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Partial Differential Equations

Organized by
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23 July – 28 July 2023

Abstract. This workshop focused on nonlinear elliptic and parabolic partial differential equations, touching topics such as geometric flows, geometric variational problems and minimal surfaces, free boundaries, and geometric measure theory.

Mathematics Subject Classification (2020): 35xx.

Introduction by the Organizers

The workshop *Partial Differential Equations*, organized by Guido De Philippis (Courant), Ailana Fraser (UBC), and Felix Schulze (Warwick) was held July 23–July 28, 2023. The meeting was attended by 45 participants with broad geographic representation. The program consisted of 21 talks and left sufficient time for discussions.

As in the tradition of the workshop, a main theme of the workshop was around PDE related to geometric and variational problems. Geometric flows were the topic of several presentations. This included a talk in which existence and uniqueness results for 2-dimensional Ricci flow from non-compact rough initial data were established, and used to give a complete classification of all 2-dimensional expanding solitons. Another talk discussed an application of free boundary mean convex mean curvature flow with surgery to prove the existence of three free boundary minimal discs in any convex 3-ball with generic boundary. The classification of ancient curve shortening flows with finite entropy was discussed, and in particular, uniqueness of the tangent flow at negative infinity. It was presented that all 3-dimensional steady gradient Ricci solitons are O(2)-symmetric, by showing that all 'flying wing' solitons are O(2)-symmetric. Another talk discussed how to use mean curvature flow to obtain explicit lower bound for the density of topologically nontrivial minimal cones in terms of the Colding-Minicozzi entropy of a sphere. A new weak solution concept for mean curvature flow, which are evolving varifolds coupled to the phase volumes by a simple transport equation, and which enjoys both existence and uniqueness properties, was presented.

Several of the talks related to PDE arising from geometric variational problems, such as minimal surfaces and harmonic maps. Upper and lower bounds on the index and the nullity for sequences of harmonic maps with uniformly bounded Dirichlet energy were presented. By combining a Lojasiewicz estimate with a flow argument, which evolves a given map with a weighted flow, it was shown that any map with small energy defect is given by a collection of rational maps that describe the behaviour of the map at different scales. Geometric properties of complete minimal surfaces in $\mathbb{R}^3$ with embedded planar ends were used to get Morse index estimates of Willmore spheres and Willmore real projective planes. A gluing procedure to produce a new family of free boundary minimal surfaces in $\mathbb{B}^3$ having any sufficiently large genus and three boundary components was presented. The problem of studying area variations of surfaces under pointwise Lagrangian constraint was discussed, and in particular a new monotonicity formula for this problem was derived. A new model for soap films, based on the minimization of the Allen-Cahn energy under a suitably formulated homotopic spanning condition was presented; the Euler-Lagrange equations solved by these minimizers arise as a new class of free boundary problems with a semilinear PDE.

New progress related to regularity theory for solutions of geometric PDE was announced in several talks. One talk discussed the free boundary regularity of two phase Bernoulli problem. Generic regularity of minimizing hypersurfaces in dimensions 9 and 10 was presented. Another talk discussed generic regularity of closed embedded hypersurfaces of constant mean curvature. For minimizers of parametric elliptic functionals, a construction of nonlinear entire anisotropic minimal graphs over $\mathbb{R}^4$ was presented, completing the solution to the anisotropic Bernstein problem. Furthermore, a new approach to ε-regularity for optimal transportation maps using harmonic approximation was introduced.

Long time behaviour for vortex dynamics in the 2 dimensional Euler equations was discussed, using gluing methods to describe the global dynamics of the case of two vortex pairs traveling in opposite directions. The existence of positive critical points of the Trudinger-Moser embedding for arbitrary Dirichlet energies was presented. An optimal transport approach to timelike Ricci bounds and Einstein's theory of gravity in a non smooth setting was presented, and a new isoperimetric-type inequality in Lorentzian manifolds with non-negative timelike Ricci curvature and an application to the geometry of black holes was discussed. Another talk discussed results on quasiconformal maps and the Burkholder area inequality.

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Workshop: Partial Differential Equations

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Abstracts

Area Variations under Legendrian Constraint

Tristan Rivière

In the first part of the talk we will adress the problem of studying area variations of surfaces under pointwise Lagrangian constraint in $\mathbb{C}^2$ (or any arbitrary Kähler Surface). We will explain the challenges of performing analysis (well posedness, existence, regularity...) with the associated Euler-Lagrange Equation. Then in trying to find conserved quantities and monotonicity formula for this problem we will naturally be invited to "lift" our problem to 5 dimensions by introducing a fifth Legendrian coordinate and to work in the Heisenberg group (or any Sasakian 5-manifold). The Lagrangian constraint is then converted into a Legendrian one. The area variation under pointwise Legendrian constraint consists in looking for critical points of the area among surfaces which are horizontal. This is a model of "extreme anisotropic" variational problem where one direction is forbidden while total isotropy holds in the remaining 4 directions (which are not integrable). We will derive a new monotonicity formula for this problem. Ultimately the main result we would like to explain is the following : In any 5 dimensional closed Sasakian manifold N^5 (e.g. S^5, $S^3 \times S^2$, Heisenberg group $\mathbb{H}^2$...etc) we prove that any min-max operation on the area among Legendrian surfaces is achieved by a continuous conformal Legendrian map from a closed riemann surface S into N^5 equipped with an integer multiplicity bounded in L^∞. Moreover this map, equipped with this multiplicity, satisfies a weak version of the Hamiltonian Minimal Equation. We conjecture that any solution to this equation is a smooth branched Legendrian immersion away from isolated Schoen-Wolfson conical singularities with non zero Maslov class. If time permits we will explain our motivation for studying such question in relation with the Willmore conjecture in arbitrary co-dimensions.

On the fine structure of the two-phase free boundaries

Bozhidar Velichkov
(joint work with Guido De Philippis, Luca Spolaor)

Let u be a local minimiser of the Alt-Caffarelli-Friedman's two-phase functional

$$\mathcal{F}(u) = \int |Du|^2 dx + \lambda_1 |\{u > 0\}| + \lambda_2 |\{u < 0\}|.$$

In Spolaor-Velichkov [2] (2017, $d = 2$) and De Philippis-Spolaor-Velichkov [3] (2021, $d = 3$) we showed that around a contact point $x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\}$, the free boundaries $\partial\{u > 0\}$ and $\partial\{u < 0\}$ are $C^{1,\alpha}$-regular manifolds.

In this talk we discuss some first results (obtained recently with Luca Spolaor and Guido De Philippis) on the structure of the contact set $\partial\{u > 0\} \cap \partial\{u < 0\}$. Precisely, we consider the symmetric case in which the problem reduces to the one-phase problem studied by Chang-Lara and Savin [1], and we will show that, in dimension $d = 2$, the contact set is (locally) the union of a finite number of intervals and that the solution u is analytic at the endpoints of those intervals. We will also discuss the connection with the (non-linear) thin-obstacle problem and to a classical result of Levy about the (linear) thin obstacle problem.

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Free boundary flow with surgery and applications

Robert Haslhofer

Mean curvature flow with surgery for closed mean-convex surfaces has been constructed by Brendle-Huisken [4] and Kleiner and the author [12]. However, until recently the construction of a flow with surgery in the setting of mean-convex surfaces with free boundary seemed inaccessible, since both the approach from [4] and [12] crucially rely on the noncollapsing result of Andrews [1], which is only available in the setting without boundary. Recently, we solved this problem for mean-convex surfaces with free boundary in any strictly convex domain D:

Theorem ([10])**.** *There exists a free boundary flow with surgery starting at any smooth compact strictly mean-convex free boundary surface $M_0 \subset D$.*
Moreover, the flow either becomes extinct in finite time or for $t \to \infty$ converges to a finite collection of stable connected minimal surfaces with empty or free boundary.

Here, a free boundary flow with surgery is a free boundary $(\delta, \mathcal{H})$-flow. In particular, $\delta > 0$ is a small parameter that captures the quality of the surgery necks and half necks, and $\mathcal{H}$ is a triple of curvature scales $H_{\mathrm{trigger}} \gg H_{\mathrm{neck}} \gg H_{\mathrm{thick}} \gg 1$, which is used to specify more precisely when and how surgeries are performed.

To prove the theorem we implemented our recent new approach from [9], which is based on weak solutions rather than a priori estimates for smooth solutions. Specifically, we study sequences $\mathcal{M}^j$ of free boundary $(\delta, \mathcal{H}^j)$-flows, with the same mean-convex initial condition $M_0 \subset D$, where the curvature scales $\mathcal{H}^j$ improve along the sequence. Given any rescaling factors $\lambda_j \to \infty$, we consider the blowup sequence $\widetilde{\mathcal{M}}^j = \mathcal{D}_{\lambda_j}(\mathcal{M}^j - X_j)$. We establish a hybrid compactness theorem, which allows us to pass to a limit of $\widetilde{\mathcal{M}}^j$, which is smooth near the surgery regions but potentially singular elsewhere. Moreover, using Edelen's monotonicty formula [6] we rule out microscopic surgeries. We then generalize the theory of mean-convex Brakke flows with free boundary from [7] to our setting of hybrid limits, and in particular establish multiplicity-one. As a consequence, taking also into account the recent classification of ancient solutions from [2, 3], we then establish a canonical neighborhood theorem, which allows us to conclude.

As an application, in joint work with Ketover we prove:

Theorem ([11]). *Every strictly convex 3-ball B with nonnegative Ricci-curvature contains at least 3 embedded free-boundary minimal disks in the generic case, and at least 2 solutions even without genericity assumption. Moreover, the area of our 2nd solution is always strictly less than twice the area of the Grüter-Jost solution.*

A natural family of examples of 3-balls to illustrate this are the ellipsoids

$$E(a,b,c) := \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\} \subset \mathbb{R}^3.$$

They contain at least 3 obvious 'planar' solutions, which are obtained by intersecting $E(a,b,c)$ with the coordinates planes. On the other hand, for $a \geq 2\max(b,c)$ our theorem produces a nonplanar embedded free-boundary minimal disk $\Sigma(a)$. Moreover, for $a \to \infty$ our surfaces $\Sigma(a)$ converge in the sense of varifolds to the planar disk $\{x = 0\} \times E(b,c) \subset \mathbb{R} \times E(b,c)$ with multiplicity-two.

To outline our proof, recall that Grüter-Jost [8] already proved the existence of at least 1 solution. Moreover, by a beautiful degree theory argument of Maximo-Nunes-Smith [14] for generic metrics the number of solutions is always odd. Hence, our task is to produce a 2nd solution. To get started, sliding the Grüter-Jost disk a bit to both sides we can decompose $B = B^- \cup Z \cup B^+$, where Z is a short cylindrical region and $\partial B^\pm$ are smooth strictly mean-convex disks with free-boundary. Using the free boundary flow with surgery from above, and ideas from our earlier work with Buzano and Hershkovits [5], we produce an optimal free-boundary foliation of B, namely a foliation $\{\Sigma_t\}_{t\in[-1,1]}$ of B by free-boundary disks, such that the Grüter-Jost disk sits in the middle of the foliation as Σ_0 and all other slices have strictly less area. As an aside, we mention that these smooth foliations are of independent interest. Using our optimal foliation we can then form a certain two parameter family $\{\Sigma_{s,t}\}$. Loosely speaking, this family is constructed by joining the surfaces Σ_s and Σ_t by a thin half neck. Establishing a half version of the catenoid estimate from [13], we can suitably open up the half neck to arrange that

$$\sup_{s,t} |\Sigma_{s,t}| < 2|\Sigma_0|.$$

This guarantees that min-max for our two-parameter family does not simply produce the Grüter-Jost disk with multiplicity-two, and together with a standard Lusternik-Schnirelmann argument allows us to conclude.

