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Model Theory: Combinatorics, Groups, Valued Fields and Neostability

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Abstract. The scope of contemporary model theory has expanded enormously over the last several decades, helped by the development of new tools applicable to an ever wider range of structures. In the spirit of the previous meetings in the series, this workshop will bring together researchers from apparently separate subfields of model theory whose work is linked by common themes, with a particular emphasis on intrinsic model theoretic questions motivated by the classification of approriately 'tame' groups and fields and new developments in asymptotic combinatorics.

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Introduction by the Organizers

The workshop *Model Theory: Combinatorics, Valued Fields and Neostability*, organised by Itay Kaplan (Jerusalem), Silvain Rideau-Kikuchi (Paris), Katrin Tent (Münster) and Frank Wagner (Lyon) was attended by 46 physical and 3 virtual participants from all over the world. The meeting concentrated on the currently most active areas in model theory — generalizations of the abstract tools appropriate for an expanding class of structures, as well as its applications to the recent and fruitful interactions with asymptotic combinatorics, groups and valued fields:

(i) Neostability: NIP, NTP_2 and $NSOP_1$ theories;

(ii) Pseudofinite combinatorics;

(iii) Applications to groups and valued fields.

Since Hrushovski's proof of the Manin-Mumford conjecture which sparked the development of simplicity theory, progress in pure and in applied model theory has gone hand in hand. Shelah's forking independence notion was studied in ever wider contexts – simple, NIP (also called dependent), NTP_2 theories – and applied to the study of still somewhat tame, but unstable algebraic structures, with an increasing emphasis on identifying a stable part within. The study of pseudofinite, or more generally measurable structures, was inspired by the study of definable sets over pseudofinite fields by Chatzidakis, Macintyre and van den Dries, but really took off after Hrushovki's work on approximate subgroups (which led to their eventual classification by Breuillard, Green and Tao). In the last couple of years, another class of tame theories has been identified due to work by Chernikov, Kaplan, Kim and Ramsey, namely $NSOP_1$ theories. These carry an independence notion (Kim-independence) very similar to forking independence, and while the theoretical development now has reached a degree of maturity, applications to algebra are just starting to be developed.

There were a total of 22 talks of 50 minutes each, ranging over a wide spectrum of model theory and its applications. Seven talks were in neostability theory, developing the theory of new tame classes of structures, constructing examples and analysing their fine structure. Eight talks concerned the model theory of fields with added structure (derivation, automorphism, valuation), as well as a model-theoretic approach to tropical and real geometry. Two talks were about the interactions with combinatorics, and two talks about model-theoretic dynamics, leading to a new proof of Hrushovski's classification of infinite approximate subgroups. Two talks concerned o-minimality, and one talk the Tarski problem for hyperbolic groups.

Neostability. The talks in this section ranged from revisiting the old problem of classifying reducts of algebraically closed fields (Hasson) via a description of the automorphism groups of certain saturated structures (Chatzidakis) and a new examples of properly $NSOP_1$ and positively NIP algebraic theories (d'Elbée, Dobrowolski) to higher order stability theory (Chernikov, Valentin), where the usual binary decomposition of variables into parameter and type variables is replaced by a decomposition into three or more parts, thus replacing graphs in the underlying combinatorics by hypergraphs. A particular highlight in this section was the talk by Ramsey, who defined a new independence notion, GS-independence (which, in $NSOP_1$ theories coincides with Kim-independence) and a new class of theories where GS-independence is very well-behaved. This should give new insights both to Kim-independence in $NSOP_1$ theories, as well as extend neostabily-theoretic methods to the new class of treeless structures.

o-minimality. The two talks in this section concerned the shatter function in an o-minimal expansion of the ordered additive group of the reals, partially confirming a conjecture of Chernikov (Tran), and a characterization of those Lie groups which are definable in some o-minimal expansion of $\mathbb{R}$ (Onshuus).

Model-theoretic dynamics. In this section, there were two talks, one relating the Ellis group of the G-flow of finitely satisfiable types of a group G with the Ellis group of the G^*-flow of an elementary extension (Newelski). Krupinski gave a very impressive talk, where he used topological dynamics on locally compact flows to give a much shorter and entirely new proof of Hrushovski's Lie Model Theorem for arbitrary approximate subgroups, viz. the existence of a quasi-homomorphism from any approximate subgroup to some Lie group.

Combinatorics. There were two talks in this line of research, concerned with generalizing and improving bounds on combinatorial results. One used an Elekes-Szabó type theorem for algebraic group actions in order to characterize when there are maximally (quadratically) many triple lines in a finite subset of a reducible projective cubic surface with smooth components in the projective three space over the complex numbers (Zou); the other relaxed the notion of stability to allow perturbations by subsets of Loeb measure 0 in a pseudofinite group, and deduced the existence of squares and corners in dense subsets of Cartesian squares (Palacín).

Fields. From early on, the model theory of (various expansions of) fields has been closely linked both to the development of pure model theory and to the applications of model theory in arithmetic and geometry. Talks in this section covered a large spectrum of topics: a classification of definably semisimple groups interpretable in p-adically closed fields as linear up to finite kernel (Halevi), the complex p-adic field with the roots of the unit (Scanlon), Lang-Weil type bounds for the number of rational points of difference varieties over finite difference fields (Hils), residue field domination in henselian valued fields of equicharacteristic 0 (Haskell), transfer theorems for NIP valued fields and a classification of complete NIP henselian fields (Anscombe), a new and definable notion of stratification for real or algebraiclaly closed fields of characteristic zero, finer than the usual stratification (Halupczok), and a finiteness statement for algebras of functions over skeleta in Berkovich analytifications of algebraic varieties via stable completions of algebraic varieties (Loeser). A particular highlight was the talk of Jahnke, who presented a model-theoretic approach to Scholze's tilting method via ultraproducts, allowing for a transfer of many first-order properties between a perfectoid field and its tilt (and conversely) and thus opening up a very promising direction towards arithmetic geometry.

Hyperbolic groups. The model theory of free, and more generally hyperbolic groups is an area where many model-theoretic statements — such as Tarski's problem — so far have only been shown using deep methods from geometric group theory. In his talk André managed to present one such result in a very comprehensible way, namely a classification of finitely generated hyperbolic groups up to $\forall\exists$-elementary equivalence, together with an algorithm deciding whether two finitely presented hyperbolic groups have the same $\forall\exists$-theory or not.

Workshop: Model Theory: Combinatorics, Groups, Valued Fields and Neostability

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Abstracts

Generic multiplicative endomorphism of fields

Christian d'Elbée

Let L be the language consisting of symbols $+, -, \cdot, ^{-1}, 0, 1$ together with θ, a unary function symbol. We consider the L-theory T of L-structures (K, θ) where K is an algebraically closed field and $\theta : K \to K$ is a *multiplicative map*, that is a map satisfying $\theta(x) = 0$ if and only if $x = 0$ and θ restricted to the multiplicative group $K^\times$ is a group endomorphism. Typical examples of such multiplicative maps are the power functions $\mathrm{pw}^n : x \mapsto x^n$. In [3], the author proved that the class of *existentially closed* models of T is elementary by giving a geometric characterization of those and expressing this characterization by a set of first-order sentences, denoted ACFH. In this talk, the author presented various features shared by models of ACFH.

Recall that NSOP_1 theories were introduced Džamonja and Shelah in [5] and received a considerable amount of attention lately both in the abstract development of a suited notion close to Shelah's forking (called *Kim-forking*) and in finding new examples of strict (i.e. not simple) NSOP_1 theories. The motivation for studying ACFH is two-fold. First, the theory ACFH was constructed to cumulate genericity from previous examples of strict NSOP_1 theories (such as fields with generic subgroups, fields with random maps), to feed in Shelah's classification project as a new strict NSOP_1 theory. Second, ACFH was meant to be a candidate for the theory of *nonstandard power functions* $(\mathbb{C}, \mathrm{pw}^\infty)$, i.e. non-principal ultraproducts of $(\mathbb{F}_p^{\mathrm{alg}}, \mathrm{pw}^{n_p})$ considered as models of T, ranging over some primes $p \in P$. Although ACFH *does not* capture the theory of nonstandard power functions, one of the features shared by both theories is the ubiquity of definable *pseudofinite-cyclic* groups (see below).

Denote by $\mathop{\smile\hskip-0.9em\vert\hskip0.5em}^{\mathrm{alg}}$ the algebraic independence in algebraically closed fields and cl_θ the model-theoretic algebraic closure in ACFH, which is obtained by iteratively taking field-theoretic algebraic closure and direct image by θ. An analysis of types and higher amalgamation in ACFH yield that the ternary relation defined by

$$A \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^{\theta}_{C} B \iff \mathrm{cl}_\theta(AC) \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^{\mathrm{alg}}_{\mathrm{cl}_\theta(C)} \mathrm{cl}_\theta(BC)$$

satisfies the hypotheses of the characterisation of NSOP_1 theories given by the work of Chernikov-Kaplan-Ramsey in [1, 6]. Over models, the relation $\mathop{\smile\hskip-0.9em\vert\hskip0.5em}^{\theta}$ coincides with Kim-independence, an analogous of Shelah's forking independence in the realm of NSOP_1 theories.

In simple theories, Shelah's forking and Kim-forking coincide, but this is no longer true in strict NSOP_1 theories. Nonetheless, forking independence $\mathop{\smile\hskip-0.9em\vert\hskip0.5em}^{f}$ can be identified in the theory ACFH, it is obtained by 'forcing' successively the *base monotonicity* (on the right) and the *extension* property, denoted $(\mathop{\smile\hskip-0.9em\vert\hskip0.5em}^{\theta m})^* = \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^{f}$. In recent work, [4] an extension of the abstract theory of Kim-forking was developed in NSOP_1 theories where $\mathop{\smile\hskip-0.9em\vert\hskip0.5em}^{f}$ satisfies the *existence* axiom, $A \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^{f}_{C} C$ for all C. It is an open problem whether the existence axiom always holds for forking in NSOP_1 theories. The author gave indications that $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies existence in ACFH. Under existence for forking, ACFH has elimination of imaginaries.

Given a model (K, θ) of T and a polynomial $P(X) = k_0 + k_1 X + \dots k_n X^n \in \mathbb{Z}[X]$, one defines $P(\theta)$ to be the map $x \mapsto x^{k_0}\theta(x^{k_1})\dots\theta^{(n)}(x^{k_n})$. We assume now that (K, θ) is a model of ACFH. Then the map $P(X) \mapsto P(\theta)$ is an embedding of the ring $\mathbb{Z}[X]$ into the ring of definable multiplicative endomorphisms of (K, θ), whose image is denoted by $\mathbb{Z}[\theta]$. For any $\phi \in \mathbb{Z}[\theta]$, ϕ is onto $K \to K$, the kernel $G = \ker\phi$ behaves like a *generic* multiplicative subgroup of K (see [2]) in particular, it satisfies the equation $K = G + G$ (as sets). The question of whether every definable endomorphism of (K, θ) is in $\mathbb{Z}[\theta]$ was raised by the author. Answering this question seems challenging in light of currently available methods in NSOP_1 theories.

A *pseudofinite-cyclic* group G is one which is elementary equivalent to an ultraproduct of finite cyclic groups. The following criterion was developed jointly with I. Herzog:

Fact (d'Elbée-Herzog, [3, Appendix]) An abelian group G is *pseudofinite-cyclic* if and only if for all prime p

$$\#(G[p]) = \#(G/pG) \leq p.$$

With this criterion, an application of Snake's Lemma yields: for any field K with divisible multiplicative group, if $\phi : K^\times \to K^\times$ is a surjective multiplicative group endomorphism, then $\ker\phi$ is pseudofinite-cyclic. In particular, in any model (K, θ) of ACFH, $\ker\phi$ is a pseudofinite-cyclic subgroup of $K^\times$, for $\phi \in \mathbb{Z}[\theta] \setminus \{0\}$.

Of course, the latter is also true for any nonstandard power function $(\mathbb{C}, \mathrm{pw}^\infty)$. However $G = \ker \mathrm{pw}^\infty$ does not satisfy the equation $G + G = \mathbb{C}$, so $(\mathbb{C}, \mathrm{pw}^\infty)$ is not a model of ACFH, regardless of the choice of $(n_p)_p \in \mathbb{N}^P$.

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A finiteness result for tropical functions on skeleta

François Loeser
(joint work with Antoine Ducros, Ehud Hrushovski, Jinhe Ye)

Let F be a complete non-archimedean field. Among the several frameworks available for doing analytic geometry over F, Berkovich's is the one that encapsulates in the most natural way the deep links between non-archimedean and tropical (or polyhedral) geometry. Indeed, every Berkovich space X over F contains plenty of natural "tropical" subspaces, which are called *skeleta*. Roughly speaking, a skeleton of X is a subset S of X on which the sheaf of functions of the form $\log|f|$ with f a section of $\mathcal{O}_X^\times$ induces a piecewise linear structure; i.e., using such functions one can equip S with a piecewise linear atlas, whose charts are modelled on (rational) polyhedra and whose transition maps are piecewise affine (with rational linear part).

This definition is rather abstract, but there are plenty of concrete examples of skeleta. The prototype of such objects is the "standard skeleton" S_n of $(\mathbb{G}_m^n)^{an}$, that consists of all Gauss norms with arbitrary real parameters; the family

$$(\log|T_1|, \ldots, \log|T_n|)$$

induces a piecewise-linear isomorphism $S_n \simeq \mathbb{R}^n$.

Now if X is an arbitrary analytic space and if $\phi_1, \ldots, \phi_m$ are quasi-finite maps from X to $(\mathbb{G}_m^n)^{an}$, then $\bigcup_j \phi_j^{-1}(S_n)$ is a skeleton by [1], Theorem 5.1 (it consists only of points whose Zariski-closure is n-dimensional, so it is empty if $\dim X < n$), and $\phi_j^{-1}(S_n) \to S_n$ is a piecewise immersion for all j; of course, every piecewise-linear subspace of $\bigcup_j \phi_j^{-1}(S_n)$ is still a skeleton.

If S is a skeleton of an analytic space X and if f is a regular invertible functions defined on a neighborhood of S, then $\log|f|$ is a piecewise-linear function on S, and our purpose is to understand what are the piecewise linear functions on S that can arise this way in the *algebraic* situation.

Let us make precise what we mean. Let X be an *algebraic* variety over F, say irreducible of dimension n; let us call *log-rational* any real-valued function of the form $\log|f|$ for f a non-zero rational function on X, viewed as defined over U^{an} for U the maximal open subset of X on which f is well-defined and invertible. Let $\phi_1, \ldots, \phi_m$ be (algebraic) quasi-finite maps from X to $\mathbb{G}_m^n$ (the corresponding analytic maps will also be denoted $\phi_1, \ldots, \phi_m$). Let S be a subset of the skeleton $\bigcup \phi_j^{-1}(S_n)$ defined by a Boolean combination of inequalities between log-rational functions. Our main theorem is the following finiteness result.

Theorem 1 (Berkovich setting)**.** *Let X be an irreducible algebraic variety over F of dimension n and assume F is algebraically closed. Let S be as above. Then there exists finitely many non-zero rational functions $f_1, \ldots, f_\ell$ on X such that the group of restrictions of log-rational functions to S is stable under* min *and* max *and is generated under addition, substraction,* min *and* max *by the (restrictions of the) functions* $\log|f_i|$ *and the constants* $\log|a|$ *for* $a \in F^\times$.

Let us insist on the assumption that F is algebraically closed: for a general F the theorem does not hold, as shown by a counter-example due to Michael Temkin.

In fact, we do not work directly with Berkovich spaces but with the model-theoretic avatar of this geometry, namely the theory of *stable completions* of algebraic varieties which was introduced by Hrushovski and the speaker in [2]. Thus, what we actually prove is Theorem 2 which is a version of the result above in this model-theoretic framework – the final transfer to Berkovich spaces being straightforward.

