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Recent Trends in Algebraic Geometry

Organized by
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Abstract. Algebraic geometry has grown into a broad subject, with many different streams often advancing quite independently of each other. Nonetheless, important advances have often come from visionary applications of ideas in one part of the subject to another. This workshop brought together leaders and future leaders in different areas of the subject, centered on geometric methods or geometric problems. It also brought together groups from different regions of the globe, in order to bridge communities of different sorts, and help new ideas quickly spread throughout algebraic geometry. Some of the best freshly-minted algebraic geometers were deliberately invited, so that they could meet their peers from around the world and learn about different perspectives on the subject.

Mathematics Subject Classification (2020): 14-XX.

Introduction by the Organizers

The workshop *Recent Trends in Algebraic Geometry* was organised by Olivier Debarre (Paris), Gavril Farkas (Berlin), Yuri Tschinkel (New York), and Ravi Vakil (Stanford). There were 18 one-hour talks with a maximum of four talks a day, and an evening session of short presentations allowing young participants to introduce their current work (and themselves). The schedule deliberately left plenty of room for informal discussion and work in smaller groups. The extended abstracts give a detailed account of the broad variety of topics of the meeting. (It should be noted that six of the lectures were given by recent Ph.D.'s.) We focus on a representative sample here:

Alexander Perry (with Hotchkiss and de Jong): The period-index problem. Perry's talk seemed designed to illustrate the importance of making sure there are workshops such as this to bring together very different parts of algebraic geometry. He reported on the first significant new progress in a long time on the period-index conjecture, on the relationship between two measures of complexity of the Brauer group. This question comes from the more arithmetic part of the subject. One of the two main results Perry described, joint with recent Ph.D. Hotchkiss, is the first nontrivial case of the unramified period-index conjecture in dimemnsion greater than 2: for a complex abelian threefold, for any Brauer class, the index divides the square of the period. The method of proof was ingenious and dramatic, pulling in the recent theory of derived categories, as well as Hodge theory. Part of the argument is interpreted as a case of the integreal Hodge conjecture "for categories". Perry then pulls in the theory of Donaldson-Thomas invariants (a development originally motivated from physics), develops the theory for Calabi-Yau-3 categories, and then uses the existence of an abelian threefold with a nonvanishing Donaldson-Thomas invariant. The connections came as a surprise to all concerned.

Isabel Vogt (joint with Eric Larson): Interpolation for Brill Noether curves. A real highlight of the wrkshop was the talk of Isabel Vogt who reported on the resolution of a century old problem obtained jointly with Eric Larson. Precisely, one asks when given n general points in the projective space $\mathbb{P}^r$, one can pass a Brill-Noether general curve C of genus g and degree d throuh them? Simple cases of this question go back to the very beginnings of algebraic geometry, but a complete answer to this daunting problem depending on four parameters seemed out of reach.

An obvious necessary condition is that the parameter space of pairs consisting of Brill-Noether general curves together with n points should exceed the dimension of the parameter space of n points in $\mathbb{P}^r$, that is, rn. This translates into an inequality $(r-1)n \leq (r+1)d-(r-3)(g-1)$, and the obvious expectation is that for g, r, d, n satisfying this inequality, one should be able to construct a Brill-Noether general curve through n points. However, an easy inspection reveals four trivial counterexamples, the simplest being when $(g, r, d) = (2, 3, 5)$. The rather amazing result of E. Larson and Vogt is that with the exception of these pathological counterexamples, the interpolation problem always holds! Their result, which holds even in positive characteristics reduces to a sophisticated interpolation property of the normal bundle of the curve in projective space. The main merit of this work, which makes use of important techniques developed in the course of the resolution by E. Larson of the Maximal Rank Conjecture (another milestone in the tejory of algebraic curves), is that it presents a clear degeneration path capable of handling all the possible cases.

Serge Cantat: Dynamics of Large Groups of Automorphisms of K3 Surfaces. A large group Γ of automorphisms of a (complex projective) K3 surface X is a nonabelian free group whose nontrivial elements act with positive topological entropy on S. Classical examples include:

- The surface S is a divisor of multidegree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the group Γ is generated by the three involutions of X induced by the three double covers $S \to \mathbb{P}^1 \times \mathbb{P}^1$ given by projections.
- The set of pentagons p_0, p_1, p_2, p_3, p_4 in $\mathbf{R}^2$ modulo the action of the group of direct isometries of $\mathbf{R}^2$ is a (real) K3 surface X and the group Γ is generated by the five involutions of X given by reflecting p_i along the line $\langle p_{i-1}, p_{i+1} \rangle$.

In this situation, using a combination of Hodge theory, complex analysis and pluripotential theory, dynamical systems, and arithmetic dynamics, Cantat and Dujardin studied stationary measures, equidistribution of random orbits, and finite orbits.

Sam Payne: Motivic structures in the cohomology of moduli spaces of curves. Payne reported on recent work with a number of different collaborators, leading to dramatic increases in our understanding of the cohomology of moduli spaces of curves in significnat new regions of moduli. He proposed the study of collections of subrings of $H^*(\overline{\mathcal{M}}_{g,n})$ that are closed under the tautological operations that map cohomology classes on moduli spaces of smaller dimension to those on moduli spaces of larger dimension and contain the tautological subrings. These definitions are well-suited for inductive arguments a number of applications. Among a long list of advances, Payne described confirmation of predictions of Chenevier and Lannes for the ℓ-adic Galois representations and Hodge structures that appear in $H^k(\overline{\mathcal{M}}_{g,n})$ for $k = 13$, 14, and 15. He also explained why $H^4(\overline{\mathcal{M}}_{g,n})$ is generated by tautological classes for all g and n, confirming a prediction of Arbarello and Cornalba from the 1990s.

Introductory talks from (and conversations with) younger (or not) participants. On Tuesday evening, volunteering participants had the opportunity to present snapshots of their research in the form of five minute, one blackboard talks. The list of speakers was established by Sara Torelli, and the session was moderated by David Eisenbud. The presentations, listed below, covered a similarly wide range of topics. This speakers included those younger participants in the workshop who did not have the opportunity to give a one-hour talk. As with previous years' young participants, we expect these researchers to quickly establish themselves as leaders in their areas. Here is a list of the presentations that were given. The first half featured:

Carl Lian: *Counting curves in* $\mathbb{P}^r$
Samir Canning: *Pixton's conjecture and the Gorenstein question*
Isabel Stenger: *Movable cones of Calabi–Yau pairs*
Eric Larson: *The Maximal Rank Theorem*
Andrés Rojas: *Brill–Noether theory for k-gonal curves on K3 surfaces*

After a short break, we resumed for the second half with:

Sara Torelli: *Constant cycle curves on K3 surfaces*
Cécile Gachet: *Varieties with positive exterior powers of the tangent bundle*
Nick Addington: *Birational invariants from higher K-theory?*
Daniele Agostini: *Ulrich bundles and arithmetic degree*
Ekaterina Amerik: *Algebraically coisotropic submanifolds*

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Workshop: Recent Trends in Algebraic Geometry

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Abstracts

Maps to a moving elliptic curve

Rahul Pandharipande
(joint work with Samir Canning, Dragos Oprea, Hsian-Hua Tseng)

Moduli spaces of genus g Hurwitz covers of $\mathbb{CP}^1$ yield cycles on the moduli space of genus g curves. The study of these classes is a very well-developed subject. Much less studied is the moduli space of Hurwitz covers of elliptic curves. I presented results and open questions related to ramified covers of a moving elliptic curve (connected mainly to tautological classes on moduli spaces (of curves, Abelian varieties, and K3 surfaces), but also to quantum cohomology, Hilbert schemes, and Hodge integrals).

More specifically, there are 8 variations of the moduli problem which I wish to consider given by the following 3 binary choices:

(i) The target elliptic curve may be fixed or moving.
(ii) The moduli space may be of stable maps or of admissible covers.
(iii) The class in the moduli space of domain curves may be taken in cohomology or Chow.

1. Modularity for a stable maps to a fixed elliptic curve E in cohomology.

The central result here is due to Oberdieck-Pixton: the quasi-modularity of the cohomological cycle,

$$\sum_{d=0}^{\infty} q^d [\overline{M}_{g,n}(E,d)]^{vir} \ \in \ H^*(\overline{M}_{g,n}) \otimes \mathsf{QMod}\,.$$

In the above result, the virtual class defines (via push-forward) a cohomology class on the moduli space of stable curves. The proof uses the magical polynomiality of the double ramification cycle (a property proven from Pixton's formula). The algebra of quasi-modular forms, QMod, is generated by the Eisenstein series $E_2(q), E_4(q), E_6(q)$.

A fundamental open question concerns the parallel claim for a moving elliptic target. Let

$$\pi : \mathcal{E} \to \overline{M}_{1,1}$$

be the universal family of stable curves of genus 1 with 1 marked point.

Question 1: Does quasi-modularity hold in the moving case:

$$\sum_{d=0}^{\infty} q^d [\overline{M}_{g,n}(\pi,d)]^{vir} \ \in \ H^*(\overline{M}_{g,n}) \otimes \mathsf{QMod} \ ?$$

Calculations by Carl Lian in genus 2 and 3 appear to suggest that an affirmative answer is not impossible.

2. Quantum cohomology of the Hilbert scheme of points of $\mathbb{C}^2$ and Hodge integrals.

The GW/DT and Crepant Resolution correspondences connect a web of mathematical theories (by the work of many authors). The genus 0 quantum cohomology of $\mathsf{Hilb}(\mathsf{C}^2)$ was calculated by Okounkov and myself almost 20 years ago. The genus 1 quantum cohomology of $\mathsf{Hilb}(\mathsf{C}^2)$ precisely yields results about Hodge integrals over the moduli spaces of maps to a moving elliptic curve. A simple nontrivial example (from more recent joint work with H.-H.Tseng) is

$$\sum_{n=1}^{\infty} \frac{u^{2n-1}}{(2n-1)!} \int_{\mathsf{Adm}_1^{n+1}((2)^{2n})_{d=2}} \lambda_{n+1}\lambda_{n-1} \;=\; \frac{i}{24} \cdot \frac{1-e^{iu}}{1+e^{iu}}\,.$$

Here, $\mathsf{Adm}_1^{n+1}((2)^{2n})_{d=2}$ is the moduli of admissible covers of degree 2 of a genus 1 curve with $2n$ simple branch points (so the domain has genus $n+1$).

Question 2: Can a closed form be found for the parallel integral over $\mathsf{Adm}_1^{n+1}((2)^{2n})_d$ for higher d? The answer is known to be a rational function in $q = -e^{iu}$, but which rational function?

3. The cycle theory of the Torelli map.

Consider the Torelli map from the moduli space of curves of compact type to the moduli space of principally polarized abelian varieties:

$$\mathsf{Tor} : M_g^{ct} \to A_g\,.$$

The Jacobians of curves which admit a d-cover of an elliptic curve define a Noether-Lefschetz locus

$$\mathsf{NL}_d \subset A_g$$

of codimension $g-1$. In joint work with Canning and Oprea in 2023, we studied the class

$$\Delta_g = \mathsf{Tor}^*[\mathsf{NL}_1] - \frac{(-1)^{g+1}g}{6B_{2g}}\lambda_{g-1}\,.$$

For all g, Δ_g lies in the Gorenstein kernel of the tautological ring $\mathsf{R}^\star(M_g^{ct})$. The study is related to the recent verification of Pixton's conjecture for M_6^{ct} by Canning-Larson-Schmitt, which shows $\Delta_6 \neq 0$. To control the higher d cases, $\mathsf{Tor}^*[\mathsf{NL}_d]$, the answers to Questions 1 and 2 are required.

Question 3: Can we explain all of the Gorenstein kernel of $\mathsf{R}^\star(M_g^{ct})$ using the cycles $\mathsf{Tor}^*[\mathsf{NL}_d]$?

While Questions 1 and 2 are likely solvable in some way using ideas not so far ahead of where we are in the field, Question 3 would require a larger leap.

The moduli spaces of bi-elliptic curves determine non-tautological cycles on the moduli spaces of curves and Abelian varieties. I ran out of time, but I also wanted to propose a construction of non-tautological classes on the projective bundle of linear sections over the moduli of quasi-polarized $K3$ surfaces: the locus of sections which yield bi-elliptic curves. However, I do not know how to prove these cycles are non-tautological for any moduli space of $K3$ surfaces.

Cycle conjectures and birational invariants over finite fields

Stefan Schreieder
(joint work with Samet Balkan)

Throughout this extended abstract $\mathbb{F}$ denotes a finite field, $\bar{\mathbb{F}}$ denotes an algebraic closure of $\mathbb{F}$, $G_{\mathbb{F}}$ denotes the absolute Galois group of $\mathbb{F}$ and ℓ denotes a prime invertible in $\mathbb{F}$. For a $\mathbb{F}$-scheme X, we denote by $\bar{X}$ the base change of X to $\bar{\mathbb{F}}$.

1. Cycle conjectures

Let X be a smooth projective variety over a finite field $\mathbb{F}$. The cycle class map

$$\mathrm{cl}^i_X : \mathrm{CH}^i(X)_{\mathbb{Q}_\ell} \to H^{2i}(\bar{X}, \mathbb{Q}_\ell(i)) \tag{1}$$

sends a codimension-i cycle with $\mathbb{Q}_\ell$-coefficients to a degree-$2i$ class in the ℓ-adic étale cohomology group with suitable Tate twist. The image of this cycle class map lands in the subspace of $G_{\mathbb{F}}$-invariant classes $H^{2i}(\bar{X}, \mathbb{Q}_\ell(i))^{G_{\mathbb{F}}} \subset H^{2i}(\bar{X}, \mathbb{Q}_\ell(i))$. In what follows we recall some of the most basic conjectures concerning the cycle class map over finite fields, see e.g. [Jan90, §12].

Conjecture 1 (Tate Conjecture). *We have* $\mathrm{im}(\mathrm{cl}^i_X) = H^{2i}(\bar{X}, \mathbb{Q}_\ell(i))^{G_{\mathbb{F}}}$.

The Tate conjecture is known for divisors on abelian varieties [Ta66], as well as for some varieties whose cohomology can be related to that of abelian varieties, such as K3 surfaces, by Maulik, Charles, and Madapusi-Pera, see [Mau14, Cha13, MP15]. In general the Tate conjecture is however open even for divisors on surfaces.

Conjecture 2 (Beilinson Conjecture). *The cycle class map* cl^i_X *in (1) is injective.*

Together, Conjecture 1 and 2 completely describe the rational Chow groups of smooth projective varieties over finite fields. The conjecture is known for divisors, because $\mathrm{Pic}^0(X)$ is a finite group, hence torsion, but it is wide open in codimension ≥ 2 in general.

For technical reasons, the Tate conjecture is often considered in conjunction with a semi-simplicity conjecture for the action of $G_{\mathbb{F}}$ on $H^{2i}(\bar{X}, \mathbb{Q}_\ell(i))$. One version of this is as follows:

Conjecture 3 (1-semi-simplicity Conjecture). *The* $G_{\mathbb{F}}$*-action on* $H^{2i}(\bar{X}, \mathbb{Q}_\ell(i))$ *is* 1*-semi-simple, i.e. the Jordan Block for eigenvalue* 1 *of the action of Frobenius is diagonalizable.*

2. Birational invariants

Let X be a variety over a field k and let A be an étale (or pro-étale) sheaf on $\operatorname{Spec} k$. Following the notation in [Sch23], we define

$$H^i(F_0X, A) := \varinjlim_{U\subset X} H^i_{\text{proét}}(U, A),$$

where $U \subset X$ runs through all dense open subsets of X. This can be thought of as the cohomology of the generic point of X. If A is torsion, then the above group coincides with the Galois cohomology group of k with coefficients in A.