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Improved generic regularity of minimizing hypersurfaces

Christos Mantoulidis
(joint work with Otis Chodosh, Felix Schulze)

Let Γ be a smooth, closed, oriented, $(n-1)$-dimensional submanifold of $\mathbf{R}^{n+1}$. Among all smooth, compact, oriented hypersurfaces $M \subset \mathbf{R}^{n+1}$ with $\partial M = \Gamma$, does there exist one with *least area*?

Foundational results in geometric measure theory can be used to produce an integral n-current T with least mass ("minimizing") among all those with boundary equal to the multiplicity-one current represented by Γ. When $n+1 \leq 7$, it is known that T is supported on a smooth, compact, oriented hypersurface that solves the original differential geometric problem (see [1, 2, 3, 4, 5]). When $n+1 \geq 8$, smooth minimizers can fail to exist (see [6]) but it is nevertheless known that away from a compact set $\operatorname{sing} T \subset \mathbf{R}^{n+1} \setminus \Gamma$ of Hausdorff dimension $\leq n-7$, the support of T will be a smooth precompact hypersurface with boundary Γ (see [7, 5]).

A fundamental result of Hardt–Simon [8] shows that the singularities of 7-dimensional minimizing currents in $\mathbf{R}^8$, which are necessarily isolated points, can be eliminated by a perturbation of the prescribed boundary Γ, thus yielding solutions to the original geometric problem in $\mathbf{R}^8$ for the perturbed boundary.

In recent work motivated from our past results on mean curvature flow (see, e.g., [9, 10]) we obtained a generic regularity result for minimizers in higher ambient dimensions:

Theorem ([11], [12]) *Let $\Gamma^{n-1} \subset \mathbf{R}^{n+1}$ be a smooth, closed, oriented, submanifold. There exist arbitrarily small perturbations Γ' of Γ such that every minimizing integral n-current with boundary $[\![\Gamma']\!]$ is of the form $[\![M']\!]$ for a smooth, precompact, oriented hypersurface M' with $\partial M' = \Gamma'$ and* $\operatorname{sing} M' = \bar{M}' \setminus M'$ *satisfies*

$$\operatorname{sing} M' = \emptyset \text{ if } n+1 \leq 10, \text{ otherwise } \dim_H \operatorname{sing} M' \leq n - 9 - \varepsilon_n$$

where $\varepsilon_n \in (0,1]$ *is an explicit dimensional constant.*

Let us discuss what goes into the proof of this theorem. Let us denote

$$\mathcal{M}(\Gamma) = \{\text{minimizing integral } n\text{-currents in } \mathbf{R}^{n+1} \text{ with boundary } [\![\Gamma]\!]\}.$$

We agree to the following simplifying assumptions (see [12] for the general case):

- Γ is connected.
- $\mathcal{M}(\Gamma)$ is a singleton.

The above and the standard regularity theory guarantee that $\mathcal{M}(\Gamma) = \{[\![M]\!]\}$ for a smooth, precompact, oriented hypersurface $M \subset \mathbf{R}^{n+1}$ with $\partial M = \Gamma$, $\operatorname{sing} M = \bar{M} \setminus M \subset\subset \mathbf{R}^{n+1} \setminus \Gamma$, and $\dim_H \operatorname{sing} M \leq n - 7$.

Now set $\Gamma_0 := \Gamma$ and perturb Γ smoothly to $(\Gamma_s)_{s\in(-\delta,\delta)}$ by s times the unit normal to M along Γ (recall that $\operatorname{sing} M \cap \Gamma = \emptyset$) for some small $\delta > 0$. Accordingly, for each $s \in (-\delta, \delta)$, let $\mathcal{M}(\Gamma_s)$ be the set of all minimizers with boundary data Γ_s; each such is still of the form $[\![M_s]\!]$, with M_s enjoying similar a priori regularity as M. A cut-and-paste argument implies that

$$(\ddagger) \qquad [\![M_s]\!] \in \mathcal{M}(\Gamma_s),\ [\![M_{s'}]\!] \in \mathcal{M}(\Gamma_{s'}),\ s \neq s' \implies \bar{M}_s \cap \bar{M}_{s'} = \emptyset.$$

Define

$$\mathcal{L} = \cup_{s\in(-\delta,\delta)} \cup_{[\![M_s]\!]\in\mathcal{M}(\Gamma_s)} \bar{M}_s,$$

$$\mathcal{S} = \cup_{s\in(-\delta,\delta)} \cup_{[\![M_s]\!]\in\mathcal{M}(\Gamma_s)} \operatorname{sing} M_s.$$

In view of ($\ddagger$), the following "timestamp" function is well-defined:

$$\mathbf{t} : \mathcal{L} \to (-\delta, \delta),$$

$$\mathbf{t}(x) = s \text{ for all } x \in \bar{M}_s,\ [\![M_s]\!] \in \mathcal{M}(\Gamma_s),\ s \in (-\delta, \delta).$$

We are now ready to state the two main tools required for our main theorem.

Tool A ([12]) *It holds that* $\dim_H \mathcal{S} \leq n - 7$.

Tool B ([12]) *The timestamp function* $\mathbf{t} : \mathcal{L} \to (-\delta, \delta)$ *above is* α*-Hölder on* $\mathcal{S}$ *for every* $\alpha \in (0, 2 + \varepsilon_n)$*, where* $\varepsilon_n \in (0,1]$ *is an explicit dimensional constant.*

To obtain the **Theorem** from **Tools A**, **B** one can invoke a Sard-type covering argument of Figalli–Ros-Oton–Serra, who successfully proved a generic regularity result for free boundary singularities in the obstacle problem using tools similar to **A**, **B**.

Proposition ([13, Proposition 7.7]) *Let* $S \subset \mathbf{R}^n$*,* $0 < d \leq n$*, and* $0 < \beta < \alpha$*. Assume that* $\mathcal{H}^d(S) < \infty$ *and that* $f : S \to (-1,1)$ *is* α*-Hölder continuous.*

(1) If $d \leq \beta$*, then* $\mathcal{H}^{d/\beta}(f(S)) = 0$*.*
(2) If $d > \beta$*, then for a.e.* $t \in (-1,1)$ *we have* $\mathcal{H}^{d-\beta}(f^{-1}(t)) = 0$*.*

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A mean curvature flow approach to density of minimal cones

Lu Wang

(joint work with Jacob Bernstein)

A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is *minimal*, if the mean curvature $\mathbf{H}_\Sigma = 0$. Locally, Σ can be written as a graph of a function u over the tangent plane at p so

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

which is a quasi-linear elliptic equation. By the monotonicity of area ratios, any minimal hypersurface is asymptotic at infinity to a minimal cone. Similarly, minimal cones also model the behavior of singularities arising in limits of sequences of minimal hypersurfaces.

Given a minimal cone $\mathcal{C} \subset \mathbb{R}^{n+1}$, define the density $\theta(\mathcal{C})$ of $\mathcal{C}$ to be

$$\theta(C) = \frac{\operatorname{Area}(C \cap B_1)}{\omega_n}.$$

Here, without loss of generality assume the vertex of the cone is the origin, $B_1 \subset \mathbb{R}^{n+1}$ is the (open) unit ball, and ω_n is the volume of the unit n-ball. By Allard's regularity (or ϵ-regularity), [1], there exists $\epsilon = \epsilon(n) > 0$ such that $\theta(C) > 1 + \epsilon$

unless C is flat. Thus it is very natural to seek more quantitative information of ϵ. For simplicity we restrict our attention to minimal regular cones, i.e., minimal cones with isolated singularities. Indeed, this has been explored from different perspectives, such as application of heat kernel estimates by Cheng-Li-Yau [3], min-max method for minimal surfaces in sphere by Marques-Neves [7], and mean curvature flow by J. Zhu [9].

Here we utilize self-expanding solutions to the mean curvature flow (i.e., the negative L^2-gradient flow for area functional) to derive explicit lower bounds on densities for minimal regular cones. To state the results, we need to recall the Colding-Minicozzi [4] entropy for hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$ which is given by

$$\lambda(\Sigma) = \sup_{\mathbf{x}_0 \in \mathbb{R}^{n+1}, t_0 > 0} (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4t_0}}.$$

Entropy is instrumental in the study of singularities for mean curvature flow since it is monotone decreasing along the flow. It is invariant under rigid motions and dilations. Stone [8] computes

$$\lambda(\mathbb{S}^1) > \lambda(\mathbb{S}^2) > \lambda(\mathbb{S}^3) > \cdots \to \sqrt{2}.$$

As observed by Ilmanen-White [5], for a minimal cone $\mathcal{C}$ one has $\theta(C) = \lambda(C)$. With Jacob Bernstein, we prove in [2] that

Theorem 1. *For $3 \le n \le 6$, let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a minimal regular cone.*

(1) If at least one of the components of $\mathbb{R}^{n+1} \setminus \mathcal{C}$ is not contractible, then $\theta(C) \ge \lambda(\mathbb{S}^{n-1})$.

(2) If at least one of the components of $\mathbb{R}^{n+1} \setminus \mathcal{C}$ is not a homology ball, then $\theta(C) \ge \lambda(\mathbb{S}^{n-2})$.

To certain extent, the theorem complements the analogous results of Ilmanen-White [5] for area-minimizing cones. Note that minimal cones in $\mathbb{R}^2$ are unions of rays and minimal cones in $\mathbb{R}^3$ are planes. When $n = 3$, both topological restrictions are equivalent to that the link of the cone has genus at least 1, and the bound given in the theorem is ≈ 1.52, compared to the optimal one given by Marques-Neves [7] is ≈ 1.57. The restriction on the upper bound of dimension is closely related to the regularity for minimal hypersurfaces, however, it would be very interesting to see if the theorem is true in all dimensions, in light of the recent developments in minimal hypersurfaces in higher dimensions, e.g., [6].

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Uniqueness of Ricci flows from rough surfaces

Peter M. Topping
(joint work with Hao Yin, Luke T. Peachey)

A Ricci flow on a smooth surface is a time-dependent family of Riemannian metrics $g(t)$ satisfying the PDE
$$\frac{\partial g}{\partial t} = -2K_g g,$$
where K_g is the Gauss curvature of g. With respect to local isothermal coordinates x, y, the conformal factor u then satisfies the logarithmic fast diffusion equation
$$\frac{\partial u}{\partial t} = \Delta \log u,$$
where Δ is the Laplacian with respect to x and y.

This equation has been studied by many authors in many different contexts. Hamilton and Chow [8, 1] developed a theory that gave existence of a solution for a maximal existence time, after a starting metric has been specified. Their theory also described how the solution became of constant curvature as the final time was approached. In the case that the domain is $\mathbb{R}^2$, there is a large literature focussed on the logarithmic fast diffusion equation. Particularly relevant to us are the references [2, 4, 3, 10]. In the general case of arbitrary smooth initial data, a complete theory has been developed, partly in collaboration with Giesen [11, 6, 12].

In this talk we were concerned with the case of rough initial data. Because the Ricci flow in 2D evolves the metric $g(t)$ within the same conformal class, we can adjust our viewpoint and see the the flow as a combination of a fixed smooth surface M equipped with a conformal structure, plus an evolving Riemannian volume measure. The volume measure is effectively determining the conformal factor. We then attempt to start the Ricci flow with the fixed M, together with a Radon measure as initial data. Together with Hao Yin, we have recently completed the proof of the following result.

