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Group Actions and Harmonic Analysis in Number Theory

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Abstract. This workshop focuses on new problems and new methods at the interface of harmonic analysis (taken in a very broad sense) and ergodic theory, with applications focused on number theory. Special emphasis is put on equidistribution problems on arithmetic symmetric spaces, effective methods in homogeneous dynamics, periods of automorphic forms, families of L-functions over number fields and function fields, and applications of Fourier uniqueness.

Mathematics Subject Classification (2020): 11Fxx, 11Mxx, 11T23, 22Exx, 37A17, 42Axx.

Introduction by the Organizers

The workshop "Group actions and harmonic analysis in number theory" was well attended by 55 participants (14 online) with a total of 16 female participants, ranging from finishing PhD students to senior faculty and including two Fields Medalists. We heard 23 excellent presentations.

Already the title indicates that the conference theme had a somewhat interdisciplinary flavour. It brought together leading researchers with backgrounds in number theory, harmonic analysis, ergodic theory and automorphic forms. The strong focus on recent breakthroughs at the intersection of these areas created a very stimulating atmosphere and many fruitful interactions that happened mostly during the afternoon and evening breaks.

Modern analytic number theory owes many of its most important breakthrough results to an eclectic variety of powerful methods ranging from harmonic analysis and ergodic theory to ℓ-adic cohomology. This workshop took a methodological perspective and focused on new developments that were made possible by a combination of analytic, ergodic and algebraic methods. The methodological richness is well reflected in the presentations, and in the following we highlight some important results that were reported on.

A recurring topic, viewed from different angles, was the statistical distribution of number theoretically relevant sequences. The first example concerns integer values of a positive binary quadratic form satisfying a diophantine condition. Mohammadi opened the conference by explaining how one can obtain the Poissonian pair correlation with effective error terms using quantitative equidistribution results for unipotent flows combined with ideas from the geometry of numbers. Marklof reported on a recent result on the fine scale distribution of roots of quadratic congruences modulo 1, combining algebraic ideas with techniques from dynamics. Radziwiłł contributed a beautiful new proof in the spirit of analytic number theory on the distribution of $\sqrt{n}$ modulo 1 with a number of extensions. Small values of certain quadratic sequences modulo 1 were considered by Lindenstrauss as an attractive corollary of an extremely powerful new classification result for measures invariant under any higher rank action of an irreducible arithmetic quotient of a form of SL(2) over a number field assuming positive entropy.

The concept of equidistribution was also featured prominently in talks centered on problems of Linnik type. These concern the equidistribution of special subvarieties of arithmetic significance inside certain ambient locally symmetric spaces, the most famous example being integer points on the 2-sphere. A new generation of such problems was put forward in the 2006 ICM address of Michel and Venkatesh which concern the mixing properties of such equidistribution problems. Brumley presented a new analytic approach based on period formulae and ideas from sieve theory to solve the mixing conjecture under GRH. Michel showed how a clever combination of number theoretic, ergodic and representation theoretic methods yields an unconditional proof of the mixing conjecture in the split case. Matz considered a very challenging generalization to higher rank featuring the group GSp(4), and reported on progress towards a mixing conjecture in this set-up.

In recent years, powerful tools have been developed to obtain equidistribution results with effective rates by ergodic methods which for a long seemed impossible in most situations. A key example in this direction is the work of Einsiedler, Margulis and Venkatesh from 2009. Wieser showed how to generalize this by removing some important assumptions. Strömbergsson obtained an effective rate of convergence for the asymptotic distribution of rational points on expanding closed horospheres. The proof features matrix Kloosterman sums and hence imports rather deep algebro-geometric input. A connection to Manin's conjecture was established by Horesh for effective equidistribution of primitive lattices and their orthogonal lattices.

While the results stated in the previous paragraphs used arithmetic, analytic and dynamical properties of locally symmetric spaces of finite volume, several presentations focused on the case of infinite volume. Oh considered in detail the bottom of the L^2-spectrum of higher rank infinite volume symmetric spaces and identified in many interesting cases the smallest Laplace eigenvalue. Pohl provided insight, heuristics and new results on counting quite accurately resonances in infinite volume situations via transfer operators, thereby connecting the spectral theory of hyperbolic surfaces with their dynamical properties. Kontorovich focused on beautiful arithmetic applications of infinite volume quotients in the context of Apollonian circle packings and presented new counting techniques for lattices and non-lattices.

A central topic at the interface of analytic number theory and automorphic forms is the investigation of analytic properties of L-functions. This is equally challenging and important in higher rank situations, and we had two presentations presenting important advances in this field. Harcos established a new zero-free region for Rankin-Selberg L-functions for arbitrary rank and provided a vast generalization of Siegel's classical theorem. Nelson presented his breakthrough work on subconvexity for $\mathrm{GL}(n)$ which is based on a combination of integral representations, deep methods to choose and analyze suitable test vectors, and counting problems that arise from the pretrace formula. The pretrace formula in a non-spherical situation was the main tool in Milićević's talk on the sup-norm problem in the dimension aspect which opened up a rather different view on mass equidistribution problems for eigenfunctions. Moving more into classical aspects of analytic number theory and the anatomy of integers, Matthiesen explained how linear equations in smooth numbers can be solved, based on a technique of Green and Tao and enhanced by input from modern analytic number theory. Soundararajan established sharp results on the number of principal quadratic forms needed to cover almost all integers. The latter problem was inspired by a discussion in a previous MFO workshop and the solution involves a careful study of the ideal class group of imaginary quadratic number fields.

That even classical partial differential equations can have a close connection to number theory was impressively presented by Hedenmalm and Pierce. Hedenmalm used the Klein-Gordan equation as a starting point to establish Fourier uniqueness in very interesting cases, while Pierce showed that the Weil bound for algebraic exponential sums has deep consequences on regularity properties of certain dispersive PDEs, an idea that goes famously back to Fields Medalist Bourgain.

The workshop was rounded off by talks on algebraic trace functions by Fresán, ergodic properties of flows on Veech surfaces by Burrin and new bounds for rational points close to manifolds whose curvature can be controlled presented by Schindler.

Thursday night featured an exciting and lively problem session. All participants confirmed that this was a very enjoyable and productive workshop.

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Workshop: Group Actions and Harmonic Analysis in Number Theory

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Abstracts

Quantitative equidistribution and spectrum of 2 dimensional tori

Amir Mohammadi
(joint work with E. Lindenstrauss and Z. Wang)

Let M be a two dimensional flat torus, i.e., $M = \mathbb{R}^2/\Delta$ where $\Delta \subset \mathbb{R}^2$ is a lattice. The eigenvalues of the negative Laplacian of M, which we list as

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots,$$

are the values of the quadratic form

$$B_M(x,y) = 4\pi^2 \|xv_1 + yv_2\|^2$$

at integer points, where $\{v_1, v_2\}$ is a basis for the dual lattice Δ^*.

The Weyl law implies

$$\#\{j : \lambda_j \leq T\} \sim \tfrac{\mathrm{vol}(M)}{4\pi} T.$$

Let $\alpha < \beta$, and define the *pair correlation function*

$$R_M(\alpha,\beta,T) = \frac{\#\{(j,k) : j \neq k, \lambda_j, \lambda_k \leq T,\ \alpha \leq \lambda_j - \lambda_k \leq \beta\}}{T}.$$

It is predicted by Berry and Tabor that, under the correct genericity condition on M, $R_M(\alpha,\beta,T)$ is the pair correlation density of a Poisson point process. Indeed this was proved by Sarnak [9] for almost every torus in the Moduli space of flat tori. Later, Eskin, Margulis, and Mozes [3] proved this result under explicit Diophantine conditions on the coefficients of the form B_M, see Theorem below. The case of inhomogeneous forms, which correspond to eigenvalues of quasi-periodic eigenfunctions, was also studied by Marklof [7, 6], and by Margulis and Mohammadi [8]. More recently, Blomer, Bourgain, Radziwill, and Rudnick [1] studied consecutive spacing for certain families of rectangular tori. See also the work of Strömbergsson and Vishe [10] where an effective version of [7] is obtained.

In [5], we prove the following theorem.

Theorem. *Let M be a two dimensional flat torus,*

$$B_M(x,y) = ax^2 + 2bxy + cy^2$$

the associated quadratic form giving the Laplacian spectrum of M normalized so that $ac - b^2 = 1$, and let $A \geq 10^3$. Then there are absolute constants δ_0 and N, some A' depending on A, and C and T_0 depending on A, a, b, and c, and for every $0 < \delta \leq \delta_0$, a $\kappa = \kappa(\delta, A)$ so that the following holds.

Let $T \geq T_0$, assume that for all $(p_1,p_2,q) \in \mathbb{Z}^3$ with $T^{\delta/A'} < q < T^\delta$ we have

$$\left|\tfrac{b}{a} - \tfrac{p_1}{q}\right| + \left|\tfrac{c}{a} - \tfrac{p_2}{q}\right| > q^{-A}.$$

Then if

$$|R_M(\alpha,\beta,T) - \pi^2(\beta-\alpha)| > C(1 + |\alpha| + |\beta|)^N T^{-\kappa},$$

then there are two primitive vectors $u_1, u_2 \in \mathbb{Z}^2$ *so that*

$$\|u_1\|, \|u_2\| \leq T^{\delta/A} \qquad \textit{and} \qquad |B_M(u_1, u_2)| \leq T^{-1+\delta}$$

and moreover

$$R_M(\alpha, \beta, T) - \pi^2(\beta - \alpha) = \frac{\mathcal{M}_T(u_1, u_2)}{T} + O\Big((1 + |\alpha| + |\beta|)^N T^{-\kappa}\Big)$$

with

$$\mathcal{M}_T(u_1, u_2) = \# \left\{ (\ell_1, \ell_2) \in \tfrac{1}{2}\mathbb{Z}^2 : \begin{array}{c} \ell_1 u_1 \pm \ell_2 u_2 \in \mathbb{Z}^2, \\ B_M(\ell_1 u_1 \pm \ell_2 u_2) \leq T, \\ 4B_M(u_1, u_2)\ell_1\ell_2 \in [\alpha, \beta] \end{array} \right\}.$$

The proof is based on recent effective equidistribution theorems for unioptent flows on homogeneous spaces [4], and tools from geometry of numbers á la [2, 3]. Note that the analysis in [2, 3] takes place in the space of unimodular lattice in $\mathbb{R}^n$, i.e., $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. However, the proof of Theorem exploits the structure of the problem which makes it possible to work in a quotient of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ by a certain finite index subgroup of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$. This makes the results in [4] applicable, moreover, it simplifies the required analysis of the cusp when compared with [3].

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Analytic approaches to some simultaneous equidistribution problems

Farrell Brumley
(joint work with Valentin Blomer and Ilya Khayutin)

Let $\mathbf{G} = \mathrm{PB}^\times$ be the projective unit group of a quaternion algebra B over a number field. The Linnik problems, solved in many cases by Duke [5] over thirty years ago, are concerned with the equidistribution of periodic torus orbits of large discriminant on certain (locally) homogeneous spaces associated with $\mathbf{G}$. Concrete examples over $\mathbb{Q}$ include the uniform distribution of integers points on the sphere and CM points on the modular surface. The original approach of Linnik [14] was ergodic theoretic, and required fixing an auxiliary prime p at which the tori $\mathbf{T}_E$ in the sequence are split. The automorphic approach of Duke removed this congruence condition, while obtaining an effective rate of convergence. The proof is based on a non-trivial bound on the Fourier coefficients of half-integral weight modular forms [11]. Alternatively, one can appeal to the formula of Waldspurger [17] and reduce the problem to subconvex bounds on an associated twisted L-function. The full resolution of the Linnik problems was achieved by Michel and Venkatesh [16], and marked a fruitful period of exchange between ergodic theory and automorphic forms.

There has been a great deal of recent interest in refining Duke's theorem, initiated by Bourgain–Rudnick–Sarnak [4]. One can, for example, investigate the pair correlation of integer points on spheres of large radius, or the variance for the number of such points in small spherical caps or annuli. Analytic approaches to these problems generally reduce to more subtle information on *families* of L-functions than the subconvex bounds on a single L-function that underlay the proof of Duke's theorem. Adding more evidence to this emerging viewpoint is our joint work with Blomer [2] and subsequently with Blomer and Khayutin [3], in which we have considered two higher rank extensions of Duke's theorem, of increasing difficulty, that were originally suggested by Michel and Venkatesh in their contribution to the 2006 Proceedings of the ICM [15]. Both problems are motivated by Furstenberg's notion of a *joining* of two dynamical systems [9].

In the first extension, one takes two groups $\mathbf{G}_1 = \mathrm{PB}_1^\times$ and $\mathbf{G}_2 = \mathrm{PB}_2^\times$ coming from non-isomorphic quaternion algebras $\mathrm{B}_1 \not\simeq \mathrm{B}_2$. The idea is to consider simultaneously the Duke problem coming from $\mathbf{G}_1$ with that coming from $\mathbf{G}_2$. Working over $\mathbb{Q}$, one could take, for example, the Hamiltonian quaternions for B_1 and the split matrix algebra for B_2. If one additionally imposes maximal level structure, the standard procedure of taking adelic double quotients yields the sphere S^2 and the modular surface $Y_0(1)$, respectively. With this particular choice of data, the question is then whether the collection of pairs

$$\{(d^{-1/2}v, [\Lambda_v]) : v = (a, b, c) \in \mathbb{Z}^3, \|v\|^2 = d\}$$

equidistributes on the product space $S^2 \times Y_0(1)$ as $d \to \infty$. Here, $[\Lambda_v]$ is the "shape" of the orthogonal lattice $\Lambda_v = v^\perp \cap \mathbb{Z}^3$ to the integer solution v, famously considered by Gauss [10]. The convergence to the product measure was first established by Aka–Einsiedler–Shapira [1], under the assumption of a double Linnik condition, relative to two auxiliary distinct primes $p_1, p_2 > 2$. Their proof makes critical use of the higher rank joinings theorem of Einsiedler–Lindenstrauss [6]. Returning to the more general setting, in our work with Blomer [2], we prove the equidistribution of periodic toric orbits of large discriminant, for the diagonally embedded torus $\Delta\mathbf{T}_E$ inside $\mathbf{G}_1 \times \mathbf{G}_2$ ($\mathbf{G}_1 \not\simeq \mathbf{G}_2$), without recourse to the double Linnik condition, but at the price of assuming GRH. The argument passes through a delicate estimation of a fractional moment of central L-values, twisted by class group characters, using ideas inspired by [13].

When $\mathbf{G}_1 = \mathbf{G}_2 = \mathbf{G}$, one can no longer hope for the periodic orbits associated with $\Delta\mathbf{T}_E$ to equidistribute in the homogeneous space associated with the product $\mathbf{G} \times \mathbf{G}$; indeed, as E varies, $\Delta\mathbf{T}_E$ will forever remain in the diagonally embedded copy $\Delta\mathbf{G}$. However, if one shifts $\Delta\mathbf{T}_E$ by a non-trivial coset $(1, s)$ in $\Delta\mathbf{T}_E\backslash(\mathbf{T}_E \times \mathbf{T}_E)$, one arrives at the *mixing conjecture* of Michel–Venkatesh [15]. Taking $\mathbf{G} = \mathrm{PGL}_2$ and E imaginary quadratic, this can be stated more classically in terms of the class group action on CM points on the modular curve as follows. For every fundamental discriminant $d \in \mathbb{N}$, we pick an ideal class $c \in \mathrm{Cl}_d$ and consider

$$\Delta\mathrm{CM}_d(c) := \{(x, c.x) : x \text{ a CM point on } Y_0(1) \text{ of discriminant } d\}\,.$$

If $c \neq 1$, the shifted set $\Delta\mathrm{CM}_d(c)$ is, happily, no longer trapped inside $\Delta Y_0(1)$. Note, however, that for every integral ideal $\mathfrak{n}$ representing the class c, $\Delta\mathrm{CM}_d(c)$ is now contained in the Hecke correspondence $Y_0(N)$, where $N = \mathrm{N}_{E/\mathbb{Q}}(\mathfrak{n})$. The latter are known to themselves equidistribute in $Y_0(1) \times Y_0(1)$ as N gets large. The mixing conjecture states that $\Delta\mathrm{CM}_d(c)$ should equidistribute in $Y_0(1) \times Y_0(1)$ as long as the sequence $\Delta\mathrm{CM}_d(c)$ is *strict*, in the sense that any given $Y_0(N)$ contains only finitely many elements in the sequence, as both d and c vary. In a major breakthrough, Khayutin [12] resolved the mixing conjecture under a double Linnik condition, along with a “no Siegel zero” assumption. His approach relied on previous work by Ellenberg–Michel–Venkatesh [8] in certain parameter ranges, and the joinings theorem of Einsiedler–Lindenstrauss in another. In subsequent work [3], Blomer, Khayutin, and I together removed the double Linnik condition, once again under the assumption of GRH. Our methods do not invoke measure rigidity, but do make use of many other tools: we combine the work of [2] with the classical analytic number theoretic tools appearing in [12], while giving a spectral theoretic version of [8] in a complementary parameter range.

Lastly, in work in progress with Blomer and Nordentoft, we consider a newly formulated *non-split* mixing conjecture, in which we replace the product group $\mathrm{GL}_{2,\mathbb{Q}} \times \mathrm{GL}_{2,\mathbb{Q}}$ with $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$, with F a real quadratic field extension of $\mathbb{Q}$, the diagonal torus $\Delta\mathbf{T}_E$ with a rank one (non-maximal) anisotropic $\mathbb{Q}$-torus $\mathbf{T}_E$ in $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$, and the shifting element $(1, s) \in \mathbf{T}_E \times \mathbf{T}_E$ with one coming from the quotient $\mathbf{T}_E\backslash\mathbf{T}_{EF}$, with $\mathbf{T}_{EF} \supset \mathbf{T}_E$ maximal anisotropic.

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Rigidity of non-maximal torus actions, unipotent quantitative recurrence, and Diophantine approximations

Elon Lindenstrauss

(joint work with Manfred Einsiedler)

We present a new argument in the study of positive entropy measures for higher rank diagonalisable actions. The argument relies on a quantitative form of recurrence along unipotent directions (that are not known to preserve the measure). Using this argument we prove a classification of positive entropy measures for any higher rank action on an irreducible arithmetic quotient of a form of SL_2.

This measure classification result allows us to give an Adelic measure classification result where no entropy assumption is needed. A special case of this is the following:

Theorem 1 (Adelic measure classification)**.** *Let $\mathbb{K}$ be a number field, $G = \mathrm{SL}_2(\mathbb{A}_\mathbb{K})$ and $\Gamma = \mathrm{SL}_2(\mathbb{K})$. Set*

$$X_{\mathbb{A}_\mathbb{K}} = \mathrm{SL}_2(\mathbb{K})\backslash \mathrm{SL}_2(\mathbb{A}_\mathbb{K}).$$

Let $\mathbb{T} < \mathrm{SL}_2$ be the group of diagonal matrices; we identify $\mathbb{T}(\mathbb{K})\backslash\mathbb{T}(\mathbb{A}_\mathbb{K})$ with its image in $X_{\mathbb{A}_\mathbb{K}}$. Let $\ell > 1$ be an integer and let

$$A = \left\{ \begin{pmatrix} k^\ell & \\ & k^{-\ell} \end{pmatrix} : k \in \mathbb{K}^\times \right\}.$$

Let μ be an A-invariant and ergodic probability measure on $X_{\mathbb{A}_\mathbb{K}}$. Then one of the following must hold:

(1) *μ is the uniform Haar measure on $X_{\mathbb{A}_\mathbb{K}}$.*
(2) *μ is the uniform Haar measure on the closed orbit $\Gamma\mathbb{U}(\mathbb{A}_\mathbb{K})a$, where $a \in \mathbb{T}(\mathbb{A}_\mathbb{K})$ and $\mathbb{U}$ is is one of the two unipotent $\mathbb{K}$-subgroups $\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & \\ * & 1 \end{smallmatrix}\right)$.*
(3) *There is a $x_0 \in \mathbb{T}(\mathbb{K})\backslash\mathbb{T}(\mathbb{A}_\mathbb{K})$ so that $\mu = \delta_{x_0}$.*

This implies a new result on diophantine approximations on the line:
Let $\alpha \in \mathbb{R}$ and let us denote by

$$\langle \alpha \rangle = \min_{m \in \mathbb{Z}} |\alpha - m|$$

the distance to the nearest integer. Then Dirichlet's theorem states that

$$Q \cdot \min_{q \in \mathbb{N} \cap [1,Q]} \langle q\alpha \rangle \leq 1$$

for all $Q \geq 1$. Moreover, for a.e. $\alpha \in \mathbb{R}$ Khintchin's theorem gives that this quantity approaches zero along a subsequence of Q. On the other hand there is a large set of numbers α, known as the set of badly approximable numbers, for which this quantity stays bounded away from zero.

