1$ is the ℓ_1 distance on the torus. These two examples correspond to a local repulsive interaction and to a long-range attractive interaction which decays exponentially or polynomially with the distance. Such potentials are tempered if α is large enough with respect to β.

We now state our first main theorem. Let Γ_x be the first loop visiting the vertex x, namely for any $\omega = (\ell_1, \ldots, \ell_{|\omega|}) \in \Omega$, we let $w_x(\omega) := \inf\{i \in \{1,\ldots,|\omega|\} : x \in \ell_{w_i}\}$ be the smallest index of the loops visiting x, where $|\omega|$ is the total number of loops in ω. We then define for any $\omega = (\ell_1, \ldots, \ell_{|\omega|}) \in \Omega$,

$$\Gamma_x(\omega) \;:=\; \begin{cases} \ell_{w_x(\omega)} & \text{if } w_x(\omega) < \infty \\ \emptyset & \text{otherwise.} \end{cases}$$

Furthermore, we let $E_{L,U,v,N,\lambda}$ be the expectation with respect to (1). Finally, we say that the weight function $U\colon \mathbb{N}_0 \to \mathbb{R}_0^+$ has *range* R if

$$R \;:=\; \sup\{n \in \mathbb{N}_0 \,:\, U(n) > 0\}.$$

We also let $\mathbb{T}_L^o$ be the set of sites $x = (x_1, \ldots, x_d) \in \mathbb{T}_L$ such that $x_i \in 2\mathbb{N}_0 + 1$ for every coordinate $i \in [d]$ and denote the origin by $o \in \mathbb{T}_L$. The main result of this presentation is the following theorem, which has been proved in [2] (see also [3] for a further extension involving the double dimer model).

Theorem 1. *Let d, $N \in \mathbb{N}$, be such that $d \geq 3$ and $N \geq 2$, let R be a large enough integer depending on d and N, suppose that $v\colon \mathbb{Z}^d \to \mathbb{R}$ is tempered and separable, and let U be a good weight function with range at least R. Then, there exists $\lambda_0 < \infty$ such that, for any $\lambda > \lambda_0$, the following two properties hold:*

(i) *There exists $c_1 \in (0,\infty)$, which does not depend on L, such that,*

$$\liminf_{\substack{L\to\infty:\\ L\in 2\mathbb{N}}} \frac{E_{L,U,v,N,\lambda}(|\Gamma_x|)}{L^d} \;>\; c_1,$$

for any vertex $x \in \mathbb{T}_L$.

(ii) *There exist $c_2, c_3 \in (0,1)$, which do not depend on L, such that, for any $L \in 2\mathbb{N}$, any $x \in \mathbb{T}_L^o$ such that $d_L(o,x) \leq c_2\, L$,*

$$\mathcal{P}_{L,U,v,N,\lambda}\big(\exists n \in \{1,\ldots,|\omega|\} \,:\, o, x \in \ell_n\big) \;>\; c_3, \tag{2}$$

where $d_L(x,y)$ is the torus graph distance.

On the contrary, if λ is sufficiently small the model exhibits a quite different behaviour, indeed $E_{L,U,v,N,\lambda}(|\Gamma_x|) = O(1)$ in the limit as $L \to \infty$ and the quantity in the left-hand side of (2) decays exponentially with the distance between x and y (with exponential moments uniformly bounded in L). This can be proved using the cluster expansion method. Hence, the combination of these facts and of our theorem imply the occurrence of a phase transition with respect to the variation of the parameter λ.

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Grassmann variables and their connection with reinforced random walks

Margherita Disertori

Edge reinforced random walk (ERRW)[1, 2] and vertex reinforced jump processes (VRJP)[3, 4, 8] are history dependent stochastic processes where the particle tends to come back more often on sites it has already visited in the past. In both cases, on a finite undirected graph $G = (V, E)$ the process can be described via a random walk in random conductances $c_{ij} = W_{ij} e^{u_i+u_j}$, where $\{i, j\} \in E$ and $W_{ij} > 0$ is an edge weight that may be deterministic or random. Finally $\{u_i\}_{i\in V}$ is a family of real random variables with probability measure

(1)

$$\mu^V_{W,i_0}(\mathrm{d}u) = \sqrt{\sum_{S\in\mathcal{S}} \prod_{\{i,j\}\in S} W_{ij}\mathrm{e}^{u_i+u_j}} \prod_{\{i,j\}\in E} \mathrm{e}^{-W_{ij}(\cosh(u_i-u_j)-1)} \prod_{i\in V\setminus\{i_0\}} \frac{\mathrm{e}^{-u_i}}{\sqrt{2\pi}} \,\mathrm{d}u_i \,\mathrm{d}\delta(u_{i_0})$$

where i_0 is the starting point of the process and $\mathcal{S}$ is the set of spanning trees over the graph G. We denote the corresponding average by $\mathbb{E}^V_{W,i_0}[\cdot]$. Key properties of this measure were obtained [5, 6, 8, 9, 7] using its relation to a certain non-linear sigma model where the spin has both real and Grassmann components. The goal of this presentation is to introduce Grassmann variables and their use to derive the above measure.