The above group is obviously a birational invariant of X, but it is in practice to large to be of any direct use. For applications to rationality questions, it turned out to be very useful to consider the subgroup

$$H^i_{nr}(k(X)/k, A) \subset H^i(F_0X, A)$$

of unramified classes, see [CTO89] or the survey [Sch21]. An important property of unramnified cohomology is its triviality on projective space over algebraically closed fields:

$$H^i_{nr}(\bar{k}(\mathbb{P}^n)/\bar{k}, A) = 0 \quad \text{for all } i \geq 1.$$

In this talk we address the question of considering another subgroup of the above group, namely the subgroup

$$H^i(F_0\bar{X}, A)^{G_k} \subset H^i(F_0\bar{X}, A)$$

of $H^i(F_0\bar{X}, A)$ that are invariant under the action of the absolute Galois group G_k. It is easy to check that this is a k-birational invariant of X.

While applications to rationality problems tend to use integral or torsion coefficients, we want to restrict for simplicity to $\mathbb{Q}_\ell$-coefficients. For any k-variety X, there is a natural map

$$K^M_i(k(X)) \longrightarrow H^i(F_0\bar{X}, \mathbb{Q}_\ell(i))^{G_k}$$

whose image is nonzero for all $0 \leq i \leq \dim X$. On the other hand, by Deligne's theory of weights [Del80],

$$H^i(F_0\bar{X}, \mathbb{Q}_\ell(j))^{G_k} = 0 \quad \text{for all } i - 2j > 0.$$

Hence it is interesting to consider $H^i(F_0\bar{X}, \mathbb{Q}_\ell(j))^{G_k}$ for $\frac{1}{2}i \leq j < i$ and we may ask whether these groups vanish, say for $X = \mathbb{P}^n$. We can prove this vanishing for $X = \mathbb{P}^n$ whenever i is even and $j = i/2$. We further make the following conjectures.

Conjecture 4 (Vanishing conjecture for projective space in even degree)**.** *For any finite field $\mathbb{F}$ of characteristic p and any prime ℓ invertible in $\mathbb{F}$,*

$$H^{2i}(F_0\mathbb{P}^n_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell(i+1))^{G_{\mathbb{F}}} = 0 \quad \textit{for all } i, n \geq 2.$$

Conjecture 5 (Vanishing conjecture for projective space in odd degree)**.** *For any finite field $\mathbb{F}$ of characteristic p and any prime ℓ invertible in $\mathbb{F}$,*

$$H^{2i-1}(F_0\mathbb{P}^n_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell(i))^{G_{\mathbb{F}}} = 0 \quad \textit{for all } i, n \geq 2.$$

Exterior product with a nontrivial Galois-invariant element in $H^1(F_0\mathbb{P}^n_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell(1))$ easily shows that Conjecture 4 implies Conjecture 5.

3. Results

Our main results are as follows.

Theorem 1. *Let p be a prime. Then the following are equivalent:*

(1) *Conjectures 1, 2, and 3 for all smooth projective varieties over finite fields of characteristic p.*
(2) *Conjecture 4.*

As aforementioned, Conjecture 5 is weaker than Conjecture 4. This is reflected by the following precise statement.

Theorem 2. *Let p be a prime. Then the following are equivalent:*

(1) *Conjectures 1 and 3 for all smooth projective varieties over finite fields of characteristic p.*
(2) *Conjecture 5.*

The proof of the above result will show more precise results, see [BS23]. For instance, we will see that the vanishing of $H^4(F_0\mathbb{P}^4_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell(3))^{G_{\mathbb{F}}}$ for all finite fields $\mathbb{F}$ of characteristic p implies the Tate conjecture and 1-semi-simplicity conjecture for surfaces, as well as the Beilinson conjecture for threefolds over any finite field of characteristic p. This corresponds to the first open cases of these conjectures, because the Beilinson conjecture is known for surfaces.

As a consequence of our proof of Theorem 1, we obtain the following result of independent interest.

Theorem 3. *Let p be a prime and fix an integer n. If for all smooth projective varieties X of dimension $d \leq n$, the Tate, Beilinson, and 1-semi-simplicity conjectures hold for cycles of codimension $i \leq \lceil d/2 \rceil$, then they hold for cycles of any codimension on smooth projective varieties of dimension at most d.*

Roughly speaking, the above theorem says that one has to prove only half of the cases of conjectures Conjecture 1, 2, and 3. For the Tate and 1-semi-simplicity conjecture, this is a direct consequence of the hard Lefschetz theorem, proven by Deligne [Del80]. The new content of the above theorem is the respective assertion for Beilinson's conjecture.

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On Chern classes of Lagrangian fibered hyper-Kähler manifolds

Claire Voisin

A hyper-Kähler manifold X is a compact Kähler manifold which is simply connected and has a unique (up to coefficient) holomorphic 2-form, which is nowhere degenerate. Thus X is symplectic holomorphic, hence of even complex dimension, and has trivial canonical bundle. The study of the deformations of the complex structure of X shows that there are countably many families of projective hyper-Kähler manifolds. From now on, X will be projective (over the complex numbers).

Hyper-Kähler manifolds of dimension 2 are the $K3$ surfaces. In 2002, Beauville and myself proved

Theorem 1. *Let X be a projective $K3$ surface. Then there exists a canonical 0-cycle $o_X \in \mathrm{CH}_0(X)$ such that*
(i) For any $D \in \mathrm{CH}^1(X)$, $D^2 = (\deg D^2)\, o_X$ in $\mathrm{CH}_0(X)$.
(ii) $c_2(X) = 24\, o_X$ in $\mathrm{CH}_0(X)$.

The analogous statement for higher dimensional hyper-Kähler manifolds is open

Conjecture 1. *(Beauville-Voisin) Let X be a projective hyper-Kähler manifold. Let $z \in \mathrm{CH}^k(X)$ be a codimension k cycle on X which can be written as a polynomial in divisor classes $D_i \in \mathrm{CH}^1(X)$ and Chern classes $c_{2j}(X) \in \mathrm{CH}^{2j}(X)$. Then if z is cohomologous to 0, it is rationally equivalent to 0 modulo torsion.*

We observe here that the odd Chern classes of X vanish in $\mathrm{CH}(X)$ since the cotangent bundle of X is isomorphic to its dual. This conjecture is known for the Fano varieties of lines of cubic fourfolds by [9]. It was proved by Lie Fu for generalized Kummer varieties in [2] and by Maulik and Negut for punctual Hilbert schemes of $K3$ surfaces in [7].

This talk is devoted to the study of Conjecture 1 for Lagrangian fibered projective hyper-Kähler manifolds $f : X \to B$. More precisely, by work of Verbitsky [1] and Huybrechts [3], one has the following relations in the cohomology algebra of X, for any isotropic class $\alpha \in H^2(X, \mathbb{Q})$, and for any $i \geq k$

$$c_I^{\mathrm{top}}(X)\alpha^{n+1-k} = 0 \text{ in } H^*(X, \mathbb{Q}), \tag{1}$$

where $c_I^{\mathrm{top}}(X) \in H^{4i}(X)$ is any polynomial of weighted degree $4i$ in the topological Chern classes $c_{2j}^{\mathrm{top}}(X)$ (of weighted degree $4j$).

The notion of isotropy for a degree 2 cohomology class α on X refers to the Beauville-Bogomolov quadratic form and it is equivalent to the fact that $\int_X \alpha^{2n} = 0$ by the Beauville-Fujiki relations. Suppose now that X has a Lagrangian fibration $f : X \to B$, where the basis can be assumed to be normal and is known to be projective of dimension n of Picard rank n by work of Matsushita [5], [6]. Let $L \in \operatorname{Pic} X \cong \mathrm{CH}^1(X)$ be the pull-back to X of a generator of $\operatorname{Pic} B$. Then $l = c_1(L)$ is isotropic since $\int_X l^{2n} = 0$. Hence Conjecture 1 predicts

Conjecture 2. *Let X, L be as above, with X projective hyper-Kähler of dimension n. Then for $i \geq k$,*

$$c_I(X)L^{n+1-k} = 0 \text{ in } \mathrm{CH}(X)_{\mathbb{Q}}), \tag{2}$$

where $c_I(X) \in \mathrm{CH}(X)$ is any weighted polynomial of weighted degree $2i$ in the $c_{2j}(X) \in \mathrm{CH}^{2j}(X)$ (of degree $2j$).

We prove in [10] the following

Theorem 2. *Let X, L be as above, then for $i \geq k$*

$$c_{2i}(X)L^{n+1-k} = 0 \text{ in } \mathrm{CH}(X)_{\mathbb{Q}}. \tag{3}$$

Furthermore, if (X, L) is a LSV manifold constructed in [4] as a compactification of the intermediate Jacobian fibration associated to a cubic fourfold, with its canonical Jacobian fibration, one has

$$c_2^2(X)L^{n-1} = 0 \text{ in } \mathrm{CH}(X)_{\mathbb{Q}},$$

(where $n = 5$ in this case).

The relation (3) is formal and proved by induction on i. For $k = 1$, the result follows from the fact that the general fiber of f, whose class is proportional to L^n in $\mathrm{CH}(X)$, is an abelian variety, which has trivial tangent bundle and normal bundle in X. (3) does not prove all the relations expected from Conjecture 2 because it involves only the Chern classes themselves, not the higher degree monomials in the Chern classes. The second statement to the contrary involves the monomial c_2^2, whose study is much more subtle. The proof in that case is not formal at all and it uses as a geometric ingredient the fact that the discriminant hypersurface for the LSV Lagrangian fibration is the dual hypersurface in $(\mathbb{P}^5)^*$ of the cubic fourfold $Y \subset \mathbb{P}^5$. In particular, it is rationally connected and has $\mathrm{CH}_0 = \mathbb{Z}$. This property does not hold in general.

In the course of the proof, we establish some results on the rank loci $X_{\leq k} \subset X$, resp. $X_k \subset X$, where a Lagrangian fibration $f : X \to B$ has rank $\leq k$, resp. equal to k, and relate them to the classes $c_{2j}(X)$. The following results are established in [10].

Theorem 3.

(i) One has $\dim B_k \leq k$, where $B_k := f(X_k)$.

(ii) One has $\dim X_{k/B_k} \geq k$.

(iii) Assume equalities hold in (i) and (ii). Then, the general fiber of X_k over B_k is a finite disjoint union of k-dimensional complex tori. Furthermore, the normal bundle $N_{X_k/X}$ restricted to each of these tori is a homogeneous vector bundle.

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Non-abelian big monodromy

Daniel Litt
(joint work with Joshua Lam, Aaron Landesman)

Let $\Sigma_{g,n}$ be a smooth orientable surface of genus g, with n punctures, and let $\mathrm{Mod}_{g,n}$ be its mapping class group, i.e.

$$\mathrm{Mod}_{g,n} = \pi_0(\mathrm{Homeo}^+(\Sigma_{g,n})),$$

the component group of the space of orientation-preserving homeomorphisms of $\Sigma_{g,n}$. The natural action of $\mathrm{Homeo}^+(\Sigma_{g,n})$ on $\Sigma_{g,n}$ induces an outer action of $\mathrm{Mod}_{g,n}$ on $\pi_1(\Sigma_{g,n})$, and hence an action on the character varieties

$$X_{g,n}(r) = \mathrm{Hom}(\pi_1(\Sigma_{g,n}), GL_r(\mathbb{C})) /\!\!/ GL_r(\mathbb{C}).$$

The goal of this work is to study the finite orbits of this action; the main theorem is:

Theorem 1 (Landesman-L–, [LL22a]). *Let*

$$\rho : \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$$

be a representation such that $[\rho] \in X_{g,n}(r)$ *has finite orbit under* $\mathrm{Mod}_{g,n}$. *If* $g > r^2 - 1$, *then* ρ *has finite image.*

This resolves conjectures of Kisin and Whang [BKMS18, Sin10, LL19] positively, and conjectures of Esnault-Kerz [EK21] and Budur-Wang [BW17] negatively.

This work is part of a broader program which aims

(1) to classify finite orbits of the $\mathrm{Mod}_{g,n}$-action on $X_{g,n}(r)$ in general, and, as part of this,
(2) to classify local systems of geometric origin on generic n-punctured curves of genus g.

Point (2) above naturally arises as part of (1), as local systems of geometric origin on a generic n-punctured curves of genus g necessarily have finite orbit under $\mathrm{Mod}_{g,n}$. This program began in [LL22b] and continued in [LL23a], with additional applications to some questions in geometric topology obtained in [LL23b].

The analysis of the $\mathrm{Mod}_{g,n}$-action on $X_{g,n}(r)$ has a long history—for example, in the case $(g, n, r) = (0, 4, 2)$, classifying finite orbits is the same as classifying algebraic solutions to the Painlevé VI equation; this classification was completed by Lisovyy and Tykhyy via a computer-aided proof in 2014 [LT14].

Fix conjugacy classes $C_1, \cdots, C_n \subset GL_r(\mathbb{C})$, and let $X_{g,n}(r)(C_1, \cdots, C_n)$ be the subvariety of $X_{g,n}(r)$ consisting of representations whose local monodromy about the i-th puncture lies in C_i, for all i. In the forthcoming work [LLL23], we will completely classify finite orbits of $\mathrm{Mod}_{0,n}$ on $X_{0,n}(2)(C_1, \cdots, C_n)$, as long as some C_i has infinite order. The classification is in two parts: the finite orbits that lie in positive-dimensional families, which by results of Corlette-Simpson [CS08] are of "pullback type," and can easily be classified, and the rigid finite orbits, which are of geometric origin. The main result of the classification is:

Theorem 2 (Lam-Landesman-L– [LLL23])**.** *Suppose some C_i as above has infinite order. Let*

$$\rho : \pi_1(\Sigma_{0,n}) \to \mathrm{SL}_2(\mathbb{C})$$

be a representation with Zariski-dense image such that

$$[\rho] \in X_{0,n}(2)(C_1, \cdots, C_n)$$

has finite orbit under $\mathrm{Mod}_{0,n}$*. Then ρ arises via Katz's middle convolution operation from a finite complex reflection group.*

We now sketch the proof of Theorem 1, which is loosely inspired by Katz's proof of the p-curvature conjecture for the Gauss-Manin connection [Kat72].

Step 1. The unitary case. Every unitary representation is a direct sum of irreducible unitary representations. Therefore, we suppose ρ is unitary, irreducible, and has finite orbit under $\mathrm{Mod}_{g,n}$, with $\mathrm{rk}\,\rho < \sqrt{g+1}$. We construct from ρ a finite étale cover $\mathcal{M}$ of $\mathcal{M}_{g,n}$, with associated family of punctured curves $\pi^\circ : \mathcal{C}^\circ \to \mathcal{M}$, and a projective unitary local system $\mathbb{V}$ on $\mathcal{C}^\circ$ whose restriction to a fiber C° of π° has monodromy given by ρ. An analysis of the period map associated to $R^1\pi^\circ_* \operatorname{ad}(\mathbb{V})$ shows that $\mathbb{V}$ is cohomologically rigid. The main result of [KP20] then gives that ρ is defined over the ring of integers $\mathcal{O}_K$ of some number field K. Moreover, by replacing $\mathcal{M}$ by a dominant étale scheme over it, along which certain cohomological lifting obstructions vanish, we can assume $\mathbb{V}$ lifts from a projective local system to a bona fide local system.

By compactness of the unitary group and discreteness of $\mathcal{O}_K$, it suffices to show that for each embedding $\iota : \mathcal{O}_K \hookrightarrow \mathbb{C}$, $\rho \otimes_{\mathcal{O}_K,\iota} \mathbb{C}$ is unitary. (We know this unitarity for one such ι by assumption, but not the others.) The rigidity of $\mathbb{V}$ implies by non-abelian Hodge theory that these $\rho \otimes_{\mathcal{O}_K,\iota} \mathbb{C}$ underlie complex polarizable variations of Hodge structure for any complex structure on $\Sigma_{g,n}$. Hence, given their low rank, these local systems are unitary by [LL22b, Theorem 1.2.12].

Step 2. The semisimple case. Now suppose ρ is an arbitrary semisimple representation, with finite orbit under $\mathrm{Mod}_{g,n}$ and with $\mathrm{rk}\,\rho < \sqrt{g+1}$. Again, we associate to ρ a local system $\mathbb{V}$ on a family of curves $\pi^\circ : \mathcal{C}^\circ \to \mathcal{M}$, so that $\mathcal{M}$ has a dominant étale map to $\mathcal{M}_{g,n}$ and whose fibral monodromy is given by ρ. By non-abelian Hodge theory, we may deform $\mathbb{V}$ to a local system $\mathbb{V}_0$ underlying a complex polarizable variation of Hodge structure. By [LL22b, Theorem 1.2.12], $\mathbb{V}_0$ has unitary monodromy when restricted to a fiber of π°. By the unitary case, $\mathbb{V}_0$ thus has finite monodromy when restricted to a fiber of π°. Note that even if $n = 0$, i.e. π° is proper, we here need to use non-abelian Hodge theory for non-proper varieties, as the total space $\mathcal{C}^\circ$ will not be proper.

