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Dynamische Systeme

Organized by
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Abstract. This workshop continues a series of workshops whose current format originated in 1981 under then-organizers Moser and Zehnder, and whose latest iteration took place in July 2023. The general goal of this series of workshops is to discuss the latest developments in the field of dynamical systems, broadly construed, and its connections with neighboring areas of mathematics such as differential geometry, partial differential equations, and more recently contact and symplectic geometry. We continued this tradition, bringing in new participants working in areas of dynamical systems and its connections with other areas of mathematics that are currently highly active and/or showing great promise for future development. Key focus areas for the 2023 workshop include spectral rigidity for planar domains, chaotic and oscillatory motions in celestial mechanics, conformal symplectic dynamics, and relations between dynamics.

Mathematics Subject Classification (2020): 37, 53D, 35, 70F, 70H.

Introduction by the Organizers

The workshop *Dynamische Systeme*, organised by M.-C. Arnaud (Paris), M. Hutchings (Berkeley) and V. Kaloshin (Vienna), was well attended with 46 participants with broad geographic representation from 12 countries. The workshop covered a diverse range of topics in dynamical systems and related areas, with a special emphasis on various kinds of spectra and their applications to dynamics.

Different kinds of results on rigidity were presented. Alena Erchenko proved that if two smooth compact connected oriented surfaces with boundary of Anosov type have the same marked boundary distance, then they are isometric. Konstantin Drach proved several results concerning the Lyapunov rigidity of expanding maps of the circle. Ilya Koval proved that almost all billiard maps in an ellipse are such that their perturbations that have rotational caustics near the boundary are also ellipses. Alfonso Sorrentino proved some rigidity results concerning the completely periodic Lagrangian tori of higher dimensional twist maps.

Variational methods were used by several speakers. Using Levi-Civita regularization, Kai Cieliebak proved the existence of periodic orbits for the electrons of a helium atom. Considering the restricted 3-body problem, Susanna Terracini proved the existence of orbits having prescribed behavior in the past and the future for almost all angular momenta.

Using normal forms in infinite dimension, Jessica Elisa Massetti proved stability in long time for the beam equation and the non linear Schrödinger equation.

Maxime Zavidovique studied the discounted Hamilton-Jacobi PDE, that is associated to a conformally Hamiltonian dynamics, and proved that it selected one particular weak KAM solution when the conformal factor tends to 1, and extended this result to a degenerate setting.

Using qualitative methods and horseshoes, in a problem with 4 planets, Jacques Fejoz showed the existence of orbits such that the semimajor axis of the outer planet has very large variations.

Sylvain Crovisier presented results on the relations between Lyapunov exponents and entropy for smooth diffeomorphisms of surfaces. Answering a conjecture of Viana, he proved that the existence of an empirical Lyapunov exponent almost everywhere implies the existence of a physical measure. Patrice Le Calvez stated two results concerning periodic points in conservative surface dynamics. The first one is that an area-preserving homeomorphism of a hyperbolic closed surface, whose rotation vector has a nonzero rational direction, has infinitely many periodic orbits with nonzero rotation vector. The second result is that a C^∞ generic Hamiltonian diffeomorphism of a closed surface of genus at least 1 has infinitely many periodic orbits with nonzero rotation vector. This answers a question of Viktor Ginzburg and Basak Gurel.

Dmitry Turaev studied reversible vector fields in $\mathbb{R}^{2n}$ such that the set of fixed points of the involutory reversing symmetry is n-dimensional. He proved that for such systems that have a smooth one-parameter family of symmetric periodic orbits which is of saddle type in normal directions, the topological entropy is positive when the stable and unstable manifolds of this family of periodic orbits have a strongly-transverse intersection. Using Birkhoff sections, Ana Rechtman explained why every hyperbolic periodic orbit of every C^∞ generic Reeb flow has heteroclinic intersections.

In more symplectic dynamics, Gabriele Benedetti constructed Zoll magnetic systems on the two-torus by a Nash-Moser construction, generalizing a result of Guillemin for the two-sphere. Barney Bramham presented a dynamical interpretation of the Calabi invariant in higher dimensions, generalizing a result of Fathi in the two-dimensional case. Jo Nelson explained a computation of knot-filtered embedded contact homology for torus knots, with applications to the dynamics of surface diffeomorphisms in mapping classes arising from fibered knots. Leonid

Polterovich presented a general theory of big fiber theorems and ideal-valued measures, with applications to non-displaceability results in symplectic geometry. Rohil Prasad studied the behavior of low energy holomorphic curves with applications to the dynamics of Reeb pseudorotations.

The meeting was held in an informal and stimulating atmosphere. The weather was nice and the traditional walk to St. Roman, took place on Wednesday.

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Workshop: Dynamische Systeme

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Abstracts

Physical measures of smooth surface diffeomorphisms

Sylvain Crovisier
(joint work with Jérôme Buzzi, Omri Sarig)

The dynamics of a diffeomorphism f of a closed manifold M can be described through its invariant probability measures. Different quantities may be associated to such a measure μ: its basin $\mathcal{B}(\mu) = \{x \in M, \frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)} \to \mu\}$, its entropy $h(\mu)$, its upper Lyapunov exponent $\lambda^+(\mu) := \lim_{+\infty} \frac{1}{n}\int \log \|Df^n\| d\mu$.

Some particular measures play an important role: the *measures maximizing the entropy* (those satisfying $h(\mu) = \sup_\nu h(\nu)$), and the *physical measures* (those satisfying $\mathrm{Vol}(\mathcal{B}(\mu)) > 0$).

We present two results about these measures whose proofs are similar and are based on Yomdin's technique [7]:

Theorem. *Let f be a C^∞ diffeomorphism of a closed surface and (ν_k) be a sequence of ergodic measures converging towards an ergodic measure μ. Then,*

$$h(\nu_k) \underset{k}{\to} h(\mu) > 0 \qquad \Longrightarrow \qquad \lambda^+(\nu_k) \underset{k}{\to} \lambda^+(\mu).$$

This result (and more precise versions, including the C^r case) has appeared in [2]. It has strong ergodic consequences, which will be discussed in [4]. In particular it implies that for smooth surface diffeomorphisms with positive topological entropy, (up to considering a suitable iterate) the measures maximizing the entropy are exponentially mixing and satisfy a central limit theorem.

Whereas measures maximizing the entropy do exist for any smooth diffeomorphisms (as shown by Newhouse [5]), this is not the case of physical measures: proving that their existence for a given dynamical system is a major problem. Viana has conjectured that a smooth diffeomorphism admits a physical measure if all the Lyapunov exponents are defined and do not vanish for points belonging to a subset with full volume. We have proved that this is indeed the case on surfaces:

Theorem. *Let f be a C^∞ diffeomorphism of a closed surface with positive topological entropy. If the set $\{x \in M, \limsup_{+\infty} \frac{1}{n}\log \|Df^n(x)\|\}$ has positive volume, then f admits a physical measure.*

This result has appeared in [3]. Another proof, which also states a C^r-version, has been given by Burguet in [1].

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Marked boundary rigidity for Anosov type surfaces

Alena Erchenko
(joint work with Thibault Lefeuvre)

Consider a smooth compact connected oriented Riemannian manifold (Σ, g) with boundary of Anosov type, meaning that the following conditions are satisfied:

- The boundary $\partial\Sigma$ of Σ is strictly convex, i.e., the second fundamental form of g is strictly positive on the boundary $\partial\Sigma$;
- The metric g has no conjugate points in Σ, i.e., for any two points $p, q \in \Sigma$ there is no non-zero Jacobi field along a geodesic piece connecting p and q such that it vanishes at those points;
- Let K be the maximal geodesic flow-invariant set in the interior of the unit tangent bundle $S\Sigma$ of (Σ, g). Then, K is hyperbolic, i.e., there exists a geodesic flow-invariant continuous splitting

  $$T(S\Sigma)|_K = \mathbb{R}X \oplus E^s \oplus E^u,$$

  and uniform constants $C, \lambda > 0$ such that

  $$\begin{aligned} &|d\varphi_t(w)| \leq Ce^{-\lambda t}|w|, && \forall t \geq 0, \forall w \in E^s, \\ &|d\varphi_{-t}(w)| \leq Ce^{-\lambda t}|w|, && \forall t \geq 0, \forall w \in E^u, \end{aligned} \tag{1}$$

  where $\varphi_t \colon S\Sigma \to S\Sigma$ is the geodesic flow.

For any $x, y \in \partial\Sigma$, let $\mathcal{C}_{x,y}$ be the set of all homotopy classes of curves with fixed endpoints x and y. Since g is a metric of Anosov type, for every $x, y \in \partial\Sigma$ and for every homotopy class of curves $c \in \mathcal{C}_{x,y}$, there exists a unique g-geodesic $\gamma_{x,y}(c)$ joining x to y [5, Lemma 2.2]. We define $\mathcal{P} := \{(x, y, c) \mid x, y \in \partial\Sigma, c \in \mathcal{C}_{x,y}\}$. The marked boundary distance function is then defined as

$$d_g \colon \mathcal{P} \to [0, \infty), \qquad d_g(x, y, c) := \ell_g(\gamma_{x,y}(c)). \tag{2}$$

An interesting question is if d_g determines the "geometry" of (Σ, g) (see, for instance, [1, Conjecture 1.6])).

Conjecture 1 (Marked Boundary Rigidity Conjecture)**.** *Let g_1, g_2 be two metrics of Anosov type on Σ. If g_1 and g_2 have the same marked boundary distance function, that is $d_{g_1} = d_{g_2}$, then there exists a smooth diffeomorphism $\phi \colon \Sigma \to \Sigma$ such that $g_1 = \phi^* g_2$ and $\phi|_{\partial\Sigma}$ is the identity map.*

When Σ is diffeomorphic to a ball, Conjecture 1 is more concisely known as the boundary rigidity conjecture [7, 2]. The boundary rigidity conjecture is proved in dimension 2 [9], and the most general result in dimension ≥ 3 is that boundary rigidity holds under the extra assumption that the manifold is foliated by strictly convex hypersurfaces [10]. We note that this problem is closely related to the Caldéron problem for the Dirichlet-to-Neumann map (see [8, Section 11.1]).

We prove Conjecture 1 for all surfaces.

Theorem 1. [6, Theorem 1.1] *Let Σ be a smooth compact connected oriented surface with boundary. Let g_1, g_2 be two metrics of Anosov type on Σ. If g_1 and g_2 have the same marked boundary distance function, that is, $d_{g_1} = d_{g_2}$, then there exists a smooth diffeomorphism $\phi \in \mathrm{Diff}_0(\Sigma, \partial\Sigma)$ such that $g_1 = \phi^* g_2$, where $\mathrm{Diff}_0(\Sigma, \partial\Sigma)$ is defined as the set of all diffeomorphisms of Σ fixing the boundary, and isotopic to the identity through a path of diffeomorphisms preserving the boundary.*

The main ingredient of the proof is a new transfer principle showing that, in any dimension, the marked length spectrum rigidity conjecture implies the marked boundary distance rigidity conjecture under the existence of a suitable isometric embedding into a closed Anosov manifold.

To formulate the transfer principle, we first introduce some terminology. We say that (Σ, g) is extendable if there exists a codimension 0 isometric embedding of (Σ, g) into a smooth closed connected oriented Rimenannian manifold (M, g') with Anosov geodesic flow. Two metrics g_1 and g_2 of Anosov type are consistently extendable if they are both extendable to the same manifold M and the extensions g_1' and g_2' coincide on $M \setminus \Sigma$.

Theorem 2. [6, Theorem 1.4] *Let Σ be a smooth compact connected oriented manifold with boundary. Let g_1, g_2 be two smooth metrics of Anosov type on Σ. Assume that g_1 and g_2 have the same marked boundary distance function, that is, $d_{g_1} = d_{g_2}$, and that the metrics are consistently extendable to a closed manifold M. Further assume that the marked length spectrum is injective on M for Anosov metrics of finite regularity. Then, there exists a smooth diffeomorphism $\phi\colon \Sigma \to \Sigma$ such that $\phi|_{\partial\Sigma} = \mathbf{1}_{\partial\Sigma}$ and $g_1 = \phi^* g_2$.*

The conditions in the above theorem are guaranteed for surfaces by the following two facts.

Theorem 3. [3, Theorem A] *Let (Σ, g) be a smooth compact connected oriented Riemannian manifold with boundary of Anosov type. Further assume that each component of the boundary is diffeomorphic to a sphere. Then, (Σ, g) is extendable.*

Theorem 4. [4, Theorem 1.1, Remark 3.12] *Let M be a smooth closed connected oriented surface. Let g_1, g_2 be two C^k-metrics with $k \geq 4$, Anosov geodesic flow on M, and the same marked length spectrum. Then, there exists a C^{k-1} diffeomorphism $\phi\colon M \to M$, isotopic to the identity, such that $g_1 = \phi^* g_2$.*

We note that the condition on the boundary in Theorem 3 is satisfied for surfaces. The theorem can be extended to the case with a boundary component diffeomorphic to $S^1 \times S^{n-2}$ if Σ is n-dimensional. Moreover, Guedes Bonthonneau, in work in progress, is able to fully remove the restriction on the topology of the boundary components when $n = 3$.

Finally, recall that boundary rigidity is the case of Conjecture 1 where Σ is diffeomorphic to a ball. As a corollary of Theorems 2 and 3, we obtain that marked length spectrum rigidity of manifolds with Anosov geodesic flows implies boundary rigidity.

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A variational approach to frozen planet orbits in helium

Kai Cieliebak

(joint work with Urs Frauenfelder, Evgeny Volkov)

Frozen planet orbits are periodic orbits of the classical helium atom in which both electrons move on a line on the same side of the nucleus. Such orbits play a role in the semiclassical treatment of the helium atom, and numerical results suggest that they exist and are stable for all negative energy values.

In joint work with Urs Frauenfelder and Evgeny Volkov [2, 3], we develop a variational framework to algebraically count frozen planet orbits of given energy (or equivalently, of given period). To regularize collisions of the electrons with the nucleus, we apply a method by Barutello, Ortega and Verzini [1] separately to both electrons. This leads to different time reparametrizations for the two electrons, and thus to a nonlocal functional $\mathcal{B}$ which is not smooth in the usual sense. Nonetheless, this functional has an L^2-gradient vector field $\nabla\mathcal{B}$ with the following properties:

(1) $\nabla\mathcal{B}$ defines a self-adjoint Fredholm section of class C^1 whose spectrum is uniformly bounded from below;
(2) the zero set of $\nabla\mathcal{B}$ (i.e., the critical point set of $\mathcal{B}$) is compact.

For such a vector field one can define an integer valued Euler number $\chi(\nabla\mathcal{B}) \in \mathbb{Z}$, counting its zeroes with appropriate signs. This is based on the observation that the determinant line bundle over the space of essentially positive self-adjoint Fredholm operators has a canonical orientation.

To compute $\chi(\nabla\mathcal{B})$, we deform $\mathcal{B}$ to the functional $\mathcal{B}_{av}$ in which the two electrons interact only by their average positions. It turns out that, restricted to a suitable space of symmetric orbits, the functional $\mathcal{B}_{av}$ has a unique critical point, which is nondegenerate of Morse index zero. Homotopy invariance of the Euler number now gives $\chi(\nabla\mathcal{B}) = \chi(\nabla\mathcal{B}_{av}) = 1 \in \mathbb{Z}$. In particular, there exists a frozen planet orbit of given energy.

The existence of a frozen planet orbit of given energy can be proved more directly, for example by a Birkhoff type shooting method as pointed out by Lei Zhao. The preceding results should rather be seen as a proof of concept that variational techniques are applicable to nonlocal functionals such as the one above. Interesting directions for further research include the application of these techniques to other physical problems, and the development of a Floer theory for nonlocal functionals.

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Local strong Birkhoff conjecture of almost every ellipse

Illya Koval

Billiard dynamics can be defined on the smooth convex planar domain as follows. Assume there is a point inside of the domain, that moves with unit speed along a straight line. Whenever the ball hits the boundary, it reflects, such that the angle of incidence is equal to that of reflection.

There are many interesting tables one can consider. One of them is an ellipse. In ellipses, the billiard dynamics allow for the first integral. Moreover, if one draws another smaller ellipse with the same foci, it would be a caustic for the bigger one. Namely, if a billiard ball was tangent to the smaller ellipse before the reflection, it will stay tangent after. This shows that billiards in ellipses are integrable.