Let us give some explanations. Let X be an algebraic variety over a valued field F. We denote by $\widehat{X}$ the stable completion of X. The standard skeleton S_n of $(\mathbb{G}_m^n)^{an}$ has a natural counterpart $\widehat{S}_n$ and $\bigcup \phi_j^{-1}(\widehat{S}_n)$ makes sense as a subset of $\widehat{X}$; moreover, the inequalities between log-regular functions that cut out S inside $\bigcup \phi_j^{-1}(S_n)$ also make sense here, and cut out a subset $\widehat{S}$ of $\bigcup \phi_j^{-1}(\widehat{S}_n)$. This subset is F-definably homeomorphic to an F-definable subset of Γ^N for some N. It follows moreover from its construction that $\widehat{S}$ is contained in the subset $X^{\#}$ of $\widehat{X}$ consisting of strongly stably dominated types (or, otherwise said, of Abhyankar valuations), and even in its subset $X^{\#}_{\text{gen}}$ of Zariski-generic points. And now the model-theoretic version of Theorem 1 is the following:

Theorem 2 (Model-theoretic setting)**.** *Let F be an algebraically closed field endowed with a non-trivial valuation* $\text{val} : F \to \Gamma \cup \{\infty\}$. *Let X be an irreducible algebraic variety over F. Let Υ be an iso-definable subset of $X^{\#}_{\text{gen}}$ which is Γ-internal, that is, F-definably isomorphic to an F-definable subset of Γ^N for some N. There exists finitely many non-zero rational functions $f_1, \ldots, f_\ell$ on X such that the group of restrictions of val-rational functions to Υ is stable under* min *and* max *and generated under addition, substraction,* min *and* max *by the (restrictions of the) functions* $\text{val}(f_i)$ *and the constants* $\text{val}(a)$ *for $a \in F^{\times}$ (as the terminology suggests, a val-rational function is a Γ-valued function of the form* $\text{val}(f)$ *with f rational, defined on the stable completion of the invertibility locus of f).*

Let us start with a remark. The Γ-internal subsets we are really interested in for application to Berkovich theory seem to be of a very specific form (they are definable subsets of $\bigcup \phi_j^{-1}(\widehat{S}_n)$ for some family (ϕ_j) of quasi-finite maps from X to $\mathbb{G}_m^n$) and our main theorem deals at first sight with fare more general Γ-internal subsets. But this is somehow delusive; indeed, we show that every Γ-internal subset of $X^{\#}_{\text{gen}}$ is contained in some finite union $\bigcup \phi_j^{-1}(\widehat{S}_n)$ as above.

We are now going to describe roughly the main steps of the proof of our main theorem.

Step 1. This first step has nothing to do with valued fields and concerns general divisible abelian ordered groups. Basically, one proves the following. Let D be an M-definable closed subset of Γ^n for some divisible ordered group M contained in a model Γ of DOAG, let $g_1, \ldots, g_m$ be $\mathbb{Q}$-affine M-definable functions on Γ^n, and let f be any continuous *and Lipschitz* M-definable map from D to Γ, such that for every x in D there is some index i with $f(x) = g_i(x)$. Then under these assumptions, f lies in the set of functions from D to Γ generated under addition, substraction, min and max by the g_i, the coordinate functions and M: Here the Lipschitz condition refers to a Lipschitz constant in $\mathbb{Z}_{\geq 0}$, so that it is a void condition when M has no non-trivial convex subgroup and D is definably compact, but meaningful in general.

Step 2. We start with proving a finiteness result in the spirit of our theorem under a weaker notion of generation. More precisely, we show the existence of $f_1, \ldots, f_\ell$ such the following weak version of our statement holds, with H denoting the group of Γ-valued functions on Υ generated by the $\mathrm{val}(f_i)$ and the constants $\mathrm{val}(a)$ for $a \in F^\times$: *for every non-zero rational function g on X there exist finitely many elements $h_1, \ldots, h_r$ of H such that Υ is covered by its definable subsets $\{\mathrm{val}(g) = \mathrm{val}(h_i)\}$ for $i = 1, \ldots, r$.*

The key point for this step is the purely valuation-theoretic fact that an Abhyankar extension of a defectless valued field is still defectless, a result proved by F.-V. Kuhlmann [3].

Step 3. One strengthens the statement of Step 2 by showing that the f_i can even be chosen so that all functions $(\mathrm{val}(g))|_\Upsilon$ as above are Lipschitz, when seen as functions on $\mathrm{val}(f)(\Upsilon) \subset \Gamma^m$. This is done by using an interpretation of the Lipschitz property in terms of coarsenings and refinements of valuations.

Step 4. One proves that the set of functions on Υ of the form $\mathrm{val}(g)$ is stable under min and max. This follows from orthogonality between the residue field and the value group sorts in ACVF.

Step 5. By the very choice of the f_i, every function $\mathrm{val}(g)|_\Upsilon$ gives rise *via* the embedding $\mathrm{val}(f)|_\Upsilon$ to a definable function on $\mathrm{val}(f)(\Upsilon)$ that is piecewise equal to one of the coordinate functions $x_1, \ldots, x_\ell$ (Step 2) and is moreover Lipschitz (Step 3); it is thus (Step 1) equal to $t(x_1, \ldots, x_\ell, a)$ where t is a term in $\{+, -, \min, \max\}$ and a a tuple of elements of $\mathrm{val}(F^\times)$. Then $\mathrm{val}(g)|_\Upsilon = t(\mathrm{val}(f_1)|_\Upsilon, \ldots, \mathrm{val}(f_\ell)|_\Upsilon, a)$ and we are done.

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Very ampleness in strongly minimal sets

Assaf Hasson
(joint work with Benjamin Castle)

The present work studies structures interpretable in algebraically closed field, and how far they can be from constructible sets (with their full induced structure). More precisely, the question we are studying is:

Problem. Let K be an algebraically closed field, $X \subseteq K^n$ a definable set. Let $\mathcal{X}^{\mathrm{Zar}}$ be the full K-induced structure on X and $\mathcal{X}$ a reduct of $\mathcal{X}^{\mathrm{Zar}}$. When is $\mathcal{X}$ a proper reduct of $\mathcal{X}^{\mathrm{Zar}}$?

To simplify the discussion we use the following terminology:

Definition.

(i) Let $\mathcal{K}$ be a structure. A $\mathcal{K}$-relic is a structure $\mathcal{X}$ whose universe, X, is a definable subset of K^n and whose atomic relations and functions are $\mathcal{K}$-definable.

(ii) If T is a theory then a T-relic is a $\mathcal{K}$-relic for some $\mathcal{K} \models T$.

(iii) A $\mathcal{K}$-relic, $\mathcal{X}$, is *full* if every $\mathcal{K}$-definable subset of X^n (for all n) is $\mathcal{X}$-definable.

We will focus on the case of ACF-relics.

Let us consider some examples:

Examples.

(i) A special case: when is a reduct, $\mathcal{X}$, of an algebraically closed field full (i.e., field addition and multiplication are $\mathcal{X}$-definable).
- If $\mathcal{X}$ is a full reduct of an algebraically closed field, then $\mathcal{X}$ is not locally modular.
- The ACF-reduct consisting of the two ternary relations $A(x,y,z) \equiv x^2+y^2=z^2$ and $M(x,y,z) \equiv' x^2y^2 = z^2$ is not locally modular (X/E is an algebraically closed field for $E(x,y) \equiv x^2 = y^2$). It is, however, easy to check that $\mathcal{X}$ has many automorphisms that do not preserve the underlying algebraically closed field. Thus $\mathcal{X}$ is not a full relic.
- The ACF-reduct consisting of the binary function $A(x,y) := x^2+y^2$ is easily seen to be full (in any characteristic other than 2).

(ii) If $K \models$ ACF then the K-relic on $K\times\{0\} \cup K \times \{1\}$ equipped with copies of field addition and multiplication in each copy of K (but no other structure) is, clearly, not full, since the function $(x,0) \mapsto (x,1)$ is not definable.

The following question of Martin's appears in [5]:

Question. Let $K \models \mathrm{ACF}_0$. Let $\mathcal{X}$ be a relic properly expanding $(K,\cdot)$ or the K-vector space structure on $(K,+)$. Is $\mathcal{X}$ full?

For the additive group the question was answered (positively) by Marker and Pillay, [4]. We give (among others) a positive answer for the multiplicative case as well.

Historically, Martin's question arose as part of the study of Zilber's restricted trichotomy conjecture:

Conjecture. If $K \models \mathrm{ACF}$ and $\mathcal{X}$ is a non-locally modular strongly minimal ACF-relic then $\mathcal{X}$ interprets a copy of K.

As can be seen from the above examples, Zilber's conjecture alone does not suffice for addressing our main question (and does not immediately address Martin's question). It is, however, key to the answer.

From now on **we assume Zilber's restricted trichotomy conjecture in its formulation above**[1].

The main step in answering our main question is addressing it for strongly minimal relics. As we have seen, non-local modularity is a necessary but not sufficient condition. We now isolate a condition that is necessary and sufficient. The definition (as well as the terminology) is inspired by the work of Hrushovski and Zilber [3][2]:

Definition. A strongly minimal structure $\mathcal{M}$ is a *very ample* if there exists an almost faithful family of plane curves $\{C_t : t \in T\}$ such that for any distinct $x, y \in M^2$ we have $\mathrm{MR}(C^x \cap C^y) \leq \mathrm{MR}(T) - 2$. Where:

- **(i)** By a plane curve we mean a definable subset of M^2 of Morley rank 1.
- **(ii)** By an almost faithful family of plane curve we mean a family where for all $t \in T$ the set $\{s \in T : |C_s \cap C_t| = \infty\}$ is finite. And,
- **(iii)** $C^x := \{t \in T : x \in C_t\}$.

Our first theorem is then:

Theorem. Let K be an algebraically closed field, $\mathcal{M}$ a strongly minimal K-relic. Then the following are equivalent:

- **(i)** $\mathcal{M}$ is very ample.
- **(ii)** $\mathcal{M}$ is isomorphic, outside a finite set, to the full structure induced on some irreducible algebraic curve over K.
- **(iii)** $\mathcal{M}$ is full.

From this we conclude, by a non-trivial induction on dimension:

[1] This a very mild assumption since the conjecture has been proved by Castle [1] in characteristic 0, by Hasson and Sustretov for 1-dimensional relics ([2]) in all characteristics, and in a yet unpublished work by Castle and Ye for high-dimensional ACVF-relics.

[2] It can be shown that in the context of Zariski Geometries the above definition of very ampleness coincides with the one of Hrushovski and Zilber. The proof uses the main result of [3].

Theorem. Let K be an algebraically closed field, Then a K-relic, $\mathcal{M}$, is full if and only if it is almost strongly minimal and and every strongly minimal set in $\mathcal{M}^{eq}$ is very ample.

Remarks.

(i) Any definable family of plane curves witnessing very ampleness of a strongly minimal set obviously shows that this strongly minimal set is not locally modular. So the very ampleness assumption (with the fact that all ACF relics have finite Morley Rank) implies that a relic satisfying the assumptions is not 1-based.

(ii) As we have seen in the above examples, very ampleness is not preserved under non-orthogonality. So the requirement that every strongly minimal set is very ample is necessary.

(iii) Note that for ACF-relics unidimensionality does not imply almost strong minimality.

(iv) In full generality, it is not clear how to verify that all strongly minimal sets in $\mathcal{M}^{eq}$ are very ample. There are, however, some special cases where this can be verified:

- **a**. Any strongly minimal expansion of a very ample strongly minimal set is very ample.
- **b**. Any non-locally modular strongly minimal set with elimination of imaginaries (in the home sort) can be shown to be very ample. In particular any strongly minimal expansion of a field is very ample.
- **c**. A strongly minimal expansion of a divisible groups is very ample if and only if it is not locally modular.
- **d**. If X, Y are strongly minimal, X is very ample and Y is internal to X then Y is very ample.

With all of the above we can answer Martin's question:
Theorem. Let K be an algebraically closed field, $(G, \cdot)$ a divisible 1-dimensional algebraic group over K. Let G^{lin} be the expansion of $(G, \cdot)$ by all K-definable endomorphisms of G. Then any K-relic properly expanding G^{lin} is full.

We leave open the following question:
Question. Let $\mathcal{M}$ be a non-locally modular strongly minimal set. Is there a very ample strongly minimal set internal to M? More specifically: is there a definable equivalence relation E on M such that M/E is very ample.

We suspect that Hrushovski's amalgamation construction technique may provide a counter-example, at least, to the weaker form of the above question.

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Simplicity of the automorphism group of some saturated structures

Zoé Chatzidakis

(joint work with Thomas Blossier, Charlotte Hardouin, Amador Martin Pizarro)

1. Introduction

The talk given in this workshop was very similar to the talk given two months before during the Mini workshop 2248a *Topological and Differential Expansions of o-minimal Structures* (27 November - 3 December 2022). The reader interested in further details is referred to this report [4], or to the ArXiv preprint [2].

D. Lascar proved in 1995 a very striking and surprising result: $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}^{alg})$ is simple ([7]). It was actually the continuation of an earlier paper (1992, [6]), on automorphism groups of countable saturated strongly minimal structures, and where the result was announced assuming $\aleph_1 = 2^{\aleph_0}$. The proof given in the 1992 paper used topology (Polish group, Baire category), the proof in the 1995 paper was much more combinatorial. These results were later extended to other types of structures, using both methods - topological or combinatorial.

We wanted to extract from the second proof of Lascar what made things works, and how this can be used to extend the existing results to other fields with operators. This is done by listing several fundamental properties, and proving a few lemmas. It also builds on existing work by the four authors (in various combinations).

2. Fields with operators

Our theory T is a complete theory of fields (in the language $\{+, \cdot, 0, 1, ^{-1}, \ldots\}$), with a good notion of dimension or rank, and with some operators. Here are the structures we will discuss more in detail:

(i) The theory of algebraically closed fields of a given characteristic. No operator. ACF_p with $p = 0$ or p a prime.

(ii) The theory of differentially closed fields of characteristic 0, one or several commuting derivations are the operators. DCF_0, $\mathrm{DCF}_{0,m}$.

(iii) The theory of existentially closed difference fields of characteristic 0, with prescribed action of the automorphism σ on $\bar{\mathbb{Q}}$. The operators are σ and σ^{-1}. ACFA.

(iv) Separably closed fields, together with the λ-functions as operators: (a) finite degree of imperfection; (b) infinite degree of imperfection.

(v) The theories $T(X)$ of differential fields of characteristic 0 introduced by Hrushovski and Itai [5], the derivation is the operator.

We very much use the existence (and uniqueness) of the generic type of the additive group, and its precise description in the five examples. The generics of examples 1-3 and 5 are regular, but not those of example 4. There are good notions of bases in examples (1-3), (4b) and 5.

We use these types to define a notion of closure, denoted cl. This notion, in contrast with algebraic closure, depends on the ambient model, and the closure of $\emptyset$ may be uncountable.

The theories of examples 1,2 and 4,5 are stable, so that uncountably saturated models exist (with some restriction on the cardinalities in example 4). The theories of example 3 are however unstable. Under suitable saturation hypotheses on the fixed field of the algebraically closed difference field K, the results of Shelah on existence and uniqueness of κ-prime models over K do extend ([3]).

3. The result

Theorem 1. *Let T be one of the theories (1-3), M a model of T, and $\kappa \geq \aleph_1$. Assume that M is κ-prime over $A := \mathrm{cl}_M(\emptyset)$. Then $\mathrm{Aut}(M/A)$ is simple.*

In particular we have:

Corollary. *Let T be one of the theories (1-3), M an uncountable model of T which is saturated. Then $\mathrm{Aut}(M/\mathrm{cl_M}(\emptyset))$ is simple.*

A notion playing an important role in the proof is that of *unbounded* automorphism. Lascar [7], then Blossier, Hardouin and Martin-Pizarro [1], show that the only bounded automorphism of a κ-saturated model M are the identity and powers of the Frobenius; so the only bounded automorphism which fixes $\mathrm{cl}(M)$ is the identity. Both results are then direct consequences of

Theorem 2. *Let $\mathcal{U}$ be κ-prime over $\mathrm{cl}_{\mathcal{U}}(\emptyset)$, with T as in (1-3). Let $\tau \in \mathrm{Aut}(\mathcal{U}/\mathrm{cl}(\emptyset))$ be unbounded. Then every $\nu \in \mathrm{Aut}(\mathcal{U}/\mathrm{cl}(\emptyset))$ can be written as the product of four conjugates of τ and τ^{-1}.*

I also discussed the possible extension to examples 4b and 5. Other examples may arise.

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Ellis groups

Ludomir Newelski

(joint work with Adam Malinowski)

How does the (definable) topological dynamics of a group G change when we extend the group elementarily? In particular are the related Ellis (semi)groups related algebraically?

Assume G is an infinite group. By a G-algebra we mean an algebra of subsets of G closed under left translation by elements of G. Assume $\mathcal{A}$ is a G-algebra. Then its Stone space $S(\mathcal{A})$ is naturally a G-flow and we consider the associated Ellis semigroup $E(S(\mathcal{A}))$ and its Ellis subgroups.

Definition.
(1) Given $p \in S(\mathcal{A})$ we define a homomorphism of G-algebras
$d_p : \mathcal{A} \to \mathcal{P}(G)$ by $d_p(U) = \{g \in G : g^{-1}A \in p\}$.
(2) We say that $\mathcal{A}$ is d-closed if $\mathcal{A}$ is closed under d_p for every $p \in S(\mathcal{A})$.
(3) When $\mathcal{A}$ is d-closed we define a binary operation $*$ on $S(\mathcal{A})$ by

$$U \in p * q \iff d_q U \in p$$

When $\mathcal{A}$ is d-closed, then $(S(\mathcal{A}), *)$ is a left-continuous semigroup isomorphic to the Ellis semigroup $E(S(\mathcal{A}))$.

