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Hypergroups and Twin Buildings, I

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Abstract

We discuss a conjecture on thick twin buildings the verification of which is needed in order to show that thick twin buildings are mathematically equivalent to regular actions of certain twin Coxeter hypergroups. (A corresponding result for buildings is shown in [5; Sections 10.2, 10.3].) We prove that the conjecture holds in the case where the support of its sagittal has cardinality 2 and in the case where its sagittal has length at most 3. (Sagittals are defined in Section 1.) Our exposition is based on an earlier treatment of the subject; cf. [3].

1. Introduction

A hypergroup (as defined in [5]) is an algebraic concept which generalizes the notion of a group. A building (as defined in [4]) is a geometric concept which generalizes the notion of a projective space. Many buildings can be identified with their automorphism group. The Fano plane, for instance, is mathematically equivalent with the simple group $PSL_3(2)$. Tits' result [4] on thick buildings of spherical type and rank at least 3 is a far-reaching generalization of this simple observation.

Tits' result needs to be considered under the observation that the class of thick buildings of spherical type and rank at least 3 is a quite small subclass of the class of all buildings. However, his identification of thick buildings of spherical type and rank at least 3 with their automorphism groups admits a fairly straightforward generalization to an identification of *all* buildings with the members of a specific class of hypergroups, the class of the so-called Coxeter hypergroups; cf. [5; Chapter 9].

The situation is more delicate when it comes to twin buildings. The analog of the above identification of buildings with Coxeter hypergroups within the theory of twin buildings (as suggested in [5; Section 10.3]) depends on the verification of a pure building theoretic question which we phrase here as a conjecture.

Throughout this note, (W, I) stands for a Coxeter system, T for a twin building of type (W, I), X for the set of chambers of T. For each element w in W, we define

$$f_w := \{(y, z) \in X_- \times X_- \mid \delta_-(y, z) = w\} \cup \{(y, z) \in X_+ \times X_+ \mid \delta_+(y, z) = w\}$$

and

$$r_w := \{(y,z) \in X_- \times X_+ \mid \delta^*(y,z) = w^{-1}\} \cup \{(y,z) \in X_+ \times X_- \mid \delta^*(y,z) = w^{-1}\}.$$

(The symbols X_-, X_+, δ_-, δ_+, and δ^* are standard in the theory of twin buildings; they are introduced and used in [1].)

Conjecture C

Let t, u, and v be elements in W such that $f_v \cap (r_t \circ r_u)$ is not empty. Then $f_v \subseteq r_t \circ r_u$.

Of course, the question can be answered in the affirmative if B possesses a strongly transitive group of automorphisms. In particular, the question has a positive answer if B satisfies Condition (co) and/or is 2-spherical. (Condition (co) is defined on page 290 of [1] as well as the restriction to 2-spherical twin buildings are standard hypotheses in the study of automorphism groups of twin buildings.) However, these conditions seem to have little to do with a general approach to Conjecture C.

To state the main results of this note we introduce the following terminology.

Let w be an element in W. We define supp(w) to be the set of the elements i in I such that $w \notin \langle I \setminus \{i\}\rangle$. Note that supp$(w)$ is the smallest subset J of I with $v \in \langle J\rangle$. The set supp(w) is called the *support* of w.

The elements f_w with $w \in W$ will be called the *sagittals* of T, the elements r_w with $w \in W$ will be called the *transversals* of T. By the *support* of a sagittal f_w with $w \in W$ we mean the support of w (as defined in Section 8.) By the *length* of a sagittal f_w with $w \in W$ we mean the I-length of w.

The element f_v in Conjecture C is called its *sagittal*. It is the goal of this note to verify Conjecture C in the case where the support of its sagittal has cardinality 2 and in the case where its sagittal has I-length at most 3; cf. Theorems 8.3 and 10.2.

The following notation will be used throughout the remainder of this note.

For each element w in W, the I-length of w will be denoted by $\ell_I(w)$.

For each element u in W, we write $I_{-1}(u)$ to denote the set of all elements v in W such that $\ell_I(v) = \ell_I(vu^{-1}) + \ell_I(u)$ and $I_1(u)$ for the set of all elements t in W satisfying $\ell_I(tu) = \ell_I(t) + \ell_I(u)$.

For each element w in W, we write f^*_w to denote the set of all pairs (y,z) with $(z,y) \in f_w$. Similarly, r^*_w stands for the set of all pairs (y,z) with $(z,y) \in r_w$. This notation implies that, for each element w in W, $f^*_w = f_{w^{-1}}$ and $r^*_w = r_{w^{-1}}$.

2. Composing Sagittals

In this section, we compile results from [5; Section 10.4].

Lemma 2.1

For each element i in I, we have $h_1 \subseteq h_i \circ h_i \subseteq h_1 \cup h_i$.

Proof. This is [5; Lemma 10.1.2(ii)]. □

Let i be an element in I. In Lemma 2.1, we saw that $h_1 \subseteq h_i \circ h_i \subseteq h_1 \cup h_i$. In the following, we will say that h_i is *of first type* if $h_1 = h_i \circ h_i$, and we will say that h_i is *of second type* if $h_i \circ h_i = h_1 \cup h_i$. Of course, h_i may be neither of first nor of second type.

We notice that T is thick if and only if each element f_i with $i \in I$ is of second type.

Lemma 2.2

Let w be an element in W, and let i be an element in I. Then the following hold.

(i) *If $w \in I_1(i)$, $f_w \circ f_i = f_{wi}$.*

(ii) *Assume that $w \in I_{-1}(i)$ and that f_i is of first type. Then $f_w \circ f_i = f_{wi}$.*

(iii) *Assume that $w \in I_{-1}(i)$ and that f_i is of second type. Then $f_w \circ f_i = f_{wi} \cup f_w$.*

Proof. This is [5; Lemma 10.4.3]. □

Lemma 2.3

Let u and v be elements in W, and assume that $u \in I_1(v)$. Then $f_u \circ f_v = f_{uv}$.

Proof. This is [5; Corollary 10.4.4]. □

3. Composing Sagittals With Transversals

In this section, we compile results from [5; Section 10.5] and related facts.

Lemma 3.1

Let w be an element in W, and let i be an element in I. Then the following hold.

(i) *If $w^{-1} \in I_{-1}(i)$, $r_w \circ f_i = r_{iw}$.*

(ii) *Assume that $w^{-1} \in I_1(i)$ and that f_i is of first type. Then $r_w \circ f_i = r_{iw}$.*

(iii) *Assume that $w^{-1} \in I_1(i)$ and that f_i is of second type. Then $r_w \circ f_i = r_{iw} \cup r_w$.*

Proof. This is [5; Lemma 10.5.2]. □

Induction now allows us to generalize Lemma 3.1(i).

Lemma 3.2

Let t and v be elements in W with $t^{-1} \in I_{-1}(v^{-1})$. Then $r_t \circ f_v = r_{v^{-1}t}$.

Proof. There is nothing to show if $v = 1$. Thus, we assume that $v \neq 1$. In this case, there exist elements i in I and v' in W such that

$$v = iv' \quad \text{and} \quad \ell_I(v) = \ell_I(v') + 1.$$

It follows that $i \in I_1(v')$. Thus, as $v = iv'$, Lemma 2.3 yields

$$f_v = f_i \circ f_{v'}.$$

From $t^{-1} \in I_{-1}(v^{-1})$ we obtain an element s in W such that

$$t = vs \quad \text{and} \quad \ell_I(t) = \ell_I(v) + \ell_I(s).$$

Thus, setting $t' := v's$ we have

$$t = it', \quad \ell_I(t') = \ell_I(v') + \ell_I(s), \quad \text{and} \quad \ell_I(t) = \ell_I(t') + 1;$$

cf. [5; Lemma 2.3.8(ii)].

From $t = it'$ and $\ell_I(t) = \ell_I(t') + 1$ we obtain that $t^{-1} \in I_{-1}(i)$. Thus, as $t = it'$, Lemma 3.1(i) yields

$$r_t \circ f_i = r_{t'}.$$

From $t' = v's$ and $\ell_I(t') = \ell_I(v') + \ell_I(s)$ we obtain that $(t')^{-1} \in I_{-1}((v')^{-1})$. Thus, by induction,

$$r_{t'} \circ f_{v'} = r_{(v')^{-1}t'}.$$

From $f_v = f_i \circ f_{v'}$, $r_t \circ f_i = r_{t'}$, and $r_{t'} \circ f_{v'} = r_{(v')^{-1}t'}$ we obtain that

$$r_t \circ f_v = r_t \circ f_i \circ f_{v'} = r_{t'} \circ f_{v'} = r_{(v')^{-1}t'} = r_{v^{-1}t},$$

as wanted. □

Corollary 3.3

Let t and u be elements in W, and assume that $t \in I_1(u)$. Then the following hold.

(i) *We have $r_{tu} \circ f_t = r_u$.*

(ii) *We have $f_u \circ r_{tu} = r_t$.*

Proof. (i) We are assuming that $t \in I_1(u)$. Thus, by [5; Lemma 6.5.2(i)], $u^{-1} \in I_1(t^{-1})$. From this we obtain that $u^{-1}t^{-1} \in I_{-1}(t^{-1})$, whence $(tu)^{-1} \in I_{-1}(t^{-1})$. Thus, by Lemma 3.2, $r_{tu} \circ f_t = r_u$.

(ii) We are assuming that $t \in I_1(u)$. Thus, $u^{-1} \in I_1(t^{-1})$, so that, by (i),

$$r_{(tu)^{-1}} \circ f_{u^{-1}} = r_{u^{-1}t^{-1}} \circ f_{u^{-1}} = r_{t^{-1}}.$$

It follows that $f^*_{u^{-1}} \circ r^*_{(tu)^{-1}} = r^*_{t^{-1}}$. This is equivalent to $f_u \circ r_{tu} = r_t$. □

Corollary 3.4

The following hold.

(i) *For each element w in W, we have $r_w \circ f_w = r_1$.*

(ii) *For each element w in W, we have $f_w \subseteq r^*_w \circ r_1$.*

(iii) *For each element w in W, we have $f_w \circ r_w = r_1$.*

Proof. (i) This the case $(t, u) = (w, 1)$ in Corollary 3.3(i).[1]

(ii) From (i) we obtain that $f_w \subseteq r^*_w \circ r_w \circ f_w = r^*_w \circ r_1$.

(iii) Since $r^*_1 = r_1$, this follows from (i). □

[1] Alternately: If $w = 1$, $f_w = f_1$, so that f_w is the identity on X. It follows that $r_w \circ f_w = r_w = r_1$. If $w \neq 1$, I contains an element i with $w^{-1} \in I_{-1}(i)$, and we may apply [5; Lemma 10.3.3(i)] to w in place of u and v. Then we obtain that $r_w \circ f_w = r_{iw} \circ f_{iw}$, so that our claim follows by induction.

Lemma 3.5
Assume T to be thick, and let t, u, and v be elements in W. Assume that $r_v \cap (f_u \circ r_t)$ is not empty. Then $r_v \subseteq f_u \circ r_t$.

Proof. This is [5; Lemma 10.5.5(ii)]. □

4. Composing Transversals

In this section, we look at composites of tranversals.

Lemma 4.1
Let t, u, and v be elements in W, and let i be an element in I. Assume that $i \in I_1(t)$. Then the following hold.