Birkhoff conjecture is one of the most famous open problems in billiard dynamics. This conjecture states, that the only integrable billiard domains are ellipses. In a sense, the conjecture claims that the existence of so many caustics in a single domain is a very rare phenomenon.

Before tackling the conjecture itself, one should define what exactly does integrability mean. It turns out that the definition of it is not unique. It should involve the existence of many invariant curves or caustics, and, in the most canonical way, should only consider the dynamics near the boundary of the domain. Particularly, only orbits with arbitrary small reflection angles can be studied.

However, there was no result on Birkhoff conjecture, that worked with integrability arbitrary close to the boundary. There were many local and global theorems proven by various authors, but all of them considered the fixed neighborhoods of the boundary.

For example, the result of Kaloshin and Sorrentino [3], that shows that the only integrable deformations of ellipses are ellipses themselves, requires the existence of a caustic of 3-periodic points, while the new result by Bialy and Mironov [2] that proves Birkhoff conjecture in the class of centrally symmetric domains, requires an invariant curve of period 4. Since the period should approach infinity close to the boundary, these results are not "canonical".

Attempts were made to generalize these results to work close to the boundary. For example, [4] tried to prove the local Birkhoff conjecture for nearly-circular ellipses near the boundary. This means that one considers an integrable small deformation of a nearly-circular ellipse and proves that it must be an ellipse itself. However, this attempt ran into a problem and was only able to reduce the size of the neighborhood by a fixed amount.

In the talk, we are going to consider the following notion of integrability. We call a caustic a rational one, if tangent to it orbits are all periodic. Particularly, the dynamics, restricted to the caustic should have a rational rotation number $\omega = p/q$, where q is the period of orbits, and p is the number of times they wind around the boundary. In general, the rotation number can be considered to be from 0 to $1/2$, where smaller rotation numbers correspond to dynamics near the boundary.

As such, we will call a domain q_0-rationally integrable for some $q_0 \geq 3$ if it has all the rational caustics with rotation numbers lower than $1/q_0$. Since ellipses have all the rational caustics, except the $1/2$ one, they satisfy this definition. One should note that by increasing q_0, one makes Birkhoff conjecture harder to prove, since one requires less caustics to exist.

The main theorem of the talk, stated in [1], finally provides a result arbitrary close to the boundary. Specifically, it shows that for every $q_0 \geq 3$, for every ellipse, every small q_0-rationally integrable deformation of it is an ellipse itself, provided the eccentricity of original ellipse lies outside of a locally finite set in $[0, 1)$.

The proof has two distinct parts. In the first part, the result is proven for nearly-circular ellipses. The main section of the talk will be devoted to it. There, we consider a system of linear conditions on a linear part of the deformation to preserve our family of caustics. The goal would be to prove that this infinite dimensional system has trivial kernel, since then in order to preserve all the caustics and hence have all functionals be 0, one would have to choose a trivial deformation. The proof will involve several techniques related to geometry, Jacobi elliptic functions and algebraic field theory.

However, when eccentricity is away from 0, these methods wouldn't work, since we have used asymptotic expansions of various objects, when eccentricity goes to 0, and hence we were in the perturbative regime. Instead, we consider the aforementioned system of functionals as a linear operator on the space of deformations, that has the eccentricity as a parameter. We claim that this operator is holomorphic in eccentricity in certain sense. This allows us to say that the set of eccentricities, where the operator has non-trivial kernel behaves like the set of zeros of an analytic function, namely it is either the whole complex domain or some locally finite set. Since we already know that the kernel is trivial near 0, we have that the latter option is true.

There are many interesting open questions, associated with this talk. First of all, this bad set of eccentricities, described in the main theorem, contains 0, so the local Birkhoff conjecture near the disc remains open. It would be nice to see what happens near the disc, but this may prove challenging. Secondly, it would be interesting to know if there are other points in this bad set, except 0. We provide fast-converging formulas for the entries of that operator, so it is feasible to do some numerical analysis to check if they actually do exist.

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Zoll magnetic systems on the two-torus: a Nash–Moser approach

Gabriele Benedetti

(joint work with Luca Asselle, Massimiliano Berti)

An autonomous Hamiltonian flow at a given energy is called Zoll if all orbits are periodic and have the same period (after a global smooth time reparametrization). Classical examples of Zoll flows are given by the geodesic flow on the round two-sphere, the magnetic flow induced by a constant field on the flat two-torus, and the flow of the harmonic and gravitational potential in negative energy. Such totally resonant flows exhibit the simplest type of dynamics and yet they play a central role in contact and symplectic geometry, as they are optimal objects for systolic-type quantities [1, 5].

For this reason, a natural question is to understand how abundant Zoll flows are within a given class of Hamiltonian systems, e.g., among geodesic flows, magnetic systems, or central potentials. Two seminal results show the rigid and flexible side of this question. Firstly, Joseph Bertrand proved in 1873 that the harmonic and gravitational potentials are the only central potentials for which all bounded motions are periodic [6]. Secondly, Otto Zoll constructed in 1903 an explicit infinite-dimensional family of two-spheres of revolution, parameterized by the space of odd functions on an interval, whose geodesic flow is Zoll [10]. This result was later extended to Riemannian metrics on the two-sphere which are close to the round metric g_0 and are not necessarily of revolution: given a function $u\colon S^2 \to \mathbb{R}$, there is a one-parameter family of Riemannian metrics $g_\tau = (1 + \tau u + o(\tau))g_0$ which have fixed volume and are Zoll for every $\tau \in (-\delta, \delta)$ if and only if u is odd. The necessity of an odd function was shown by Funk in 1913 [7], while the sufficiency was proved by Guillemin in 1973 [8]. In particular, Guillemin's construction of g_τ is not explicit but relies on a beautiful application of the Nash–Moser implicit function theorem.

Given this background, the focus of our talk is the existence of magnetic flows on two-tori of revolution which are Zoll at a given positive energy h. These systems are parametrized by a pair of periodic functions $a, b \in C_0^\infty(\mathbb{T})$ with zero average and possess an integral of motion I with values in $\mathbb{T}$, where $\mathbb{T}$ denotes the circle. For example, the trivial magnetic system, in which the torus is flat and the magnetic field is constant, corresponds to the pair $(0, 0)$. In this case, the system is Zoll at every energy, trajectories are Euclidean circles of radius $1/\sqrt{2h}$, and the integral of motion I yields the horizontal coordinate of the center of the circle. Non-trivial magnetic systems can be Zoll at energy h only if h is bigger than a certain positive constant depending on (a, b) [3]. On the other hand, there are explicit examples on the flat torus ($a = 0$) which are Zoll for a sequence of energies diverging to infinity [3], where the sequence is given by the zeros of the first Bessel function J_1.

Based on [4], the main result of this talk is the construction, for each fixed energy h and for each (α, β) belonging to an infinite dimensional linear subspace

$$V_h \subset C_0^\infty(\mathbb{T}) \times C_0^\infty(\mathbb{T}),$$

of a one-parameter family of magnetic systems

$$(a_\tau, b_\tau) = \big(\tau\alpha + o(\tau), \tau\beta + o(\tau)\big), \quad \tau \in (-\delta, \delta)$$

which are Zoll at energy h.

To this purpose, we exploit the integral of motion I and a global torus-like surface of section for the magnetic flow to define a finite-dimensional reduction of the action functional $S_h(a, b) \in C_0^\infty(\mathbb{T})$ with the property that $S_h(a, b) = 0$ if and only if (a, b) yields a Zoll flow at energy h. Following Guillemin, we use the implicit function theorem to find zeros of S_h close to the trivial pair $(0, 0)$. Although the differential $\mathrm{d}S_h(0, 0)$ is surjective thanks to the properties of the function J_1, the map S_h is not of class C^1 and therefore the standard implicit function theorem cannot be applied. This problem originates from the fact that S_h and, hence also $\mathrm{d}S_h(a, b)$, involves composition operators and such operators lose regularity when differentiated. To overcome this difficulty the Nash–Moser implicit function theorem can be applied, provided one can show that $\mathrm{d}S_h(a, b)$ is surjective and satisfies the so-called tame estimates. We can show that these conditions are indeed met by analyzing the normal operator $N_h(a,b) := \mathrm{d}S_h(a,b)\mathrm{d}S_h(a,b)^*$ with respect to the L^2-product. Indeed, thanks to the properties of the function J_1, the operator $N_h(a,b)$ is of multiplication-type at the highest order and of composition-type only at the lower orders, a fact that ensures the necessary regularity.

Now that the existence of exotic Zoll magnetic systems at a given energy is settled, it will be interesting to understand how large can the set of Zoll energies of a magnetic system be. In the analytic category, this question seems related to the work presented by Illya Koval [9] and by Alfonso Sorrentino [2] at this workshop.

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Existence of Birkhoff sections for C^∞-generic Reeb flows and geodesic flows

Ana Rechtman

(joint work with Vincent Colin, Pierre Dehornoy, Umberto Hryniewicz)

On a closed 3-manifold, a Reeb vector field is a non-singular vector field defined by a contact form. Geodesic flows oriented closed surfaces are examples of Reeb vector fields on closed 3-manifolds. The aim of this report is to present the main result of [3], about the existence of Birkhoff sections as well as a result using these surfaces to conclude that *entropy is everywhere*, for C^∞-generic Reeb vector fields. The generic parts of the following results are explicit hypothesis, that are presented after each statement. These results are the continuation of the results presented in the Oberwolfach report [2].

1. **Definitions and results.** Consider a closed 3-manifold M. A contact structure on M is a plane field ξ that is nowhere integrable, as such it can be defined by the kernel of a differential 1-form α, that is $\xi = \ker \alpha$. The non-integrability condition implies that $\alpha \wedge d\alpha \neq 0$. The 1-form α is a contact form and its Reeb vector field X is defined by the equations

$$\iota_X d\alpha = 0 \qquad \text{and} \qquad \alpha(X) = 1.$$

Observe that there are many contact forms for a given contact structure, if f is a function on M that is never equal to zero, then $f\alpha$ is a contact form for ξ. The Reeb vector field depends on the form.

Definition 1. Let (M, X) be a closed 3-manifold with a non-singular vector field. A Birkhoff section of X is an immersed surface S in M such that:

- the interior of S is embedded and transverse to X;
- the boundary of S is mapped to a collection of periodic orbits of X;
- every orbit intersects S in bounded time.

Birkhoff sections allow one to transform the 3-dimensional dynamics of the flow of X on a problem of a homeomorphism or diffeomorphism of the surface S (given by the first return map to the surface). These type of sections appear in the works by H. Poincaré on the restricted circular 3-body problem and were constructed by Birkhoff for some geodesic flows [1]. The existence of a Birkhoff section implies that the flow is supported by an open book decomposition: the boundary of S is the binding of the open book and the pages are diffeomorphic to S. E. Giroux's correspondance implies that given a contact structure there is (at least) one of its Reeb vector fields that admits a Birkhoff section. One can ask if every Reeb vector field of a given contact structure admits a Birkhoff section, these question remains unanswered at this point.

The results I want to present are the following:

Theorem 1 (Colin-Dehornoy-Hryniewicz-Rechtman, Contreras-Mazzucchelli). *The set of Reeb vector fields on a closed 3-manifold M that admit a Birkhoff section contains a C^∞-generic set.*

Theorem 2 (Colin-Hryniewicz-Rechtman, work in progress). *For a C^∞-generic Reeb vector field on a closed 3-manifold M, every hyperbolic periodic orbit has a homoclinic orbit.*

In both cases, the genericity can be considerer for a fixed contact structure. The Reeb vector fields considered in these theorems are non-degenerate meaning that all its periodic orbits are either hyperbolic or irrationally elliptic. A hyperbolic periodic orbit has a homoclinic orbit if there is an orbit contained in the stable and in the unstable manifolds of the periodic orbit.

2. **Comments on Theorem 1.** In the C^∞-topology, there are now two proofs of Theorem 1 that are both based in the existence of broken book decompositions for non-degenerate Reeb vector field [4].

Definition 2 (Broken book decomposition, informal definition). A broken book decomposition is given by a link K and a 2D foliation $\mathcal{F}$ of $M \setminus K$, with the following properties:

- the leaves of $\mathcal{F}$ are properly embedded in $M \setminus K$ and hence their boundary is contained in K;
- $K = K_r \sqcup K_b$. The tubular neighborhood of a knot $k \in K_r$ is foliated radially by $\mathcal{F}$. If $k \in K_b$, there is a tubular neighborhood U of k such that the intersection of any leaf with U is a collection of annuli, there are two types of annuli in $\mathcal{F} \cap U$: either one boundary component contains k, or both boundary components are in ∂U. In the first case we speak of a *radial leaf*, and in the second of a *hyperbolic leaf*. We ask further that there are four sectors of radial leaves and four sectors of hyperbolic leaves as in Figure 1.

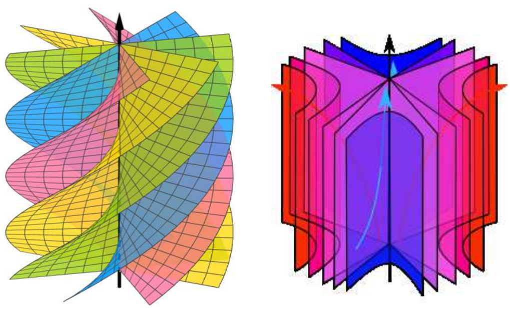

Figure 1. A local picture of a radial component and of a broken component of the binding.

We say that K is the binding of the broken book decomposition, K_r is the radial part of the binding and K_b is the broken part of the binding. The leaves of $\mathcal{F}$ are called the pages.

A broken book decomposition $(K, \mathcal{F})$ carries a vector field X if X is tangent to K and transverse to the leaves of the foliation $\mathcal{F}$. If K has no broken components, that is $K_b = \emptyset$, then one has an *open book decomposition* and any leaf of $\mathcal{F}$ is a Birkhoff section of X. From a broken book decomposition with $K_b \neq \emptyset$, in order to prove Theorem 1, it is enough to find an immersed compact oriented surface with boundary S' such that $\partial S'$ is mapped to a collection of periodic orbits of X disjoint from K_b and whose intersection number with each connected component of K_b is positive. The interior of S' is assumed to be embedded. Assume, for a moment, that one finds such a surface S' and that its interior is always transitive to X. A process introduced by D. Fried [7], allows one to add this surface to the foliation $\mathcal{F}$ to obtain an open book decomposition whose binding is contained in

$K \cup \partial S'^1$. This idea is presented in [4] and accomplished in the two proofs of Theorem 1 Let me review some differences between the proofs:

(1) G. Contreras and M. Mazzucchelli [6], assume that the Reeb vector field is *strongly* non-degenerate meaning that the intersections between stable and unstable manifolds of the hyperbolic periodic orbits are transverse. Since the periodic orbits in K_b are hyperbolic, they prove that this orbits have homoclinic orbits in each branch of their stable and unstable manifolds. This is one of the conditions needed to then apply the strategy explained in Section 4.6 of [4].

   Thus the C^∞-generic here is strongly non-degenerate.

(2) In [3], we employ a new strategy. One can find S' from null-homologous link made of periodic orbits of X that links positively with each connected component of K_b. Using that for C^∞-generic Reeb vector fields periodic orbits are equidistributed [8] and that every invariant measure links positively with the invariant volume, one can find such a link.

   Thus the C^∞ generic in this case is non-degenerate plus the equidistribution of periodic orbits with respect to the volume.

The advantage of the strongly non-degenerate hypothesis is that it is also generic among geodesic flows. There is a proof for geodesic flows using only broken book decompositions and Birkhoff annuli, that can be achieved from the observations in [4]. The advantage of the second proof, that is the conditions explained in item (2) above, is that the linking condition might be computable in explicit examples. As always, having two proofs of the same results can have advantages.

So the set of Reeb vector fields admitting a Birkhoff section is hence C^∞-generic, and one can prove using the implicit function theorem that is C^1-open (see Section 5 of [3]). The obvious open question is: are there 3D Reeb vector fields that do not admit a Birkhoff section?