We work in the following combinatorial set-up. $G \prec H$ are infinite group structures, $\mathcal{A}$ is a d-closed G-subalgebra of the algebra $\mathrm{Def}(G)$ of definable subsets of G and $\mathcal{B}$ is a d-closed H-algebra containing $A(H)$ for every $A \in \mathcal{A}$. Also we assume that $\mathcal{B}|_G := \{B \cap G : B \in \mathcal{B}\}$ equals $\mathcal{A}$. We prove the following theorems.

Theorem 1. *Assume there are generic points in $S(\mathcal{B})$. Then the Ellis groups of $S(\mathcal{A})$ are homomorphic images of some subgroups of the Ellis groups of $S(\mathcal{B})$.*

Theorem 2. *Assume every minimal left ideal in $S(\mathcal{B})$ is a group. Then the Ellis groups of $S(\mathcal{A})$ are isomorphic to some closed subgroups of the Ellis groups of $S(\mathcal{B})$.*

The main motivation for the combinatorial set-up is the following model-theoretic set-up, which is its special case. Assume G is an infinite group definable in a model M and N is a $*$-elementary extension of M. Then G, $H = G(N)$, $\mathcal{A} = \mathrm{Def}_{ext,G}(M)$ and $\mathcal{B} = \mathrm{Def}_{ext,G}(N)$ satify the assumptions of the combinatorial set-up. We compare the G-flow $S_{ext,G}(M) = S(\mathcal{A})$ and and the H-flow $S_{ext,G}(N) = S(\mathcal{B})$. Theorems 1 and 2 translate to:

Theorem 1′. *Assume there are generic types in $S_{ext,G}(N)$. Then the Ellis groups of $S_{ext,G}(M)$ are homomorphic images of some subgroups of the Ellis groups of $S_{ext,G}(N)$.*

Theorem 2′. *Assume every minimal left ideal in $S_{ext,G}(N)$ is a group. Then the Ellis groups of $S_{ext,G}(M)$ are isomorphic to some closed subgroups of the Ellis groups of $S_{ext,G}(N)$.*

The assumptions of Theorems 1 and 2 are dual to each other, just like their proofs. Theorem 1' was already proved in [2]. The proofs of Theorems 1 and 2 use the dual notions of weak heirs and weak coheirs. We elaborate on them. In the stable case we provide a characterization in terms of local forking. Involved in the proof of Theorem 1 is a variant of the Ellis structure theorem for some left-continuous semigroups that are not necessarily compact. The results of this talk appear in [1].

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Number of triple lines on reducible cubic surfaces and Elekes-Szabó for algebraic group actions

Tingxiang Zou

(joint work with Martin Bays and Jan Dobrowolski)

The classical Orchard Problem asks the following question: given a set of n points on a plane, what is the maximal number of lines each contains at least three distinct points of this set. These lines are called triple lines. When n is large enough, the best solution is given by choosing finite subgroups on an elliptic curve [4]. Elekes and Szabó also studied this problem restricted to subsets on plane curves [3]. They proved that if a curve contains arbitrary large finite sets with quadratically many triples lines, then this curve must be a cubic curve. We investigated the spacial version of this question.

Let X be a (possibly reducible) cubic surface with smooth irreducible components in $\mathbb{P}^3(\mathbb{C})$: the projective three-space over complex numbers. We asked if it is possible to have a sequence of finite sets $A_n \subseteq X(\mathbb{C})$ such that the size of A_n goes to infinity, and the number of distinct collinear triples in A_n^3 is $c_n|A_n|^2$ with $\lim_{n\to\infty} \log_{|A_n|} c_n = 0$. Such a configuration is possible when $(A_n)_n$ are concentrated on some projective plane π. For example when π is not contained in X, then $\pi \cap X$ is a cubic curve, and one can find quadratically many collinear triples on a cubic curve as mentioned above. We asked if there are other possibilities. And the answer is no, unless X is a union of three parallel planes.

The formulation of this problem can be simplified by taking ultraproducts and define the coarse dimension based on non-standard counting, which captures the exponent r in n^r when n tends to infinity.

Given an ultraproduct $M := \prod_{i\to\mathcal{U}} M_i$, where $\mathcal{U}$ is a non-principal ultrafilter on some countable index set I. A set $A \subseteq M^n$ is called *internal* if $A = \prod_{i\to\mathcal{U}} A_i$ for $A_i \subseteq M_i^n$. We consider the nonstandard reals $\mathbb{R}^{\mathcal{U}}$ over the same ultrafilter and the standard part map st : $\mathbb{R}^{\mathcal{U}} \cup \{-\infty, \infty\} \to \mathbb{R} \cup \{-\infty, \infty\}$. Let $\xi \in \mathbb{R}^{\mathcal{U}}$ with $\xi > r$ for all $r \in \mathbb{R}$, namely $\xi \in \mathbb{R}^{\mathcal{U}}_{\geq 0} \setminus \mathbb{R}$. Define the *coarse dimension with respect to* ξ, $\boldsymbol{\delta}_\xi$, as a function on internal sets of M given by $\boldsymbol{\delta}_\xi(A) := \operatorname{st}\log_\xi |A|$, where $|A|$ is the non-standard cardinality of A. An internal set A is called *broad* if $0 < \boldsymbol{\delta}_\xi(A) < \infty$.

Let K be an ultrapower of the complex numbers. We will fix some $\xi \in \mathbb{R}^{\mathcal{U}}$ from now on and simply denote $\boldsymbol{\delta}_\xi$ as $\boldsymbol{\delta}$. We have proved the following:

Proposition 1. *Let $X \subseteq \mathbb{P}^3(K)$ be a cubic surface with smooth irreducible components. Suppose X is not a union of three planes intersecting on a common projective line. Let $R_X \subseteq X^3$ be the relation defined as: $(a, b, c) \in R_X$ if a, b, c are three distinct collinear points on X which are not contained in a line contained in X. Suppose there are $A, B, C \subseteq X$ such that $\boldsymbol{\delta}(A) = \boldsymbol{\delta}(B) = \boldsymbol{\delta}(C) = 1$ and $\boldsymbol{\delta}(R_X \cap (A \times B \times C)) = 2$ then there is a projective plane $\pi \not\subseteq X$ such that $\boldsymbol{\delta}(R_X \cap (A \times B \times C) \cap \pi^3) = 2$.*

This result shows that any large enough finite set on X which contains quadratically many collinear triples must concentrate on a plane not contained in X.

As a corollary, we get the finitary version of Proposition 1.

Proposition 2. *For any $\epsilon > 0$, there exists $\eta > 0$ and $N_0 \in \mathbb{N}$ with the following properties. Let $X \subseteq \mathbb{P}^3(\mathbb{C})$ be a cubic surface with smooth irreducible components which is not the union of three planes intersecting on a common projective line. Let R_X be the collinearity relation on X^3. Suppose $A \subseteq X$ is a finite subset with $|A| =: N \geq N_0$ such that $|R_X \cap \pi^3 \cap A^3| < N^{2-\epsilon}$ for all projective planes π. Then $|R_X \cap A^3| < N^{2-\eta}$.*

To prove Proposition 1 in the case when X is reducible, i.e. a union of three planes or a union of a smooth quadric surface and a plane, we established an Elekes-Szabó type theorem about algebraic group actions defined over fields of characteristic 0.

To state the result, we need to define a condition called wgp on broad internal sets, which is a tool for analyzing internal broad subsets of algebraic varieties.

A broad internal subset A of an absolutely irreducible variety $V = V(K)$ is called *in weak general position* (wgp) if $\boldsymbol{\delta}(A \cap W) < \boldsymbol{\delta}(A)$ for any proper subvariety W of V.

Proposition 3. *Let K be an ultrapower of $\mathbb{C}$. Let $G = G(K)$ be an algebraic group acting on an irreducible variety $X = X(K)$ by regular maps defined over some algebraically closed countable subfield $F \leq K$. Let Γ be the graph of the action, namely,*

$$\Gamma = \{(x, g, x') : x, x' \in X, g \in G, x' = g(x)\}.$$

Let Y and Z be irreducible (over F) subvarieties of G and X respectively, and H be the connected algebraic subgroup of G generated by Y. Suppose further that no non-trivial element of H fixes Z pointwise.

If there are broad internal sets A wgp in Y and B wgp in Z, such that

$$\boldsymbol{\delta}(\Gamma \cap (B \times A \times B)) = \boldsymbol{\delta}(A) + \boldsymbol{\delta}(B),$$

then H is nilpotent.

The key step to prove the above proposition is to build a coarse approximate subgroup in H. By the understanding of coarse approximate subgroups in algebraic groups over fields of characteristic 0 [2], we know A is essentially contained in a nilpotent group. More precisely, by the results about wgp coarse approximate subgroups studied in [1], we deduce that H must be nilpotent.

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Definably semisimple groups interpretable in p-adically closed fields

Yatir Halevi

(joint work with Assaf Hasson and Kobi Peterzil)

A valued field (K, v) is p-adically closed if it is elementarily equivalent to a finite extension of $\mathbb{Q}_p$.

It is well known [4], that every definable field in a p-adically closed field is definably isomorphic to a finite extension of the field. In previous papers [2, 3], we have shown that every interpretable infinite field is definably isomorphic to a finite extension of the field. Furthermore, we show that every infinite interpretable group has unbounded exponent and that if it is dp-minimal then it is abelian-by-finite.

We present here a general result on definably semisimple groups interpretable in a p-adically closed field. We say that a definable group G is *definably semisimple* if it has no infinite abelian definable normal subgroup.

Our main result is the following.

Theorem.[1] *Let G be an interpretable group in a p-adically closed field K. If G is definably semisimple then it has a finite normal subgroup $H \trianglelefteq G$ for which G/H is definably K-linear.*

Our results pass through local analysis of interpretable groups which we outline here.

Proposition. *Let G be an infinite interpretable group. Then there exists a finite normal subgroup $H \trianglelefteq G$, an infinite definable subset $X \subseteq G/H$ and a definable injection $f : X \to D^n$, where D is either K, Γ or $K/\mathcal{O}$, where $\mathcal{O}$ is the valuation ring and Γ is the value group.*

Thus, up to a quotient by a finite normal subgroup, we may pull back (local) algebraic and topological properties from D to G.

Assume for simplicity that $H = \{e\}$. Using the function f we may pull back certain *infinitesimal* type-definable subgroups from D to G. If D is either Γ or $K/\mathcal{O}$ we can use this to find an infinite abelian definable subgroup of G. Thus if G is definably semisimple necessarily $D = K$. Now using the infinitesimal subgroup and the fact that in K every definable function is locally differentiable, we can find a definable local differentiable Lie group in G. Now, using Lie theory and the adjoint map we find a homomorphism from G to $M_n(K)$ with finite kernel.

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On the Tarski problem for all hyperbolic groups

Simon André

Around 1945, Tarski asked whether non-abelian free groups are elementarily equivalent. This famous problem remained open for more than fifty years, and was finally solved positively by Sela in [8] (see also [5] by Kharlampovich and Myasnikov). Then, Sela extended his work to *torsion-free* hyperbolic groups, and classified the finitely generated groups that are elementarily equivalent to a given non-abelian torsion-free hyperbolic group G: a finitely generated group G' is elementarily equivalent to G if and only if there exist two isomorphic subgroups $H \subset G$ and $H' \subset G'$ such that G and G' are hyperbolic towers over H and H' respectively (see for instance [4, 6, 9] for the definition of a tower).

The main goal of the talk was to present a partial generalization of this classification to *all* hyperbolic groups, possibly with torsion. The notion of a hyperbolic group was introduced by Gromov in [3], and has been a fundamental object of study in infinite group theory ever since. Recall that a geodesic metric space is said to be *hyperbolic* if there exists a constant $\delta \geq 0$ such that any geodesic triangle $\Delta(x,y,z)$ is *δ-slim*, meaning that for any point p on the side $[x,y]$, the distance from p to $[x,z] \cup [z,y]$ is at most δ (i.e. $[x,y]$ is contained in the union of the δ-neighborhoods of $[x,z]$ and $[z,y]$). A group G is said to be *hyperbolic* if it acts by isometries, properly discontinuously and cocompactly (i.e. with compact quotient) on a proper geodesic hyperbolic metric space. Note that this definition implies that the group G is finitely generated (by the famous Milnor-Švarc lemma). Equivalently, a finitely generated group G is hyperbolic if its Cayley graph (with respect to some, or equivalently any, finite generating set of G) is a hyperbolic metric space.

We say that two groups are *$\forall\exists$-equivalent* if they satisfy the same *$\forall\exists$-sentences*, i.e. the same first-order sentences of the form

$$\forall x_1 \cdots \forall x_m \exists y_1 \cdots \exists y_n \ \psi(x_1, \ldots, x_m, y_1, \ldots, y_n),$$

where ψ is a quantifier-free formula with $m+n$ free variables. Informally, our main result (Theorem 1 below) states that a hyperbolic group G and a finitely generated group G' are $\forall\exists$-equivalent if and only if there is a finite sequence of moves joining them, each move consisting in adding or removing generators and relations from the normalizer of some finite subgroup in a very specific manner. These moves are of three types, which we call HNN-equivalence, O-equivalence and QH-equivalence (see Definition 1).

Theorem 1. *Let G be a hyperbolic group, and let G' be a finitely generated group. Then G and G' are $\forall\exists$-equivalent if and only if there exists a finite sequence of groups $G_0 = G, \ldots, G_n \simeq G'$ for some integer $n \geq 0$ such that for every $i < n$, the groups G_i and G_{i+1} are HNN-equivalent or O-equivalent or QH-equivalent.*

Remark 2. *The groups G_i and G_{i+1} are $\forall\exists$-equivalent for every $i < n$.*

Remark 3. *It can be seen that HNN-equivalence, O-equivalence and QH-equivalence preserve hyperbolicity. Hence, as a corollary of Theorem 1, we recover the main result of* [1] *(previously proved by Sela in the absence of torsion, see* [9, Theorem 7.10]*): if a finitely generated group G' is $\forall\exists$-equivalent to a hyperbolic group G, then G' is hyperbolic.*

It is worth pointing out that, for torsion-free groups, we recover Sela's classification. Indeed, in the absence of torsion, an HNN-equivalence simply consists in doing a free product with $\mathbb{Z}$ or collapsing a free factor isomorphic to $\mathbb{Z}$ in a free product decomposition, and a QH-equivalence consists in building or collapsing a hyperbolic floor in the sense of [4, 6, 9]. Moreover, O-equivalence is a new phenomenon that only appears in the presence of torsion.

We give below a slightly imprecise definition of the three types of equivalence that appear in the statement of Theorem 1.

Definition 1. *Let G be a finitely generated group and let H be a subgroup of G. We say that G is a* floor *over H if there exists a finite subgroup F of H such that:*

— *G splits as an amalgamated product of the form $H *_{N_H(F)} N_G(F)$;*
— *$N_H(F)$ is infinite;*
— *the set of finite subgroups of H normalized by $N_H(F)$ has a maximum for inclusion (this is always the case when G is a hyperbolic group), and this maximum is F;*
— *one of the following three conditions holds:*

(i) *$N_H(F)$ has a subgroup isomorphic to the free group of rank 2, and $N_G(F)$ is obtained from $N_H(F)$ by adding a stable letter centralizing F (in other words, G is simply the HNN extension of H over the identity of F). We say that G is an* HNN-floor *over H.*

(ii) *$N_H(F)$ has a subgroup isomorphic to the free group of rank 2, and $N_G(F)$ splits as a graph of groups Δ with two vertices, one of which is $N_H(F)$, and the other is a finite extension*

$$1 \to F \to Q \to \pi_1^{\mathrm{orb}}(\mathcal{O}) \to 1$$

where $\mathcal{O}$ denotes a two-dimensional hyperbolic compact connected orbifold without mirrors, and the edge groups of Δ coincide with the preimages in Q of the boundary and conical subgroups of $\pi_1^{\mathrm{orb}}(\mathcal{O})$; some additional technical conditions are required (such as the existence of a particular retraction $r : G \twoheadrightarrow H$). We say that G is a QH-floor *over H.*

(iii) *$N_H(F)$ is virtually cyclic infinite, the embedding $N_H(F) \hookrightarrow N_G(F)$ used to define the amalgamated product $G = H *_{N_H(F)} N_G(F)$ maps non-conjugate finite subgroups to non-conjugate finite subgroups, and moreover there exists an embedding $N_H(F) \hookrightarrow N_G(F)$ that maps non-conjugate finite subgroups to non-conjugate finite subgroups. In addition, we suppose that there exists a splitting of H in which $N_H(F)$ is not elliptic. We say that G is an* O-floor *over H.*

If G' is a finitely generated group, we say that G and G' are HNN-equivalent *(respectively* QH-equivalent, O-equivalent*) if G is an HNN-floor (respectively a QH-floor, an O-floor) over a subgroup $H \subset G$ isomorphic to G', or vice versa.*

Sela proved in [8] that the first-order theory of a non-abelian free group F_n admits a *uniform* quantifier elimination down to the Boolean algebra of $\forall\exists$-sentences: for any first-order sentence ψ in the language of groups, there exists a Boolean combination φ of $\forall\exists$-sentences, independent of the rank n, such that ψ is true in F_n if and only if φ is true in F_n. Since non-abelian free groups are $\forall\exists$-equivalent (see [2], [7] or [8, Theorem 3]), it follows that they are elementarily equivalent. For a fixed non-elementary torsion-free hyperbolic group H, Sela proved a similar uniform quantifier elimination (we refer to the discussion after the proof of Propostion 7.8 in [9]): for every first-order sentence ψ, there exists a Boolean combination φ of $\forall\exists$-sentences such that, for every finitely generated group G which is either a free product $G = H * F_n$ (for some $n \geq 0$) or a hyperbolic tower over H, ψ is true in G if and only if φ is true in G. Since H and G are $\forall\exists$-equivalent (see [9]), they are elementarily equivalent. In particular, it follows that two torsion-free hyperbolic groups are $\forall\exists$-equivalent if and only if they are elementarily equivalent. For hyperbolic groups with torsion, we expect a similar uniform elimination of quantifiers, and we formulate the following conjecture.