In personal communications Bourgain asked whether this statement could be improved by allowing a denominator that is a product of two numbers $q_1, q_2 \leq Q$. In other words, what is the behavior of the following function

$$Q \in [0,\infty) \mapsto f(Q) = Q \cdot \min_{q_1,q_2 \in \mathbb{N}\cap[1,Q]} \langle q_1 q_2 \alpha \rangle$$

for an arbitrary $\alpha \in \mathbb{R}$?

If $Q + 1 = p$ is a prime number and $|\alpha - \frac{m}{p}| < \frac{1}{2}Q^{-3}$ for some nonzero $m \in \mathbb{Z}$, then $q_1q_2\alpha$ has distance less than $\frac{1}{2}Q^{-1}$ to $\frac{q_1q_2m}{p} \in \mathbb{Q}\backslash\mathbb{Z}$ and so distance at least $\gg Q^{-1}$ to the nearest integer. This implies $f(Q) \gg 1$. Using the Baire Category Theorem it is now easy to see that there exists a dense G_δ-set of real number α for which $\limsup_{Q\to\infty} f(Q) > 0$. For $\alpha = \sqrt{r}$ with r rational or for almost every α Blomer, Bourgain, Radzwiłł and Rudnick [1] show that $f(Q) \ll_\epsilon Q^{-1+\epsilon}$ which is essentially optimal.

Even thought one cannot guarantee that $\limsup_{Q\to\infty} f(Q) > 0$ for *all* $\alpha \in \mathbb{R}$, allowing products of denominators does give something: Theorem 2 implies that for every real α we have $f(e^t) \to 0$ "for almost all scales" $t \to \infty$. Indeed, we prove the following significantly stronger uniform statement:

Theorem 2 (An improved Dirichlet-type theorem). *For any $\varepsilon > 0$, and integer $\ell > 0$ there exists an $N \in \mathbb{N}$ so that for every $\alpha \in \mathbb{R}$, for every $Q > N$ there exist some integers $n \leq N$ and $q \leq Q$ for which*

$$\langle qn^{2\ell}\alpha\rangle \leq \min\left(\tfrac{\varepsilon}{q}, \tfrac{1+\varepsilon}{Q}\right)$$

We note that Theorem 2 is a strengthening of the following theorem [2, Thm. 1.11] by Einsiedler, Fishman, and Shapira.

Theorem 3 (Einsieder-Fishman-Shapira). *Let $\alpha \in \mathbb{R}$. Then*

$$\liminf_{n\to\infty} \liminf_{q\to\infty} q\langle qn\alpha\rangle = 0.$$

We also note that both Theorem 3 and Theorem 2 rely on adelic measure classification results. Theorem 3 uses [3, Thm. 1.4] by Lindenstrauss while Theorem 2 instead uses Theorem 1.

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Effective equidistribution of orbits of semisimple groups

Andreas Wieser

Let $X = \Gamma\backslash G$ be a congruence quotient of a semisimple group $G < \mathrm{SL}_N(\mathbb{R})$ defined over $\mathbb{Q}$ and let $H < G$ be a semisimple subgroup without compact factors. By a theorem of Ratner, H-invariant and ergodic measures on X are homogeneous and by subsequent work of Mozes-Shah limits of such measures are again homogeneous (or escape to infinity). Here, we aim to discuss effective versions of this equidistribution theorem.

Given a closed orbit xH, the rate of equidistribution (in X) is certainly never faster than the rate of equidistribution of orbits containing xH (under larger groups). The 'size' of such an intermediate orbit xM can be measured by its volume or, as we prefer here, by its arithmetic counterpart, the discriminant $\mathrm{disc}(xM)$.

Theorem. *There exists $\delta > 0$ depending only on G, H with the following property. Let $xH \subset X$ be a closed orbit and let $D > 0$ be such that* $\mathrm{disc}(xM) \geq D$ *for all proper periodic orbits $X \supset xM \supset xH$. Then for any $f \in C_c^\infty(X)$*

$$\Big|\int_X f - \int_{xH} f\Big| \ll_f D^{-\delta}.$$

One may significantly reduce the set of orbits for which to check $\mathrm{disc}(xM) \geq D$ (e.g. M reductive but not semisimple need not be considered). Also, the dependence of the implicit constant on the function f can be made completely explicit with a Sobolev norm in a large number of derivatives.

Work in this direction was initiated by Einsiedler, Margulis, and Venkatesh [2] who proved a slightly more general version of the above theorem assuming that H has finite centralizer in G. Subsequent works have removed the centralizer assumption in this breakthrough result in some special cases [1, 5], but not in full. The theorem above accomplishes this goal when one is interested in equidistribution in the ambient space X. In principle, one ought to obtain by the same method a statement as in the previous works [2, 1, 5] asserting in particular that the invariant measure on xH is polynomially close (in terms of $\mathrm{vol}(xH)$) to the invariant measure on the orbit of a larger group through x.

We remark that the analogue of the theorem for adelic periods would be more suitable for arithmetic applications such as the integral Hasse principle. Ongoing work with Einsiedler, Lindenstrauss, and Mohammadi aims to address this. Previous works for adelic periods consider mainly the case of maximal subgroups H [3, 4].

The proof of the above theorem follows in principle the scheme in [2] based on polynomial divergence for unipotent flows as well as an effective ergodic theorem. The effective ergodic theorem crucially uses uniformity of spectral gap for the action of H on $L^2(xH)$ where xH varies through all closed H-orbits (Clozel's property (τ)). The main new ingredient is an effective closing lemma established by Lindenstrauss, Margulis, Mohammadi, and Shah following their work in [6]. In particular, that closing lemma asserts that we can find a pair of points on xH whose displacement is transversal to a given proper Lie subalgebra containing $\mathrm{Lie}(H)$. Shearing these two points under a unipotent flow we either see sufficiently fast divergence or the displacement was almost centralized. In either case, one can obtain 'almost invariance' of the measure on xH if the points are Birkhoff generic in a suitable effective sense.

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Effective equidistribution of primitive rational points on expanding horospheres

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(joint work with Daniel El-Baz and Min Lee)

We prove an effective version of a result due to Einsiedler, Mozes, Shah and Shapira [2] on the equidistribution of primitive rational points on expanding closed horospheres in the space of lattices (see arXiv:2212.07408). Special cases of this result were obtained by Lee and Marklof [5] and by El-Baz, Huang and Lee [3].

In order to state the result, let $1 \le n < d$ (we also have a result in the case $n = d$, which is easier). Set $G = \mathrm{SL}_{d+n}(\mathbb{R})$, $\Gamma = \mathrm{SL}_{d+n}(\mathbb{Z})$ and $X = \Gamma\backslash G$. For each $q \in \mathbb{Z}^+$, write $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$, and call a matrix $R \in M_{d\times n}(\mathbb{Z}_q)$ *q-primitive* if its rows generate $(\mathbb{Z}_q)^n$. Let $\mathcal{R}_q$ be the set of q-primitive matrices. For each $R \in \mathcal{R}_q$, define

$$L_{R,q} := \Gamma \begin{pmatrix} q^{-n/d} I_d & R \\ 0 & qI_n \end{pmatrix} \in X.$$

(Here, by abuse of notation, the top right "R" really denotes a lift of R in $M_{d\times n}(\mathbb{Z})$; note that $L_{R,q}$ is independent of the choice of this lift.) Also set

$$H := \left\{ \begin{pmatrix} A & 0 \\ U & I_n \end{pmatrix} \,:\, A \in \mathrm{SL}_d(\mathbb{R}),\, U \in M_{n\times d}(\mathbb{R}) \right\}.$$

It turns out that $L_{R,q} \in \Gamma\backslash\Gamma H$ for all $R \in \mathcal{R}_q$, and the result by Einsiedler, Mozes, Shah and Shapira [2] states that as $q \to \infty$, the sequence

$$S_q := \left\{ \langle q^{-1}R, L_{R,q}\rangle \,:\, R \in \mathcal{R}_q \right\}$$

becomes equidistributed in $(\mathbb{R}/\mathbb{Z})^{dn} \times (\Gamma\backslash\Gamma H)$ with respect to the measure Leb $\times$ μ_H, where μ_H is the H-invariant probability measure on $\Gamma\backslash\Gamma H$.

Our main theorem is the following effective version of the above equidistribution result: For any C^k-function f on $(\mathbb{R}/\mathbb{Z})^{dn} \times (\Gamma\backslash\Gamma H)$, all of whose k first (Lie) derivatives are bounded,

$$\frac{1}{|\mathcal{R}_q|} \sum_{R\in\mathcal{R}_q} f(q^{-1}R, L_{R,q}) = \int_{(\mathbb{R}/\mathbb{Z})^{dn}} \int_{\Gamma\backslash\Gamma H} f(T,h)\, d\mu_H(h)\, dT$$
$$+ O\Big(S_{\infty,\kappa}(f)\, q^{-\vartheta} + S_{2,\kappa'}(f)\, q^{-\vartheta'}\Big),$$

for certain explicit positive constants $\kappa, \kappa', \vartheta, \vartheta'$. Here $S_{p,k}$ denotes the standard Sobolev L^p norm of order k on $(\mathbb{R}/\mathbb{Z})^{dn} \times (\Gamma\backslash\Gamma H)$. The papers [5] and [3] proved such effective results in the special case $n = 1$, but our result improves on those result both in terms of the power rate decay (ϑ and ϑ') and in terms of the Sobolev order (κ and κ') required in the error term.

Our proof makes use of recent bounds due to Erdélyi and Tóth giving bounds on so called matrix Kloosterman sums [4]; we also use Rogers' integration formula [6] in the geometry of numbers.

As a by-product of our proof, we obtain the following bound on a cardinality counting matrices modulo a prime p: Let $1 \le r \le n < d$. For any prime p and any $1 \le H \le \frac{p-1}{2}$, let

$$N_{p,H} = \left|\left\{X \in M_{d\times n}(\mathbb{Z}) \,:\, \|X\|_\infty \le H,\ \mathrm{rank}(X \bmod p) = r\right\}\right|.$$

Then

$$N_{p,H} \asymp \max\left(H^{dr}, \frac{H^{dn}}{p^{(d-r)(n-r)}}\right).$$

This asymptotic estimate complements an asymptotic formula due to Ahmadi and Shparlinski [1] of the form

$$N_{p,H} = \frac{(2H+1)^{dn}}{p^{(d-r)(n-r)}} + O(\cdots)$$

which is non-trivial in the case $H \ge p^\gamma$ for a certain exponent $\gamma = \gamma(r,n,d) > \frac{1}{2}$. (Note that our bound on $N_{p,H}$ is valid also for significantly smaller H.)

We also discuss an application on counting small solutions to a system of linear congruences, where we prove a variant of a result by Strömbergsson and Venkatesh [7].

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Lattices: Counting and Equidistribution

Tal Horesh

(joint work with Y. Karasik)

An integer vector $v = (a_1, \dots, a_n) \in \mathbb{Z}^n$ is called *primitive* if $\gcd(a_1, \dots, a_n) = 1$; let $\mathbb{Z}^n_{\mathrm{prim}}$ denote the set of all primitive vectors in $\mathbb{R}^n$. Recalling the Gauss Circle Problem, concerning the asymptotics of the number of integer vectors that lie in a ball of radius $R > 0$ as $R \to \infty$, the first natural counting problem concerning primitive vectors would be:

The primitive sphere problem. What is the asymptotics of

$$\#(\mathbb{Z}^n_{\mathrm{prim}} \cap \{x \in \mathbb{R}^n : \|x\| \le R\}),$$

where $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$, $x = (x_1, \ldots, x_n)$. The main term is known to be

$$\frac{1}{\zeta(n)} \operatorname{vol}(\{x \in \mathbb{R}^n : \|x\| \le R\}) = \frac{1}{\zeta(n)} \operatorname{vol}(\{x \in \mathbb{R}^n : \|x\| \le 1\}) R^n,$$

where $\operatorname{vol}(\cdot)$ stands for the Lebesgue measure on $\mathbb{R}^n$, and the optimal main term is identical to the one for the "standard" sphere problem when $n \ge 4$ (and conjecturally also for $n = 2, 3$). The primitive sphere problem can be refined as follows. Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, and let $\Phi \subseteq \mathbb{S}^{n-1}$ be a "nice" measurable subset. Then one may ask, how many primitive vectors are there in the ball of radius R that project radially to Φ, namely what is the asymptotics of

$$\#(\mathbb{Z}^n_{\mathrm{prim}} \cap \{x \in \mathbb{R}^n : \|x\| \le R,\ \frac{x}{\|x\|} \in \Phi\}).$$

As expected, the main term becomes

$$\frac{1}{\zeta(n)} \operatorname{vol}(\Phi) \operatorname{vol}(\{x \in \mathbb{R}^n : \|x\| \le 1\}) R^n,$$

where we abuse notation and use $\operatorname{vol}(\cdot)$ also for the Lebesgue measure on $\mathbb{S}^{n-1}$. In what follows, we will use it for all the invariant measures on the subspaces in question. The "refined" primitive sphere problem relates to the distribution of the discrete set $\{v/\|v\| : v \in \mathbb{Z}^n_{\mathrm{prim}}, \|v\| \le R\}$ in $\mathbb{S}^{n-1}$ as $R \to \infty$.

The orthogonal lattice. Let us consider another counting question related to primitive vectors. For a primitive vector v, let $v^\perp$ denote the orthogonal hyperplane to v, and define the *orthogonal lattice* to v as

$$\Lambda_v = \mathbb{Z}^n \cap v^\perp.$$

Notice that the rank of Λ_v is $n-1$. When aiming to study the distribution of Λ_v, it is rather standart to look at its *shape*, denoted $[\Lambda_v]$, which is the equivalence class of all lattices of rank $n-1$ modulo rescaling and rotation. Explicitely,

$$[\Lambda_v] = \mathrm{SO}(n-1) g \operatorname{SL}_{n-1}(\mathbb{Z}) \in \mathrm{SO}(n-1)\backslash \operatorname{SL}_{n-1}(\mathbb{R}) / \operatorname{SL}_{n-1}(\mathbb{Z}) = \mathcal{L}_{n-1},$$

where $g \in \operatorname{SL}_{n-1}(\mathbb{R})$ is a matrix whose columns span the covolume 1 lattice $(\operatorname{cov}\Lambda_v)^{\frac{1}{n-1}} \Lambda_v$. Here, one might ask what is the asymptotics of

$$\#(\{v \in \mathbb{Z}^n_{\mathrm{prim}} : \|v\| \le R,\ \frac{v}{\|v\|} \in \Phi,\ [\Lambda_v] \in \mathcal{E}\}), \tag{1}$$

where $\mathcal{E} \subseteq \mathcal{L}_{n-1}$ is some "nice" measurable subset. As one would expect, the main term is

$$\frac{1}{\zeta(n)} \operatorname{vol}(\Phi) \operatorname{vol}(\mathcal{E}) \operatorname{vol}(\{x \in \mathbb{R}^n : \|x\| \le 1\}) R^n,$$

which has been established in several works (e.g. [11, 10, 1], see also [5, 6, 8, 7] for the more delicate case of equidistribution as $\|v\| = R$).

From primitive vectors to primitive lattices. The next step after primitive vectors is *primitive lattices*; given $1 \le d \le n$, a d-lattice in $\mathbb{R}^n$ is a discrete subgroup of rank d, namely

$$\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_d$$

where $v_1, \ldots, v_d \in \mathbb{R}^n$ are linearly independent over $\mathbb{R}$. We say that a d-lattice Λ is *primitive* if $\Lambda = V_\Lambda \cap \mathbb{Z}^n$ where $V_\Lambda = \operatorname{span}_{\mathbb{R}}(\Lambda)$. Let $\mathcal{P}(d,n)$ denote the set of primitive d-lattices in $\mathbb{R}^n$. The primitive sphere problem naturally generalizes to the question of how many primitive d-lattices are there of covolume up to R as $R \to \infty$, which was answered by Schmidt [4] as follows:

$$\#\{\Lambda \in \mathcal{P}(d,n) : \operatorname{cov}\Lambda \le R\} = c_{d,n}R^n + O(R^{n-\max\{\frac{1}{d},\frac{1}{n-d}\}})$$

where $c_{d,n}$ is an explicit constant. Given a d-lattice Λ, its orthogonal lattice is $\Lambda^\perp = \mathbb{Z}^n \cap V_\Lambda^\perp$, where $V_\Lambda^\perp$ is the orthogonal complement of V_Λ in $\mathbb{R}^n$. Notice that there is a $1:2$ correspondence between primitive 1-lattices and primitive vectors, given by $\mathbb{Z}v \longleftrightarrow \pm v$ and so the $d > 1$ analog of the counting problem in (1) becomes estimating

$$\#(\{\Lambda \in \mathcal{P}(d,n) : \operatorname{cov}\Lambda \le R,\ V_\Lambda \in \Phi,\ [\Lambda] \in \mathcal{E},\ [\Lambda^\perp] \in \mathcal{E}'\}),$$

where $\Phi \subseteq \operatorname{Gr}(d,n)$ (the grassmannian of d-dimensional subspaces in $\mathbb{R}^n$), $\mathcal{E} \subseteq \mathcal{L}_d$ and $\mathcal{E}' \subseteq \mathcal{L}_{n-d}$. This question was solved in a quantitative manner in [2] (and later refined in [9]). The result which is the topic of the present talk is a generalization of the above to lattices in flag varieties.

Rational points in flag varieties. Since there is a $1:1$ correspondence between the elements of $\mathcal{P}(d,n)$ and the rational subspaces in $\operatorname{Gr}(d,n)$,

$$\Lambda \mapsto V_\Lambda$$
$$\mathbb{Z}^n \cap V \mapsfrom V,$$

then the primitive lattices are the rational points in the projective variety $\operatorname{Gr}(d,n)$. The guiding conjecture in the area of counting rational points in projective variety is due to Manin and is given by:

Conjecture. *Let X be a Fano variety of dimension m defined over $\mathbb{Q}$. Then there exists a thin set Z such that*

$$\#(\{x \in X(\mathbb{Q}) \setminus Z\} : H(x) \le B) \sim c_X B P(\log B)$$

where $X(\mathbb{Q})$ is the set of rational points in X, P is a polynomial of degree $r_X - 1$ (r_X is the rank of the Picard group of X), $c_X > 0$ is a constant that depends on the variety and $H : X(\mathbb{Q}) \to \mathbb{R}_{>0}$ is the anticanonical height function.