3. **Comments on Theorem 2.** A Birkhoff section allows to change the study of a 3D flow, to the study of the dynamics of a 2D diffeomorphism or homeomorphism, that is a priory a simpler problem. In [3], we used the existence of Birkhoff sections for zero entropy Reeb vector fields (see Theorem 1.4 in [4]) to prove that C^∞-generically a Reeb vector field has positive topological entropy. Having positive entropy is an open condition by results of A. Katok [9], and this holds true among Reeb vector fields. The proof of Theorem 2 relies in a fundamental way, in the techniques developped by P. Le Calvez and M. Sambarino for finding homoclinic orbits among strongly non-degenerate homeomorphisms of closed surfaces [10]. Again, by A. Katok's result, the existence of a homoclinic orbit is equivalent to having positive entropy.

Using the full strength of the techniques in [10], adapted to our setting (the surface we consider has boundary and the homeomorphism might be degenerate along it), we prove Theorem 2. To finish this report, I want to make a few comments:

[1]Some technicalities are hidden in this description, for example one needs the surfaces to be δ-strong along their boundary (see [4] or [3])

(1) The hypothesis hidden behind the C^∞-genericity are the following (condition (1a) and (1b) are both important):
   (a) The Reeb vector field has to be strongly non-degenerate;
   (b) Equidistribution of periodic orbits with respect to the invariant volume;
   (c) Zehnder condition around elliptic periodic orbits: every tubular neighborhood of the periodic orbit contains another tubular neighborhood whose boundary is made of finitely pieces of stable and unstable manifolds of a hyperbolic periodic orbit ([11]).

(2) Our proof can be adapted for geodesic flows, hence gives another proof to the main theorem in [5] and to previous results on the existence of homoclinic orbits for every hyperbolic periodic orbit of a geodesic flow.

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Degenerate discounted Hamilton-Jacobi equations

Maxime Zavidovique

(joint work with Q. Chen, A. Fathi, J. Zhang)

If M is a compact, connected smooth manifold without boundary, we consider a Hamiltonian $H : T^*M \to \mathbb{R}$ that is continuous and verifies the following properties:

- convexity: for all $x \in M$, the function $p \mapsto H(x, p)$ is convex,
- superlinearity: the limit $\lim\limits_{\|p\|_x \to +\infty} H(x, p)/\|p\|_x = +\infty$ holds.

The superlinearity condition is stated with the use of an auxiliary riemannian metric, but the property is independent on the metric.

Given this Hamiltonian comes the Lagrangian function $L : TM \to \mathbb{R}$ defined by

$$\forall (x,v) \in TM, \quad L(x,v) = \max_{p \in T_x^* M} p(v) - H(x,p).$$

It is also convex and superlinear in v.

1. Non-degenerate Hamilton-Jacobi equations

In [1] we obtained the following result:

Theorem 1. *For all $\lambda > 0$, there exists a unique viscosity solution $u_\lambda : M \to \mathbb{R}$ to the discounted Hamilton-Jacobi equation;*

$$\lambda u_\lambda(x) + H(x, D_x u_\lambda) = 0. \tag{1}$$

Moreover, there is a unique constant c_0 and function $u_0 : M \to \mathbb{R}$ such that $u_\lambda + c_0/\lambda$ uniformly converges to u_0 as $\lambda \to 0$. The function u_0 is a weak KAM solution, that is a viscosity solution of $H(x, d_x u_0) = c_0$.

- The real novelty in the previous result is the convergence one. The rest was known since the 80's and the convergence was known to hold, up to subsequences. Actually, Lions, Papanicolaou and Varadhan introduced this vanishing discount method to prove the existence of weak KAM solutions.
- The set of weak KAM solutions is never reduced to a single function. For example one easily checks that this set is invariant by addition of constant functions.
- All the functions at stake here are automatically Lipschitz, hence differentiable almost everywhere.
- The proof heavily relies on Mather minimizing measures, that are Borel probability measures μ on TM that are
  1. closed: for all $f \in C^1(M, \mathbb{R})$, $\int_{TM} D_x f(v) d\mu = 0$,
  2. minimizing: $\int_{TM} L(x,v) d\mu = -c_0$.

  The limit function u_0 is actually expressed in terms of those measures.
- When the Hamiltonian H is Tonelli (smooth and strictly convex in the C^2 sense), then all the above objects have dynamical meanings.
  - The Mather measures are invariant by the Euler-Lagrange flow of L.
  - If $\lambda > 0$, setting $\mathcal{G}(du_\lambda) = \{(x, D_x u_\lambda), x \in \mathcal{D}(Du_\lambda)\}$ where $\mathcal{D}(Du_\lambda)$ is set of differentiability points of u_λ, then for $t > 0$ the inclusion $\varphi_{H,\lambda}^{-t}\left(\overline{\mathcal{G}(du_\lambda)}\right) \subset \mathcal{G}(du_\lambda)$ holds, where $\varphi_{H,\lambda}$ is the conformally symplectic flow generated by the equations

    $$\begin{cases} \dot{x} = \partial_p H(x,p), \\ \dot{p} = -\partial_x H(x,p) - \lambda p. \end{cases}$$
  - Similarly, for $t > 0$ it holds $\varphi_H^{-t}\left(\overline{\mathcal{G}(du_0)}\right) \subset \mathcal{G}(du_0)$ where φ_H is the Hamiltonian flow associated to H.

- The functions u_λ are given by the following formula (that can be taken as their definition in the present case)

$$\forall x \in M, \quad u_\lambda(x) = \min_{\gamma(0)=x} \int_{-\infty}^{0} e^{\lambda s} L\big(\gamma(s), \dot\gamma(s)\big) ds,$$

where the infimum is taken amongst the absolutely continuous curves $\gamma : (-\infty, 0] \to M$ such that $\gamma(0) = x$.
- the function u_0 verifies a similar property (that characterizes weak KAM solutions):

$$\forall x \in M, \quad \forall t > 0, \quad u_0(x) = \min_{\gamma(0)=x} u_0\big(\gamma(-t)\big) + \int_{-t}^{0} L\big(\gamma(s), \dot\gamma(s)\big) ds + tc_0.$$

2. Degenerate Hamilton-Jacobi equations

Having in mind the previous results, one may ask what other kind of perturbations of the stationary Hamilton-Jacobi equation (defining weak KAM solutions) select a unique weak KAM solution. It can be seen from the theory of viscosity solutions that it is important to have an equation with a non-decreasing dependance on the value of the unknown $u_\lambda(x)$. Therefore we will focus here on equations of the form

$$\lambda\alpha(x)u_\lambda(x) + H(x, D_x u_\lambda) = c_0, \tag{2}$$

where $\alpha : M \to [0, +\infty)$ is a given continuous function. If α is identically 0 then there is no perturbation and no reasonable result can be expected. On the countrary, if $\alpha > 0$ everywhere, then, dividing by α one reduces to the results of the previous section. Hence one needs to find an appropriate intermediate condition. We introduce the following:

Non-degeneracy condition: for all Mather measures μ, one has $\int_{TM} \alpha(x) d\mu > 0$.

Note that a Theorem of Mañé asserts that for a generic H, there exists a unique Mather measure. Hence for most Hamiltonians, the above condition allows α to vanish on very large sets, hence the equations to be rather degenerate.

Building on the results of [2] and [3], we prove in [4] a generalization of the following:

Theorem 2. *Assume $\alpha : M \to [0, +\infty)$ verifies the non-degeneracy condition and H is convex and superlinear as before. For all $\lambda > 0$, there exists a unique viscosity solution $\tilde u_\lambda : M \to \mathbb{R}$ to the degenerate discounted Hamilton-Jacobi equation;*

$$\lambda\alpha(x)\tilde u_\lambda(x) + H(x, D_x \tilde u_\lambda) = c_0. \tag{3}$$

Moreover, there is a unique constant c_0 and function $\tilde u_0 : M \to \mathbb{R}$ such that $\tilde u_\lambda + c_0/\lambda$ uniformly converges to $\tilde u_0$ as $\lambda \to 0$. The function $\tilde u_0$ is a weak KAM solution, that is a viscosity solution of $H(x, d_x \tilde u_0) = c_0$.

Most of the previous Theorem in new, including uniqueness of the $\tilde u_\lambda$ that requires new, dynamically inspired, methods. The functions $\tilde u_\lambda$ no longer verify a nice explicit representation formula as before. However one recovers (with Gronwall's inequality) properties closer to that of weak KAM solutions: for all $x \in M$ and $t > 0$,

$$\tilde{u}_\lambda(x) = \min_{\gamma(0)=x} e^{A_\gamma(-t)}\tilde{u}_\lambda\big(\gamma(-t)\big) + \int_{-t}^{0} e^{A_\gamma(s)}\left[L\big(\gamma(s),\dot{\gamma}(s)\big) + c_0\right] ds,$$

where $A_\gamma(s) = \int_0^s \alpha\circ\gamma(\sigma)d\sigma$. It can be guessed from the above formula that a crucial point will be to ensure that minimizing curves γ spend enough time in the regions where $\alpha > 0$ to ensure that $A_\gamma(s)$ goes to $-\infty$ as $s \to -\infty$.

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Torus knotted Reeb dynamics in the standard tight contact 3-sphere

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(joint work with Morgan Weiler)

Recall that a 1-form λ on a 3-manifold Y is a *contact form* whenever $\lambda\wedge d\lambda$ is a volume form. The associated *Reeb vector field* is uniquely determined by

$$\lambda(R) = 1, \quad d\lambda(R,\cdot) = 0.$$

A *closed Reeb orbit* is a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$, modulo reparametrization. Denote the set of all closed Reeb orbits of λ by $\mathcal{P}(\lambda)$. Consider the unit 3-sphere S^3 in $\mathbb{C}^2$ and let $J_{\mathbb{C}^2}$ be the standard complex structure on $\mathbb{C}^2$. Then the standard tight contact structure is given by

$$(\xi_{std})\,|_p = T_pS^3 \cap J_{\mathbb{C}^2}(T_pS^3)$$

and may expressed as the kernel of the contact form

$$\lambda_0 = \frac{i}{2}\left(z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2\right).$$

We can realize the right handed torus knot $T(p,q)$ in S^3 as

$$T(p,q) = \left\{(z_1,z_2)\in S^3 \subset \mathbb{C}^2 \mid z_1^p + z_2^q = 0\right\};$$

the projection map $\pi : S^3 \setminus T(p,q) \to S^1$ is the Milnor fibration. Etnyre shows in [2], that positive transverse torus knots are transversely isotopic if and only if they have the same topological knot type and the same self-linking number. We establish the following quantitative existence result.

Theorem 1. *Let λ be a contact form on (S^3, ξ_{std}) with* $\mathrm{Vol}(\lambda) \leq \frac{1}{pq+\delta}$*, which admits the maximal self-linking transverse $T(p,q)$ torus knot as an elliptic Reeb orbit, denoted by b, with rotation number $pq+\delta$ and $\mathcal{A}(b) := \int_b \lambda = 1$, where δ is either 0 or a sufficiently small positive irrational number. Then there exists a second Reeb orbit distinct from b and*

$$\inf\left\{ \frac{\mathcal{A}(\gamma)}{\ell(\gamma, b)} \;\middle|\; \gamma \in \mathcal{P}(\lambda) \setminus \{b\} \right\} \leq \sqrt{\frac{Vol(\lambda)}{pq+\delta}}.$$

Our result follows from the ECH Weyl law[1] and our computation of knot filtered embedded contact homology of (S^3, ξ_{std}) with respect to transverse positive $T(p,q)$ torus knots having rotation number $pq+\delta$, where the rotation number is well-defined when using a trivialization which induces the orbit to have push off linking number zero.

Theorem 2. *Let ξ_{std} be the standard tight contact structure on S^3. Let b_0 be the standard transverse positive $T(p,q)$ torus knot. Then for $k \in \mathbb{N}$,*

$$ECH_{2k}^{\mathcal{F}_b \leq K}(S^3, \xi_{std}, b_0, pq) = \begin{cases} \mathbb{Z}/2 & K \geq N_k(p,q), \\ 0 & \textit{otherwise}, \end{cases}$$

and in all other gradings $*$, $ECH_*^{\mathcal{F}_b \leq K}(S^3, \xi_{std}, b_0, pq) = 0$. If δ is a sufficiently small positive irrational number, then up to grading $k \in \mathbb{N}$ and knot filtration threshold K inversely proportional to δ,

$$ECH_{2k}^{\mathcal{F}_b \leq K}(S^3, \xi_{std}, b_0, pq+\delta) = \begin{cases} \mathbb{Z}/2 & K \geq N_k(p,q) + \delta(\$N_k(p,q) - 1), \\ 0 & \textit{otherwise}, \end{cases}$$

where $\$N_k(p,q)$ is the number of repeats in $\{N_j(p,q)\}_{j \leq k}$ with value $N_k(p,q)$, and in all other gradings $*$, $ECH_*^{\mathcal{F}_b \leq K}(S^3, \xi_{std}, b_0, pq+\delta) = 0$.

Here $N_k(p,q) = \{pm + qn \mid m, n \in \mathbb{Z}_{\geq 0}\}_k$ and δ has to be small enough so that $N_k(2,q) + \delta(\$N_k(p,q) - 1) \leq N_{k+1}(p,q)$ for all k. We proved Theorem 2 in [5]. To do so, we generalized the definition and invariance of knot filtered embedded contact homology to allow for degenerate knots with rational rotation numbers and developed Morse-Bott methods for understanding the embedded contact homology chain complex of positive torus knotted fibrations of the standard tight contact 3-sphere in terms of their presentation as open books and as Seifert fiber spaces.

Using Theorem 1 for Reeb flows, we generalize work on the relation between mean action of periodic orbits and the Calabi invariant of area preserving diffeomorphisms of the unit disk, to higher genus surfaces, by repackaging the surface dynamics into of an open book decomposition of (S^3, ξ_{std}) along $T(p,q)$. Given an area preserving diffeomorphism of a surface which is rotation near the boundary, one can define an action function which agrees with the rotation number on the

[1]The ECH Weyl law [1], states that if (Y, λ) is a closed contact 3-manifold with nonvanishing contact invariant such that the ECH action spectrum $c_k(Y,\lambda) < \infty$ for all k, then $\lim_{k\to\infty} \frac{c_k(Y,\lambda)^2}{2k} = \mathrm{Vol}(Y, \lambda)$.

boundary, and is a measure of how much the diffeomorphism distorts curves. The Calabi invariant of the diffeomorphism is the average of the action function over the surface. Before stating our results, we provide some context.

Hutchings developed unknot filtered embedded contact homology for planar open book decompositions of (S^3, ξ_{std}) in [3], to show that for symplectomorphisms of the unit disk which are rotation near the boundary[2] whose Calabi invariant is less than the rotation number, that there exists a periodic orbit so that the infimum of its mean action is less than or equal to the Calabi invariant. Le Calvez established the same result for C^1 area preserving diffeomorphisms of the unit disk using generating functions and foliations [4]. Weiler [8, 9] established results for annular symplectomorphisms subject to a twist condition using Hopf link filtered embedded contact homology. Pirnapasov and Prasad established analogous results for C^∞-generic Hamiltonian symplectomorphisms of surfaces of arbitrary genus and an arbitrary number of boundary components without a rotation condition on the boundary using a Weyl law for periodic Floer homology [7].

Given an exact symplectomorphisms $\psi : (\mathring{\Sigma}_g, \omega = d\beta)$, where $\partial\mathring{\Sigma}_g$ is the positive $T(p,q)$ torus knot and $g = (p-1)(q-1)/2$ such that ψ is freely isotopic to the positive pq-periodic Nielsen-Thurston representative of the mapping class group of $\mathring{\Sigma}_g$ and rotation near the boundary by $\frac{2\pi}{pq+\delta}$, where δ is either 0 or a sufficiently small positive irrational number. Since ψ is not Hamiltonian, one must appeal to a topological argument to show that there exists a primitive β of ω for which ψ is exact, e.g. $[\psi^*\beta - \beta] = 0 \in H^1(\mathring{\Sigma}_g; \mathbb{R})$; any two such primitives β and β' differ by dh such that $h \equiv c$ near $\partial\mathring{\Sigma}_g$.

The *action function* of (ψ, β, θ_0) is the unique function $f = f_{\psi,\beta,\theta_0}$ for which

$$df = \psi^*\beta - \beta \text{ and } f|_{\partial\mathring{\Sigma}_g} = \theta_0.$$

Usually, the Calabi invariant is defined for Hamiltonian symplectomorphisms. Our definition of the action function f, drops the requirement that ψ be Hamiltonian (or even isotopic to the identity, although that requirement depends on the free isotopy class of ψ). We define the *Calabi invariant* of ψ by

$$\mathcal{V}(\psi) := \int_{\mathring{\Sigma}_g} f\omega.$$

In general, the Calabi invariant depends on β (e.g. see [7]). However, the variance in β is controlled by the homotopy class of ψ, and so in the cases under consideration, all $\mathcal{V}_\beta(\psi)$ are equal.