Conjecture 4. *Any two hyperbolic groups are $\forall\exists$-equivalent if and only if they are elementarily equivalent.*

Note that if one believes this conjecture, then the classification 1 is in fact a classification up to elementary equivalence.

Our proof of the classification 1 builds on our previous works [1] and [2], in which we proved that hyperbolicity is preserved by elementary equivalence among finitely generated groups, and classified the finitely generated virtually free groups up to $\forall\exists$-equivalence.

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Riso-stratifications

Immanuel Halupczok

(joint work with David Bradley-Williams)

1. Motivation / Goal

The surface from Figure 1 is smooth at a_1, singular at a_2 and, in some sense, "even more" singular at a_3: near a_2, the surface is at least "roughly translation invariant" in direction of the x-axis, whereas near a_3, it is not roughly translation invariant in any direction. There are various ways in which one can make "roughly translation invariant" precise, though the most natural definitions have serious drawbacks (which we will recall below). In this talk, a new way to make this notion precise which avoids those drawbacks was presented. This is joint work with David Bradley-Williams; see [2].

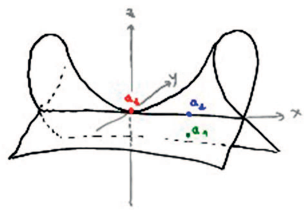

FIGURE 1. A surface and points with "different amounts of singularity"

Most of the following works in many different topological fields (e.g. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$), but for simplicity, let us just work over $\mathbb{R}$. Let us fix a set $X \subset \mathbb{R}^n$ whose singularities we want to consider. We assume X to be definable in a suitable language L, e.g. the ring language. (Another suitable language is the subanalytic one. More generally, the following works for any polynomially bounded o-minmal language on $\mathbb{R}$.)

In general, given a neighbourhood $B \subset \mathbb{R}^n$ of some point $x \in X$, we are interested in whether $X \cap B$, after applying some "small perturbation" $\phi\colon B \to B' \subset \mathbb{R}^n$, becomes translation invariant in d-dimensions (for some $d \leq n$). Here, the "small perturbation" should be a bijection of a certain kind $\mathcal{K}$, e.g. a homeomorphism or a bi-Lipschitz[1] map. Let us formulate this a bit more precisely:

Definition 1. *Fix $x \in X$ and let $B \subset \mathbb{R}^n$ be a neighbourhood of x. We say that X is d-$\mathcal{K}$-trivial on B if there exists a $\mathcal{K}$-bijection $\phi\colon B \to B' \subset \mathbb{R}^n$ and a d-dimensional vector sub-space $V \subset \mathbb{R}^n$ such that for $x_1, x_2 \in B'$ satisfying $x_1 - x_2 \in V$, we have $x_1 \in \phi(X \cap B)$ if and only if $x_2 \in \phi(X \cap B)$.*

For a fixed notion $\mathcal{K}$ of perturbation map, we can then classify the singularities of X, partitioning it into sets $S_0, \dots, S_{\dim X}$ as follows:

Definition 2. *For each $d \leq \dim X$, let S_d be the set of those $x \in X$ which have a neighbourhood B on which X is d-$\mathcal{K}$-trivial, but which have no neighbourhood B on which X is $(d+1)$-$\mathcal{K}$-trivial.*

Unfortunately, if $\mathcal{K}$ means "homeomorphism" or "bi-Lipschitz", then the resulting sets S_d are not definable in general, making this entire approach not very useful. A classical remedy to this is to introduce a suitable notion of *stratification*: a partition of X into sets S_d which in particular has the property that every $x \in S_d$ has a neighbourhood on which X is d-$\mathcal{K}$-trivial. Those S_d are definable, but they are less precise in the sense that X might also be $(d+1)$-$\mathcal{K}$-trivial on a neighbourhood of an $x \in S_d$. Moreover, the S_d are usually not canonically defined; in

[1]Bi-Lipschitz means that both the map and its inverse are Lipschitz continuous.

Figure 1 for example, while it would be natural to set $S_0 = \{(0,0,0)\}$, any bigger finite set would also yield a stratification.

Our goal now is to replace the notion $\mathcal{K}$ by another one which directly yields definable sets S_d.

2. Passing to a non-standard model

A key idea is that we should look at *infinitesimal* neighbourhood of points $x \in X$. To this end, we need to work in a non-standard model, so fix a proper elementary extension $\mathbb{R}^* \supsetneq \mathbb{R}$. Recall that $\mathbb{R}^*$ is naturally a valued field, where the valuation ring $\mathcal{O}_{\mathbb{R}^*}$ is the convex closure of $\mathbb{R}$, and the maximal ideal $\mathcal{M}_{\mathbb{R}^*} \subset \mathcal{O}_{\mathbb{R}^*}$ consists of the infinitesimal elements of $\mathbb{R}^*$. We denote the valuation by $|\cdot|_v$ and write it multiplicatively, i.e., $|\cdot|_v$ is the canonical map $(\mathbb{R}^*)^\times \to \Gamma := (\mathbb{R}^*)^\times/\mathcal{O}_{\mathbb{R}^*}^\times \cup \{0\}$, extended by $0 \mapsto 0$.

We will need to put an additional assumption on $\mathbb{R}^*$, namely that it is *spherically complete*, meaning that every nested chain of valuative balls $B_i \subset \mathbb{R}^*$ has non-empty intersection. Note that such elementary extensions exist: If L is the ring language, we can take $\mathbb{R}^*$ to be a Hahn field like $\mathbb{R}^* = \mathbb{R}((t^{\mathbb{Q}}))$; for the more general languages L mentioned above, the existence of such $\mathbb{R}^*$ has been proven in [3] (and reproven in bigger generality in [1]).

Let us write X^* for the subset of $(\mathbb{R}^*)^n$ defined by the same formula as X. Given $x \in X$ (a standard point), we now have a notion of "*the* infinitesimal neighbourhood of x", namely:

Definition 3. *Given $x \in X$, set $B_x := x + \mathcal{M}_{\mathbb{R}^*}^n \subset (\mathbb{R}^*)^n$ (where $\mathcal{M}_{\mathbb{R}^*}^n$ is the cartesian power of $\mathcal{M}_{\mathbb{R}^*}$).*

We will apply Definition 1 within $\mathbb{R}^*$, with $B = B_x$. It turns out that we can also set $B' := B_x$ (due to the ultrametric nature of the valuation). Given that the neighbourhood is infinitesimal, it makes sense to also impose that the map $\phi\colon B_x \to B_x$ produces only an infinitesimal perturbation. Here is the property we want it to have:

Definition 4. *A bijection $\phi\colon B_x \to B_x$ is a* risometry *if, for every pair $x_1, x_2 \in B_x$ of distinct points, we have*

$$|\phi(x_1) - \phi(x_2) - (x_1 - x_2)|_v < |x_1 - x_2|_v. \tag{1}$$

The intuition behind this definition is as follows: Set $d := x_1 - x_2$ and $d' := \phi(x_1) - \phi(x_2)$. Imposing $d = d'$ would mean that ϕ has to be a translation, i.e., it would not perturb $X \cap B_x$ at all. Instead, we want to allow d and d' to differ by something infinitesimal. The right way to make this precise is that the difference $d - d'$ should be infinitesimal compared to d (or, equivalently, compared to d'). This is exactly what (1) is saying.

We now plug those notions into Definitions 1 and 2:

Definition 5. *Fix $d \leq n$.*

(i) *Given $x \in X$ (a standard point), we say that X^* is d-riso-trivial on B_x if there exists a risometry $\phi\colon B_x \to B_x$ and a d-dimensional vector sub-space $V \subset (\mathbb{R}^*)^n$ such that for $x_1, x_2 \in B_x$ satisfying $x_1 - x_2 \in V$, we have $x_1 \in \phi(X \cap B_x)$ if and only if $x_2 \in \phi(X \cap B_x)$.*

(ii) *Let S_d be the set of those $x \in X$ such that on B_x, X^* is d-riso-trivial but not $(d+1)$-riso-trivial.*

Our main result is:

Theorem 6. *For each d, S_d is L-definable.*

To prove this, we first extend the language L to a language L' which includes the valuation. In L', the infinitesimal neighbourhoods B_x are definable and we prove more genereally that the set of open valuative balls (considered as living in some imaginary sort) on which X^* is d-riso-trivial is L'-definable.[2] The main ingredient to get back to the smaller language L is the fact that the structure on the residue field $\mathbb{R}$ of $\mathbb{R}^*$ induced by the language L' is only the L-structure on $\mathbb{R}$.

Now that we have our canonical partition of X into definable sets S_d, a natural question is: do these sets form a stratification of X, in some of the classical senses? The first property to check is the following:

Theorem 7. *For each d, if S_d is non-empty, then* $\dim S_d = d$.

The next required property would be that the topological closure of S_d is contained in $S_0 \cup \cdots \cup S_d$. In general, this is not the case for the definition of the sets S_d given above, but one can modify the sets S_d to obtain sets S'_d which still satisfy Theorems 7 and 6 and which additionally satisfy the topological condition. We call this partition $S'_0 \cup \cdots \cup S'_{\dim X}$ of X the *riso-stratification* of X. To finish relating this to classical stratifications, we also prove that this riso-stratification satisfies Whitney's regularity conditions, meaning that the riso-stratification of X is in particular essentially[3] a Whitney stratification of X.

While this is nice to know, note that the riso-stratification does not only have the advantage of being canonical, but it also has some stronger regularity properties (that neither Whitney nor Verdier stratifications have), which allow us to get some information about Poincaré series. (However, explaining this would require an entire second talk.)

References

[1] David Bradley-Williams and Immanuel Halupczok, *Spherically complete models of hensel minimal valued fields*, 2021, Preprint.

[2] The main work lies in this proof. Maybe a bit surprisingly, for this result to be true, it is important to *not* impose that the risometries ϕ from Definition 5 are definable. To be able to nevertheless get some control of those ϕ, we need $\mathbb{R}^*$ to be spherically complete.

[3] Only "essentially" because one usually requires the strata of a Whitney stratification to be C^∞-manifolds; our S'_d are only C^1-manifolds.

[2] David Bradley-Williams and Immanuel Halupczok, *Riso-stratifications and a tree invariant*, 2022, Preprint.
[3] Elliot Kaplan, *T-convex T-differential fields and their immediate extensions*, arXiv preprint arXiv:2104.10802 (2021).

Lang-Weil type bounds in finite difference fields

Martin Hils
(joint work with Ehud Hrushovski, Jinhe Ye and Tingxiang Zou)

We establish coarse estimates for the number of rational points of difference varieties in finite difference fields. For p prime and $0 \leq m < n$, we denote by $DF(p, n, m)$ the finite difference field $(\mathbb{F}_{p^n}, \mathrm{Frob}_{p^m})$, where $\mathrm{Frob}_{p^m}(x) = x^{p^m}$. Let $\{DF(p_i, n_i, m_i)\}_{i\in\mathbb{N}}$ be an enumeration of all finite difference fields. We consider difference fields in the language of difference rings $\mathcal{L} := \{0, 1, +, -, \times, \sigma\}$.

We fix a pseudofinite difference field

$$(K, \sigma) := \prod_{i\to\mathcal{U}} DF(p_i, n_i, m_i),$$

where $\mathcal{U}$ is a (non-principal) ultrafilter on $\mathbb{N}$ satisfying

$$\lim_{i\to\mathcal{U}} m_i = \lim_{i\to\mathcal{U}} n_i/m_i = \infty.$$

Definition. **(i)** *Let $D = \prod_{i\to\mathcal{U}} D_i \subseteq K^n$ be an internal subset of K^n (e.g., a definable set). The* (normalized) coarse dimension *of D is defined as*

$$\boldsymbol{\delta}(D) := st.\frac{\log |D|}{\log |K|} := \lim_{i\to\mathcal{U}} \frac{\log |D_i|}{\log p_i^{n_i}} \in [0, n] \cup \{-\infty\}.$$

(ii) *Let $\phi(x_1, \ldots, x_n)$ be an $\mathcal{L}(K)$-formula and $D = \phi[K] \subseteq K^n$. The* transformal dimension *of D is defined as*

$$\mathrm{trf.dim}(D) := \max\{\mathrm{trf.deg}(K(a)_\sigma/K) \mid a \in (K', \sigma) \succcurlyeq (K, \sigma) \text{ with } \models \phi(a)\},$$

where $K(a)_\sigma$ denotes the difference field generated by the tuple a over K and where trf.deg *denotes the transformal transcendence degree.*

The following is (the non-standard version of) our main result.

Theorem. *Let $D \subseteq K^n$ be a quantifier-free $\mathcal{L}(K)$-definable set. Then*

$$\boldsymbol{\delta}(D) = \mathrm{trf.dim}(D).$$

In particular, the coarse dimension of any quantifier-free definable set in K is an integer.

This confirms a conjecture by Zou [4, Conjecture 3.1]. Moreover, Zou proved that the equality between coarse dimension and transformal dimension for quantifier-free definable sets implies the same equality for existentially definable sets ([4, Theorem 3.1]).

To prove the theorem, we use the following qualitative version of strong Lang-Weil bounds due to Cafure-Matera [1] in order to estimate the number of rational points of difference varieties in finite difference fields.

Fact 1. *Given $n, \ell \in \mathbb{N}$, there exists a constant $C_n(\ell)$, which depends polynomially on ℓ, such that whenever $V \subseteq \mathbb{A}^n$ is an absolutely irreducible variety defined over the finite field $\mathbb{F}_q$, with $d = \dim(V)$ and $\deg(V) \leq \ell$, then*

$$|\#V(\mathbb{F}_q) - q^d| \leq C_n(\ell) q^{d-1/2}.$$

The main difficulty is to control the algebraic dimension of the irreducible components of the Frobenius specializations (i.e. the algebraic varieties one obtains when replacing σ by a Frobenius automorphism Frob_{p^m}) of a difference variety X. If X is of transformal dimension d, we show that there is a difference subvariety X_s of transformal dimension $< d$ such that all Frobenius specializations of $X \setminus X_s$ (for p^m large enough) are equidimensional of dimension d. We prove the existence of X_s using the model theory of contractive valued difference field, where the induced automorphism σ on the value group Γ is assumed to satisfy $\sigma(\gamma) \geq n\gamma$ for any $\gamma \in \Gamma_{>0}$ and n$\in \mathbb{N}$. By the recent work [2] of Dor and Hrushovski, the corresponding theory has a model-companion, denoted by $\widetilde{\omega VFA}$. It is not hard to see that for quantifier-free $\mathcal{L}$-definable sets in models of $\widetilde{\omega \mathrm{VFA}}$, the topological dimension (with respect to the valuation topology) and the transformal dimension agree. We may thus use the following key fact, together with standard techniques for difference fields, like the primitive element theorem, to show the existence of X_s. (By ACFA we denote the model-companion of the theory of difference fields.)

Fact 2 ([2, 3])**.** *Let $\mathcal{L}_{val}$ be the language of valued difference fields, and let Q be the set of prime powers. For $q \in Q$, let $L_q := (\mathbb{F}_q(t)^{alg}, v_t, \mathrm{Frob}_q)$, where v_t denotes the t-adic valuation. Then the following holds.*

- **(i)** $\widetilde{\omega\mathrm{VFA}} = \{\phi \ \mathcal{L}_v\text{-sentence} \mid L_q \models \phi \text{ for all } q \in Q \text{ with } q \gg 0\}$. *In particular, $\prod_{q \to \mathcal{U}} L_q \models \widetilde{\omega\mathrm{VFA}}$ for any non-principal ultrafilter $\mathcal{U}$ on Q.*
- **(ii)** *Let $L \models \widetilde{\omega\mathrm{VFA}}$, with residue field k_L.*
  - **a**. *Both L and k_L (with the induced automorphism) are models of ACFA.*
  - **b**. *The residue field k_L is stably embedded in L, with induced structure a pure model of ACFA.*
  - **c**. *The partial type over L of an element x from the valuation ring $\mathcal{O}$ asserting that the residue of x is transformally transcendental over k_L, is complete (and thus also definable).*

References

[1] A. Cafure and G. Matera, *Improved explicit estimates on the number of solutions of equations over a finite field*, Finite Fields Appl. **12**(2) (2006), 155–185.