(i) *Assume that $i \in I_1(v)$ and that $f_v \subseteq r^*_{it} \circ r_u$. Then $f_{iv} \subseteq r^*_t \circ r_u$.*

(ii) *Assume that T is thick and that $f_{iv} \subseteq r^*_{it} \circ r_u$. Then $f_{iv} \subseteq r^*_t \circ r_u$.*

Proof. (i) We are assuming that $i \in I_1(v)$. Thus, by Lemma 2.3,

$$f_i \circ f_v = f_{iv}.$$

From $i \in I_1(t)$ we also obtain $r_{it} \circ f_i = r_t$; cf. Corollary 3.3(i). Thus, we also have

$$f_i \circ r^*_{it} = r^*_t.$$

Since we are assuming that $f_v \subseteq r^*_{it} \circ r_u$, we now obtain that

$$f_{iv} = f_i \circ f_v \subseteq f_i \circ r^*_{it} \circ r_u = r^*_t \circ r_u,$$

as wanted.
(ii) We are assuming that T is thick. Thus, by Lemma 2.2(iii),

$$f_{iv} \subseteq f_i \circ f_{iv}.$$

We are assuming that $i \in I_1(t)$. Thus, by Corollary 3.3(i), $r_{it} \circ f_i = r_t$, so that

$$f_i \circ r^*_{it} = r^*_t.$$

Since we are assuming that $f_{iv} \subseteq r^*_{it} \circ r_u$, we now obtain that

$$f_{iv} \subseteq f_i \circ f_{iv} \subseteq f_i \circ r^*_{it} \circ r_u = r^*_t \circ r_u,$$

as wanted. □

Lemma 4.2
Let t, u, and v be elements in W, and assume that $t \in I_1(v)$ and that $u \in I_1(v^{-1})$. Then $r^*_t \circ r_{uv^{-1}} = r^*_{tv} \circ r_u$.

Proof. We are assuming that $t \in I_1(v)$. Thus, $v^{-1} \in I_1(t^{-1})$. Thus, by Corollary 3.3(i), $r_{v^{-1}t^{-1}} \circ f_{v^{-1}} = r_{t^{-1}}$; equivalently,

$$r^*_{tv} \circ f^*_v = r^*_t.$$

We are also assuming that $u \in I_1(v^{-1})$. Thus, by [5; Lemma 6.5.2(i)], $v \in I_1(u^{-1})$, so that, again by Corollary 3.3(i), $r_{vu^{-1}} \circ f_v = r_{u^{-1}}$; equivalently

$$f^*_v \circ r_{uv^{-1}} = r_u.$$

From $r^*_{tv} \circ f^*_v = r^*_t$ and $f^*_v \circ r_{uv^{-1}} = r_u$ we obtain that

$$r^*_t \circ r_{uv^{-1}} = r^*_{tv} \circ f^*_v \circ r_{uv^{-1}} = r^*_{tv} \circ r_u,$$

as wanted. □

Lemma 4.2 has two applications which will not be needed in the remainder of this note, but seem to be appealing.

Let t and u be elements in W, and assume that $t \in I_1(u^{-1})$. In this case, we may apply Lemma 4.2 to t, 1, and u^{-1} in place of t, u, and v. We obtain that $r^*_t \circ r_u = r^*_{tu^{-1}} \circ r_1$. On the other hand, by Corollary 3.4(ii), $f_{tu^{-1}} \subseteq r^*_{tu^{-1}} \circ r_1$. Thus, we obtain that $f_{tu^{-1}} \subseteq r^*_t \circ r_u$.

Setting $t = 1$, $u = 1$, and $v = w^{-1}$ in Lemma 4.2 we obtain that $r^*_1 \circ r_w = r^*_{w^{-1}} \circ r_1$, equivalently, $r_1 \circ r_w = r_w \circ r_1$.

Lemma 4.3

Let t and u be elements in W, and let i be an element in I. Then the following hold.

(i) *Assume that $t \in I_{-1}(i)$ or $u \in I_{-1}(i)$. Then $r^*_t \circ r_u \subseteq r^*_{ti} \circ r_{ui}$.*

(ii) *Assume that $t \in I_1(i)$ or $u \in I_1(i)$. Then $r^*_{ti} \circ r_{ui} \subseteq r^*_t \circ r_u$.*

Proof. (i) From Lemma 3.1 we know that $r_{u^{-1}} \subseteq r_{(ui)^{-1}} \circ f_i$. Thus, $r^*_u \subseteq r^*_{ui} \circ f_i$; equivalently $r_u \subseteq f_i \circ r_{ui}$.

Assume first that $t \in I_{-1}(i)$. Then, by Lemma 3.1(i), $r_{t^{-1}} \circ f_i = r_{(ti)^{-1}}$; equivalently, $r^*_t \circ f_i = r^*_{ti}$.

From $r_u \subseteq f_i \circ r_{ui}$ and $r^*_t \circ f_i = r^*_{ti}$ we obtain that

$$r^*_t \circ r_u \subseteq r^*_t \circ f_i \circ r_{ui} = r^*_{ti} \circ r_{ui}.$$

Assume now that $u \in I_{-1}(i)$. Then interchanging the roles of t and u the above reasoning shows that $r^*_u \circ r_t \subseteq r^*_{ui} \circ r_{ti}$, and that is equivalent to $r^*_t \circ r_u \subseteq r^*_{ti} \circ r_{ui}$.

(ii) We are assuming that $t \in I_{-1}(i)$ or $u \in I_{-1}(i)$. Thus, we have $ti \in I_1(i)$ or $u \in I_1(i)$. Applying (i) to ti and ui in place of t and u we now obtain that $r^*_t \circ r_u \subseteq r^*_{ti} \circ r_{ui}$. □

Corollary 4.4

Let t, u, and v be elements in W, and assume that $u \in I_1(v)$. Then $r^*_{tv} \circ r_{uv} \subseteq r^*_t \circ r_u$.

Proof. We proceed by induction with respect to $\ell_I(v)$.

If $\ell_I(v) = 0$, there is nothing to show if $v = 1$. Thus, we assume that $1 \leq \ell_I(v)$. In this case, we find elements v' in W and i in I such that

$$v = v'i \quad \text{and} \quad \ell_I(v) = \ell_I(v') + 1.$$

Note that $v' \in I_1(i)$. Thus, as $u \in I_1(v)$, we obtain from [5; Lemma 2.3.8(i)] that $uv' \in I_1(i)$. Thus, by Lemma 4.3(ii), $r^*_{tv'i} \circ r_{uv'i} \subseteq r^*_{tv'} \circ r_{uv'}$. Thus, as $v'i = v$, we obtain that $r^*_{tv} \circ r_{uv} \subseteq r^*_{tv'} \circ r_{uv'}$.

From $u \in I_1(v)$ and $v' \in I_1(i)$ we obtain that $u \in I_1(v')$; cf. [5; Lemma 2.3.8(i)]. Thus, by induction, $r^*_{tv'} \circ r_{uv'} \subseteq r^*_t \circ r_u$.

From $r^*_{tv} \circ r_{uv} \subseteq r^*_{tv'} \circ r_{uv'}$ and $r^*_{tv'} \circ r_{uv'} \subseteq r^*_t \circ r_u$ we obtain that $r^*_{tv} \circ r_{uv} \subseteq r^*_t \circ r_u$. □

We conclude this section with two results on the case where T is thick.

Lemma 4.5

Assume that T is thick. Let t and u be elements in W, let i be an element in I, and assume that $\{t, u\} \subseteq I_1(i)$. Then $r^*_{ti} \circ r_u \subseteq r^*_t \circ r_u$.

Proof. We are assuming that $u \in I_1(i)$. Thus, by Lemma 3.1(iii), $r_{u^{-1}} \subseteq r_{u^{-1}} \circ f_i$. Thus, $r^*_u \subseteq r^*_u \circ f_i$; equivalently, $r_u \subseteq f_i \circ r_u$.

We are assuming that $t \in I_1(i)$. Thus, $ti \in I_{-1}(i)$, so that, by Lemma 3.1(i), $r_{(ti)^{-1}} \circ f_i = r_{t^{-1}}$; equivalently, $r^*_{ti} \circ f_i = r^*_t$.

From $r_u \subseteq f_i \circ r_u$ and $r^*_{ti} \circ f_i = r^*_t$ we obtain that

$$r^*_{ti} \circ r_u \subseteq r^*_{ti} \circ f_i \circ r_u = r^*_t \circ r_u,$$

as wanted. □

Corollary 4.6

Assume that T is thick. Let w be an element in W. Then the following hold.

(i) *We have $r^*_w \circ r_1 \subseteq r^*_1 \circ r_1$.*

(ii) *We have $f_w \subseteq r^*_1 \circ r_1$.*

Proof. (i) There is nothing to show if $w = 1$. Thus, we assume that $w \neq 1$. In this case, we find elements v in W and i in I such that $v \in I_1(i)$.

Since $1 \in I_1(i)$, we may apply Lemma 4.5 to v and 1 in place of t and u. We obtain that $r^*_w \circ r_1 \subseteq r^*_v \circ r_1$. By induction, we also may assume that $r^*_v \circ r_1 \subseteq r^*_1 \circ r_1$. Thus, we have $r^*_w \circ r_1 \subseteq r^*_1 \circ r_1$.

(ii) From Corollary 3.4(ii) we know that $f_w \subseteq r^*_w \circ r_1$, from (i) that $r^*_w \circ r_1 \subseteq r^*_1 \circ r_1$. Thus, $f_w \subseteq r^*_1 \circ r_1$. □

Proposition 4.7

Assume that T is thick. Let t and u be elements in W, and assume that $t \in I_1(u)$. Then $r^*_{tu} \circ r_1 \subseteq r^*_t \circ r_1$.

Proof. There is nothing to show if $u = 1$. Therefore, we assume that $u \neq 1$. In this case, we find elements u' in W and i in I such that $u'i = u$ and $u' \in I_1(i)$.

From $t \in I_1(u)$ and $u' \in I_1(i)$ we obtain that $t \in I_1(u')$; cf. [5; Lemma 2.3.8(i)]. Thus, as $\ell_I(u') = \ell_I(u) - 1$, induction yields

$$r^*_{tu'} \circ r_1 \subseteq r^*_t \circ r_1.$$

From $t \in I_1(u)$ and $u' \in I_1(i)$ we obtain that $tu' \in I_1(i)$; cf. [5; Lemma 2.3.8(i)]. Since $1 \in I_1(i)$, we may apply Lemma 4.5 to tu' and 1 in place of t and u. We obtain that

$$r^*_{tu} \circ r_1 \subseteq r^*_{tu'} \circ r_1.$$

Summarizing we obtain that $r^*_{tu} \circ r_1 \subseteq r^*_t \circ r_1$. □

5. Several Facts About the Bruhat Order

We define

$$T := \{w^{-1}iw \mid w \in W,\ i \in I\}.$$

Given u and v in W, we say that u is a *subelement* of v if $u = v$ or if W contains elements $w_0, \ldots, w_n$ with $u = w_0$, $v = w_n$, and n a positive integer such that, for each element i in $\{1, \ldots, n\}$,

$$w_{i-1}^{-1}w_i \in T \quad \text{and} \quad \ell_I(w_{i-1}) \leq \ell_I(w_i).$$

Being a subelement is an order on W. This order is called the *Bruhat order* on W.

Note that 1 is a subelement of each element in W.

The following lemma is the key to all the subsequent results in this section.

Lemma 5.1

Let u and v be elements in W, and assume that $u \neq 1$. Then the following conditions are equivalent.

- (a) *The elements u is a subelement of v.*
- (b) *There exist elements $i_1, \ldots, i_n$ in I with $n = \ell_I(v)$ and $v = i_1 \cdots i_n$ and elements $j_1, \ldots, j_m$ in $\{1, \ldots, n\}$ such that ($m = \ell_I(u)$,) $j_1 \leq \ldots \leq j_m$, and $u = i_{j_1} \cdots i_{j_m}$.*
- (c) *For any n elements $i_1, \ldots, i_n$ in I with $n = \ell_I(v)$ and $v = i_1 \cdots i_n$, there exist elements $j_1, \ldots, j_m$ in $\{1, \ldots, n\}$ such that ($m = \ell_I(u)$,) $j_1 \leq \ldots \leq j_m$, and $u = i_{j_1} \cdots i_{j_m}$.*

Proof. This is [2; Corollary 2.2.3]. □

Corollary 5.2

Let t and u be elements in W. Then the following hold.