It remains to argue that the restriction $\mathbb{V}_0|_{C^\circ}$ of $\mathbb{V}_0$ to a fiber C° of π° agrees with the restriction $\mathbb{V}|_{C^\circ}$, corresponding to ρ. Recall that $\mathbb{V}_0$ and $\mathbb{V}$ were only deformation equivalent, so it may be surprising that they necessarily restrict to the same local system on fibers. We verify this agreement via another analysis of the period map associated to $R^1\pi^\circ_* \operatorname{ad}(\mathbb{V}_0)$, which provides sufficient rigidity to conclude that $\mathbb{V}_0$ has the same fibral monodromy as $\mathbb{V}$.

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On the field of meromorphic functions on a Stein compact set

Olivier Benoist

Let F be a field with algebraic closure $\overline{F}$ and let $\Gamma_F := \mathrm{Aut}(\overline{F}/F)$ be its absolute Galois group. The *cohomological dimension* $\mathrm{cd}(F)$ of F is defined to be the smallest integer such that $H^q(F, A) := H^q(\Gamma_F, A) = 0$ for all $q > \mathrm{cd}(F)$ and all finite Γ_F-modules A. It is a measure of the arithmetic complexity of the field F.

When $F = \mathbb{C}(X)$ is the function field of an integral complex algebraic variety X of dimension n, one has $\mathrm{cd}(F) = n$ as a consequence of Tsen's theorem (see [Ser94, II.4, Proposition 11]). The main goal of this talk is an analogue of this result in complex analytic geometry, more precisely in Stein geometry.

Stein spaces are the analogues of affine algebraic varieties in complex analytic geometry. They are characterized among all (possibly singular) complex spaces by the vanishing of the higher cohomology groups of all coherent sheaves. If S is a connected normal Stein space, its ring of holomorphic functions $\mathcal{O}(S)$ is an integral domain. One can therefore consider the field $\mathcal{M}(S) := \mathrm{Frac}(\mathcal{O}(S))$ of meromorphic functions on S and investigate its cohomological dimension.

Question 1. *Let S be a connected normal Stein space of dimension n. Is it true that* $\mathrm{cd}(\mathcal{M}(S)) = n$*?*

Question 1 is known to have a positive answer in dimension 1, thanks to an argument attributed to Artin by Gurlanick [Gur88]. It is open in dimension $n \geq 2$, even for $S = \mathbb{C}^n$. We will answer positively a variant of this question, for germs of meromorphic functions along Stein compact subsets.

Recall that a compact subset K in a Stein space S is said to be *Stein* if it admits a basis of Stein open neighborhoods. In this situation, we let $\mathcal{O}(K)$ denote the ring of germs of holomorphic functions in a neighborhood of K (which depends on the germ of S along K). When S is normal and K is connected, the ring $\mathcal{O}(K)$ is an integral domain, and we let $\mathcal{M}(K) := \mathrm{Frac}(\mathcal{O}(K))$ denote the field of germs of meromorphic functions in a neighborhood of K. The following theorem is proved in [Ben, Theorem 0.4].

Theorem 2. *Let S be a connected normal Stein space of dimension n and let $K \subset S$ be a connected Stein compact subset. Then* $\mathrm{cd}(\mathcal{M}(K)) = n$.

Theorem 2 is new in dimension $n \geq 2$, already when $S = \mathbb{C}^n$ and $K \subset S$ is the closed unit ball. It should be useful to solve problems controlled by Galois cohomology (related to quadratic forms, principal homogeneous spaces...) arising in analytic contexts. Here is an example of such an application, to a quantitative result on Hilbert's 17th problem in the real-analytic setting (see [Ben, Theorem 0.1]).

Theorem 3. *Let M be a compact real-analytic manifold of dimension n. Then any nonnegative real-analytic function $f : M \to \mathbb{R}$ is a sum of 2^n squares of real-analytic meromorphic functions on M.*

The qualitative variant of Theorem 3 that disregards the number of squares is due to Jaworski [Jaw86, Theorem 1]. What is new in Theorem 3 is the quantitative bound on the number squares, in the spirit of Pfister's quantitative solution to the original Hilbert's 17th problem [Pfi67, Theorem 1].

Here is how Theorem 3 is deduced from Theorem 2. The real-analytic manifold M may be realized as the fixed locus of an antiholomorphic involution on a complex manifold S. By Grauert's solution to the Levi problem [Gra58], one may assume that S is a Stein manifold and that $M \subset S$ is a Stein compact subset. Over any field of characteristic $\neq 2$, Voevodsky's proof of the Milnor conjectures [Voe03] provides a strong link between quadratic forms and Galois cohomology. Applied to sums of squares quadratic forms, these results show that the validity of Theorem 3 is controlled by the vanishing of Galois cohomology classes in $H^q(\mathcal{M}(M), \mathbb{Z}/2)$ for $q \geq n+1$. But these classes do vanish by Theorem 2.

Theorem 2 is an algebraic statement about a field of analytic origin. Its proof is correspondingly a mixture of algebraic and analytic tools. The basic idea is to exploit the weak Lefschetz theorem of Andreotti and Frankel [AF59]: a Stein manifold of dimension n has the homotopy type of a CW-complex of dimension $\leq n$, and hence its singular cohomology groups vanish in degree $> n$.

To deduce Theorem 2, which is a vanishing theorem for Galois cohomology, from this vanishing result for singular cohomology, one must bridge the gap between singular and Galois cohomology. The standard tool to do so is étale cohomology. On the one hand, Galois cohomology is a particular case of étale cohomology. On the other hand, Artin's comparison theorem shows that the étale cohomology of a complex algebraic variety computes the singular cohomology of its analytification. The only missing piece for the proof is the following extension of Artin's comparison theorem in Stein geometry (more precisely, in relative algebraic geometry over a base Stein compactum K).

Theorem 4. *Let S be a Stein manifold. Let X be an $\mathcal{O}(S)$-scheme of finite type and let $\mathbb{L}$ be a constructible torsion étale abelian sheaf on X. If one lets U run over all Stein open neighborhoods of a Stein compact subset K of S, the change of topology morphisms*

$$\operatorname*{colim}_{K\subset U} H^k_{\acute{e}t}(X_{\mathcal{O}(U)}, \mathbb{L}_{\mathcal{O}(U)}) \to \operatorname*{colim}_{K\subset U} H^k((X_{\mathcal{O}(U)})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}})$$

are isomorphisms for $k \geq 0$.

Appropriate dévissages allow us to reduce to the case where $X = \mathrm{Spec}(\mathcal{O}(S))$ and $\mathbb{L} = \mathbb{Z}/m$. One then has to show that the morphisms

$$\operatorname*{colim}_{K\subset U} H^k_{\text{ét}}(\mathrm{Spec}(\mathcal{O}(U)), \mathbb{Z}/m) \to \operatorname*{colim}_{K\subset U} H^k(U, \mathbb{Z}/m)$$

are isomorphisms for $k \geq 0$. To compare the étale topology on $\mathrm{Spec}(\mathcal{O}(U))$ and the classical topology on U, we make use of the Leray spectral sequences associated to the morphisms of sites $\varepsilon_U : (X_{\mathcal{O}(U)})^{\mathrm{an}} \to (X_{\mathcal{O}(U)})_{\text{ét}}$. We need to show that the morphism $\mathbb{Z}/m \to \varepsilon_{U,*}\mathbb{Z}$ is an isomorphism and that $\mathrm{R}^q\varepsilon_{U,*}\mathbb{Z}/m = 0$ for $q \geq 1$, at least after taking the colimit over all possible U. The first assertion follows from Bingener's relative GAGA theorem over a Stein compactum [Bin76, Theorem 4.2]. The second assertion requires to show that singular cohomology classes on U (or more generally on analytifications of étale $\mathcal{O}(U)$-schemes) are étale-locally trivial. This is the heart of the proof. Our strategy to attack it is to use Grauert's bump method for exhausting U by Stein compacta with controlled topological and analytical properties.

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Algebraic structures on moduli of curves from vertex operator algebras

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(joint work with Chiara Damiolini, Daniel Krashen)

The moduli stack $\overline{\mathcal{M}}_{g,n}$, parametrizing families of Deligne-Mumford stable n-pointed curves of genus g, has proved crucial for the study of smooth curves and their degenerations. Vertex operator algebras (VOAs) have played a similar role in the study of conformal field theories, finite group theory, the construction of knot invariants, and 3-manifold invariants. By fixing coordinates at marked points, representations of VOAs may be used to study $\overline{\mathcal{M}}_{g,n}$. Namely, by associating to every stable n-pointed and coordinitized curve, a collection of V-modules, one can construct a vector space of coinvariants, the largest quotient of the tensor product of the modules on which a natural Lie algebra acts trivially. Varying these data gives sheaves on $\widehat{\mathcal{M}}_{g,n}$, the space parametrizing families of stable n-pointed coordinatized curves. Changing coordinates gives $\widehat{\mathcal{M}}_{g,n}$ the structure of a torsor over $\overline{\mathcal{M}}_{g,n}$. Assumptions on modules ensure sheaves descend to $\overline{\mathcal{M}}_{g,n}$.

After work in mathematical physics, the first examples of sheaves of coinvariants studied in algebraic geometry came from certain affine VOAs, derived from a simple Lie algebra $\mathfrak{g}$, and a positive integer ℓ. By [TK87, TUY89] these give vector bundles that support a projectively flat connection, and satisfy factorization, allowing ranks to be computed recursively. Analogous results were shown to hold for sheaves from (discrete series) Virasoro VOAs [BFM91], more general VOAs on smooth pointed coordinitized curves [FBZ04], and for VOAs with finite and semi-simplicity properties on rational curves [NT05]. For examples given by affine VOAs more was shown: Chern characters form cohomological field theories and Chern classes lie in the tautological ring [MOP15, MOP$^+$17]. If $g = 0$, they are globally generated [Fak12]. There are canonical isomorphisms of (dual) fibers with generalized theta functions [Ber93, Tha94, BL94, Fal94, Pau96, LS97, BG19, BF19].

Much of this theory has recently been extended: Sheaves defined by VOAs of CFT-type are quasi-coherent, carrying a projectively flat connection with logarithmic singularities on the boundary of $\widehat{\mathcal{M}}_{g,n}$ [DGT21]. If V is C_2-cofinite and rational, these satisfy factorization, and are vector bundles [DGT22a]. If also simple and self-dual, Chern characters form cohomological field theories, and Chern classes are tautological [DGT22b]. If V is generated in degree 1, and $g = 0$, are globally generated [DG23]. Many examples of C_2-cofinite and rational VOAs giving such vector bundles are known [DGT22a, §9], [DGT22b, §5], [DG23, §7-9].

Sheaves of coinvariants are known to be coherent in various cases [DGK22], [DG23]. As vector bundles are critical to the study of moduli, it is natural to ask:

Question. *When are coherent sheaves of coinvariants locally free?*

To answer the question, in [DGK23] we define a series of associative algebras $\mathfrak{A}_d(V)$ for $d \in \mathbb{Z}_{\geq 0}$. To describe these and our main geometric application, [DGK23, Corollary 5.2.6] (stated below) we start with a small amount of notation.

Background: VOAs, and V-modules, and sheaves of coinvariants. The input used to define a VOA includes a vector space, a vertex operator, together with the vacuum and conformal vectors. These satisfy a long list of axioms (see eg. [DGK23, §1]). We consider VOAs satisfying finiteness assumptions C_1-cofiniteness, and the stronger C_2-cofiniteness, as well as rationality. By [Liu22], a VOA is C_1 if and only if it is (strongly) finitely generated. A VOA is said to be rational if every finitely generated module is a finite sum of simple modules.

There is a one-to one correspondence between simple V-modules and simple modules over $A(V)$, an associative algebra constructed as a quotient of V. Important properties of V are reflected in $A(V)$. For instance, if V is C_1-cofinite, then $A(V)$ is finitely generated, If V is C_2-cofinite, then $A(V)$ is finite dimensional, and if V is rational, then $A(V)$ is finite and semi-simple (for references, see [DGK23]).

As in [DGK23, §2], one may construct (left and right) Verma modules for V via a normed, associative algebra $\mathcal{U} = \mathcal{U}(V)$ as follows. A left A-module W_0

is a module over a subring $\mathcal{U}_0 \subset \mathcal{U}$. We set $\Phi^{\mathsf{L}}(W_0) = \left(\mathcal{U}/\,\mathrm{N}^1_{\mathsf{L}}\mathcal{U}\right) \otimes_{\mathcal{U}_0} W_0 = \oplus_d \left(\mathcal{U}/\,\mathrm{N}^1_{\mathsf{L}}\mathcal{U}\right)_d \otimes_{\mathcal{U}_0} W_0$, an $\mathbb{N}$-graded module. Given a right A-module Z_0, we set $\Phi^{\mathsf{R}}(Z_0) = Z_0 \otimes_{\mathcal{U}_0} \left(\mathcal{U}/\,\mathrm{N}^1_{\mathsf{R}}\mathcal{U}\right)$. Here $\mathrm{N}^1_{\mathsf{L}}\mathcal{U}$ and $\mathrm{N}^1_{\mathsf{R}}\mathcal{U} \subset \mathcal{U}$ are neighborhoods of 0.

To describe the sheaf of coinvariants, let $\pi\colon \mathcal{C} \to S$ be a projective curve, with n distinct smooth sections $P_i\colon S \to \mathcal{C}$ and formal coordinates t_i at P_i, and let $W^\bullet = W^1 \otimes \cdots \otimes W^n$ be the tensor product of an n-tuple of V-modules. We set $\mathcal{W} := W^\bullet \otimes \mathcal{O}_S$, and $\mathcal{L} := \mathcal{L}_{\mathcal{C}\setminus P_\bullet}(V)$ the sheaf of Chiral Lie algebras [DGT21, DGT22a], explicitly described in [DGK23, §4.4]. The *sheaf of coinvariants* $[\mathcal{W}]_{\mathcal{L}}$ on S is defined to be the cokernel $\mathcal{L} \otimes_{\mathcal{O}_S} \mathcal{W} \to \mathcal{W} \to [\mathcal{W}]_{\mathcal{L}} \to 0$.

Mode transition algebras. The mode transition algebra, is defined as $\mathfrak{A} = \Phi^{\mathsf{R}}(\Phi^{\mathsf{L}}(\mathsf{A})) = \Phi^{\mathsf{L}}(\Phi^{\mathsf{R}}(\mathsf{A})) = \left(\mathcal{U}/\,\mathrm{N}^1_{\mathsf{L}}\mathcal{U}\right) \otimes_{\mathcal{U}_0} \mathsf{A} \otimes_{\mathcal{U}_0} \left(\mathcal{U}/\,\mathrm{N}^1_{\mathsf{R}}\mathcal{U}\right)$, a bigraded vector space. By [DGK23, Def. 3.2.5] $\mathfrak{A}$ has an algebra structure, and $\Phi^{\mathsf{L}}(W_0) = \oplus_d \Phi^{\mathsf{L}}(W_0)_d$ is a left $\mathfrak{A}$-module. The subalgebra $\mathfrak{A}_d := \mathfrak{A}_{d,-d}$ of $\mathfrak{A}$ acts on $\Phi^{\mathsf{L}}(W_0)_d$.

Geometric application. Mode transition algebras reflect both the algebraic structure of V and the geometry of coinvariants. On the geometric side, we show coherent sheaves of coinvariants are locally free when the mode transition algebras admit unities that act as identities on modules (we call these strong unities).

A sheaf of $\mathcal{O}_S$-modules is locally free if and only if is coherent and flat. So when $[\mathcal{W}]_{\mathcal{L}}$ is coherent, to show it is locally free, it suffices to show it is flat. For this one may reduce to showing that vector spaces of coinvariants have the same dimension over all pointed and coordinatized curves.