[2] Y. Dor and E. Hrushovski, *Specialization of difference equations and high Frobenius powers*, preprint, 2022 (`arXiv.2212.05366`).

[3] E. Hrushovski, *The elementary theory of the Frobenius automorphism*, preprint, 2022, (`arXiv:math/0406514v2 [math.LO]`)
[4] T. Zou, *Pseudofinite difference fields and counting dimensions* J. Math. Log. **21**(1) (2021), Paper No. 2050022, 28 pp.

Automorphisms of ordered abelian groups

Jan Dobrowolski
(joint work with Rosario Mennuni)

The goal of the talk is to present the main result of [1]:

Theorem 1. *The category of ordered abelian groups with an automorphism has the Amalgamation Property.*

While this is a purely algebraic statement, our proof uses model-theoretic tools such as the compactness theorem, existentially closed models, and o-minimality.

Moreover, noting that any quantifier-free formula in the language $L_\sigma := \{+, -, 0, <, \sigma, \sigma^{-1}\}$ is NIP in any ordered abelian group with an automorphism σ, one can conclude from Theorem 1 the following:

Corollary 2. *The theory of ordered abelian groups with an automorphism is NIP in the sense of positive logic.*

Namely, there do not exist contradictory existential formulas $\phi(x; y)$ and $\psi(x; y)$ and parameters $(a_i)_{i<\omega}$ and $(b_W)_{W\subseteq\omega}$ in some model such that $\phi(a_i; b_W)$ holds for $i \in W$ and $\psi(a_i; b)$ holds for $i \notin W$.

Below we sketch the proof of Theorem 1.

Step 1: Realification. Given an amalgamation problem $B \leftarrow A \rightarrow C$, we find extensions of A', B', C' of A, B, C, respectively, and we equip each of them with a structure of an ordered real vector space with an automorphism. For this, we use compactness and the following fact (see [2, Corollaries 0.5 and 0.14, Proposition 0.25]).

Fact 3. *Every countable-dimensional ordered vector space is isomorphic to a Hahn sum of Archimedean ordered vector spaces.*

A', B', and C' are obtained as some sufficiently rich Hahn groups over $\mathbb{R}$ equipped with suitably constructed automorphisms. Moreover, we obtain commutativity of suitable diagrams, which yield that amalgamating $B' \leftarrow A' \rightarrow C'$ is enough to amalgamate $B \leftarrow A \rightarrow C$.

Hence, from now on we are working with ordered real vector spaces with an automorphism.

Step 2: Intermediate Value Property (IVP). In this step we prove that σ-polynomials have the IVP on existentially closed models.

Definition 4.

— *A σ-polynomial is an expression of the form $\Sigma_{i=0}^n r_i\sigma^i$ for some $n < \omega$ and $r_i \in \mathbb{R}$.*
— *If A is an ordered real vector space with an automorphism, then a σ-polynomial over A is an expression of the form $\Sigma_{i=0}^n r_i\sigma^i + a$ with $a \in A$.*
— *The polynomial associated to $\Sigma_{i=0}^n r_i\sigma^i + a$ is the polynomial $\Sigma_{i=0}^n r_i x^i$.*
— *If $b \in B \supseteq A$ where A and B are ordered real vector spaces with an automorphism, then we say that b is σ-algebraic over A when b is a zero of a non-trivial σ-polynomial over A.*
— *A σ-polynomial over A is absolutely monotone if it defines a monotone function in every extension of A.*

As pointed out to us by Martin Hils, a σ polynomial is absolutely monotone if and only if the polynomial associated to it has no positive real roots. This allows to decompose an arbitrary σ-polynomial as a product of absolutely monotone σ-polynomials and σ-polynomials of degree 1. If f is absolutely monotone, we show that f has the IVP on any maximally complete (with respect to the Archimedean valuation) model, hence also on any existentially closed model. We deal with the degree 1 case by constructing suitable 1-generated extensions.

Step 3: IVP for minima. With considerable technical effort, we extend the IVP on existentially closed models to functions of the form $\min(f_1 + a_1, \ldots, f_n + a_n)$ for σ-polynomials $f_1, \ldots, f_n$. This step is crucial for amalgamating σ-transcendental points.

Step 4: Amalgamating σ-algebraic points. We solve amalgamation problems of the form $B \leftarrow A \rightarrow C$ where $B = \langle A, b\rangle_\sigma$ is generated over A by an element b which is σ-algebraic over b. As in Step 2, we deal separately with the absolutely monotone case and the degree 1 case, and we conclude by decomposing the extension $A \rightarrow B$ into a suitable tower of extensions.

Step 5: Amalgamating σ-transcendental points. Consider $B \leftarrow A \rightarrow C$ with $B = \langle A, b\rangle_\sigma$ generated over A by an element b σ-transcendental over A. Let M be an existentially closed model containing B, and let A' be the set of all elements of M which are σ-algebraic over A. Let B' be the substructure of M generated by $A' \cup \{b\}$. By Step 4, we can amalgamate $A' \leftarrow A \rightarrow C$ into some C'. Now it is enough to amalgamate $B' \leftarrow A' \rightarrow C'$. For this, is suffices to show that $\mathrm{qftp}(b/A')$ is finitely satisfiable in A' (hence consistent with the atomic diagram of C'). So let $\phi(x) \in \mathrm{qftp}(b/A')$. We may assume $\phi(x) \equiv ((x, \sigma(x), \ldots, \sigma^n(x)) \in Z)$ for some cell Z definable in the ordered real vector space A'. As b is non-algebraic over A', we get that Z is an open cell, hence $\phi(x) \equiv (h(x) > 0)$ where $h = \min(f_1, \ldots, f_m)$ for some σ-polynomials f_i over A'. As $b \in Z$, we have $h(b) > 0$, so there is some $m \in M$ with $h(b) > m > 0$. Since M is existentially closed, we can choose m so that $\sigma(m) = m$, hence $m \in A'$. If $h(a') \geq 0$ for some $a' \in A'$ then a' satisfies ϕ; otherwise, by the IVP for h on M we can find $m' \in M$ with $h(m') = 0$. But then m' is σ-algebraic over A', so $m' \in A'$, and $h(m') = m > 0$, so m' satisfies ϕ.

References

[1] J. Dobrowolski, R. Mennuni, *The Amalgamation Property for automorphisms of ordered abelian groups*, arXiv:2209.03944 v2, 2023.

[2] S. Kuhlmann. *Ordered Exponential Fields*, volume 12 of Fields Institute Monographs. American Mathematical Society, 2000.

Corners in stability

Daniel Palacín

(joint work with Amador Martin-Pizarro, Julia Wolf)

1. Introduction

Given an abelian group G, written additively, we say that a subset $S \subset G \times G$ contains a (non-trivial) *corner* if there exists a triple $(x, y, g) \in G^3$, with $g \neq 0_G$, such that $(x, y), (x+g, y), (x, y+g) \in S$. It is well-known that a non-trivial corner $(x_0, y_0), (x_0 + g, y_0), (x_0, y_0 + g)$ in the Cayley graph of a subset $A \subset G$ yields a 3-arithmetic progression $x_0 - y_0 - g, x_0 - y_0, x_0 - y_0 + g$ in A. Hence, for finite groups of odd order, the existence of non-trivial corners for dense subsets of $G \times G$ implies Roth's Theorem on arithmetic progressions of length 3 in dense subsets of G. The existence of non-trivial corners for finite cyclic groups of odd order is due to Ajtai and Szemerédi [1]. Nowadays, several proofs of this result are known, some of them with reasonable bounds [5, 7].

Concerning non-abelian groups, one may distinguish between *naive corners*, a configuration of the form $(x, y), (gx, y), (x, gy)$ with $g \neq 1_G$, and *BMZ corners* which are configurations of the form $(x, y), (gx, y), (gx, gy)$ with $g \neq 1_G$. Here the term BMZ stands for Bergelson, McCutcheon and Zhang, who proved in [2] the existence of BMZ corners for sets of positive upper density in amenable groups. One can also define other 2-dimensional shapes for abelian groups such as squares, *i.e.* patterns of the form $(x, y), (x + g, y), (x, y + g), (x + g, y + g)$, or L-shapes which consist of 4 pairs of the form $(x, y), (x + g, y), (x, y + g), (x, y + 2g)$. Peluse [8] has obtained the first reasonable bounds for the existence of L-shapes in dense subsets of $\mathbb{F}_p^n \times \mathbb{F}_p^n$.

We state the existence of some 2-dimensional shapes for definable relations which are robustly stable (see Definition 2.3) in non-principal ultraproducts of finite groups, not necessarily abelian.

2. Ultraproducts and robust stability

2.1. **Set-up.** Fix a non-principal ultraproduct $G = \prod_{n \to \mathfrak{U}} G_n$ of finite groups $(G_n)_{n \in \mathbb{N}}$ of strictly increasing order. We say that a subset $X \subset G$ is *internal* if $X = \prod_{n \to \mathfrak{U}} X_n$ for some subsets $X_n \subset G_n$ for each $n \in \mathbb{N}$. Internal sets form a Boolean algebra which comes naturally equipped with the so-called Loeb measure, which is defined for an internal subset $X = \prod_{n\to\mathfrak{U}} X_n$ of G as follows:

$$\mu_G(X) = \lim_{n\to\mathfrak{U}} \frac{|X_n|}{|G_n|}.$$

This is a finitely additive probability measure on the Boolean algebra of internal sets. Working within a suitable language, expanding the language of groups, we may assume that all (relevant) internal sets are definable and that the Loeb measure is continuous in the following sense: for every $\delta \in [0,1]$ and every internal subset $Z \subset G^n \times G^m$, there is some internal set $Y_\delta \subset G^m$ such that

$$\{y \in G^m \mid \mu_{G^n}(Z_y) > \delta\} \subset Y_\delta \subset \{y \in G^m \mid \mu_{G^n}(Z_y) \geq \delta\},$$

where $Z_y = \{x \in G^n \mid (x,y) \in Z\}$ is the fiber of Z at y.

2.2. **Dense sets.** Definable sets of positive Loeb measure are called *dense*. We can extend this terminology to types by saying that a partial type is dense if it only implies dense definable sets. A standard compactness argument shows that every dense partial type can be completed to a dense complete one.

In this context, dense types play the role of generic types in definable groups of stable or simple theories. Bearing this idea in mind, we introduce the following terminology to capture a measure-theoretic idea of independence. Given two dense types $p, q \in S_G(A)$, define

$$\mathrm{gp}(p,q) = \{(a,b) \models p \times q \mid \mathrm{tp}(a/A,b) \text{ or } \mathrm{tp}(b/A,a) \text{ is dense}\},$$

where the notation gp stands for *good position.* By the discussion above, note that $\mathrm{gp}(p,q) \neq \emptyset$. So, there are always pairs in good position.

2.3. **Robust stability.** Let $k \geq 1$. Given a subset $S \subset G \times G$, we set $\mathcal{H}_k(S)$ to denote the collection of all half-graphs of height k induced by S. That is,

$$\mathcal{H}_k(S) := \left\{(a_1, b_1, \ldots, a_k, b_k) \in G^{2k} \;\middle|\; (a_i, b_j) \in S \text{ iff } i \leq j\right\}.$$

With this notation, recall that a subset $S \subset G \times G$ is *k-stable* if $\mathcal{H}_k(S) = \emptyset$. In our setting, in the presence of a measure, we can introduce the following notion:

Definition. *A definable subset $S \subset G\times G$ is* robustly k-stable *if $\mu_{G^{2k}}(\mathcal{H}_k(S)) = 0$.*

Robust stability can be defined in a more general context but to ease the framework we restrict our attention to subsets of the group $G\times G$. In fact, similar notions have been considered in a more graph-theoretical context, see [3, 9].

We show for pairs in good position that the value of a robustly stable relation is constant:

Theorem 1. *Let $S \subset G\times G$ be a definable dense set and suppose that it is robustly k-stable for some $k \geq 1$. Let $p, q \in S_G(M)$ be two complete dense types over a countable elementary substructure M. The set S is homogeneous on $\mathrm{gp}(p,q)$, i.e. either $\mathrm{gp}(p,q) \subset S$ or $\mathrm{gp}(p,q) \subset (G\times G) \setminus S$.*

Hence, robustly stable relations are also stationary, as the stable ones.

3. Existence of 2-dimensional shapes

We state our main results:

Theorem 2. *Let $S \subset G \times G$ be a definable dense set and suppose that it is robustly k-stable for some $k \geq 1$. Then S contains a dense collection of both naive and of BMZ corners, i.e. the definable set*

$$\left\{(x, y, g) \in G^3 \;\middle|\; (x, y), (xg, y), (x, yg), (xg, yg) \in S\right\}$$

is dense. In particular, if G is abelian, then there is a dense set of triples which form a square in S.

The proof relies on Theorem 1, as well as on methods suggested by Hrushovski [4] such as the Stabilizer Theorem. The model-theoretic version of Roth's Theorem proved in [6] allows us to obtain the following:

Theorem 3. *Let $S \subset G \times G$ be a definable dense set and suppose that it is robustly k-stable for some $k \geq 1$. If G does not have elements of order 2, then the definable set*

$$\left\{(x, y, g) \in G^3 \;\middle|\; (x, y), (xg, y), (x, yg), (xg, yg), (x, gyg), (xg, gyg) \in S\right\}$$

is dense. In particular, if G is abelian, then there is a dense set of triples which form (non-trivial) squares and L-shapes.

As a consequence, by a standard application of Łoś's Theorem we obtain the following finitary version.

Corollary 1. *Given an integer $k \geq 1$ and a real number $\delta > 0$, there is an integer $\ell_0(k, \delta) \geq 1$ and real numbers $\theta = \theta(k, \delta) > 0$ and $\epsilon = \epsilon(k, \delta) > 0$ with the following property. Let G be a finite group of odd order $|G| \geq \ell_0$ and let $S \subset G \times G$ be a relation of size $|S| \geq \delta |G|^2$ such that $|\mathcal{H}_k(S)| \leq \theta |G|^{2k}$. Then the set*

$$\Lambda(S) = \left\{(x, y, g) \in G^3 \mid (x, y), (xg, y), (x, yg), (xg, yg), (x, gyg), (xg, gyg) \in S\right\}$$

has size $|\Lambda(S)| \geq \epsilon |G|^3$. In particular, if G is abelian, then S contains a positive density of (non-trivial) squares and L-shapes.

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Imperfection in NIP fields

Sylvy Anscombe
(joint work with Franziska Jahnke)

In a theory T, a formula $\varphi(\bar{x}, \bar{y})$ has the *independence property* if in some model $\mathbb{M}$ there exist families $(\bar{a}_i)_{i\in\mathbb{N}}$ and $(\bar{b}_I)_{I\subseteq\mathbb{N}}$, of tuples from $\mathbb{M}$, where $|\bar{a}_i| = |\bar{x}|$ and $|\bar{b}_I| = |\bar{y}|$, such that $\varphi(\bar{a}_i, \bar{b}_I)$ is true in $\mathbb{M}$ if and only if $i \in I$. A theory T is NIP (or is *dependent*) if no formula has the independence property in T. Moreover a structure is said to be NIP if its complete theory is.[1]

Finite fields are trivially NIP; separable closed fields are stable ([13]), thus NIP; and real closed fields are o-minimal, by the results of Tarski, thus NIP. The *Conjecture on NIP Fields* — sometimes called *Shelah's Conjecture*[2] — proposes that infinite NIP fields are either separably closed, real closed, or *henselian*, i.e. they admit a non-trivial henselian valuation.

Delon, in 1981, proved a "NIP transfer theorem" in equal characteristic zero:

Fact 1 (Delon, [6]). *Let (K, v) be a henselian valued field of equicharacteristic 0. Then,*

$$(K, v) \text{ is NIP in } \mathcal{L}_{\mathrm{val}} \Longleftrightarrow Kv \text{ is NIP in } \mathcal{L}_{\mathrm{ring}}.$$

For the language of rings we take $\mathcal{L}_{\mathrm{ring}} = \{+, \cdot, 0, 1\}$ and for the language of valued fields we may take one of the usual options, for example the language of rings expanded by a unary predicate symbol that is interpreted by the valuation ring. Delon's theorem originally included the additional assumption that the value group vK is NIP as an ordered abelian group, but it was later shown by Gurevich and Schmidt that this holds for any ordered abelian group ([7, Theorem 3.1]).