Theorem 1 (Main existence and uniqueness theorem [13, 14])**.** *Let M be a two-dimensional smooth manifold equipped with a conformal structure, and let μ be a Radon measure on M that is nonatomic in the sense that*
$$\mu(\{x\}) = 0 \quad \textit{for all } x \in M.$$

Writing $\tilde{M}$ *for the universal cover of* M*, and* $\tilde{\mu}$ *for the lift of* μ *to* $\tilde{M}$*, define* $T \in [0, \infty]$ *by*

- $T = \infty$ *if* $\tilde{M} = D$*;*
- $T = \frac{1}{4\pi}\tilde{\mu}(\tilde{M})$ *if* $\tilde{M} = \mathbb{R}^2$*;*
- $T = \frac{1}{8\pi}\tilde{\mu}(\tilde{M})$ *if* $\tilde{M} = S^2$*.*

Then there exists a smooth complete conformal Ricci flow $g(t)$ *on* M*, for* $t \in (0, T)$*, attaining* μ *as initial data in the sense that the volume measure* $\mu_{g(t)}$ *of* $g(t)$ *satisfies*

$$\mu_{g(t)} \rightharpoonup \mu \text{ as } t \searrow 0$$

and so that if $\tilde{g}(t)$*,* $t \in (0, \tilde{T})$*, is any other smooth complete conformal Ricci flow on* M *that attains* μ *as initial data in the same sense, then* $\tilde{T} \leq T$ *and*

$$g(t) \equiv \tilde{g}(t) \qquad \text{for all } t \in (0, \tilde{T}).$$

If $T \in (0, \infty)$ *then* $\mu_{g(t)}(M) = (1 - \frac{t}{T})\mu(M)$ *for all* $t \in (0, T)$*.*

To clarify, the condition that $\mu_{g(t)} \rightharpoonup \mu$ as $t \searrow 0$ is saying that for every $\varphi \in C_c^0(M)$, we have

$$\int_M \varphi d\mu_{g(t)} \to \int_M \varphi d\mu$$

as $t \searrow 0$.

In even more recent work together with Luke Peachey [9], we demonstrated that the nonatomic hypothesis is the best one could ever hope to achieve in the sense that every complete 2D Ricci flow defined on a time interval $(0, \epsilon)$ arises from Theorem 1 for some nonatomic Radon measure μ (and the same conformal structure as the flow):

Theorem 2 (Time zero limits of complete Ricci flows)**.** *Suppose* M *is a smooth surface and* $g(t)$ *is any smooth complete Ricci flow on* M *for* $t \in (0, \epsilon)$*, for some* $\epsilon > 0$*. Then there exists a nonatomic Radon measure* μ *on* M *such that*

$$\mu_{g(t)} \rightharpoonup \mu \tag{1}$$

as $t \searrow 0$*. The measure* μ *is nontrivial unless the universal cover of* M *is the disc and* $g(t) = 2th$ *for* h *a complete hyperbolic metric on* M*.*

This result is giving us a converse of short time existence, and implies a one-to-one correspondence between complete Ricci flows and initial data as in Theorem 1, in a sense that is made precise in [9].

A theory covering Theorem 1 can be extracted from the Kähler Ricci flow theory of Guedj-Zeriahi [7] and Di Nezza and Lu [5] in the case that M is a *closed* manifold. However, noncompactness is essential in the applications we have in mind. For example, as a first application of Theorem 1, we give a complete classification of 2D expanding Ricci solitons that was started with Yin in [13] and completed with Peachey in [9].

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The long time behaviour for vortex dynamics in the 2-dimensional Euler equations

Monica Musso

(joint work with Juan Dávila, Manuel del Pino, Shrish Parmeshwar)

The evolution of a two dimensional incompressible inviscid ideal fluid with smooth initial velocity concentrated in small regions is well understood on finite intervals of time: it converges to a superposition of Dirac deltas centered at collision-less solutions to the point vortex system, in the limit of vanishing regions. Even though for generic initial conditions the vortex point system has a global smooth solution, much less is known on the long time behaviour of the fluid vorticity.

We consider the case of two vortex pairs traveling in opposite directions. Using the inner-outergluing method adapted to this context, we describe the global dynamics of this configuration. This work is in collaboration with J.Dávila (U.Bath), M.del Pino (U.Bath) and S.Parmeshwar (Imperial College London).

Free boundary minimal surfaces: advances and perspectives

Alessandro Carlotto
(joint work with Giada Franz, Mario B. Schulz, David Wiygul)

In my lecture I have tried to describe some of the things we have learnt, over the past few years, about various questions in the global theory of free boundary minimal surfaces in the Euclidean unit ball (henceforth denoted by $\mathbb{B}^3$). In essence, I have discussed how to construct such surfaces and how to distinguish them by virtue of the fine analysis of their (equivariant or absolute) Morse index. As a byproduct, there starts to emerge a comparative picture of variational versus perturbative methods, although a lot is still to be understood in that direction.

1. The realization problem

We shall begin our journey with the natural "existence problem" which, in this specific context, we like to refer to as *realization problem*:

Given any $g \geq 0$ and $b \geq 1$ can we embed the surface $\Sigma_{g,b}$, having genus g and b boundary components, as a free boundary minimal surface in the unit ball?

Said that till 2010 the only known examples were the rotationally symmetric ones (i. e. the flat disc ($g = 0, b = 1$), and the critical catenoid ($g = 0, b = 2$)), the question above has lately received great attention, and we have witnessed a number of significant new results through various methods, that can be grouped in three classes as follows: **optimization** of the first Steklov eigenvalue, **min-max** methods for the area functional, **gluing** and/or desigularization methods. Despite such significant advances, developing techniques to generate examples with "low topological complexity" i. e. with low genus and few boundary components turned out to be a very hard problem. In particular, the case of genus one and connected boundary was considered by most experts to be especially delicate; we first learnt about this matter at Stanford in 2012 in the context of a graduate class taught by Schoen, although we later found out this specific question (or variations thereof) had been mentioned by a number of authors, starting at least with Lin in 1987 [19] (cf. [22]) and ending with the influential survey [17]. After significant efforts, the problem was finally resolved in 2020 in joint work with Franz and Schulz [3]:

Theorem A. *For each $g \geq 1$ there exists an embedded free boundary minimal surface M_g in $\mathbb{B}^3$ with connected boundary, genus g and dihedral symmetry $\mathbb{D}_{g+1}$.*

Each surface M_g is constructed via equivariant min-max methods, along lines initiated by Ketover in [16], building on earlier work by Simon-Smith and Colding-De Lellis. A well-known drawback of min-max methods for the area functional is that, since one needs to go through weak notions of convergence (i. e. those typical of geometric measure theory), they are not topologically effective: they do not allow to control the topology of the critical points they produce. We showed, through a (somewhat surprising) chain of *ad hoc* arguments that the topological type is actually preserved in the limit, namely that one can first single out geometric reasons why the boundary need be connected in the limit, and then in turn the genus is precisely that imposed at the level of sweepouts we designed.

The idea of extracting a "general theory" from this work has very recently been developed - at least to some extent - by Franz and Schulz in [12], where they proved a lower semicontinuity result for the first Betti number (at least in the multiplicity one case), leading - among other applications - to the variational construction of free boundary minimal surfaces in $\mathbb{B}^3$ having genus zero and any assigned number $b \geq 3$ of boundary components; this is to be compared with [7] (doubling of the equatorial disk, which only holds for $b \gg 1$) and with [11] (which is an abstract existence result, and thus has the drawback of bearing limited information on the resulting surfaces).

2. The topological uniqueness problem

Let us move to the natural companion question, that is the topological uniqueness problem for free boundary minimal surfaces in $\mathbb{B}^3$:

Given any $g \geq 0$ and $b \geq 1$ is there at most (only) one embedding of $\Sigma_{g,b}$ as a free boundary minimal surface in the unit ball (up to ambient isometry)?

We first note that to date the only unconditional uniqueness theorem was proven by Nitsche [21] for $g = 0, b = 1$: the flat disc is the only free boundary minimal disc in $\mathbb{B}^3$. It is widely conjectured (cf. [8]) although still open that the critical catenoid is the only (embedded, else see [6]) free boundary minimal annulus in $\mathbb{B}^3$. This is partly supported by the theory of minimal cycles in round $\mathbb{S}^3$, based on the uniqueness of the Clifford torus established by Brendle in [2]. In very recent work, we show that if one does not restrict to "low topological complexity" then the topological uniqueness question can be answered in the strongest negative terms:

"The topology and symmetry group of a free boundary minimal surface in the Euclidean unit ball do not determine the surface uniquely."

More precisely, in [4] we proved the following statement:

Theorem B. *For any sufficiently large integer g there exist in the unit ball of Euclidean $\mathbb{R}^3$ two distinct, properly embedded, free boundary minimal surfaces having genus g, three boundary components and symmetry group coinciding with the antiprismatic group $\mathbb{A}_{g+1}$ of order $4(g+1)$.*

This results follows from combining the existence theorem in [13] (which is the desingularization of the union of the equatorial disk and the critical catenoid) with our own construction of a second sequence, whose limit varifold is instead the union of the equatorial disk and two catenoidal annuli, symmetric with respect to the origin, that are singled out by the conditions of passing through the equatorial circle and meeting again the round unit sphere along another circle (respectively near the north and south poles) at a right angle. To that aim, we had to develop adequate machinery for desingularizing sationary varifolds *at the free boundary*, which is a significant difference with all earlier literature on the subject.

One important motivation, that lies behind the precise formulation of this theorem, is the recent result by Kapouleas-Wiygul (see [15]), asserting the uniqueness of each Lawson surface, in the round three-dimensional sphere, given its topology and symmetry group. It is equally unclear whether this polymorphism also happens for complete, embedded, minimal surfaces in $\mathbb{R}^3$ having, say, finite total curvature: we are unable to tell whether one can construct a non-isometric twin for each surface belonging to the Costa-Hoffman-Meeks family. In this sense, the theorem suggests some sort of additional **flexibility** of free boundary minimal surfaces in the Euclidean unit ball compared both to the closed case in round $\mathbb{S}^3$ and the complete case in Euclidean $\mathbb{R}^3$.

3. Morse index, open questions and some results

It is very hard to effectively estimate the Morse index of minimal hypersurfaces, for whichever class (closed, complete, free boundary ...). In $\mathbb{B}^3$ it has only been computed for rotationally symmetric examples, the equatorial disc (= 1) and the critical catenoid (= 4). Very recently, in the aforementioned paper [12] Franz and Schulz proved the existence of an index **5** free boundary minimal surfaces in $\mathbb{B}^3$ that has *either* $\boldsymbol{g=1}$ and $\boldsymbol{b=1}$ *or* $\boldsymbol{g=0}$ and $\boldsymbol{b=2}$. There is in fact ample evidence for the former alternative to hold, and for conjecturing this to be our M_1, mentioned in the main theorem of Section 1.

With Schulz and Wiygul we are currently carrying through a systematic investigation around this theme, both in the "absolute" case and in the **G-equivariant** case (for **G** a group of isometries of the ambient manifold); our first paper on this theme, that is [5], is an attempt to vastly generalize ideas that originated in work by Montiel-Ros [20]. As a sample application, we are able to distinguish (so to say on purely variational grounds) the two families that came in play in Section 2:

Theorem C. *There exist g_0 such that for all integers $g > g_0$ the (maximal) equivariant Morse index and nullity of the free boundary minimal surfaces $\Sigma_g^{KL}, \Sigma_g^{CSW} \subset \mathbb{B}^3$ satisfy*

$$\mathit{index}_{\mathbb{A}_{g+1}}(\Sigma_g^{KL}) = 1, \qquad \mathit{nullity}_{\mathbb{A}_{g+1}}(\Sigma_g^{KL}) = 0,$$
$$\mathit{index}_{\mathbb{A}_{g+1}}(\Sigma_g^{CSW}) = 2, \qquad \mathit{nullity}_{\mathbb{A}_{g+1}}(\Sigma_g^{CSW}) = 0.$$

Here g denotes the genus of the surfaces in question.