- (i) *Assume that t is a subelement of u. Then, for each element v in W with $\{t, u\} \subseteq I_1(v)$, tv is a subelement of uv.*
- (ii) *Assume that $t \in I_1(u)$. Then t is a subelement of tu.*

Proof. (i) This follows from Lemma 5.1.

(ii) We are assuming that $t \in I_1(u)$. thus, by [5; Lemma 6.5.2(i)], $u^{-1} \in I_1(t^{-1})$. Thus, as $1 \in I_1(t^{-1})$, we obtain from (i) that t^{-1} is a subelement of $u^{-1}t^{-1}$. It follows that t is a subelement of tu. □

Lemma 5.3

Let v be an element in W, set $n := \ell_I(v)$, and let $i_1, \ldots, i_n$ be elements in I with $v = i_1 \cdots i_n$. Let u be a subelement of v with $u \neq 1$, and set $m := \ell_I(u)$. Then $\{1, \ldots, n\}$ contains elements $j_1, \ldots, j_m$ with $j_1 \leq \ldots \leq j_m$ and $u = i_{j_1} \cdots i_{j_m}$ such that the following conditions hold.

(i) *For any two integers l with $1 \leq l \leq m-1$ and j with $j_l + 1 \leq j \leq j_{l+1} - 1$, we have $i_{j_1} \cdots i_{j_l} \in I_1(i_j)$.*

(ii) *For each integer j with $j_m + 1 \leq j \leq n$, we have $u \in I_1(i_j)$.*

Proof. Since u is a subelement of v, $\{1, \ldots, n\}$ contains elements $j_1, \ldots, j_m$ with

$$j_1 \leq \ldots \leq j_m \quad \text{and} \quad u = i_{j_1} \cdots i_{j_m};$$

cf. Lemma 5.1. Among the m-tuples $(j_1, \ldots, j_m)$ with $j_1 \leq \ldots \leq j_m$ and $u = i_{j_1} \cdots i_{j_m}$ we choose $(j_1, \ldots, j_m)$ such that $j_1 + \ldots + j_m$ is as large as possible. We will see that both conditions (i) and (ii) hold.

We first show that (i) holds. To do so we assume, by way of contradiction, that there exist integers l in $\{1, \ldots, m-1\}$ and j in $\{j_l + 1, \ldots, j_{l+1} - 1\}$ such that $i_{j_1} \cdots i_{j_l} \in I_{-1}(i_j)$. Then, by [2; Theorem 1.5.1], $\{1, \ldots, l\}$ contains an element k with

$$i_{j_1} \cdots i_{j_{k-1}} i_{j_{k+1}} \cdots i_{j_l} = i_{j_1} \cdots i_{j_l} i_j.$$

It follows that

$$i_{j_1} \cdots i_{j_{k-1}} i_{j_{k+1}} \cdots i_{j_l} i_j = i_{j_1} \cdots i_{j_l},$$

and then

$$i_{j_1} \cdots i_{j_{k-1}} i_{j_{k+1}} \cdots i_{j_l} i_j i_{j_{l+1}} \cdots i_{j_m} = i_{j_1} \cdots i_{j_m} = u.$$

However, since $j_k + 1 < j$, we have

$$j_1 + \ldots + j_m + 1 \leq j_1 + \cdots + j_{k-1} + j_{k+1} + \cdots + j_l + j + j_{l+1} + \cdots + j_m,$$

and that contradicts the choice of $(j_1, \ldots, j_m)$.

To show that (ii) holds, we assume, by way of contradiction, that there exists an integer j with $j_m + 1 \leq j \leq n$ and $u \in I_{-1}(i_j)$. Then, by [2; Theorem 1.5.1], $\{1, \ldots, m\}$ contains an element k with $i_{j_1} \cdots i_{j_{k-1}} i_{j_{k+1}} \cdots i_{j_m} = u i_j$. It follows that

$$i_{j_1} \cdots i_{j_{k-1}} i_{j_{k+1}} \cdots i_{j_m} i_j = u.$$

However, since $j_k + 1 \leq j$, we have

$$j_1 + \ldots + j_m + 1 \leq j_1 + \cdots + j_{k-1} + j_{k+1} + \cdots + j_m + j$$

which, again, contradicts the choice of $(j_1, \ldots, j_m)$. □

Lemma 5.4

Let j be an element in I, let v' be an element in $I_1(j)$, and let u be a subelement of $v'j$. Then at least one of the elements u and uj is subelement of v'.

Proof. Set $n := \ell_I(v'j)$. Then, since $v' \in I_1(j)$, $\ell_I(v') = n - 1$. Thus, I contains elements i_1, …, i_{n-1} such that $v' = i_1 \ldots i_{n-1}$. Set $i_n := j$. Then

$$v'j = i_1 \cdots i_n.$$

We are assuming that u is a subelement of $v'j$. If $u = 1$, u is a subelement of $v'j$, and we are done. Thus, we assume that $u \neq 1$. Thus, by Lemma 5.1, $\{1, \ldots, n\}$ contains elements j_1, …, j_m with

$$j_1 \leq \ldots \leq j_m \quad \text{and} \quad u = i_{j_1} \cdots i_{j_m}.$$

Suppose that $j_m \neq n$. Then, by Lemma 5.1, u is a subelement of v', and we are done.

Suppose that $j_m = n$. Then, as $i_n = j$, $i_{j_m} = j$. Thus, as $u = i_{j_1} \cdots i_{j_m}$, $uj = i_{j_1} \cdots i_{j_{m-1}}$. Since $\{j_1, \ldots, j_{m-1}\} \subseteq \{1, \ldots, n-1\}$, this implies that uj is a subelement of v'. Again, we are done. □

Let v be an element in W, let u be a subelement of v, assume that $u \neq 1$, and set $n := \ell_I(v)$. We say that u is an *isolated* subelement of v if I contains elements i_1, …, i_n such that $v = i_1 \cdots i_n$ and $\{1, \ldots, n\}$ contains uniquely determined elements j_1, …, j_m with $j_1 \leq \cdots \leq j_m$ and $u = i_{j_1} \cdots i_{j_m}$.

The above definition says that $u = i_{j_1} \cdots i_{j_m}$, but not that $m = \ell_I(u)$. From [2; Proposition 1.4.7], however, one obtains that $m = \ell_I(u)$.

Does the definiton depend on the choice of the elements i_1, …, i_n?

Note also that, by Lemma 5.3, the elements j_1, …, j_m in the above definition satisfy the conditions (i) and (ii) in Lemma 5.3.

We say that 1 is an *isolated* subelement of v if $\ell_I(v) = |\mathrm{supp}(v)|$.

Lemma 5.5

Let j be an element in I, let v' be an element in $I_1(j)$, and let u be an isolated subelement of $v'j$. Then at most one of the elements u and uj is subelement of v'.

Proof. We proceed by induction with respect to $\ell_I(u)$.

Assume first that $\ell_I(u) = 0$. Then $u = 1$. Thus, as u is assumed to be an isolated subelement of $v'j$, 1 is an isolated subelement of $v'j$. This means that $\ell_I(v'j) = |\mathrm{supp}(v'j)|$.

We are assuming that $v' \in I_1(j)$. Thus, as $\ell_I(v'j) = |\mathrm{supp}(v'j)|$, j is not subelement of v', so that we are done in this case.

Set $n := \ell_I(v'j)$. Then, since $v' \in I_1(j)$, $\ell_I(v') = n - 1$. Thus, I contains elements i_1, …, i_{n-1} such that $v' = i_1 \ldots i_{n-1}$. Set $i_n := j$. Then

$$v'j = i_1 \cdots i_n.$$

Now assume that $1 \leq \ell_I(u)$. Thus, $u \neq 1$. Thus, as u is assumed to be an isolated subelement of v, we obtain from Lemma 5.1 uniquely determined elements $j_1, \ldots, j_m$ in $\{1, \ldots, n\}$ with

$$j_1 \leq \ldots \leq j_m \quad \text{and} \quad u = i_{j_1} \cdots i_{j_m}.$$

Assume that uj is a subelement of v'. We shall be done if we succeed in showing that u is not a subelement of v'.

Since uj is a subelement of v', $uj = 1$ or $\{1, \ldots, n-1\}$ contains elements $j'_1, \ldots, j'_{m'}$ with

$$j'_1 \leq \ldots \leq j'_{m'} \quad \text{and} \quad uj = i_{j'_1} \cdots i_{j'_{m'}}.$$

Suppose that $uj \neq 1$. Then, since $j = i_n$, we obtain from $uj = i_{j'_1} \cdots i_{j'_{m'}}$ that

$$u = i_{j'_1} \cdots i_{j'_{m'}} i_n.$$

Since u is assumed to be an isolated subelement of $v'j$ we conclude from $j_1 \leq \ldots \leq j_m$ and $u = i_{j_1} \cdots i_{j_m}$ that $(j_1, \ldots, j_m) = (j'_1, \ldots, j'_{m'}, n)$. This is impossible, since $j_m \leq n-1$.

This shows that $uj = 1$. As a consequence, $u = j$. Since $j = i_n$, this implies that $u = i_n$. Since u is assumed to be an isolated subelement of $v'j$, this implies that $i_n \notin \{i_1, \ldots, i_{n-1}\}$. It follows that u is not a subelement of v'. □

Lemma 5.6

Let j be an element in I, let v' be an element in $I_1(j)$, and let u be an isolated subelement of $v'j$. Then the following hold.

(i) *If u is a subelement of v', u is an isolated subelement of v'.*

(ii) *If uj is a subelement of v', uj is an isolated subelement of v'.*

Proof. (i) If u is a subelement of v', u is an isolated subelement of v', since u is an isolated subelement of $v'j$.

(ii) We are assuming that $v' \in I_1(i)$. Thus, $\ell_I(v'j) = \ell_I(v') + 1$. Set $n := \ell_I(v'j)$. Then $\ell_I(v') = n - 1$. Thus, I contains elements $i_1, \ldots, i_{n-1}$ such that

$$v' = i_1 \cdots i_{n-1}.$$

Assume that ui is a subelement of v'. We will see that uj is an isolated subelement of v'.

Since uj is a subelement of v' and $v' = i_1 \cdots i_{n-1}$, $uj = 1$ or $\{1, \ldots, n-1\}$ contains elements $j_1, \ldots, j_m$ with

$$j_1 \leq \ldots \leq j_m \quad \text{and} \quad uj = i_{j_1} \cdots i_{j_m}.$$

Suppose first that $uj = 1$. Then $u = j$. Since u is an isolated subelement of v, this implies that $\ell_I(v') = |\mathrm{supp}(v')|$. Thus, by definition, 1 is an isolated subelement of v'. Thus, as $uj = 1$, uj is an isolated subelement of v', as wanted.

Suppose now that $uj \neq 1$. Then we obtain from $uj = i_{j_1} \cdots i_{j_m}$ that

$$u = i_{j_1} \cdots i_{j_m} j.$$

Since u is assumed to be an isolated subelement of $v'j$, this shows that $(j_1, \ldots, j_m)$ is the only finite sequence of elements in $\{1, \ldots, n-1\}$ satisfying

$$j_1 \leq \ldots \leq j_m \quad \text{and} \quad uj = i_{j_1} \cdots i_{j_m}.$$

Again, we have shown that uj is an isolated subelement of v'. □

6. The First Two Structure Theorems and Some Consequences

Throughout this section, we assume T to be thick.