Transfer theorems of this kind were proved in other contexts by Bélair in [4] (for unramified henselian valued fields with perfect residue fields), and by Jahnke and Simon in [8] (for separably tame fields of finite imperfection degree). Bélair's approach builds on Delon's, and it shows in particular that p-adically closed fields are NIP (cf. [4, Corollaire 7.5]). On the other hand the results of Jahnke and Simon extended a strategy developed by Chernikov and Hils ([5]) in the context of NTP2 valued fields. In the NIP case the strategy is: if (K, v) satisfies two properties 'IM' and 'SE', and Kv is NIP, then (K, v) is NIP.

This talk presented parts of joint work with Franziska Jahnke ([1, 2]). This parts consisted of (i) a discussion a transfer theorem for finitely ramified henselian valued fields of mixed characteristic, and (ii) a completion of the analysis of separably tame valued fields from [8]. This division into (i) and (ii) reflects a precise

[1] For a general reference on NIP theories, the reader is encouraged to consult [12].
[2] This conjecture is often named after Shelah, who made a related conjecture in [11]).

dichotomy: Recall the 'standard decomposition' of a valued field of mixed characteristic:[3]

$$K \xrightarrow{vK/\Delta_0} Kv_0 \xrightarrow{\Delta_0/\Delta_p} Kv_p \xrightarrow{\Delta_p} Kv$$

where Δ_0 is the smallest convex subgroup of the value group vK of v that contains $v(p)$, and Δ_p is the largest convex subgroup not containing $v(p)$. Assuming $\aleph_1$-saturation of (K,v), $(Kv_0,\bar{v}_p)$ is maximal and the value group Δ_0/Δ_p is isomorphic to $\mathbb{Z}$ or $\mathbb{R}$. See also [3].

Bélair's method does not include finitely ramified henselian valued fields of mixed characteristic with imperfect residue fields. In this case we verify the SE condition of Jahnke and Simon (i.e. stable embeddedness of the value group and residue field) by applying an embedding lemma that we proved in [2] and that was closely based on [10].

Turning to (ii): Bélair also showed in [4] that an algebraically maximal Kaplansky field (K,v) of positive characteristic is NIP in $\mathcal{L}_{\text{val}}$ if and only if its residue field Kv is NIP in $\mathcal{L}_{\text{ring}}$, and that the same holds if (K,v) is finitely ramified with perfect residue field. Jahnke and Simon generalized Bélair's result to separably algebraically maximal Kaplansky fields of finite degree of imperfection and arbitrary characteristic. Thus the gap in the separably algebraically maximal Kaplansky setting is for infinite imperfection degree.

All of these transfer theorems have the spirit of Ax-Kochen/Ershov: under certain algebraic assumptions (including for example henselianity), if (K,v) is a valued field such that the residue field Kv is NIP, then (K,v) is NIP. In fact our main theorem is the following.

Theorem 1 (Main Theorem, [1])**.** *Let (K,v) be a henselian valued field. Then (K,v) is NIP in $\mathcal{L}_{\text{val}}$ if and only if both of the following hold:*

(1) *Kv is NIP.*

(2) *Either*

(a) $\begin{cases} \textbf{(a.i)} & (K,v) \textit{ is of equal characteristic, and} \\ \textbf{(a.ii)} & (K,v) \textit{ is trivial or separably defectless Kaplansky;} \end{cases}$

or

(b) $\begin{cases} \textbf{(b.i)} & (K,v) \textit{ has mixed characteristic } (0,p)\textit{, and} \\ \textbf{(b.ii)} & (K,v_p) \textit{ is finitely ramified, and} \\ \textbf{(b.iii)} & (Kv_p,\bar{v}) \textit{ is trivial or separably defectless Kaplansky;} \end{cases}$

or

(c) $\begin{cases} \textbf{(c.i)} & (K,v) \textit{ has mixed characteristic } (0,p)\textit{, and} \\ \textbf{(c.ii)} & (Kv_0,\bar{v}) \textit{ is defectless Kaplansky.} \end{cases}$

[3]Note that this diagram illustrates the composition of places associated to the composition of valuations.

To prove the implications in the 'forward' direction we use a theorem of Kaplan, Scanlon, and Wagner ([9]): infinite NIP fields of positive characteristic admit no propert Artin-Schreier extensions.

As a corollary we prove that if (K, v) is NIP then so is its henselization. Moreover, we obtain a classification of the complete theories of NIP fields, assuming that Shelah's conjecture is true.

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On groups definable in an o-minimal expansion of $(\mathbb{R}, +, \cdot)$

Alf Onshuus

It has been known since [3] that any group definable in an *o*-minimal expansion of the real field can be equipped with a Lie group structure. It is therefore natural to ask when a Lie group is Lie isomorphic to a group definable in such an expansion.

In this talk we give a complete classification of which Lie groups admit such a "definable representation". This is, we characterize, up to Lie isomorphism, the real Lie groups that are definable in an o-minimal expansion of the real field. For any such group, we find a Lie-isomorphic group definable in $\mathbb{R}_{\exp}$ for which any Lie automorphism is definable.

The talk will be divided as follows: We will first recall the torsion-free solvable case (proved by Coversano, Starchenko and Onshuus in [1]): any connected torsion-free

Lie group admits a definable representation if and only if it is *completely solvable*, where:

Definition. *A connected torsion-free solvable Lie group G is* completely solvable *if and only if there exist a sequence of subgroups*

$$G = G_n > G_{n-1} > \cdots G_0 = \{e\}$$

such that each G_i is normal in G and G_{i+1}/G_i is a one-dimensional simply connected Lie group for $i < n$.

We will then sketch the proof that in order for a connected Lie group G to be Lie isomorphic to a definable group it is necessary and sufficient that its solvable radical is completely solvable.

Finally, we will sketch the proof (by Conversano, Post and Onshuus in [2]) that a Lie group G has a Lie isomorphic definable copy in an o-minimal expansion of the real field if and only if:

— G has finitely many connected components,
— the center $\mathcal{Z}(G)$ of G has finitely many connected components, and
— there is a normal simply-connected completely solvable subgroup N such that $\mathcal{R}(G)/N$ is compact.

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On the shatter functions of semilinear families

Chieu-Minh Tran

(joint work with Abdul Basit)

For a collection $\mathcal{S}$ of subsets of an ambient set V (also called a set system on V), its shatter function $\pi_{\mathcal{S}} : \mathbb{N} \to \mathbb{N}$ is a natural measure of its complexity. Setting $\mathcal{S} \cap A := \{S \cap A : S \in \mathcal{S}\}$ for finite $A \subseteq V$, the function $\pi_{\mathcal{S}}$ is given by

$$\pi_{\mathcal{S}}(n) := \max\{|\mathcal{S} \cap A| : A \subseteq V, |A| = n\}.$$

A celebrated result by Sauer and Shelah establishes that either $\pi_{\mathcal{S}}(n) = 2^n$ for all n or $\pi_{\mathcal{S}}$ is polynomially bounded. It is well known that structure $\mathcal{M}$ is NIP if and only if the collection of definable sets given by an arbitrary formula has polynomially bounded shatter function.

A conjecture by Chernikov suggests that in "sufficiently geometric" settings (e.g., o-minimal), a structure $\mathcal{M}$ is "linear/modular" NIP if and only if the collection of definable sets given by an arbitrary formula has polynomially assymptotic shatter function. In this talk, a joint work with Abdul Basit is discussed, where we verify this belief for o-minimal expansions of $(\mathbb{R}; +)$. More precisely, we show that if $(X_b)_{b\in Y}$ is a definable family in $(\mathbb{R}; +)$, and $\mathcal{S}$ is the collection $\{X_b\}_{b\in Y}$, then $\pi_{\mathcal{S}}$ is assymptotic to a polynomial. In other words, there are constants $C_1, C_2 > 0$, and a natural number s such that

$$C_1 n^s < \pi_{\mathcal{S}}(t) < C_2 n^s \quad \text{for every } n \geq 1.$$

The aforementioned belief then follows from this statement through a standard application of the trichotomy principle for o-minimal structures.

The proof of our result consists of three steps. In the first step, we reduce to the problem to the case where $(X_b)_{b\in Y}$ as above is "uniform". The precise notion of uniform is what is needed to carry out the later steps, but, intuitively, it is easier to handle the case where $(X_b)_{b\in Y}$ does not contain sets of different dimensions or sets of different "shapes" (e.g., some are triangles and some are rectangles).

In the second step, we relate the family $(X_b)_{b\in Y}$ to another definable family $(X'_b)_{b\in Y}$ which is geometrically simpler, but at the cost that X'_b is a subset of disjoint Euclidean spaces. The challenges include finding suitable $(X'_b)_{b\in Y}$ through an induction on dimension procedure and establish that the shatter functions of the set systems corresponding to $(X_b)_{b\in Y}$ and $(X'_b)_{b\in Y}$ are appropriately related.

In the last step, we first perform projections to reduce the problem to the case where each member of $(X'_b)_{b\in Y}$ is a disjoint union of single points and rays. Techniques from a paper by Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko combined with another result by Basit, Chernikov, Starchenko, Tao, and Tran already allow us to handle the case where each member of $(X'_b)_{b\in Y}$ is a union of two single points. Modifications of these techniques involving variations of shatter functions and using linear programming are needed to handle the general case that we need.

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Towards higher classification theory

Artem Chernikov

1. Introduction

Model theory provides, among other things, methods of converting asymptotic quantitative questions about properties of finite hypergraphs into qualitative questions about the "shape", "volume" or "dimension" of certain limiting infinite objects to which the infinitary model-theoretic machinery can be applied. Shelah's classification program [8] isolates several combinatorial dividing lines (stability, NIP, distality, etc.) separating mathematical structures exhibiting various degrees of wild, or Gödelian, behavior, from the tame ones in which one develops a "geometric" theory akin to algebraic or semi-algebraic geometry for definable sets in such structures. These dividing lines are amazingly robust, and have been rediscovered in various branches of mathematics.

These tameness notions in Shelah's classification theory are typically given by restrictions on the combinatorial complexity of definable binary relations, by forbidding certain induced subgraphs (e.g. T is *stable* if no definable binary relation can contain arbitrary large finite half-graphs; and *NIP* if sufficiently large random bipartite graphs are omitted). A typical result then demonstrates that binary relations are "approximated" by the unary ones in some form, up to a "small" error. For example, stationarity of forking in stable theories says that given $p\,(x)\,, q\,(y)$ types over a model M, there exists a *unique type* $r(x, y)$ over M so that if $(a, b) \models r$ then $a \models p, b \models q$ and $a \mathop{\smile\!\!\!\!\!\!\;|}_M b$ — that is, there is a unique type $r\,(x, y)$ extending $p\,(x) \cup q\,(y)$, up to the forking formulas $\varphi\,(x, y) \in \mathcal{L}\,(M)$. Another example: T is *distal* if and only if for any $p\,(x)\,, q\,(y)$ global invariant types that commute, there is a unique global type $r\,(x, y)$ extending $p\,(x) \cup q\,(y)$.

Recently a number of results began to emerge concerning the higher arity generalizations of these phenomena, both in the context of pure model theory and in connection to hypergraph combinatorics: under some restricting assumption on the definable relations of arity $n+1$, demonstrate an "approximation" by relations each involving at most n out of $n+1$ variables, up to a "small error". Mirroring the passage from graphs to hypergraphs in combinatorics, this leads to significant growth in complexity of the occurring phenomena. We overview some of these developments focusing on of n-dependent theories (with the case $n = 1$ corresponding to NIP) introduced by Shelah, n-stability (several possible definitions have recently emerged in the literature, but very much remain to be explored), n-distality (recently introduced by Walker), and connections to higher amalgamation and stationarity, as well as implications for the algebraic structures definable in such theories.

2. N-dependence

A higher order generalization of NIP, the class of *k-dependent theories*, was introduced by Shelah in [9], with the 1-dependent case corresponding to the class of NIP theories, and basic properties of k-dependent theories were investigated in [4], in particular making an explicit definition of the VC_k-dimension.

We fix a complete theory T in a language $\mathcal{L}$. For $k \geq 1$, a formula $\varphi\,(x; y_1, \ldots, y_k)$ is *k-dependent* if there are no infinite sets $A_i = \{a_{i,j} : j \in \omega\} \subseteq M_{y_i}, i \in \{1, \ldots, k\}$ in a model $\mathcal{M}$ of T such that $A = \prod_{i=1}^{n} A_i$ is shattered by φ: for any $s \subseteq \omega^k$, there is some $b_s \in M_x$ s.t. $\mathcal{M} \models \varphi\,(b_s; a_{1,j_1}, \ldots, a_{k,j_k}) \iff (j_1, \ldots, j_k) \in s$. T is *k-dependent* if all formulas are k-dependent. T is *strictly k-dependent* if it is k-dependent, but not $(k-1)$-dependent. We have: 1-dependent = NIP $\subseteq$ 2-dependent $\subseteq \ldots$, with all inclusions strict as witnessed e.g. by the theory of the generic k-hypergraph.

In some sense all currently known "algebraic" examples of k-dependent theories are built from multilinear forms over NIP fields. By Cherlin-Hrushovski, smoothly approximable structures are 2-dependent, and coordinatizable via bilinear forms over finite fields. Infinite extra-special p-groups are strictly 2-dependent [7], and strictly k-dependent pure groups constructed using Mekler's construction [1] are essentially of this form as well, using Baudisch's interpretation in alternating bilinear maps. More generally:

Theorem 1. *([2] for $k = 2$, [3] in general) If the field K is NIP, then the theory T of alternating n-linear forms over K (with sorts for the field and for the vector space, generalizing Granger) is (strictly) n-dependent.*

This leads one to speculate that if T is k-dependent, then it is "linear, or 1-based" relative to its NIP part. One precise version of this conjecture is:

Conjecture 1. *If K is an k-dependent field (pure, or with valuation, derivation, etc.), then K is NIP.*

There is some mounting evidence for this conjecture: k-dependent fields are Artin-Schreier closed ([7], generalizing Kaplan-Scanlon-Wagner for $k = 1$), valued fields of positive characteristic are Henselian ([2], generalizing Johnson for $k = 1$), the question for valued fields reduces to pure fields (Boissonneau). A key general result used in the proof of Theorem 1 is:

Theorem 2 (Composition Lemma)**.** *Let $\mathcal{M}$ be an $\mathcal{L}'$-structure such that its reduct to a language $\mathcal{L} \subseteq \mathcal{L}'$ is NIP. Let $d, k \in \mathbb{N}$, $\varphi(x_1, \ldots, x_d)$ be an $\mathcal{L}$-formula, and $(y_0, \ldots, y_k)$ be arbitrary $k + 1$ tuples of variables. For each $1 \leq t \leq d$, let $0 \leq i_1^t, \ldots, i_k^t \leq k$ be arbitrary, and let $f_t : M_{y_{i_1^t}} \times \ldots \times M_{y_{i_k^t}} \to M_{x_t}$ be an arbitrary $\mathcal{L}'$-definable k-ary function. Then the formula*

$$\psi\left(y_0; y_1, \ldots, y_k\right) := \varphi\left(f_1(y_{i_1^1}, \ldots, y_{i_k^1}), \ldots, f_d(y_{i_1^d}, \ldots, y_{i_k^d})\right)$$

is k-dependent.

The following is a characterization of k-dependence in terms of a "hypergraph regularity lemma", generalizing the $k = 1$ case from [5]:

Theorem 3. *(C., Towsner [6]) Assume that T is k-dependent, $k' \geq k + 1$, $\mathbb{M} \models T$ and let $\mu_1, \ldots, \mu_{k'}$ be global Keisler measures on the definable subsets of the sorts $\mathbb{M}^{x_1}, \ldots, \mathbb{M}^{x_{k'}}$ respectively, such that each μ_i is Borel-definable and all these measures commute, i.e. $\mu_i \otimes \mu_j$ for all $i, j \in [k']$. Then for every formula $\varphi(x_1, \ldots, x_{k'}) \in \mathcal{L}(\mathbb{M})$ and $\varepsilon \in \mathbb{R}_{>0}$ there exist some formula $\psi(x_1, \ldots, x_{k'})$ which is a Boolean combination of finitely many ($\leq k$)-ary formulas each given by an instances of φ with some parameters placed in all but at most k variables, so that taking $\mu := \mu_1 \otimes \ldots \otimes \mu_{k'}$ we have $\mu\left(\varphi \triangle \psi\right) < \varepsilon$.*

It is also proved in that paper that if T is a k-dependent first-order theory (classical or continuous), then its Keisler randomization T^R is also k-dependent, generalizing Ben Yaacov for $k = 1$.