We note that, a posteriori, this result indicates the way towards a variational construction of the families; in particular, a variational construction of $\{\Sigma_g^{CSW}\}$ *beyond the asymptotic regime* (i. e. for all values of g) would necessarily need to build upon the design of $\mathbb{A}_{g+1}$-equivariant 2-sweepouts.

Also, let us remark how (with limited effort) the maximally equivariant calculation above allows to determine effective, two-sided linear bounds on the Morse index and nullity of the same families of free boundary minimal surfaces mentioned above, which significantly improve on the general lower bounds obtained in [1] and the (ineffective) upper bounds obtained in [18] with other methods.

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Sharp quantitative rigidity results for maps from S^2 to S^2 of general degree

Melanie Rupflin

We discussed how carefully constructed parabolic PDEs, in the present case a suitably weighted version

$$\partial_t u = \tau_{\rho^2 g_{S^2}}(u) = \rho^{-2}(\Delta_{g_{S^2}} u + |\nabla u|^2 u) \tag{1}$$

of the harmonic map flow, can be combined with Łojasiewicz estimates in order to prove quantitative rigidity results that give a precise answer to the natural question of whether a map whose energy is nearly minimal can be expected to be close to a minimiser of the energy.

We considered these questions for the Dirichlet energy $E(u) := \frac{1}{2}\int_{S^2} |\nabla u|^2 dv_{g_S^2}$ of maps $u : S^2 \to S^2$ for which we know that $E(u) \geq 4\pi|\deg(u)|$ with equality if and only if u is a rational map, i.e. given by a meromorphic function from $\hat{\mathbb{C}}$ to itself in stereographic coordinates.

For maps with degree ± 1 the results of [1] establish that to any such u there is a degree ± 1 rational map ω with

$$\int_{S^2} |\nabla(u-\omega)|^2 dv_{g_{S^2}} \leq C\delta_u,$$

see [3] and [6] for simplified proofs and [2] for an extension to higher dimensions.

The case of degree ± 1 maps is special as for such maps with $E(u) \approx 4\pi$ energy cannot concentrate on multiple scales or at multiple points. It is precisely this feature of different behaviour at different scales which means that a higher degree map u is *not* forced to be close to any rational map even if its energy defect is very small: We can e.g. construct degree 2 maps $u_{a,\mu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $a \neq 0$ and $\mu \gg 1$, with energy defect $\delta_{u_{a,\mu}} \sim \frac{a^2}{\log \mu}$ so that $u_{a,\mu}(z) \approx z + \frac{1}{\mu z}$ for $|z| \gtrsim 1$ but which are shifted by a for $|z| \lesssim \mu^{-1}$ and whose distance to any meromorphic map is hence at least of order $|a|$.

The natural question is hence whether for maps of general degree smallness of the energy defect δ_u implies that u is essentially given by a *collection of rational maps which represent u at a very different scales* and in [4] we indeed proved

Theorem 1. *For any $\alpha < \infty$ and any $k \in \mathbb{N}$ there exists a constant C so that for any map $u \in H^1(S^2, S^2)$ of degree k there exists a collection of rational maps $\omega_1, \ldots, \omega_n$ from S^2 to S^2 with $\deg(\omega_i) \geq 1$ and $\sum_{i=1}^n \deg(\omega_i) = k$ and a corresponding partition of S^2 into disjoint subsets Ω_i so that u is essentially given by ω_i on Ω_i in the sense that*

$$\int_{\Omega_i} |\nabla(u-\omega_i)|^2 dv_{g_{S^2}} \leq C\delta_u(|\log(\delta_u)|+1) \text{ for every } i$$

and so that ω_i is essentially constant outside of Ω_i in the sense that

$$\int_{S^2\setminus\Omega_i} |\nabla\omega_i|^2 dv_{g_{S^2}} \leq C(\delta_u)^{2\alpha} \text{ and } osc_U\, \omega_i \leq C(\delta_u)^\alpha$$

for every connected component U of $S^2 \setminus \Omega_i$ and every $i = 1, \ldots, n$.

This result is sharp as for $a = \mu^{-1}$ the maps $u_{a,\mu}$ described above have distance of at least $c\mu^{-1}$ from every collection of rational maps and $\delta_u \sim \mu^{-2}(\log\mu)^{-1}$.

We also note that these domains Ω_i are all obtained from balls by cutting out a (potentially empty) collection of far smaller balls and that they correspond to vastly different scales as in any gauge in which at least some of the energy of ω_i appears at scale 1 all other sets Ω_j correspond to sets with diameter of order $O(\delta_u^\alpha)$. In such a viewpoint u is hence essentially described by the corresponding ω_i while the other maps ω_j look like they are constant.

We note that while the energy E is conformally invariant with respect to the domain metric, this symmetry is not present for the corresponding L^2-gradient flow. In the talk we explained how one can exploit the resulting freedom of how to weigh different parts of the domain in order to design a gradient flow that allows us to extract the behaviour of a map at different scales. The basic idea is to consider the domain S^2 equipped with a metric $(1+\sum_j \rho_j^2)g_{s^2}$ that blows up all regions that contain a certain amount of energy to unit size unless they are extremely concentrated and to use additional weights $\rho_{j,*}^2$ to separate the extremely concentrated regions from the rest of the domain by annuli of order 1. Using such weights prevents not only the formation of singularities away from the highly concentrated set, but furthermore deters energy from flowing in to or out of highly concentrated regions and allows us to obtain pointwise bounds on the limit on the bulk of the domain.

As Łojasiewicz estimates can be used to establish a priori bounds on the L^2-distance that a map can travel along a gradient flow, see [5] and [6], we can hence extract the rational maps ω_i that describe u at different scales by carrying out a three-step procedure in each relevant gauge: we first flow the map with the flow (1) with weight $\rho^2 = 1 + \sum_j \rho_j^2 + \sum_j \rho_{j,*}^2$, then cut out all highly concentrated regions of the resulting limit and finally flow this new map again, now with weight $\rho^2 = 1 + \sum_j \rho_j^2$, to obtain ω_i in the limit $t \to \infty$.

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Quasiconformal maps and the Burkholder Area Inequality

André Guerra
(joint work with K. Astala, D. Faraco, A. Koski, J. Kristensen)

For a map $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C};\mathbb{C})$ holomorphic outside the unit disc $\mathbb{D}$ and with a Laurent series of the form

$$f(z) = z + \sum_{j=1}^{\infty} \frac{b_j}{z^j}, \quad \text{if } |z| > 1, \tag{1}$$

the classical Grönwall–Bieberbach Area formula asserts that

$$\sum_{j=2}^{\infty} j|b_j|^2 = \frac{1}{\pi}\int_{\mathbb{D}} [\det A_f - \det \mathrm{D}f]\,\mathrm{d}x,$$

where $A_f(z) \equiv z + b_1\bar{z}$ is a linear map, which can be identified with

$$A_f = \frac{1}{\pi}\int_{\mathbb{D}} \mathrm{D}f\,\mathrm{d}x.$$

In [2], we found an L^p-version of the area formula, found by replacing the determinant with the so-called Burkholder function

$$B_p(\mathrm{D}f) \equiv \left(1 - \frac{p|\mu_f|}{1+|\mu_f|}\right)(|f_z| + |f_{\bar{z}}|)^p, \qquad \mu_f \equiv \frac{f_{\bar{z}}}{f_z},$$

introduced in [5]. Following [3], we think of B_p as a *weighted L^p-norm* (with weight depending on μ_f and p) and one easily verifies that $B_2 = \det$. Recalling that a homeomorphism $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C};\mathbb{C})$ is K-quasiconformal if $|\mu_f| \le \frac{K-1}{K+1}$ a.e., we then have:

Theorem 1. *Let f be a K-quasiconformal map with the expansion* (1) *outside $\mathbb{D}$. For $2 \le p \le \frac{2K}{K-1} \equiv p_K$, we have*

$$\frac{p}{2}\frac{B_p(A_f)}{\det A_f}\sum_{j=2}^{\infty} j|b_j|^2 \le \frac{1}{\pi}\int_{\mathbb{D}} [B_p(A_f) - B_p(\mathrm{D}f)]\,\mathrm{d}x.$$

This result has interesting consequences concerning sharp higher integrability results. Indeed, we have:

Corollary 1. *Let f be a map as in the theorem. Then, with $2 \le p \le \frac{2K}{K-1} \equiv p_K$,*

$$\frac{p}{2}\frac{B_p(A_f)}{\det A_f}\sum_{j=2}^{\infty} j|b_j|^2 + \frac{1}{\pi}\int_{\mathbb{D}} |\mathrm{D}f|^p\,\mathrm{d}x \le \frac{p_K}{p_K - p} B_p(A_f),$$

This inequality is sharp.

This result gives a sharp form of Astala's higher integrability theorem [1], which asserts that a K-quasiconformal map, originally assumed to be in $W^{1,2}_{\mathrm{loc}}$, is actually in $W^{1,p}_{\mathrm{loc}}$ for all $p < p_K$. The simple example of the map $f(z) = |z|^{1/K-1}z$ shows that this exponent is optimal.

As another consequence of the theorem, we obtain an optimal form of the higher integrability property of Jacobians, first discovered by Müller [8]:

Corollary 2. *Let $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C};\mathbb{C})$ be a map as in* (1)*, which is also orientation-preserving, i.e. it satisfies a.e.* $\det \mathrm{D}f \geq 0$*. Then*

$$\text{[explicit function of } (b_j)_{j\geq 1}] + \int_{\mathbb{D}} \det(\mathrm{D}f) \log |\mathrm{D}f|^2 \,\mathrm{d}x \leq \int_{\mathbb{D}} |\mathrm{D}f|^2 \,\mathrm{d}x.$$

This inequality is sharp.

In fact, equality in both corollaries is attained for an infinite-dimensional family of piecewise radial mappings, cf. [3, 4] for further details. This is a manifestation of the fact that the energy landscape corresponding to the Burkholder function is *highly degenerate* and, in particular, extremely non-concave. In fact, the Burkholder function is rank-one concave and, conjecturally, quasiconcave.

Recall that a function $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ is said to be *rank-one concave* if W is concave along any segment parallel to a rank one matrix, while it is said to be *quasiconcave* if

$$\frac{1}{|\mathbb{B}^n|}\int_{\mathbb{B}^n} W(\mathrm{D}f)\,\mathrm{d}x \leq W(A_f), \qquad A_f \equiv \frac{1}{|\mathbb{B}^n|}\int_{\mathbb{B}^n} \mathrm{D}f\,\mathrm{d}x,$$

for all $f \in W^{1,\infty}(\mathbb{B}^n;\mathbb{R}^n)$ which are linear (and therefore equal to $z \mapsto A_f(z)$) on $\mathbb{S}^{n-1}$. Note that the theorem is nothing but a *quasiconcavity inequality* for B_p, albeit with two important differences: on the one hand, the inequality only holds for a suitable class of quasiconformal maps; on the other, we allow for boundary conditions substantially more general than linear. Indeed, for f as in (1), we have that

$$0 = b_2 = b_3 = \cdots \iff f(z) = z + \frac{b_1}{z} = z + b_1\bar{z} = A_f(z) \text{ on } \mathbb{S}^1.$$

In general, when looking at functions $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ for $n \geq 3$, rank-one concavity does not imply quasiconcavity, according to Šverák's example [9]. The case $n = 2$, however, remains open, and the Burkholder function is the prime example of a rank-one concave function which we hope to be quasiconcave. Together with the results of [6], the theorem gives evidence in this direction. The importance of deciding on the quasiconcavity of B_p stems from the fact that

$$B_p \text{ is quasiconcave } \implies \|\mathcal{S}\|_{L^p\to L^p} = p-1 \text{ for } p \geq 2,$$

where $\mathcal{S}$ is the Beurling–Ahlfors transform; that the latter should be the case is an outstanding conjecture due to Iwaniec [7].