Theorem 6.1 [First Structure Theorem]
Let t, u, and v be elements in W. Assume that $f_v \cap (r_t^* \circ r_u)$ is not empty. Then tu^{-1} is a subelement of v.

Proof. Set $q := tu^{-1}$. We will see that q is a subelement of v and proceed by induction with respect to $\ell_I(v)$.

Assume first that $\ell_I(v) = 0$. Then $v = 1$. Since $f_v \cap (r_t^* \circ r_u)$ is assumed not to be empty, this means that $f_1 \cap (r_t^* \circ r_u)$ is not empty. It follows that $r_t = r_u$, and then that $t = u$. Thus, as $q = tu^{-1}$, $q = 1$, and that implies that q is a subelement of v.

Assume now that $1 \leq \ell_I(v)$. Then we find elements v' in W and i in I such that $v = v'i$ and $\ell_I(v) = \ell_I(v') + 1$. Note that $v' \in I_1(i)$.

From $v' \in I_1(i)$ and $v'i = v$ we obtain that $f_{v'} \circ f_i = f_v$; cf. Lemma 2.2(i). Thus, since $f_v \cap (r_t^* \circ r_u)$ is assumed not to be empty, $(f_{v'} \circ f_i) \cap (r_t^* \circ r_u)$ is not empty. It follows that

$$(r_t \circ f_{v'}) \cap (r_u \circ f_i)$$

is not empty. Let u' be an element in W such that

$$r_{u'} \cap (r_t \circ f_{v'}) \quad \text{and} \quad r_{u'} \cap (r_u \circ f_i)$$

both are not empty.

Since $r_{u'} \cap (r_t \circ f_{v'})$ is not empty, so is $f_{v'} \cap (r_t^* \circ r_{u'})$. Thus, by induction, tu'^{-1} is a subelement of v'. Set $q' := tu'^{-1}$. Then q' is a subelement of v'.

Since $r_{u'} \cap (r_u \circ f_i)$ is not empty, $u' \in \{iu, u\}$; cf. Lemma 3.1. Thus, as $q' = tu'^{-1}$, we have $q' \in \{tu^{-1}i, tu^{-1}\}$. It follows that

$$q' \in \{qi, q\}.$$

Assume first that $q' = qi$ and $\ell_I(q) = \ell_I(q') - 1$. From $\ell_I(q) = \ell_I(q') - 1$ we obtain that $\ell_I(q') = \ell_I(q) + 1$. Then, as $q' = qi$, $q \in I_1(i)$. Thus, as $q' = qi$, we obtain from Corollary 5.2(i) that q is a subelement of q'. From $v' \in I_1(i)$ and $v = v'i$ we also obtain that v' is a subelement of v; again, by Corollary 5.2(i). Now, as q is a subelement of q', q' is a subelement of v', and v' is a subelement of v, we obtain from the transitivity of the Bruhat order that q is a subelement of v.

Next, assume that $q' = qi$ and $\ell_I(q) = \ell_I(q') + 1$. From $q' = qi$ we obtain that $q = q'i$. Thus, as $\ell_I(q) = \ell_I(q') + 1$, $q' \in I_1(i)$. Recall also that $v' \in I_1(i)$ and that q' is a subelement of v'. Thus, applying Corollary 5.2(ii) to q' and v' in place of u and v we obtain that q is a subelement of v.

Assume, finally, that $q' = q$. Then, as q' is a subelement of v', q is a subelement of v'. Recall that $v = v'i$ and that $v' \in I_1(i)$. Thus, by Corollary 5.2(i), v' is a subelement of v. Now, as q is a subelement of v' and v' is a subelement of v, we obtain from the transitivity of the Bruhat order that q is a subelement of v. □

Proposition 6.2

Let v be an element in W, and let u be a subelement of v. Then $r_v^* \circ r_1 \subseteq r_u^* \circ r_1$.

Proof. From Proposition 4.7 we know that $r_v \circ r_1 \subseteq r_1 \circ r_1$, so that we are done if $u = 1$.

Assume that $u \neq 1$. In this case, we set $n := \ell_I(v)$, and we let $i_1, \ldots, i_n$ be elements in I with $v = i_1 \cdots i_n$.

Since u is a subelement of v with $u \neq 1$ and $m = \ell_I(u)$, Lemma 5.3 provides elements $j_1, \ldots, j_m$ in $\{1, \ldots, n\}$ contains with $j_1 \leq \ldots \leq j_m$ and $u = i_{j_1} \cdots i_{j_m}$ such that $i_{j_1} \cdots i_{j_l} \in I_1(i_j)$ for any two integers l with $1 \leq l \leq m-1$ and j with $j_l + 1 \leq j \leq j_{l+1} - 1$ and $u \in I_1(i_j)$ for each integer j with $j_m + 1 \leq j \leq n$.

We now define two $(n+1)$-tuples of elements of W,

$$(u_0, \ldots, u_n) \quad \text{and} \quad (v_0, \ldots, v_n).$$

Let k be an element in $\{0, \ldots, n\}$.

If $0 \leq k \leq j_1 - 1$, we set $u_k = 1$. (Note that $u_0 = 1$.)

If $j_1 \leq k \leq n$, we define $u_k := i_{j_1} \cdots i_{j_l}$, where l is the largest integer in $\{1, \ldots, k\}$ with $j_l \leq k$. (Note that $u_n = u$.)

Let k be an element in $\{1, \ldots, n\}$, and let l denote the largest integer in $\{1, \ldots, k\}$ with $j_l \leq k$. Then we have

$$j_l = k \quad \text{or} \quad j_l + 1 \leq k.$$

If $j_l = k$, we have $j_{l-1} \leq k - 1$, so $u_{k-1} = i_{j_1} \cdots i_{j_{l-1}}$, and then $u_k = u_{k-1} i_{j_l}$.

If $j_l + 1 \leq k$, we have $u_{k-1} = i_{j_1} \cdots i_{j_l} = u_k$. Thus, as $i_{j_1} \cdots i_{j_l} \in I_1(i_k)$, $u_{k-1} \in I_1(i_k)$.

Thus, we have

$$u_k = u_{k-1} i_k \quad \text{or} \quad u_k = u_{k-1} \in I_1(i_k).$$

We set $v_n = 1$ and, for each element k in $\{0, \ldots, n-1\}$, we define $v_k := i_n \cdots i_{k+1}$. (Note that $v_0 = v^{-1}$.)

From $\ell_I(v) = n$ and $v = i_1 \cdots i_n$ we obtain that

$$v_{k-1} = v_k i_k \quad \text{and} \quad v_k \in I_1(i_k).$$

If $u_k = u_{k-1} i_k$, we apply Lemma 4.3(ii) to $u_{k-1} i_k$, v_k, and i_k in place of t, u, and i to obtain

$$r^*_{u_{k-1}} \circ r_{v_{k-1}} = r^*_{u_{k-1}} \circ r_{v_k i_k} \subseteq r^*_{u_{k-1} i_k} \circ r_{v_k} = r^*_{u_k} \circ r_{v_k}.$$

If $u_k = u_{k-1} \in I_1(i_k)$, Lemma 4.5 yields (since $u_{k-1} \in I_1(i_k)$ and $v_k \in I_1(i_k)$)

$$r^*_{u_{k-1}} \circ r_{v_{k-1}} = r^*_{u_{k-1}} \circ r_{v_k i_k} \subseteq r^*_{u_{k-1}} \circ r_{v_k} = r^*_{u_k} \circ r_{v_k}.$$

Thus, we have

$$r^*_{u_{k-1}} \circ r_{v_{k-1}} \subseteq r^*_{u_k} \circ r_{v_k}$$

in both cases.

Now recall that $u_0 = 1$, $u_n = u$, $v_0 = v^{-1}$, and $v_n = 1$. Thus, by induction,

$$r_1 \circ r^*_v = r^*_{u_0} \circ r_{v_0} \subseteq r^*_{u_n} \circ r_{v_n} = r^*_u \circ r_1.$$

On the other hand, applying Lemma 4.2 to v, v^{-1}, and v^{-1} in place of t, u, and v we obtain that $r^*_v \circ r_1 = r^*_1 \circ r_{v^{-1}}$; equivalently, $r^*_v \circ r_1 = r_1 \circ r^*_v$. Thus, $r^*_v \circ r_1 \subseteq r^*_u \circ r_1$. □

Theorem 6.3 [Second Structure Theorem]

Let t and v be elements in W. Then t is a subelement of v if and only if $f_v \subseteq r^*_t \circ r_1$.

Proof. Assume first that t is a subelement of v. Then, by Proposition 6.2, $r^*_v \circ r_1 \subseteq r^*_t \circ r_1$. In Corollary 3.4(ii), we saw already that $f_v \subseteq r^*_v \circ r_1$. It follows that $f_v \subseteq r^*_t \circ r_1$.

Assume, conversely, that $f_v \subseteq r^*_t \circ r_1$. Then, by Theorem 6.1, t is a subelement of v. □

Corollary 6.4

Let t, u, and v be elements in W. Then the following hold.

(i) *Assume that $t \in I_1(u^{-1})$ and that tu^{-1} is a subelement of v. Then $f_v \subseteq r^*_t \circ r_u$.*

(ii) *Let i be an element in I. Assume that $\ell_I(tiu^{-1}) = \ell_I(t) + 1 + \ell_I(u^{-1})$ and that tiu^{-1} is a subelement of v. Then $f_v \subseteq r^*_t \circ r_u$.*

Proof. (i) We are assuming that $t \in I_1(u^{-1})$. Thus, applying Lemma 4.2 to 1 and u^{-1} in place of u and v we obtain that $r^*_t \circ r_u = r^*_{tu^{-1}} \circ r_1$.

We are assuming that tu^{-1} is a subelement of v. Thus, by Theorem 6.3, $f_v \subseteq r^*_{tu^{-1}} \circ r_1$.

From $f_v \subseteq r^*_{tu^{-1}} \circ r_1$ and $r^*_t \circ r_u = r^*_{tu^{-1}} \circ r_1$ we obtain that $f_v \subseteq r^*_t \circ r_u$.

(ii) From $\ell_I(tiu^{-1}) = \ell_I(t) + 1 + \ell_I(u^{-1})$ we obtain that $ti \in I_1(u^{-1})$. Thus, as tiu^{-1} is assumed to be a subelement of v, we obtain from (i) that $f_v \subseteq r^*_{ti} \circ r_u$.

Since $\{t, u\} \subseteq I_1(i)$, we obtain from Lemma 4.5 that $r^*_{ti} \circ r_u \subseteq r^*_t \circ r_u$.

From $f_v \subseteq r^*_{ti} \circ r_u$ and $r^*_{ti} \circ r_u \subseteq r^*_t \circ r_u$ we obtain that $f_v \subseteq r^*_t \circ r_u$. □

Corollary 6.5

Let t, u, and v be elements in W, and assume that $f_v \cap (r^*_t \circ r_u)$ is not empty. Then the following hold.

(i) *We have $f_v \subseteq r^*_{tu^{-1}} \circ r_1$.*

(ii) *Assume that $t \in I_1(u^{-1})$. Then $f_v \subseteq r^*_t \circ r_u$.*

Proof. We are assuming that $f_v \cap (r^*_t \circ r_u)$ is not empty. Thus, by Theorem 6.1, tu^{-1} is a subelement of v.

(i) Since tu^{-1} is a subelement of v, the claim follows from Theorem 6.3.

(ii) Since tu^{-1} is a subelement of v, the claim follows from Corollary 6.4(i). □

Corollary 6.6

Let t and v be elements in W, and assume that $f_v \cap (r_t^* \circ r_1)$ is not empty. Then $f_v \subseteq r_t^* \circ r_1$.