In homogeneous varieties $Z = \emptyset$, so Schmidt's theorem confirms Manin's conjecture for the Grassmannian $\operatorname{Gr}(d,n)$, where $H(V_\Lambda) = \operatorname{cov}\Lambda^n$. Franke, Manin and Tchinkel [3] proved this conjecture for the more general case of flag varieties. Given a flag of rational subspaces of $\mathbb{R}^n$,

$$\mathbf{F} = (\{0\} = V_0 < V_1 < \cdots < V_{\ell-1} < V_\ell = \mathbb{R}^n),$$

we consider the corresponding flag of primitive lattices

$$\mathbf{F}(\mathbb{Z}) = (\{0\} = \Lambda_0 < \Lambda_1 < \cdots < \Lambda_{\ell-1} < \Lambda_\ell = \mathbb{Z}^n). \tag{2}$$

Since the lattices Λ_i are primitive, the quotients Λ_{i+1}/Λ_i are free, and are therefore lattices in the corresponding quotient spaces. Let

$$d_i = \operatorname{rank}(\Lambda_{i+1}/\Lambda_i)$$

for all $i = 0, \ldots, \ell - 1$. If X is the variety consisting of all the flags of subspaces of dimensions

$$\underline{d} = (0, d_1, d_1 + d_2, \ldots, \sum d_i = n),$$

then the rational points are flags of rational subspaces, and these rational flags $\mathbf{F}$ correspond to flags of rational lattices $\mathbf{F}(\mathbb{Z})$. The anticanonical height function on $X(\mathbb{Q})$ is given by

$$H(\mathbf{F}) = \prod_{i=0}^{\ell-1} \operatorname{cov} \Lambda_i{}^{d_i + d_{i+1}}.$$

The result presented in the talk is the following.

Theorem 1 (H-Karasik, 2022)**.** *Let $\underline{d} = (d_1, \ldots, d_\ell)$ be a vector of positive integers such that $d_1 + \cdots + d_\ell = n$, and let X the variety of all $\underline{d}$-flags. Let $\mathcal{E}_i \subseteq \mathcal{L}_i$ and $\Phi \subseteq X$ by subsets with C^1 boundary. Then, the number of $\underline{d}$-flags of primitive lattice (2) of height at most B and with $\mathbf{F} \in \Phi$ and $[\Lambda_{i+1}/\Lambda_i] \in \mathcal{E}_i$ is*

$$c_{\underline{d}} \operatorname{vol}(\Phi) \prod_{i=0}^{\ell-1} \operatorname{vol}(\mathcal{E}_i) \cdot B \sum_{i=0}^{\ell-2} \frac{(-1)^{\ell-2-i}}{i!} (\log B)^i + O(B^{1-\delta})$$

where $c_{\underline{d}}$ and δ are explicit positive constants.

The above theorem is an extension of the result of Franke, Manin and Tchinkel so that it includes equidistribution of the rational points in X (because of the restriction to Φ), equidistribution of the shapes of the quotient lattices, the explicit polynomial in the main term, and a bound on the error term.

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A new zero-free region for Rankin–Selberg L-functions

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(joint work with Jesse Thorner)

Establishing zero-free regions for automorphic L-functions is a central problem of number theory. We report about the recent preprint of the same title [3], which extends Siegel's celebrated lower bound for Dirichlet L-functions [15] to all GL_1-twists of $\mathrm{GL}_n \times \mathrm{GL}_{n'}$ Rankin–Selberg L-functions. The result is meant over an arbitrary number field, but for simplicity we restrict the present summary to the rational field.

Let $\mathfrak{F}_n$ be the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. Let $\mathfrak{F}_n^* \subset \mathfrak{F}_n$ be the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ whose central character is trivial on $\mathbb{R}_{>0}$. In particular, $\mathfrak{F}_1$ is the group of unitary Hecke characters of $\mathbb{A}_\mathbb{Q}^\times \cong \mathbb{R}_{>0} \times \mathbb{A}_\mathbb{Q}^1$, which acts on $\mathfrak{F}_n$ via

$$(\pi \otimes \chi)(g) = \pi(g)\chi(\det g).$$

It follows that, for each $\pi \in \mathfrak{F}_n$, there is a unique pair $(t_\pi, \pi^*) \in \mathbb{R} \times \mathfrak{F}_n^*$ satisfying

$$\pi = \pi^* \otimes |\cdot|^{it_\pi}, \qquad L(s,\pi) = L(s + it_\pi, \pi^*).$$

Moreover, for all $(\pi, \pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$,

$$L(s, \pi \times \pi') = L(s + it_\pi + it_{\pi'}, \pi^* \times \pi'^*).$$

The point of this normalization is that $L(s,\pi^*)$ and $L(s,\pi^* \times \pi'^*)$ can only have a pole at $s = 1$, hence $L(s,\pi)$ and $L(s,\pi\times\pi')$ can only have a pole at $s = 1 - it_\pi$ and $s = 1 - it_\pi - it_{\pi'}$, respectively. Finally, we shall denote the analytic conductor of $L(s,\pi)$ by $C(\pi)$.

The classical results of de la Vallée Poussin [12] and Siegel [15] have been extended to the standard L-functions $L(s,\pi)$ considered above. Important milestones include Jacquet–Shalika [8], Moreno [11], and Hoffstein–Ramakrishnan [4]. See Brumley [5, Theorem A.1] and Jiang–Lü–Thorner–Wang [9, Section 4] for important recent results.

According to the Langlands functoriality conjecture, the Rankin–Selberg L-function $L(s,\pi\times\pi')$ equals a product of standard L-functions $L(s,\Pi)$. However, we are far from knowing this, and correspondingly, we understand the analytic properties of Rankin–Selberg L-functions much less than those of standard L-functions. Concerning non-vanishing, Shahidi [14, Theorem 5.2] proved that $L(s,\pi\times\pi') \neq 0$ for $\Re(s) \geq 1$. The first uniform improvement over this basic result is due to

Brumley (see [1] and [10, Theorem A.1]); we display slightly weaker exponents for notational simplicity:

Theorem 1. *There exists a constant $c_1 = c_1(n,n') > 0$ with the following property. If $(\pi,\pi') \in \mathfrak{F}_n^* \times \mathfrak{F}_{n'}^*$, then $L(s,\pi\times\pi')$ has no zero in the region*

$$\Re(s) \geq 1 - c_1(C(\pi)C(\pi'))^{-n-n'}(|\mathrm{Im}(s)|+1)^{-nn'}.$$

The classical results of de la Vallée Poussin [12] and Siegel [15] have also been extended to special Rankin–Selberg L-functions. Important milestones include Moreno [11], Sarnak [13], Goldfeld–Li [2], and Humphries [5]. As a combination of Brumley [5, Theorem A.1] and Humphries–Thorner [7, Theorem 2.1], we have

Theorem 2. *There exists $c_2 = c_2(n,n') > 0$ with the following property. If $(\pi,\pi') \in \mathfrak{F}_n^* \times \mathfrak{F}_{n'}^*$ satisfies $\pi = \tilde{\pi}$ or $\pi' = \tilde{\pi}'$ or $\pi' = \tilde{\pi}$, then $L(s,\pi\times\pi')$ has at most one zero β (necessarily real and simple) in*

$$\Re(s) \geq 1 - c_2/\log(C(\pi)C(\pi')(|\mathrm{Im}(s)|+3)).$$

If the exceptional zero β exists, then $(\pi,\pi') = (\tilde{\pi},\tilde{\pi}')$ or $\pi' = \tilde{\pi}$.

The next result we highlight is due to Humphries–Thorner [6, Theorem 2.4].

Theorem 3. *For every $\pi \in \mathfrak{F}_n^*$ and $\varepsilon > 0$, there exists $c_3 = c_3(\pi,\varepsilon) > 0$ with the following property. If $\chi \in \mathfrak{F}_1^*$ is quadratic, then*

$$L(\sigma, \pi\otimes(\tilde{\pi}\otimes\chi)) \neq 0, \qquad \sigma \geq 1 - c_3 C(\chi)^{-\varepsilon}.$$

In the recent preprint [3], we extended Siegel's celebrated lower bound for Dirichlet L-functions [15] to all GL_1-twists of $L(s,\pi\times\pi')$. The proof relies crucially on the group structure of $\mathfrak{F}_1$, not just $\mathfrak{F}_1^*$. As a result, we substantially improved on Brumley's lower bound in the GL_1-twist aspect, but the dependence on (π,π') is no longer effective.

Theorem 4. *Let $(\pi,\pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$. For all $\varepsilon > 0$, there exists an ineffective constant $c_4 = c_4(\pi,\pi',\varepsilon) > 0$ such that if $\chi \in \mathfrak{F}_1$, then*

$$|L(\sigma,\pi\times(\pi'\otimes\chi))| \geq c_4 C(\chi)^{-\varepsilon}, \qquad \sigma \geq 1 - c_4 C(\chi)^{-\varepsilon}.$$

In particular, there exists $c_5 = c_5(\pi,\pi',\varepsilon) > 0$ such that

$$|L(\sigma+it,\pi\times\pi')| \geq c_5(|t|+1)^{-\varepsilon}, \qquad \sigma \geq 1 - c_5(|t|+1)^{-\varepsilon}.$$

The new zero-free region allows us to prove an analogue of the Siegel–Walfisz theorem for Rankin–Selberg L-functions.

Theorem 5. *For $(\pi,\pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$, let $\Lambda_{\pi\times\pi'}(m)$ denote the m-th Dirichlet coefficient of $-L'(s,\pi\times\pi')/L(s,\pi\times\pi')$, and let*

$$\mathcal{M}_{\pi\times\pi'}(x) = \begin{cases} x^{1-iu}/(1-iu), & \pi' = \tilde{\pi}\otimes|\cdot|^{iu}; \\ 0, & \textit{otherwise.} \end{cases}$$

If $q \le (\log x)^A$ *is a positive integer coprime to the conductors of* π *and* π'*, and* $\gcd(a,q) = 1$*, then*

$$\sum_{\substack{m \le x \\ m \equiv a \pmod q}} \Lambda_{\pi\times\pi'}(m) = \frac{\mathcal{M}_{\pi\times\pi'}(x)}{\varphi(q)} + O_{\pi,\pi',A}\left(\frac{x}{(\log x)^A}\right).$$

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Beyond the spherical sup-norm problem

Djordje Milićević

(joint work with Valentin Blomer, Gergely Harcos, Péter Maga)

The sup-norm problem on arithmetic Riemannian manifolds occupies a prominent place at the intersection of harmonic analysis, number theory, and quantum mechanics. It asks about the sup-norm of L^2-normalized joint eigenfunctions of invariant differential operators and Hecke operators, most classically in terms of their Laplace eigenvalues (as in the QUE problem for high-energy eigenstates), but also in terms of the volume of the manifold and other parameters; see [5, 3, 7, 1] for a sample of this rich landscape. The sup-norm problem sheds light on how much high-energy eigenstates can localize ("scarring") and, in its various incarnations, has connections to the multiplicity problem, zero sets and nodal lines of automorphic functions, bounds for Faltings' delta function, and more.

In the paper presented [2] we open a new perspective on the sup-norm problem and propose a version of higher complexity. The sup-norm problem makes perfect sense not only on the level of symmetric spaces, but also on the level of groups. For an irreducible unitary representation of a maximal compact subgroup $K \subset G$ on some finite-dimensional complex vector space V^τ, a cross-section of the homogeneous vector bundle over G/K defined by τ may be identified with a vector-valued function $f : G \to V^\tau$ which transforms on the right by K with respect to τ:

$$f(gk) = \tau(k^{-1})f(g), \quad g \in G,\ k \in K.$$

Then it is interesting to bound the sup-norm of (an L^2-normalized) f or, more delicately, its components as the *dimension* of V^τ gets large. In our paper, we present a detailed investigation of the first nontrivial case $G = \mathrm{SL}_2(\mathbb{C})$, taking for concreteness $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$.

A complementary viewpoint is provided by representation theory. Nontrivial irreducible unitary representations of G are principal series representations $\pi_{\nu,p}$ indexed by certain pairs $(\nu, p) \in \mathfrak{a}^*_{\mathbb{C}} \times \frac{1}{2}\mathbb{Z}$, with the corresponding irreducible representation space V of G decomposing as a Hilbert space orthogonal sum

$$V = \bigoplus_{\substack{\ell \geqslant |p| \\ \ell \equiv p \,(\mathrm{mod}\, 1)}} V^\ell = \bigoplus_{\substack{\ell \geqslant |p| \\ \ell \equiv p \,(\mathrm{mod}\, 1)}} \bigoplus_{\substack{|q| \leqslant \ell \\ q \equiv \ell \,(\mathrm{mod}\, 1)}} \mathbb{C}\phi_{\ell,q}.$$

Here, $\ell \in \frac{1}{2}\mathbb{Z}_{\ell \geqslant 0}$ parametrizes the K-type, that is, the $(2\ell + 1)$-dimensional representation τ_ℓ of K such that $V^\ell = V^{\tau_\ell}$, and the diagonal matrices $\mathrm{diag}(e^{i\varrho}, e^{-i\varrho}) \in K$ act on the L^2-normalized $\phi_{\ell,q}$ by $e^{2qi\varrho}$. Given a discrete, finite covolume subgroup $\Gamma \leqslant G$, the non-spherical sup-norm problem asks, for every $V \simeq \pi_{\nu,p}$ appearing in $L^2(\Gamma\backslash G)$, about the size of

$$\|\phi_{\ell,q}\|_\infty \quad \text{and} \quad \|\Phi_\ell\|_\infty, \quad \Phi_\ell = \Bigg(\sum_{|q| \leqslant \ell} |\phi_{\ell,q}|^2 \Bigg)^{1/2}.$$

Here we are interested in the case of large $p = \ell$ ("lowest weight/new vectors" in a high-dimensional K-type) and $\nu \ll 1$, a completely new problem for which we establish essentially ("up to ε") the most optimistic baseline estimates for the maximum size on a fixed compact set $\Omega \subset G$:

$$\|\Phi_\ell|_\Omega\|_\infty \ll \ell^{3/2}, \quad \|\phi_{\ell,q}|_\Omega\|_\infty \ll_\varepsilon \ell^{1+\varepsilon},$$

noting that especially the latter one is not trivial in any sense other than that no arithmetic input is required.

Our main results are the following power-saving bounds in the high-dimensional sup-norm problem on the arithmetic quotient $\Gamma\backslash G = \mathrm{SL}_2(\mathbb{Z}[i])\backslash\mathrm{SL}_2(\mathbb{C})$.

Theorem 1. *Let $\ell \geqslant 1$ be an integer, $I \subset i\mathbb{R}$ and $\Omega \subset G$ be compact sets. Then, for every cuspidal automorphic representation $V \subset L^2(\Gamma\backslash G)$ with minimal K-type τ_ℓ and spectral parameter $\nu_V \in I$,*

$$\|\Phi_\ell|_\Omega\|_\infty \ll_{\varepsilon,I,\Omega} \ell^{4/3+\varepsilon}.$$

Theorem 2. *Under the assumptions of Theorem 1,*

$$\max_{|q|\leqslant \ell} \|\phi_{\ell,q}|_\Omega\|_\infty \ll_{\varepsilon,I,\Omega} \ell^{26/27+\varepsilon}.$$

Theorem 3. *Under the assumptions of Theorem 1:*

- *For $q = 0$, we have*

$$\|\phi_{\ell,0}|_\Omega\|_\infty \ll_{\varepsilon,I,\Omega} \ell^{7/8+\varepsilon}.$$

- *Assuming additionally that V lifts to an automorphic representation for $\mathrm{PGL}_2(\mathbb{Z}[i])\backslash\mathrm{PGL}_2(\mathbb{C})$, for $q = \pm\ell$ we have*

$$\|\phi_{\ell,\pm\ell}|_\Omega\|_\infty \ll_{\varepsilon,I,\Omega} \ell^{1/2+\varepsilon}.$$

We get a handle on the pointwise values of $|\phi(g)|$ via the amplified pre-trace formula, with the smooth weight function and the amplifier chosen so as to emphasize the contributions of our chosen automorphic form(s) ϕ. To analyze the corresponding weights (that is, the test function) on the geometric side, we require an understanding of the image and the inverse of the spherical transform given for f in the τ_ℓ-isotypical subspace $\mathcal{H}(\tau_\ell) \subset L^2(G)$ by

$$\mathcal{H}(\tau_\ell) \to \mathbb{C} \times \tfrac{1}{2}\mathbb{Z}, \quad \widehat{f}(\nu,p) = \int_G f(g)\varphi^\ell_{\nu,p}(g)\,\mathrm{d}g,$$

with the spherical trace function $\varphi^\ell_{\nu,p}(g) = \mathrm{tr}(\pi_{\nu,p}(\overline{\chi_\ell})\pi_{\nu,p}(g)\pi_{\nu,p}(\overline{\chi_\ell}))$. We build on the L^2 theory of Gelfand–Naimark [4] and the work of Wang [9] to develop a Schwartz class Paley–Wiener theorem that shows that the spherical transform is an isomorphism between the class of all infinitely differentiable $f \in \mathcal{H}(\tau_\ell)$ with all derivatives rapidly decreasing and the class of all functions $h(\nu,p)$ holomorphic on $\mathbb{C} \times \frac{1}{2}\mathbb{Z}$, satisfying the Weyl group symmetry $h(\nu,p) = h(-\nu,-p)$, the "extra symmetry" $h(\nu,p) = h(p,\nu)$ when $\nu \equiv p \pmod 1$, $|\nu|,|p| \leqslant \ell$, and rapidly decreasing in every vertical strip $|\mathfrak{Re}\,\nu| \leqslant B$.

To understand concentration on the geometric side after the inverse spherical transform, we prove estimates on the spherical trace functions $\varphi^\ell_{\nu,\ell}(g)$ and (for the individual Wigner components) the generalized spherical trace functions $\varphi^{\ell,q}_{\nu,\ell}(g)$. These estimates are at the analytic core of our work and involve elaborate stationary phase analysis as well as arguments from classical inequalities and combinatorics. We show that the (generalized) spherical trace functions concentrate close to various submanifolds, including the identity, the centralizer of A in the maximal compact subgroup $K = \mathrm{SU}(2)$, the subgroup of diagonal matrices in G, the maximal compact subgroup K itself, or the set of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $|a| = |d|$ and $|b| = |c|$. This in turn gives rise to the number-theoretic problem of Diophantine nature of counting matrices (or pairs of matrices) of determinant n such that (for a fixed but arbitrary $g \in \Omega$) $g^{-1}(\gamma/\sqrt{n})g$ is close to one of these distinguished submanifolds, in a precise quantitative sense. Solving this explicitly leads us to counting problems for lattice points in $\mathbb{C}^4$ or $\mathbb{C}^8$ in intersections of (randomly centered) small balls, "thin wafers", and "thin cylinders".

Finally, in the case $q = \pm\ell$, we go beyond the amplified pre-trace formula, motivated by the recent brilliant idea of Steiner and Khayutin–Steiner in the weight aspect on $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ [6, 8]. The very strong and uniform localization in the Harish-Chandra transform and the corresponding counting problem allow us to take the amplifier so long that it works in effect as a fourth moment that can be realized as the diagonal term in the double pre-trace formula. We avoid the theta correspondence and instead detect this diagonal term by an argument reminiscent of the Voronoi formula for Rankin–Selberg L-functions over $\mathbb{Q}(i)$. In fact, following a suggestion of J. Buttcane we have realized that the very strong $q = \pm\ell$ estimate of Theorem 3 can be bootstrapped to yield an even stronger estimate $\|\Phi_\ell|_\Omega\|_\infty \ll_\varepsilon \ell^{1+\varepsilon}$ for the vector-valued forms in Theorem 1, demonstrating that the Wigner basis vectors with $q = \pm\ell$ play a very special role.