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Harmonic approximation in Optimal transport

Felix Otto

We revisit the variational ε-regularity theory that provides a robust alternative to the comparison-principle based theory developed by Figalli, Klein & De Philippis. At the core is an approximation of the displacement by the gradient of a harmonic function, as obtained from a suitable Neumann-Poisson problem. In recent work with L.Koch, we simplified the arguments from the work with M.Goldman and M.Huesmann, working directly on the level of rough measures and transfer plans.

Index estimates for sequences of harmonic maps

Jonas Hirsch

(joint work with Tobias Lamm)

For a closed Riemann surface (M, g) and a closed Riemannian manifold (N, h), which we assume to be isometrically embedded into some euclidean space $\mathbb{R}^m$, the Dirichlet energy for maps $u \in W^{1,2}(M, N)$ is given by

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 \, dv_g.$$

Critical points of E are called harmonic maps and they are solutions of the elliptic partial differential equation

$$-\Delta u = A(u)(\nabla u, \nabla u),$$

where A is the second fundamental form of the embedding of $N \hookrightarrow \mathbb{R}^m$.

We study sequences of harmonic maps $u_k \in C^\infty(M,N)$, $k \in \mathbb{N}$, with uniformly bounded Dirichlet energy. By now it is well-known that such a sequence has a weak limit $u_0 \in C^\infty(M,N)$ which is again a harmonic map and that we actually have local smooth convergence away from at most finitely many singular points, where the energy concentrates. Around these finitely many points one can perform a suitable blow-up and in the limit one finds at most finitely many harmonic spheres i.e. maps $\omega_i \in C^\infty(S^2,N)$, $i = 1,\dots,L$, the so called bubbles. Hence one has a clear understanding of the convergence behaviour of such a sequence of harmonic maps away from the finitely many singular points and also close to the singular points via the suitably chosen blow-up. The difficult part is to understand what is going on in the intermediate region which we call the neck region.

Over the last 25 years many people have contributed to a better understanding of the convergence in the neck region and it has been shown that the so called energy identity and the no-neck property hold true, see for example [1, 4, 6, 7, 8, 9, 12, 13, 14, 15, 18, 19]. Here the energy identity corresponds to the fact that in the limit there is no energy in the neck region and the no-neck property shows that the weak limit u_0 and the bubbles ω_i are actually all pointwise connected. Thus there is a very satisfying theory available for the convergence of the sequence u_k.

In contrast to this, we are interested in upper and lower bounds on "the index of the limiting bubble tree" in fact giving a more precise picture of the convergence of the "spectrum" of the index form.

Recall that the index $\mathrm{Ind}(u)$ is defined to be the dimension of the maximal subspace on which the second variation of the Dirichlet energy E is negative definite. Recall that the second variation of E is given by

$$D^2E(u)(X,X) = \int_M (|\nabla X|^2 - \langle A_u^2(X), X\rangle \, dv_g,$$

where $X \in W^{1,2}(M, u^*TN) := \{X \in W^{1,2}(M,\mathbb{R}^m) : X(x) \in T_{u(x)}N \text{ for a.e. } x \in M\}$ and

$$\langle A_u^2(X), Y\rangle = \langle A_u(\nabla u, \nabla u), A_u(X,Y)\rangle$$

for $X, Y \in W^{1,2}(M, u^*TN)$. Additionally, the nullity $\mathrm{Nul}(u)$ is defined to be the dimension of the kernel of the bilinear form associated to $D^2E(u)$. We remark that harmonic maps with a controlled index have been constructed respectively studied in [11, 16, 17].

To be more precise, the goal of this paper is to show bounds for the index of the sequence u_k in terms of the index (and nullity) of the limiting objects u_0 respectively ω_i with $1 \le i \le L$. In a first Theorem we recall a lower bound that is relies on a rather standard capacity argument and has already be observe in [5]:

Theorem 1. *Let $u_k \in W^{1,2}(M,N)$ be a sequence of harmonic maps with uniformly bounded energy $E(u_k) \le C$ and with a weak limit $u_0 \in C^\infty(M,N)$ and finitely many bubbles $\omega_i \in C^\infty(M,N)$, $1 \le i \le L$, as described above. Then we have the estimate*

$$\mathrm{Ind}(u_0) + \sum_{i=1}^{l} \mathrm{Ind}(\omega_i) \leq \liminf_{k\to\infty} \mathrm{Ind}(u_k),$$

where $\mathrm{Ind}(\cdot)$ *denotes the index of the corresponding map.*

The much harder result is the upper bound for the index along the sequence u_k. The main result of our paper is the following.

Theorem 2. *Let* u_k*,* u_0 *and* ω_i*,* $1 \leq i \leq L$ *be as in Theorem . Then we have the estimate*

$$\limsup_{k\to\infty}(\mathrm{Ind}(u_k) + \mathrm{Nul}(u_k)) \leq \mathrm{Ind}(u_0) + \mathrm{Nul}(u_0) + \sum_{i=1}^{L}(\mathrm{Ind}(\omega_i) + \mathrm{Nul}(\omega_i)).$$

Related results also in the context of harmonic maps [18, 19] and critical points of conformally invariant variational problems [3] have been obtained recently. See Remark for further comments on these works.

We note that a similar result has been obtained in the very influential work of Chodosh and Mantoulidis [2] on the Allen-Cahn approximation of minimal surfaces and by Marques and Neves [10] in the context of minimal surfaces.

We also remark that in this Theorem we can not get rid of the nullity in general, since there is apriori no reason which excludes that we can have a sequence of eigenvalues $\lambda_k < 0$ converging to zero.

The argument for Theorem involves a detailed analysis of the convergence of eigenfunctions of $D^2E(u_k)$. In order to do this we first define a family of bilinear forms associated and varying along the sequence u_k. More precisely, we let

$$\langle\langle X, Y\rangle\rangle_{u_k} = \int_M (\langle X, Y\rangle + \langle A^2_{u_k}(X), Y\rangle)\, dv_g.$$

We show that this bilinear form is actually a scalar product once a suitable isometric immersion $N \hookrightarrow \mathbb{R}^m$ has been fixed, which we do from then on. It then follows from the fact that the index and the nullity are independent of the underlying vector space and the scalar product, that we can diagonalize $D^2E(u_k)$ with respect to the scalar product $\langle\langle\cdot,\cdot\rangle\rangle_{u_k}$. In order to study the index and the nullity along the sequence u_k we then study sequences of eigenfunctions $X_k \in W^{1,2}(M, u_k^*TN)$ corresponding to an arbitrary eigenvalue λ_k, i.e. solutions of the linear PDE

$$P(u_k)(-\Delta X_k - A^2_{u_k}(X_k)) = \lambda_k(X_k + A^2_{u_k}(X_k)),$$

where $P(u_k)$ denotes the orthogonal projection of $\mathbb{R}^m$ onto $T_{u_k}N$. Without loss of generality we assume the X_k to be normalized in the sense that $\langle\langle X_k, X_k\rangle\rangle_{u_k} = 1$.

The reason why we choose $\langle\langle\cdot,\cdot\rangle\rangle_{u_k}$ as our underlying scalar product has two main reasons. The first one is that by multiplying the equation for X_k with X_k itself and using our normalization one directly obtains a uniform lower bound on the eigenvalues, namely

$$\lambda_k \geq -1$$

and a uniform upper bound for the $W^{1,2}$-norm of X_k, namely

$$\|X_k\|_{W^{1,2}(M,u_k^*TN)} \leq \sqrt{2}.$$

These estimates directly imply good convergence properties of the X_k in the region where u_k strongly converges to u_0. The second reason for the choice of the scalar product is the conformal invariance of the second term in its definition. This fact turns out to be crucial when studying the convergence of the X_k in the bubble region.

Finally, we show that the energy of the X_k converges to zero in the neck region. In order to do this we adjust and simplify some of the arguments of [7, 14, 15]. To conclude, we are able to show a variant of the energy identity for the sequence X_k which together with the fact that we can show that orthogonality is preserved along the sequence, implies the upper bound which we claim in Theorem .

Remark. *While we were finishing our paper we became aware of a Preprint on the arXiv by Francesca Da Lio, Matilde Gianocca and Tristan Rivière* [3]*, in which the authors obtain a related result as our Theorem for sequences of critical points of general conformally invariant quadratic variational problems. The main difference between the two results is that in* [3] *the authors assume less regularity on the data (such as the target manifold and the differential forms involved). The two results were obtained independently with a rather different proof. We believe that our argument, especially because of the choice of the globally defined bilinear form* $\langle\langle \cdot, \cdot \rangle\rangle_u$*, will be of independent interest. We want to highlight again that it is because of this choice that we get uniform lower bounds for the eigenvalues and uniform upper bounds for the* $W^{1,2}$*-norm of the eigenfunctions basically for free.*

Only recently, we also became aware of two papers of Hao Yin [18, 19] who also proved Theorem for sequences of harmonic maps. His argument relies on a rather delicate analysis of the neck region and is also of independent interest. His analysis was focused on the behavior of harmonic maps in the neck region and in contrast, we provide a self-contained analysis on all three regions arising via the bubbling process on M.

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O(2)-symmetry of 3D steady gradient Ricci solitons

Yi Lai

In 1982, Hamilton introduced the Ricci flow equation $\partial_t g(t) = -2\mathrm{Ric}$. Ricci solitons are Riemannian manifolds satisfying

$$\mathrm{Ric} = \frac{1}{2}\mathcal{L}_X g + \lambda\, g,$$

where X is a vector field and $\lambda \in \mathbb{R}$. Ricci solitons generate self-similar Ricci slows, and often arise as singularity models of compact Ricci flows. In particular, a steady gradient soliton is a soliton with $\lambda = 0$ and $X = \nabla f$ for some smooth function f, and thus satisfies

$$\mathrm{Ric} = \nabla^2 f.$$

In dimension 2, the only steady gradient Ricci soliton is Hamilton's cigar soliton, which is rotationally symmetric. In dimension $n \geq 3$, Bryant constructed a steady gradient Ricci soliton which is rotationally symmetric. In dimension 3, all steady gradient Ricci solitons are non-negatively curved and they are asymptotic to sectors of angle $\alpha \in [0, \pi]$. For instance, the Bryant soliton is asymptotic to a ray ($\alpha = 0$), and the soliton $\mathbb{R} \times$ Cigar is asymptotic to a half-plane ($\alpha = \pi$).

In dimension 3, Hamilton conjectured that there exists a 3D steady gradient Ricci soliton that is asymptotic to a sector with angle in $(0, \pi)$, which is called a *3D flying wing*. We constructed a family of $\mathbb{Z}_2 \times O(2)$-symmetric flying wings, which confirmed Hamilton's conjecture [5], and the asymptotic cone angles of these flying wings can take any value in $(0, \pi)$ [6]. In dimension $n \geq 4$, we constructed a family of non-collapsed, $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient solitons with positive curvature operators, which are not rotationally symmetric [5]. In dimension 3, we prove

Theorem ([7])**.** *All 3D steady gradient Ricci solitons are $O(2)$-symmetric.*

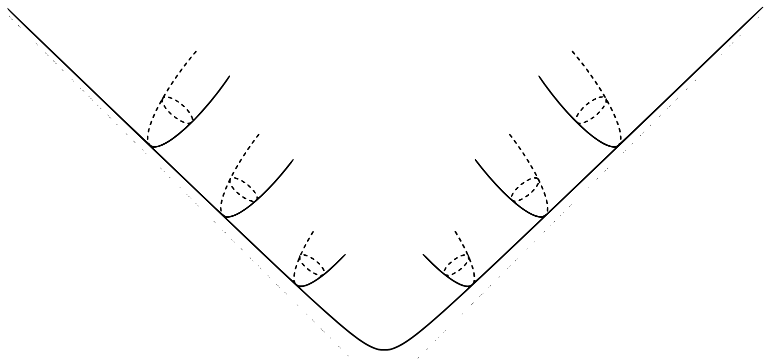

Figure 1. A 3d flying wing

We give a sketch of the $O(2)$-symmetry theorem. We may assume without loss of generality that it is not a Bryant soliton. Then first step is to understand the asymptotic geometry at infinity. We show there are two "edges", two limits ($\mathbb{R} \times$ Cigar and $\mathbb{R}^2 \times S^1$), one critical point, an "almost" $\mathbb{Z}_2$-isometry, including the equality

$$\lim_{s\to\infty} R(\Gamma(s)) = \lim_{s\to-\infty} R(\Gamma(s)) = 4,$$

where R is the scalar curvature and Γ is an integral curve of ∇f. In this step we used Brendle's uniqueness of non-collapsed 3D steady solitons [1]. A corollary of this step is the uniqueness of the Bryant soliton among all 3D steady gradient solitons on $\mathbb{R}^3$ asymptotic to rays.