An ℓ-tuple $\gamma = (\gamma_1, \dots, \gamma_\ell)$ of points in $\mathring{\Sigma}_g$ is a *periodic orbit* of ψ if $\gamma_{i+1 \mod \ell} = \psi(\gamma_i)$. It is *simple* if $\gamma_i \neq \gamma_j$ for $i \neq j$. Its *total action* is

$$\mathcal{A}(\gamma) := \sum_{i=1}^{\ell} f(\gamma_i).$$

[2]Work by Pirnapasov [6] allows one to remove the Hutchings' condition in [3] that the map be rotation near the boundary.

If $\ell(\gamma) = |(\gamma_1, \ldots, \gamma_\ell)|$ then the *mean action* of γ is the ratio $\mathcal{A}(\gamma)/\ell(\gamma)$. As in the case of the Calabi invariant, the total action and mean action do not depend on β. Let $\mathcal{P}(\psi)$ denote the simple periodic orbits of ψ. Using a suspension construction and Theorem 1 we establish the following.

Theorem 3. *Let ψ be as described above, $f > 0$, and $\mathcal{V}(\psi) < \frac{1}{pq+\delta}$. Then we have*

$$\inf \left\{ \frac{\mathcal{A}(\gamma)}{\ell(\gamma)} \middle| \gamma \in \mathcal{P}(\psi) \right\} \leq \mathcal{V}(\psi).$$

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Big fiber theorems and ideal-valued measures

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(joint work with Adi Dickstein, Yaniv Ganor, Frol Zapolsky)

In various fields of mathematics there exist big fiber theorems:

For any map $f : X \to Y$ in a suitable category there is $y_0 \in Y$ such that the fiber $f^{-1}(y_0)$ is "big".

The notion of being "big" depends on the specific situation. We focus on the following examples:

(A.) Topological Centerpoint Theorem (Rado, Karasev);
(B.) Maximal fiber theorem for maps of the torus (Gromov);
(C.) Non-displaceable fiber theorem in symplectic topology (Entov–Polterovich).

Theorems A and B can be proved by using cohomological ideal-valued measures (IVMs), an algebraic tool introduced by Gromov in [3]. Roughly speaking, an IVM associates to each open subset of a manifold an ideal of a given associative skew-commutative algebra, and this correspondence behaves nicely under certain natural operations on subsets, which is manifested in a number of axioms. In the talk I discussed an adaptation of this tool to symplectic topology called an IVQM (*an ideal-valued quasi-measure*), see [1]. The main feature of IVQMs is that some of the axioms entering the definition of an IVM are satisfied only for pairs of Poisson-commuting (in a suitable sense) subsets. Additionally, IVQMs are invariant under the action of the identity component of the symplectomorphism group, and vanish on displaceable subsets. The construction of IVQMs is based on relative symplectic cohomology theory recently introduced by Varolgunes [6]. IVQMs lead to a unified viewpoint at Theorems A,B,C above and have a number of applications to symplectic rigidity. I presented some of them, following [1]. Furthermore, I discussed some recent results from [4]: a generalization of Theorem C in terms of relative symplectic cohomology, as well as an application of this result to the theory of symplectic quasi-states. I concluded with a brief overview of the theory of quasi-states and its link to the problem of hidden variables in quantum mechanics [2, 5].

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On the fragility of periodic tori for families of symplectic twist maps

Alfonso Sorrentino

(joint work with Marie-Claude Arnaud, Jessica E. Massetti)

In the study of Hamiltonian systems an important role is played by so-called *integrable systems*. These systems – whose dynamics is quite simple to describe due to the presence of a large number of conserved quantities, *i.e.*, symmetries – arise quite naturally in many physical and geometric problems.

Integrability appears to be a very fragile property that is not expected to persist under generic small perturbations: understanding the essence of this fragility is a very compelling task, which is of interest in various contexts, and provides the ground for some of the foremost conjectures in dynamics.

In this work we aim to shed more light on this issue in the setting of symplectic twist maps of the $2d$-dimensional annulus $\mathbb{T}^d \times \mathbb{R}^d$, where $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, and $d \geq 1$.

Definition 1 (Symplectic twist maps)**.** *A symplectic twist map of the 2d-dimensional annulus is a C^1 diffeomorphism $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ that admits a lift $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$, $F(q,p) =: (Q(q,p), P(q,p))$ satisfying*

(i) $F(q+m,p) = F(q,p) + (m,0)\ \forall m \in \mathbb{Z}^d$;

(ii) (Twist condition) the map $(q,p) \mapsto (q, Q(q,p))$ is a diffeomorphism of $\mathbb{R}^d \times \mathbb{R}^d$;

(iii) (Exactness) there exists a generating function of the map F, namely a function $S : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that
- *$S(q+m, Q+m) = S(q,Q)$, $\forall m \in \mathbb{Z}^d$,*
- *$PdQ - pdq = dS(q,Q)$.*

Moreover, a symplectic twist map f is said to be strongly positive if there exists $\alpha, \beta > 0$ such that

$$-\beta\|v\|^2 \leq \partial_q \partial_Q S(q,Q)(v,v) \leq -\alpha\|v\|^2 \quad \forall\, q, Q, v \in \mathbb{R}^d.$$

In our investigation we will focus on two related issues:

- The persistence and the properties of invariant Lagrangian tori that are foliated by periodic points. See Theorem 1.
- The rigidity of completely integrable twist maps, namely, to which extent it is possible to deform them in a non-trivial way, preserving some (or all) of their features. See Theorem 2.

Let us first introduce our main dynamical objects of interest.

Definition 2 (Periodic and completely-periodic tori)**.** *Let $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be a lift of a symplectic twist map $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$. Let $\gamma : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a $\mathbb{Z}^d$-periodic and continuous function, and let $\mathcal{L} := \mathrm{graph}(\gamma)$. For $(m,n) \in \mathbb{Z}^d \times \mathbb{N}^*$ with m and n coprime, we say that:*

- *$\mathcal{L}$ is a (m,n)-periodic graph of F, if*
$$F^n(q, \gamma(q)) = (q+m, \gamma(q)) \qquad \forall\, q \in \mathbb{R}^d;$$
- *$\mathcal{L}$ is a (m,n)-completely periodic graph of F, if it is invariant by F and a (m,n)-periodic graph of F.*

We refer to the projection of $\mathcal{L}$ to $\mathbb{T}^d \times \mathbb{R}^d$ as, respectively, (m,n)-periodic torus or (m,n)-completely periodic torus of f.

Remark 1. *One can prove that for strongly positive symplectic twist maps, if one considers Lipschitz Lagrangian graphs, then the notions of periodic and completely periodic graphs coincide. See* [1, Proposition 2.8].

Given a twist map f and a periodic potential G, we will consider a one-parameter families of a twist maps obtained by deforming f by G in the following way.

Definition 3 (Symplectic deformation by a potential)**.** *Let $G \in C^2(\mathbb{T}^d, \mathbb{R})$ and f be a symplectic twist map with generating function $S(q,Q)$. A symplectic deformation of f by the potential G is given by the family of twist maps f_ε whose* *generating functions* S_ε *are*

$$(q,Q) \mapsto S_\varepsilon(q,Q) := S(q,Q) + \varepsilon G(q), \quad \varepsilon \in \mathbb{R}.$$

In particular, $f_\varepsilon(q,p) := f(q, p + \varepsilon \nabla G(q))$.

Let us also specify a regularity assumption on the twist map that will be assumed in our main results.

Definition 4 (Analyticity property)**.** *A symplectic twist map* $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ *satisfies the analyticity property if there exists a holomorphic map* $\mathcal{F} : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d \times \mathbb{C}^d$*, where* $\mathcal{F}(q,p) =: (Q(q,p), P(q,p))$*, such that:*

- *(i)* $\mathcal{F}$ *is a holomorphic diffeomorphism of* $\mathbb{C}^d \times \mathbb{C}^d$*;*
- *(ii)* $\mathcal{F}_{|\mathbb{R}^d \times \mathbb{R}^d}$ *is a lift of* f*;*
- *(iii) (Twist condition) the map* $(q,p) \mapsto (q, Q(q,p))$ *is a diffeomorphism of* $\mathbb{C}^d \times \mathbb{C}^d$*;*
- *(iv) (Exactness) there exists a generating function* $S : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$ *such that*
  - $S(q+m, Q+m) = S(q,Q)\ \forall m \in \mathbb{Z}^d$*;*
  - $PdQ - pdq = dS(q,Q)$.

We can now state our two main results.

Theorem 1. *Let* $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ *be symplectic twist map,* $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ *denote its lift and* $S : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ *its generating function. Let* $G : \mathbb{T}^d \to \mathbb{R}$ *be a potential function.*
Consider the family of symplectic twist maps $f_\varepsilon : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$*, with* $\varepsilon \in \mathbb{R}$*, obtained as symplectic deformation of* f *by* G*, and denote by* F_ε *a continuous family of lifts of* f_ε*.*

Assume that:

- *(i)* f *is strongly positive,*
- *(ii)* f *satisfies the analyticity property,*
- *(iii)* G *admits a holomorphic extension to* $\mathbb{C}^d$*.*

Then, for every $(m,n) \in \mathbb{Z}^d \times \mathbb{N}^*$*, with* m *and* n *coprime, the set*

$$\mathcal{I}_{m,n}(f,G) := \{\varepsilon \in \mathbb{R} :\ F_\varepsilon \text{ has a Lipschitz Lagrangian } (m,n)\text{-periodic graph}\}$$

is either the whole $\mathbb{R}$ *or consists of isolated points.*

If, in addition, G *is non-constant and*

(iv) $\|\partial_q\partial_q S\|_\infty + \|\partial_Q\partial_Q S\|_\infty$ *is bounded (i.e.,* f *is said to have bounded rate),*
then $\mathcal{I}_{m,n}(f,G)$ *consists of at most finitely many points.*

Theorem 2. *Let* $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ *be symplectic twist map,* $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ *be its lift, and* $S : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ *its generating function. Let* $G \in C^2(\mathbb{T}^d)$*.*

Consider the family of symplectic twist maps $f_\varepsilon : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$*, with* $\varepsilon \in \mathbb{R}$*, obtained as symplectic deformation of* f *by* G*, and denote by* F_ε *a continuous family of lifts of* f_ε*.*

Assume that:

- *(i) f is completely integrable (e.g., $S(q,Q) := h(q-Q)$ for some $h : \mathbb{R} \to \mathbb{R}$),*
- *(ii) f is strongly positive,*
- *(iii) f satisfies the analyticity property,*
- *(iv) there exist a basis $(\rho_1, \dots, \rho_d)$ of $\mathbb{Q}^d$ and $I_1, \dots, I_d \subset \mathbb{R}$ open intervals, such that for any $\frac{m}{n} \in \bigcup_{j=1}^d I_j \rho_j \cap \mathbb{Q}^d$, there are infinitely many values of $\varepsilon \in \mathbb{R}$, accumulating to 0, such that the corresponding F_ε admits a Lipschitz Lagrangian (m,n)-periodic graph.*

Then, G must be identically constant.

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Low-action holomorphic curves and invariant sets

Rohil Prasad

Fix a closed, smooth, oriented, and connected manifold Y of odd dimension $2n+1 \geq 3$. A *framed Hamiltonian structure* on Y is a pair $\eta = (\lambda, \omega)$ of a smooth 1-form and smooth 2-form such that $d\omega = 0$ and $\lambda \wedge \omega^n > 0$. The framed Hamiltonian structure η has an associated *Hamiltonian vector field* R_η defined implicitly by the equations

$$\lambda(R_\eta) \equiv 1, \qquad \omega(R_\eta, -) \equiv 0.$$

This formalism covers most symplectic dynamical systems of interest, including Reeb flows and suspension flows of symplectic diffeomorphisms. Groundbreaking work of Hofer [4] introduced the use of *holomorphic curves* in $\mathbb{R} \times Y$ to detect periodic orbits of the vector field R_η. In this context, a J-holomorphic curve is a smooth, proper map $u : C \to \mathbb{R} \times Y$ where (C, j) is a punctured Riemann surface and the map u satisfies the non-linear Cauchy–Riemann equation

$$Du \circ j = J \circ Du.$$

Here J denotes a "η-adapted" almost-complex structure on $\mathbb{R} \times Y$. The geometry of a J-holomorphic curve is controlled by two non-negative quantities called its *action* and *Hofer energy*, defined respectively as

$$\mathcal{A}(u) := \int_C u^*\omega, \qquad \mathcal{E}(u) := \sup_{t \in \mathrm{Reg}(a \circ u)} \int_{(a \circ u)^{-1}(t)} u^*\lambda$$

where $a : \mathbb{R} \times Y \to \mathbb{R}$ is the projection map. The action controls the average distance of the tangent planes of $u(C)$ to the "vertical subbundle" $\mathrm{Span}(\partial_a, R_\eta) \subset T(\mathbb{R} \times Y)$. Our main technical result is, when C is a cylinder and $\mathcal{A}(u)$ is sufficiently small, an upgrade of this statement to a uniform pointwise bound.

Theorem 1. There exists a geometric constant $\kappa = \kappa(Y, \eta, J) \geq 1$ such that the following holds. Let $u : C \to \mathbb{R} \times Y$ be any J-holomorphic curve such that C is homeomorphic to a cylinder and $\mathcal{A}(u) \leq \kappa^{-1}$. Then u is an immersion and for any $\zeta \in C$, we have the bound

$$\operatorname{dist}(Du(T_\zeta C), \operatorname{Span}(\partial_a, R_\eta)) \leq \kappa \mathcal{A}(u)^{1/3}.$$

To give a better feel for the statement of Theorem 1, we present the following qualitative Corollary.

Corollary 1. Any J-holomorphic cylinder $u : C \to \mathbb{R} \times Y$ of sufficiently low action is transverse to the level sets of $\mathbb{R} \times Y$. Each level set of $u(C)$ is a immersed, closed ϵ-pseudo-orbit of the vector field R_η, where $\epsilon \to 0$ as $\mathcal{A}(u) \to 0$.

We also remark that the main novelty of Theorem 1 is that the estimate does not depend on $\mathcal{E}(u)$. In fact, it does not even require $\mathcal{E}(u)$ to be finite. We apply Theorem 1 to study the orbit structure of R_η when R_η $\mathbb{R} \times Y$ has plenty of J-holomorphic cylinders.

Theorem 2. Assume that for a dense set of $z \in Y$, the following holds. There exists a sequence $\{u_k : C_k \to \mathbb{R} \times Y\}_{k \geq 1}$ of J-holomorphic curves such that i) C_k is a cylinder for each k, ii) $(0, z) \in u_k(C_k)$, and iii) $\mathcal{A}(u_k) \to 0$ as $k \to \infty$. Then the flow of R_η is "nowhere minimal": for dense $y \in Y$ the orbit of y is not dense.

Work in progress aims to improve Theorem 2 by relaxing assumption i) to

i') the Euler characteristics $\chi(C_k)$ admit a finite k-independent lower bound.

We expect this improvement to significantly broaden the scope of Theorem 2. The primary examples of systems satisfying the current assumptions of Theorem 2 are *pseudorotations*. More precisely, any non-degenerate Hamiltonian pseudorotations of $\mathbb{CP}^n$ satisfies the assumptions of Theorem 2, as does any Reeb flow on a closed 3-manifold with exactly two closed orbits. This assertion follows in the former case from work of Ginzburg–Gurel [3] and in the latter case from work in preparation by the author. The latter class of systems contain in particular Katok's celebrated examples [5] of Finsler geodesic flows on S^2 with two closed orbits. We also use Theorem 1, embedded contact homology, and holomorphic intersection theory to derive further dynamical results regarding these "Reeb pseudorotations".

Theorem 3. Let Y be a smooth, closed, connected, oriented 3-manifold and λ any contact form whose Reeb flow has two closed orbits. Then there exists a sequence of vector fields $\{R_n\}_{n \geq 1}$ on Y approximating the Reeb vector field R_λ in the C^0 topology such that the flow of R_n is periodic for every n.