The standard approach, from [TUY89, Theorem 6.2.1], relies on the factorization property to argue that this holds. However, since we do not assume V is rational or C_2-cofinite, and so particular, $A(V)$ may not be finite dimensional or semi-simple, by [DGK22, Proposition 7.1], one cannot apply factorization. Instead, by smoothing (as described in [DGK23]), we show that if the mode transition algebras admit identity elements satisfying any of the equivalent properties of [DGK23, Definition/Lemma 3.3.1], then one can identify the rank of the space of coinvariants at a nodal curve at the central fiber of a degeneration with the rank of the coinvariants on the general fiber. This allows one to argue inductively, reducing the number of nodes. As a consequence of [DGK23, Theorem 5.0.3], if strong unities exist, so this smoothing process for V may be carried out, we show:

Corollary. [DGK23, Corollary 5.2.6] *Let $W^1, \ldots, W^n$ be simple modules over a C_1-cofinite vertex operator algebra V, such that coinvariants are coherent for curves of genus g, and such that $\mathfrak{A}_d(V)$ admit strong unities for all $d \in \mathbb{Z}_{\geq 0}$. Then sheaves of coinvariants are locally free. If the conformal dimensions of $W^1, \ldots, W^n$ are rational, these sheaves descend to vector bundles on $\overline{\mathcal{M}}_{g,n}$.*

Proof. (sketch) To prove the claim, we induct on the number of nodes via degenerations. The base case follows from the assumption of coherence and the fact that sheaves of coinvariants support a projectively flat connection on schemes parametrizing families of smooth curves [FBZ04, DGT21]. $\square$

Examples. One has $\mathfrak{A}_0 = \mathsf{A}$, which has a unity [DGK23, Remark 3.2.4]. By [DGK23, Example 3.3.2] if V is rational and C_2-cofinite, then $\mathfrak{A}_d$ admit strong unities for all $d \in \mathbb{N}$, and the Corollary specializes to [DGT22a, VB Corollary]. By [DGK23, Proposition 7.2.1], the Heisenberg VOA (which is C_1-cofinite, but neither C_2-cofinite, nor rational) also admits strong unities for all $d \in \mathbb{N}$. In particular, by the Corollary, associated sheaves of coinvariants which are coherent if $g = 0$ are (globally generated) vector bundles [DGK23, Corollary 7.4.1].

Questions. There are a number of questions about these sheaves, as asked in [DGT22b, §6], for V of CohFT-type, and in [DG23, §10] for V generated in degree 1. One may also ask: Can one find Chern classes of associated bundles (in this more general setting where V is not of CohFT-type) and determine when the higher Chern classes are tautological? What are sufficient conditions ensuring the Chern classes are effective? Can one realize the infinitely many extremal rays of the effective cone known to exist from [CLTU23] as first Chern classes of bundles of coinvariants? There are natural maps between different sheaves of coinvariants; what can one say about their degeneracy classes or loci? Finally, as asked in [DGT22b] for V of CohFT-type, and in [DG23] for V is generated in degree 1, are there geometric interpretations of (dual) fibers in the case where V is C_1-cofinite (and so strongly finitely generated)? For instance, for lattice VOAs V_L, where $L'/L \cong Z/mZ$, and $m = 2k$, which is generated in degree k, Ueno and Nagatomo [Uen95] gave an identification with theta functions (V_L are called lattice Heisenberg VOAs in [BD04, §4.9]). Dimensions of spaces of conformal blocks are computed as an application (see [DGT22b, Ex 5.2.5] for an alternative approach).

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The Kawamata–Viehweg vanishing theorem for schemes and applications

Takumi Murayama

Summary. Let X be a smooth complex projective variety. In 1953, Kodaira [14] proved what is now called the Kodaira vanishing theorem, which states that if $\mathcal{L}$ is an ample invertible sheaf on X, then $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$. Since then, Kodaira's theorem and its generalizations for complex projective varieties—in particular, the Kawamata–Viehweg vanishing theorem [12, 25] and its relative version due to Kawamata, Matsuda, and Matsuki [13]—have become indispensable tools in algebraic geometry over fields of characteristic zero, particularly in birational geometry and the minimal model program. However, while the goal in birational

geometry is to study birational equivalences between projective varieties, recent progress in the minimal model program (e.g., [7, 8, 11]) has shown that it would be very useful to have a version of the Kawamata–Viehweg vanishing theorem that holds for schemes that are not necessarily projective varieties.

In [19], I proved the relative Kawamata–Viehweg vanishing theorem for schemes of equal characteristic zero. My results are optimal given known counterexamples to vanishing theorems in positive characteristic (Raynaud [22]) and in mixed characteristic (Totaro, as cited in [3]) and have many applications to both algebraic geometry and commutative algebra. In this talk, I discussed my vanishing theorem and some of its applications, including to rational singularities and the relative minimal model program with scaling for algebraic spaces and both complex and non-Archimedean analytic spaces (joint with Shiji Lyu [15]). Below, I focus on my vanishing theorem [19] and the relative minimal model program [15].

Motivation. In birational geometry, even if one's primary interest is in the classification of smooth complex projective varieties up to birational equivalence, it is often necessary to consider more general objects. For example:

(I) (Vanishing theorems) Kodaira's vanishing theorem [14] is proved using complex analysis on complex manifolds.
(II) (Resolution of singularities) Hironaka's resolution of singularities [10] is proved using an inductive strategy that involves schemes of finite type over quasi-excellent local rings.
(III) (The minimal model program) In the minimal model program:
- Mori's Bend and Break [17] requires working over finite fields.
- By work of Mori [18] and Reid [23], a good theory of minimal models of complex projective varieties must allow singularities.
- Recent progress on the ACC conjecture for log canonical thresholds [7, 8] and for minimal log discrepancies [11] require working with schemes of finite type over $\mathbf{C}[\![x_1, x_2, \ldots, x_n]\!]$.

These phenomena raise the following question: *Can we do birational geometry for spaces that are not smooth complex projective varieties?* We answer this question affirmatively for various categories of spaces in equal characteristic zero.

Main results. My first main result is that a relative version of the Kawamata–Viehweg vanishing theorem [12, 25] holds for all proper morphisms of Noetherian schemes of equal characteristic zero and of arbitrary dimension.

Theorem 1 (Murayama [19]). *Let $f\colon X \to Y$ be a proper surjective morphism of integral Noetherian schemes of equal characteristic zero such that X is regular and Y has a dualizing complex $\omega_Y^\bullet$. Suppose $\mathcal{L}$ is a f-big and f-nef invertible sheaf on X. Then, we have $R^i f_*(\omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$, where ω_X is the unique cohomology sheaf of $f^!\omega_Y^\bullet$.*

Theorem 1 was previously known when X and Y are varieties [12, 25, 13], when $\dim(X) = 2$ [24], when $\dim(X) = 3$ and f is birational [2], and when $\dim(Y) = 1$

[5, 20]. Since Kodaira-type vanishing theorems are false in both positive characteristic (Raynaud [22]) and in mixed characteristic (Totaro, as cited in [3]), my methods yield the most general versions of the Kawamata–Viehweg vanishing theorem possible for proper morphisms of schemes of arbitrary dimension.

A novel aspect of my proof of Theorem 1 is that even though the result is about schemes, we must leave the world of schemes and work with the Zariski–Riemann space $\mathrm{ZR}(X)$ of X [21], which is a ringed space but not a scheme in general.

As an application of Theorem 1, Shiji Lyu and I established the relative minimal model program with scaling in various categories of spaces of equal characteristic zero. Our work gives a unified proof of the relative minimal program with scaling for quasi-projective varieties [4], algebraic spaces of finite type over a field [26], and complex analytic spaces [9, 6] with appropriate choices of scaling divisors A.

Theorem 2 (Lyu and Murayama [15])**.** *Let $\pi\colon X \to Z$ be a projective morphism in one of the following categories, where X and Z are integral and X is normal:*

- *Quasi-excellent Noetherian schemes, formal schemes, or algebraic spaces of equal characteristic zero admitting dualizing complexes.*
- *Semianalytic germs of complex analytic spaces.*
- *Berkovich or rigid-analytic spaces over a complete non-Archimedean field of characteristic zero (assumed to be non-trivial in the rigid setting).*

Suppose X is $\mathbf{Q}$-factorial over Z and Δ is a $\mathbf{Q}$-divisor on X such that (X,Δ) is klt. Let A be a π-ample $\mathbf{Q}$-invertible sheaf on X such that $K_X+\Delta+A$ is π-nef. Then, the relative minimal model program with scaling of A over Z exists, and moreover terminates in finitely many steps over every affinoid subdomain in Z if $cK_X+\Delta$ is π-big for some rational number $c \in (-\infty,1]$.

In addition to [4, 26, 9, 6], Theorem 2 was previously known for rigid-analytic surfaces [16] and for schemes of dimension ≤ 3 ([24, 3] and the references therein).

Theorem 2 illustrates the power of working in the general context of quasi-excellent schemes: It suffices to prove Theorem 2 for schemes and then use GAGA theorems to move between the algebraic and analytic worlds. These GAGA techniques were previously used for resolutions of singularities and for weak factorization of birational maps ([1] and the references therein), but we need to prove new GAGA results for Grothendieck duality. Our GAGA results also imply Theorem 1 holds for Moishezon morphisms in these other categories.

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Motivic structures in the cohomology of moduli spaces of curves

Sam Payne

Algebraic geometry endows the rational cohomology groups of moduli spaces of curves with additional structures, such as (mixed) Hodge structures and ℓ-adic Galois representations. Standard conjectures from arithmetic regarding analytic continuations of L-functions attached to these Galois representations lead to striking predictions, by Chenevier and Lannes, about which such structures can appear in motivic weights less than or equal to 22 [7, Theorem F].

In a series of recent works with Jonas Bergström and Carel Faber [3] and with Samir Canning and Hannah Larson [4, 5], we have confirmed these predictions in motivic weights less than or equal to 15. More precisely, let $\mathsf{L} := H^2(\mathbb{P}^1)$ denote the Tate structure, and let let $\mathsf{S}_{k+1} := W_k H^k(\mathcal{M}_{1,k})$ be the Hodge structure or Galois representation that corresponds, via Eichler-Shimura, to the space of weight $k+1$ cusp forms for $\mathrm{SL}_2(\mathbb{Z})$. Then, for $k \leq 15$, the semi-simplification of $H^k(\overline{\mathcal{M}}_{g,n})$ is a polynomial in L, S_{12}, and S_{16}. In particular, we showed that $H^k(\overline{\mathcal{M}}_{g,n})$ vanishes for odd $k < 11$ and is a direct sum of copies of $\mathsf{L}^{k/2}$ for even $k < 16$.

The proofs of these results build on the inductive methods used by Arbarello and Cornalba to compute $H^2(\overline{\mathcal{M}}_{g,n})$ and to prove the vanishing of $H^k(\overline{\mathcal{M}}_{g,n})$ for $k \in \{1, 3, 5\}$ [1]. The induction is on pairs (g, n), using the combinatorial structure of the boundary. To confirm the predictions of Chenevier and Lannes in a given motivic weight k via such an inductive argument, it suffices to do so for an explicit finite set of pairs (g, n). One difficulty is that the set of pairs required as base cases increases with k. For the results mentioned above, several of the needed base cases were not yet in the literature. In [3], the essential missing bases cases were $\overline{\mathcal{M}}_{4,n}$ for $n \in \{1, 2, 3\}$. To prove the vanishing of $H^k(\overline{\mathcal{M}}_{g,n})$ for odd $k < 11$, we showed that the cohomology of each of these three spaces is a polynomial in L, via point-counting over finite fields. In [4], we discovered that $H^{11}(\overline{\mathcal{M}}_{g,n})$ vanishes for $g \geq 2$, and hence the main result mentioned above can be proved for motivic weights $k \leq 12$ using inductive arguments in which all of the base cases (g, n) for $g \geq 2$ have polynomial point counts.

In weights 13, 14, and 15, new methods are needed, because the required bases cases do not all have polynomial point counts. Instead, we use the Chow–Künneth generation Property (CKgP) for open moduli spaces $\mathcal{M}_{g,n}$ and show, using a decomposition of the diagonal argument, that the cycle class map for any smooth variety or DM stack X with the CKgP over the complex numbers surjects onto the pure weight cohomology $\bigoplus_k W_k H^k(X)$ [5].

For a fixed motivic weight k, once $H^k(\overline{\mathcal{M}}_{g,n})$ is sufficiently well-understood for all (g, n), one can proceed to study the weight k graded piece of the compactly supported cohomology of the open moduli space of smooth curves $\mathcal{M}_g$, using the weight spectral sequence associated to the Deligne–Mumford compactification by stable curves. This spectral sequence degenerates at E_2, and the kth row of the E_1-page is naturally understood as a complex of decorated graphs [8, §§2.3-2.5].

The weight 0 row was studied via tropical methods with Melody Chan and Søren Galatius in [6] and shown to be quasi-isomorphic to Kontsevich's commutative graph complex. The weight 2 and 11 rows were studied with Thomas Willwacher [8, 9]. Altogether, this work provides many new insights and results regarding the cohomology of $\mathcal{M}_g$ and the mapping class group of a closed orientable surface of genus g. These applications were explained in the talk, and are presented in detail in the references cited.

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Actions of finite group schemes on curves

Michel Brion

Finite group schemes occur naturally when considering vector fields in positive characteristics. Let X be a projective variety over an algebraically closed field k, with tangent sheaf $\mathcal{T}_X$. The (finite-dimensional) space of vector fields $H^0(\mathcal{T}_X) = \mathrm{Der}(\mathcal{O}_X)$ is the Lie algebra of the automorphism group scheme Aut_X. It has an additional structure of p-Lie algebra in characteristic $p > 0$, since the p-th power of a derivation is a derivation. It is easy to show that if $H^0(\mathcal{T}_X) \neq 0$, then it contains $D \neq 0$ such that $D^p = 0$ or $D^p = D$. Via the bijective correspondence between finite-dimensional p-Lie algebras and finite group schemes killed by the relative Frobenius morphism, it follows that Aut_X contains either α_p (the Frobenius kernel of the additive group) or μ_p (that of the multiplicative group).

For example, some smooth projective surfaces of general type have non-zero vector fields (see [6]); for these, the automorphism group scheme is finite and non-reduced. Also, given a smooth projective curve C and a non-negative integer n, there exists an elliptic surface $X \to C$ such that $h^0(\mathcal{T}_X) \geq n$, and every such surface satisfies $\dim(\mathrm{Aut}_X) \leq 2$ (see [7]). Finally, if $p = 2$ then there exists a smooth projective surface X such that $\mathcal{T}_X \simeq \mathcal{O}_X^{\oplus 2}$ and $\dim(\mathrm{Aut}_X) = 1$; in particular, X is not an abelian surface and Aut_X is non-reduced (see [5]).

This motivates the study of actions of finite group schemes. Some basic properties of actions of finite groups do not extend readily to this setting. Specifically, given a finite group G acting on a variety X, the singular locus X_{sing} (viewed as a reduced subscheme) is G-stable; there exists a unique G-action on the normalization $\tilde{X}$ that lifts the G-action on X; in characteristic 0, there exists a G-equivariant desingularization. But for $G = \alpha_p$ acting on $X = \operatorname{Spec} k[x,y]/(y^p - x^{p+1})$ via $u \cdot (x,y) = (x, y+u)$, the singular point 0 is not G-stable. Moreover, the G-action on X does not lift to the normalization $\tilde{X} = \mathbb{A}^1 \to X$, $t \mapsto (t^p, t^{p+1})$ (this is readily checked in terms of the corresponding derivation $D = \partial/\partial y$, see [9] for a closely related example). Thus, X has no equivariant desingularization.

As a partial remedy, we introduce in [3] the notion of equivariant normalization. Given a finite group scheme G, we say that a G-variety X is *G-normal* if every finite birational morphism of G-varieties $Y \to X$ is an isomorphism.