Proof. This is the case $u = 1$ in Corollary 6.5(i).

7. A Reduction Theorem

Our first lemma is an inductive generalization of [1; Lemma 5.139(2)]].

Lemma 7.1

Let t and u be elements in W with $t \in I_1(u)$. Let x and y be elements in X with $(y, x) \in r_t$. Then X contains exactly one element z with $(y, z) \in f_u$ and $(z, x) \in r_{tu}$.

Proof. We proceed by induction with respect to $\ell_I(u)$.

Assume that $u = 1$. Then $(y, y) \in f_u$ and $(y, x) \in r_{tu}$. Moreover, if $(y, z) \in f_u$ and $(z, x) \in r_{tu}$, then $y = z$.

Assume that $u \neq 1$. Then there exist elements u' in $\langle J \rangle$ and j in J with $u = u'j$ and $\ell_I(u) = \ell_I(u') + 1$. From $t \in I_1(u)$, $u = u'j$, and $\ell_I(u) = \ell_I(u') + 1$ we obtain that

$$t \in I_1(u') \quad \text{and} \quad tu' \in I_1(j);$$

cf. [5; Lemma 2.3.8(i)].

Since $t \in I_1(u')$ and $\ell_I(u) = \ell_I(u') + 1$, induction yields that X contains exactly one element z' with

$$(y, z') \in f_{u'} \quad \text{and} \quad (z', x) \in r_{tu'}.$$

Since $(z', x) \in r_{tu'}$ and $tu' \in I_1(j)$, there exists exactly one element z in X such that

$$(z', z) \in f_j \quad \text{and} \quad (z, x) \in r_{tu};$$

cf. [1; Lemma 5.139(2)].

From $u' \in I_1(j)$ we obtain that $f_{u'} \circ f_j = f_u$; cf. Lemma 2.2. Thus, as $(y, z') \in f_{u'}$ and $(z', z) \in f_j$, $(y, z) \in f_u$.

So far, we have shown the existence of an element z in X satisfying $(y, z) \in f_u$ and $(z, x) \in r_{tu}$. In order to show uniqueness we choose an element $\bar{z}$ in X satisfying $(y, \bar{z}) \in f_u$ and

$$(\bar{z}, x) \in r_{tu}.$$

We will see that $\bar{z} = z$.

From $(y, \bar{z}) \in f_u$ and $f_{u'} \circ f_j = f_u$ we obtain an element $\bar{z}'$ in X with

$$(y, \bar{z}') \in f_{u'}$$

and $(\bar{z}', \bar{z}) \in f_j$.

From $(\bar{z}', \bar{z}) \in f_j$ and $(\bar{z}, x) \in r_{tu}$ we obtain that $(\bar{z}', x) \in f_j \circ r_{tu}$. Since $u = u'j$, this implies that $(\bar{z}', x) \in f_j \circ r_{tu'j}$. On the other hand, as $tu' \in I_1(j)$, Corollary 3.3(ii) yields $f_j \circ r_{tu'j} = r_{tu'}$. Thus,

$$(\bar{z}', x) \in r_{tu'}.$$

From $(y, \bar{z}') \in f_{u'}$ and $(\bar{z}', x) \in r_{tu'}$ together with the choice of z' we obtain that $z' = \bar{z}'$. Thus, as $(\bar{z}', \bar{z}) \in f_j$,

$$(z', \bar{z}) \in f_j.$$

Thus, as $(\bar{z}, x) \in r_{tu}$, the choice of z forces $\bar{z} = z$. □

Lemma 7.2

Let w be an element in W, let J be a subset of $I \cap I_1(w)$, and let t, u, and v be elements in $\langle J \rangle$. Let x' and x be elements in X with $(x', x) \in f_w$, and let y and z be elements in X with $(y, z) \in f_v$, $(x', y) \in r_t$, $(x', z) \in r_u$, and $(x, y) \in r_{tw}$. Then $(x, z) \in r_{uw}$.

Proof. We proceed by induction on $\ell_I(v) + \ell_I(w)$.

If $v = 1$, $y = z$. In this case, $t = u$. Thus, $(x, z) = (x, y) \in r_{tw} = r_{uw}$, and we are done.

If $w = 1$, $x' = x$. In this case, $(x, z) = (x', z) \in r_u = r_{uw}$, and we are done.

We assume that $v \neq 1$ and that $w \neq 1$.

Since $v \in \langle J \rangle$ and $v \neq 1$, there exist elements v' in $\langle J \rangle$ and k in J with $v = v'k$ and $\ell_I(v) = \ell_I(v') + 1$. Thus, by Lemma 2.2, $f_{v'} \circ f_k = f_v$. Thus, as $(y, z) \in f_v$, X contains an element z' such that $(y, z') \in f_{v'}$ and $(z', z) \in f_k$.

Since $w \in W$ and $w \neq 1$, there exist elements w' in W and l in I with $w = w'l$ and $\ell_I(w) = \ell_I(w') + 1$. Thus, by Lemma 2.2, $f_{w'} \circ f_l = f_w$. Thus, as $(x', x) \in f_w$, X_- contains an element x'' such that $(x', x'') \in f_{w'}$ and $(x'', x) \in f_l$.

Since $(x', z) \in r_u$ and $(z, z') \in f_k$, $(x', z') \in r_u \circ f_k$. On the other hand, by Lemma 3.1, $r_u \circ f_k \subseteq r_{ku} \cup r_u$. Let u' denote the element in W with $(x', z') \in r_{u'}$. Then $u' \in \{ku, u\}$. In particular, $u' \in \langle J \rangle$. Thus, as $(x, y) \in r_{tw}$, induction yields that $(x, z') \in r_{u'w}$.

From $(x, y) \in r_{tw}$ and $(x'', x) \in f_l$ we obtain that $(x'', y) \in f_l \circ r_{tw}$. On the other hand, as $tw \in I_{-1}(l)$ (by [5; Lemma 2.3.8(i)]) we obtain from Lemma 3.1(i) that $f_l \circ r_{tw} = r_{tw'}$. Thus, we have $(x'', y) \in r_{tw'}$, so that, by induction, $(x'', z) \in r_{uw'}$.

From $(x'', x) \in f_l$ we obtain that $(x, x'') \in f_l$. Thus, as $(x'', z) \in r_{uw'}$, $(x, z) \in f_l \circ r_{uw'}$. On the other hand, by Lemma 3.1, $f_l \circ r_{uw'} \subseteq r_{uw} \cup r_{uw'}$. Thus,

$$(x, z) \in r_{uw} \quad \text{or} \quad (x, z) \in r_{uw'}.$$

Assume that $(x, z) \in r_{uw'}$. Then, as $(x, z') \in r_{u'w}$ and $(z', z) \in f_k$, $uw' \in \{ku'w, u'w\}$. Since $\{ku', u'\} \subseteq \langle J \rangle$, this implies that $\langle J \rangle w' = \langle J \rangle w$, contradiction. Thus, $(x, z) \in r_{uw}$. □

Lemma 7.3

Let w be an element in W, let J be a subset of $I \cap I_1(w)$, and let t, u, and v be elements in $\langle J \rangle$. Then $f_v \cap (r_t^* \circ r_u) = f_v \cap (r_{tw}^* \circ r_{uw})$.

Proof. We first show that $f_v \cap (r_t^* \circ r_u) \subseteq f_v \cap (r_{tw}^* \circ r_{uw})$. To do this we let y and z be elements in X and assume that $(y, z) \in f_v \cap (r_t^* \circ r_u)$. We will see that $(y, z) \in r_{tw}^* \circ r_{uw}$.

From $(y, z) \in r_t^* \circ r_u$ we obtain an element x' in X with $(y, x') \in r_t^*$ and $(x', z) \in r_u$.

From $(y, x') \in r_t^*$ we obtain that $(x', y) \in r_t$. Thus, as $t \in I_1(w)$, X contains exactly one element x with $(x', x) \in f_w$ and $(x, y) \in r_{tw}$; cf. Lemma 7.1.

From $(x, y) \in r_{tw}$ we obtain that $(y, x) \in r_{tw}^*$. From $(x', x) \in f_w$, $(y, z) \in f_v$, $(x', y) \in r_t$, $(x', z) \in r_u$, and $(x, y) \in r_{tw}$ we obtain that $(x, z) \in r_{uw}$; cf. Lemma 7.2. From $(y, x) \in r_{tw}^*$ and $(x, z) \in r_{uw}$ we obtain that $(y, z) \in r_{tw}^* \circ r_{uw}$.

Since $\{t, u\} \subseteq I_1(w)$, we obtain from Corollary 4.4 that $r_{tw}^* \circ r_{uw} \subseteq r_t^* \circ r_u$. □

Lemma 7.3 can be used to prove an interesting generalization of Corollary 4.6(ii). In fact, we obtain that $f_v \subseteq r_u^* \circ r_u$ for any two elements u and v in W with $\mathrm{supp}(v) \subseteq I_1(u)$. To see this, we first notice that, by Corollary 4.6(ii), $f_v \subseteq r_1^* \circ r_1$. Now, applying Lemma 7.3 to $\mathrm{supp}(v)$, 1, 1, and u in place of J, t, u, and w we obtain that $f_v \cap (r_1^* \circ r_1) = f_v \cap (r_u^* \circ r_u)$. Thus, $f_v \subseteq r_u^* \circ r_u$. As a consequence of this observation we obtain that $f_i \subseteq r_w^* \circ r_w$ for any two elements w in W and i in $I_1(w)$.

Theorem 7.4 [Reduction Theorem]
Let v be an element in W. Assume that $f_v \subseteq r_{t'}^* \circ r_{u'}$ for any two elements t' and u' in W with $f_v \cap (r_{t'}^* \circ r_{u'}) \neq \emptyset$ and $\mathrm{supp}(t') \cup \mathrm{supp}(u') \subseteq \mathrm{supp}(v)$. Then $f_v \subseteq r_t^* \circ r_u$ for any two elements t and u in W with $f_v \cap (r_t^* \circ r_u) \neq \emptyset$.

Proof. Let t, u, and v be elements in W, and assume that $f_v \cap (r_t^* \circ r_u)$ is not empty. We have to show that $f_v \subseteq r_t^* \circ r_u$.

Set $J := \mathrm{supp}(v)$. Since $f_v \cap (r_t^* \circ r_u)$ is not empty, tu^{-1} is a subelement of v; cf. Theorem 6.1. Thus, $tu^{-1} \in \langle J \rangle$. It follows that $\langle J \rangle t = \langle J \rangle u$. Let q denote the uniquely determined element of shortest length in $\langle J \rangle t$ satisfying $\langle J \rangle q = \langle J \rangle t$. Then $\langle J \rangle$ contains elements t' and u' such that

$$t = t'q, \quad u = u'q, \quad t' \in I_1(q), \quad \text{and} \quad u' \in I_1(q).$$

From $u' \in I_1(q)$ we obtain that $f_q \circ r_{u'q} = r_{u'}$; cf. Corollary 3.3(ii). Similarly, $t' \in I_1(q)$ yields $f_q \circ r_{t'q} = r_{t'}$; equivalently, $r_{t'q}^* \circ f_q^* = r_{t'}^*$. Thus, as $t = t'q$ and $u = u'q$,

$$f_v \cap (r_t^* \circ r_u) \subseteq f_v \cap (r_{t'q}^* \circ r_{u'q}) \subseteq f_v \cap (r_{t'q}^* \circ f_q^* \circ f_q \circ r_{u'q}) \subseteq f_v \cap (r_{t'}^* \circ r_{u'}).$$

Since $f_v \cap (r_t^* \circ r_u)$ is assumed not to be empty, this shows that $f_v \cap (r_{t'}^* \circ r_{u'})$ is not empty. On the other hand, $\{t', u'\} \subseteq \langle J \rangle$, so that $\mathrm{supp}(t') \cup \mathrm{supp}(u') \subseteq \mathrm{supp}(v)$. Thus, by hypothesis, $f_v \subseteq r_{t'}^* \circ r_{u'}$.