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Towards a symplectic version of Duke's theorem

Jasmin Matz
(joint work with Farrell Brumley)

Let $\mathcal{F}$ denote the usual fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ acting on the hyperbolic upper half plane $\mathbb{H}$, and for $d < 0$ define the set of Heegner points

$$\Lambda_d = \{\frac{-b+\sqrt{d}}{2a} \in \mathcal{F} \mid b^{-}4ac = d,\ a,b,c \in \mathbb{Z}\} \subseteq \mathcal{F}.$$

When d is a fundamental discriminant, then $\#\Lambda_d$ equals the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$. In [2] Duke proved that Λ_d equidistributes in $\mathcal{F}$

as $|d| \longrightarrow \infty$, $d < 0$. More precisely, he showed that for every convex $\Omega \subseteq \mathcal{F}$ with piecewise smooth boundary there exists $\delta > 0$ such that

$$\frac{\#\Lambda_d \cap \Omega}{\#\Lambda_d} = \mu(\Omega) + O(|d|^{-\delta})$$

where μ denotes the hyperbolic measure on $\mathbb{H}$. He was also able to establish a similar result when $d > 0$ and Λ_d consists of geodesics with endpoints $\frac{-b\pm\sqrt{d}}{2a}$. The proof of this result made crucial use of half-integral weight Maass forms, estimates for Fourier coefficients and spectral expansion on $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbb{H}$. Generalizing those results, Cohen [1] studied, among other things, with similar methods the situation for Hilbert modular forms, that is, she showed that the analogue of the sets of Heegner points in $\mathrm{SL}_2(\mathcal{O}_F)\backslash\mathrm{SL}_2(F_\infty)$ become equidistributed for large discriminant, where F is a fixed totally real number field.

More generally, one can ask for the distribution of closed torus orbits on locally symmetric spaces. In [3] the distribution of homogeneous toral data in the adelic quotient of $\mathrm{PGL}(n)$ over a number field is studied. To define homogeneous toral data, let G be a reductive group over a number field F, T_j, $j \in \mathbb{N}$, a sequence of anisotropic (mod center) tori in G, and $\delta_j \in G(\mathbb{A})^1$. We then obtain a sequence of *homogeneous toral data* $T_j(F)\backslash T_j(\mathbb{A}_F)^1 \cdot \delta_j$ in $G(F)\backslash G(\mathbb{A}_F)^1$, and we assume that their 'discriminant' goes to ∞ as $j \to \infty$. If $G = \mathrm{SL}(2)$, and the sequence of homogeneous toral data is chosen suitably, then we are exactly in the setting of [2], when $F = \mathbb{Q}$, or [1], when F is a totally real number field.

For the group $\mathrm{PGL}(3)$ [3] showed that the sequence of probability measures on $G(F)\backslash G(\mathbb{A}_F)^1$ which are supported on $T_j(F)\backslash T_j(\mathbb{A}_F)^1 \cdot \delta_j$ and that one can canonically attach to those homogeneous toral data, converges to the Haar probability measure as $j \to \infty$ when imposing certain assumptions. The approach of [3] is fundamentally different from [2] and [1]. They use number theoretic methods, including Eisenstein series and subconvexity results (where available) or assumptions for certain L-functions to prove, among other things, positive entropy and non-escape of mass for the limiting measure. To complete the argument they use ergodic methods.

In current ongoing joint work with F. Brumley we are studying the symplectic setting. Let $G = \mathrm{GSp}(2n)/\mathbb{Q}$. This setup is particularly interesting since $G(\mathbb{Q})\backslash G(\mathbb{A})^1/K \simeq \mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathcal{H}_n$, (where K is the standard maximal compact subgroup of $G(\mathbb{A})$, and $\mathcal{H}_n$ denotes the Siegel upper half space of dimension $(n^2+n)/2$) parametrizes the principally polarized abelian varieties up to isomorphism. A particularly interesting sequence of homogeneous toral data can be obtained from sequences of CM-fields of degree $2n$ over $\mathbb{Q}$. Choosing suitable bases for those fields over $\mathbb{Q}$, the corresponding homogeneous toral data will then correspond to the analogue of the Heegner points, namely principally polarized abelian varieties having CM by the full ring of integers in those CM fields. In this setting we have been able to establish non-escape of mass for the corresponding sequence of measures, provided that the discriminants of the CM fields tend to ∞. The same way we obtain positive entropy of the limiting measure, once the existence of that is proven. In ongoing work we now focus on completing the ergodic side of the argument as well.

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Hyperbolic Fourier series and the Klein-Gordon equation

Haakan Hedenmalm

We introduce the hyperbolic Fourier series representation

$$f(t) \sim a_0 + \sum n \neq 0 a_n \, e^{i\pi nt} + b_n \, e^{-i\pi n/t},$$

where we explain that the coefficients exist uniquely, $a_n = a_n(f)$, $b_n = b_n(f)$, for any distribution f on the extended real line $\mathbb{R} \cup \{\infty\}$. The formulae for the coefficients are linear,

$$a_n(f) = \langle A_n, f \rangle_{\mathbb{R}}, \quad b_n(f) = \langle B_n, f \rangle_{\mathbb{R}},$$

where the biorthogonal system is smooth on the extended real line. We study the biorthogonal system in some detail. If we go beyond distribution theory, we may still find coefficients but we may lose uniqueness. Since the way to obtain the existence of hyperbolic Fourier series representation goes via solving a Dirichlet problem, the work [8] is useful to us. The smoothness of the biorthogonal system may be analyzed with the methods of [1]. The works [2], [3], [4], [5], [6], [7] are directly relevant to the presentation.

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Producing counterexamples for pointwise convergence of solutions to dispersive PDE's: number-theoretic methods

Lillian B. Pierce
(joint work with Rena Chu)

Given a polynomial $P(\xi) \in \mathbb{R}[\xi_1, ..., \xi_n]$ of degree $k \geq 2$, the operator

$$T_t^P f(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i(\xi \cdot x + P(\xi)t)} d\xi,$$

initially defined for f of Schwartz class on $\mathbb{R}^n$, gives a solution to the linear PDE

$$\begin{cases} \partial_t u - i\mathcal{P}(D)u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{1}$$

Here $D = \frac{1}{i}(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ and $\mathcal{P}(D)$ is defined according to its real symbol by

$$\mathcal{P}(D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} P(\xi) \hat{f}(\xi) d\xi.$$

When $P(\xi) = |\xi|^2$, (1) describes the linear Schrödinger equation. In this case, Carleson famously asked [Car80, Eqn (14) p. 24]: what is the smallest $s > 0$ such that

$$\lim_{t \to 0} T_t^P f(x) = f(x), \qquad \text{a.e. } x \in \mathbb{R}^n, \text{ for all } f \in H^s(\mathbb{R}^n)? \tag{2}$$

Here, $H^s(\mathbb{R}^n)$ is the Sobolev space, for a real parameter $s > 0$. This question was resolved for dimension $n = 1$ quite swiftly by Carleson, and Dahlberg and Kenig, who established that (2) holds if and only if $s \geq 1/4$ [Car80, DK82]. In higher dimensions, there is a long history of work by many researchers, on necessary and sufficient conditions for the Schrödinger pointwise convergence problem.

For several decades it was expected that $s = 1/4$ might be the critical threshold in all dimensions, until Bourgain developed a counterexample in sufficiently high dimensions that showed that the necessary condition could push s above $1/4$; see [Bou13]. Then in 2016, Bourgain showed that $s \geq 1/4 + \delta(n)$ with $\delta(n) = (n-1)/(4(n+1))$ is necessary for (2) to hold [Bou16]. Shortly thereafter, Du and Zhang showed that $s > 1/4 + \delta(n)$ is sufficient [DZ19]. This brought a long-standing problem to a resolution (up to the endpoint).

Bourgain's counterexample construction was interesting, and immediately attracted attention: it cleverly employed Gauss sums to force $|T_t^P f(x)|$ to be large (and hence ultimately violate (2)) for test functions f defined using exponential sums. A thorough investigation of Bourgain's construction appears in [Pie20]; the methods rely on number-theoretic constructions involving Gauss sums and simultaneous Dirichlet approximation.

Recently, in work with Chen An we elaborated on the idea of such constructions, and developed a more flexible method for producing counterexamples to pointwise convergence results of the form (2) for the initial value problem (1). In the work [ACP22], we demonstrated the new method for symbols of the form $P(X_1, \ldots, X_n) = X_1^k + \cdots + X_n^k$ for any degree $k \geq 3$. To move beyond the quadratic setting, this required new ideas to show that higher-degree analogues of Gauss sums are "often large," and to replace the role of simultaneous Dirichlet approximation, which no longer applies conveniently. Again, we were able to use number-theoretic methods, including the Weil bound for exponential sums, to construct new counterexamples. Ultimately, we proved that $s \geq 1/4 + \delta(n,k)$ is necessary for (2) to hold, for $\delta(n,k) = (n-1)/(4((k-1)n+1))$. It remains an open question what the sharp result should be for degrees $k \geq 3$.

Thereafter, [EPV22] adapted the method of [ACP22] to achieve a result of the same strength, for any polynomial whose leading form (homogeneous part of highest degree) takes the special shape

$$P_k(X_1, \dots, X_n) = X_1^k + Q_k(X_2, \dots, X_n), \tag{3}$$

where $Q_k \in \mathbb{Q}[X_2, \dots, X_n]$ is a nonsingular form of degree k that is independent of X_1. For degree 2 forms, this special shape does not entail a loss of generality, since any quadratic form can be diagonalized over $\mathbb{R}$, and as we will explain, the underlying problem allows for such changes of coordinates. However, for $k \geq 3$, forms P_k of that special shape are quite sparse among degree k forms in $\mathbb{Q}[X_1, \dots, X_n]$, and it is well-known that in arithmetic problems, a form with some diagonal structure is generally easier to handle.

This leads to a motivating question: what is the minimal regularity required for (2) when the real polynomial symbol P has leading form belonging to a generic class of degree k forms in $\mathbb{Q}[X_1, \dots, X_n]$? In particular, for any fixed $n \geq 2$ and degree $k \geq 2$, nonsingular forms are generic among degree k forms in $\mathbb{Q}[X_1, \dots, X_n]$. Hence it is reasonable to focus on any PDE where P has nonsingular leading form.

For any fixed real symbol P, the key to proving or disproving pointwise convergence as in (2) is the associated maximal operator $f \mapsto \sup_{0<t<1} |T_t^P f|$. For a given s, to prove that pointwise convergence (2) holds for all $f \in H^s(\mathbb{R}^n)$, it suffices to prove (for example) that the maximal operator maps $H^s(\mathbb{R}^n)$ to $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. In the other direction, to prove that convergence (2) fails for some functions in $H^s(\mathbb{R}^n)$, it suffices to prove that the maximal operator is unbounded from $H^s(\mathbb{R}^n)$ to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$; see for example [Pie20, Appendix A] for a summary of these standard arguments.

Now we note a key aspect of the problem, which brings in a group action: Given a value $s > 0$, the truth (or falsity) of a bound of the form

$$\| \sup_{0<t<1} |T_t^P f| \|_{L^1_{\mathrm{loc}}(\mathbb{R}^n)} \leq C_s \|f\|_{H^s(\mathbb{R}^n)} \qquad \text{for all } f \in H^s(\mathbb{R}^n) \tag{4}$$

is invariant under $\mathrm{GL}_n(\mathbb{R})$-action on the polynomial P. This means that unlike in many arithmetic problems, we can change variables to make the polynomial P take a more agreeable form. If one wishes to understand this putative bound (4) for an arbitrary polynomial P with nonsingular leading form $P_k \in \mathbb{Q}[X_1, \dots, X_n]$, it is no loss of generality to first apply a $\mathrm{GL}_n(\mathbb{Q})$ change of variable to put P_k in a convenient form. We heavily exploit the following property: for *every* nonsingular form in $L[X_1, \dots, X_n]$ for an infinite field L, there is a $\mathrm{GL}_n(L)$ change of variables under which the form becomes *Dwork-regular*. Dwork-regular forms have been extensively developed by Dwork and later Katz. By definition, P_k is Dwork-regular over $\mathbb{Q}$ in the variables $X_1, \ldots, X_n$ if there are no simultaneous solutions in $\mathbb{P}^{n-1}_{\overline{\mathbb{Q}}}$ to

$$P_k(X_1, \ldots, X_n) = 0, \qquad X_i \frac{\partial P_k}{\partial X_i}(X_1, \ldots, X_n) = 0, \qquad 1 \leq i \leq n.$$

Any Dwork-regular form over $\mathbb{Q}$ is nonsingular over $\mathbb{Q}$, and as mentioned before, any nonsingular form becomes Dwork-regular under an appropriate change of variables.

Our interest in passing to Dwork-regular forms is that they are particularly amenable to applications of the Weil bound even after fixing one or more variables. This allows us to make new progress on the convergence problem (2) despite a central difficulty encountered for generic forms, which occurs if each variable "interacts" with many other variables in the leading form P_k. The role of "interaction" is a second essential difficulty we encounter when tackling the problem for generic forms P_k.

In order to quantify how a variable "interacts" with other variables in P_k, we need an appropriate notion of "rank." The fact that diagonalization, so convenient for quadratic forms, is out of reach for most higher-degree forms, is a dominant theme in the study of symmetric tensors (which, roughly speaking, generalize the symmetric matrix associated to a quadratic form). This has led to the development of many notions of rank for degree k forms, including the Schmidt rank (or h-index), Waring rank (symmetric tensor rank), slicing rank, relative rank, the property of decomposability, and more. Each such notion of rank is motivated by specific applications in algebraic invariant theory, number theory, algebraic geometry, computational complexity, etc.

Similarly, our present work leads to a new notion of rank: we say a variable X_i intertwines with X_j in a form P_k of degree $k \geq 2$ if $(\partial^2/\partial X_i \partial X_j)P_k \not\equiv 0$. (By convention, X_i intertwines with itself.) The intertwining rank $r(X_i)$ of X_i in P_k is the number of variables with which X_i intertwines. The *intertwining rank* of the form P_k is $\min_{1 \leq i \leq n} r(X_i)$. (Thus for example, $X_1^3 + X_2^3 + X_3^3 + X_4^3$ has intertwining rank 1, while $X_1^3 + X_1 X_2^2 + X_2 X_3 X_4$ has intertwining rank 2.)

A new result presented in this talk (joint work with Rena Chu) is as follows:

Fix $n \geq 2$ and $k \geq 2$. Let $P \in \mathbb{R}[X_1, ..., X_n]$ be a polynomial whose leading form $P_k \in \mathbb{Q}[X_1, \ldots, X_n]$ is Dwork-regular in $X_1, \ldots, X_n$ over $\mathbb{Q}$ and has intertwining rank r. Suppose there is a constant C_s such that for all $f \in H^s(\mathbb{R}^n)$,

$$\| \sup_{0<t<1} |T_t^P f| \|_{L^1(B_n(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}. \tag{5}$$

Then $s \geq 1/4 + \delta(n, k, r)$ with

$$\delta(n, k, r) = \frac{n - r}{4((k-1)(n-(r-1))+1)}.$$

For intertwining rank $r = 1$, this recovers the special case of diagonal symbols considered in [ACP22], and all the symbols considered in [EPV22]. For $k = 2$, by the spectral theorem, any quadratic leading form is diagonalizable under $\mathrm{GL}_n(\mathbb{R})$, which (after further renormalization) reduces the case of quadratic forms to the case of intertwining rank $r = 1$. The novelty in our present work is that we can prove new results for all intertwining ranks $1 < r < n$ and degrees $k \geq 3$. In particular, we exhibit indecomposable forms P_k for which our method applies; that is, forms that cannot be reduced to the shape (3) by any change of variable.

By the invariance of (4) under $\mathrm{GL}_n(\mathbb{Q})$-action on P, the main result stated above furthermore applies to any polynomial P with leading form $P_k \in \mathbb{Q}[X_1, \dots, X_n]$ lying in the $\mathrm{GL}_n(\mathbb{Q})$-orbit of a Dwork-regular form of intertwining rank r. This points to an interesting type of question, which is in fact typical when one encounters a notion of rank for a higher degree form. Given a particular notion of rank, in an application one often wants to manipulate the original form (or class of forms) to make the (particular) rank more advantageous; the limits of this procedure may depend on the underlying field (for example, whether it is algebraically closed). For our particular setting, this motivates questions we are addressing in ongoing work: how big is the $\mathrm{GL}_n(\mathbb{Q})$-orbit of a Dwork-regular form of intertwining rank r, and how the does the intertwining rank behave under minimization via $\mathrm{GL}_n(\mathbb{Q})$?

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Fractal Weyl bounds via transfer operators

Anke Pohl

(joint work with Frédéric Naud and Louis Soares)

Resonances of Riemannian orbifolds play an important role in many areas of mathematics, e.g., analysis, dynamical systems, mathematical physics, and number theory. In this extended abstract we focus on geometrically finite hyperbolic orbisurfaces, i.e., on two-dimensional Riemannian orbifolds of the form $\Gamma\backslash\mathbb{H}$, where

$\mathbb{H}$ denotes the hyperbolic plane and Γ is a finitely generated Fuchsian group, acting on $\mathbb{H}$ by Riemannian isometries. We set $X := \Gamma\backslash\mathbb{H}$ and let Δ_X denote the hyperbolic Laplacian on X. The resolvent

$$R_X(s) := \big(\Delta_X - s(1-s)\big)^{-1} \colon L^2(X) \to H^2(X)$$

of Δ_X is defined for $s \in \mathbb{C}$ with $\Re s > 1/2$ and $s(1-s)$ not being an L^2-eigenvalue of Δ_X. It extends meromorphically to all of $\mathbb{C}$ as operators

$$R_X(s) \colon L^2_{\mathrm{comp}}(X) \to H^2_{\mathrm{loc}}(X)\,.$$

The resonances of X are the poles of this meromorphic family. We denote by $\mathcal{R}(X)$ the multiset of resonances of X, where each resonance is repeated according to its multiplicity (i.e., the rank of the residue operator at this resonance). We let

$$N_X(r) := \#\left\{s \in \mathcal{R}(X) : \left|s - \tfrac{1}{2}\right| \le r\right\}\,, \quad r > 0\,,$$

denote the counting function of resonances in balls (centered at $1/2$).

For *compact* hyperbolic orbisurfaces X, all resonances originate from L^2-eigenvalues. Up to finitely many exceptions, they are located at the critical axis $\Re s = 1/2$, and the Weyl law for their asymptotic counting is well-known:

$$N_X(r) \sim \frac{\mathrm{vol}(X)}{2\pi} r^2 \qquad \text{as } r \to \infty\,. \tag{1}$$

For non-compact hyperbolic orbisurfaces X of *finite area*, not all resonances originate from L^2-eigenvalues. In this situation, also scattering resonances make an appearance, which has the effect that the resonance set spreads out more. However, it is confined to the strip $\Re s \in [0, 1]$. This difference to compact spaces complicates the counting of resonances. Nevertheless, by work of Selberg [13] and W. Müller [7], the same asymptotics for the resonance set as for compact hyperbolic orbisurfaces was established. Thus, also for these orbisurfaces, the Weyl law (1) is known to be valid. We emphasize that it is a Weyl law for the resonance set, not necessarily for the L^2-eigenvalue set.

For hyperbolic surfaces X of *infinite area*, in stark contrast, it is not yet known if such a Weyl law for the resonance set should be expected. For geometrically finite hyperbolic orbisurfaces X of infinite area with at least one periodic geodesic, Guillopé and Zworski [4, 5] showed

$$N_X(r) \asymp r^2 \qquad \text{as } r \to \infty\,.$$

Thus, the order of growth of the resonance counting function is as for hyperbolic orbisurfaces of finite area, but (non-)equality of the implied constants could not yet be decided. A few results regarding the finer structure of these constants are known, e.g., as in [1, 11].