On the one hand, by a symmetry improvement argument, we can construct inductively an approximating $SO(2)$-symmetric metric $\overline{g}$ satisfying

$$|\overline{g} - g|_{C^{100}} \leq e^{-(2+\epsilon_0)\, d_g(\cdot,\Gamma)}, \tag{1}$$

for some $\epsilon_0 > 0$. Let X be the killing field of the approximate metric $\overline{g}$. Then evolve X by the heat equation $\partial_t X(t) = \Delta_t X(t) + \mathrm{Ric}(X(t))$. Then the lie derivative $\mathcal{L}_{X(t)} g(t)$ satisfies the linearized Ricci-Deturck flow $\partial_t h(t) = \Delta_{L,g(t)} h(t)$. By Anderson-Chow estimate this implies

$$\partial_t |h(t)| \leq \Delta_t |h(t)| + \frac{2|\mathrm{Ric}|^2}{R} |h(t)|.$$

The evolution equation of R implies that R is a solution to this linear heat equation, and we can moreover show that

$$R(x) \geq C(\epsilon_0)^{-1} e^{-2(1+\epsilon_0)\, d_g(x,\Gamma)}. \tag{2}$$

Therefore, by combining (1) and (2), we can use R as an upper barrier function, and show by a heat kernel argument that $|h(t)|$ decays to zero as $t \to \infty$. So $X(t)$ converges to a non-zero smooth vector field X_∞ satisfying $\mathcal{L}_{X_\infty} g = 0$. The killing field X_∞ thus generates a $SO(2)$-isometry, and we can furthermore show that it is also an $O(2)$-isometry.

We can compare 3D steady gradient Ricci solitons with convex translators in $\mathbb{R}^3$, under which the $O(2)$-symmetry is compared with the reflectional symmetry (i.e. $\mathbb{Z}_2$-symmetry). By the classification for the convex translators in $\mathbb{R}^3$ [4], I believe:

Conjecture 1. *If two 3D flying wings have the same asymptotic cone angle, then they are isometric modulo rescalings.*

It is also interesting to see whether the $O(2)$-symmetry holds for 3D ancient Ricci flows, as well as higher dimensional steady gradient solitons with positive curvature operator. In dimension 4, one can ask whether the $O(3)$-symmetry holds for all non-collapsed 4D steady gradient solitons.

Moreover, the recent breakthrough works of Choi-Haslhofer-Hershkovits classified all non-collapsed translators in $\mathbb{R}^4$ [3, 2], which inspires the following conjecture in Ricci flow:

Conjecture 2. *The only non-collapsed steady gradient solitons with non-negative curvature operator are the 4D Bryant soliton, and the family of $\mathbb{Z}_2 \times O(3)$-symmetric solitons I constructed in* [5]. *Moreover, the blow-down of each of the $\mathbb{Z}_2 \times O(3)$-symmetric soliton is a ray.*

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Critical points of the Moser-Trudinger functional

Andrea Malchiodi

(joint work with Francesca De Marchis, Luca Martinazzi, Pierre-Damien Thizy)

Consider a smooth bounded domain Ω of $\mathbf{R}^2$: it is well-known that the Sobolev space $W_0^{1,2}(\Omega)$ embeds into any $L^p(\Omega)$ with $p \in [1, \infty)$, but not in L^∞. Pohozaev and Trudinger proved that integrability holds up to exponential class, and later Moser was able to obtain a sharp inequality in [11]: setting, for $\Lambda > 0$,

$$M_\Lambda = \left\{ u \in W_0^{1,2}(\Omega) \mid \int_\Omega |\nabla u|^2 dx = \Lambda \right\},$$

he showed that

$$\sup_{\|u\|^2_{W_0^{1,2}(\Omega}} F(u) < +\infty; \qquad F(u) = \int_\Omega (e^{u^2} - 1) dx,$$

provided that $\Lambda \leq 4\pi$. The constant 4π is sharp for such an inequality to hold, and the constrained supremum of F is attained for $\Lambda < 4\pi$.

Carleson and Chang proved in [1] that the supremum of F is attained on the unit disk also for $\Lambda = 4\pi$, and the result was extended in [6] to general planar domains.

Struwe showed in [12] that for domains close to the disk in a suitable sense F possesses a local maximum for Λ slightly larger than 4π, and was able to find a second constrained critical point using a reversed mountain pass scheme (a.e. in Λ). This result was refined in [7], where the authors showed existence of a second critical point for all domains and Λ in a full right-neighborhood of 4π.

For larger values of Λ existence of constrained critical points strongly depends on the topology of the domain. In [8] it was proven that for the unit disk there exists $\Lambda^\sharp$ such that F has no constrained critical points to M_Λ for $\Lambda > \Lambda^\sharp$. On the other hand, in [4] the authors proved that if Ω is not contractible then there are constrained solutions developing two peaks if Λ converges to 8π from the right. This construction was extended to an arbitrary number of peaks with polygonal symmetry on round annuli. We have the following result.

Theorem 1. ([9]) *If Ω is non contractible and $\Lambda > 0$, F admits constrained critical points to M_Λ.*

A related result was proved in [3] for closed surfaces via a min-max method, and extending some blow-up analysis results in [5]. A crucial property is that blow-ups only may occur when Λ approaches multiples of 4π from the right, as proven in [10] for the radial case.

In [9] the Leray-Schauder degree of the resulting Euler-Lagrange equation was also computed using a homotopy argument, see [2] for Liouville equations.

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The anisotropic Bernstein problem

Connor Mooney
(joint work with Yang Yang)

A well-known theorem of Bernstein from 1915 asserts that any global solution on $\mathbb{R}^2$ to the minimal surface equation is a linear function. The Bernstein problem asks whether the same result holds with $\mathbb{R}^2$ replaced by $\mathbb{R}^n$. It is known through spectacular work of Bernstein, Fleming [12], De Giorgi [4], Almgren [1], Simons [21], and Bombieri-De Giorgi-Giusti [2] that the answer is positive if and only if $n \leq 7$. This body of work introduced a number of ideas (ϵ-regularity, monotonicity formula, stability inequality, Simons identity, etc.) that are pervasive in the study of elliptic PDEs and geometric variational problems.

It is natural to ask whether similar results hold for anisotropic minimal surfaces, which are critical points of functionals of the form

$$A_\Phi(\Sigma) = \int_\Sigma \Phi(\nu)\, dA. \tag{1}$$

Here $\Sigma \subset \mathbb{R}^{n+1}$ is an oriented hypersurface, ν is the unit normal to Σ, and Φ is an elliptic integrand, namely, a one-homogeneous function on $\mathbb{R}^{n+1}$ that is positive and smooth on $\mathbb{S}^n$, and satisfies in addition that $\{\Phi < 1\}$ is uniformly convex. Such functionals have attracted attention both for their applied and theoretical interest ([3], [5], [6], [7], [8], [9], [11], [14]). In particular, they arise in models of crystal surfaces, and they present important technical challenges not present for the area functional (especially due to the lack of a monotonicity formula).

The anisotropic Bernstein problem asks whether entire anisotropic minimal graphs in $\mathbb{R}^{n+1}$ are necessarily hyperplanes, for a general elliptic integrand Φ. The answer is positive in dimension $n = 2$ by work of Jenkins [13] and in dimension $n = 3$ by work of Simon [20]. We showed in [15] that the answer is negative in dimensions $n \geq 6$, leaving open the cases $n = 4, 5$. Finally, in recent work with Y. Yang we completed the solution to the anisotropic Bernstein problem by constructing nonlinear entire graphical minimizers in the case $n = 4$ ([17]):

Theorem 1. *There exists a smooth nonlinear function $u : \mathbb{R}^4 \to \mathbb{R}$ and an elliptic integrand Φ on $\mathbb{R}^5$ such that the graph of u in $\mathbb{R}^5$ minimizes A_Φ.*

We denote by C_{kl} the cone over $\mathbb{S}^k \times \mathbb{S}^l$ in $\mathbb{R}^{k+l+2}$. The first examples of nonlinear entire minimal graphs, constructed in [2], are asymptotic to $C_{kk} \times \mathbb{R} \subset \mathbb{R}^{2k+3}$, where $k \geq 3$. The method in [2] is to carefully construct super- and sub-solutions to the minimal surface equation, with appropriate ordering and symmetries, and then use the maximum principle. In [2] it is also shown that each side of C_{kk} ($k \geq 3$) is foliated by smooth minimal hypersurfaces. It is clear that these resemble the level sets of the minimal graphs, but no explicit connection is made.

The anisotropic minimal graph in [15] is asymptotic to $C_{22} \times \mathbb{R}$, but the approach to constructing it is completely different from [2]. In [15], the method is to first fix a choice of solution u, and then construct the integrand Φ by solving a linear *hyperbolic* equation. It turns out that for the choice $u = |x|^2 - |y|^2$, with $x, y \in \mathbb{R}^3$, the equation for Φ reduces after a change of variable to the two-variable wave equation. This makes the construction significantly shorter and more elementary than for the case of the area functional. Unfortunately the analogous choice for u in $\mathbb{R}^4$ does *not* solve an equation of minimal surface type, as shown in [15].

To solve the anisotropic Bernstein problem in lower dimensions we use the maximum principle, inspired by [2]. Our approach is to make explicit the connection between foliations of each side of the cones C_{kl} by anisotropic minimal hypersurfaces, and level sets of anisotropic minimal graphs. As a consequence we were able to construct, in the anisotropic case, examples with many different growth rates in the optimal dimension $n = 4$, and to recover the examples from [2].

Our approach consists of four steps. In the first step [16] we prove that the cones C_{kl} minimize functionals of the form (1) for all $k, l \geq 1$ by constructing foliations. Morgan [18] previously proved that C_{11} minimizes such a functional by constructing a calibration. Although constructing a foliation is more involved, the leaves in our foliation give a hint as to how to proceed in the anisotropic Bernstein problem. We showed in particular in [16] that for any $\mu \in (0, 1/2)$, there is an integrand $\overline{\Psi}$ such that the sides of C_{11} are foliated by minimizers of $A_{\overline{\Psi}}$ that closely resemble the level sets of functions that are homogeneous of degree $1 + \mu$, and smooth with non-vanishing gradient away from the origin.

The second step is to perturb the leaves in the foliation so that they have the same asymptotic behavior as before, but have strictly positive or negative anisotropic mean curvature. The new leaves then define functions $\overline{w}$ and $\underline{w}$ that are homogeneous of degree $1+\mu$, constant on the leaves, vanish on C_{11}, and by virtue of the curvature of their level sets serve as good candidates for super- and sub-solutions to the equation we eventually wish to solve.