Theorem 4. Let Y be a smooth, closed, connected, oriented 3-manifold and λ any contact form whose Reeb flow $\{\phi^t_\lambda\}_{t \in \mathbb{R}}$ has two closed orbits. Write $T_1 > T_2 > 0$ for the actions of the two closed orbits. Assume that T_1/T_2 is "super-Liouvillean", that is the denominators $\{q_n\}_{n \geq 1}$ of its continued fraction expansion satisfy the identity

$$\limsup_{n \to \infty} q_n^{-1} \log(q_{n+1}) = +\infty.$$

Then there exists a sequence of times $t_n \to +\infty$ such that the sequence of maps $\{\phi_\lambda^{t_n}\}_{n\geq 1}$ converges in the C^0 topology to the identity. As a consequence, the Reeb flow is not topologically mixing.

These results are analogous to groundbreaking results of Bramham [1, 2] for pseudorotations of the closed 2-disk.

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Long time behavior of Sobolev norms: normal forms and energy methods

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(joint work with Roberto Feola)

We discuss the problem of long time behavior of general initial data of a given PDE with an elliptic fixed point at the origin and defined on a compact manifold. This is a longstanding problem in the study of infinite dimensional dynamical systems. On such a domain, in general, no dispersive effect help to control the evolution of the Sobolev norm of solutions for long time. On the other hand, at least in the case of small initial conditions, a Birokhoff Normal Form approach reveals to be an effective tool in the understanding of the *optimal* time of stability of solutions (see [1, 3] and references therein). In contrast with KAM theory, where perpetual stability can be proved for "special" initial data evolving quasi/almost-periodically in time (see the recent [2]), normal forms techniques provide information of the evolution of all initial data, for finite but very long time. Note that it is relatively simple to prove a polynomial lifespan of solutions (w.r.t. the size of initial data), while obtaining exponential stability times turns out to be intimately related with the connection between regularity and size of initial conditions (this is carefully studied in [4]). This is due to the presence of the so-called "small divisors", which arise from (close to) resonant interactions between linear frequencies of oscillations, that one needs to control during a normal form analysis.

We shall focus on the following two equations:

$$\text{(beam)}\qquad \begin{aligned}&\psi_{tt} + \psi_{xxxx} + \mathfrak{m}\psi + f(\psi) = 0, \qquad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}\\ &\psi = \psi(x,t), \quad \mathfrak{m} \in [1,2]\end{aligned}$$

where the nonlinearity is given by $f(\psi) := \partial_\psi F(\psi)$, F being real analytic in the neighborhood of the origin and such that $F(0) = 0$, and

$$\text{(NLS)}\qquad \begin{aligned}&\mathrm{i}u_t + \Delta u \pm |u|^{2p}u = 0, \qquad x \in \mathbb{T}^d, d \geq 1\\ &u = u(x,t), \quad \mathbb{N} \ni p \geq 1,\end{aligned}$$

where Δ is the Laplacian operator.

The nice feature is that both equations[1] read as $\mathrm{i}u_t = Lu + N(u)$ where

- u belongs to some Banach space, possibly Hilbert separable as the (scale of) Sobolev one(s) $H^s(\mathbb{T}^d)$, $s \geq 0$
- L is a typically unbounded self-adjoint operator with pure point spectrum. This implies that, considering the base $\{e^{\mathrm{i}jx}\}$, the vector field reads $\mathrm{i}\dot{u}_j = \lambda_j u_j + N_j(u)\,, j \in \mathbb{Z}^d$ where λ_j are the eigenvalues of L (i.e. for each Fourier's mode $\lambda_j = \sqrt{j^4 + \mathfrak{m}}$ for the beam and $\lambda_j = |j|^2$ for the NLS respectively)
- the nonlinear term $N(u) \sim O(u^{q+1})$, $q \geq 1$

Given the Cauchy problem

$$\begin{cases}\mathrm{i}u_t = Lu + N(u)\\ u(0,x) = u^0(x) \in H^s(\mathbb{T}^d)\end{cases}$$

we are interested in how the norm $\|\cdot\|_s := \|\cdot\|_{L_2} + \|(\sqrt{-\Delta})^s \cdot\|_{L^2}$ of the corresponding solution evolves.

In general, we can phrase the result we aim at as follows: *Given $\|u^0\|_s \leq \epsilon$, then the solution satisfies $\|u(t)\|_s \leq f(t)$, for any time $|t| \leq T$, where $T > T_{good}(\epsilon)$.*

Now, our main questions then are

(1) Who is $f(t)$? Are we able to prove that $f(t) = \mathsf{c}\epsilon$, for some absolute constant c? In this case we would prove *stability* of the solution, otherwise, by determining precisely $f(t)$ for all times up to T we would get a control from above on the possible growth of its Sobolev norm.
(2) Who is T_{good}? This question goes in the direction of determining a *lower bound* on T that is strictly better than the trivial time of existence which can easily be proved to be like $T_{good} \gtrsim 1/\epsilon^q$.

The beam equation: a stability result.

Theorem 1 (Sobolev stability). *Let s be large enough and fix $0 < \gamma < 1$. There are a large measure set $\mathfrak{M}_\gamma \subset [1,2]$ and an absolute constant $c > 0$ such that $\forall\, \mathfrak{m} \in \mathfrak{M}_\gamma$ the following holds.*

[1] concerning the beam equation, after a convenient linear symplectic change of variables

For any $0 < \epsilon \le \gamma^{cs}$ *and any initial condition* $(\psi_0, \psi_1) \in H^{s+1} \times H^{s-1}$ *such that*

$$\|\psi_0\|_{s+1} + \|\psi_1\|_{s-1} \le \epsilon/8, \qquad 2^s(\|\psi_0\|_{L^2} + \|\psi_1\|_{L^2}) \le \epsilon/8, \tag{1}$$

the corresponding solution exists and satisfies

$$\|\psi(t)\|_{s+1} + \|\partial_t \psi(t)\|_{s-1} \le 8\epsilon \qquad \forall\, |t| \le T_0(\epsilon)\,,$$

where

$$T_0(\epsilon) \gtrsim T_{good} := \frac{\gamma^{cs^2}}{\epsilon}\, (1/\epsilon)^{(s-1)^{1/3}}\,.$$

Corollary 1 (Optimization)**.** *Under the hypotheses above, if one sets*

$$s = s(\epsilon) \sim_c 1 + \left(\frac{\ln 1/\epsilon}{\ln 1/\gamma}\right)^{3/5}$$

then

$$T_0(\epsilon) \gtrsim_\gamma T_{good} := 1/\epsilon \exp\{(\ln 1/\epsilon)^{1+1/5}\}.$$

The key, non trivial ingredient of the above results is the possibility of imposing the following Diophantine condition on the set of admissible frequencies $\lambda = (\lambda_j)_{j\in\mathbb{Z}}$

$$D_\gamma = \{\lambda \in \mathbb{R}^{\mathbb{Z}} \,:\, |\lambda \cdot k| \ge \gamma^{\mathtt{d}(k)} \prod_j \frac{1}{(1+\langle j\rangle^2 |k_j|^2)^{\tau(\mathtt{d}(k))}} \quad \forall k \in \Lambda\,,\, |k| < \infty\},$$

where $\langle j\rangle = \max\{1, |j|\}$, $\Lambda \subset \mathbb{Z}^{\mathbb{Z}}$ is a suitable *non-resonant* sublattice, and $\mathtt{d}(k)$ is the number of nonzero component of k. Note that we are in a degenerate situation where we have only one parameter (the mass) for tuning infinitely many frequencies and get a sufficiently non-resonant vector, so that a Birkhoff Normal Form procedure can be performed (note however that the number of steps $\mathtt{N}$ of BNF is related to the regularity as $\mathtt{N} \sim s$) . Proving that the measure of possible masses $\mathtt{m} \in [1,2]$ such that the above condition holds is of order $1 - O(\gamma)$ is highly nontrivial. Then, taking a sharp care of all the constant's dependence throughout the procedure, we are able to perform an optimization regularity-size and achieve a surprising exponential-type stability time in the Sobolev category (while usually, only a polynomial type-one is possible at best). This result should be compared with the one in [3] in the context of the 1-d NLS with convolution potential acting as a Fourier multiplier that provides infinitely many parameters for the frequency modulation $\lambda_j = j^2 + V_j$, $(V_j)_{j\in\mathbb{Z}} \in \ell^\infty$.

Control on the (possible) growth of Sobolev norms. In the case of the completely resonant NLS, the situation is drastically different: too many nontrivial resonant relations may occur in any n-wave interaction $\lambda_{j_1} \pm \cdots \pm \lambda_{j_n}$, $j_n \in \mathbb{Z}^d$ so that a Birkhoff Normal Form is out of reach.

Theorem 2. *Fix $0 < \epsilon \ll 1$, $p \geq 1$ and $s_1 > d/2 + 2$. There exist absolute constants $C \geq M > 0$ such that $\forall s \geq s_1 + 1$ and any initial datum $u^0 \in H^s$ such that*

$$\|u^0\|_{s_1} \leq \epsilon, \quad (M)^{s_1}\|u^0\|_{L^2} \leq \epsilon, \quad \|u^0\|_s < \infty \tag{2}$$

the following holds. There exist a time $T = T(u^0, M, s_1) > 0$ and a unique solution $u = u(x,t)$ s.t.

$$u \in C^0([0,T], H^s) \cap C^1([0,T], H^{s-2}) \quad \text{for} \quad T \geq T_{good} \sim \frac{2^{2ps_1}}{\epsilon^{2p}}.$$

Moreover one has $\forall\, t \in [0,T]$ that

$$\|u(t)\|_s \lesssim C^{ps(s-s_1)}\left(\|u^0\|_s + (2M)^s\|u^0\|_{L^2}\right)\left[1 + \left(M^{2p(s-s_1)}\frac{\epsilon^{2p}}{2^{2ps_1}}\, t\right)^{s-s_1}\right].$$

Of course, if restricted to low 2-3-dimensional tori and low degree p, there are even global in time results, see [4, 7] and references therein. Our aim is to propose an approach uniform in the space dimension and the degree p of the nonlinearity. In the above theorem, no normal form is involved. Note that energy estimates plus a classical Grönwall lemma would give an exponential upper bound. In order to get a *polynomial* control from above on all the scale of high norms $\|\cdot\|_s$ we provide improved energy estimates combining pseudodifferential calculus and tameness properties enjoyed by our norms. The smallness condition on the L^2-norm of the initial data, which entails that the energy is not concentrated on the low modes, enables us to determine the time of existence through suitable scaling properties of the norm combined with the stability of the s_1-norm.

The above results are part of the works [5, 6].

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The Calabi homomorphism in higher dimensions as an average rotation

Barney Bramham

In 1980 Albert Fathi found an expression for the Calabi homomorphism of compactly supported Hamiltonian disc maps as an average rotation number. In this talk we gave a generalisation of this to higher dimensions which will appear in [1].

Let $H : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function with compact support and $\tilde{\varphi} = \{\varphi_t\}_{t\in[0,1]}$ be the generated path of Hamiltonian diffeomorphisms with respect to the standard symplectic structure $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. We use the following sign convention $\iota_{X_{H^t}}\omega_0 = dH^t$ for the Hamiltonian vector field. The following space-time integral

$$\mathrm{Cal}(\tilde{\varphi}) := \int_0^1 \left(\int_{\mathbb{R}^{2n}} H(t,z)\,\omega_z^n \right) dt \ \in \mathbb{R}$$

which turns out to depend only on the time-1 map $\varphi := \varphi_1 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, see for example [7], defines a group homomorphism [3] from the group of compactly supported Hamiltonian diffeomorphisms to $(\mathbb{R}, +)$, called the *Calabi homomorphism*.

A. Fathi discovered that in two dimensions the Calabi homomorphism has an interpretation as the average amount that pairs of trajectories wind around each other. Alternative proofs were later found by Gambaudo-Ghys in [6], by Deryabin [4], Shelukhin [8], and by Bechara [2]. Here is the statement:

Theorem 1 (Fathi [5])**.** *If $n = 1$, and the Hamiltonian isotopy $\tilde{\varphi}$ is compactly supported in the open unit disc $\mathbb{D} \subset \mathbb{R}^2$, then*

$$\int_{\mathbb{D}\times\mathbb{D}} \mathrm{wind}_{\tilde{\varphi}}(z,w)\,dzdw = 2\mathrm{Cal}(\tilde{\varphi}) \tag{1}$$

where $\mathrm{wind}_{\tilde{\varphi}} : (\mathbb{D}\times\mathbb{D})\backslash\Delta \to \mathbb{R}$ is defined at (z,w) on the complement of the diagonal, as the change in argument of the continuous path of non-zero vectors $t \mapsto \varphi_t(z) - \varphi_t(w) \in \mathbb{R}^2\backslash\{0\}$.

More precisely, $\mathrm{wind}_{\tilde{\varphi}}(z,w) := (\theta(1) - \theta(0))/2\pi$, where $\theta : [0,1] \to \mathbb{R}$ is any continuous function for which $\varphi_t(z) - \varphi_t(w) = r(t)e^{i\theta(t)}$ for some continuous function $r : [0,1] \to \mathbb{R}$.

In the talk we explained a generalisation of this result to higher dimensions, that applies to any compactly supported Hamiltonian isotopy $\tilde{\varphi} = \{\varphi_t\}_{t\in[0,1]}$ on $(\mathbb{R}^{2n}, \omega_0)$. To make sense of a winding or rotation number we project onto a 2-dimensional subspace. More precisely, suppose

$$V \subset (\mathbb{R}^{2n}, \omega_0)$$

is a symplectic 2-dimensional vector subspace. Let $\pi_V : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the unique linear projection with image V and kernel the symplectic complement V^ω.

Definition 1. *We call $(z,w) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ a* collision pair *for $\tilde{\varphi} = \{\varphi_t\}$ with respect to the subspace V, if there exists $t \in [0,1]$ so that $\pi_V(\varphi_t(z)) = \pi_V(\varphi_t(w))$.*

In other words (z, w) is a collision pair if the projections onto V of their trajectories $t \mapsto \varphi_t(z)$, $t \mapsto \varphi_t(w)$ coincide at some parameter $t \in [0, 1]$. Of course the two trajectories will never coincide in $\mathbb{R}^{2n}$ unless $z = w$. One can show:

Lemma 1. *The set of collision pairs $\mathcal{C} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is a closed subset of measure zero.*

Definition 2 (The projected winding number on V)**.** *Let $\mathcal{C} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ be the collision set of $\tilde{\varphi}$ with respect to the subspace V. We define the V-*winding *function*

$$\mathrm{wind}^V_{\tilde{\varphi}} : \big(\mathbb{R}^{2n} \times \mathbb{R}^{2n}\big) \backslash \mathcal{C} \to \mathbb{R}$$

at a non-collision pair (z, w) to be the change in argument of the continuous path of non-zero vectors

$$t \mapsto \pi_V(\varphi_t(z)) - \pi_V(\varphi_t(w)) \in V \backslash \{0\}$$

with respect to Euclidean angles in V [1]*.*

Here is the main result. As mentioned, $\tilde{\varphi} = \{\varphi_t\}_{t \in [0,1]}$ is a compactly supported Hamiltonian isotopy on $\mathbb{R}^{2n}$, and $V \subset \mathbb{R}^{2n}$ is a 2-dimensional symplectic vector subspace.

Theorem 2. *The function $(z, w) \mapsto \mathrm{wind}^V_{\tilde{\varphi}}(z, w)$, defined almost everywhere on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, is locally integrable. Moreover, if the isotopy $\tilde{\varphi}$ is supported in a bounded open subset $Q \subset \mathbb{R}^{2n}$ for which each slice parallel to V^ω has ω^{n-1}-volume equal to 1, then*

$$\int_{Q \times Q} \mathrm{wind}^V_{\tilde{\varphi}}(z, w)\; \omega^n_z \times \omega^n_w \;=\; 2n\mathrm{Cal}(\tilde{\varphi}).$$

If we restrict attention to symplectic lines V that are also complex, i.e. J_0-invariant, where J_0 is the standard complex structure on $\mathbb{R}^{2n}$ after identifying with $\mathbb{C}^n$, then V^ω coincides with the orthogonal complement $V^\perp$, and one can make a statement where the support is independent of V. For example, if the isotopy $\tilde{\varphi}$ is supported in the open unit Euclidean ball $B^{2n}_1(0) \subset \mathbb{R}^{2n}$, then for each complex line $V \subset \mathbb{R}^{2n}$ we have

$$\int_{Q \times Q} \mathrm{wind}^V_{\tilde{\varphi}}(z, w)\; \omega^n_z \times \omega^n_w \;=\; 2n\pi^{n-1}\mathrm{Cal}(\tilde{\varphi})$$

where $Q = B^2_1(0) \times B^{2n-2}_1(0)$ is the product of the Euclidean open balls in V and V^ω with radius 1.