From $f_v \subseteq r_{t'}^* \circ r_{u'}$ together with Lemma 7.3 we obtain that $f_v \subseteq r_t^* \circ r_u$. □

8. The Case Where the Sagittal has Cardinality at Most 2

In this section, T is assumed to be thick. We shall see that, for any three elements t, u, and v in W, $f_v \subseteq r_{t^{-1}} \circ r_u$ if $f_v \cap (r_t^* \circ r_u)$ is not empty and $|\mathrm{supp}(v)| \leq 2$; cf. Theorem 8.3. In other words, we prove that Conjecture C holds if the support of its sagittal has cardinality at most 2. We will refer to Theorem 7.4.

Lemma 8.1

Let t, u, and v be elements in W, and assume that $f_v \cap (r_t^* \circ r_u)$ is not empty. Assume further that I contains a subset H with $|H| = 2$ and $\{t, u, v\} \subseteq \langle H \rangle$. Assume finally that H contains elements j and k with $\{t^{-1}, v^{-1}\} \subseteq I_{-1}(j)$ and $\{v, u^{-1}\} \subseteq I_{-1}(k)$. Then $f_v \subseteq r_t^* \circ r_u$.

Proof. We first claim that $\ell_I(t) + 1 \leq \ell_I(v)$.

Assume, by way of contradiction, that $\ell_I(v) \leq \ell_I(t)$. Then, since $|I| = 2$ and $\{t^{-1}, v^{-1}\} \subseteq I_{-1}(j)$, $t^{-1} \in I_{-1}(v^{-1})$. Thus, by Lemma 3.2, $r_t \circ f_v = r_{v^{-1}t}$. On the other hand, we are assuming that $f_v \cap (r_t^* \circ r_u)$ is not empty, and that implies that $r_u \cap (r_t \circ f_v)$ is not empty. Thus, $r_u = r_{v^{-1}t}$. It follows that $v^{-1}t = u$; equivalently, $t = vu$. Since we are assuming that $\ell_I(v) \leq \ell_I(t)$, this implies that $\ell_I(v) \leq \ell_I(vu)$.

On the other hand, since $\{v, u^{-1}\} \subseteq I_{-1}(k)$, we have $\ell_I(vu) \leq \ell_I(v) - 1$, contradiction.

This contradiction shows that $\ell_I(t) + 1 \leq \ell_I(v)$. Since $|I| = 2$ and $\{t^{-1}, v^{-1}\} \subseteq I_{-1}(j)$, this implies that $v^{-1} \in I_{-1}(t^{-1})$, so that

$$v = tw \quad \text{and} \quad \ell_I(v) = \ell_I(t) + \ell_I(w)$$

for some element $w \in W \setminus \{1\}$.

Recall that $v \in I_{-1}(k)$. Thus, as $v = tw$ and $\ell_I(v) = \ell_I(t) + \ell_I(w)$,

$$w \in I_{-1}(k).$$

From $v = tw$ and $\ell_I(v) = \ell_I(t) + \ell_I(w)$ we also obtain that

$$f_v = f_t \circ f_w;$$

cf. Lemma 2.3. Thus, as $f_v \cap (r_t^* \circ r_u)$ is assumed not to be empty, the intersection $(f_t \circ f_w) \cap (r_t^* \circ r_u)$ is not empty. It follows that

$$(f_t^* \circ r_t^*) \cap (f_w \circ r_u^*)$$

is not empty. On the other hand, by Corollary 3.4(iii), $f_t^* \circ r_t^* = r_1$. Thus, $r_1 \cap (f_w \circ r_u^*)$ is not empty. It follows that $f_w \cap (r_1 \circ r_u)$ is not empty. Thus, by Theorem 6.1, u^{-1} is a subelement of w.

Since $\{u^{-1}, w\} \subseteq I_{-1}(k)$ and u^{-1} is a subelement of w, we have

$$w = su^{-1} \quad \text{and} \quad \ell_I(w) = \ell_I(s) + \ell_I(u^{-1})$$

for some element s in W. Thus, as $v = tw$ and $\ell_I(v) = \ell_I(t) + \ell_I(w)$, we have

$$v = tsu^{-1} \quad \text{and} \quad \ell_I(v) = \ell_I(t) + \ell_I(s) + \ell_I(u^{-1}).$$

If $t \in I_1(u^{-1})$, we know that $f_v \subseteq r_t^* \circ r_u$ already from Corollary 6.5(ii).

Now assume that $t \notin I_1(u^{-1})$. Then, $\ell_I(s)$ is odd. Thus, there exist elements l in I and s' in W such that $s = ls'$ and $\ell_I(s) = \ell_I(s') + 1$. It follows that W contains an element q such that

$$v = tlu^{-1}q \quad \text{and} \quad \ell_I(v) = \ell_I(t) + 1 + \ell_I(u^{-1}) + \ell_I(q).$$

Thus, by Corollary 6.4(ii), $f_v \subseteq r_t^* \circ r_u$. □

Proposition 8.2

Let t, u, and v be elements in W, and assume that $f_v \cap (r_t^* \circ r_u)$ is not empty. Assume further that I contains a subset H with $|H| = 2$ and $\{t, u, v\} \subseteq \langle H \rangle$. Then $f_v \subseteq r_t^* \circ r_u$.

Proof. Assume first that $v = 1$. Then $f_1 \cap (r_t^* \circ r_u)$ is not empty, whence $t = u$. It follows that $f_v = f_1 \subseteq r_t^* \circ r_t = r_t^* \circ r_u$, so that we are done in this case. Therefore, we assume that $v \neq 1$. In this case, I contains elements j and k with

$$v^{-1} \in I_{-1}(j) \quad \text{and} \quad v \in I_{-1}(k).$$

Case I: Assume that $t^{-1} \in I_{-1}(j)$ and $u^{-1} \in I_{-1}(k)$. In this case, we are done by Lemma 8.1.

Case II: Assume that $t^{-1} \in I_1(j)$ and $u^{-1} \in I_{-1}(k)$.

Set $v' := jv$. Then, as $v^{-1} \in I_{-1}(j)$,

$$v = jv' \quad \text{and} \quad \ell_I(v) = \ell_I(v') + 1.$$

Thus, by Lemma 2.3,

$$f_j \circ f_{v'} = f_v.$$

Since $f_v \cap (r_t^* \circ r_u)$ is assumed not to be empty, this implies that $(f_j \circ f_{v'}) \cap (r_t^* \circ r_u)$ is not empty. It follows that

$$(f_j \circ r_t^*) \cap (f_{v'} \circ r_u^*)$$

is not empty. On the other hand, by Lemma 3.1, $r_t \circ f_j \subseteq r_{jt} \cup r_t$. It follows that $f_j \circ r_t^* \subseteq r_{jt}^* \cup r_t^*$. Thus, one of the intersections

$$r_{jt}^* \cap (f_{v'} \circ r_u^*) \quad \text{and} \quad r_t^* \cap (f_{v'} \circ r_u^*)$$

is not empty.

Assume first that $r_{jt}^* \cap (f_{v'} \circ r_u^*)$ is not empty. Then $f_{v'} \cap (r_{jt}^* \circ r_u)$ is not empty. Thus, as $\ell_I(v') = \ell_I(v) - 1$, induction yields

$$f_{v'} \subseteq r_{jt}^* \circ r_u.$$

Thus, applying Lemma 4.1(i) to v' and j in place of v and i we obtain that $f_v \subseteq r_t^* \circ r_u$, as wanted.

Assume now that $r_t^* \cap (f_{v'} \circ r_u^*)$ is not empty. Then, by Lemma 3.5,

$$r_t^* \subseteq f_{v'} \circ r_u^*.$$

From Lemma 3.1 we also know also that $r_{jt} \subseteq r_t \circ f_j$; equivalently,

$$r_{jt}^* \subseteq f_j \circ r_t^*.$$

Thus, since $f_j \circ f_{v'} = f_v$, we have

$$r_{jt}^* \subseteq f_j \circ f_{v'} \circ r_u^* = f_v \circ r_u^*.$$

It follows that $f_v \cap (r_{jt}^* \circ r_u)$ is not empty. Thus, as $\{(jt)^{-1}, v^{-1}\} \subseteq I_{-1}(j)$ and $\{v, u^{-1}\} \subseteq I_{-1}(k)$, we obtain from Lemma 8.1 that

$$f_v \subseteq r_{jt}^* \circ r_u.$$

Applying Lemma 4.1(ii) to v and j in place of iv and i we now obtain that $f_v \subseteq r_t^* \circ r_u$, as wanted.

Case III: Assume that $t^{-1} \in I_{-1}(j)$ and $u^{-1} \in I_1(k)$. This is Case II with u, t, and v^{-1} in place of t, u and v.

Case IV: Assume that $t^{-1} \in I_1(j)$ and $u^{-1} \in I_1(k)$.

Set $v' := jv$. Then, as $v^{-1} \in I_{-1}(j)$,

$$v = jv' \quad \text{and} \quad \ell_I(v) = \ell_I(v') + 1.$$

Thus, by Lemma 2.3,

$$f_j \circ f_{v'} = f_v.$$

Since $f_v \cap (r_t^* \circ r_u)$ is assumed not to be empty, this implies that $(f_j \circ f_{v'}) \cap (r_t^* \circ r_u)$ is not empty. It follows that

$$(f_j \circ r_t^*) \cap (f_{v'} \circ r_u^*)$$

is not empty. On the other hand, by Lemma 3.1, $r_t \circ f_j \subseteq r_{jt} \cup r_t$. It follows that $f_j \circ r_t^* \subseteq r_{jt}^* \cup r_t^*$. Thus, one of the intersections

$$r_{jt}^* \cap (f_{v'} \circ r_u^*) \quad \text{and} \quad r_t^* \cap (f_{v'} \circ r_u^*)$$

is not empty.

Assume first that $r_{jt}^* \cap (f_{v'} \circ r_u^*)$ is not empty. Then $f_{v'} \cap (r_{jt}^* \circ r_u)$ is not empty. Thus, as $\ell_I(v') = \ell_I(v) - 1$, induction yields

$$f_{v'} \subseteq r_{jt}^* \circ r_u.$$

Thus, applying Lemma 4.1(i) to v' and j in place of v and i we obtain that $f_v \subseteq r_t^* \circ r_u$, as wanted.

Assume now that $r_t^* \cap (f_{v'} \circ r_u^*)$ is not empty. Then, by Lemma 3.5,

$$r_t^* \subseteq f_{v'} \circ r_u^*.$$

From Lemma 3.1 we also know that $r_{jt} \subseteq r_t \circ f_j$; equivalently,

$$r_{jt}^* \subseteq f_j \circ r_t^*.$$

Thus, since $f_j \circ f_{v'} = f_v$, we have

$$r^*_{jt} \subseteq f_j \circ f_{v'} \circ r^*_u = f_v \circ r^*_u.$$

It follows that $f_v \cap (r^*_{jt} \circ r_u)$ is not empty. Thus, as $(jt)^{-1} \in I_{-1}(j)$, $u^{-1} \in I_1(k)$, and $v \in I_{-1}(k)$, we obtain from Case III that

$$f_v \subseteq r^*_{jt} \circ r_u.$$

Applying Lemma 4.1(ii) to v and j in place of iv and i we now obtain that $f_v \subseteq r^*_t \circ r_u$, as wanted. □

Theorem 8.3 [First Main Theorem]
Let t, u, and v be elements in W, and assume that $f_v \cap (r^*_t \circ r_u)$ is not empty. Assume further that $|\mathrm{supp}(v)| \leq 2$. Then $f_v \subseteq r^*_t \circ r_u$.