The third step is to make a choice of integrand Φ on $\mathbb{R}^5$. Our choice agrees exactly with $\overline{\Psi}$ on $\{x_5 = 0\}$, and can be viewed as way of smoothly extending $\overline{\Psi}$ to $\mathbb{S}^4 \backslash \{x_5 = 0\}$. The case of the area functional suggests taking $\Phi^2|_{\{x_5=1\}} = 1 + \overline{\Psi}^2$, and our integrand indeed resembles this choice.

The final step is to "re-stack" the level sets of the functions $\overline{w}$ and $\underline{w}$ in a way that they become legitimate super- and sub-solutions to the equation of minimal surface type defined by Φ on one side of C_{11}. This is accomplished by composing $\overline{w}$ and $\underline{w}$ with appropriate concave, resp. convex one-variable functions. We can then proceed as in [2].

We conclude with several open questions. First, our approach works equally well to construct examples asymptotic to $C_{kl} \times \mathbb{R}$ for all $k = l \geq 1$, but the approach does not work when $k \neq l$, because the argument relies on the odd symmetry over C_{kk} of solutions to the PDE associated to Φ. It would be interesting to build anisotropic minimal graphs asymptotic to any $C_{kl} \times \mathbb{R}$. In [19] Simon constructs entire minimal graphs that are asymptotic to the cylinders over all of the area-minimizing Lawson cones, and the ideas in that work may be helpful. Second, it is known that minimal graphs satisfying the controlled growth condition $|\nabla u(x)| = o(|x|)$ are necessarily linear [10]. A key tool in the argument is the Simons identity for the Laplace of the second fundamental form, which has an anisotropic analogue (see [22]). Similar arguments might thus be used to prove controlled growth Bernstein theorems in the anisotropic case. Finally, we remark that when Φ is close to the area functional on $\mathbb{S}^n$ in a strong topology, e.g. C^3, the results are the same as in the area case. For example, the Bernstein theorem holds up to dimension $n = 7$ [20]. It would be interesting to weaken the topology required for such results, and our examples may shed light on this question.

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Ancient finite-entropy curve shortening flow

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(joint work with Dong-Hwi Seo, Wei-Bo Su, Kai-Wei Zhao)

Let $\gamma : \mathbb{R}^1 \times I \to \mathbb{R}^2$ be a smooth solution to

$$\gamma_t = \gamma_{ss},$$

where s is the arclength parameter. Then, we call $\mathcal{M} := \sup_{t\in I}(M_t, t)$, where $M_t := \gamma(\mathbb{R}^1, t)$, a curve shortening flow.

Along the flow, the Colding-Minicozzi's Entropy [3] monotone decreases.

$$\mathrm{Ent}(\Gamma) := \sup_{\lambda>0, y\in\mathbb{R}^2} \int_\Gamma (4\pi\lambda)^{-\frac{1}{2}} e^{\frac{|x-y|^2}{4\lambda}} ds(x).$$

Hence, any limit flow of a compact curve shortening flow at a singularity has finite-entropy. Although we already know that a limit flow from an embedded closed curve shortening flow is a shrinking circle, the study of the classification of ancient embedded finite-entropy flows can help us to understand the limit mean curvature flow.

To study the asymptotic behavior of finite-entropy flows, we recall the rescaled flow $\overline{M}_\tau := (-t)^{-\frac{1}{2}} M_t$, where $\tau := -\log(-t)$. Then, by the Huisken's monotonicity formula [4], compactness [5], and local regularity [6], we can show that the rescaled flow converges to a shrinker with multiplicity in the $C^\infty_{\rm loc}$-topology. Then, the embeddedness yields

Lemma. *As $\tau \to -\infty$, the rescaled flow converges in the C^∞_{loc}-topology to a line with multiplicity $m \geq 2$ up to suitable rotations, unless $\mathcal{M}$ is a static line or a shrinking circle.*

Next, the linearized operator $\mathcal{L}$ of a line is the Ornstein-Uhlenbeck operator.

$$\mathcal{L} := \frac{\partial^2}{\partial x^2} - \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2},$$

which has the integrable kernel $\ker \mathcal{L} = \operatorname{span}\{x\}$. Therefore, adopting the Allard-Almgren's proof [1] for the uniqueness of tangent cones at singularities of minimal surfaces, we can obtain the uniqueness of the tangent flow.

Theorem. *As $\tau \to -\infty$, the rescaled flow exponentially fast converges in the C^∞_{loc}-topology to a line with multiplicity $m \geq 2$, unless $\mathcal{M}$ is a static line or a shrinking circle.*

The exponential convergence allows us to deal with the noncompact tangent flow as like a compact one. Thus, we can solve the dynamics between the Gaussian L^2-norm of the projections P_+, P_0, P_- of profile $\vec{u}$ to unstable, neutral, and stable spaces, where

$$P_+\vec{u} = \langle \vec{u}, 1\rangle_{\mathcal{H}}, \qquad P_+\vec{u} = \tfrac{1}{2}\langle \vec{u}, x\rangle_{\mathcal{H}} x, \qquad P_-\vec{u} = \vec{u} - P_-\vec{u} - P_0\vec{u}$$

and

$$\langle f, g\rangle_{\mathcal{H}} := \int_{\mathbb{R}^1} fg(4\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{4}}\, dx,$$

so that we obtain

$$\vec{u}(x,\tau) = \vec{a}e^{\frac{\tau}{2}} + O(e^\tau)$$

on each compact interval. Namely, a nontrivial curve shortening flow M_t converges to the parallel lines $\{x_2 = \vec{a}\}$ after taking a suitable fixed rotation.

By using this sharp asymptotic behavior of the unnormalized flow, we can show that each tip converges to the corresponding Grim Reaper curve. Then, the embeddedness of the flow concludes

Theorem. *An ancient closed embedded finite-entropy curve shortening flow is a shrinking circle or a paper clip.*

Theorem. *An ancient noncompact complete embedded finite-entropy curve shortening flow is a graph over a fixed bounded open interval.*

The above theorem does not provide a classification result for noncompact flows. However, based on the properties we discussed, we can expect that such a flow would be one of the ancient trombones which are recently constructed in [2].

Conjecture. *An ancient noncompact complete embedded finite-entropy curve shortening flow is a Grim Reaper or an ancient trombone.*

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Willmore instability of inverted complete minimal surfaces with planar ends

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(joint work with Jonas Hirsch, Rob Kusner)

Let Σ be a smooth, closed, 2-dimensional manifold and $f : \Sigma \to \mathbb{R}^2$ an immersion, $n \geq 3$. We equip Σ with the induced metric $g = f^*\delta_{\mathbb{R}^n}$. Then the *Willmore functional* is defined as $\mathcal{W}(f) := \frac{1}{4}\int_\Sigma |\vec{H}|^2 d\mu_g$, where $\vec{H}$ is the mean curvature vector of f and $d\mu_g$ the induced area measure. This functional has a non-compact (ambient) invariance goup – isometries of $\mathbb{R}^n$, scaling and inversions at spheres. As a consequence, methods from calculus of variations are harder to apply, but interesting questions arise that are related to geometric PDEs of fourth order, geometric spectral theory, conformal geometry and minimal surface theory.

In the first half of the talk we gave an overview of results and open questions about the Willmore functional using methods from these four fields. We first recall the *Li-Yau inequality* [12, 9]: For each $a \in \mathbb{R}^n$, we have $\mathcal{W}(f) \geq 4\pi\sharp\{f^{-1}(a)\}$ with equality if and only if f is the inversion (at a sphere) of a complete minimal surface X with finite total curvature. Note that ∞ is mapped to the center of the inversion which rises the question whether a Willmore surface (i.e. a critical point of $\mathcal{W}$) on a punctured disc can be extended across the singularity. The relevant PDE is of fourth order (in f), quasilinear with a cubic nonlinearity in the lower order term:

$$\Delta^\perp \vec{H} + g^{ik}g^{jl}A^0_{ij}\langle A^0_{kl}, \vec{H}\rangle = 0,$$

where $\Delta^\perp$ is the Laplacian on the normal bundle and A^0 the tracefree second fundamental form. Bryant [4] confirmed the extendability of Willmore surfaces arising as inverted complete minimal surface X in $\mathbb{R}^3$ with embedded planar ends. For general Willmore surfaces, results were obtained in [10, 14, 2] showing also that extendability is not always possible – the inverted Catenoid is $C^{1,\alpha}$, but not $C^{1,1}$ – but the surface can be extended smoothly when certain residue vanish. Bryant showed that a Willmore surface in $\mathbb{R}^3$ with the topological type of a sphere or an $\mathbb{R}P^2$ is always an inverted minimal surfaces with embedded planar ends. As a consequence, the energy of these surfaces is quantized $\mathcal{W}(f_{\mathbb{S}^2}) = 4\pi m$, where in fact only $m \in \mathbb{N} \setminus \{2, 3, 5, 7\}$ can be realized [4, 5]. Later in this report we present results about the Willmore Morse index of these surfaces (also for general genus). In the overview of results related to $\mathcal{W}$, we also presented results and conjectures about minimizers of the Willmore functional for a fixed genus. Using the stereographic projection $st : \mathbb{S}^n \setminus \{p\} \to \mathbb{R}^n$, a computation shows

$$\mathcal{W}(f) = \int_\Sigma \left(\frac{1}{4} |\vec{H}_{st^{-1} \circ f}|^2 + 1 \right) d\mu_{g_{st^{-1} \circ f}},$$

which shows that minimal surfaces in $\mathbb{S}^n$ are Willmore surfaces after stereographic projection. In fact, the above formula makes it reasonable to think that the minimizer among surfaces of fixed genus is attained by a minimal surface in a sphere (except for the $\mathbb{R}P^2$-type with codimension one because there is no such minimal immersion into $\mathbb{S}^3$ [11]). This **conjecture** remains unsolved so far. Further results and open problems are:

- For every (orientable) genus $g \geq 1$ and every $n \geq 3$ there is a smooth minimizer $f_g^n : \Sigma_g \to \mathbb{R}^n$ which attains the infimum of the Willmore functional among all competitors of the same genus [15, 1]. So far, it is **unclear** whether the (orientable) minimizers for different codimension agree.
- The surfaces $\xi_{g,1}$ of Lawson [11] are conjectured to be the minimizers for orientable fixed genus g [9]. A weaker, but still **open questions** is: Is the minimum of $\mathcal{W}$ among surfaces with fixed oriented genus monotone in g?
- The minimizer for all orientable surfaces of genus $g \geq 1$ with codimension one is the Clifford torus [13]. This is an improved version of the Willmore conjecture saying that the Clifford torus is the minimizer among tori [16].
- Li and Yau [12] proved that $\mathcal{W}(f) \geq 6\pi$ for all immersions $f : \mathbb{R}P^2 \to \mathbb{R}^n$. Equality only holds for (stereographic projections of) the Veronese surface – an embedded minimal $\mathbb{R}P^2$ in $\mathcal{S}^4$.
- For every $n \geq 4$ there is a smooth immersed Klein bottle $f^n : K \to \mathbb{R}^n$ minimizing $\mathcal{W}$ among all Klein bottles [3]. A certain minimal Klein bottle in $\mathbb{S}^4$, named $\tilde{\tau}_{3,1}$ and originally found by Lawson [11], is the unique minimizer for $\mathcal{W}$ *in its conformal class* [7]. We **conjectured** that $\tilde{\tau}_{3,1}$ is indeed the unique minimizer for $\mathcal{W}$ for arbitrary Klein bottles as competitors [7].