A further significant difference to the situation of finite-area orbisurfaces is the location of the resonance set. For infinite-area orbisurfaces it is not confined to a strip of finite width, but may distribute all over a certain right half-plane in $\mathbb{C}$ (with the Hausdorff dimension δ of the limit set of X being the right-most resonance).

This makes it interesting to consider a resonance counting function with a "vertical counting direction." More precisely, for $\sigma \in \mathbb{R}$ and $T > 0$ we define

$$N_X(\sigma, T) := \#\{s \in \mathcal{R}(X) : \Re s \geq \sigma\,,\ |\Im s| \leq T\}$$

to be the function counting the resonances in the box $[\sigma, \infty) + i[-T, T]$, with an interest of understanding its asymptotics for $T \to \infty$. Motivated by Sjöstrand's work [14] and numerical experiments, Lu–Sridhar–Zworski [6] conjectured a *fractal Weyl law* of the form

$$N_X(\sigma, T) \sim c_\sigma T^{1+\delta} \qquad \text{as } T \to \infty\,, \tag{2}$$

with δ being the right-most resonance of X (and c_σ a suitable implied constant, potentially depending on everything other than T). An important result towards this conjecture was achieved by Zworski [15] and Guillopé–Lin–Zworski [3], with two different proofs. They showed that for Schottky surfaces X (i.e., geometrically finite hyperbolic orbisurfaces of infinite area without cusps and elliptic points), for all $\sigma \in \mathbb{R}$ we have

$$N_X(\sigma, T) \ll_\sigma T^{1+\delta} \qquad \text{as } T \to \infty\,.$$

We now turned to the case of geometrically finite hyperbolic orbisurfaces of infinite area with cusps, at least one periodic geodesic and potentially elliptic points, in which we could establish the following result.

Theorem (Naud–P.–Soares)**.** *For certain geometrically finite hyperbolic orbisurfaces X of infinite area with cusps, for all $\sigma \in \mathbb{R}$, we have*

$$N_X(\sigma, T) \ll_\sigma T^{1+\delta}(\log T)^{2-\delta} \qquad \textit{as } T \to \infty\,. \tag{3}$$

The proof is based on transfer operator techniques, following the strategy of the proof in [3]. However, it is more involved due to the presence of a cusp. Every cusp has the effect that thickenings of the limit set do not have uniform contraction properties. Further, the required one-parameter transfer operator families $(\mathcal{L}_s)_s$ for orbisurfaces with cusps are *a priori* valid only for $\Re s \gg 1$, and hence we need to work within the domain of the meromorphic continuation of these families in s. The realm of this theorem heavily depends on the existence of representations of the Selberg zeta function of X as the Fredholm determinant of a well-structured transfer operator family for X. Such families are, e.g., provided in [10, 2] and, in particular by combination of [9] and [12] for a descent class of Fuchsian groups. An announcement of the theorem above with a sketch of the proof for non-cofinite Hecke triangle group appeared in [8]. The result in full detail will be available soon, including also a discussion of the extended setting involving finite-dimensional unitary representations of the fundamental group of X (i.e., a vector-valued situation).

Comparing (3) to the conjectured asymptotics (2) we notice the additional factor of $(\log T)^{2-\delta}$. It is not yet understood if this factor is immanent to the setting and, if so, if the exponent is best possible.

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Sphere Packings and Arithmetic

Alex Kontorovich

A sphere packing $\mathcal{P}$ is an infinite collection of pairwise disjoint balls on S^n which are dense. We say that $\mathcal{P}$ is *Kleinian* if there exists some $\Gamma < \mathrm{Isom}(\mathbb{H}^{n+1})$ that is discrete and geometrically finite such that Λ_Γ, the limit set of Γ, coincides with $R_{\mathcal{P}}$, the residual set of $\mathcal{P}$. Examples are the Apollonian circle packing ($n = 2$) and Soddy sphere packing ($n = 3$). Note that such a Γ is *thin*, i.e., it's a non-lattice.

Given a Kleinian packing $\mathcal{P}$, let

$$\Gamma_{\mathcal{P}} := \langle R_S : S \in \mathcal{P} \rangle$$

denote the (infinitely-generated) group generated by the reflections in the spheres.

Theorem 1 (Kontorovich–Nakamura '19, Kapovich–Kontorovich '23)**.** *The group*

$$\tilde{\Gamma} := \langle \Gamma, \Gamma_{\mathcal{P}} \rangle$$

is a lattice in $\mathbb{H}^{n+1}$.

Given an identification

$$\iota : S^n \cong \mathbb{R}^n \cup \{\infty\}$$

via stereographic projection, we get Euclidean coordinates on S^n, whence each $S \in \mathcal{P}$ has some radius $r(S)$, and corresponding "bend"

$$b(S) = \frac{1}{r(S)}.$$

We say that $\mathcal{P}$ is *integral* if there exists a choice of coordinates ι such that for each $S \in \mathcal{P}$, we have $b(S) \in \mathbb{Z}$. A *superpacking* is

$$\tilde{\mathcal{P}} = \Gamma_{\mathcal{P}} \cdot \mathcal{P}.$$

$\mathcal{P}$ is *superintegral* if there exists ι such that for each $S \in \tilde{\mathcal{P}}$, we have $b(S) \in \mathbb{Z}$.

Remark. Kontorovich–Nakamura showed that there exist packings $\mathcal{P}$ that are integral but not super-integral.

Theorem 2 (Kontorovich–Nakamura, Kapovich–Kontorovich)**.** *If $\mathcal{P}$ is Kleinian and superintegral, then the group $\tilde{\Gamma}$, which is a lattice, is actually arithmetic.*

Example. For $\mathcal{P}$ the Apollonian packing, we have

$$\tilde{\Gamma} = \langle R_0, R_1, \ldots, R_4 \rangle \cong \mathrm{SL}_2(\mathbb{Z}[i]),$$

which gives an instance of this arithmeticity theorem.

Given a (super)integral packing $\mathcal{P}$, we denote by

$$\mathcal{B} = \mathcal{B}(\mathcal{P}, \iota) = \{b(S) : S \in \mathcal{P}\}$$

the set of bends of the spheres in $\mathcal{P}$.

Conjecture (Local-Global Conjecture, Graham–Lagarias–Mallows–Wilkes–Yan '03)**.** *Given an integral packing $\mathcal{P}$, there exists q_0 such that for all n sufficiently large in terms of $\mathcal{P}$ and ι, we have*

$$n \in \mathcal{B} \iff n \in \mathcal{B} \pmod{q_0}.$$

The state of the art remains the following.

Theorem 3 (Bourgain–Kontorovich '14)**.** *Almost all n satisfy the local global conjecture when $\mathcal{P}$ is the Apollonian packing. That is to say,*

$$\frac{\#\{n \in \mathcal{B} \cap [1, X]\}}{\#\{n \in \mathcal{B} \pmod{q_0} \cap [1, X]\}} = 1 + O(X^{-\epsilon}).$$

This argument has been generalized to other circle packings in work of Zhang (2015) and and Fuchs–Strange–Zhang (2019).

In higher dimensions, more progress is possible.

Theorem 4 (Kontorovich '19)**.** *For $\mathcal{P}$ the Soddy sphere packing, the local global conjecture holds.*

A key ingredient in many of the above results is effective counting in orbits. Given a packing $\mathcal{P}$ and a choice of coordinates ι, look at

$$\mathcal{N}(T) := \#\{S \in \mathcal{P} : b(S) < T\}.$$

New progress on this asymptotic count is as follows.

Theorem 5 (Counting Theorem, Kontorovich–Lutsko '22)**.** *We have*

$$N(T) = c_{\rho,\iota} T^\delta + O\left(T^{\frac{3}{5}\delta + \frac{2}{5}\cdot 1}\right).$$

If we look at the hyperbolic Laplacian Δ acting on $L^2(\Gamma\backslash\mathbb{H}^{n+1})$, then it has a spectrum, which is purely continuous above $n^2/4$ [Lax–Phillips]. Below this value, there is the bottom eigenvalue $\lambda_0 = \delta(n-\delta)$, where δ denotes the Hausdorff dimension of Λ_Γ, due to Patterson '76 and Sullivan '82. For the Apollonian circle packing, δ is about 1.30, so $\lambda_0 = 0.91$. Then the exponent in the error term is 1.18.

Sarnak asked in his 2007 letter to Lagarias whether there was no λ_1, that is, no other discrete eigenvalues below $n^2/4 = 1$. This is solved as follows.

Theorem 6 (Kelmer–Kontorovich–Lutsko '23)**.** *For $\mathcal{P}$ the Apollonian packing, we answer Sarnak's question in the affirmative: λ_0 is the unique discrete Laplace eigenvalue.*

This is what allows the strong and explicit error estimate in the counting theorem above. The counting theorem has a long history.

- Boyd in 1973 proved that $\mathcal{N}(T) = T^{\delta+o(1)}$, and asked whether the true asymptotic might be something like $T^\delta \log^A T$, that is, with extra log factors.
- Kontorovich–Oh '11 settled this question; there are no log factors: $\mathcal{N}(T) \sim cT^\delta$.
- Vinogradov and Lee–Oh '13 independently obtained a power savings error: $\mathcal{N}(T) = cT^\delta + O(T^{\delta-\epsilon})$.
- Many others worked on this problem, Mohammadi–Oh, Oh–Shah, Kim, Edwards...

To indicate the proof of the Counting Theorem 5, let's solve the following toy problem:

Let us try to count

$$\#\left\{(c,d) = 1 : c^2 + d^2 < T\right\}.$$

Use that

$$\Im(\gamma i) = \frac{1}{c^2+d^2}, \quad \Gamma = \mathrm{SL}_2(\mathbb{Z}).$$

So we're looking at the orbit

$$\mathcal{O} = \Gamma \cdot i.$$

Then

$$\mathcal{N}(T) = F_T(i),$$

where

$$F_T(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1_{\Im \gamma z > 1/T}.$$

We have

$$N(T) \sim \langle F_T, \psi \rangle ,$$

where ψ is a bump function about i. Unfolding gives

$$\begin{aligned}\langle F_T, \psi \rangle &= \int_{\Gamma_\infty \backslash \mathbb{H}} 1_{\Im z > 1/T} \psi(x+iy) \frac{dx\, dy}{y^2} \\ &= \int_{1/T}^{\infty} \left[\int_{\mathbb{R}/\mathbb{Z}} \psi(x+iy)\, dx \right] \frac{dy}{y^2}.\end{aligned}$$

Pretend that ψ is an eigenfunction of Δ, with eigenvalue $\lambda = s(1-s)$. Then solving an ODE shows that the inner integral is

$$\left[\int_{\mathbb{R}/\mathbb{Z}} \psi(x+iy)\, dx \right] = c_\Psi y^{1-s}.$$

Inserting this in the above gives

$$\langle F_T, \psi \rangle = c \cdot \frac{T^s}{s}. \tag{1}$$

The constant c depends on the function ψ, but s depends only on the eigenvalue. We can get rid of the dependence on ψ with the following trick.

Take $T = 1$, so that

$$\langle F_1, \psi \rangle = \frac{c}{s}.$$

Then

$$\langle F_T, \psi \rangle = \langle T^s F_1, \psi \rangle . \tag{2}$$

Let $\mathfrak{S} = \frac{1}{2} + \sqrt{\Delta - \frac{1}{4}}$. Then it is possible to prove that

$$F_T = T^{\mathfrak{S}} F_1. \tag{3}$$

From this "Main Identity" it is easy to conclude the Counting Theorem from abstract spectral methods. A similar argument was used in [Kontorovich2019].

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Density of rational points near/on compact manifolds

Damaris Schindler
(joint work with Shuntaro Yamagishi)

In this talk we discuss the density of rational points 'close' to smooth manifolds. Let $\mathcal{M}$ be a compact (immersed) submanifold of $\mathbb{R}^M$ and let $R = M - \dim \mathcal{M}$ be the codimension of $\mathcal{M}$. Given $Q \in \mathbb{N}$ and $\delta \geq 0$, we let

$$N(\mathcal{M}; Q, \delta) := \#\{(\mathbf{a}, q) \in \mathbb{Z}^M \times \mathbb{N} : 1 \leq q \leq Q, \operatorname{dist}(\mathbf{a}/q, \mathcal{M}) \leq \delta/q\},$$

where $\operatorname{dist}(\cdot, \cdot)$ denotes the L^∞-distance on $\mathbb{R}^M$. What can we say about the counting function $N(\mathcal{M}; Q, \delta)$ for $Q \to \infty$ and $\delta \to 0$?

A first upper bound for $N(\mathcal{M}; Q, \delta)$ is given by

$$N(\mathcal{M}; Q, \delta) \ll Q^{\dim \mathcal{M}+1},$$

while a probabilistic heuristic suggests

$$\delta^R Q^{\dim \mathcal{M}+1} \ll N(\mathcal{M}; Q, \delta) \ll \delta^R Q^{\dim \mathcal{M}+1}.$$

We know that this heuristic estimate does not hold in complete generality. For example, if $\mathcal{M}$ is a rational hyperplane in $\mathbb{R}^M$ and $\delta \leq 1$, then we have

$$Q^{\dim \mathcal{M}+1} \ll N(\mathcal{M}; Q, \delta) \ll Q^{\dim \mathcal{M}+1}.$$

In one of the spectacular achievements in the field [1], Beresnevich established the following sharp lower bound

$$N(\mathcal{M}; Q, \delta) \gg_{\mathcal{M}} \delta^R Q^{\dim \mathcal{M}+1} \quad \text{for any } \delta \gg Q^{-\frac{1}{R}}, \tag{1}$$

assuming $\mathcal{M}$ is an analytic submanifold of $\mathbb{R}^M$ which contains at least one nondegenerate (see [1] for the definition) point. In his groundbreaking work [5], Huang proposed the following conjecture:

Let $\mathcal{M}$ be a bounded immersed submanifold of $\mathbb{R}^M$ with boundary. As above, let $R = M - \dim \mathcal{M}$. Suppose $\mathcal{M}$ satisfies 'proper' curvature conditions. Then there exists a constant $c_{\mathcal{M}} > 0$ depending only on $\mathcal{M}$ such that

$$N(\mathcal{M}; Q, \delta) \sim c_{\mathcal{M}} \delta^R Q^{\dim \mathcal{M}+1}$$

when $\delta \geq Q^{-\frac{1}{R}+\epsilon}$ for some $\epsilon > 0$ and $Q \to \infty$.

In the same article Huang established this conjecture for the case when $\mathcal{M}$ is a hypersurface with Gaussian curvature bounded away from zero. Previously, the conjecture was only known to hold for planar curves [4]. For earlier results towards the main conjecture see for example [2], [6], [7] for the case of planar curves, and [1], [3] for the more general case.

In this talk we present a generalisation of Huang's work to manifolds of higher codimension with a certain curvature condition. More precisely, consider a manifold $\mathcal{M}$ that is locally given by

$$(2) \qquad \mathcal{M} := \{(\mathbf{x}, f_1(\mathbf{x}), \dots, f_R(\mathbf{x})) \in \mathbb{R}^M : \mathbf{x} = (x_1, \dots, x_n) \in \overline{B_{\varepsilon_0}(\mathbf{x}_0)}\},$$

where $\mathbf{x}_0 \in \mathbb{R}^n$, $\varepsilon_0 > 0$ and $f_r \in C^\ell(\mathbb{R}^n)$ $(1 \leq r \leq R)$ for some $\ell \geq 2$. In particular, $\dim \mathcal{M} = n$.

Let $w \in C_0^\infty(\mathbb{R}^n)$ be a non-negative weight function with $\operatorname{supp} w$ contained in a sufficiently small (with respect to $\mathcal{M}$) open neighbourhood of $\mathbf{x}_0$. For $Q \in \mathbb{N}$ and $0 \leq \delta \leq \frac{1}{2}$, we define

$$(3) \qquad \mathcal{N}_w(Q, \delta) = \sum_{\substack{\mathbf{a} \in \mathbb{Z}^n \\ q \leq Q \\ \|q f_1(\mathbf{a}/q)\| \leq \delta \\ \vdots \\ \|q f_R(\mathbf{a}/q)\| \leq \delta}} w\left(\frac{\mathbf{a}}{q}\right),$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

Let

$$N_0 := \sum_{\substack{\mathbf{a} \in \mathbb{Z}^n \\ q \leq Q}} w\left(\frac{\mathbf{a}}{q}\right).$$

Given any $f \in C^2(\mathbb{R}^n)$ we denote by $H_f(\mathbf{x})$ the Hessian matrix of f.

Then we establish the following result.

Theorem. *Let $n \geq 2$ and $\ell > \max\{n+1, \frac{n}{2}+4\}$. Suppose that for any*

$$(t_1, \dots, t_R) \in \mathbb{R}^R \backslash \{\mathbf{0}\}$$

we have

$$\det H_{t_1 f_1 + \cdots + t_R f_R}(\mathbf{x}_0) \neq 0.$$

Assume that $\varepsilon_0 > 0$ is sufficiently small. Then we have

$$\left|\mathcal{N}_w(Q,\delta) - (2\delta)^R N_0\right| \ll \begin{cases} \delta^{\frac{(R-1)(n-2)}{n}} Q^n \mathcal{E}_n(Q) & \text{if } \delta \geq Q^{-\frac{n}{n+2(R-1)}} \\ Q^{n-\frac{(n-2)(R-1)}{n+2(R-1)}} \mathcal{E}_n(Q) & \text{if } \delta < Q^{-\frac{n}{n+2(R-1)}}, \end{cases}$$

where

$$\mathcal{E}_n(Q) = \begin{cases} \exp(\mathfrak{c}_1 \sqrt{\log Q}) & \text{if } n = 2 \\ (\log Q)^{\mathfrak{c}_2} & \text{if } n \geq 3. \end{cases}$$

for some positive constants $\mathfrak{c}_1$ and $\mathfrak{c}_2$. Here the constants $\mathfrak{c}_1$ and $\mathfrak{c}_2$ and the implicit constants depend only on $\mathcal{M}$ and w.

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Pairs of saddle connections

Claire Burrin

(joint work with Jon Chaika and Samantha Fairchild)

There is a collection of classical problems at the intersection of number theory and harmonic analysis that concern the distribution of lattice or primitive lattice points in space. The Gauss circle problem — i.e., quantifying the size of the approximation $||\mathbb{Z}^2 \cap D_R(0)| - \pi R^2|$ — stands as the most famous example. If we restrict to the set $\hat{\mathbb{Z}}^2$ of primitive lattice points, the Gauss circle problem has a geometric interpretation: we are looking at the count of cylinders of simple periodic geodesics on the standard torus $\mathbb{R}^2/\mathbb{Z}^2$ that share the same direction. Indeed, upon adding a marked point on the torus, each such cylinder is represented by a unique primitive simple periodic geodesic (passing through that point), and unfolding $\mathbb{R}^2/\mathbb{Z}^2$ to the whole plane $\mathbb{R}^2$ shows that these representatives correspond to the vectors in $\hat{\mathbb{Z}}^2$.

In my talk, I discussed a higher genus analogue of this setting. In the place of the standard torus, we take a union of finite polygons that admits a complete pairing of edges in pairs of parallel edges of equal length. Identifying these pairs yields a flat surface of genus $g \geq 2$ with conical singularities. Outside of these singularities, the surface inherits an atlas of charts for which the transition maps are Euclidean translations, and a linear flow F_θ^t determined by a direction $\theta \in S^1$. Such a construction is called a translation surface; we invite the reader to consult the broad audience survey [7]. At the singularities the linear flow is not well defined, and this leads to the definition of saddle connections, which are line segments connecting two singularities, with no further singularity in-between.