Set $\{\xi_1, \ldots, \xi_N\}$ be N abstract generators. We construct an algebra by requiring that these objects anticommute: $\xi_i\xi_j = -\xi_j\xi_i$ and $\xi_j^2 = 0$. The real Grassmann algebra generated by $\{\xi_1, \ldots, \xi_N\}$ is the set of polynomes in these variables with real coefficients. Since $\xi_j^2 = 0$ each generator can appear with power 0 or 1 only:

$$\mathcal{G}_{\mathbb{R}}\big[\xi_1, \dots, \xi_N\big] \;=\; \Big\{X = \sum_{I \subset \{1,\dots,N\}} c_I \xi^I \,\Big|\, c_I \in \mathbb{R}\Big\}$$

where c_I is an antisymmetric tensor and $\xi^I = \prod_{j\in I} \xi_j$ according to some fixed ordering. Of particular importance is the subalgebra made of polynomes containing only even degree monomials

$$\mathcal{G}_{\mathbb{R}}^{even}\big[\xi_1, \dots, \xi_N\big] \;=\; \Big\{X \in \mathcal{G}_{\mathbb{R}} \,\Big|\, X = \sum_{I \subset \{1,\dots N\},\ |I| even} c_I \xi^I\Big\}$$

Each element $X \in \mathcal{G}_{\mathbb{R}}^{even}$ commutes with all elements of the algebra $\mathcal{G}_{\mathbb{R}}$ and can be uniquely decomposed as $X = x + n$ where $x \in \mathbb{R}$ and n is nilpotent. Using this decomposition we can extend every smooth function $f \in C^\infty(\mathbb{R})$ to a function $f : \mathcal{G}_{\mathbb{R}}^{even} \to \mathcal{G}_{\mathbb{R}}^{even}$ as follows.

$$f(X) \;=\; f(x+n) \;=\; \sum_{k\geq 0} \frac{f^{(k)}(x)}{k!} n^k.$$

The sum above is finite since n is nilpotent. The operations of partial derivation and integration can be extended to elements in $\mathcal{G}_{\mathbb{R}}$ as follows. Every $X \in \mathcal{G}_{\mathbb{R}}$ can be uniquely written as $X = X_1 + \xi_i X_2$, where X_1, X_2 are independent of ξ_i. The partial derivative with respect to ξ_i is then defined via

$$\partial_{\xi_i} X \;:=\; X_2.$$

The integral with respect to ξ_i can be defined as a map $I_j : \mathcal{G}_{\mathbb{R}} \to \mathcal{G}_{\mathbb{R}}$ such that (a) $I_j(v)$ is independent of ξ_j, (b) I_j is linear and (c) $I_j(\partial_{\xi_j} X) = 0$. With these constraints we obtain $I_j(1) = 0$ so we only need to define $I_j(\xi_j)$. With the convention $I_j(\xi_j) := 1$ we obtain

$$\int \mathrm{d}\xi_j X \;=\; \partial_{\xi_j} X.$$

We are now ready to introduce the non-linear sigma model related to ERRW and VRJP. Consider the Grassmann algebra $\mathcal{G}_{\mathbb{R}}[\{\xi_j, \eta_j\}_{j\in V}]$. We associate to each point $j \in V$ a spin $v_j = (x_j, y_j, z_j, \xi_j, \eta_j)$, where x, y, z are even elements in the algebra $x, y, z \in \mathcal{G}_{\mathbb{R}}^{even}$. We introduce the non-positive definite bilinear form

$$\langle v, v' \rangle \;=\; xx' + yy' - zz' + \xi\eta' - \eta\xi'.$$

Then, by inserting the non-linear constraint $\langle v, v\rangle = -1$ we further obtain that $z = \pm\sqrt{1 + x^2 + y^2 + 2\xi\eta}$. After selecting the + part we obtain the space H^2 with two additional Grassmann components, which motivates the name $H^{2|2}$. In analogy with statistical mechanics we define the *energy* of a spin configuration $v = \{v_j\}_{j\in V}$ as

$$S_V(v) \;:=\; \sum_{\{i,j\}\in E} W_{ij} \langle v_i - v_j, v_i - v_j\rangle \;=\; -\sum_{\{i,j\}\in E} W_{ij}(1 + \langle v_i, v_i\rangle).$$