We now present some results and examples from [3]. First, *every G-variety X has a G-normalization* in an obvious sense. Indeed, for any two finite birational morphisms of G-varieties $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$, there exists a finite birational morphism of G-varieties $f : Y \to X$ which factors through f_1, f_2; moreover, the normalization morphism $g : \tilde{X} \to X$ factors through f. So the statement follows easily from the finiteness of g. Similar arguments show that *if a G-variety X is G-normal, then the quotient X/G is normal. The converse holds if the G-action on X is free.*

For instance, the above curve X is α_p-normal, since α_p acts freely on it with quotient $x : X \to \mathbb{A}^1$. This yields an example of a singular G-normal curve. One may show that every such curve is cuspidal, i.e., the normalization map is bijective.

The normalization of a variety is obtained by globalizing the process of integral closure. We do not know any direct analogue of this construction for the equivariant normalization. Still, the latter is related to the usual normalization as follows: consider again a finite group scheme G and embed it in a smooth connected algebraic group $G^{\#}$ (for example, we may view G as a subgroup scheme of GL_n via some faithful representation). Given a G-variety X, let $X^{\#} = (G^{\#} \times X)/G$, where G acts on $G^{\#} \times X$ via $g \cdot (h,x) = (hg^{-1}, g \cdot x)$. Then $X^{\#}$ is a variety equipped with an action of $G^{\#}$ via left multiplication on itself, and with a $G^{\#}$-equivariant morphism $f : X \to G^{\#}/G$ having fiber X at the base point. One may then show that *X is G-normal if and only if $X^{\#}$ is normal.*

As a consequence, *the following conditions are equivalent for a curve X equipped with a G-action:*

(i) *X is G-normal.*
(ii) *$X^{\#}$ is smooth.*
(iii) *The quotient stack $[X/G]$ is smooth.*
(iv) *The ideal sheaf $\mathcal{I}_{G\cdot x}$ is invertible for any $x \in X$.*

Here $G \cdot x$ denotes the G-orbit of x (this is a finite subscheme of X). Since the homogeneous space $G \cdot x$ is a local complete intersection, it follows that *every G-normal curve is a local complete intersection* as well. As another consequence, *when $G = \alpha_p$ or μ_p, a G-curve X is G-normal if and only if X/G is smooth and X is smooth at every G-fixed point.*

For example, let X be the curve in $\mathbb{P}^2$ with homogeneous equation $z^p = f(x,y)$, where $f \in k[x,y]$ has degree p and is not a pth power of a linear form. Also, let μ_p act on $\mathbb{P}^2$ via its linear action on $\mathbb{A}^3$ given by $t \cdot (x,y,z) = (x,y,tz)$. This action stabilizes X, and the quotient is the projection $[x : y] : X \to \mathbb{P}^1$. Using the above criterion, one checks that X is μ_p-normal if and only if f has no multiple factors. Also, X is rational, and singular if $p \geq 3$.

An open question asks *to classify (say) μ_p-normal curves X with prescribed quotient $Y = X/\mu_p$, in terms of data on Y*. This includes the well-known classification of μ_p-torsors over Y in terms of pairs (L, s), where L is a line bundle over Y, and s a trivializing section of $L^{\otimes p}$ (then X is the zero locus of $s - 1$ in the total space of the dual line bundle). Thus, the isomorphism classes of μ_p-torsors over Y correspond bijectively to the p-torsion points of the Jacobian of Y. More generally, the μ_p-normal curves over Y should correspond to line bundles over Y equipped with branching data, in analogy with the classification of abelian covers (see e.g. [8]).

Finally, equivariantly normal curves occur in some questions on smooth projective surfaces as follows. Let E be an elliptic curve, and G a finite subgroup scheme (for example, one may take $G = \mu_p$ if E is ordinary, and $G = \alpha_p$ if E is supersingular). Given a projective G-normal curve X on which G acts faithfully, the quotient $S = (E \times X)/G$ (where G acts on $E \times X$ via $g \cdot (z, x) = (g + z, g \cdot x)$) is a smooth projective surface equipped with a faithful action of E. In fact, *all such surfaces are obtained by this construction.*

This settles an issue in the classification of maximal connected algebraic subgroups of the birational automorphism groups of surfaces (see [4]). Also, each surface $S = (E \times X)/G$ comes with a morphism $f : S \to C$, where $C = E/G$ is an elliptic curve, and f is a fibration with fiber X. In particular, if X is rational then f is the Albanese morphism. Such "cuspidal fibrations" occur in the classification of (minimal smooth projective) surfaces in characteristic p. Specifically, they yield a description of quasi-hyperelliptic surfaces (a special class which only exists when $p \in \{2, 3\}$), see [1].

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Chow rings of Moduli spaces of stable curves

Hannah Larson
(joint work with Samir Canning)

The moduli spaces $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves of genus g are equipped with natural maps

$$\pi_i : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n} \qquad \text{and} \qquad \xi_\Gamma : \prod_{v \in \Gamma} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n},$$

where π_i forgets the ith marking, and ξ_Γ glues together curves as encoded in a stable graph Γ. The *tautological rings* $R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$ form the smallest system of subrings of the rational Chow ring that is closed under push forwards and pullbacks along these natural maps. The tautological ring is finitely generated. A natural set of generators is given by decorated strata classes and there are conjectures for the relations among them.

The main question we explored in this talk is

Question. *For which g, n is $A^*(\overline{\mathcal{M}}_{g,n}) = R^*(\overline{\mathcal{M}}_{g,n})$?*

This question has been fully answered in genus 0 by Keel (the Chow and cohomology rings are tautological for all $n \geq 3$ [4]) and genus 1 by Belorousski (the rings are tautological if and only if $n \leq 10$ [1]). For $g \geq 2$, the only previously known cases where $A^*(\overline{\mathcal{M}}_{g,n}) = R^*(\overline{\mathcal{M}}_{g,n})$ were $(g, n) = (2, 0), (2, 1)$ and $(3, 0)$ [5, 2, 3]. For $g \geq 2$, work of van Zelm [6] shows the Chow and cohomology rings are not tautological once $2g + n \geq 24$, leaving finitely many open cases.

Theorem 1 (Canning–Larson)**.** *The Chow and cohomology rings of $\overline{\mathcal{M}}_{g,n}$ are isomorphic and generated by tautological classes for $g = 2$ and $n \leq 9$ and for $3 \leq g \leq 7$ and $2g + n \leq 14$.*

For such (g, n), this also implies that $\overline{\mathcal{M}}_{g,n}$ has polynomial point count.

The key idea of the proof is to prove that many $\mathcal{M}_{g,n}$ for small g and n have the *Chow–Künneth generation Property (CKgP)*, which is a way of making up for the lack of a Künneth formula in Chow. We say X has the CKgP if for all Y, there is a surjection

$$A^*(X) \otimes A^*(Y) \to A^*(X \times Y).$$

Below are some important observations about the CKgP. These combine into a powerful tool when applied to moduli spaces of curves:

(1) *The CKgP plays well with stratifications, products, and finite group quotients.* Thus, using the inductive nature of its boundary, we can reduce to showing that $\overline{\mathcal{M}}_{g,n}$ has the CKgP and $A^*(\overline{\mathcal{M}}_{g,n}) = R^*(\overline{\mathcal{M}}_{g,n})$ to showing the open moduli spaces $\mathcal{M}_{g',n'}$ have the CKgP and $A^*(\mathcal{M}_{g',n'}) = R^*(\mathcal{M}_{g',n'})$ for $g' \leq g$ and $2g' + n' \leq 2g + n$.

(2) *If $X \to Y$ is proper and surjective, and X has the CKgP, then Y has the CKgP.* Thus, to attack each piece $\mathcal{M}_{g',n'}$, we can pass from studying its gonality strata to studying Hurwitz spaces of covers with marked points.

(3) *If $U \subset X$ is open and X has the CKgP, then U has the CKgP.* Often, we can construct a moduli space of marked curves (or curves with a map to $\mathbb{P}^1$ or $\mathbb{P}^2$) as a quotient of affine space minus some discriminant locus. In contrast with having polynomial point count or knowing generators in cohomology, we may throw out *arbitrary* closed subsets and still preserve CKgP and generators in Chow.

(4) *If X is smooth and proper and has the CKgP, then the cycle class map is an isomorphism.* In other words — no matter what techniques from Chow we have used in (2) and (3) — so long as we put our pieces back together into a smooth, proper space, we now also have access to cohomology.

Part (1) above naturally leads us prove results for the collection of $\overline{\mathcal{M}}_{g,n}$ that satisfy $2g + n \leq a$ for some a. From this perspective, one should hope to prove $A^*(\overline{\mathcal{M}}_{g,n}) = R^*(\overline{\mathcal{M}}_{g,n})$ for all $2g + n \leq 12$ (the largest such region that does not contain $\overline{\mathcal{M}}_{1,11}$, which is known to have non-tautological classes). Our theorem confirms this hope, and several cases beyond it. The cases with $2g + n > 12$, such as $\overline{\mathcal{M}}_7$, require more delicate, ad hoc arguments, and we view the results in this region as more surprising.

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Interpolation for Brill–Noether curves

Isabel Vogt

(joint work with Eric Larson)

The interpolation problem is one of the oldest in mathematics. Broadly it asks:

When does there exist a curve of a specified type passing through a collection of n general points in some ambient algebraic variety X?

The maximum integer n for a given type of curve should be viewed as a measure of the flexibility of that type of curve. Most classical results on the interpolation problem have focused on the case where X is the projective plane $\mathbb{P}^2$. These include results from classical antiquity (such as the fact that a line passes through 2 general points, and a conic passes through 5 general points) and from the dawn of algebraic geometry (such as that a smooth degree d plane curve interpolates

$d(d+3)/2$ general points). Here we will allow $X = \mathbb{P}^r$ for any $r \geq 2$. We'd like to consider curves of degree d and genus g, however this presents several issues. First, it is not known, in general, when there exists a smooth curve of degree d and genus g in $\mathbb{P}^r$. And even when it is known that such a curve exists, the moduli space $\mathcal{M}_g(\mathbb{P}^r, d)$ of degree d maps from a genus g curve to $\mathbb{P}^r$ can be badly behaved: it can have multiple components of different dimensions which do not have to be generically smooth. Hence the tuple (d, g, r) alone is not an adequate notion of type. For this reason, and also because of intrinsic interest, we focus on Brill–Noether curves defined by the following Theorem.

Theorem 1 (The Brill–Noether Theorem [2, 3, 4, 5]). *Suppose that*

$$\rho(d, g, r) := g - (r+1)(g - d + r) \geq 0.$$

Then there exists a unique irreducible component $\mathcal{M}_g(\mathbb{P}^r, d)^{\mathrm{BN}}$ of $\mathcal{M}_g(\mathbb{P}^r, d)$ parameterizing nondegenerate maps such that the map $\mathcal{M}_g(\mathbb{P}^r, d)^{\mathrm{BN}} \to \mathcal{M}_g$ is dominant. This component is generically smooth of dimension $(r+1)d - (r-3)(g-1)$ and the general point corresponds to an embedding when $r \geq 3$. Conversely, when $\rho(d, g, r) < 0$ no such component exists.

We call this component the Brill–Noether component and the curves $f : C \to \mathbb{P}^r$ parameterized by it Brill–Noether curves. This provides a precise and natural interpolation problem:

Question 2. *What is the maximum number of general points through which a Brill–Noether curve of degree d and genus g in $\mathbb{P}^r$ passes?*

Equivalently, we are asking what the maximum n is for which the evaluation map

$$\mathcal{M}_{g,n}(\mathbb{P}^r, d)^{\mathrm{BN}} \xrightarrow{\mathrm{ev}_n} (\mathbb{P}^r)^n$$

taking a marked map $(f : C \to \mathbb{P}^r, p_1, \ldots, p_n \in C)$ to the images of the points $(f(p_1), \ldots, f(p_n))$ is dominant. By counting dimensions, one sees that this is only possible when

$$n \leq \left\lfloor \frac{(r+1)d - (r-3)(g-1)}{r-1} \right\rfloor. \tag{1}$$

Example 3. Consider curves of degree 5 and genus 2 in $\mathbb{P}^3$. By the Riemann–Roch theorem,

$$h^0(C, \mathcal{O}_C(2)) := \dim H^0(C, \mathcal{O}_C(2)) = 9 < 10 = \dim H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)),$$

so every such curve lies on a quadric surface. The naive dimension count (1) would predict that curves of degree 5 and genus 2 pass through 10 general points. But quadric surfaces only pass through 9 general points! So such curves are a counterexample to the naive expectation based on dimension counts.

There are a total of four similar counterexamples,

$$(d, g, r) \in \{(5,2,3), (6,4,3), (7,2,5), (10,6,5)\}$$

all of which are obstructed by a particular surface containing the curve. The presence of counterexamples makes this a subtle problem that has attracted substantial interest as well as previous partial results in the special cases of rational curves [8, 9], canonical curves [10, 11], nonspecial curves [1], and curves in low-dimensional projective spaces [7, 12]. Our main theorem completely solves this problem for all Brill–Noether curves.

Theorem 4 (E. Larson–V. [6])**.** *A Brill–Noether curve of degree d and genus g in $\mathbb{P}^r$ passes through the expected number*

$$n \leq \left\lfloor \frac{(r+1)d-(r-3)(g-1)}{r-1} \right\rfloor$$

of general points if and only if

$$(d,g,r) \notin \big\{(5,2,3),(6,4,3),(7,2,5),(10,6,5)\big\}.$$

This theorem is proved by showing that the evaluation map ev_n is generically smooth when n satisfies (1). This is equivalent to dominance in characteristic 0, but is a stronger condition in positive characteristic. By deformation theory, the obstructions to smoothness of ev_n at a general point $(f\colon C \to \mathbb{P}^r, p_1, \dots, p_n)$ lie in $H^1(N_C(-p_1 - \cdots - p_n))$. Using the Euler exact sequence, one computes that

$$\chi(N_C(-p_1 - \cdots - p_n)) = (r+1)d - (r-3)(g-1) - (r-1)n,$$

and so $\chi(N_C(-p_1 - \cdots - p_n)) \geq 0$ is equivalent to (1). This motivates the following definition.

Definition 5. A vector bundle E on a curve C satisfies interpolation if

- $h^1(E) = 0$, and
- For all $n \geq 1$ and D a general effective divisor of degree n on C, the twist $E(-D)$ has at most one nonzero cohomology group. More explicitly:
  - If $n \geq \chi(E)/\operatorname{rk}(E)$, then $H^0(E(-D)) = 0$.
  - If $n \leq \chi(E)/\operatorname{rk}(E)$, then $H^1(E(-D)) = 0$.

This is a stability-like property that generalizes the notion of balanced vector bundles on $\mathbb{P}^1$ and implies semistability when the slope is integral. By deformation theory, if the normal bundle N_C of a general BN-curve of degree d and genus g in $\mathbb{P}^r$ satisfies interpolation, then ev_n is generically smooth and thus dominant for n satisfying (1). Thus Theorem 4 is implied by the following more precise result:

Theorem 6 (E. Larson–V. [6])**.** *The normal bundle N_C of a general Brill–Noether curve C of degree d and genus g in $\mathbb{P}^r$ satisfies interpolation if and only if*

(1)
$$(d,g,r) \notin \big\{(5,2,3),(6,4,3),(6,2,4),(7,2,5),(10,6,5)\big\}, \textit{and}$$

(2) *If the characteristic of the ground field is 2 and $g = 0$, then $d \equiv 1 \bmod r-1$.*

The extra counterexample $(d, g, r) = (6, 2, 4)$ in all characteristics arises because interpolation for N_C is a priori a stronger condition than ev_n being dominant when $\mu(N_C) \notin \mathbb{Z}$. On the other hand, the extra counterexamples in characteristic 2 arise because generic smoothness is a priori a stronger condition than ev_n being dominant when the characteristic is positive. In this case the evaluation map is dominant but inseparable because $N_C(-1)$ is a Frobenius pullback.