In the spirit of our analogy with the genus 1 setting, we are more particularly interested in translation surfaces that would showcase 'optimal dynamics' in the form of a Weyl-type dichotomy: for any given direction θ the linear flow F_θ^t is either uniquely ergodic or *completely periodic*, meaning that the surface may be cut out in cylinders of periodic geodesics along the saddle connections in the direction of θ. It is known that on every translation surface the flow is uniquely ergodic in almost every direction θ [4] and that translation surfaces with optimal dynamics exist, with the most prominent class of examples being given by *Veech surfaces* (also sometimes called *lattice surfaces*). In fact, in genus 2, Veech surfaces are the only translation surfaces showcasing optimal dynamics.

Set $G := \mathrm{SL}_2(\mathbb{R})$. For the purpose of this abstract, we will only say (and use) that Veech surfaces are characterized by the existence of a non-cocompact lattice $\Gamma < G$ and that the 'parametrizing (discrete) set'[1] $\Lambda \subset \mathbb{R}^2$ of saddle connections can be written as a finite disjoint union $\Lambda = \Gamma v_1 \cup \cdots \cup \Gamma v_k$ of discrete lattice orbits [5].[2] Note that in genus 1, this set is precisely $\hat{\mathbb{Z}}^2 = \mathrm{SL}(2,\mathbb{Z})\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

The basic counting problem for saddle connections on Veech surfaces is by now well understood. If X is a Veech surface then there is a constant $c_\Lambda > 0$ such that

$$|\Lambda \cap D_R(0)| \sim c_\Lambda |D_R(0)| := c_\Lambda \pi R^2$$

as $R \to \infty$ [5]. Effective counts and specializations to more general counting regions are presented in [3]. We view these counting results as the higher genus analogues of the classical asymptotic, again as $R \to \infty$,

$$|\hat{\mathbb{Z}}^2 \cap D_R(0)| \sim \frac{1}{\zeta(2)} |D_R(0)|,$$

where the density constant $\frac{1}{\zeta(2)}$ is the probability of two integers to be coprime.

In this talk I reported on recent joint work [2] on the distribution of *pairs of saddle connections*. The main result (below) had two motivations. The first one stems from the fact [8] that when X is a *non-arithmetic* Veech surface — meaning

[1] We expand a little on the construction of this set. Every translation surface X can be described as a compact Riemann surface equipped with an abelian differential ω whose zero set coincides with the set of singularities of X. The set Λ is the set of holonomy vectors $v_\gamma = \int_\gamma \omega$ of the saddle connections γ on X.

[2] There exists a dictionary according to which the number k of orbits equals the number of inequivalent cusps of Γ and the stabilizer subgroup Γ_v of the vector v for the linear action of Γ on the Euclidean plane corresponds to the stabilizer subgroup $\Gamma_{\mathfrak{a}}$ of the cusp $\mathfrak{a}$ for the action of Γ by fractional linear transformation on the hyperbolic upper half-plane; see [2, Section 3].

that the lattice Γ is not commensurable to $\mathrm{SL}(2,\mathbb{Z})$ — the set Λ is discrete but not *uniformly* discrete. That is, for any $\varepsilon > 0$ there exist a pair of distinct saddle connections $x, y \in \Lambda$ such that $|x-y| < \varepsilon$. The following result asserts that upon controlling the separation constant ε, we can guarantee that the density of saddle connections with close-by neighbors is negligible.

Theorem 1. *For any $\varepsilon > 0$, we have*

$$\limsup_{R\to\infty} \frac{|\{(x,y) \in \Lambda^2 : x \in D_R(0),\, |x-y| \in (0,\varepsilon)\}|}{R^2} \leq C \cdot \varepsilon^2, \tag{1}$$

where the constant C is uniform.

Our second motivation was based on a disjointness criterium developed by Chaika. Via this criterium, we can play the estimate (1) against the weak mixing of the flow F_θ^t in almost every direction θ [1] to show

Theorem 2. *For every Veech surface, and almost every pair of directions $(\theta, \phi) \in S^1 \times S^1$, the flows F_θ^t and F_ϕ^t are disjoint.*

Corollary. *For every Veech surface, the product flow $F_\theta^t \times F_\phi^t$ is uniquely ergodic.*

The proof of Theorem 1 builds on an integral formula (inspired by the integral formulas of Siegel, Rogers, and Schmidt in the geometry of numbers), a measure classification argument, and the analytic theory of Eisenstein series. Here is a rough sketch of the proof. Since X is a Veech surface, we can equip the non-compact homogeneous space G/Γ with a probability measure μ. Following a measure-classification argument of Veech [6], we derive the integral formula

$$\begin{aligned}\int_{G/\Gamma} \sum_{x,y\in\Lambda} f(gx, gy)\, d\mu(g) = c_\Lambda \sum_{c\in\mathcal{N}} \frac{\varphi(c)}{|c|} \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(x, tx + cx^*)\, dt\, dx \\ + c_\Lambda \int_{\mathbb{R}^2} f(x,x) dx + c_\Lambda \int_{\mathbb{R}^2} f(x,-x) dx.\end{aligned}$$

for all semicontinous function f on $\mathbb{R}^2 \times \mathbb{R}^2$. The sum on the right hand-side is indexed over the discrete set of real determinants

$$\mathcal{N} = \{\det(x|y) : x, y \in \Lambda\}.$$

Each determinant in the discrete set $\mathcal{N}$ appears with multiplicity $\varphi(c)$ and x^* is chosen such that $\det(x|tx + cx^*) = c$. A reparametrization shows that the set $\mathcal{N}$ is in bijection with the set of double cosets $\Gamma_{\mathfrak{a}}\backslash\Gamma/\Gamma_{\mathfrak{b}}$. Such double cosets are known to also index the Fourier expansion of non-holomorphic Eisenstein series. We use the provided access to the spectral theory of Eisenstein series to control the large-scale statistics of the set $\mathcal{N}$.

To prove Theorem 1, we choose $f_{R,\varepsilon}(x,y) = \chi_{D_R(0)}(x)\chi_{D_\varepsilon(x)}(y)$. For a neighborhood $\mathcal{U}$ of identity in G, we can extend the support with R^+, ε^+ such that

$$\sum_{x,y\in\Lambda} f_{R,\varepsilon}(x,y) \leq \sum_{x,y\in\Lambda} f_{R^+,\varepsilon^+}(gx, gy)$$

for all $g \in \mathcal{U}$. Integrating both sides over $\mathcal{U}\Gamma/\Gamma$, bounding by positivity and evaluating the right hand-side via the integral formula above[3] yields the statement.

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Statistics of the roots of quadratic congruences and geodesic random line processes

Jens Marklof

(joint work with Matthew Welsh)

Conider the roots μ of the quadratic congruence $\mu^2 \equiv D \bmod m$, where D is a fixed non-zero integer and m runs through the positive integers. We are interested in the distribution of the sequence of $\frac{\mu}{m}$ modulo one, ordered by increasing denominator $m = 1, 2, \ldots$ with arbitrary ordering of the terms with the same m (our results will not depend on the choice made). Hooley proved in [2] that this sequence is uniformly distributed modulo one if D is not a perfect square. Thus, for any $0 \leq a < b \leq 1$ we have that

$$\lim_{N\to\infty} \frac{1}{N} \#\big\{ j \leq N : \xi_j \in [a,b) + \mathbb{Z} \big\} = b - a.$$

It is natural to ask for finer statistical tests that capture the pseudorandom properties of the sequence. The simplest of these is the pair (or two-point) correlation measure $R_{2,N}$, which for a finite interval I is defined by

$$R_{2,N}(I) = \frac{1}{N} \#\big\{ i,j \leq N : i \neq j,\ \xi_i - \xi_j \in N^{-1} I + \mathbb{Z} \big\}.$$

In [5] we prove the following statement when $D \not\equiv 1 \bmod 4$. This has recently been extended by Li and Welsh [3] to the remaining case $D \equiv 1 \bmod 4$.

[3] The control of the evaluation is in practice obtained via a second averaging over the cone of G/Γ; we refer the interested reader to [2].

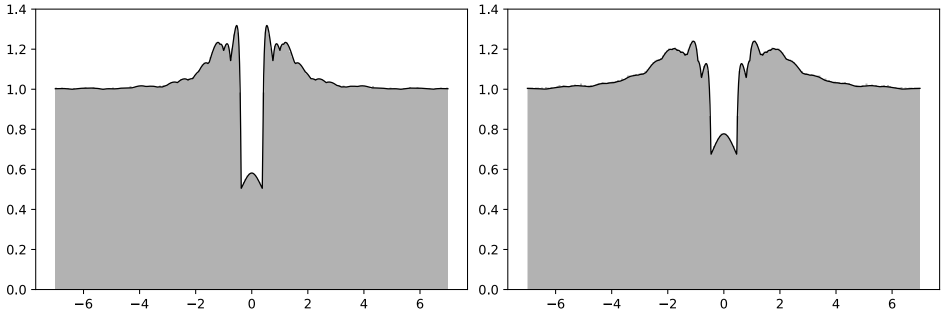

FIGURE 1. The pair correlation density w_D with $D = 2$ and 3 ($\not\equiv 1 \bmod 4$) compared to numerical data. See [5] for details.

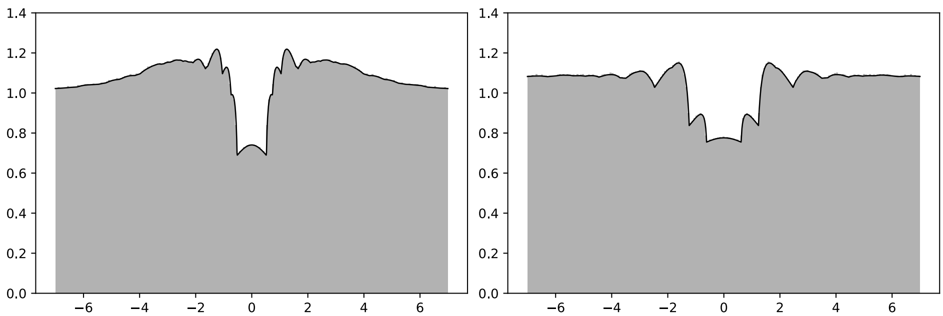

FIGURE 2. The pair correlation density w_D with $D = 5$ and 17 ($\equiv 1 \bmod 4$) compared to numerical data. Reproduced from [3].

Theorem. *Assume $D \neq 0$ is square-free. Then there is an even and continuous function $w_D : \mathbb{R} \to \mathbb{R}_{\geq 0}$, such that for every finite interval I we have*

$$\lim_{N\to\infty} R_{2,N}(I) = \int_I w_D(v)\mathrm{d}v.$$

In fact, we prove in [5] the convergence of all fine-scale distributions, including gap statistics and higher order correlation functions. The limiting density w_D is shown in Figures 1 and 2 for four different values of D and compared with numerical data.

The key to the proof of the theorem is that the roots can be represented as certain point sets, which are generated by the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on the upper half plane. If D is negative, then these points are of the form

$$\gamma z_l \equiv \frac{\mu}{m} + \mathrm{i}\frac{\sqrt{-D}}{m} \mod 1,$$

where $z_1, \ldots, z_h$ are points in the complex upper half plane determined by D, and γ runs over the modular group. If D is positive, then the relationship of the roots is more complicated, we have

$$z_{\gamma c_l} \equiv \frac{\mu}{m} + \mathrm{i}\frac{\sqrt{D}}{m} \mod 1,$$

where $c_1, \ldots, c_h$ are geodesics in the complex upper half plane specified by the choice of D, z_{c_l} is the "top" of the corresponding geodesic, and γ again runs over the modular group. Note that in general $z_{\gamma c_l} \neq \gamma z_{c_l}$, so the cases of negative and positive D have genuinely different geometric constructions. An important feature is that the geodesics c_l project to closed geodesics on the modular surface. This implies that the stabiliser $Z(c_l)$ of c_l in $\mathrm{SL}(2,\mathbb{Z})$ is an infinite subgroup.

For negative D, we can use the limit theorems for the fine-scale statistics of directions in hyperbolic lattices [4] to prove the theorem. A new phenomena arises for positive D, where we have developed analogous results for the fine-scale statistics of the corresponding tops of geodesics [5]. The limit distribution is given by a geodesic random line process in the hyperbolic plane, which is constructed by taking the union of geodesics

$$\bigcup_{l=1}^{h} \bigcup_{\gamma \in \mathrm{SL}(2,\mathbb{Z})/Z(c_l)} \gamma c_l$$

and then applying a random isometry drawn from the Haar probability measure on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. We conclude with two open questions:

Question A: As $D \to \infty$, are the fine-scale statistics given by a Poisson point process? Specifically, do we have that $\int_I w_D(v)\mathrm{d}v \to |I|$ for any finite interval I? Convergence might hold only along a subsequence or require additional averaging over D.

Question B: Duke, Friedlander and Iwaniec [1] have established uniform distribution mod 1, if we select only those $\frac{\mu}{m}$ with prime denominators m. Do the pair correlation function and other fine-scale statistics converge when restricted to this sparse subsequence, and what are their limits?

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Linear equations in smooth numbers

Lilian Matthiesen
(joint work with Mengdi Wang)

Let $y > 0$ be a real number. A positive integer n is called y-smooth if its largest prime factor is at most y. The y-smooth numbers below N form a subset of the integers below N which is, in general, sparse but enjoys good equidistribution properties in arithmetic progressions and short intervals. These distributional properties turn y-smooth numbers into an important technical tool for many arithmetic questions. As an example for one of the striking applications of smooth numbers within analytic number theory, we mention Vaughan's work [5] which introduced smooth numbers in combination with a new iterative method to the study of bounds in Waring's problem. Wooley extended these methods in [6] and achieved substantial improvements on Waring's problem by working with smooth numbers.

This talk is based on the preprint [4], which is part of an ongoing research project. The aim of this project is to investigate to what extent the methods used by Green and Tao [1] to study linear equations in primes can be applied in the extremely spare setting of smooth numbers. The ultimate goal is to prove, for values of y as small as possible in terms of x, non-trivial lower bounds on the frequency with which a system $\psi_1, \dots, \psi_r \in \mathbb{Z}[x_1, \dots, x_s]$ of pairwise $\mathbb{Q}$-linearly independent linear forms with integral coefficients simultaneously take y-smooth values when the variables x_i are all restricted to integers of size about x.

The preprint [4] settles one integral part of this project on an essentially optimal range for the smoothness parameter y. We prove new results on the equidistribution of smooth numbers in short intervals and arithmetic progressions and use these results to prove "higher uniformity" for a certain subset of the set of y-smooth numbers. In order to state the main results precisely, we start by introducing the relevant subset of the y-smooth numbers as well as a weighted version of its characteristic function. Given any real numbers $0 < y' \leq y$, we may consider the set of y-smooth numbers that are free from prime factors smaller than y'. We call such numbers $[y', y]$-smooth and denote their set by

$$S([y', y]) := \{n \in \mathbb{N} : p \mid n \Rightarrow p \in [y', y]\}.$$

Given any $x > 0$, the subset of $[y', y]$-smooth numbers $\leq x$ and its cardinality are denoted by

$$S(x, [y', y]) := S([y', y]) \cap [1, x] \quad \text{and} \quad \Psi(x, [y', y]) := |S(x, [y', y])| \ .$$

Our notation extends the following standard notation for y-smooth numbers:

$$S(y) := S([1, y]), \quad S(x, y) := S(x, [1, y]), \quad \text{and} \quad \Psi(x, y) = \Psi(x, [1, y]).$$

With this notation, we define the weighted characteristic function

$$g_{[y',y]}(n) = \frac{n}{\alpha(n, y)\Psi(n, [y', y])} \mathbf{1}_{S([y',y])}(n), \qquad (n \in \mathbb{N}),$$

of $[y', y]$-smooth numbers, where $\alpha(n, y)$ denotes the saddle point associated to $S(n, y)$. If $1 \leq A < W$ are coprime integers, we further define a *W-tricked* version of $g_{[y',y]}$ by setting

$$g^{(W,A)}_{[y',y]}(m) = \frac{\phi(W)}{W} g_{[y',y]}(Wm + A), \qquad (m \in \mathbb{N}).$$

With these preparations, the main result of the preprint [4] is the following. (For notation and definitions around nilsequences, we refer to [4, Section 8].)

Theorem 1 (Higher uniformity)**.** *Let N be a large positive parameter and let $K' \geq 1$, $K > 2K'$ and $d \geq 0$ be integers. Let $y' = (\log N)^{K'}$ and suppose that $(\log N)^K < y < N^\eta$ for some sufficiently small $\eta \in (0,1)$ depending the value of d. Let $(G/\Gamma, G_\bullet)$ be a filtered nilmanifold of complexity Q_0 and degree d. Finally, let $w(N) = \frac{1}{2}\log_3 N$, $W = \prod_{p<w(N)} p$ and define $\delta(N) = \exp(-\sqrt{\log_4 N})$.*

If K' and K/K' are sufficiently large, where the lower bound on K' depends on the degree d of $G_\bullet$, then the estimate

$$\left| \frac{W}{N} \sum_{n \leq (N-A)/W} (g^{(W,A)}_{[y',y]}(n) - 1) F(g(n)\Gamma) \right| \ll_d (1 + \|F\|_{\mathrm{Lip}}) \delta(N) Q_0 + \frac{1}{\log w(N)}$$

holds uniformly for all $1 \leq A \leq W$ with $\gcd(A,W) = 1$, all polynomial sequences $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ and all 1-bounded Lipschitz functions $F : G/\Gamma \to \mathbb{C}$.

As an illustration, we mention the following fairly easy consequence of the above result combined with the Green–Tao–Ziegler inverse theorem [2] for the U^k-norms in the quantitative version given by Manners [3].

Corollary. *With the notation and under the assumptions of the theorem above, the following holds. If K' is sufficiently large depending on r, s and L, and if $y \leq N^\eta$ is sufficiently large to ensure that $\Psi(N,y) > N/\log_8 N$, then the number $\mathcal{N}(N,\mathfrak{K})$ of $\mathbf{n} \in \mathbb{Z}^s \cap N\mathfrak{K}$ for which the given system of linear polynomials $(\psi_j(\mathbf{n}) + a_j)_{1\leq j\leq r}$ takes simultaneous $[y',y]$-smooth values satisfies:*

$$\mathcal{N}(N,\mathfrak{K}) := \sum_{\mathbf{n}\in\mathbb{Z}^s\cap N\mathfrak{K}} \prod_{j=1}^{r} \mathbf{1}_{S([y',y])}(\psi_j(\mathbf{n}) + a_j) \gg \mathrm{vol}(\mathfrak{K}) N^{s-r} \Psi(N,[y',y])^r \prod_p \beta_p$$

for all sufficiently large N, where

$$\beta_p = \frac{1}{p^s} \sum_{\mathbf{u}\in(\mathbb{Z}/p\mathbb{Z})^s} \prod_{j=1}^{r} \frac{p}{p-1} \mathbf{1}_{\psi_j(\mathbf{u})+a_j \not\equiv 0 \ (\mathrm{mod}\, p)}.$$

Asymptotic results on the quantity $\mathcal{N}(N,\mathfrak{K})$ fall into the broader context of counting smooth values of polynomials or forms and that of studying smooth solutions to Diophantine equations. Our results contribute to a vast literature examining such questions, and we refer to the introduction of the preprint [4] for a more detailed discussion relating the new results to previous work.

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Covering integers by quadratic forms

Kannan Soundararajan
(joint work with Ben Green)

How large must Δ be so that almost all integers n up to N may be represented by some binary quadratic form $x^2 + dy^2$ with $d \leq \Delta$? This problem was recently considered by Hanson and Vaughan [5] who showed that $\Delta \geq (\log N)(\log\log N)^4$ suffices, and established a related result when only a positive proportion of the integers below N are to be represented. Their work was based on the circle method, and using a different approach using class group characters, Diao [4] established that a positive proportion of integers below N may be represented using the quadratic forms $x^2 + dy^2$ with d being a prime number below $\log N(\log\log N)$.