Note that $S_V(v) \in \mathcal{G}_{\mathbb{R}}^{even}$ and hence $e^{-S_V(v)}$ is well defined. To break the non-compact symmetry we introduce the analog of a magnetic field by setting $v_{i_0} = o := (0,0,1,0,0)$. The 'Gibbs' measure can be written as

$$\langle f(v)\rangle_{W,i_0}^{V} := \int_{(H^{2|2})^{V\setminus\{i_0\}}} \mathrm{d}\delta(v_{i_0}-o)\prod_{j\neq i_0}\left(\frac{\mathrm{d}x_j\mathrm{d}y_j}{(2\pi)}\partial_{\xi_j}\partial_{\eta_j}\frac{1}{z_j}\right)\mathrm{e}^{-S_V(v)}f(v)$$

for regular enough functions. The factor $\frac{1}{z}$ appears because we are integrating on the non-linear manifold $H^{2|2}$. This model is connected to the probability measure $\mu_{W,i_0}^{V}(u)$ (cf. (1)) as follows

$$\mathbb{E}_{W,i_0}^{V}\left[f(e^u)\right] = \langle f(x+z)\rangle_{W,i_0}^{V}.$$

This can be seen by performing the change of coordinates $(x,y,\xi,\eta)\to(u,s,\bar\psi,\psi)$ (horospherical coordinates) defined via

$$x = \sinh u - \mathrm{e}^u\left(\frac{s^2}{2}+\bar\psi,\psi\right), \quad y = \mathrm{e}^u s, \quad \xi = \mathrm{e}^u\bar\psi, \quad \eta = \mathrm{e}^u\psi.$$

Conditioned on u, the variables $s,\bar\psi,\psi$ are Gaussian distributed with covariance $C(u)$, hence the corresponding integral can be performed exactly. The result is formula (1).

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The vertex-reinforced jump processes on $\mathbb{Z}^d$ with long range interactions

Silke Rolles

(joint work with Margherita Disertori, Franz Merkl)

Consider the vertex-reinforced jump process on the complete graph with vertex set $\mathbb{Z}^d$, $d \geq 1$, and edge weights

$$W_{ij} \ = \ w(\|i-j\|_\infty), \qquad i,j \in \mathbb{Z}^d,$$

with a decreasing weight function $w : [1,\infty) \to (0,\infty)$. Assume that

$$\sum_{i\in\mathbb{Z}^d\setminus\{0\}} w(\|i\|_\infty) \ < \ \infty \qquad \text{and} \qquad w(x) \ \geq \ \overline{W}\frac{(\log_2 x)^\alpha}{x^{2d}} \quad \text{for all } x \geq 1$$

for some $\alpha > 3$ and $\overline{W}$ large. In an ongoing joint project, we show that under these conditions the vertex-reinforced jump process is a.s. transient.

Consider the discrete time process associated to the vertex-reinforced jump process on finite boxes $\Lambda_N = \{0,1,\dots,2^N-1\}^d \cup \{\rho\}$, $N \in \mathbb{N}$, with wired boundary conditions and wiring point ρ. The transience proof uses the representation of this process as a random walk in random conductances, which was discovered by Sabot and Tarrès [4]. The random conductances are given by

$$W_{ij}\, \mathrm{e}^{u_i+u_j}, \qquad i,j \in \Lambda_N \cup \{\rho\},$$

where $u_\rho = 0$ and u_i, $i \in \Lambda_N$, are distributed according to the non-linear hyperbolic supersymmetric sigma model, also called $H^{2|2}$ model, which was introduced by Zirnbauer in [5]. Our key estimate shows that there exists $c > 0$ such that for all N, all $i \in \Lambda_N$, and all $m \in [0, c\overline{W}]$, one has

$$\mathrm{E}^{\Lambda_N}\left[(\cosh u_i)^m\right] \ \leq \ 2,$$

where the expectation is with respect to the $H^{2|2}$ model on $\Lambda_N \cup \{\rho\}$. Using a monotonicity result of Poudevigne [3], an upper bound for this expectation is given by the expectation of $(\cosh u_i)^m$ with respect to another $H^{2|2}$ model with hierarchical interactions. The task of studying $\cosh u_i$ in the last model can be reduced to studying it in an effective $H^{2|2}$ model as is shown in [1]. Using some monotonicity, we can compare the effective model with an $H^{2|2}$ model on a finite piece of a line graph with vertex set $\{1,\dots,N\}$ and nearest neighbor edges only. We analyse this one-dimensional $H^{2|2}$ model using methods inspired by the work of Disertori, Spencer, and Zirnbauer [2].

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