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The stability of normal bundles of curves

Izzet Coskun

(joint work with Eric Larson, Geoffrey Smith, Isabel Vogt)

The normal bundle $N_{C/\mathbb{P}^r}$ of a smooth projective curve $C \subset \mathbb{P}^r$ controls the deformations of C in $\mathbb{P}^r$ and plays a central role in questions of arithmetic and moduli. Let $\mu(V) := \frac{\deg(V)}{\mathrm{rk}(V)}$ be the *slope* of a vector bundle V on a curve C. The bundle V is *(semi)stable* if $\mu(W) \underset{(-)}{<} \mu(V)$ for every proper subbundle W of V. Stable bundles satisfy nice cohomological and metric properties. Hence, it is important to know the (semi)stability of naturally defined bundles such as $N_{C/\mathbb{P}^r}$.

By the Brill-Noether Theorem, a general smooth projective curve C of genus g admits a nondegenerate map of degree d to $\mathbb{P}^r$ if and only if the Brill-Noether number $\rho(d, g, r) := g - (r + 1)(g - d + r) \geq 0$. We call a triple of nonnegative integers (d, g, r) a *BN triple* if $\rho(d, g, r) \geq 0$. When $r \geq 3$, the general such map is an embedding and there is a unique component of the Hilbert scheme that dominates the moduli space $\overline{M}_g$ and whose general member parameterizes nondegenerate genus g curves of degree d in $\mathbb{P}^r$. A member of this component is called a *BN-curve*. A BN triple (d, g, r) is *(semi)stable* if the general BN-curve of degree d and genus g in $\mathbb{P}^r$ has a (semi)stable normal bundle. Otherwise, (d, g, r) is *unstable*.

Example 1 (Canonical curves)**.** The triples $(6, 4, 3)$ and $(10, 6, 5)$ corresponding to canonical curves of genus 4 and 6 are unstable. A canonical curve C of genus 4 is a $(2, 3)$ complete intersection in $\mathbb{P}^3$ and the normal bundle of the curve in the quadric destabilizes $N_{C/\mathbb{P}^3}$. A general canonical curve C of genus 6 lies in a del Pezzo surface X of degree 5 and $N_{C/X}$ destabilizes $N_{C/\mathbb{P}^5}$. The triple $(8, 5, 4)$ corresponding to a canonical curve of genus 5 is semistable but not stable. The general canonical curve C of genus 5 is a $(2, 2, 2)$ complete intersection in $\mathbb{P}^5$, hence $N_{C/\mathbb{P}^5} = \mathcal{O}_C(2)^{\oplus 3}$ and is strictly semistable.

Example 2 (Genus 2 curves)**.** The triples $(5, 2, 3)$, $(6, 2, 4)$, $(7, 2, 5)$ and $(8, 2, 6)$ are unstable. Every genus 2 curve C is hyperelliptic and lies in a rational surface scroll X swept out by the lines spanned by the fibers of the g^1_2. The normal bundle $N_{C/X}$ is a line bundle of degree 12 which destabilizes $N_{C/\mathbb{P}^r}$ for these 4 triples.

I do not know any other examples of unstable triples with $g > 0$. Our first theorem characterizes the stable triples in $\mathbb{P}^3$. We always work over an algebraically closed field of arbitrary characteristic.

Theorem 3. [CLV22, Theorem 1] *Let C be a general BN space curve in $\mathbb{P}^3$ of degree d and genus $g \geq 2$, then $N_{C/\mathbb{P}^3}$ is stable except when $(d, g) \in \{(5, 2), (6, 4)\}$.*

In the 1980s several authors investigated the stability of normal bundles of BN-curves in $\mathbb{P}^3$. The stability was proved for $(d, g) = (6, 2)$ by Sacchiero [S83], for $(d, g) = (9, 9)$ by Newstead [N83], for $(d, g) = (6, 3)$ by Ellia [E83], and for $(d, g) = (7, 5)$ by Ballico and Ellia [BE84]. Ellingsrud and Hirschowitz [ElH84] announced a proof of stability in an asymptotic range of degrees and genera.

Our next theorem characterizes the stable canonical triples.

Theorem 4. [CLV23, Theorem 1.1] *Let C be a general canonical curve of genus at least 7. Then $N_{C/\mathbb{P}^{g-1}}$ is semistable. In particular, if $g \equiv 1$ or $3 \pmod 6$, then $N_{C/\mathbb{P}^{g-1}}$ is stable.*

Aprodu, Farkas and Ortega [AFO16] conjectured that the normal bundle of a general canonical curve of genus at least 7 is stable and settled the case $g = 7$. Bruns [B17] proved the stability when $g = 8$.

Even when (d, g, r) is a (semi)stable BN triple, there may be special BN-curves whose normal bundles are unstable. For example, normal bundles of trigonal or tetragonal canonical curves of $g = 5$ or $g \geq 7$ are unstable [AFO16, CLV23]. Given an arbitrary BN-curve $C \subset \mathbb{P}^r$, determining the Harder-Narasimhan filtration of $N_{C/\mathbb{P}^r}$ is an interesting question, especially when C is a multi-canonical curve.

In view of these theorems, it is natural to conjecture the following.

Conjecture 1. [CLV22, Conjecture 1.1] *All but finitely many BN triples* (d, g, r) *with* $g \geq 2$ *are stable.*

In fact, one may conjecture that except for the triples in Examples 1 and 2, every BN triple with $g \geq 2$ is stable (at least in characteristic 0). The following two theorems provide evidence for the conjecture.

Theorem 5. [CG23, Theorem 1.1] *Let* $C \subset \mathbb{P}^r$ *be a general BN-curve of degree* d *and genus* g*. Let* $u = \left\lfloor \frac{(r+1)^2}{2(r-1)} \right\rfloor$*. Let* k *be the integer such that* $(k-1)u < g-1 \leq ku$*. Then* $N_{C/\mathbb{P}^r}$ *is semistable if one of the following holds:*

(1) $g = 1$,
(2) $g \geq \binom{r-1}{2} + 2 + \left\lceil \frac{5r^2-7r}{2(r-1)} \right\rceil r(r-1)$,
(3) $d \geq \min\left(g + \frac{r^2}{4} + 2r - 3, (k+1)(r+1), (g-1)(2r-3) + r + 1\right)$

Let $b_2(r)$ be least integer such that for all $d \geq b_2(r)$, there exists positive integers d_1, d_2 such that $d = d_1 + d_2$, $d_1, d_2 \geq r+1$ and $\gcd(r-1, 2d_1+1) = 1$. If $p \geq 5$ is the smallest prime number that does not divide $r-1$, then $b_2(r) \leq 2r + \frac{p-1}{2}$. In general, $b_2(r) = 2r + O(\log(r))$ and $b_2(r) \leq \frac{5r-3}{2}$. In fact, if $r \geq 1636$, then $b_2(r) \leq 2.01r + 2.015$. Then preserving the notation from Theorem 5, we have:

Theorem 6. [CG23, Theorem 1.2] *Let* $C \subset \mathbb{P}^r$ *be a general BN-curve of degree* d *and genus* $g \geq 2$*. Then* $N_{C/\mathbb{P}^r}$ *is stable if one of the following holds:*

(1) $g \geq \binom{r-1}{2} + 3 + \left\lceil \frac{2(r+1)b_2(r)+3r^2-13r-2}{2(r-1)} \right\rceil r(r-1)$,
(2) $d \geq b_2(r) + \min\left(g + \frac{r^2}{4} + r - 5, b_2(r) + (k-2)(r+1), (g-1)(2r-3)\right)$.

In particular, for every r, there exists at most finitely many BN triples (d, g, r) which are not (semi)stable.

Normal bundles of genus one curves have been studied in [EiL92, ElH84, ElL81]. Larson and Vogt [LV22] solved the interpolation problem for normal bundles of BN-curves. When the rank divides the degree, interpolation implies semistability. Ballico and Ramella [BR99] and Ran [R23] have some partial results on asymptotic (semi)stability of the normal bundles of BN-curves in $\mathbb{P}^r$. Our results significantly improve these earlier results. One can obtain sharper results when $r = 4$.

Theorem 7. [CG23, Theorem 1.5] *If* $C \subset \mathbb{P}^4$ *is a general BN-curve of degree 7 and genus 2, and the ground field does not have characteristic 2, then* $N_{C/\mathbb{P}^4}$ *is stable.*

Combining this theorem with sharper forms of earlier theorems, one can prove the semistability for all but 48 BN triples and the stability for all but an additional 64 BN triples in $\mathbb{P}^4$ (in characteristic not equal to 2).

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From exceptional sets to non-free sections

Sho Tanimoto

(joint work with Brian Lehmann, Eric Riedl)

This is an extended abstract of a talk based on [LRT23] which is joint work with Brian Lehmann and Eric Riedl.

Introduction. We assume that our ground field k is $\mathbb{C}$. Let B be a smooth projective curve defined over k. We say that $\pi : \mathcal{X} \to B$ is a Fano fibration if (1) $\mathcal{X}$ is a smooth projective complex variety; (2) π is flat with connected fibers; and (3) the generic fiber $\mathcal{X}_\eta$ of π is a Fano variety. One may consider a Fano fibration as a regular integral model of a Fano variety $\mathcal{X}_\eta$ defined over the function field $k(B)$. One of central questions in arithmetic algebraic geometry is to understand the space of sections $\mathrm{Sec}(\mathcal{X}/B)$ which is a scheme locally finite type over k and it consists of countably many irreducible components. The questions we ask about this space are what the dimensions of irreducible components are or how many components of a fixed numerical class we have in this moduli space. It was observed by Victor Batyrev that questions like these might be answered by using ideas from arithmetic geometry, i.e., Manin's conjecture.

Manin's conjecture is a conjetural asymptotic formula for the counting function of rational points of bounded height on a smooth Fano variety defined over a number field. There is a version of Manin's conjecture over global function fields, and it was noticed by Batyrev in [Bat88] that such a conjecture can be justified by assuming certain properties of moduli spaces of sections of Fano fibrations. (Batyrev's heuristics) These assumptions have been refined as Geometric Manin's conjecture by Lehmann and myself [LT19] and Ellenberg–Venkatesh [EV05].

Conjecture 1 (Geometric Manin's conjecture)**.** *Let $\pi : \mathcal{X} \to B$ be a Fano fibration.*

(1) *(Exceptional set) "Pathological" components of sections in* $\mathrm{Sec}(\mathcal{X}/B)$ *are controlled by the Fujita invariant.*
(2) *(Uniqueness) There is a unique non-pathological component of* $\mathrm{Sec}(\mathcal{X}/B, \alpha)$*, which should be counted in Manin's conjecture, for sufficiently positive algebraic class α of sections. We call this unique component as the Manin component.*
(3) *(Stability) Manin components of sections exhibit homological or motivic stability as the degree increases.*

Here is the definition of the Fujita invariant:

Definition 1. Let X be a smooth projective variety defined over a field F of characteristic 0 and L be a big and nef $\mathbb{Q}$-divisor on X. The Fujita invariant or the a-invariant is

$$a(X, L) = \min\{t \in \mathbb{R} \mid tL + K_X \text{ is pseudo-effecitve}\}.$$

When L is not big, we define $a(X, L) = +\infty$. When X is singular, we take a smooth resolution $\beta : \widetilde{X} \to X$ and define $a(X, L) = a(\widetilde{X}, \beta^* L)$.

Next let us explain what we mean by pathological components in GMC (1). A key definition to this is freeness of sections:

Definition 2. Let $\pi : \mathcal{X} \to B$ be a Fano fibration. A section C of π is relatively free if $T_{\mathcal{X}/B}|_C$ is globally generated and $H^1(C, T_{\mathcal{X}/B}|_C) = 0$.

Deformation theory of a section C is controlled by the restricted tangent bundle $T_{\mathcal{X}/B}|_C$, i.e., the tangent space of the moduli space at C is given by $H^0(C, T_{\mathcal{X}/B}|_C)$ and the obstruction space is given by $H^1(C, T_{\mathcal{X}/B}|_C)$. In particular, if a section C is relatively free, then the dimension of the moduli space at C is given by expected dimension $-K_{\mathcal{X}/B} + (\dim \mathcal{X} - 1)(1 - g(B))$ and C is a smooth point of the moduli space. Thus irreducible components of $\mathrm{Sec}(\mathcal{X}/B)$ generically parametrizing relatively free sections are easier to understand. On the other hand, components parametrizing only non-relatively free sections are difficult to understand, and Geometric Manin's conjecture (1) gives a way to access to these pathological components using the Fujita invariants. In my joint work with Brian Lehmann and

Eric Riedl [LRT23], we solved GMC (1) in full generality over the field of complex numbers.

Theorem 3 ([LRT23, Theorem 1.3]). *Let $\pi : \mathcal{X} \to B$ be a Fano fibration. Then there exist positive constants $\xi = \xi(\pi)$ and $T = T(\pi)$ with the following properties: let M be an irreducible component of $\mathrm{Sec}(\mathcal{X}/B)$ parametrizing a family of non-relatively-free sections C which satisfy $-K_{\mathcal{X}/B} \cdot C \geq \xi$. Let $\mathcal{U}^\nu$ be the normalization of the universal family over M and let $ev : \mathcal{U}^\nu \to \mathcal{X}$ denote the evaluation map. Then either:*

(1) *ev is not dominant. In this case the subvariety $\mathcal{Y}$ swept out by the sections parametrized by M satisfies*
$$a(\mathcal{Y}_\eta, -K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}/B}).$$
(2) *ev is dominant. Let $f : \mathcal{Y} \to \mathcal{X}$ be the finite part of the Stein factorization of ev. Then we have*
$$a(\mathcal{Y}_\eta, -f^* K_{\mathcal{X}/B}) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/B}).$$
Furthermore, there is a rational map $\phi : \mathcal{Y} \dashrightarrow \mathcal{Z}$ with the following properties: let C be a general section on $\mathcal{Y}$ and $\mathcal{W}$ be a resolution of the main component of the closure of $\phi^{-1}(\phi(C))$.
(a) *We have $a(\mathcal{W}_\eta, -f^* K_{\mathcal{X}/B}|_{\mathcal{W}_\eta}) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/B})$.*
(b) *The Iitaka dimension of $K_{\mathcal{W}_\eta} - a(\mathcal{W}_\eta, -f^*K_{\mathcal{X}/B}|_{\mathcal{W}_\eta}) f^* K_{\mathcal{X}/B}|_{\mathcal{W}_\eta}$ is 0.*
(c) *the strict transform of a general deformation of C in $\mathcal{W}$ is relatively free in $\mathcal{W}$.*
(d) *The sublocus of M parametrizing deformations of C in $\mathcal{W}$ has codimension at most T in M.*

In particular this theorem shows that non-relatively free sections of sufficiently large anticanonical degree come from generically finite B-maps $f : \mathcal{Y} \to \mathcal{X}$ such that $a(\mathcal{Y}_\eta, -f^* K_{\mathcal{X}/B}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}/B})$. We call such maps as accumulating maps. These accumulating maps are easier to understand using techniques from higher dimensional algebraic geometry such as the minimal model program. Let us give you an example of this kind:

Example 4 (Cubic hypersurfaces). Let $\pi : \mathcal{X} \to B$ be a Fano fibration such that $\mathcal{X}_\eta$ is isomorphic to a smooth cubic hypersurface of dimension ≥ 5. Then using adjunction theory, one can prove that there is no accumulating map. Thus Theorem 3 tells us that a general section of sufficiently large anticanonical degree is relatively free.

The main ingredient of the proof of Theorem 3 is theory of foliations and slope stability. The second main theorem in [LRT23] is a certain boundedness of non-relatively free sections. It claims that non-relatively free sections of sufficiently large degree are coming from a bounded family of accumulating maps.

Theorem 5 ([LRT23, Theorem 1.6]). *Let $\pi : \mathcal{X} \to B$ be a Fano fibration. Then we have*

(1) *There is a proper closed subset $\mathcal{V} \subsetneq \mathcal{X}$ such that if $M \subset \mathrm{Sec}(\mathcal{X}/B)$ is an irreducible component parametrizing a non-dominant family of sections then the sections parametrized by M are contained in $\mathcal{V}$.*

(2) *There are a proper closed subset $\mathcal{V} \subsetneq \mathcal{X}$ and a bounded family of smooth projective B-varieties $\mathcal{Y}$ equipped with B-morphisms $f : \mathcal{Y} \to \mathcal{X}$ satisfying the following properties:*
   (a) *f is generically finite onto its image but not birational;*
   (b) $a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}|_{\mathcal{Y}_\eta}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}_\eta})$*;*
   (c) *if equality of Fujita invariants is achieved, then the Iitaka dimension of $K_{\mathcal{Y}_\eta} - a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}|_{\mathcal{Y}_\eta})f^*K_{\mathcal{X}_\eta}|_{\mathcal{Y}_\eta}$ is zero;*

   If $M \subset \mathrm{Sec}(\mathcal{X}/B)$ is a component that generically parametrizes non-relatively free sections of sufficiently large degree, then a general section C parametrized by M satisfies either (i) $C \subset \mathcal{V}$ or (ii) $C = f(C')$ where $f : \mathcal{Y} \to \mathcal{X}$ is in our family, and C' is a relatively free section in $\mathcal{Y}$.