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Lyapunov spectral rigidity of expanding circle maps

Kostiantyn Drach

(joint work with Vadim Kaloshin)

In 1990, Croke [1] and Otal [2] proved a remarkable result on rigidity of negatively curved metrics in dimension 2. They showed that a smooth metric g of negative curvature on a closed surface is uniquely defined (up to smooth coordinate changes) by its *marked length spectrum*, i.e., by the lengths of closed geodesics for the metric g 'marked' by their respective homotopy types. As it turns out, knowing just the *length spectrum* of g, i.e., the set of lengths of all closed geodesics and 'forgetting' about their homotopy types is not enough to reconstruct the metric, as the examples of Sunada [4] and Vignéras [5] show. However, the *local (unmarked) length spectral rigidity* question for *nearby* negatively curved metrics is still widely open. We study a one-dimensional analog of this question for expanding circle endomorphisms. Our setup is the following.

Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, be a $C^{r,1}$-smooth, $r \geqslant 1$, expanding circle endomorphism of degree $d \geqslant 2$ normalized so that $f(0) = 0$ and $f'(x) > 1$ for all $x \in \mathbb{S}^1$. (Here, $C^{r,1}$-smooth means that f has r derivatives and the r^{th} derivative is Lipshitz.) For brevity, we write $\mathcal{E}^r_d$ for the class of such maps.

Denote by $\mathbb{P}^f_n$ the set of all periodic points of period n for $f \in \mathcal{E}^r_d$. We assume that n is the smallest period. The log-*multiplier* of a periodic point $p \in \mathbb{P}^f_n$ is defined as

$$\lambda^f(p) := \log{(f^n)'}(p).$$

For each $n \in \mathbb{N}$, we define the *Lyapunov spectrum for period* n as the set

$$\mathrm{Lyap}_n(f) := \left\{\lambda^f(p) : p \in \mathbb{P}^f_n\right\}.$$

The union

$$\mathrm{Lyap}(f) := \bigcup_{n\in\mathbb{N}} \mathrm{Lyap}_n(f)$$

of all these sets yields the *Lyapunov spectrum* of the expanding circle map f.

There is a natural *marked* counterpart of the Lyapunov spectrum. Namely, it is known that any two expanding circle maps $f, g \in \mathcal{E}_d^r$ are topologically conjugate via an orientation-preserving homeomorphism $\varphi\colon \mathbb{S}^1 \to \mathbb{S}^1$ as follows:

$$g = \varphi \circ f \circ \varphi^{-1}.$$

This homeomorphism respects the symbolic dynamics and hence provides a natural marking: we say that $p^f \in \mathbb{P}_n^f$ and $p^g \in \mathbb{P}_n^g$ are *corresponding periodic points* if $\varphi(p^f) = p^g$. We call φ the *marking conjugacy* and say that f and g have the same *marked Lyapunov spectra* if $\lambda^f(p^f) = \lambda^g(p^g)$ for every pair of corresponding periodic points. By the classical result of Shub and Sullivan [3], the marked Lyapunov spectrum defines an expanding circle map up to a *smooth* change of coordinates, namely, if f and g have the same marked Lyapunov spectra, then the marking conjugacy φ is $C^{r,1}$-smooth.

We are interested in the following question: *does the (unmarked) Lyapunov spectrum of an expanding circle map uniquely define the smooth conjugacy class of the map?* Similarly to the unmarked length spectrum setup for negatively curved metrics, in general the answer to the question above is 'no':

Proposition 1 (A counterexample to general Lyapunov spectral rigidity)**.** *For every $\epsilon > 0$ there exists a non-linear map $f \in \mathcal{E}_d^r$ (that depends on ϵ) and there exists $g \in \mathcal{E}_d^r$ (that depends on ϵ and f) such that*

$$\|f - g\|_{C^{r,1}} \leqslant \epsilon \quad \textit{and} \quad \mathrm{Lyap}(f) = \mathrm{Lyap}(g),$$

but the marking conjugacy φ is not C^1. (Here, $\|\cdot\|_{C^{r,1}}$ denotes the $C^{r,1}$-norm.)

Nonetheless, the following local rigidity result holds. Before stating this result, let us introduce two notions. We say that the Lyapunov spectrum of $f \in \mathcal{E}_d^r$ is *β-sparse* if there exist $\beta > 0$ and $C > 0$ such that for every $n \in \mathbb{N}$ and for all $\ell_1, \ell_2 \in \mathrm{Lyap}_n(f)$,

$$|\ell_1 - \ell_2| \geqslant C \cdot e^{-\beta \cdot n}.$$

We will also say that $\mathrm{Lyap}(f)$ is *simple* if the *log*-multipliers of periodic orbits are pairwise distinct.

Theorem 1 (Local Lyapunov spectral rigidity)**.** *Let $f \in \mathcal{E}_d^r$ be an expanding circle endomorphism. Assume that the Lyapunov spectrum of f is simple and β-sparse. Then there exists $\epsilon = \epsilon(f) > 0$ with the following property:*

If $g \in \mathcal{E}_d^r$ is another expanding circle map such that

$$\|g - f\|_{C^{r,1}} \leqslant \epsilon \quad \textit{and} \quad \mathrm{Lyap}_n(g) = \mathrm{Lyap}_n(f) \ \ \forall n \in \mathbb{N},$$

then g is $C^{r,1}$-smoothly conjugate to f, i.e., the marking conjugacy φ is a $C^{r,1}$-smooth diffeomorphism.

The proof of Theorem 1 is based on a novel KAM-type iterative scheme which, in turn, employs a Livsic-type theorem and the Whitney extension theorem as the main ingredients.

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A counterexample to the theorem of Laplace and Lagrange on the stability of semi major axes

Jacques Fejoz
(joint work with Andrew Clark, Marcel Guardia)

Consider the Newtonian 4-body problem in space, with positions $x_0, ..., x_3 \in \mathbf{R}^3$ and masses $m_0, ..., m_3 > 0$. For the sake of simplicity, let us focus on this region of the phase space which is called the "hierarchical planetary problem": bodies 0 and 1 revolve around their center of mass, body 2 revolves around and far away from bodies 0 and 1, and body 3 revolves around and even farther away from bodies 0, 1 and 2. Each body thus primarily undergoes the attraction of one other body: bodies 0 and 1 are close to being isolated, body 2 primarily undergoes the attraction of a fictitious body located at the center of mass of 0 and 1, and body 3 primarily undergoes the attraction of a fictitious body located at the center of mass of 0, 1 and 2. We think of body 0 as the Sun and of the three other bodies as planets. The position of the Sun may be recovered from the positions of the planets and from the conservation of the center of mass in an Galilean frame of reference attached to it.

The 18-dimensional phase space is the product of the phase spaces of the three planets, each diffeomorphic to $\mathbf{T} \times \mathbf{R} \times S^2 \times S^2$, where $\mathbf{T} \times \mathbf{R}$ is the symplectic Kepler space (with coordinates the mean anomaly[1] ℓ and the semi major axis a) and where $S^2 \times S^2$ is the symplectic secular space (with coordinate s, determining the oriented plane of the ellipse and the polar angle of the ellipse in its plane). Since the Kepler space is a symplectic submanifold, third Kepler law (the period of revolution depends only on the energy, or, equivalently, on the semi major axis) may be recast by saying that there are Darboux coordinates (ℓ, L) such that L depends only on the semi major axis (and not on the other elliptical elements).

In the first approximation, our problem consists of three uncoupled Kepler problems and is integrable. At the next order of approximation, because of the

[1] The mean anomaly is the angle determining the position of the planet on its Keplerian ellipse, which increases linearly with time in the Keplerian dynamics, and which vanishes at the perihelion.

mutual attraction of planets 1 and 2, the secular dynamics of the two inner planets is non trivial anymore. It is described by the first term in the expansion with respect to $\|x_1\|/\|x_2\|$ of the average

$$\int_{\mathbf{T}^2} \frac{d\ell_1\, d\ell_2}{\|x_1 - x_2\|},$$

which can be thought of as a function on the secular space of planets 1 and 2 in the open set where Keplerian ellipses do not intersect one another. This kind of dynamics had been extensively studied by Lagrange, Laplace and many others in the neighborhood of circular and coplanar Keplerian ellipses, but much less globally on the secular space. Surprisingly, as noticed by Harrington in 1966, it is integrable too. The typical secular motion is that each Keplerian plane rotates around the total angular momentum vector, and each Keplerian ellipse rotates in its plane. Computation shows that if the two Keplerian planes are mutually inclined, there is a hyperbolic singularity, where the inner ellipse instead has its argument of pericenter blocked. This singularity, in the full phase space, gives rise to a symplectic, normally hyperbolic, invariant cylinder which is 16-dimensional or, after the symplectic reduction by the symmetry of rotations, 12-dimensional.

We will focus on instabilities in 5 dimensions, namely the s_2 and a_3 directions, over a time interval which is polynomially small with respect to the small distances. Other directions are either

- s_1 (we need to localize at the hyperbolic cylinder, which determines at least the adiabatic components of s_1)
- angles
- or directions in which instabilities would be exponentially slow (e.g. the semi major axes of the two inner planets)
- or stable directions due to the conservation of the angular momentum, e.g. e_3 (a function of the angular momentum of the third planet and a_3).

The s_2 direction contains both adiabatic invariants and angles. We could also control the other angles but the main point is to control adiabatic invariants.

Theorem (A. Clark-J. F.-M. Guardia). *Assume $m_0 \neq m_1$.[2] For every finite itinerary $s_2^1, \dots s_2^k \in S^2 \times S^2$, $a_3^1, \dots, a_3^k \in]0, +\infty[$ and every $\epsilon > 0$, there exists an open set of initial conditions whose trajectories realise the prescribed itinerary up to precision ϵ.*

This theorem proves the existence of Arnold diffusion in "celestial mechanics", as conjectured by Arnold in 1964.

Some notations: Let e_j be the eccentricities and C_j be the angular momenta. In the hierarchical regime, for eccentricities bounded away from 1, $a_1 \ll a_2 \ll a_3$. Even further, we consider a *strongly hierarchical regime*, where not only the semimajor axes ratios $\alpha_i = a_i/a_{i+1}$ are small, but even the ratios of the ratios α_i/α_{i+1} are

[2]Conjecturally, if $m_0 = m_1$, the conclusion of the theorem holds. But the proof would require additional, significant computations.

small, in the following quantitative manner:

(1) $$a_1 = O(1) \ll a_2 \ll a_3^{1/3}.$$

Here is the rough description of the scales of times where the trajectories of the theorem will be found:

- The fastest frequencies are the mean motions (Keplerian frequencies) of the two inner planets. Since $a_1 \ll a_2$, these inner mean motions do not interfere, which allows us to average out the mean anomalies, without resonances. As a consequence, the conjugate variables L_1 and L_2, or, equivalently, the semi major axes a_1 and a_2, are constant; this is the content of the Laplace-Lagrange theorem on the stability of semi major axes, whose conclusion will not extend to a_3 (due to the irrelevance of averaging out ℓ_3 in the strongly hierarchical regime).
- The next frequencies are the secular frequencies of the two inner planets. They govern the rotation of the plane of the ellipses around their angular momentum vector $C_1 + C_2$, and the rotation of the ellipses in their plane, as well as the quasiperiodic oscillations of the corresponding inclinations and eccentricities. The dynamics of the truncated relevant normal form ("quadrupolar dynamics" of planets 1 and 2) is still integrable, as already mentioned, due to the fact that the quadrupolar Hamiltonian does not depend on the argument of the outer pericenter g_2.
- In the strongly hierarchical regime, the outer semimajor axis is so large that the mean motion of planet 3 is slower than secular frequencies of the two inner planets.
- Then come the secular frequencies of the (outer) planet 3, approximately determined by the quadrupolar Hamiltonian of planets 2 and 3. The conservation of the total angular momentum vector $C = C_1 + C_2 + C_3 \simeq C_3$ prevents significant changes in the plane of the outer ellipse, or of the product $a_3\sqrt{1-e_3^2}$. In contrast, it does not prevent major (joint) changes in a_3 and e_3, nor changes in $C_1 + C_2$ since C_3 is an infinite source of angular momentum.

Along the orbits we prove the existence of, the two inner planets are close to the hyperbolic secular singularity of the quadrupolar Hamiltonian or to the associated stable and unstable manifolds. In particular, their mutual inclination will be large.

Some comments are in order.

- The drifting time needed to follow the prescribed itinerary in the theorem satisfies

(2) $$0 < T < C(m_0, m_1, m_2, m_3)\,\frac{N}{\delta^\kappa},$$

  where C is a constant depending only on the masses and the exponent $\kappa > 0$ does not depend on N nor on the itinerary. To be more precise, call $\alpha_i = a_i/a_{i+1}$, $i = 1, 2$, the semimajor axis ratios. As δ tends to zero, the α_i's will be chosen polynomially smaller, and the drifting time itself depends polynomially on the α_i's.

- As stated, the theorem assumes small semi major axis ratios, for fixed masses. But a refinement shows that the instability mechanism continues when we let the masses of the planets simultaneously tend to 0, i.e. in the planetary regime where $m_j = \rho\,\tilde{m}_j$ for $j = 1, 2, 3$ with $\rho > 0$ small. If planets 1 and 2 are located at a uniform distance (with respect to ρ) from the Sun and place planet 3 very far away, so that $a_3 \sim \rho^{-2/3}$, the instability time is $O(N/\delta/\rho^{35/3})$.

  Note that the instability time is polynomial with respect to the masses of the planets. This is consistent with Nekhroshev theory, because the standard hypotheses of this theory are not met (due in particular to the lack of uniform convexity or steepness).
- Let us briefly describe what the changes in $\tilde{C}_2$ imply in terms of the orbital elements of the second planet. Our prescribed itinerary in particular determines an itinerary in: the eccentricity e_2^k, the mutual inclination θ_{23}^k between planets 2 and 3, and the longitude h_2^k of the node of planet 2, for $k = 0 \dots N$. Then, we can construct an orbit and times $t_0 < t_1 < \cdots < t_N$ such that the osculating orbital elements satisfy

$$|e_2(t_k)-e_2^k| \leq \delta, \quad |\theta_{23}(t_k)-\theta_{23}^k| \leq \delta, \quad |h_2(t_k)-h_2^k| \leq \delta \quad \text{for} \quad k = 0, 1, ..., N. \tag{3}$$

  As already mentioned, the angular momentum of the third body is almost constant and therefore, the evolution of e_3 is determined by the evolution of a_3.

  Finally, the evolution of the eccentricity e_1 of the first planet, and the mutual inclination θ_{12} between planets 1 and 2, cannot be controled since they are prescribed by the diffusion mechanism. Let us briefly mention that:
  - The eccentricity e_1 does change but it can start arbitrarily close to 0. That is, the initial configuration can have all planets performing close to circular motion.
  - The mutual inclination i_{12} always stays above 55 degrees.
- In our Solar System, semimajor axes seem very stable. There are some exceptions. Notably, the semimajor axis of the Moon is drifting. But this is due to non-Hamiltonian, tidal effects. Also, at the early stages of our Solar System, planets migrated towards the exterior of the Solar System. But this migration too is a non-conservative phenomenon, explained by the interaction with the planetesimal disk.

  Orbits described in theorem show wild variations of various elliptical elements, and, plausibly, subsequent collisions of neighboring planets and their accretion. We may conjecture that only the observation of many extra-solar systems might exhibit one day such transient behavior.

The proof consists in

- analyzing the “inner dynamics” carried on the hyperbolic cylonder
- proving that the invariant manifolds of the hyperbolic cylinder cross transversally along a so-called homoclinic chanel (there are actually two of them)
- analyzing the “outer dynamics” (or scattering map) obtained by following the unstable and stable foliations of the cylinder
- showing that any finite random iteration of the the inner and outer dynamics are shadowed by integral curves, following the initial idea of Moeckel.