3. N-distality

Definition 1. *(Walker [10]) A theory is* n-distal *if it satisfies the following condition. Assume that* $(a_i : i \in I)$ *is an indiscernible sequence indexed by a dense linear order* I, $I = I_0 + I_1 + \ldots + I_{n+1}$ *with each* I_j *dense without endpoints, and* $b_1, \ldots, b_{n+1}$ *are so that: for any* $0 \leq t \leq n$, *we have that the sequence* $I_0 + b_0 + \ldots + I_{t-1} + b_{t-1} + I_t + I_{t+1} + b_{t+1} + \ldots + I_n + b_n + I_{n+1}$ *is indiscernible (i.e. we are omitting* b_t *here). Then the sequence* $I_0 + b_0 + \ldots + b_n + I_{n+1}$ *is indiscernible (with all* $b_t, 0 \leq t \leq n$ *placed in the corresponding cuts).*

The following generalizes a standard characterization of distality:

Fact 1. [10] *If* T *is* n*-distal, then for any global invariant types* $p_i(x_i), i \leq n$ *that are* pairwise commuting, *we have* $\bigcup_{0 \leq t \leq n} \bigotimes_{0 \leq i \leq n, i \neq t} p_i(x_i) \vdash \bigotimes_{0 \leq i \leq n} p_i(x_i)$. *That is, the type* $\bigotimes_{0 \leq i \leq n} p_i$ *in* $n+1$ *variables is determined by all of its restrictions to* n *variables.*

Turns out that n-distality is connected to certain notions of triviality of forking (as studied by Poizat and others) between generically stable types (for $k = 1$, in the sense that they are all realize).

Definition 2. *Let* T *be a stable theory and* $k \geq 1$. *We say that* T *is*

- **(i)** k-trivial *if for any tuples* $(a_i : i < k+2)$ *and a set* A, *if every* $k+1$ *of the* a_i*'s form an independent set over* A *(in the sense of forking), then every* $\{a_i : i < k+2\}$ *is also an independent set over* A.
- **(ii)** totally k-trivial *if for any tuples* a, $(b_i : i < k+1)$ *and a set* A, *if* a *is independent from any* k *of the* b_i*'s over* A, *then it is also independent from all* $k+1$ *of them over* A *(note that we are not requiring the* b_i*'s to be independent over* A*).*
- **(iii)** *For* $k \geq 1$, *a theory* T *is* indiscernibly k-trivial *if for any infinite sequence* $\mathcal{I}$ *and tuples* $(a_t : t < k+1)$, *if* $\mathcal{I}$ *is indiscernible over* $(a_t : t \in s)$ *for every* $s \subseteq \{0, 1, \ldots, k\}$ *with* $|s| = k$, *then* $\mathcal{I}$ *is indiscernible over* $(a_t : t < k+1)$.

Fact 2. [10] *Let* T *be a stable theory and* $k > 0$. *Then* T *is* k*-trivial if and only if* T *is* $(k+1)$*-distal.*

A theory T is *strongly* 2*-distal* if for any sequence $\mathcal{I}_0 + b_0 + \mathcal{I}_1$ and tuples a_0, a_1, if $\mathcal{I}_0 + \mathcal{I}_1$ is indiscernible over $a_0 a_1$, $\mathcal{I}_0 + b_0 + \mathcal{I}_1$ is indiscernible over a_0 and $\mathcal{I}_0 + b_0 + \mathcal{I}_1$ is indiscernible over a_1, then $\mathcal{I}_0 + b_0 + \mathcal{I}_1$ is indiscernible over $a_0 a_1$. We observe:

Theorem 4. *If* T *is stable, then the following are equivalent:*

- **(i)** T *is strongly* 2*-distal,*
- **(ii)** T *is indiscernibly trivial,*
- **(iii)** T *is totally trivial.*

Whether triviality is equivalent to k-triviality (equivalently, k-distality implies 2-distality) in stable theories is an old question of Poizat (known to hold in superstable theories). Theorem 4 combined with Poizat's examples answers Walker's question [10]: there exist stable 2-distal not strongly 2-distal theories.

4. Connections to higher amalgamation and stationarity

Higher amalgamation was studied by a number of authors, starting with Shelah's work on stability in AEC's, Hrushovski in the study of the saturation spectrum and of generalized imaginaries, continued in a series of papers by Goodrick, Kim, Kolesnikov and others.

Definition 3. *For $n \in \omega$, let $[n] = \{1, \dots, n\} \in \omega$. For a set X, we let $\mathcal{P}(X)$ be the set of all subsets of X, $\mathcal{P}_{<n}(X)$ ($\mathcal{P}_{\leq n}(X)$) the set of all subsets of X of size less (respectively, less or equal) than n, and $\mathcal{P}^-(X) := \mathcal{P}(X) \setminus \{X\}$.*

We let T be a complete *simple* first-order theory in a language $\mathcal{L}$, and we work in $\mathbb{M}^{\mathrm{heq}}$, the expansion of $\mathbb{M}$ by the hyper-imaginaries. As usual, $\downarrow$ denotes forking independence and $\mathrm{bdd}(A)$ is the bounded closure of the set A in $\mathbb{M}^{\mathrm{heq}}$.

Definition 4. **(i)** *For $n \geq 1$, T satisfies (*independent*) n-*amalgamation* if for every independent system of types $\{r_s(x_s) : s \in \mathcal{P}^-([n])\}$ there exists a complete type $r_n(x_n)$ such that $\{r_s(x_s) : s \in \mathcal{P}([n])\}$ is an independent system of types.*

(ii) *T satisfies (*independent*) n-*uniqueness* if for every independent system of types $\{r_s(x_s) : s \in \mathcal{P}^-([n])\}$ there exists at most one complete type $r_n(x_n)$ such that $\{r_s(x_s) : s \in \mathcal{P}([n])\}$ is an independent system of types.*

(iii) *T satisfies n-amalgamation (n-uniqueness)* over a set $A \subseteq \mathbb{M}$ *if (i) (respectively, (ii)) holds for every independent system of types with $r_\emptyset = \mathrm{tp}(\mathrm{bdd}(A))$.*

(iv) *T satisfies* complete n-amalgamation *(or $\leq n$-*amalgamation*) if T satisfies m-amalgamation for all $1 \leq m \leq n$.*

Theorem 5. *Given $n \geq 1$, let T be a simple theory with $\leq (n+2)$-amalgamation. Then T is n-dependent if and only if T has $(n+1)$-uniqueness (over models).*

For $n = 1$ this corresponds to the well-known fact that if T is simple (hence satisfies ≤ 3-amalgamation over models) and there exists a non-stationary type (i.e. 2-stationarity fails), then T is not NIP. Theorem 5 also gives us a collapse of 2-dependence and several notions of 2-stability considered in the literature.

Definition 5 (Takeuchi). *A partitioned formula $\varphi(x; y_1, y_2)$ has OP_2 if there exist sequences $(a_i)_{i\in\omega}, (b_j)_{j\in\omega}$ with $a_i \in \mathbb{M}^{y_1}, b_j \in \mathbb{M}^{y_2}$ so that for every strictly increasing $f : \omega \to \omega$ there exists $c_f \in \mathbb{M}^x$ satisfying $\models \varphi(c_f, a_i, b_j) \iff i \leq f(j)$ for all $(i, j) \in \omega^2$.*

A related property FOP_2 with increasing functions replaced by arbitrary functions $f : \omega \to \omega$ is considered by Terry and Wolf. We let $\mathcal{C}_\prec := (\mathbb{L}, C, \prec)$ be the *generic countable convexly ordered binary branching C-relation*, i.e. the Fraïssé limit of all finite convexly ordered binary branching C-relations. The following is considered by Kaplan, Ramsey, Simon:

Definition 6. *A theory T is* treeless *if there is no formula $\varphi(x, y, z)$ and $(a_g : g \in \mathbb{L})$ such that $\models \varphi(a_f, a_g, a_h) \iff \mathcal{C}_{\prec} \models C(f, g, h)$.*

Theorem 6. *The following are equivalent:*

(i) *T is not treeless;*

(ii) *there exists a $\mathcal{C}_{\prec}$-indiscernible which is not $(\mathbb{L}, \prec)$-indiscernible;*

(iii) *there exists a $\mathcal{C}_{\prec}$-indiscernible $(a_g : g \in \mathbb{L})$ and $\varphi(x, y, z)$ so that*

$$\models \varphi(a_f, a_g, a_h) \iff \mathcal{C} \models C(f, g, h).$$

It is easy to see that each of treeless, no OP_2 and no FOP_2 imply 2-dependence, and under 4-amalgamation we get a converse:

Theorem 7. *If T is simple with ≤ 4-amalgamation, then the following are equivalent:*

(i) *T satisfies 3-uniqueness;*

(ii) *T is 2-dependent;*

(iii) *T has no OP_2;*

(iv) *T has no FOP_2;*

(v) *T is treeless.*

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Generalized locally compact models of approximate subgroups via topological dynamics

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(joint work with Anand Pillay)

A subset X of a group is called an *approximate subgroup* if it is symmetric (i.e. $e \in X$ and $X^{-1} = X$) and $XX \subseteq FX$ for some finite $F \subseteq \langle X \rangle$. Approximate subgroups were introduced by Tao in [8] and play a central role in additive combinatorics.

A compact neighborhood of the neutral element in a locally compact group is always an approximate subgroup. Let X be an approximate subgroup and $G := \langle X \rangle$. By a *locally compact [resp. Lie] model* of X we mean a group homomorphism $f : \langle X \rangle \to H$ for some locally compact [resp. Lie] group H such that $f[X]$ is relatively compact in H and there is a neighborhood U of the neutral element in H with $f^{-1}[U] \subseteq X^m$ for some $m < \omega$. It is easy to show that if $f : \langle X \rangle \to H$ is a locally compact model of X, then X can be recovered up to commensurability as the preimage of any compact neighborhood of the identity in H.

A breakthrough in the study of the structure of approximate subgroups was obtained by Hrushovski in [3], where a locally compact (and in consequence Lie) model for any pseudofinite approximate subgroup (more generally, *near-subgroup*) X was obtained by using model-theoretic tools. This paved the way for Breuillard, Green, and Tao to give a full classification of all finite approximate subgroups in [1].

By a *definable* (in some structure M) *approximate subgroup* we mean an approximate subgroup X of some group such that $X, X^2, X^3, \dots$ are all definable in M and $\cdot|_{X^n \times X^n} : X^n \times X^n \to X^{2n}$ is definable in M as well. If the approximate subgroup X is definable in M, then in the definition of a locally compact model one usually additionally requires *definability* of f in the sense that for any open $U \subseteq H$ and compact $C \subseteq H$ such that $C \subseteq U$, there exists a definable (in M) subset Y of G such that $f^{-1}[C] \subseteq Y \subseteq f^{-1}[U]$. Note that in the abstract situation of an arbitrary approximate subgroup X, we can always equip the ambient group with the *full structure* (i.e. add all subsets of all finite Cartesian powers as predicates), and then X becomes definable and the additional requirement of definability of locally compact models is automatically satisfied. In other words, definable approximate subgroups generalize abstract approximate subgroups.

Massicot and Wagner [7] proved the existence of definable locally compact (and in consequence Lie) models for all definably amenable definable approximate subgroups, and Wagner conjectured that a Lie model exists for an arbitrary approximate subgroup. Literally, this conjecture is false; a counter-example can be found for example in [5, Section 4]. However, in another breakthrough paper [4], Hrushovski weakened the notion of locally compact [and Lie] model by replacing a homomorphism by a quasi-homomorphism with compact error set S whose preimage is contained in a (small) power of X, and he proved the existence of such *generalized definable locally compact models* for arbitrary approximate subgroups (where the notion of definability is also weakened appropriately). This allowed him to deduce the existence of suitable generalized Lie models and obtain full classifications of approximate lattices, e.g., in $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{Q}_p)$.

The proof in [4] of the existence of generalized locally compact (and Lie) models is based on a new theory developed by Hrushovski including *definability patterns structures* and *local logics*, which may be very difficult to non model theorists.

We prove the existence of generalized definable locally compact models via topological dynamics methods in a model-theoretic context. The main idea is to extend the fundamental theory of Ellis groups to the context of suitable locally compact flows (in fact, locally compact left topological semigroups), and then the desired generalized locally compact model is a certain (explicitly defined) quasi-homomorphism to the canonical Hausdorff quotient of the Ellis group. Our proof is much shorter and uses only standard model theory (e.g. types, externally definable sets). We also prove universality of our generalized definable locally compact model in a suitable category. It is also interesting to consider the special case when the approximate subgroup X in question generates a group G in finitely many steps. Then the target space of our generalized [definable] locally compact model is compact, and it is in fact the classical [resp. externally definable] generalized Bohr compactification of G defined by Glasner (see [2] and [6]). This special case can be seen as a structural result on arbitrary definable generic subsets of definable groups.

To give a few details of our construction, let X be an approximate subgroup definable in a structure M, and $G := \langle X \rangle$. Let $N \succ M$ be an $|M|^+$-saturated extension of M. By $S_{G,M}(N)$ we denote the set of all complete types over N concentrated on G and finitely satisfiable in M. Then $S_{G,M}(N) = \bigcup_n S_{X^n,M}(N)$, and we introduce a topology on $S_{G,M}(N)$ by declaring that a subset $F \subseteq S_{G,M}(N)$ is closed if $F \cap S_{X^n,M}(N)$ is closed (in the usual type space topology on $S_{X^n,M}(N)$) for every $n \in \mathbb{N}$. Then $S_{G,M}(N)$ is a "locally compact G-flow", with the natural left action of G. We show that it is in fact a left topological semigroup with suitably defined semigroup operation. Then we prove the existence of minimal left ideals and an extension of Ellis theorem which presents each minimal left ideal $\mathcal{M}$ as a disjoint union of groups $u\mathcal{M}$ (where u ranges over the idempotents in $\mathcal{M}$), which are all isomorphic and which we call *Ellis groups*. Then we show that a certain topology on Ellis groups (called the *τ-topology*) is quasi locally compact, T_1, and separately continuous. The maximal Hausdorff quotient $u\mathcal{M}/H(u\mathcal{M})$ is a locally compact group. Finally, our main theorem says that the function $f\colon G \to u\mathcal{M}/H(u\mathcal{M})$ given by $f(g) := ugu/H(u\mathcal{M})$ is a generalized definable locally compact model of X with suitably defined compact error set. This model has some additional good properties, and our second theorem says that it is universal in a suitable category.

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Residue field domination

Deirdre Haskell
(joint work with Clifton Ealy, Pierre Simon)

Let K be a valued field, with valuation ring $\mathcal{O}$, maximal ideal $\mathfrak{m}$ and residue field $k = \mathcal{O}/\mathfrak{m}$. Many of the imaginaries in K are internal to the residue field. These include the obvious ones, like the k-vector spaces $\mathcal{O}_\gamma/\mathfrak{m}\mathcal{O}_\gamma$, where $\mathcal{O}_\gamma = \{x \in \mathcal{O} : v(x) \geq \gamma\}$, and less obvious ones like the fibers of $\mathrm{RV} = K^\times/(1+\mathfrak{m})$ given by $\mathrm{RV}_\gamma = \{x(1+\mathfrak{m}) : v(x) = \gamma\}$. In an algebraically closed valued field, these are all stable, stably embedded sets. In [4], we were motivated by the coincidence of the k-internal sets with the collection of stable, stably embedded sets to define the notion of *stably dominated* for types in general, and to study the consequences of the notion in an algebraically closed valued field. In particular, we proved [4] that a type is stably dominated if and only if it is orthogonal to the value group. In the recent paper [3], we give a more purely algebraic criterion for a type in an algebraically closed valued field to be stably dominated over a set of parameters in the field sort. In the process, we observe that it is useful to isolate the concept of a good separated basis (also observed in [1], where the property is called vs-defectless).

Definition 1. *The field L has the* good separated basis property *over the subfield C if, for every finite-dimensional subspace of L as a vector space over C, there is a basis $\ell_1, \dots, \ell_n$ such that for all $c_1, \dots, c_n$ in C,*

$$v(\sum_{i=1}^{n} c_i\ell_i) = \min_{1\leq i\leq n} \{v(c_i\ell_i)\}$$

and for all $1 \leq i, j \leq n$, if $v(c_i) \neq v(c_j)$ then $v(\ell_i/\ell_j) \notin \Gamma_C$.

Theorem 2. *Work inside a large saturated model of the theory of algebraically closed valued fields. Let C be a field, a a field element and write $L = \mathrm{dcl}(Ca)$. Assume that L is a regular extension of C. Then $\mathrm{tp}(a/C)$ is stably dominated if and only if $\Gamma(L) = \Gamma(C)$ and L has the good separated basis property over C.*

A henselian valued field in general cannot be assumed to have a non-trivial stable part, and yet still has all of the k-internal sets described above. We are therefore inspired to define a notion of *residue field domination*. This was first studied in [2] for real closed valued fields, and then in [3] for any henselian valued field of residue characteristic 0, provided the sets of nth powers in RV have finite index. Some useful examples are given to illustrate how neither of residue field domination and stable domination implies the other, especially when the parameters lie outside of the valued field sort. We do get the following theorem, which is analogous to Theorem 2 above.