The main ingredients of the proof of this theorem are boundedness of singular Fano varieties [Bir21] as well as universal families of accumulating maps up to twisting constructed in [LST22]. We also systematically use the space of twists of a fixed generically finite and dominant morphisms which is constructed in [LRT23].

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Ramified double covers of curves on non-standard Nikulin surfaces

Margherita Lelli–Chiesa
(joint work with D'Evangelista, Andreas Leopold Knutsen, Alessandro Verra)

1. Introduction

Étale double covers of complex algebraic curves (or equivalently, Prym curves) and their moduli space

$$\mathcal{R}_g := \{(C,\eta)\,|\, C \text{ smooth curve of genus } g,\ \eta \in J(C)[2]\,,\, \eta \not\simeq \mathcal{O}_C\}\,,$$

have received much attention in the last century primarily because of their connection to Prym varieties, which are particular types of principally polarized abelian varieties. In the last decades *Nikulin surfaces* entered this picture; these are particular types of $K3$ surfaces named after Nikulin, who classified symplectic involutions on $K3$s [Ni].

We recall that a polarized Nikulin surface of genus h is, by definition, a triple (S,M,H) where (S,H) is a primitively genus h polarized $K3$ surface, $M \in \operatorname{Pic}(S)$ satisfies $H \cdot M = 0$ and S contains 8 disjoint (-2)-curves $N_1,\dots,N_8$ such that $2M \sim N_1+\cdots+N_8$. In particular, M defines a double cover $\pi : \hat{S} \to S$ branched along the curves $N_1,\dots,N_8$ and, by contracting the (-1)-curves on $\hat{S}$ arising as inverse images of the N_i, one obtains another $K3$ surface $\widetilde{S}$ endowed with a symplectic involution having 8 fixed points. It is worth stressing that π restricts to an étale double cover of any curve $C \in |H|$, that is, the pair $(C, M|_C) \in \mathcal{R}_h$ is a Prym curve.

The minimal Picard rank of a polarized Nikulin surface (S,M,H) is at least 9 as $\operatorname{Pic}(S)$ always contains the lattice

$$\Lambda_h := \mathbb{Z}[H] \oplus_\perp \langle N_1,\dots N_8, M\rangle.$$

Garbagnati and Sarti [GS] proved that the embedding of Λ_h in $\operatorname{Pic}(S)$ either is primitive or has index 2, with the last case only occurring for odd values of h. According to the terminology introduced in [KLV1], S is called a Nikulin surface of *standard type* in the former case, and of *non-standard type* in the latter. The moduli spaces of genus h polarized Nikulin surfaces of standard and non-standard type are denoted by $\mathcal{F}_h^{\mathbf{N},s}$ and $\mathcal{F}_h^{\mathbf{N},ns}$, respectively, and are both irreducible of dimension 11 by [Do, §3].

In the last decade Nikulin surface of standard type found many applications to the study of Prym curves. In particular, Farkas and Verra [FV1] proved that a general Prym curve of genus h lies on a Nikulin surface of standard type if and only if $h \leq 7$ and $h \neq 6$ and this was used by the same authors in order to study the birational geometry of $\mathcal{R}_h$ in low genera [FV2]. Specialization to Prym curves on Nikulin surfaces of standard type also lead to the proof by Farkas-Kemeny [FK] of the Prym-Green Conjecture on the syzygies of Prym canonical curves in odd genus.

We now turn to the non-standard case where, by [VGS, GS], h is odd and S carries two line bundles R, R' whose classes, up to renumbering the curves N_i, can be written as follows:

$$R \sim \frac{H - N_1 - N_2}{2}, \quad R' \sim \frac{H - N_3 - \cdots - N_8}{2}, \quad \text{if } h \equiv 3 \bmod 4;$$
$$R \sim \frac{H - N_1 - N_2 - N_3 - N_4}{2}, \quad R' \sim \frac{H - N_5 - N_6 - N_7 - N_8}{2}, \quad \text{if } h \equiv 1 \bmod 4.$$

In particular, the restrictions of R and R' to any smooth curve $C \in |H|$ define two theta characteristics on C with many sections: the Prym curve $(C, M|_C)$ is thus quite special. This is perhaps the main reason why Nikulin surfaces of non-standard type have been almost ignored sofar.

Quite recently, Bud [Bu] initiated the study of the birational geometry of the moduli spaces of ramified double covers of curves

$$\mathcal{R}_{g,2n} := \left\{ f : \widetilde{C} \to C \,|\, C \text{ smooth curve of genus } g,\ f \text{ a } 2:1 \text{ cover ramified at } 2n \text{ points} \right\}$$

proving that $\mathcal{R}_{g,2}$ is of general type for $g \geq 16$ and uniruled for $3 \leq g \leq 6$. In the same paper he discovered that, if $\pi : \widetilde{C} \to C$ defines a general point of $\mathcal{R}_{g,2}$ then $\widetilde{C}$ is Prym-Brill-Noether general. This highlights a deep unexpected difference with the étale case where, if the genus g of C is even and C is general, the curve $\widetilde{C}$ has genus $2g-1$ and gonality g by [AF] and is thus Brill-Noether special.

I am here reporting on the role that Nikulin surfaces of non-standard type may play in relation to ramified double covers of curves. The key observation is that, if C lies in the linear system $|R|$ or $|R'|$ and $\widetilde{C} := \pi^{-1}(C)$, then the restriction $\pi|_{\widetilde{C}} : \widetilde{C} \to C$ is a double cover ramified at at 2, 4 or 6 points depending on the congruence of h modulo 4 and on the linear system where C lies. Furthermore, by varying h one obtains all possible values for the genus g of C. As a first application, the following result was obtained jointly with D'Evangelista:

Theorem 2. ([DL]) *Let $f : \widetilde{C} \to C$ be a general double cover of a genus $g \geq 2$ curve ramified at 2, 4, or 6 points. Then the curve $\widetilde{C}$ is Brill-Noether general.*

The above theorem provides a generalization of Bud's result and it is a direct application of the proof of the Gieseker-Petri Theorem provided by Lazarsfeld [La] using specialization to $K3$ surfaces. Indeed, if S is a non-standard Nikulin surface and $C \in |R|$ or $C \in |R'|$, the double cover $\widetilde{C} = \pi^{-1}(C) \subset \hat{S}$ of C can be identified with its image in the $K3$ surface $\tilde{S}$. By the description of the $\mathrm{Pic}(\hat{S})$ provided in [VGS], the linear system $|\widetilde{R}|$ where $\widetilde{C}$ lies contains no reducible members; Lazarsfeld's Theorem [La] can thus be applied to conclude that $\widetilde{C}$ is Brill-Noether general.

It is natural to ask whether for $n = 1, 2, 3$ a general element of $\mathcal{R}_{g,2n}$ can be realized on a Nikulin surface of non-standard type. Since $\dim \mathcal{R}_{g,2n} = 3g - 3 + 2n$, $\dim \mathcal{F}_h^{\mathbf{N},ns} = 11$ and a genus g linear system on a $K3$ surface has dimension g, this may only occur for $g \leq 6$ if $n = 1$, for $g \leq 5$ when $n = 2$ and for $g \leq 4$ if $n = 3$. An ongoing project with Knutsen and Verra aims to verify this expectation and use it to provide unirational parametrizations of the moduli space $\mathcal{R}_{g,2}$, $\mathcal{R}_{g,4}$ and $\mathcal{R}_{g,6}$ in low genera.

The main result I discussed focused on the case of $\mathcal{R}_{5,4}$ and on the proof of the following theorem.

Theorem 3. *(*[KLV2]*) A general element of $\mathcal{R}_{5,4}$ can be realized on a Nikulin surface of non-standard type and genus $h = 21$. Furthermore, both the moduli spaces $\mathcal{F}_{21}^{\mathbf{N},ns}$ and $\mathcal{R}_{5,4}$ are unirational.*

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The period-index problem

Alexander Perry

(joint work with James Hotchkiss, Aise Johan de Jong)

The complexity of a Brauer class $\alpha \in \mathrm{Br}(K)$ over a field K can be measured by two integers: the *period* $\mathrm{per}(\alpha)$, which is simply the order of α in $\mathrm{Br}(K)$; and the *index* $\mathrm{ind}(\alpha)$, equal to $\sqrt{\dim_K(D)}$ where D is the unique central division algebra of class α. The integers $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ share the same prime factors, and $\mathrm{per}(\alpha)$ divides $\mathrm{ind}(\alpha)$. The *period-index problem* is to determine an integer e such that $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^e$. The following longstanding conjecture, first raised in print by Colliot-Thélène [2], predicts a precise value of e for function fields of varieties.

Conjecture 1 (Period-index conjecture)**.** *Let K be a field of finite transcendence degree d over an algebraically closed field k. For all $\alpha \in \mathrm{Br}(K)$,* $\mathrm{ind}(\alpha)$ *divides* $\mathrm{per}(\alpha)^{d-1}$*.*

This is vacuously true for $d \leq 1$, as then $\mathrm{Br}(K) = 0$ by Tsen's theorem. It is also true for $d = 2$ by de Jong [3] when $\mathrm{per}(\alpha)$ is prime to the characteristic of k, and by [9, 5] in general. In higher dimensions, the conjecture is wide open: it is not even known for a single field K of transcendence degree $d \geq 3$ that there exists an integer e such that $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^e$ for all $\alpha \in \mathrm{Br}(K)$.

If X is a smooth projective model for K, then the restriction $\mathrm{Br}(X) \to \mathrm{Br}(K)$ is an isomorphism onto the subgroup of unramified Brauer classes. By [5], Conjecture 1 would follow in general if it were known for all unramified Brauer classes on all K. In this way, the conjecture can be viewed as a global problem. I reported on recent progress on the period-index problem from this perspective, restricting for simplicity to the case where the base field $k = \mathbf{C}$ is the complex numbers. The first result, joint with de Jong, gives evidence that the integer e in the period-index problem can be chosen uniformly in α.

Theorem 1 ([4])**.** *Let $X \to S$ be a smooth proper morphism of complex varieties. Assume that the very general fiber of $X \to S$ is projective and satisfies the Lefschetz standard conjecture in degree 2. Then there exists a positive integer e such that for all $s \in S(\mathbf{C})$ and $\alpha \in \mathrm{Br}(X_s)$,* $\mathrm{ind}(\alpha)$ *divides* $\mathrm{per}(\alpha)^e$*.*

Recall that for a smooth projective d-dimensional variety Y with an ample divisor h, the Lefschetz standard conjecture in degree 2 says that the inverse of the hard Lefschetz isomorphism $(-) \cup h^{d-2} \colon \mathrm{H}^2(Y, \mathbf{Q}) \xrightarrow{\sim} \mathrm{H}^{2d-2}(Y, \mathbf{Q})$ is algebraic. This conjecture is known for some interesting classes of varieties, including threefolds of Kodaira dimension less than 3 [11, 12] and holomorphic symplectic varieties of $\mathrm{K3}^{[n]}$ or Kummer type [1, 6].

The idea behind the proof of Theorem 1 is to use the algebraicity of the inverse Lefschetz isomorphism to reduce to studying classes in the image of a correspondence from a surface, and to use the known period-index conjecture for surfaces to bound the index of such classes.

The second result, joint with Hotchkiss, establishes the first nontrivial case of the unramified period-index conjecture in dimension greater than 2.

Theorem 2 ([8])**.** *Let X be a complex abelian threefold. For all $\alpha \in \mathrm{Br}(X)$,* $\mathrm{ind}(\alpha)$ *divides* $\mathrm{per}(\alpha)^2$*.*

The proof relies on the Hodge theory and enumerative geometry of categories. Suppose that $\mathcal{C}$ is an enhanced triangulated category that admits an embedding as a semiorthogonal component into the derived category of a smooth proper variety. Then by [10] there is an associated finitely generated abelian group $\mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$ equipped with a weight 0 Hodge structure and a natural map $\mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$ from the Grothendieck group factoring through the subgroup $\mathrm{Hdg}(\mathcal{C}) \subset \mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$ of integral Hodge classes. When $\mathcal{C}$ is the derived category of α-twisted sheaves for a class $\alpha \in \mathrm{Br}(X)$ on a smooth proper variety, we write $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$, $\mathrm{Hdg}(X, \alpha)$, and $\mathrm{K}_0(X, \alpha)$ for these invariants. Since $\mathrm{ind}(\alpha)$ can be computed as the minimal positive rank of an element of $\mathrm{K}_0(X, \alpha)$, the period-index conjecture for α factors into two steps:

(1) Construct a Hodge class $v \in \mathrm{Hdg}(X, \alpha)$ of rank $\mathrm{per}(\alpha)^{\dim(X)-1}$.
(2) Show that v is algebraic, i.e. in the image of $\mathrm{K}_0(X, \alpha) \to \mathrm{Hdg}(X, \alpha)$.

Step (1) was solved by Hotchkiss [7] when $\mathrm{per}(\alpha)$ is prime to $(\dim X - 1)!$. The key ingredient is an explicit description of the Hodge structure $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$ when α is topologically trivial, in terms of a twist by a B-field. In special cases, like when X is an abelian variety, this also allows Step (1) to be solved for all $\alpha \in \mathrm{Br}(X)$.

Step (2) can be regarded as a case of the integral Hodge conjecture for categories. In [8] we develop a theory of reduced Donaldson–Thomas invariants for CY3 categories, with the feature that the variational integral Hodge conjecture holds for classes with nonvanishing invariant. Theorem 2 is then proved by specializing (X, α) within the Hodge locus for the class v from Step (1) to an untwisted abelian threefold with nonvanishing invariant.

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The Hilbert scheme in logarithmic geometry, after Kennedy-Hunt

Dhruv Ranganathan

In this talk, I discussed shortly forthcoming work of Patrick Kennedy-Hunt (Cambridge) concerning the Hilbert scheme in logarithmic geometry [1].

Degenerations. The motivation for studying such an object are as follows. Suppose

$$\pi : \mathcal{X} \to B,$$

is a simple normal crossings degeneration over a smooth curve B. That is, a flat and proper morphism with smooth domain, with π a smooth fibration away from a single point $0 \in B$, and such that $X_0 = \pi^{-1}(0)$ is a reduced snc variety. Let $\mathcal{X}^\circ$ denote the complement of the singular fiber. We might then ask the following slightly vague question:

How does the relative Hilbert scheme $\mathsf{Hilb}(\mathcal{X}^\circ/B^\circ)$ *degenerate?*

The question being asked is really "what should we put in the special fiber of such family?" There is an obvious candidate, which is the relative Hilbert scheme $\mathsf{Hilb}(\mathcal{X}/B)$. However, this is rather poorly behaved. For example, if π has relative dimension 2, and if we consider Hilbert scheme of points, the family $\mathsf{Hilb}(\mathcal{X}^\circ/B^\circ)$ is smooth over B° – so in particular, very nice. One might expect that the proposed mystery family $\mathsf{L}(X/B) \to B$ that completes this should be correspondingly very nice. For example, one might ask that this family is again simple normal crossings. At the very least, one could ask for $\mathsf{L}(X/B) \to B$ to be flat over B. The relative Hilbert scheme achieves neither.

Pairs. An intimately tied mystery to the one above is encapsulated by the following, equally vague question. Let Y be a smooth projective variety and let $D \subset Y$ be a simple normal crossings divisor.