Proof. From Proposition 8.2 we know that $f_v \subseteq r^*_{t'} \circ r_{u'}$ for any two elements t' and u' in W such that $f_v \cap (r^*_{t'} \circ r_{u'})$ is not empty and $\mathrm{supp}(t') \cup \mathrm{supp}(u') \subseteq \mathrm{supp}(v)$. Thus, the claim follows from Theorem 7.4. □

9. The Third Structure Theorem

In this section, we show that Condition C holds if the quotient of the subscripts of its transversals is an isolated subelement of its sagittal.

We begin with an application of Lemma 3.1.

Lemma 9.1
Let w be an element in W, let i be an element in I, and let x, y, and z be elements in X. Then the following hold.

(i) *Assume that $(y, z) \in f_i$ and that $(x, z) \in r_w$. Then $(x, y) \in r_{iw} \cup r_w$.*

(ii) *Assume that $(y, z) \in f_i$ and that $(z, x) \in r_w$. Then $(y, x) \in r_{wi} \cup r_w$.*

Proof. (i) From $(y, z) \in f_i$ we obtain that $(z, y) \in f_i$. Thus, as $(x, z) \in r_w$, $(x, y) \in r_w \circ f_i$. On the other hand, by Lemma 3.1, $r_w \circ f_i \subseteq r_{iw} \cup r_w$. It follows that $(x, y) \in r_{iw} \cup r_w$.

(ii) We are assuming that $(z, x) \in r_w$. Thus, we have $(x, z) \in r_{w^{-1}}$. Thus, by (i), $(x, y) \in r_{iw^{-1}} \cup r_{w^{-1}}$. It follows that $(y, x) \in r_{wi} \cup r_w$. □

Lemma 9.2
Let t, u, and v' be elements in W, and let j be an element in I. Let x, z', and z be elements in X with $(z', z) \in f_j$ and $(x, z) \in r_u$. Suppose that X contains an element y with $(y, z') \in f_{v'}$ and $(y, x) \in r^*_t$. Then the following hold.

(i) *Assume that $t(ju)^{-1}$ is not a subelement of v'. Then $(x, z') \notin r_{ju}$.*

(ii) *Assume that tu^{-1} is not a subelement of v'. Then $(x, z') \notin r_u$.*

Proof. (i) Assume that $(x, z') \in r_{ju}$. Then, as $(y, x) \in r_t^*$, we have $(y, z') \in r_t^* \circ r_{ju}$. Since $(y, z') \in f_{v'}$, this implies that $f_{v'} \cap (r_t^* \circ r_{ju})$ is not empty, so that, by Theorem 6.1, $t(ju)^{-1}$ is a subelement of v'.

(ii) Assume that $(x, z') \in r_u$. Then, as $(y, x) \in r_t^*$, we have $(y, z') \in r_t^* \circ r_u$. Since $(y, z') \in f_{v'}$, this implies that $f_{v'} \cap (r_t^* \circ r_u)$ is not empty, so that, by Theorem 6.1, tu^{-1} is a subelement of v'. □

Lemma 9.3

Let t, u, and v' be elements in W, and let j be an element in I. Assume that $v' \in I_1(j)$. Assume further that $f_{v'j} \cap (r_t^* \circ r_u)$ is not empty. Then the following hold.

(i) *Assume that $t(ju)^{-1}$ is not a subelement of v'. Then $f_{v'} \cap (r_t^* \circ r_u)$ is not empty.*

(ii) *Assume that tu^{-1} is not a subelement of v'. Then $f_{v'} \cap (r_t^* \circ r_{ju})$ is not empty.*

Proof. We are assuming that $f_{v'j} \cap (r_t^* \circ r_u)$ is not empty. Thus, X contains elements x, y, and z with $(y, z) \in f_{v'j}$,

$$(y, x) \in r_t^*, \quad \text{and} \quad (x, z) \in r_u.$$

We are assuming that $v' \in I_1(j)$. Thus, by Lemma 2.2(i), $f_{v'} \circ f_j = f_{v'j}$. Since $(y, z) \in f_{v'j}$, this implies that $(y, z) \in f_{v'} \circ f_j$. Thus, X contains an element z' with

$$(y, z') \in f_{v'} \quad \text{and} \quad (z', z) \in f_j.$$

Applying Lemma 9.1(i) to u, j, and z' in place of w, i, and y we obtain from $(z', z) \in f_j$ and $(x, z) \in r_u$ that

$$(x, z') \in r_{ju} \cup r_u.$$

(i) We are assuming that $t(ju)^{-1}$ is not a subelement of v'. Thus, by Lemma 9.2(i), $(x, z') \notin r_{ju}$. It follows that

$$(x, z') \in r_u.$$

From $(y, z') \in f_{v'}$, $(y, x) \in r_t^*$, and $(x, z') \in r_u$ we obtain that $f_{v'} \cap (r_t^* \circ r_u)$ is not empty.

(ii) We are assuming that tu^{-1} is not a subelement of v'. Thus, by Lemma 9.2(ii), $(x, z') \notin r_u$. It follows that

$$(x, z') \in r_{ju}.$$

From $(y, z') \in f_{v'}$, $(y, x) \in r_t^*$, and $(x, z') \in r_{ju}$ we obtain that $f_{v'} \cap (r_t^* \circ r_{ju})$ is not empty. □

Lemma 9.4

Assume that T is thick. Let t, u, and v' be elements in W, and let i and j be elements in I. Assume that $\{t, u\} \subseteq I_1(i)$ and that $v' \in I_1(j)$. Assume further that $ju = ui$, and that $f_{v'j} \cap (r_{ti}^* \circ r_{ui})$ is not empty. Then $t(ju)^{-1}$ and tu^{-1} both are subelements of v'.

Proof. We are assuming that $f_{v'j} \cap (r_{ti}^* \circ r_{ui})$ is not empty. Thus, X contains elements x, y, and z with $(y, z) \in f_{v'j}$,

$$(y, x) \in r_{ti}^*, \quad \text{and} \quad (x, z) \in r_{ui}.$$

We are assuming that $v' \in I_1(j)$. Thus, by Lemma 2.2(i), $f_{v'} \circ f_j = f_{v'j}$. Since $(y, z) \in f_{v'j}$, this implies that $(y, z) \in f_{v'} \circ f_j$. Thus, X contains an element z' with

$$(y, z') \in f_{v'} \quad \text{and} \quad (z', z) \in f_j.$$

We are assuming that $ju = ui$. Thus, as $(x, z) \in r_{ui}$, $(x, z) \in r_{ju}$. Moreover, since $(z', z) \in f_j$, we have $(z, z') \in f_j$. It follows that

$$(x, z') \in r_{ju} \circ f_j.$$

We are assuming that $u \in I_1(i)$. Thus, $\ell_I(ui) = \ell_I(u) + 1$. Since $ui = ju$, this implies that $\ell_I(ju) = \ell_I(u) + 1$. It follows that $(ju)^{-1} \in I_{-1}(j)$, so that, by Lemma 3.1(i),

$$r_{ju} \circ f_j = r_u.$$

From $(x, z') \in r_{ju} \circ f_j$ and $r_{ju} \circ f_j = r_u$ we obtain that $(x, z') \in r_u$. Thus, as $(y, x) \in r^*_{ti}$, we conclude that $(y, z') \in r^*_{ti} \circ r_u$. Now recall that $(y, z') \in f_{v'}$. Thus, $f_{v'} \cap (r^*_{ti} \circ r_u)$ is not empty.

Since $f_{v'} \cap (r^*_{ti} \circ r_u)$ is not empty, we obtain from Theorem 6.1 that tiu^{-1} is a subelement of v'. Since we are assuming that $ju = ui$, this implies that $t(ju)^{-1}$ is a subelement of v'.

We are assuming that T is thick and that $\{t, u\} \subseteq I_1(i)$. Thus, we obtain from Lemma 4.5 that $r^*_{ti} \circ r_u \subseteq r^*_t \circ r_u$. On the other hand, we have seen that $f_{v'} \cap (r^*_{ti} \circ r_u)$ is not empty. Thus, $f_{v'} \cap (r^*_t \circ r_u)$ is not empty, so that, by Theorem 6.1, tu^{-1} is a subelement of v'. □

Proposition 9.5

Assume that T is thick. Let t, u, and v be elements in W, and assume that tu^{-1} is an isolated subelement of v. Let i be an element in I with $\{t, u\} \subseteq I_1(i)$, and assume that $f_v \cap (r^*_{ti} \circ r_{ui})$ is not empty. Let x, y, and z be elements in X with $(y, z) \in f_v$, $(y, x) \in r^*_t$, and $(x, z) \in r_u$. Then X contains an element x' with $(x', x) \in f_i$, $(y, x') \in r^*_{ti}$, and $(x', z) \in r_{ui}$.

Proof. From Lemma 3.1 we know that $r_{t^{-1}} \subseteq r_{it^{-1}} \circ f_i$; equivalently, $r^*_t \subseteq r^*_{ti} \circ f_i$. Since we are assuming that $(y, x) \in r^*_t$, this implies that $(y, x) \in r^*_{ti} \circ f_i$. Thus, X contains an element x' with

$$(y, x') \in r^*_{ti} \quad \text{and} \quad (x', x) \in f_i.$$

Applying Lemma 9.1(ii) to u, z, x', and x in place of w, x, y, and z we obtain from $(x', x) \in f_i$ and $(x, z) \in r_u$ that

$$(x', z) \in r_{ui} \cup r_u.$$

We shall be done if we succeed in showing that $(x', z) \in r_{ui}$.

We proceed by induction with respect to $\ell_I(v)$.

If $\ell_I(v) = 0$, $v = 1$. Since we are assuming that $(y, z) \in f_v$, this implies that $y = z$ and that $t = u$. Thus, as $(y, x') \in r^*_{ti}$, we obtain that $(z, x') \in r^*_{ui}$; equivalently, $(x', z) \in r_{ui}$.

Assume that $1 \leq \ell_I(v)$. Then there exist elements v' in W and j in I such that

$$v = v'j \quad \text{and} \quad \ell_I(v) = \ell_I(v') + 1.$$

From $v = v'j$ and $\ell_I(v) = \ell_I(v') + 1$ we obtain that $v' \in I_1(j)$. Thus, as $v = v'j$, Lemma 2.2(i) yields that $f_{v'} \circ f_j = f_v$. Since we are assuming that $(y, z) \in f_v$, this implies that $(y, z) \in f_{v'} \circ f_j$, so that X contains an element z' with

$$(y, z') \in f_{v'} \quad \text{and} \quad (z', z) \in f_j.$$

We claim that $ju \neq ui$. Assume, by way of contradiction, that $ju = ui$. Then, as $\{t, u\} \subseteq I_1(i)$, $v' \in I_1(j)$, and $f_v \cap (r^*_{ti} \circ r_{ui})$ is assumed to be not empty, Lemma 9.4 yields that $t(ju)^{-1}$ and tu^{-1} both are subelements of v'. Since tu^{-1} is assumed to be an isolated subelement of v, this is impossible; cf. Lemma 5.5. Thus, we have shown that

$$ju \neq ui.$$

We claim that $ju \in I_1(i)$. If $u^{-1} \in I_{-1}(j)$, this follows from $u \in I_1(i)$; cf. [5; Lemma 2.3.8(ii)]. Assume that $u^{-1} \in I_1(j)$. Then, by [5; Lemma 6.5.2(i)], $j \in I_1(u)$. Thus, as $u \in I_1(i)$, $ju = ui$ or $ju \in I_1(i)$. Since $ju \neq ui$, this shows that

$$ju \in I_1(i).$$

We are assuming that tu^{-1} is an isolated subelement of v. Since tu^{-1} is a subelement of v, we obtain from Lemma 5.4 that one of the elements tu^{-1} and $t(ju)^{-1}$ is a subelement of vj.