We refer to the three articles below for further details and references.

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A functional analytic approach to unbounded and oscillating solutions to the N-body problem

Susanna Terracini

(joint work with Jaime Paradela Díaz, Davide Polimeni)

We report on the functional-analytic approach to the search of unbounded trajectories in the N-Body problem (hyperbolic, parabolic, parabolic-hypebolic, oscillating etc.). We explore the use of renormalised energies in various contexts together with other global variational and topological methods. The same approach is pursued in the search for symbolic dynamics in various relevant models of celestial mechanics.

At first, we deal with half entire solutions to the N-body problem of Celestial Mechanics in the Euclidean space $\mathbb{R}^d$ of hyperbolic, parabolic or mixed type. We consider N point masses $m_1, ..., m_N > 0$ moving under the action of the mutual attraction, with the inverse-square law of universal gravitation. We denote the components of the configuration vector $x = (r_1, ..., r_N) \in \mathbb{R}^{dN}$ of the positions of the bodies and by $|r_i - r_j|$ the Euclidean distance between two bodies i and j. Newton's equation of motion for the i-th body of the N-body problem reads as

$$m_i \ddot{r}_i = - \sum_{j=1,\ldots,N,\ j\neq i}^{N} m_i m_j \frac{r_i - r_j}{|r_i - r_j|^3}.$$

Since these equations are invariant by translation, we can fix the origin of our inertial frame at the center of mass of the system. We can thus define the configuration space of the system as

$$\mathcal{X} = \left\{ x = (r_1, ..., r_N) \in \mathbb{R}^{dN},\ \sum_{i=1}^{N} m_i r_i = 0 \right\}$$

and denote by $\Omega = \{x \in \mathcal{X} \mid r_i \neq r_j \ \forall\, i \neq j\} \subset \mathcal{X}$ the set of configurations without collisions, which is open and dense in $\mathcal{X}$, and with Δ its complement, that is the collision set. Now we can write the equations of motion as

$$\ddot{x} = \nabla U(x), \tag{1}$$

where the function $U : \Omega \to \mathbb{R} \cup \{+\infty\}$ is the Newtonian potential

$$U(x) = \sum_{i<j} \frac{m_i m_j}{|r_i - r_j|} \tag{2}$$

and the gradient is taken with respect to the mass scalar product $\langle \cdot, \cdot \rangle_M$, which is defined as

$$\langle x, y \rangle_M = \sum_{i=1}^{N} m_i \langle r_i, s_i \rangle$$

for any $x = (r_1, ..., r_N)$, $y = (s_1, ..., s_N) \in \mathcal{X}$. Newton's equations define an analytic local flow on $\Omega \times \mathbb{R}^{dN}$ with a first integral given by the mechanical energy:

$$h = \frac{1}{2} \|\dot{x}\|_M^2 - U(x),$$

where $\| \cdot \|_M$ is the norm induced by the mass scalar product and h represents the energy of the motion.

We will be concerned with the class of expansive motions, which is defined in the following way.

Definition 1. A motion $x : [0, +\infty) \to \Omega$ is said to be expansive when all the mutual distances diverge, that is, when $|r_i(t) - r_j(t)| \to +\infty$ as $t \to +\infty$ for all $i < j$. Equivalently, the motion is expansive if $U(x(t)) \to 0$ as $t \to +\infty$.

From the conservation of the energy, we observe that since $U(x(t)) \to 0$ implies $\|\dot{x}(t)\|_M \to \sqrt{2h}$ as $t \to +\infty$, expansive motions can only occur at nonnegative energies.

For a given motion, we introduce the minimum and the maximum separation between the bodies at time t as the two functions

$$r(t) = \min_{i<j} |r_i(t) - r_j(t)| \quad \text{and} \quad R(t) = \max_{i<j} |r_i(t) - r_j(t)|.$$

The next fundamental theorems give us a more accurate description of the system's expansion.

Theorem 1 (Pollard, 1967 [17]). *Let x be a motion defined for all $t > t_0$. If r is bounded away from zero, then we have that $R = O(t)$ as $t \to +\infty$. In addition, $R(t)/t \to +\infty$ if and only if $r(t) \to 0$.*

Theorem 2 (Marchal-Saari, 1976 [12]). *Let x be a motion defined for all $t > t_0$. Then either $R(t)/t \to +\infty$ and $r(t) \to 0$, or there is a configuration $a \in \mathcal{X}$ such that $x(t) = at + O(t^{2/3})$. In particular, for superhyperbolic motions (i.e. motions such that $\limsup_{t\to+\infty} R(t)/t = +\infty$) the quotient $R(t)/t$ diverges.*

Theorem 3 (Marchal-Saari, 1976 [12]). *Suppose that $x(t) = at + O(t^{2/3})$ for some $a \in \mathcal{X}$ and that the motion is expansive. Then, for each pair $i < j$ such that $a_i = a_j$, we have $|r_i(t) - r_j(t)| \approx t^{2/3}$.*

Now, let us recall the well known Chazy classification of the expansive motions for the three-body problem, based on the asymptotic order of growth of the distances between the bodies. This prevents expansive motion to be superhyperbolic, so we can assume that it is of the form $x(t) = at + O(t^{2/3})$ for some limit $a \in \mathcal{X}$. Assuming that the center of mass of the system is at rest, Chazy classified these motions as follows.

Theorem 4 (Chazy [5]). *Every solution of the Restricted 3-body Problem defined for all (future) times belongs to one of the following classes*

- *B (bounded):* $\sup_{t\geq 0} |q(t)| < \infty$.
- *P (parabolic)* $|q(t)| \to \infty$ *and* $|\dot{q}(t)| \to 0$ *as* $t \to \infty$.
- *H (hyperbolic):* $|q(t)| \to \infty$ *and* $|\dot{q}(t)| \to c > 0$ *as* $t \to \infty$.
- *O (oscillatory)* $\limsup_{t\to\infty} |q(t)| = \infty$ *and* $\liminf_{t\to\infty} |q(t)| < \infty$.

Notice that this classification also applies for $t \to -\infty$. We distinguish both cases adding a superindex + or − to each of the cases, e.g. H^+ and H^-.

In fact, we can more precisely distinguish between:

- *Hyperbolic*: $a \in \Omega$ and $|r_i(t) - r_j(t)| \approx t$ for all $i < j$;
- *Partially hyperbolic*: $a \in \Delta$ but $a \neq 0$;
- *Completely parabolic*: $a = 0$ and $|r_i(t) - r_j(t)| \approx$ [1] $t^{2/3}$ for all $i < j$.

The following definition is in order.

Definition 2. A motion $x(t)$ is said to have limit shape when there is a time dependent similarity $S(t)$ of the space $\mathbb{R}^d$ such that $S(t)x(t)$ converges to some configuration $a \neq 0$.

In our case, there is a diagonal action of $S(t)$, which means that $S(t)x = (S(t)r_1, ..., S(t)r_N)$ for $x = (r_1, ..., r_N) \in \mathcal{X}$. In particular, for the case of (half) hyperbolic motions, we can say that the limit shape of such a motion is its asymptotic velocity $a = \lim_{t\to+\infty} \frac{x(t)}{t}$. Similaily, (half) parabolic motions also possess a limit shape, which is now bound to be a central configuration, that is, a critical point of the potential U constrained on the inertia ellipsoid $\mathcal{E} = \{x \in \mathcal{X} \,:\, \|x\|_M^2 = 1\}$.

In this talk we are going to tackle the existence of half entire expansive solutions for the Newtonian N-body problem by a global variational approach, using a renormalized action functional, as the Lagrangian is not expected to be integrable on the half line. In particular, referring to Chazy's classification, we will show a

[1] Given positive functions f and g, we write $f \approx g$ when there exist two positive constants α and β such that $\alpha \leq \frac{f}{g} \leq \beta$.

proof of existence for each one of the previous three classes of motions. At first, we shall revisit recent works by E. Maderna and A. Venturelli about the existence of half hyperbolic and parabolic trajectories.

Theorem 5 (Maderna and Venturelli 2020, [10])**.** *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a hyperbolic motion $x : [1, +\infty) \to \mathcal{X}$ of the form*

$$x(t) = at - \log(t)\nabla U(a) + o(1) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = x(1) \in \mathcal{X}$, for any collisionless configuration $a \in \Omega$.

Theorem 6 (Maderna and Venturelli 2009, [9])**.** *Let us consider $d \in \mathbb{N}$, $d \geq 2$, and a Keplerian potential $U : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$. For Newton's equations $\ddot{x} = \nabla U(x)$ in $\mathbb{R}^d$ there is a parabolic solution $x : [1, +\infty) \to \mathcal{X}$ of the form*

$$x(t) = \beta b_m t^{2/3} + o(t^{2/3}) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = x(1) \in \mathcal{X}$, for any minimizing normalized central configuration b_m and for $\beta = \sqrt[3]{\frac{9}{2}U(b_m)}$.

Here a minimal central configuration is a minimizer of the potential U constrained on the inertia ellipsoid $\mathcal{E} = \{x \in \mathcal{X} : \|x\|_M^2 = 1\}$. The existence of hyperbolic and parabolic solutions for the Newtonian N-body problem has already been proved by Maderna and Venturelli in 2020 and 2009, respectively. In [10], they proved the existence of hyperbolic motions for any prescribed limit shape, any initial configuration of the bodies and any positive value of the energy by constructing global viscosity solutions for the Hamilton-Jacobi equation $H(x, d_x u) = h$. In addition, they showed that these solutions are fixed points of the associated Lax-Oleinik semigroup. In [9], for any starting configuration they proved the existence of parabolic arcs asymptotic to any prescribed minimizing normalized central configuration. These solutions, whose actions are infinite, were found as the limits of converging subsequences in families of minimizing motions, where the existence of the approximate solutions follows from the application of the Direct Method of the Calculus of Variations More specifically, these solutions were obtained as the limits of solutions of sequences of approximating two-point boundary value problems. Both proofs in [10] and [9] can be seen as applications of Marchal's Theorem.

Compared to Maderna and Venturelli's articles, we show alternative and simpler proofs for the existence of hyperbolic and parabolic solutions in a unitary context, which are both based on a straightforward application of the Direct Method of the Calculus of Variations to minimize the renormalized Lagrangian actions associated to the problem.

After proving Theorems 5 and 6, we are also able to similarly prove the existence of partially hyperbolic solutions for the N-body problem. In order to state our main result we need to introduce the *a-cluster partition* associated with $a \in \Delta \setminus \{0\}$, where clusters are the equivalence classes of the relation $i \simeq j \iff a_i - a_j = 0$. Given a cluster K, we consider the potential U_K, where the sum in (2) restricted to the cluster K. The a-clustered potential U_a is the sum of all the clustered potentials of the partition. Now we can state our main theorem:

Theorem 7 (Polimeni and Terracini, 2023)**.** *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a partially hyperbolic motion $x : [1, +\infty) \to \mathcal{X}$ of the form*

$$x(t) = at + \beta b_m t^{2/3} + o(t^{1/3^-}) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = x(1) \in \mathcal{X}$, for any collision configuration $a \in \Delta$, for any minimizing normalized central configuration $b_m \in \mathcal{X}$ of the a-clustered potential, and for any choice of the energy constant $h > 0$.

Partially hyperbolic motions are those expansive motions of the form $x(t) = at + O(t^{2/3})$, for $t \to +\infty$, such that their limit shapes have collisions, that is, $a \in \Delta \setminus \{0\}$, and $a \neq 0$. For the Newtonian N-body problem, the existence of partially hyperbolic solutions for any prescribed positive energy and any given initial configuration of the bodies has already been proved by Burgos in [2], where his proof follows from and application of Marchal's Theorem and Maderna and Venturelli's Theorem on the existence of hyperbolic motions. With respect to Burgos' result, our approach gives us much more information about the asymptotic behaviour of the solution and a better description of the motion of the bodies. Indeed, to prove Theorem 7, we partition the set of bodies following the natural cluster partition that was presented by Burgos and Maderna in [3] and is defined as follows: if $x(t) = (r_1(t), ..., r_N(t))$ and $a = (a_1, ..., a_N)$, then $a_i = a_j$ if and only if $|r_i(t) - r_j(t)| = O(t^{2/3})$, and the partition of the set of bodies is defined by this equivalence relation. This means that partially hyperbolic motions can be viewed as clusters of bodies moving asymptotically with a linear growth, while the distances of the bodies inside each clusters grow with a rate of order $t^{2/3}$. Using this particular partition, we are able to decompose the Lagrangian action into two terms: one of them is related to the hyperbolic motion of the clusters and the other one is related to the parabolic motion of the bodies inside the clusters. Through similar proofs to the ones in Theorems 5 and 6, we can thus apply the Direct Method of the Calculus of Variation and Marchal's Theorem also to the case of partially hyperbolic motions.

Next, we discuss the problem of oscillatory motion in a particular configuration of the Restricted 3-body Problem known as the Restricted Isosceles 3-body Problem. In this configuration, the two primaries have equal masses $m_0 = m_1 = 1/2$ and move periodically on a degenerate ellipse of eccentricity one (a line), according to the Kepler laws for the motion of the 2-body Problem. The massless particle moves on the plane perpendicular to the line along which the primaries move. In polar coordinates, the Hamiltonian of the Restricted Isosceles 3-body Problem reads

$$H_G(r, t, y) = \frac{y^2}{2} + \frac{G^2}{2r^2} - V(r, t) \qquad V(r, t) = \frac{1}{\sqrt{r^2 + \rho^2(t)}}. \tag{3}$$

where G is the modulus of the angular momentum (which is preserved) and $2\rho(t)$ is the disctance beteween the two primaries .

In [8], M. Guardia, J. Paradela, T. Seara and C.Vidal, proved the following result.

Theorem 8 (Guardia, Paradela Díaz, Seara and Vidal, [8])**.** *Consider the Hamiltonian system H_G defined in* (3)*. Denote by X^+ (respectively Y^-) either H^+, P^+, B^+ or OS^+ (respectively H^-, P^-, B^- or OS^-) according to Chazy's classification in Theorem 4. Then, there exists $G_* \gg 1$ such that for all $G \in \mathbb{R}$ such that $|G| \geq G_*$, the Hamiltonian system H_G satisfies*

$$X^+ \cap Y^- \neq \emptyset$$

for all possible combinations of X^+ and Y^-.

Theorem 8 is proved by exploiting the fact that for G large enough, in a suitable region of the phase space, the Hamiltonian H_G can be studied as a perturbation of the (integrable) 2-body Problem. This allowed the authors to prove that the periodic orbit γ_∞ posses global stable and unstable invariant manifolds which intersect transversally. As a corollary of this result, a rather straightforward implementation of Moser's ideas shows the truth of Theorem 8.

The following is the first main obtained in collaboration with Jaime Paradela Díaz.

Theorem 9 (Paradela Díaz and Terracini 2022)**.** *Consider the Hamiltonian system H_G defined in* (3)*. Denote by X^+ (respectively Y^-) either H^+, P^+, B^+ or OS^+ (respectively H^-, P^-, B^- or OS^-) according to Chazy's classification in Theorem 4. Then, for almost all $G \in \mathbb{R}$ the Hamiltonian system H_G satisfies*

$$X^+ \cap Y^- \neq \emptyset$$

for all possible combinations of X^+ and Y^-.

To the best of our knowledge, Theorem 9 is the first complete analytic proof of the existence of oscillatory motions relying upon a global analytical approach rather than on perturbative techniques. Some interesting related works, where the existence of oscillatory motions is obtained in a setting which is not close to integrable, are [13] and [4]. While in [13] the author shows the existence of oscillatory motions in the 3-body Problem close to triple collision (small values of the total angular momentum), in [4] the authors obtain a computer assisted proof of the existence of oscillatory motions in the Restricted Circular 3-body Problem for small values of the Jacobi constant.

Theorem 9 is indeed obtained as a consequence of the following result.