I described work in progress with Ben Green which gives a complete resolution of this problem.

Theorem. *Let N be large, and write for some real number α*

$$\Delta = (\log N)^{\log 2} 2^{\alpha\sqrt{\log\log N}}.$$

Then

$$\#\{n \leq N : \quad n = x^2 + dy^2 \qquad \text{for some } 1 \leq d \leq \Delta\} = (\Phi(\alpha) + o(1))N,$$

where Φ is the Gaussian distribution function

$$\Phi(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx.$$

A consequence of this is a phase transition: almost none of the integers $n \leq N$ can be represented by $x^2 + dy^2$ with $d \leq (\log N)^{\log 2 - \epsilon}$, but almost all of them can be represented by $x^2 + dy^2$ with $d \leq (\log N)^{\log 2 + \epsilon}$.

In fact, we establish a much more precise version of this result. Recall that most integers below N have about $\log\log N$ prime factors, and if k is suitably close to $\log\log N$ then the number of integers below N with exactly k prime factors (counted with multiplicity) is

$$\mathcal{A}(N, k) \sim \frac{N}{\log N} \frac{(\log\log N)^{k-1}}{(k-1)!}.$$

Our work shows that almost all such integers with k prime factors may be represented using the quadratic forms $x^2 + dy^2$ with $d \leq k^3 2^k$, and that almost none of them are represented if $d \leq 2^k/k^3$. This readily implies the main theorem stated above. The truth, which could likely be established through a refinement of our work, is that the transition here takes place at the threshold $\Delta = \sqrt{k}2^k$.

The techniques involved in our proof involve the second moment method, class group characters and related L-functions, and the Selberg–Delange method. These ideas are related to earlier advances in this classical field, including for example the work of Blomer [1, 2] on sums of two square-full numbers, and work of Blomer and Granville [3] on the moments of representation numbers of quadratic forms.

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Equidistribution of discrete Fourier transforms of trace functions on commutative algebraic groups over finite fields

Javier Fresán

(joint work with Arthur Forey and Emmanuel Kowalski)

Let k be a finite field, and $\overline{k}$ an algebraic closure of k. For each integer $n \geq 1$, we denote by k_n the extension of k of degree n inside $\overline{k}$. Let ℓ be a prime number invertible in k, and $\overline{\mathbf{Q}}_\ell$ an algebraic closure of the field of ℓ-adic numbers along with a fixed embedding $\iota \colon \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$. Let G be a connected commutative algebraic group over k (for example, an additive group $\mathbb{G}_a^d$, a split torus $\mathbb{G}_m^d$, an abelian variety, $\mathbb{G}_a \times \mathbb{G}_m$, etc.). We consider the trace function

$$t_M \colon G(k_n) \longrightarrow \mathbf{C}$$

of a semisimple $\overline{\mathbf{Q}}_\ell$-perverse sheaf M on G that is pure of weight 0 (here, we use the embedding ι to speak of weights and view t_M as taking complex values). The easiest example of such a trace function is a character on $G(k_n)$, but those will not be of interest for us. A more interesting example is a trace function supported at a smooth subvariety $Z \hookrightarrow G$, for instance the normalised characteristic function

$$t_M(x) = \begin{cases} (-1)^{\dim Z}|k_n|^{-\dim Z/2} & x \in Z(k_n), \\ 0 & x \notin Z(k_n). \end{cases}$$

The *discrete Fourier transform* of t_M is the function

$$\begin{aligned} \widehat{t_M}\colon \widehat{G(k_n)} &\longrightarrow \mathbf{C} \\ \chi &\longmapsto \widehat{t_M}(\chi) = \sum_{x \in G(k_n)} t_M(x)\chi(x), \end{aligned}$$

where $\widehat{G(k_n)} = \mathrm{Hom}(G(k_n), \mathbf{C}^\times)$ is the group of characters of $G(k_n)$. Many problems in analytic number theory naturally lead to the question: how the multisets

$$(\widehat{t_M}(\chi))_{\chi \in \widehat{G(k_n)}}$$

are distributed when n tends to infinity?

Theorem 1 (Equidistribution Theorem)**.** *There exists an integer $r \geq 1$ and a compact subgroup $K \subset \mathrm{U}_r$ of the unitary group such that $(\widehat{t_M}(\chi))_{\chi \in \widehat{G(k_n)}}$ equidistributes on average like traces of random matrices of K. That is, the equality*

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{|G(k_n)|} \sum_{\chi \in \widehat{G(k_n)}} f(\widehat{t_M}(\chi)) = \int_K f(\mathrm{tr}(g)) d\mu(g)$$

holds for each continuous bounded function $f\colon \mathbf{C} \to \mathbf{C}$, where μ denotes the normalized Haar measure on K.

If G is unipotent, then $\widehat{t_M}$ is itself the trace function of a perverse sheaf on the dual unipotent group: the Fourier transform $\mathrm{FT}(M)$ of M. In that case, the result is covered by Deligne's equidistribution theorem [1], and K arises as a maximal compact subgroup of the complex points of the arithmetic monodromy group of the restriction of $\mathrm{FT}(M)$ to an open subset on which it is a shifted ℓ-adic local system (the integer r is the generic rank of the Fourier transform). However, if G is not unipotent, then there is no algebraic variety naturally parameterising characters of $G(\overline{k})$, and one can even show that $\widehat{t_M}$ is *not* the trace function of a constructible complex of $\overline{\mathbf{Q}}_\ell$-sheaves. This was the main obstacle to make progress until Katz [4] brought the breakthrough, in the case $G = \mathbb{G}_m$, of using the tannakian formalism to define an analogue of the arithmetic monodromy group and prove an equidistribution theorem as above. In the monograph [2], we generalise his ideas to any connected commutative algebraic group.

The Grothendieck–Lefschetz trace formula, Deligne's theory of weights, and Weyl's equidistribution criterion hint at how to construct this "monodromy group". On the one hand, letting $\mathcal{L}_\chi$ denote the rank-one ℓ-adic sheaf $\mathcal{L}_\chi$ on G with trace function the character χ, the expression

$$\widehat{t_M}(\chi) = \sum_{i=-\dim G}^{\dim G} (-1)^i \mathrm{tr}(\mathrm{Frob}_k \mid \mathrm{H}^i_c(G_{\bar{k}}, M \otimes \mathcal{L}_\chi))$$

as an alternating sum of traces of Frobenius acting on the étale cohomology with compact support of $M \otimes \mathcal{L}_\chi$ and the fact that the absolute value of the eigenvalues of Frobenius on $\mathrm{H}^i_c(G_{\bar{k}}, M \otimes \mathcal{L}_\chi)$ is at most $|k|^{i/2}$ suggest that the discrete Fourier transforms cannot be distributed like traces of random matrices in a *compact* group unless the higher cohomology groups vanish for most characters. In [2, Thm. 2.1], we prove this in the following quantitative form:

Theorem 2 (Generic Vanishing Theorem)**.** *Let $\mathcal{X}(k_n)$ be the subset of characters χ such that $\mathrm{H}^i_c(G_{\bar{k}}, M \otimes \mathcal{L}_\chi)$ and $\mathrm{H}^i(G_{\bar{k}}, M \otimes \mathcal{L}_\chi)$ vanish for all $i \neq 0$, and the "forget of supports" map $\mathrm{H}^0_c(G_{\bar{k}}, M \otimes \mathcal{L}_\chi) \to \mathrm{H}^0(G_{\bar{k}}, M \otimes \mathcal{L}_\chi)$ is an isomorphism. Then*

$$|\widehat{G(k_n)} \setminus \mathcal{X}(k_n)| \ll |k_n|^{\dim G - 1}$$

as n tends to infinity, with an implicit constant that only depends on M.

The integer r is the dimension of $\mathrm{H}^0(G_{\bar{k}}, M \otimes \mathcal{L}_\chi)$ for a generic character χ. In order to prove equidistribution, one also needs a uniform bound on the dimension of the higher cohomology groups for non-generic χ; at this point, we make crucial use of Sawin's *quantitative sheaf theory* [5]. On the other hand, the product of $\widehat{t_M}$ and $\widehat{t_N}$ is again the Fourier transform of a trace function, namely that of

$$M *_! N = \mathrm{R}m_!(\mathrm{pr}_1^* \otimes \mathrm{pr}_2^* N), \tag{1}$$

where $m\colon G \times G \to G$ denotes the group law, and $\mathrm{pr}_i\colon G \times G \to G$ the projections to the first and the second factor. This suggests to endow perverse sheaves on G with the structure of a tannakian category in which convolution (1) is the tensor product, and fibre functors look like $M \mapsto \mathrm{H}^0(G_{\bar{k}}, M \otimes \mathcal{L}_\chi)$. Once we overcome all the technical complications that arise in trying to do so, the smallest tannakian subcategory containing a given perverse sheaf M is equivalent to the category of representations of a reductive (since M is supposed semisimple) algebraic group G_M over $\overline{\mathbf{Q}}_\ell$, and K is a maximal compact subgroup of $G_M(\mathbf{C})$.

To compute K in practice, a very useful tool is *Larsen's alternative*, which in its simplest form is the statement that a compact subgroup $K \subset \mathrm{SU}_r$ of the special unitary group with fourth moment $M_4(K) = \int_K |\mathrm{tr}(g)|^4 d\mu(g)$ equal to 2 is either finite or the whole of SU_r. Thanks to the equidistribution theorem, this fourth moment can be computed as the limit

$$M_4(K) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{|G(k_n)|} \sum_{\chi \in \widehat{G(k_n)}} |\widehat{t_M}(\chi)|^4, \tag{2}$$

so in situations where one is able to prove that the right-hand side is equal to 2 and exclude finite groups, Larsen's alternative gives $K = \mathrm{SU}_r$.

When the perverse sheaf M is supported on a subvariety $Z \hookrightarrow G$, the computation of the right-hand side of (2) involves the number of solutions in $Z(k_n)^4$ to the equation $x_1 + x_2 = x_3 + x_4$ in $G(k_n)$. We say that Z is a *Sidon subvariety* of G if all solutions are trivial, in the sense that either $x_1 = x_3$ or $x_1 = x_4$ holds; these trivial solutions contribute 2 to $M_4(K)$. If M has tannakian rank at least 2, then there are no extra contributions, so that K is either finite or contains SU_r. Two examples of Sidon subvarieties are $\mathbb{G}_m$ embedded diagonally into $\mathbb{G}_a \times \mathbb{G}_m$, and a non-hyperelliptic curve C embedded into its jacobian [3]. They give rise to concrete equidistribution statements for the *Kloosterman–Salié sums*

$$\frac{1}{\sqrt{|k_n|}} \sum_{x \in k_n^\times} \chi(x)\psi(\mathrm{tr}_{k_n/k}(ax + 1/x)),$$

for a given non-trivial additive character ψ of k, as the pair $(a, \chi) \in k_n \times \widehat{k_n^\times}$ varies, and the normalized Artin L-functions

$$L(\rho, T/\sqrt{|k|}) = \prod_{x \in C(\bar{k})} \det(1 - (T/\sqrt{|k|})^{\deg(x)}\rho(\mathrm{Frob}_x)),$$

as ρ varies through non-trivial characters of finite order on $\pi_1(C)^{\mathrm{ab}}$, normalised so that they take the value 1 on a canonical divisor of C.

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The Bottom of the L^2-spectrum

Hee Oh

(joint work with S. Edwards, M. Fraczyk and M. Lee)

Let $X = G/K$ be a Riemannian symmetric space where G is a connected semisimple real algebraic group and $K < G$ is a maximal compact subgroup. Let $\Gamma < G$ be a torsion-free discrete subgroup and let $\mathcal{M} = \Gamma\backslash X$ be the associated locally symmetric space. Consider the Laplacian operator Δ on $\mathcal{M}$. Let $\lambda_0 = \lambda_0(\mathcal{M})$ denote the bottom of the L^2-spectrum of $\mathcal{M}$ for the negative Laplacian:

$$\lambda_0 := \inf\left\{ \frac{\int_{\mathcal{M}} \|\,\mathrm{grad}\ f\|^2\, d\,\mathrm{vol}}{\int_{\mathcal{M}} |f|^2\, d\,\mathrm{vol}} \,:\, f \in C_c^\infty(\mathcal{M}) \right\},$$

where $C_c^\infty(\mathcal{M})$ denotes the space of all smooth functions with compact supports. By Sullivan [4], λ_0 divides the positive and the L^2-spectrum of the negative Laplacian. In joint work with Edwards, Fraczyk and Lee [3], we prove the following: Suppose that G has no rank one factor and $\Gamma < G$ is a Zariski dense discrete subgroup. Then Γ is a lattice in G if and only $\lambda_0(\mathcal{M})$ is not an atom of the L^2-spectrum. In other words, if $\mathrm{vol}(\mathcal{M}) = \infty$, there is no positive L^2-Laplace eigenfunction on $\mathcal{M}$.

We prove this theorem by combining results of [1] and [2].

We pose the following two questions: let G be a semisimple real algebraic group with no rank one factors and $\Gamma < G$ be a Zariski dense discrete subgroup.

(1) When $\Gamma < G$ is not a lattice, can there exist any Laplace eigenfunction in $L^2(\Gamma\backslash X)$?
(2) Is there an example of a non-lattice Γ such that $L^2(\Gamma\backslash G)$ is non-tempered?

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Gap distribution of $\sqrt{n}$ (mod 1) and the circle method

Maksym Radziwill
(joint work with Niclas Technau)

By work of Fejér and of Csillag, it is known that the sequence $\{n^\alpha\}_n$, with $\alpha > 0$ non-integer, is uniformly distributed modulo 1. A much finer question asks for the *gap distribution* of $\{n^\alpha \pmod 1\}_n$; that is after ordering the values $n^\alpha \pmod 1$ with $n \in [N, 2N)$ in increasing order we ask for the gap distribution (at the scale $1/N$) between the ordered values. A striking folklore conjecture in this area is the following.

Conjecture. *Let $\alpha > 0$ be non-integer. Then the gap distribution of*

$$\{n^\alpha \pmod 1\}_n$$

exists. Furthermore the gap distribution is exponential if and only if $\alpha \neq \frac{1}{2}$.

The only case in which this conjecture is known is for $\alpha = \frac{1}{2}$, a result of Elkies-McMullen [3]. For the remaining exponents α it is confirmed that the pair correlation is Poissonian when $\alpha < \frac{43}{117}$ (see the work of Shubin and the first named author [7] and previous work of Sourmelidis, Lutsko and the second named author [4]), and that higher correlations are Poissonian when α shrinks to zero (see [5]) or when α is randomized and large (see [8]). Coincidentally the pair correlation of $\{\sqrt{n} \pmod 1\}_n$ is also Poissonian after removal of squares [2]; however higher correlations do not exist even after removal of the squares. We refer to [6] for related pigeon-hole statistics.

The proof of Elkies and McMullen relies on a connection with ergodic theory; their proof crucially uses a special case of Ratner's theorem (see [1] for an effective version in this special case). We aim to give a new proof of the Elkies-McMullen theorem based on the circle method, our proof can be viewed as a form of Kloosterman refinement in the context of this problem.

Theorem 1. *Let $N > 1$ be given. Let δ be the Dirac delta function at the origin. Let $\{s_{n,N}\}_{N \le n < 2N}$ denote the (numerical) ordering of $\{\sqrt{n} \pmod 1\}_{N \le n < 2N}$. Then, the measure*

$$\frac{1}{N} \sum_{N \le n < 2N} \delta(N(s_{n+1,N} - s_{n,N}))$$

converges weakly to a limit distribution $P(u)$ as $N \to \infty$, in other words, for every smooth compactly supported f,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{N \le n < 2N} f(N(s_{n+1,N} - s_{n,N})) = \int_{\mathbb{R}} f(u) P(u) du.$$

The main disadvantage of our method is that identifying the limit explicitly requires an additional effort (that we do not undertake here); however implicit in the proof is a way to rigorously evaluate the limit to any required precision. In particular one could verify with a computer calculation that the limit is indeed not exponential.

As for the advantages, the main highlights of our approach are that:

(1) We can show that all moments can be computed on a "changed measure"; in other words all the higher correlations exist as long as we insist to consider only those $\sqrt{n} \pmod 1$ that lie in the so-called minor arcs. This is new, since the standard higher correlations blow up already at the third correlation.

(2) The proof can be adapted to show that there exists a transcendental α such that on a subsequence of N the sequence $\alpha\sqrt{n} \pmod 1$ with $n < N$ is Poisson distributed thus addressing an open problem of Elkies and McMullen (albeit not fully).

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The mixing conjecture in the split case

Philippe Michel
(joint work with Valentin Blomer)

Let given $y > 0$ and let H_y be the closed horocycle of height y on the modular surface

$$\mathrm{H}_y = \{x + iy,\ x \in [0, 1)\} \subset \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = X_0(1).$$

A celebrated result of Sarnak states that as H_y equidistributes on X_Γ with respect to the hyperbolic measure. This equidistribution result admits a discrete version: given $q \geq 2$, let

$$H_q = \{\frac{a+i}{q},\ a = 0, \cdots, q-1\} \subset \mathrm{H}_{\frac{1}{q}} \subset X_\Gamma,$$

then, as $q \to +\infty$ the set H_q is also equidistributed: this is a consequence of non-trivial bounds for the q-th Fourier coefficients of Maaßforms. One can consider further variation by imposing some further conditions on the integers $a \in [0, q-1]$. In [3], Ubis and Sarnak considered the equidistribution problem of the subset $H_{q,\mathcal{P}} \subset H_q$ where the integers a are varying along the primes in $[0, q-1]$. They were able to prove under the Ramanujan-Petersson conjecture that any weak-$\star$ limit of the probability measures supported by the $H_{q,\mathcal{P}}$ dominates a positive multiple of the hyperbolic measure. Their analysis, and especially the treatment of the "type II" sums, lead naturally to the problem of the equidistribution of the set of pairs

$$H_{q,b_1,b_2,N} = \{(\frac{ab_1+i}{q}, \frac{ab_2+i}{q})\ a = 1, \cdots, N\} \subset X_0(1) \times X_0(1),$$

for $N = q^{1-\eta},\ 1 \leq b_1, b_2 \leq q^\eta$ for suitable $\eta > 0$. For η not too large methods from harmonic analysis (the shifted convolution problem) allows for satisfactory answers to this equidistribution problem and the main obstacle is the size of the shifting parameters b. In this talk we describe the proof of the following result in which the size of b is unrestricted (see [1]) :

Theorem 1. *Let q be a prime and b an integer such that $(q, b) = 1$ and let $\Lambda_{q,b}$ be the lattice*

$$\Lambda_{q,b} = \{(m, n) \in \mathbb{Z}^2,\ m + bn \equiv 0 \,(\mathrm{mod}\, q)\}.$$

As $\min_{(m,n)\in\Lambda_{q,b}-\{0\}} \|(m, n)\| \to \infty$ the set

$$H_{q,b} := \{(\frac{a+i}{q}, \frac{ab+i}{q})\ a = 1, \cdots, q-1\} \subset X_0(1) \times X_0(1)$$

equidistribute relative to the product of hyperbolic measures.

This result can be seen as an analog of the *mixing conjecture* of Michel-Venkatesh for CM points discussed by Farrell Brumley during this workshop but when the (varying) quadratic field K is replaced by the (fixed) *split* algebra $\mathbb{Q} \times \mathbb{Q}$. Unlike the quadratic field case, this theorem is unconditional.