Assume first that tu^{-1} is a subelement of v'. Then, by Lemma 5.6(i),

1. tu^{-1} is an isolated subelement of v'.

Since tu^{-1} is assumed to be a subelement of v', we obtain from Lemma 5.5 that $t(ju)^{-1}$ is not a subelement of v'. Recall also that $f_v \cap (r^*_{ti} \circ r_{ui})$ is assumed not to be empty. Thus, applying Lemma 9.3(i) to ti and ui in place of t and u we obtain that

2. $f_{v'} \cap (r^*_{ti} \circ r_{ui})$ is not empty.

From

$$(y, z') \in f_{v'}, \quad (z', z) \in f_j, \quad (y, x) \in r^*_t, \quad \text{and} \quad (x, z) \in r_u$$

together with the fact that $t(ju)^{-1}$ is not a subelement of v' we obtain that $(x, z') \notin r_{ju}$; cf. Lemma 9.2(i). Thus, applying Lemma 9.1(i) to u, j, and z' in place of w, i, and y we obtain from $(z', z) \in f_j$ and $(x, z) \in r_u$ that

3. $(x, z') \in r_u$.

Recall also that

4. $(y, z') \in f_{v'}$ and $(y, x) \in r^*_t$.

Thus, by **induction**, X contains an element x'' with

$$(x'', x) \in f_i, \quad (y, x'') \in r^*_{ti}, \quad \text{and} \quad (x'', z') \in r_{ui}.$$

From $(x', x) \in f_i$ and $(x'', x) \in f_i$ we obtain that

$$(x', x'') \in f_1 \quad \text{or} \quad (x', x'') \in f_i;$$

cf. Lemma 2.2.

Suppose that $(x', x'') \in f_i$. Then, as $(y, x') \in r_{ti}^*$, we have $(y, x'') \in r_{ti}^* \circ f_i$.

Since $t \in I_1(i)$, we have $ti \in I_{-1}(i)$. Thus, by Lemma 3.1(i), $r_{(ti)^{-1}} \circ f_i = r_{t^{-1}}$; equivalently, $r_{ti}^* \circ f_i = r_t^*$. Thus, as $(y, x'') \in r_{ti}^* \circ f_i$, we conclude that $(y, x'') \in r_t^*$, contrary to $(y, x'') \in r_{ti}^*$.

Thus, we have $(x', x'') \in f_1$ which means that $x' = x''$. Since $(x'', z') \in r_{ui}$, this implies that

$$(x', z') \in r_{ui}.$$

Since $(z', z) \in f_j$, we have $(z, z') \in f_j$. Applying Lemma 9.1(i) to ui, j, x', z, and z' in place of w, i, x, y, and z we obtain from $(z, z') \in f_j$ and $(x', z') \in r_{ui}$ that

$$(x', z) \in r_{jui} \cup r_{ui}.$$

Applying Lemma 9.1(ii) to u, z, x', and x in place of w, x, y, and z we obtain from $(x', x) \in f_i$ and $(x, z) \in r_u$ that

$$(x', z) \in r_{ui} \cup r_u.$$

Since $ju \neq ui$, $jui \notin \{ui, u\}$. Thus, $(x', z) \in r_{ui}$, so that we are done in this case.

Assume now that $t(ju)^{-1}$ is a subelement of v'. Then, by Lemma 5.6(ii),

1. $t(ju)^{-1}$ is an isolated subelement of v'.

Since $t(ju)^{-1}$ is assumed to be a subelement of v', we obtain from Lemma 5.5 that tu^{-1} is not a subelement of v'. Recall also that $f_v \cap (r_{ti}^* \circ r_{ui})$ is assumed not to be empty. Thus, applying Lemma 9.3(ii) to ti and ui in place of t and u we obtain that

2. $f_{v'} \cap (r_{ti}^* \circ r_{jui})$ is not empty.

From

$$(y, z') \in f_{v'}, \quad (z', z) \in f_j, \quad (y, x) \in r_t^*, \quad \text{and} \quad (x, z) \in r_u$$

together with the fact that tu^{-1} is not a subelement of v' we obtain that $(x, z') \notin r_u$; cf. Lemma 9.2(ii). Thus, applying Lemma 9.1(i) to u, j, and z' in place of w, i, and y we obtain from $(z', z) \in f_j$ and $(x, z) \in r_u$ that

3. $(x, z') \in r_{ju}$.

Recall also that

4. $(y, z') \in f_{v'}$, $(y, x) \in r_t^*$, and $ju \in I_1(i)$.

Thus, by **induction**, X contains an element x'' with

$$(x'', x) \in f_i, \quad (y, x'') \in r_{ti}^*, \quad \text{and} \quad (x'', z') \in r_{jui}.$$

From $(x', x) \in f_i$ and $(x'', x) \in f_i$ we obtain that

$$(x', x'') \in f_1 \quad \text{or} \quad (x', x'') \in f_i;$$

cf. Lemma 2.2.

Suppose that $(x', x'') \in f_i$. Then, as $(y, x') \in r_{ti}^*$, we have $(y, x'') \in r_{ti}^* \circ f_i$.

Since $t \in I_1(i)$, we have $ti \in I_{-1}(i)$. Thus, by Lemma 3.1(i), $r_{(ti)^{-1}} \circ f_i = r_{t^{-1}}$; equivalently, $r_{ti}^* \circ f_i = r_t^*$. Thus, as $(y, x'') \in r_{ti}^* \circ f_i$, we conclude that $(y, x'') \in r_t^*$, contrary to $(y, x'') \in r_{ti}^*$.

Thus, we have $(x', x'') \in f_1$ which means that $x' = x''$. Since $(x'', z') \in r_{jui}$, this implies that

$$(x', z') \in r_{jui}.$$

Since $(z', z) \in f_j$, we have $(z, z') \in f_j$. Applying Lemma 9.1(i) to jui, j, x', z, and z' in place of w, i, x, y, and z we obtain from $(z, z') \in f_j$ and $(x', z') \in r_{jui}$ that

$$(x', z) \in r_{ui} \cup r_{jui}.$$

Applying Lemma 9.1(ii) to u, z, x', and x in place of w, x, y, and z we obtain from $(x', x) \in f_i$ and $(x, z) \in r_u$ that

$$(x', z) \in r_{ui} \cup r_u.$$

Since $ju \neq ui$, $jui \notin \{ui, u\}$. Thus, $(x', z) \in r_{ui}$, so that we are done also in this case. □

Corollary 9.6

Assume that T is thick. Let t, u, and v be elements in W, and assume that tu^{-1} is an isolated subelement of v. Let i be an element in I, and assume that $f_v \cap (r^*_{ti} \circ r_{ui})$ is not empty. Then we have $f_v \cap (r^*_t \circ r_u) \subseteq f_v \cap (r^*_{ti} \circ r_{ui})$.

Proof. Assume first that $t \in I_{-1}(i)$ or $u \in I_{-1}(i)$. In this case, we know from Lemma 4.3(i) that $r^*_t \circ r_u \subseteq r^*_{ti} \circ r_{ui}$. It follows that $f_v \cap (r^*_t \circ r_u) \subseteq f_v \cap (r^*_{ti} \circ r_{ui})$, so that we are done in this case.

Assume now that $\{t, u\} \subseteq I_1(i)$. In this case, we let y and z be elements in X, and we assume that $(y, z) \in f_v \cap (r^*_t \circ r_u)$. We have to show $(y, z) \in r^*_{ti} \circ r_{ui}$.

Since $(y, z) \in r^*_t \circ r_u$, X contains an element x such that $(y, x) \in r^*_t$ and $(x, z) \in r_u$. Thus, as $(y, z) \in f_v$, X contains an element x' with $(x', x) \in f_i$, $(y, x') \in r^*_{ti}$, and $(x', z) \in r_{ui}$; cf. Proposition 9.5.

From $(y, x') \in r^*_{ti}$ and $(x', z) \in r_{ui}$ we obtain that $(y, z) \in r^*_{ti} \circ r_{ui}$. □

Theorem 9.7 [Third Structure Theorem]

Assume that T is thick. Let t, u, and v be elements in W, and assume that $f_v \cap (r^*_t \circ r_u)$ is not empty. Assume further that tu^{-1} is an isolated subelement of v. Then $f_v \subseteq r^*_t \circ r_u$.

Proof. We proceed by induction with respect to $\ell_I(u)$. If $\ell_I(u) = 0$, $u = 1$. In this case, the claim follows from Corollary 6.6. Therefore, we assume that $1 \leq \ell_I(u)$. In this case, I contains an element i with $u \in I_{-1}(i)$.

From $u \in I_{-1}(i)$ we obtain that $r^*_t \circ r_u \subseteq r^*_{ti} \circ r_{ui}$; cf. Lemma 4.3(i). Thus, as $f_v \cap (r^*_t \circ r_u)$ is assumed to be not empty, $f_v \cap (r^*_{ti} \circ r_{ui})$ is not empty. Moreover, since tu^{-1} is assumed to be an isolated subelement of v, $(ti)(ui)^{-1}$ is an isolated subelement of v. From $u \in I_{-1}(i)$ we also obtain that $\ell_I(ui) = \ell_I(u) - 1$. Thus, by induction,

$$f_v \subseteq r^*_{ti} \circ r_{ui}.$$

Since $(ti)(ui)^{-1}$ is an isolated subelement of v and $f_v \cap (r^*_t \circ r_u)$ is assumed not to be empty, we may apply Corollary 9.6 to ti and ui in place of t and u. We obtain that

$$f_v \cap (r^*_{ti} \circ r_{ui}) \subseteq f_v \cap (r^*_t \circ r_u).$$

From $f_v \cap (r^*_{ti} \circ r_{ui}) \subseteq f_v \cap (r^*_t \circ r_u)$ and $f_v \subseteq r^*_{ti} \circ r_{ui}$ we obtain that $f_v \subseteq r^*_t \circ r_u$. □

10. The Case Where the Sagittal has Length at Most 3

Considering Theorem 6.1 the following theorem is an immediate consequence of Theorem 9.7.

Theorem 10.1
Let t, u, and v be elements in W, and assume that $f_v \cap (r^*_t \circ r_u)$ is not empty. Assume further that $\ell_I(v) = |\mathrm{supp}(v)|$. Then $f_v \subseteq r^*_t \circ r_u$.

Proof. Since $\ell_I(v) = |\mathrm{supp}(v)|$, each subelement of v is isolated. On the other hand, since $f_v \cap (r_t \circ r_u)$ is assumed to be not empty, tu^{-1} is a subelement of v; cf. Theorem 6.1. Thus, by Theorem 9.7, $f_v \subseteq r_t \circ r_u$. □

Theorem 10.2 [Second Main Theorem]
Let t, u, and v be elements in W, and assume that $f_v \cap (r^*_t \circ r_u)$ is not empty. Assume further that $\ell_I(v) \leq 3$. Then $f_v \subseteq r^*_t \circ r_u$.

Proof. If $\ell_I(v) = 2$, we have $|\mathrm{supp}(v)| \leq 2$, so that the claim follows from Theorem 8.3. If $\ell_I(v) = 3$, we have $|\mathrm{supp}(v)| \leq 2$ or $\ell_I(v) = |\mathrm{supp}(v)|$. In the first case, the claim follows from Theorem 8.3, in the second case, the claim follows from Theorem 10.1. □

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