Theorem 10 (Symbolic Dynamics)**.** *Let $\{l_j\} \subset \mathbb{Z}$ be an increasing sequence and define the time intervals $I_j = [(l_j - l_{j-1})/2,\ (l_{j+1} - l_j)/2]$. Then, for almost all $G \in \mathbb{R}$, all $\varepsilon > 0$ and all R sufficiently large, there exists an orbit $r_{\mathrm{h}}(s) : \mathbb{R} \to \mathbb{R}_+$ of* (3) *homoclinic to γ_∞ and a constant $L > 0$ such that if the sequence $\{l_j\} \subset \mathbb{Z}$ satisfies $l_{j+1} - l_j \geq L$, then, for any sequence $\sigma = \{\sigma_j\} \subset \{0,1\}^{\mathbb{Z}}$ there exists an orbit $r_\sigma(s) : \mathbb{R} \to \mathbb{R}_+$ of* (3) *such that , if $\sigma_j = 0$*

$$|r_\sigma|_{C^1(I_j)} \geq R$$

and if $\sigma_j = 1$

$$|r_\sigma - r_\mathrm{h}|_{C^1(I_j)} \le \varepsilon,$$

Moreover, if σ *has only a finite number of non zero entries, then* r_σ *is a homoclinic solution.*

Theorem 10 can be read as follows. For almost all $G \in \mathbb{R}$ there exist an orbit r_h of (3) homoclinic to γ_∞ such that the following holds. Let $z_* = (r, y, t) = (r_h(0), \dot r_h(0), 0) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}$, let $z_\infty = (r, y, t) = (\infty, 0, 0) = \gamma_\infty \cap \{t = 0\} \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}$ and denote by Φ the Poincaré map induced on the section $\{t = 0\}$ by the flow to the Hamiltonian (3). Then, for any $\delta > 0$ and any sequence $\{z_k\}_{k\in\mathbb{Z}} \subset \{z_\infty, z_*\}^{\mathbb{Z}}$ there exists a point $z \in B_\delta(z_0)$ and a sequence $\{n_k\}_{k\in\mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}$ such that $\Phi^{n_k}(z_0) \in B_\delta(z_k)$ [2]. The statement in Theorem 10 is indeed stronger since it also provides control on the orbit in all the intervals $[(n_k - n_{k-1})/2, (n_k + n_{k+1})/2]$.

The following corollary of Theorem 10 can obtained by nowadays well known arguments.

Corollary 1. *For almost all* $G \in \mathbb{R}$ *the Restricted Isosceles* 3*-body Problem is not* C^ω *integrable and has positive topological entropy.*

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Non contractible periodic points for area preserving surface homeomorphisms

Patrice Le Calvez

Let S be a smooth connected closed orientable surface of genus $g \geq 2$, furnished with a normalized smooth area form ω. We denote $\mathrm{Homeo}_*(S)$ the space of homeomorphisms of S isotopic to the identity. A continuous path $I = (f_t)_{t\in[0,1]}$ joining the identity to a map $f \in \mathrm{Homeo}_*(S)$ is called an *identity isotopy* of f and the *trajectory* of a point $z \in S$ (defined by I) is the path $I(z) : t \mapsto f_t(z)$ joining z to $f(z)$. We write $\mathcal{M}(f)$ for the set of f-invariant Borel probability measures. The *rotation vector* $\mathrm{rot}_f(\mu) \in H_1(S,\mathbb{R})$ of a measure $\mu \in \mathcal{M}(f)$ is defined by the equality

$$\int_S \left(\int_{I(z)} \alpha\right) d\mu(z) = \langle [\alpha], \mathrm{rot}_f(\mu)\rangle,$$

where $[\alpha] \in H^1(S,\mathbb{R})$ is the cohomology class of a given closed 1-form α and I is an identity isotopy of f. The term on the left is well defined, where $\int_{I(z)} \alpha = \int_\gamma \alpha$ for every smooth path homotopic to $I(z)$ (relative to the ends). It does not depend on the choice of I because we suppose that $g \geq 2$, which implies that all identity isotopies of f are homotopic. It depends linearly on α and vanishes when α is exact. An interesting case is the case where $\mu = \mu_\omega$ is naturally associated to ω and $f \in \mathrm{Symp}^r_*(S,\omega)$, $1 \leq r \leq +\infty$, the space of C^r-diffeomorphisms of S preserving ω and isotopic to the identity. In that case we define

$$\mathrm{Ham}^r(S,\omega) = \{f \in \mathrm{Symp}^r_*(S,\omega) \,|\, \mathrm{rot}_f(\mu_\omega) = 0\}.$$

Another interesting case is the case where O is a q-periodic orbit of f and $\mu_O = \frac{1}{q}\sum_{z\in O}\delta_z$. We write $\mathrm{rot}_f(O)$ instead of $\mathrm{rot}_f(\mu_O)$, noting that $\mathrm{rot}_f(O) = \frac{1}{q}[I^q(z)]$ if $z \in O$. Here $[\Gamma] \in H_1(S,\mathbb{Z})$ is the homology class of a loop Γ.

Let us state the first result proved in [3]:

Theorem 1. *For $1 \leq r \leq \infty$, there exists an open and dense set $\mathcal{O}_r \subset \mathrm{Ham}^r(S,\omega)$ such that if $f \in \mathcal{O}_r$, there exist $p \geq g$ and $\kappa_1, \ldots, \kappa_p$ in $H_1(S,\mathbb{Q})$ linearly independent such that*

1. *(1) the space $H = \mathrm{Vect}(\kappa_1,\ldots,\kappa_p)$ is a coisotropic subspace of $H_1(S,\mathbb{R})$ (for the natural intersection form $\wedge$);*

(2) *for every* $i \in \{1, \dots, p\}$, *there exists a positive integer* n_i *and for every* $p/q \in [0,1] \cap \mathbb{Q}$ *a* qn_i*-periodic orbit* $O^i_{p/q}$ *such that* $\mathrm{rot}(O^i_{p/q}) = \frac{p}{q}\kappa_i$.

The proof uses the following result (see [2])

Theorem 2. *Suppose that* $f \in \mathrm{Symp}^r_*(S, \omega)$ *satisfies the following conditions.*

(1) *Every periodic point is non degenerate.*
(2) *The branches of hyperbolic points intersect transversally.*
(3) *If* U *is a neighborhood of an elliptic periodic point* z, *then there is a topological closed disk* D *containing* z, *contained in* U, *and bordered by finitely many pieces of stable and unstable manifolds of some hyperbolic periodic point* z'.
(4) *We have* $\mathrm{Per}(f) > 2g - 2$.

Then every hyperbolic point has transverse homoclinic intersection.

To obtain Theorem 1 one must go further in the study of maps satisfying the previous properties. Denote $\widetilde{S}$ the universal covering space of S and G the group of covering automorphisms. Denote also $\tilde{f}$ the canonical lift of f to $\tilde{S}$. Under the hypothesis of Theorem 2, denote $\mathcal{X}$ the set of f-invariant open sets V that contain all positive hyperbolic contractible fixed points, and define

$$H = \min\{\iota_*(H_1(V, \mathbb{R})) \,|\, V \in \mathcal{X}\},$$

where $\iota_* : H_1(V, \mathbb{R}) \to H_1(S, \mathbb{R})$ is induced by the inclusion map $\iota : V \to S$. We can prove that

- H is coisotropic;
- there exists $T_1, \dots, T_p$ in G such that $H = \mathrm{Vect}([T_1], \dots [T_p])$ and such that for every $i \in \{1, \dots, p\}$, there exists a positive hyperbolic point $\tilde{z}_i$ of $\tilde{f}$ and an unstable branch of $\tilde{z}_i$ that intersects a table branch of $T_i(\tilde{z}_i)$.

We deduce that the conclusion of Theorem 1 occurs because we have found rotational horseshoes. It becomes easy to get Theorem 1 because a generic Hamiltonian diffeomorphism has at least 2g+2 fixed points.

Let us now state the second result proved in [1].

Theorem 3. *If* $f \in \mathrm{Homeo}_*(S)$ *preserves a Borel probability measure* λ *such that* $\mathrm{supp}(\lambda) = S$ *and* $\mathrm{rot}_f(\lambda) \in \mathbb{R}H_1(S, \mathbb{Z})$, *then* f *has infinitely many periodic points.*

More precisely, for every ergodic measure $\nu \in \mathcal{M}(f)$ *that is not a Dirac measure at a contractible fixed point and every neighborhood* $\mathcal{U}$ *of* $\mathrm{rot}_f(\nu)$ *in* $H_1(S, \mathbb{R})$, *there exists* $\kappa \in H_1(S, \mathbb{Q}) \cap \mathcal{U}$ *and* $n \geq 1$ *such that for every* $p/q \in [0,1] \cap \mathbb{Q}$ *there exists a* qn*-periodic orbit* $O_{p/q}$ *such that* $\mathrm{rot}(O_{p/q}) = \frac{p}{q}\kappa$.

The first statement of Theorem 3 was proved independently by Rohil Prasad [5] using very strong new results of symplectic topology. The proof given in [1] uses ergodic arguments and the forcing theory on transverse foliations. In fact it is a continuation of the works of Gabriel Lellouch [4] who proved that the conclusion of Theorem 3 occurs if there exists $\mu \in \mathcal{M}(f)$ such that $\mathrm{rot}_f(\nu) \wedge \mathrm{rot}_f(\mu) \neq 0$. In this situation he proved that there exist topological rotational horseshoes (which generalize the rotational horseshoes seen previously). Under the hypothesis of Theorem 3 one can construct topological rotational horseshoes, except in a very special situation, where a generalization of the Poincaré-Birkhoff theorem in a suitable annulus is needed. This situation concerns the “integrable case” very close to the case where f is the time one map of a flow induced by a time independent symplectic vector field X. It must be noted that Theorem 3 is obvious in this last case because the non trivial dynamics is supported on invariant annuli foliated by invariant curves whose rotation numbers tend to zero when approaching the ends of the annulus.

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Chaos in reversible homoclinic tangles

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(joint work with Ale Jan Homburg, Jeroen Lamb)

It is a classical result in the theory of dynamical systems that homoclinic tangles give rise to hyperbolic horseshoes and thus positive topological entropy. The history of chaotic dynamics started with the discovery by Poincaré [1] that the stable and unstable manifolds of a saddle periodic orbit may have a transverse intersection along a homoclinic orbit. For a sufficiently small neighborhood of the union of a hyperbolic periodic orbit and its transverse homoclinic, the invariant set that consists of all orbits that stay entirely in this neighborhood is uniformly hyperbolic and admits a symbolic representation by a full shift on two symbols [2, 3]. This result, the Shilnikov-Smale theorem, provides the most fundamental criterion for chaos in a dynamical system. The fact that the Poincare's homoclinic tangle implies positive topological entropy holds true also in the original Hamiltonian setting. A subtle point here is that the Hamiltonian function is a first integral, and saddle periodic orbits of a Hamiltonian system arise in families, parameterized by the value of the Hamiltonian. Such family is a normally-hyperbolic invariant manifold; the homoclinic tangle corresponds to an intersection of its stable and unstable manifolds. Formally speaking, each periodic orbit in the family is not hyperbolic. However, inside any dynamically invariant level set of the Hamiltonian, the saddle periodic orbit is isolated and hyperbolic with a transverse homoclinic, so the Shilnikov-Smale theorem is applied and the positivity of the topological entropy follows.

Normally-hyperbolic one-parameter families of periodic orbits with transversely intersecting stable and unstable manifolds also naturally arise in reversible systems [4]. Despite the substantial interest in reversible dynamical systems, a concise characterization of a reversible homoclinic tangle, which we believe deserves to be central to the theory of chaotic dynamics in reversible systems, has been lacking.

The core issue here is that reversible systems do not need to be Hamiltonian and, typically, there exists no first integral. For example, if a perturbation of a reversible Hamiltonian system preserves the reversibility but breaks the Hamiltonian structure, then a given family of symmetric periodic orbits and their symmetric homoclinics survives the perturbation. However, the dynamically invariant foliation by energy levels gets, typically, destroyed, as the energy is no longer conserved. This provides the possibility that many orbits leave a neighborhood of the homoclinic tangle due to the drift in energy, which makes the dynamics near a reversible homoclinic tangle very much different from those in the Hamiltonian setting. The a priori non-controllable drift along the central direction means that one should go beyond the standard hyperbolicity techniques to resolve even the most basic question – whether the dynamics near the reversible tangle are chaotic?

We answer this question affirmatively for reversible flows for which the dimension of the set of fixed points of the involutory reversing symmetry is exactly half the dimension of the phase space. Namely, we prove that the set of orbits that remain in any given neighborhood of the reversible homoclinic tangle (satisfying transversality conditions) has *positive topological entropy*.

Note that we do not establish the existence of finite-type shift dynamics which are often associated with positive topological entropy. In fact, one can build examples where there is no semi-conjugacy to a non-trivial Markov chain on any invariant subset - in such examples no invariant measure with all non-zero Lyapunov exponents exist in the reversible homoclinic tangle, in spite of the positivity of the entropy.

As an example of an application of our result, we mention that symmetric homoclinic tangles of the type we consider arise locally near homoclinic loops to symmetric equilibria of reversible flows. This includes homoclinic bellows [5] and a homoclinic loop to a saddle-focus [7, 6] – in both cases there exists a symmetric homoclinic tangle, which implies the positivity of the topological entropy.

Non-Hamiltonian reversible vector fields with symmetric homoclinic tangles arise in the study of pattern formation in many classes of partial differential equations [8, 9] with one spatial variable. For example, for the partial differential equations of the reaction-diffusion type

$$u_t = Au_{xx} + N(u), \qquad x \in R^1,$$

a stationary solution satisfies the ODE

$$u''(x) = -A^{-1}N(u(x)).$$

This equation is invariant under the transformation $x \to -x$, i.e., it is reversible. The time-reversal symmetry acts as $u' \to -u'$, its set of fixed points is given by $\{u' = 0\}$ and its dimension is half of the dimension of the phase space of the ODE (the space of pairs (u, u')). Thus, our theorem is applicable. It provides a characterization of the complexity of the set of solutions near a family of reflection-symmetric solutions that are asymptotically spatially periodic with a localized "defect": the number of different patterns that materialize in a finite spatial window grows exponentially with the windows size.

Can a similar result be obtained for stationary in time and asymptotically spatially-periodic solutions of reaction-diffusion systems defined for $x \in R^m$ with $m > 1$? This question is open.

Another natural setting of non-Hamiltonian reversible dynamical systems where our theorem may be applied, is that of mechanical systems with non-holonomic constraints. If the system is defined by a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ with a single constraint $\mathbf{a}(\mathbf{q}) \cdot \dot{\mathbf{q}} = 0$, then the equations of motion derived from the d'Alembert principle are

$$\frac{d}{dt}\partial_{\dot{\mathbf{q}}}L - \partial_{\mathbf{q}}L = \mu(t)\, \mathbf{a}(\mathbf{q}),$$

where the factor μ is such that the equations are consistent with the constraint at each moment of time. This system preserves the energy $E = \partial_{\dot{\mathbf{q}}}L \cdot \dot{\mathbf{q}} - L$, but it is not Hamiltonian in general (e.g., the phase volume does not need to be preserved). However, when the Lagrangian L is an even function of the velocity vector $\dot{\mathbf{q}}$, the imposition of the constraint keeps the reversibility in tact. If the space of coordinates $\mathbf{q}$ is $(n+1)$-dimensional, then we have $(n+1)$ coordinates and $(n+1)$ velocity components subject to 2 constraints – the velocity constraint and the energy constraint. Thus, the dimension of the phase space for the system at a fixed energy level is $2n$. The set of the fixed points of the involution $R : \dot{\mathbf{q}} \to -\dot{\mathbf{q}}$ is given by the equation $\{\dot{\mathbf{q}} = 0,\ L(\mathbf{q}, 0) = E\}$ and has dimension n (i.e., half of the dimension of the phase space) if the energy E is in the range of values of $L(\mathbf{q}, 0)$. One concludes that generic reversible Lagrangian systems with one velocity constraint fall in the class we consider.

An example where our theorem 1 may be applicable is given by a Chaplygin sleigh [10, 11] moving on a generic surface. If a non-holonomic mechanical system is symmetric with respect to a continuous group acting on the configuration space, the symmetry reduction decreases the dimension of the configuration space and, hence, the dimension of $Fix(R)$, as one can see in the examples of rattlebacks [12]. Adding more velocity constraints increases the dimension of $Fix(R)$ relative to the dimension of the phase space. Thus, one obtains examples of mechanical systems where $\dim Fix(R)$ is strictly less or greater than half of the phase space dimension. In the latter case, the symmetric periodic orbits go in families that depend on more than one parameter. The question of whether symmetric homoclinic tangles involving such families of periodic orbits always yield positive topological entropy remains open.

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