For the proof, we lift the situation to a product of S-arithmetic quotients of PGL_2 in order to use techniques from ergodic theory. Specifically we use a deep theorem of Einsiedler-Lindenstrauss classifying *joinings* which are invariants under diagonalizable actions of higher rank [2]. The invariance is a consequence of the following simple multiplicative invariance property: for q_1, q_2 two fixed primes (coprime with q as q gets large) and $k, l \in \mathbb{Z}$, one has

$$H_{q,b} = \{(\frac{aq_1^k q_2^l + i}{q}, \frac{abq_1^k q_2^l + i}{q})\ a = 1, \cdots, q-1\}.$$

The classification insures that any ergodic component of any weak-$\star$ limit of the uniform probability measures supported by the $H_{q,b}$ is algebraic and therefore either the product of the Haar measures or the Haar measure supported along a diagonal PSL_2-orbit. To eliminate the later possibility, we show that for suitable pairs (φ_1, φ_2) of cuspforms generating the same representation, the limits of the corresponding Weyl sum, are zero (which is then not compatible with a measure supported along a diagonal orbit).

Evaluating the Weyl sums lead to shifted convolution problems of the shape

$$\frac{1}{q} \sum_{\substack{(m,n)\in\Lambda_{q,b} \\ m,n \ll q}} \lambda_1(m)\lambda_2(n) \to 0 \tag{1}$$

as $\min_{(m,n)\in\Lambda_{q,b}-\{0\}} \|(m,n)\| \to \infty$; here $\lambda_1(m),\ \lambda_2(n)$ denote the Hecke eigenvalues.

For this, inspired by techniques initiated by Holowinsky in the context of the QUE conjecture, we prove instead that

$$\frac{1}{q} \sum_{\substack{(m,n)\in\Lambda_{q,b} \\ m,n \ll q}} |\lambda_1(m)||\lambda_2(n)| \to 0.$$

The proof use sieve methods, multiplicative function theory and partial results toward the Sato-Tate conjecture. This step, a priori, requires the Ramanujan-Petersson conjecture for the Hecke eigenvalues of φ_1, φ_2. Fortunately, it is sufficient for us to chose φ_1, φ_2 to be shifts of a (well adjusted) twist of a fixed CM form (attached to Hecke character of a real quadratic field). The twist is adjusted so that, at all the places of S (we eventually chose $S = \{q_1, q_2, \infty\}$), the underlying local representations are unramified principal series with *trivial* parameters; this in turn insures that (1) is not compatible with measures supported along diagonal orbits.

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Orbits and analysis of automorphic forms

Paul D. Nelson

We discussed recent estimates for higher rank L-functions, on average and individually, obtained by analyzing automorphic forms using the method of coadjoint orbits and localized vectors. Our main result is that standard L-functions enjoy a spectral aspect subconvex bound in the case of uniform parameter growth: for each $n \in \mathbb{Z}_{\geq 1}$ and $c_0 > 0$, there exists $C_1 \geq 0$ with the following property. Let π be a unitary cuspidal automorphic representation for GL_n over $\mathbb{Q}$. Let $L(\pi, s)$ denote the finite part of its standard L-function and $L_\infty(\pi, s)$ the archimedean factor, so that $L_\infty(\pi, s) = \prod_{j=1}^n \Gamma_{\mathbb{R}}(s + \lambda_j)$, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ for some complex "parameters" $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Assume that for all j and k, the parameters satisfy the "uniform growth" assumption

$$|\lambda_j| \geq c_0 |\lambda_k|. \tag{1}$$

Then we establish the following estimate:

$$|L(\pi, \tfrac{1}{2})| \leq C_1 N_\pi^{\frac{1}{2}} \prod_{j=1}^n (1 + |\lambda_j|)^{\frac{1}{4} - \frac{1}{6n^5}}, \tag{2}$$

where $N_\pi \in \mathbb{Z}_{\geq 1}$ denotes the finite conductor of π. This estimate improves upon the convexity bound when N_π is fixed and the parameters λ_j vary subject to (1). It may be understood as generalizing breakthrough results of Iwaniec [5] and Blomer–Buttcane [2] for the cases $n = 2, 3$ to general n, but is a bit more general, even for $n \leq 3$, in that it does not require the parameters λ_j to be well-spaced. In particular, we obtain a t-aspect subconvex bound for standard L-functions:

$$|L(\pi, \tfrac{1}{2} + it)| \leq C_\pi (1 + |t|)^{\frac{1}{4} - \frac{1}{6n^5}}.$$

The proof is based on the integral representation of Jacquet–Piatetski-Shapiro–Shalika [14] for Rankin–Selberg L-functions on $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$, generalizing Hecke. We consider the quotients $[G] = \Gamma \backslash G = \mathrm{GL}_{n+1}\mathbb{Z} \backslash \mathrm{GL}_{n+1}\mathbb{R}$ and $[H] = \Gamma_H \backslash H = \mathrm{GL}_n\mathbb{Z} \backslash \mathrm{GL}_n\mathbb{R}$, and a pair of automorphic representations $\pi \subseteq C^\infty([G])$ and $\sigma \subseteq C^\infty([H])$. For automorphic forms $v \in \pi$ and $u \in \sigma$, we have

$$\int_{[H]} vu = L(\pi \times \sigma, \tfrac{1}{2}) Z(u, v) \tag{3}$$

for some local integral $Z(u, v)$ defined in terms of the archimedean Whittaker functions of u and v. For quantitative purposes, an important point (emphasized in low rank examples by Sarnak [10], Bernstein–Reznikov [1], Venkatesh [13] and Michel–Venkatesh [12]) is to choose the vectors v and u so that $Z(u, v)$ is not merely nonzero, but quantitatively not small. Our paper [7] with Venkatesh developed a systematic approach to making such a choice under the assumption that, for some asymptotic parameter $T \to \infty$, the parameters $\{\lambda_1, \dots, \lambda_{n+1}\}$ for π and $\{\mu_1, \dots, \mu_n\}$ for σ satisfy the assumption (related to (1))

$$\lambda_j \ll T, \quad \mu_k \ll T, \quad \lambda_j + \mu_k \asymp T. \tag{4}$$

The approach is roughly as follows. Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{g}^*, \mathfrak{h}^*$ denote the Lie algebras and their duals, which we may identify with spaces of matrices. There is a natural restriction map $\mathfrak{g}^* \to \mathfrak{h}^*$, $\xi \mapsto \xi_H$, that sends a matrix to its upper-left $n \times n$ block. The coadjoint orbits $\mathcal{O}_\pi \subseteq \mathfrak{g}^*$ and $\mathcal{O}_\sigma \subseteq \mathfrak{h}^*$ attached to π and σ describe their characters. We verify that there is an element $\tau \in \mathfrak{g}^*$, of magnitude comparable to T and lying in a fixed cone of regular elements, such that $\tau \in \mathcal{O}_\pi$ and $-\tau_H \in \mathcal{O}_\sigma$. Informally, we say that a T-dependent unit vector $v = v_T \in \pi = \pi_T$ is *localized* at the parameter τ if, with the informal asymptotic notation

$$J_\tau := G \cap (1 + o(1)) \cap \Big(G_\tau + o(T^{-1/2})\Big),$$

$$\chi_\tau(\exp x) = \exp(i\langle x, \tau\rangle),$$

we have

$$\pi(g)v \approx \chi_\tau(g)v \quad \text{for all } g \in J_\tau.$$

The results of [7] imply that there are unit vectors $v \in \pi$ and $u \in \sigma$, localized at τ and $-\tau_H$, such that the integral representation (3) specializes to

$$\int_{[H]} vu \approx T^{-n^2/4} L(\pi \times \sigma, \tfrac{1}{2}). \tag{5}$$

One can apply such integral representations in a few ways. A first application [7] is to study averages of $L(\pi \times \sigma, \frac{1}{2})$ as σ varies in a large, dyadic family, with π held fixed, using the Parseval-type identity

$$\int_{[H]} |v|^2 = \sum_\sigma \sum_{u \in \mathcal{B}(\sigma)} \left| \int vu \right|^2 = \sum_\sigma w(\sigma) |L(\pi \times \sigma, \tfrac{1}{2})|^2$$

applied to localized vectors v. The limits of the L^2-masses of such localized vectors $v = v_T$ have the property that in the $T \to \infty$ limit, the measure $|v_T|^2 \, d\mu_{[G]}$ becomes G_τ-invariant. When π is fixed, the parameter τ lies close to the nilcone, so Ratner's theorem then forces the limit measures to be uniform.

One can also average over π, as in the pioneering work of Iwaniec–Sarnak [6]. Let ϕ_τ be a smoothened normalized characteristic function of J_τ, weighted by χ_τ^{-1}. Then, heuristically, the operator $\pi(\phi_\tau)$ vanishes unless π lies in some "short" family $\mathcal{F}_\tau$, in which case $\pi(\phi_\tau)$ projects onto the span of a vector $v_\pi \in \pi$ localized at τ. With this heuristic, we obtain from (5) that

$$\int_{x,y \in [H]} u(x)\overline{u(y)} \sum_{\gamma \in \Gamma} \phi_\tau(x^{-1}\gamma y) \, dx \, dy \approx T^{-n^2/2} \sum_{\pi \in \mathcal{F}_\tau} \left| L(\pi \times \sigma, \tfrac{1}{2}) \right|^2. \tag{6}$$

Here "Lindelöf on average" recovers the convexity bound for the individual L-values (assuming (4)), so a subconvex bound follows from the amplification method of Duke–Friedlander–Iwaniec and a sufficiently robust proof of an asymptotic formula for (6). This approach was initiated by Simon Marshall, who shared some ideas in talks starting in early 2018 and tentatively announced some results (in a related p-adic aspect) under the further assumption that

$$\lambda_j - \lambda_k \asymp T, \qquad \mu_j - \mu_k \asymp T \qquad (j \neq k). \tag{7}$$

Marshall emphasized the importance, for the purpose of estimating the left hand side of (6), of the volume bound

$$\mathbb{P}(z \in J_{\tau_H} : \gamma z \in H J_\tau) \ll T^{-\delta} \qquad (\gamma \notin HZ, \text{ fixed}).$$

A general proof (without assuming (7)) was given in [8], in the the following form obtained by restricting to central directions inside the n-dimensional centralizer H_{τ_H}:

$$\mathbb{P}(z \in Z_H \cap (1 + o(1)) : \gamma z \in H J_\tau) \ll T^{-\delta}.$$

The paper [8] also made use of the quantitative amplifier for GL_n developed by Blomer–Maga [4, §4] in their work on the sup norm problem. Additional ideas were developed in [9] to handle the non-compactness of the quotient $[H]$. These made use of an observation of Blomer–Harcos–Maga [3, §3.3].

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Problem session

Chaired by Jens Marklof. Notes taken by Paul Nelson.

1. Felipe Goncalves, *Linear domino size problem*

Definition 1. A linear domino game is a triple $D = (\Sigma, T, s)$, where Σ is a set (of "symbols"), T is a set (of "tiles") consisting of ordered pairs of elements of Σ, denoted $[a, b]$, and $s : \Sigma \to \mathbb{R}_+$ is a ("size") function. A *linear domino sequence* is a sequence $w = [a_1, b_1][a_2, b_2][a_3, b_3]...$, such that $[a_i, b_i] \in T$ and $b_i = a_{i+1}$ for all i. The *average size* w is defined to be

$$A(w) := \limsup_{N\to\infty} \frac{s(a_1) + \sum_{i=1}^{N} s(b_i)}{N}.$$

Theorem 2. If Σ is finite, then there exists a periodic sequence w for which $A(w)$ is minimal.

What if Σ is infinite? Must there exist a periodic sequence w for which $A(w)$ is minimal?

Example 3. If no structure on Σ is given, then the answer is no. For instance, consider $\Sigma = \{1 + 1/n : n \geq 1\}$ and $T = \{[1 + 1/n, 1 + 1/(n + 1)] : n \geq 1\}$. Then there is only one possible sequence (the obvious one), which is not periodic, and has average size 1. However, if we compactify Σ, that is, we include 1 in Σ then there is a periodic minimizer: $w = [1, 1][1, 1]....$

Question 1. Find a linear domino game $D = (\Sigma, T, s)$, where Σ is a infinite compact topological space and $s : \Sigma \to \mathbb{R}_+$ is continuous, such that

$$A(w) > \inf_{\omega} A(\omega)$$

for any periodic linear domino sequence w. That is, there is a game for which minimizers are never periodic. If this is impossible, prove it is.

Remark 4. A generic version of the following problem reduces to Question 1 Suppose given an interval $I \subseteq \mathbb{R}$ of size one. Suppose we try to tile the line by translates $a_i + I$ of that interval for which the distances $a_i - a_j$ lie in some given set, such as

$$\{0, 1, e, \pi\} \cup [100, \infty).$$

Maximize density of the set $\cup_i(a_i + I)$. Can a maximizer be periodic? .

2. Javier Fresán, *Pseudopolynomial permutation approximation problem*

Let $f \in \mathbb{Z}[T]$ be an irreducible polynomial of degree n whose Galois group is the full symmetric group S_d. Let $0 \leq k \leq n$. The Chebotarev density theorem implies that

$$\frac{1}{\pi(X)} \# \{p \leq X : f \pmod{p} \text{ has } k \text{ zeros}\}$$

converges as $X \to \infty$ to the probability that a random permutation on S_n has k fixed points, denoted

$$\mathbb{P}\left(|\mathrm{Fix}(X_n)| = k\right).$$

The function

$$f(n) := \lfloor en! \rfloor$$

is an example of a *pseudopolynomial* (a function with the property that $m - n$ divides $f(m) - f(n)$). It has the property that $f(x) \mod p$ is well-defined for $x \in \mathbb{Z}/p$.

Conjecture 1 (Kowalski—Soundararajan)**.**

$$\lim_{X\to\infty} \frac{|\{p \le X : \#\{0 \le a \le p-1 : f(x) \mod p = 0\} = k\}|}{\pi(X)} = \frac{1}{e}\frac{1}{k!}.$$

This is supported by numerics.

3. Anke Pohl, *SOS ("surface of section") for the geodesic flow roots problem*

Question 2. Jens Marklof and Matthew Walsh have a coding for the horocycle flow on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ that gives the gaps they study. Is there also a coding for the geodesic flow on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ that gives the same?

Remark 5. Why might we want to have this? For the geodesic flow, there is also an expectation that there is a connection to Laplace eigenfunctions, due to the correspondence principle. It's not clear whether Laplace eigenfunctions are in any sense a quantization of the horocycle flow. There remain open questions in the setting of Jens and Matthew's talks, so maybe a coding of the geodesic flow would allow one to combine these two approaches to give a different view on the same problem.

4. Gergely Harcos, *Global sup-norm problem for* $\mathrm{Sp}_4(\mathbb{R})$

This problem is connected to work by Anke Pohl and Valentin Blomer. They gave a sup-norm bound for Hecke–Maaß cusp forms on $\mathrm{Sp}_4(\mathbb{R})$, restricted to a compact set. Gergely's question is whether one can do this uniformly over the compact set. The Whittaker functions are not fully explicated, and there is the deeper issue that the Whittaker expansion doesn't just involve the Hecke eigenvalues. As a side project, estimate the Whittaker functions that arise.

5. Abhishek Saha, *Beyond GRH barrier for Fourier coefficients for* Sp_4

Let F be a holomorphic Siegel cusp form of weight k for $\mathrm{Sp}_4(\mathbb{Z})$, and a Hecke eigenform. Let

$$\Lambda := \left\{ \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} : a, b, c \in \mathbb{Z},\, b^2 - 4ac < 0 \right\}.$$

This identifies with the set of binary quadratic forms. We denote by $\Lambda_0 \subseteq \Lambda$ the subset for which $b^2 - 4ac$ is a fundamental discriminant.

The Fourier expansion looks like

$$F(Z) = \sum_{S \in \Lambda} a(F,S) e\left(\mathrm{trace}(SZ)\right), \quad e(x) := e^{2\pi i x}.$$

Question 3. Show that there exists a fixed $\delta > 0$ so that for all $S \in \Lambda_0$,

$$a(F,S) \ll_F \det(S)^{\frac{k-1}{2}-\delta}.$$

Feel free to assume any standard conjecture on L-functions (e.g., GRH), as well as any expected period identities.

The current best known unconditional bound is due to Kohnen.

Theorem 6 (Kohnen)**.** We have

$$a(F,S) \ll_{F,\epsilon} \det(S)^{\frac{k}{2}-\frac{13}{36}+\epsilon}$$

for all $S \in \Lambda_0$.

Conditionally, we have

Theorem 7 (Jaasaari–Lester–Saha)**.** Assuming GRH, we have

$$a(F,S) \ll_{F,\epsilon} \frac{\det(S)^{\frac{k}{2}-\frac{1}{2}}}{\log(\det S)^{\frac{1}{8}}}.$$

for all $S \in \Lambda_0$.

This is still far from the conjectured truth, which is stated next.

Conjecture 2 (Reznikov—Saldana)**.** We have

$$a(F,S) \ll_{F,\epsilon} \det(S)^{\frac{k}{2}-\frac{3}{4}+\epsilon}$$

for all $S \in \Lambda_0$.

The Reznikov—Saldana conjecture should also hold for $S \in \Lambda$ provided one excludes the Saito–Kurokawa lifts.

To explain where these bounds come from, here's a period identity that follows from the Gan—Gross—Prasad conjectures: for a character Λ of the class group,

$$\left| \sum_{\substack{S \in \mathrm{SL}_2(\mathbb{Z}) \backslash \Lambda_0 \\ \mathrm{disc}(S) = d}} \Lambda(S) a(F,S) \right|^2 \sim d^{k-1} L\left(\tfrac{1}{2}, F \times \theta_\Lambda\right). \tag{1}$$

where $\sim$ denotes equality up to some understood constant. (We hope we got the constant right here.)

From this one can easily see that, assuming GLH, one has

$$a(F,S) \ll \det(S)^{\frac{k}{2}-\frac{1}{2}+\epsilon}.$$

Assuming GRH, one can do slightly better, as noted earlier. Note that the Reznikov—Saldana conjecture essentially says that there is square-root cancellation in the left side of (1).

The question posed challenges us to break the $\frac{k}{2} - \frac{1}{2}$ barrier assuming any standard conjectures one wishes to.

Maksym Radziwiłł: is this similar to bounding a holomorphic cusp form at a Heegner point?

6. Jens Marklof, *Gap distribution of $p\alpha$ modulo 1*

Gap statistics. Everyone knows the three gap theorem, right? Look at

$$\alpha, \quad 2\alpha, \quad 3\alpha, \quad \dots, \quad N\alpha,$$

taken modulo 1. Then there are at most three different gap lengths.

Question 4. Suppose now that instead of looking at $n\alpha$ modulo 1, we look at $p\alpha$ modulo 1, where p runs over the primes. What does the gap length distribution look like? We know that this seequence is uniformly distributed modulo 1, but we would like to understand its statistics. How about the gap lengths?

- Maksym Radziwiłł: does it have to be primes, or we could look at squares?
- Jens Marklof: see the theorem of Aled Walker.

7. Hakan Hedenmalm, *Zeros of Dirichlet series*

Definition 8. A Dirichlet polynomial is a function of the form

$$P(s) = \sum_{n=1}^{N} a_n n^{-s}.$$

Question 5. Which sets

$$Z(P) = \{s \in \mathbb{C} : P(s) = 0\}$$

are possible?

Question 6. Given a set Z, construct a Dirichlet series with zeros at Z.

Suppose we look for a Dirichlet series that converges in some half-plane. Is there such a Dirichlet series that has one single zero in that half-plane?

Given a half-plane, is there a Dirichlet series that converges there (not necessarily absolutely) and which has a single zero in that half-plane.

Hilberdink and Sais have an arxiv preprint.

Reporter: Radu Toma

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