Theorem 3. *Now work inside a large saturated model of the chosen theory of henselian valued fields. Let C be a field, a a field element and write $L = \mathrm{dcl}(Ca)$. Assume that k_L is a regular extension of k_C, that $\Gamma_L = \Gamma_C$ and that L has the good separated basis property over C. Then $\mathrm{tp}(a/C)$ is residue field dominated.*

The assumption that the value group does not increase is essential in this theorem. The given examples lead one to realise that residue field domination is too strong of a property when there are parameters from the value group. This leads to a definition of RV-domination in a natural way, and the following theorem.

Theorem 4. *Use the assumptions of Theorem 3 except now assume that Γ_L is a torsion-free extension of Γ_C. Then $\mathrm{tp}(a/C\Gamma_L)$ is* RV*-dominated.*

Similar results are proved by M. Vicaria in [5].

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Treelessness, independence and generically stable partial types

Nicholas Ramsey

In joint work with Itay Kaplan and Pierre Simon, we introduce a notion of independence called GS-independence, defined in terms of generically stable partial types. We will describe a context, the class of treeless theories, in which this notion is particularly well-behaved and give applications for both the SOP_n hierarchy and in NIP.

In more detail, a global partial type π is called *generically stable* over A if it is Ind-definable over A and whenever $\varphi(x;b) \in \pi$ and $a_i \models \pi|_{Aa_{<i}}$ for all i, then $\{i :\models \neg\varphi(a_i, b)\}$ is finite. We show that if p is a complete type over A, then p has a unique maximal extension to a global partial type that is generically stable over A, which is a kind of canonical generic extension of p. Using this, we define

a notion of independence as follows: $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_A^{\mathrm{GS}} b$ if $b \models \pi|_{Aa}$, where π is the unique maximal global partial type extending $\mathrm{tp}(b/A)$ which is generically stable over A.

This notion of independence, *GS-independence*, is reasonably well-behaved in arbitrary theories. For example, it always satisfies extension, left transitivity, and local character and always agrees with non-forking independence in simple theories (and with Kim-independence over models in NSOP_1 theories). However, we isolate a class of theories, the *treeless theories*, in which *GS-independence* is particularly useful, satisfying both base monotonicity and symmetry. The treeless theories are defined in terms of the collapse of certain indiscernible trees called *treetop indiscernibles.* We show that the treeless theories contain both the stable and binary theories and deduce many consequences of treelessness. In particular, we show that a treeless NSOP_3 theory with indiscernible triviality is NSOP_2 and treeless NSOP_1 theories are simple, from which we deduce that binary NSOP_3 theories are simple as a corollary. We additionally find applications for NIP theories, showing that a treeless NIP theory has an interpretable linear order.

Taming perfectoid fields

Franziska Jahnke
(joint work with Konstantinos Kartas)

Given two henselian valued fields (K, v) and (L, w) with the same value group ($vK = wL$) and residue field ($Kv = Lw$).

Question. *Which arithmetic properties do (K, v) and (L, w) share?*

This fundamental question has found many answers over the last 60 years:

(i) For $\mathrm{char}(Kv) = \mathrm{char}(Lw) = 0$, any first-order property (in the language $\mathcal{L}_{\mathrm{val}} = \{0, 1, +, \cdot, \mathcal{O}\}$) holds in (K, v) if and only if it holds in (L, w),[1] i.e.,

$$(K, v) \equiv_{\mathcal{L}_{\mathrm{val}}} (L, w).$$

This was shown independently by Ax and Kochen [2] and by Ershov [4].

(ii) By taking non-principal ultraproducts over the set of all primes, the theorem above allows to transfer properties between $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ *asymtoptically*: for every $\mathcal{L}_{\mathrm{val}}$-property ϕ there is an $N \in \mathbb{N}$ such that we have for all $p \geq N$

$$\mathbb{Q}_p \models \phi \Longleftrightarrow \mathbb{F}_p((t)) \models \phi.$$

This result was applied in particular to show that Artin's Conjecture holds *eventually*: for every degree d, there is $N \geq 0$ such that for $p > N$, every homogeneous polynomial over $\mathbb{Q}_p$ of degree d in d^2 many variables has a nontrivial solution in $\mathbb{Q}_p$. However, Terjanian showed that for every prime p, this property fails for some degree, i.e., no $\mathbb{Q}_p$ is a C_2-field [9].

[1] In fact, this only requires the residue fields and value groups to be elementarily equivalent in $\mathcal{L}_{\mathrm{ring}} = \{0, 1, +, \cdot\}$ and $\mathcal{L}_{\mathrm{oag}} = \{0, +, <\}$ respectively. The same is true for the fields discussed in (ii)-(iv).

(iii) If both (K, v) and (L, w) are unramified, i.e., $\operatorname{char}(K, Kv) = \operatorname{char}(L, Lw) = (0, p)$ and the value of p is the least positive element of the value group in each case, then once more we have

$$(K, v) \equiv_{\mathcal{L}_{\mathrm{val}}} (L, w).$$

For the case of perfect residue field, this was shown by Ax and Kochen [3], for imperfect residue field in joint work of Anscombe and the speaker [1].

(iv) For $(K, v), (L, w)$ tame fields of positive characteristic, there is once again an Ax-Kochen/Ershov principle by a result of Kuhlmann [6]:

$$(K, v) \equiv_{\mathcal{L}_{\mathrm{val}}} (L, w).$$

This principle fails in mixed characteristic tame fields, but a version holds nonetheless [10]: For $(K, v) \subseteq (L, w)$ tame fields, one has

$$(K, v) \prec (L, w) \iff vK \prec wL \text{ and } Kv \prec Lw$$

Here, a valued field (K, v) of residue characteristic $p > 0$ is called *tame* if
— the value group is p-divisible and
— the residue field is perfect and
— for all $(F, u) \supseteq (K, v)$ finite, one has $[F : K] = (uF : vK) \cdot [Fu : Kv]$.
Moreover, these principles can be extended to separably tame fields [7].

(v) In the realm of perfectoid fields, Scholze [8] extended work by Fontaine and Wintenberger [5], to show that the absolute Galois groups of a perfectoid field (K, v) and its tilt $(K^\flat, v^\flat)$ are canonically isomorphic. Here, a valued field (K, v) of residue characteristic $p > 0$ is called *perfectoid* if
— the value group is archimedean and p-divisible and
— (K, v) is complete and
— the Frobenius map $\Phi : \mathcal{O}/(p) \to \mathcal{O}/(p)$, $x \mapsto x^p$ is surjective.
The tilting operator turns a perfectoid field of mixed characteristic (K, v) into a perfectoid field of positive characteristic $(K^\flat, v^\flat)$ with $Kv = K^\flat v^\flat$ and $vK = v^\flat K^\flat$, such that there is some $t \in \mathfrak{m}_{v^\flat}$ with $\mathcal{O}/(p) \cong \mathcal{O}^\flat/(t)$. This ring is called an *infinitesimal thickening* of the residue field.

Note that as perfectoid field may have immediate extensions (proper algebraic extensions where the value group and residue field extensions are trivial), there is no hope for a similar Ax-Kochen/Ershov principle as for tame fields.

We show that despite the existence of immediate extensions, there is an Ax-Kochen/Ershov principle for perfectoid fields, when we replace 'residue field' by 'infinitesimal thickening of the residue field':

Theorem 1 (Perfectoid AKE, Jahnke-Kartas 2023$^+$)**.** *Let $(K, v) \subseteq (K', v')$ be two perfectoid fields. Then*

$$(K, v) \preceq (K', v') \iff \mathcal{O}/\varpi \preceq \mathcal{O}'/\varpi$$

for any $\varpi \in \mathfrak{m}_v$ with $0 < v(\varpi) \leq v(p)$.

Moreover, we show that - up to taking ultrapowers - the tilt of of a perfectoid field occurs as the residue field of a coarsening of the valuation which is tame with divisible value group:

Theorem 2 (Taming Theorem (Jahnke-Kartas, 2023^+)). *Let (K, v) be a perfectoid field of mixed characteristic and (K^*, v^*) a nonprincipal ultrapower. Then, there is a coarsening w of v^* such that*

(1) (K^*, w) is tame with divisible value group (in other words: any finite extension of (K^*, w) is unramified)

(2) $(K^\flat, v^\flat) \preceq (K^*w, \bar{v}^*)$

As an immediate consequence, we obtain that the absolute Galois groups of ultrapowers of K and $K^\flat$ are canonically isomorphic. This isomorphism restricts to a canonical isomorphism between the absolute Galois groups of K and $K^\flat$, i.e., we recover the classical Fontaine-Wintenberger Theorem.

However, our approach gives more than that: it allows one to understand which arithmetic properties (un)tilt: An $\mathcal{L}_{\mathrm{ring}}$-elementary property tilts if it passes from a tame valued field with divisible value group to its residue field, and it untilts if it lifts from the residue field of a tame valuation with divisible value group to the valued field.

Applying Kuhlmann's Ax-Kochen/Ershov results for tame fields, we also obtain that elementary equivalence is preserved under tilting:

Corollary 3. *Let (K, v) and (K', v') be perfectoid fields. Then*

— $(K, v) \preceq (K', v') \iff (K^\flat, v^\flat) \preceq ((K')^\flat, (v')^\flat)$ *and*
— $(K, v) \equiv (K', v') \implies (K^\flat, v^\flat) \equiv ((K')^\flat, (v')^\flat)$

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Non-degenerate n-linear forms and n-dependence

Nadja Hempel
(joint work with Artem Chernikov)

In Neostability (stable, simpel, NIP, $\mathrm{NTP}_2, \dots$) we mostly study classes of theories with tame binary relations. We are interested in studying theories with tame n-ary relations. One obivious candidate are the so called n-dependent theories. In the quest of finding natural algebraic examples, we looked at the theory T of symmetric or alternating nondegenerate bilinear forms on vector spaces, in a language with a separate sort for the field, i. e.

$$(V, K, +_V, \cdot_S, +_K, \cdot_K, \langle \cdot, \cdot \rangle)$$

where V and K are two different sorts with the field language $(+_K, \cdot_K)$ on K, the vector space language $+_V$ on V, scalar multiplication function $\cdot_S : K \times V \to V$ and the bilinear form function $\langle x, y\rangle : V \times V \to K$. Their basic model theory was studied by Granger in [5], and more recently in [3, 4] from the point of view of generalized stability theory. To obtain quantifer elimination, we add for each $p \in \omega$ a (definable) p-ary predicate $\theta_p(x_1, \dots, x_p)$ which holds if and only if $x_1, \dots, x_p \in V$ are linearly independent over K as well as for each $p \in \omega$ and $i \leq p$ an $(p+1)$-ary function $f_i^p : V^{p+1} \to K$ such that

$$f_i^p(v; v_1, \dots, v_p) = \begin{cases} \lambda_i & \text{if } \models \theta_p(v_1, \dots, v_p) \text{ and } v = \sum_{i=1}^p \lambda_i v_i \text{ for some } \lambda_i \in K \\ 0 & \text{otherwise} \end{cases}$$

Moreover we expand the language by relations on $K^n, n \in \omega$ definable in the language of rings such that K eliminates quantifiers in (e.g. we can always take Morleyzation of K) and call this language $\mathcal{L}_\theta^K$. The theory T has quantifer elimination in $\mathcal{L}_\theta^K$ and if K is NIP, then T is 2-dependent [1].

In this talk we considered alternating n-linear spaces, generalize nondegeneracy, and show that they are n-dependent when the underlying field is NIP. A naive way to generalize nondegeneracy to n-linear forms $\langle, \dots, \rangle_n$ would be: for any $v_1, \dots, v_{n-1}$ nonzero, there is $w \in V$ such that $\langle v_1, \dots, v_n, w\rangle_n \neq 0$. The obstacle which arises is, that particular expression are forced to be zero while working in n-linear forms satisfying additional properties (e. g. alternating or symmetric). In the case of alternating forms, we have for example that $\langle v, v, v_3 \dots, v_{n-1}, w\rangle_n = 0$ regardless of the choice of $v, v, v_3 \dots, v_{n-1}, w \in V$. To circumvent this issue, we work in $\bigoplus^{n-1} V$ modulo the subspace of $\bigoplus^{n-1} V$ containing all elements $v_1 \oplus \dots \oplus v_{n-1}$ for which $V \to K,\ w \mapsto (v_1, \dots, v_{n-1}, w)$ should be the zero map. In the case of alternating n-linear forms, we work with

$$\mathrm{Alt} = \mathrm{Span}\{v_1 \oplus \dots \oplus v_{n-1} | v_1, \dots, v_{n-1} \text{ are linearly dependent}\}.$$

and obtain

$$\Big(\bigoplus^{n-1} V\Big)/\mathrm{Alt} \ = \bigwedge^{n-1} V \quad \text{the } (n-1)\text{th exterior power of } V.$$

Any n-linear form $\langle\,,\dots,\,\rangle_n$ gives rise to a bilinear form $\langle\,,\,\rangle_2$ on $\bigoplus^{n-1} V \times V$ defined by

$$\langle v_1 \oplus \cdots \oplus v_{n-1}, v\rangle_2 = \langle v_1, \dots, v_{n-1}, v\rangle_n.$$

We say that $\langle\,,\dots,\,\rangle_n$ is an n-linear form of type Alt in V if $t/\mathrm{Alt} = s/\mathrm{Alt}$ in $(\bigoplus^{n-1} V)/\mathrm{Alt}$ implies that $\langle t, v\rangle_2 = \langle s, v\rangle_2$ for all $v \in V$.

Definition. *Let V be an n-linear space of type Alt. We say that $\langle\,,\dots,\,\rangle$ is*

- — non-dengerate *if for any non-zero $t \in (\bigoplus^{n-1} V)/Alt$ there is $w \in V$ such that $(t, w)_2 \neq 0$.*
- — generic *if for any linearly independent elements $t_1, \dots, t_m \in \bigoplus^{n-1} V/Alt$ and $k_1, \dots, k_m$ there is $w \in V$ such that $(t_i, w)_2 = k_i$.*

Then we obtain the following

- **(i)** For any n-linear space $(U, \langle\,,\dots,\,\rangle_n)$ of type Alt, there is a vector space V of dimension $\aleph_0 + \dim(U)$ containing U and an n-linear form $(\,,\dots,\,)_n$ on V of type Alt extending $\langle\,,\dots,\,\rangle_n$ and such that $(V, (\,,\dots,\,)_n)$ is a non-dengenerate.
- **(ii)** Let V be an infinite-dimensional. Then $(V, \langle\,,\dots,\,\rangle_n)$ is nondegenerate if and only if $(V, \langle\,,\dots,\,\rangle_n)$ is generic.

So let T_n be the theory of alternating nondegenerate n-linear forms on vector spaces, in a language with a separate sort for the field, i. e. of

$$(V, K, +_V, \cdot_S, +_K, \cdot_K, \langle\,,\dots,\,\rangle_n)$$

as above. The main obstacle to show n-dependence for such structure when the underlying field is NIP are formulas of the form

$$\psi(y_1, \dots, y_q) = \phi(\langle y_*^V, \dots, y_*\rangle, \dots, \langle y_*^V, \dots, y_*^V\rangle, y_*^K, \dots, y_*^K)$$

where ϕ is a formula in the field language, y_*^V is a variable from $(y_1, \dots, y_q)$ of the vector space sort, and y_*^K is a variable from $(y_1, \dots, y_q)$ of the field sort. To treat this case we showed the following Composition Lemma which is of independent interest and can be used in other situations to show n-dependence of particular theories.

Theorem 1 (Composition Lemma [2].)**.** *Let $\mathcal{M}$ be an $\mathcal{L}'$-structure such that its reduct to a language $\mathcal{L} \subseteq \mathcal{L}'$ is NIP. Let $d, k \in \mathbb{N}$, $\varphi(x_1, \dots, x_d)$ be an $\mathcal{L}$-formula, and $(y_0, \dots, y_k)$ be arbitrary $k+1$ tuples of variables. For each $1 \leq t \leq d$, let $0 \leq i_1^t, \dots, i_k^t \leq k$ be arbitrary, and let $f_t : M_{y_{i_1^t}} \times \dots \times M_{y_{i_k^t}} \to M_{x_t}$ be an arbitrary $\mathcal{L}'$-definable k-ary function. Then the formula*

$$\psi(y_0; y_1, \dots, y_k) := \varphi\left(f_1(y_{i_1^1}, \dots, y_{i_k^1}), \dots, f_d(y_{i_1^d}, \dots, y_{i_k^d})\right)$$

is k-dependent.

Finally we obtain:

Theorem 2 (Chernikov, H. [2].).

(i) *The theory T_n has quantifer elimination in $\mathcal{L}_\theta^K$.*

(ii) *If K is NIP, then T_n is n-dependent.*

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