How does the presence of D affect the Hilbert scheme $\mathsf{Hilb}(Y)$*?*

Again, the simplest answer is that "it doesn't". But also, it clearly does e.g. we specify that subschemes must intersect D or its strata in particular dimensions, one ends up stratifying the Hilbert scheme of Y. This is not dissimilar to the construction of Schubert cells, or the matroid stratification of the Grassmannian. Our version of this is as follows. Define

$$\mathsf{Hilb}^\circ(Y|D) \subset \mathsf{Hilb}(Y)$$

to be the subfunctor parameterizing points $[Z \subset Y]$ of $\mathsf{Hilb}(Y)$ such that the pullback of D to Y is *regular crossings* – the pullbacks of equations for the different irreducible components of D form regular sequences on Z. In other words, from the point of view of algebra, the subschemes $Z \subset Y$ are transverse. We call such subschemes *algebraically transverse*. Since the transversality condition can be phrased in terms of the vanishing of higher Tor's, the subfunctor above is *open*.

This transversality condition is natural from the point of view of logarithmic geometry. It is precisely the condition that Z is *log flat over a a point* when equipped with the pullback log structure from Y to Z. This is the first hint that the questions above should really be asked, and answered, in logarithmic geometry.

A pair of paragraphs on logarithmic geometry. Logarithmic schemes are enhancements of schemes by combinatorial information. While a scheme comes with a notion of a *polynomial* function, logarithmic schemes come also with the notion of a *monomial* function. Precisely, it is a scheme Y equipped with a sheaf of monoids $\mathcal{M}_Y$ that record the "monomials". Part of the data is a map

$$\mathcal{M}_Y \to \mathcal{O}_Y$$

of sheaves of monoids, that tells us how to take an element of the monomial sheaf and think about it as a polynomial. An artefect of the theory is that this map is merely a morphism of sheaves of monoids – it does not have to be injective. In particular, *there can be more monomials than polynomials.* Logarithmic schemes can be assembled into a category with good geometric properties.

Numerous schemes come with natural notions of monomial, and give examples of logarithmic schemes. If Y is a toric variety, the monomials on Y for a sheaf of monoids. A pair (Y, D) as above of a variety and an snc divisor also has a sheaf of monoids – the functions locally defined by a monomial in defining equations for the components of D. The "more monomials than polynomials" phenomena come via taking pullback in this category.

The solution of Kennedy-Hunt. In his thesis, Kennedy-Hunt proposes a moduli functor on the category of logarithmic schemes $\mathsf{Hilb}^{\mathsf{log}}(Y|D)$ that contains $\mathsf{Hilb}^{\circ}(Y|D)$ as a subfunctor. He establishes several of its basic properties. The moduli space is reverse engineered from an "imagined proof" of properness via the valuative criterion, which comes from a 2007 theorem of Tevelev [4].

Tevelev's Theorem. *Let $Z \hookrightarrow Y$ be a subvariety of a smooth toric variety. There exists a toric blowup $Y' \to Y$ such that the strict transform $Z' \hookrightarrow Y'$ is algebraically transverse.*

A particular example of the theorem applies in the following context. Consider the toric variety $Y \times \mathbb{A}^1$, and an algebraically transverse subscheme

$$Z \hookrightarrow Y \times \mathbb{G}_m,$$

flat over $\mathbb{G}_m$. The theorem explains how to complete such families to ones that remain algebraically transverse. The special fibers in these families are not simply blowups of Y, but reducible varieties obtained by gluing together torus bundles over strata in Y together. We call such an an object an *expansion* of Y along D.

As a simple example, the deformation to the normal cone of a union of strata of D, inside Y, is an example of such a special fiber. There is some ambiguity in the limit, because there is not always a *most efficient blowup.*

A *logarithmic subscheme of* (Y, D) is a subscheme of an expansion of Y along D that is algebraically transverse.

By using this definition in families, and imposing an appropriate equivalence relation among expansions, Kennedy-Hunt arrives at a moduli functor:

$$\mathsf{Hilb}^{\mathsf{log}}(Y|D) : \mathsf{LogSch} \to \mathsf{Sets}.$$

He proves that it satisfies the existence and uniqueness parts of the valuative criterion. He also proves a certain representability result. Precisely, he defines another functor

$$\mathsf{Supp}(X|D) : \mathsf{ConeComplexes} \to \mathsf{Sets},$$

a polyhedral geometry construction. Some readers may want to think of this as a *tropicalization* for the functor.

The fans of toric geometry are examples of functors on cone complexes, as are conical dual complexes of boundary divisors in snc compactifications. What Kennedy-Hunt constructs is slightly more general – what he calls a *piecewise linear space*. In any event, the functor $\mathsf{Supp}(X|D)$ determines a topological space just like a fan does. He then proves that there is a natural bijection:

$$\{\text{Polyhedral decompositions of } \mathsf{Supp}(X|D)\}$$
$$\updownarrow$$
$$\{\text{Subfunctors of } \mathsf{Hilb}^{\mathsf{log}}(Y|D) \text{ representable by stacks with logarithmic structures}\}.$$

The representable subfunctors still satisfy the valuative criterion. The situation is analogous to toric geometry: different fan structures give rise to a different, equivariant birational toric compactifications of a torus.

In the lecture, I described examples that demonstrate that the spaces, with some combinatorial burden, are essentially as well-behaved as a one could hope and lead to a good theory of degenerations for the Hilbert scheme:

(1) The logarithmic Hilbert scheme of hypersurfaces in a toric variety is itself a toric variety, via the "secondary fan" of Gelfand–Kapranov–Zelevinsky.
(2) The logarithmic Hilbert scheme of a surface pair (Y, D) is itself an snc pair. As a consequence, the relative logarithmic Hilbert scheme of an snc surface degeneration is itself an snc degeneration.
(3) The theory coincides with work of Maulik–R for curves in a threefold. This space had been constructed using different methods earlier; the space has a virtual fundamental class and leads to *logarithmic DT invariants*.

Finally, I note that in the special case when D is smooth, the construction above reduces to a theory developed by Li–Wu [2]

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Dynamics of Large Groups of Automorphisms of K3 Surfaces

Serge Cantat
(joint work with Romain Dujardin)

There are interesting examples of projective surfaces X with a large group of automorphisms, by which I mean a group $\Gamma \subset Aut(X)$ such that the image of Γ in $H^2(X; \mathbf{Z})$ contains a non-abelian free group. By the Hodge index theorem and Yomdin's theorem, a subgroup $\Gamma \subset Aut(X)$ is large if and only if it contains a non-abelian free group, the element of which have positive topological entropy (except the identity). Here are two examples described during the talk:

- By definition, a pentagon in $\mathbf{R}^2$ is just a sequence of five points p_0, p_1, ..., p_4 in the euclidean plane. The distances $\ell_i = dist(p_i, p_{i+1})$ (with i taken mod (5)) are invariant under the action of the group of direct isometries of $\mathbf{R}^2$. The quotient space $Pent(\ell_0, \ldots, \ell_4)$ is a real algebraic surface, and if this surface is (geometrically) smooth, then it is a K3 surface. Now, given three consecutive vertices p_{i-1}, p_i, p_{i+1}, there is a second point p_i' in the plane such that the distances ℓ_j are unchanged if p_i is replaced by p_i'. This defines an involution s_i on $Pent(\ell_0, \ldots, \ell_4)$. For a general choic of the ℓ_j, the group generated by the s_i is large.
- Consider a smooth surface X in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2, 2, 2)$. The three natural projections on $\mathbb{P}^1 \times \mathbb{P}^1$ are double (ramified) covers; the deck transformations are regular involutions; the group they generate is large (for a general choice of X in its linear system).

I also discussed some analogies with the Markov cubic surfaces $x^2 + y^2 + z^2 = xyz + D$ and character varieties.

The main goal of this talk was the description of the combination of Hodge theory, complex analysis and pluripotential theory, dynamicals systems (in particular ergodic theory, Pesin theory and random dynamics), and arithmetic dynamics to study stationary measures, equidistribution of random orbits, and finite orbits.

Let us assume that $\Gamma \subset Aut(X)$ is defined over $\mathbf{R}$, is large, and contains parabolic elements. This means that Γ contains an automorphism g that preserves a genus 1 fibration and has infinite order (if X were a torus and not a K3, one should also assume that g has infinite order on the cohomology). Then, the results obtained with Dujardin include the following statements:

(1) If X and Γ are defined over a number field, Γ has at most finitely many finite orbits.

(2) If ν is a probability measure on Γ and the support of ν is finite and generates Γ, then every ν-stationary measure on $X(\mathbf{R})$ with a Zariski dense support is Γ-invariant and is equal to the Lebesgue measure defined by the canonical form on the K3 surface X.

(3) Under mild assumption, one can exclude the existence of proper, Zariski closed, Γ-invariant set. Then, for every point x in $X(\mathbf{R})$, the ν-random trajectories converge almost surely towards this Lebesgue measure.

Measuring when a line bundle on a curve is trivial

David Holmes

Let $\pi\colon C \to S$ be a family of curves; smooth, proper, with connected geometric fibres. Let $\mathcal{L}$ be a line bundle on C. We are interested in the locus $\mathrm{DR}(\mathcal{L})$ of points $s \in S$ over which $\mathcal{L}$ is trivial. Sometimes this is empty (for example, if $\mathcal{L}$ has relative degree nowhere 0), and sometimes it is the whole of S (for example, if $\mathcal{L}$ is the trivial bundle). From now on we restrict to considering the case where $\mathcal{L}$ has relative degree zero.

For a more interesting example, let $S = \mathcal{M}_{1,2}$, the moduli space of genus 1 curves with 2 markings, which we can also see as the moduli space of pointed elliptic curves. Let C be the universal curve, and p_1, p_2 the sections. Setting $\mathcal{L} = \mathcal{O}_C(2p_1 - 2p_2)$, the locus $\mathrm{DR}(\mathcal{L})$ is exactly the locus where the 'point' on the universal pointed elliptic curve is 2-torsion. This forms a divisor on $\mathcal{M}_{1,2}$.

For a more refined construction of $\mathrm{DR}(\mathcal{L})$, let J/S denote the relative jacobian of C/S. This is a group scheme, so comes with a section $e\colon S \to J$. The line bundle $\mathcal{L}$ also defines an 'Abel-Jacobi' section

$$AJ_{\mathcal{L}}\colon S \to J. \tag{1}$$

The locus $\mathrm{DR}(\mathcal{L})$ is exactly the locus of points in S where these two sections e and $AJ_{\mathcal{L}}$ coincide; in particular this shows that $\mathrm{DR}(\mathcal{L})$ is closed and algebraic (and carries a natural scheme structure). We can upgrade $\mathrm{DR}(\mathcal{L})$ to an algebraic cycle $AJ_{\mathcal{L}}^! e \in \mathrm{CH}^*(S)$ in the Chow ring of S; it is necessarily of codimension g, as J/S has relative dimension g.

A natural question is how to extend these constructions to families of (pre)stable curves $\pi\colon C \to S$, and how to compute the resulting class $\mathrm{DR}(\mathcal{L})$ in terms of standard ('tautological') classes on S. The naive definition we gave above still makes sense, but in general defines only a *locally* closed subscheme of S, and so does not define a Chow class on S. Perhaps the quickest way to explain this is to say that the Abel-Jacobi map $AJ_{\mathcal{L}}\colon S \dashrightarrow J$ is only a *rational* map in this more general setting, so we cannot easily pull back the class of the zero section.

A natural response is to try to resolve the indeterminacies of the rational map $AJ_{\mathcal{L}}$ by blowing up S. This almost works, except that (because $C \to S$ is not necessarily smooth) the jacobian J/S is not in general proper. However, it turns out that the locus where we cannot resolve the indeterminacies is small enough that the locus $\mathrm{DR}(\mathcal{L})$ misses it completely, and we obtain a well-defined cycle on some blowup of S; see [6], [11] for details.

This cycle can be pushed forward to S itself, and the resulting class (the 'double ramification cycle') coincides with that defined by the theory of rubber maps (see [10], [4]). A beautiful formula for this class in terms of tautological classes was proposed by Pixton, and proven in [9], [1].

However, what if one does not push forward to S, but continues to work on the blowup of S? Is this cycle interesting? Can we compute it, in some way?

First there is a subtlety of definition: the blowup resolving the Abel-Jacobi map was not unique, and a different choice will yield a cycle on a different blowup.

However, these are not arbitrary blowups; rather they can be chosen to be log (toroidal) blowups; roughly speaking, iterated blowups in boundary strata. Moreover, any two choices of log blowup will be dominated by a third log blowup, and the corresponding DR cycles will pull back to the same class. This defines a cycle

$$\mathrm{logDR}(\mathcal{L}) \in \mathrm{logCH}^*(S) := \mathrm{colim}_{\tilde{S}\to S}\, \mathrm{CH}^*(\tilde{S}), \tag{2}$$

where the colimit runs over smooth log blowups of S (certain assumptions on S are needed for this to make sense, or one must work with slightly more careful definitions, but we suppress this point for the purposes of this abstract).

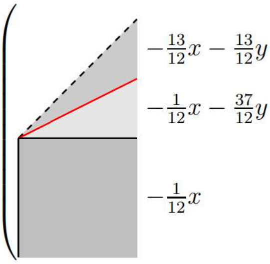
This class $\mathrm{logDR}(\mathcal{L})$ pushes forward to the usual double ramification cycle $\mathrm{DR}(\mathcal{L})$, but it is *not* in general equal to the pullback of $\mathrm{DR}(\mathcal{L})$. The class $\mathrm{DR}(\mathcal{L})$ has so far seen more detailed study, but for some applications $\mathrm{logDR}(\mathcal{L})$ has significant advantages. For example:

(1) The classes $\mathrm{logDR}(\mathcal{L})$ determine all logarithmic Gromov-Witten invariants of toric varieties (see [13]), which (by degeneration arguments) are very useful for computing classical Gromov-Witten invariants.
(2) The intersections of the classes $\mathrm{logDR}(\mathcal{L})$ satisfy a $GL(\mathbb{Z})$-equivariance property which fails for $\mathrm{DR}(\mathcal{L})$, see [7], [8].

Now, to compute the cycle seems a bit intimidating. To keep things concrete, we restrict to the 'universal' case, where $S = \overline{\mathcal{M}}_{g,n}$ and C is the universal stable curve. One might imagine describing a blowup by giving an explicit ideal sheaf on $\overline{\mathcal{M}}_{g,n}$, and then trying to find some way to write an algebraic cycle on the blowup, that is somehow 'tautological' - this sounds rather involved. However, here toric and log geometry come to the rescue! Firstly, just as toric blowups and modifications of a toric variety can be described by subdivisions of the corresponding fan, so log blowups and modifications of $\overline{\mathcal{M}}_{g,n}$ can be described by giving subdivisions of the the tropical moduli space $\mathcal{M}_{g,n}^{\mathrm{trop}}$ (which is roughly speaking a cone complex, see [3]).

How do we write cycles on these blowups? Again we take our inspiration from toric geometry, where the equivariant Chow ring is isomorphic to the ring of piecewise-polynomial functions on the fan [2], [12]. In this context we cannot hope for an isomorphism, but any piecewise-polynomial function on a subdivision of $\mathcal{M}_{g,n}^{\mathrm{trop}}$ induces a cycle class on the corresponding blowup of $\overline{\mathcal{M}}_{g,n}$. Using this language we prove in [5] an analogue of Pixton's formula for the cycle $\mathrm{logDR}(\mathcal{L})$. I will not state the formula here, but in Figure 1 an illustration is given for $\overline{\mathcal{M}}_{1,2}$, with an explicit subdivision and an explicit piecewise-polynomial (in this case, piecewise linear) function. Sage code for these computations can be found at `https://modulispaces.gitlab.io/admcycles/notebooks/Logarithmicdoubleramificationcycles.html`.

$$\mathsf{logDR}_{1,(3,-3)} = \frac{9}{2} \cdot (\psi_1 + \psi_2) + \Phi \left(\vphantom{\int} \right. $$



$$\left. \vphantom{\int} \right) \in \mathsf{logR}^1(\overline{\mathcal{M}}_{1,2}) \,.$$

FIGURE 1. The cycle logDR($\mathcal{O}(3p_1 - 3p_2)$ on $\overline{\mathcal{M}}_{1,2}$.

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