In the second half of the talk we presented results from [8] and [6]. Recall the inverted minimal surfaces with planar ends are closed Willmore surfaces after compactification [4]. The Morse index $ind(f)$ is the number of negative eigenvalues of the Jacobi operator (corresponding to the second variation of $\mathcal{W}$ at a Willmore surface). Our three main theorems are the following:

Theorem [8]: *Let $X : \Sigma \setminus \{p_1, ..., p_m\} \to \mathbb{R}^3$ be a conformal complete minimal surface with embedded planar ends with $0 \notin image(X)$. Then the Willmore Morse index of the compactified $f := \frac{X}{|X|^2}$ is bounded above by*

$$ind(f) \leq m - d,$$

where $d := \dim\operatorname{span}\{n_X(p_1), ..., n_X(p_m)\}$ *and* $n_X : \Sigma \to \mathbb{S}^2$ *is the holomorphically extended Gauß map of* X*.*

Theorem [6]: *Let* $X : \Sigma \setminus \{p_1, ..., p_m\} \to \mathbb{R}^3$ *be a conformal complete minimal surface with embedded planar ends with* $0 \notin image(X)$ *and denote* $g_X := X^*\delta_{\mathbb{R}^3}$*. If there is a logarithmically growing area-Jacobi field of* X*, i.e. there is a function* $u \in C^{2,\alpha}(\Sigma\setminus\{p_1, ..., p_m\})$ *such that* $L_{g_X} u := \Delta_{g_X} u - 2K_{g_X} u = 0$ *on* $\Sigma\setminus\{p_1, ..., p_m\}$ *and* $u(z) = \beta_i \log|z - p_i| + \tilde{u}_i(z)$ *near* p_i *in local conformal coordinates* z *(*$\tilde{u}_i$ *is smooth and bounded across* p_i *and not all* $\beta_i \in \mathbb{R}$ *vanish), then the Willmore Morse index of the compactification of* $\frac{X}{|X|^2}$ *is at least one.*

Theorem [8, 6]: *Let* $f : \Sigma \to \mathbb{R}^3$ *be a Willmore sphere which is of the topological type of a sphere or an* $\mathbb{R}P^2$*. Then its Willmore Morse index is*

$$ind(f) = m - d = m - 3 = \frac{\mathcal{W}(f)}{4\pi} - 3.$$

In particular $d = \dim\operatorname{span}\{n_f(p_1), ..., n_f(p_m)\} = 3$ *for Willmore spheres and* $\mathbb{R}P^2$*s.*

Note that $\dim\operatorname{span}\{n_f(p_1), ..., n_f(p_m)\} = \dim\operatorname{span}\{n_X(p_1), ..., n_X(p_m)\}$ due to the conformality of the inversion $x \mapsto \frac{x}{|x|^2}$. Note also that upper index bounds on inverted minimal surfaces with planar ends are available in [8, 6].

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A diffused interface model for soap films

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(joint work with Michael Novack, Daniel Restrepo)

The Plateau laws postulate that soap films at equilibrium can be modeled as closed sets $S \subset \mathbb{R}^3$ that can be locally described either as smooth minimal surfaces or as diffeomorphic images of Y-cones or T-cones. A seminal regularity result of Taylor [5] shows that if S is an Almgren minimizing set, that is, if S is minimizing the 2-dimensional Hausdorff measure of $\mathbb{R}^3$ with respect to local Lipschitz deformations, then S satisfies the Plateau laws. The problem of proving the existence of Almgren minimizing sets with a prescribed boundary datum has been recently solved by Harrison and Pugh [3] with the introduction of a notion of "homotopic spanning", which makes precise what it means for a closed set to "span" another closed set, and which is suitable for the application of the Direct Method.

The motivation of our work [4] is a well-known result of Tonegawa and Wicramasekera [6] concerning the limit behaviors of solutions to the Allen–Cahn equation $\varepsilon^2 \Delta u_\varepsilon = W'(u_\varepsilon)$ (defined by a double-well potential W). Their theorem states that stable solutions to the Allen–Cahn equation with bounded Allen–Cahn energy

$$\mathcal{AC}_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon} \int_\Omega W(u)$$

converge, in the $\varepsilon \to 0$ limit, to *smooth* minimal surfaces (outside of a singular set with codimension greater or equal than 8). In particular, Plateau-type singularities cannot arise as limits of stable solutions to the Allen–Cahn equation.

Our main result is that this approximation is possible with the introduction of a natural volume constraint. More precisely we construct a three-levels hierarchy of Plateau-type problems, which consists: at the base level, of the Harrison–Pugh formulation of the Plateau problem; at the intermediate level, of Guass' sharp interface capillarity model under a small volume constraint and with an homotopic spanning satisfied by the "bulk part" of the region bound by the considered soap-films; and, at the top level, of a diffused interface capillarity model featuring the minimization of the Allen–Cahn energy of soap-film solution densities under a small volume constraint and with an homotopic spanning condition formulated on the super-level sets of the density.

Concretely, given a boundary wire frame $\mathbf{W}$ (a compact subset of $\mathbb{R}^{n+1}$ such that $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$ in an open set with smooth boundary), and given positive parameters v and ε such that $\varepsilon \ll v$, our approach leads to the construction of solutions u_ε to the free-boundary problem

$$\begin{cases} \varepsilon^2 \, \Delta u_\varepsilon = W'(u_\varepsilon) - \lambda(\varepsilon) \, V'(u_\varepsilon) \,, & \text{on } \Omega \setminus \{u_\varepsilon = 1\} \,, \\ |\nabla^+ u_\varepsilon| = |\nabla^- u_\varepsilon| \,, & \text{on } \Omega \cap \{u_\varepsilon = 1\} \end{cases}$$

that minimize the Allen–Cahn energy among functions u satisfying the volume constraint

$$v = \int_\Omega V(u) \,, \qquad V(t) = \Big(\int_0^t \sqrt{W} \Big)^{(n+1)/n} \,,$$

and such that $\{u \geq t\}$ is homotopically spanning $\mathbf{W}$ for every $t \in (1/2, 1)$.

We prove that, up to extracting subsequences, given a sequence of minimizers $u_j = u_\varepsilon$ corresponding to $\varepsilon = \varepsilon_j \to 0$ as $j \to \infty$ and such that $\int_\Omega V(u_j) = v$ for each j, there is a pair (K_v, E_v) where K_v is a relatively closed set in Ω, E_v is an open subset of Ω with $|E_v| = v$, $\Omega \cap \partial E_v$ is contained in K_v, $K \cap E_v = \emptyset$, and $K \cup E_v^{(1)}$ is homotopically spanning $\mathbf{W}$ (where $E^{(1)}$ denotes the set of points of density one of a set E). The convergence takes place in the sense that, as Radon measures in Ω,

$$\Big\{ \varepsilon_j \, |\nabla u_j|^2 + \frac{W(u_j)}{\varepsilon_j} \Big\} \, d\mathcal{L}^{n+1} \llcorner \Omega \rightharpoonup 2 \, \mathcal{H}^n \llcorner (K_v \setminus \partial^* E_v) + \mathcal{H}^n \llcorner (\Omega \cap \partial^* E_v)$$

as $j \to \infty$, where $\partial^* E$ denotes the reduced boundary of E. Moreover, (K_v, E_v) is a minimizer of the relaxed capillarity energy

$$\mathcal{F}(K, E) = 2 \, \mathcal{H}^n (K \setminus \partial^* E) + \mathcal{H}^n (\Omega \cap \partial^* E)$$

among pairs of Borel sets (K, E) such that $\Omega \cap \partial^* E$ is $\mathcal{H}^n$-contained in K, $|E| = v$, and $K \cup E^{(1)}$ is homotopically spanning $\mathbf{W}$. Finally, it is proved that, up to extracting subsequences, given a sequence of minimizers $(K_i, E_i) = (K_v, E_v)$ corresponding to $v = v_i \to 0$ as $i \to \infty$, there is a minimizer S in the Harrison–Pugh formulation of the Plateau problem for $\mathbf{W}$ such that, in these sense of Radon measures in Ω,

$$2 \, \mathcal{H}^n \llcorner (K_i \setminus \partial^* E_i) + \mathcal{H}^n \llcorner (\Omega \cap \partial^* E_i) \rightharpoonup 2 \, \mathcal{H}^n \llcorner S \,,$$

as $i \to \infty$. By extracting a diagonal subsequence along (j, i) in the above theorems, one obtains the desired diffused interface approximation of area minimizing surfaces with Plateau-type singularities.

From the technical viewpoint, the use of the Harrison–Pugh homotopic spanning condition on Sobolev spaces (which constitute the natural energy spaces for the Allen–Cahn energy) is particularly delicate, and it calls for a deep revision of the original notion of homotopic spanning condition which is stable under modifications by $\mathcal{H}^n$-null sets, and still amenable to the implementation of the Direct Method. A key role in formulating such measure-theoretic notion of homotopic spanning is played by the concept of essential connectedness introduced in [1, 2] in the study of rigidity theorems for symmetrization inequalities.

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Timelike Ricci bounds and Einstein's theory of gravity in a non smooth setting: an optimal transport approach

Andrea Mondino

(joint work with Fabio Cavalletti, Stefan Suhr)

Optimal transport tools have been extremely powerful to study Ricci curvature, in particular Ricci lower bounds. The characterization of Ricci curvature lower bounds for smooth Riemannian manifolds in terms of optimal transport was obtained by CorderoErausquin-McCann-Schmuckenschäger [3] and vonRenesse-Sturm [11], after work by McCann [7] and Otto-Villani [10]. A key feature of such a characterization is that it does not appeal to the smooth structure structure, and it makes sense in general metric measure spaces. The theory of metric measure spaces with lower Ricci bounds in a synthetic sense via optimal transport was pioneered independently by Sturm [12, 13] and Lott-Villani [6], and has been flourishing in the last 15 years.

Since the geometric framework of General Relativity is the one of Lorentzian manifolds (or space-times), and the Ricci curvature plays a prominent role in Einstein's theory of gravity, it is natural to expect that optimal transport tools may be useful also in this setting.

The goal of the talk is to introduce the topic and to report on recent progress.

The non-smooth Lorentzian setting is provided by Lorentzian pre-length spaces (introduced by Kunzinger-Sämann [5] after work of Kronheimer-Penrose [4]). The characterization of timelike Ricci lower bounds for smooth Lorentzian spacetimes in terms of optimal transport was obtained in parallel works by McCann [8] and Mondino-Suhr [9]; the motivation of both [8, 9] was to obtain an optimal transport formulation of timelike Ricci bounds that does not appeal to a smooth structure to be stated, so to provide a starting point for developing a theory of timelike Ricci bounds in Lorentzian pre-length spaces. Such a program was developed in a joint paper with Cavalletti [1], where:

- We set foundational results on optimal transport in Lorentzian pre-length spaces (including cyclical monotonicity, stability of optimal couplings and Kantorovich duality); several results are new even for smooth Lorentzian manifolds.
- We give a synthetic notion of "timelike Ricci curvature bounded below and dimension bounded above" for a measured Lorentzian pre-length space using optimal transport. The key idea, from the aforementioned smooth works [8, 9], is to analyse convexity properties of the Boltzmann-Shannon entropy functional along future directed timelike geodesics of probability measures. Such a condition is called $\mathsf{TCD}(K,N)$ standing for "Timelike Curvature Dimension" condition; $K \in \mathbb{R}$ plays the role of lower bound on the Ricci curvature in the timelike directions and $N \in (0,\infty)$ plays the role of upper bound on the dimension, both to be intended in a synthetic sense.
- This notion is proved to be stable under a Lorentzian analog of measured-Gromov-Hausdorff convergence, for measured Lorentzian pre-length spaces.
- As applications, we establish a synthetic version of Hawking singularity Theorem (in sharp form), and we extend to the non-smooth setting several volume comparison results (of Bishop-Gromov type).

In the final part of the talk, I presented some very recent developments obtained in collaboration with Cavalletti [2]. Most notably, a new isoperimetric-type inequality for timelike non-branching $\mathsf{TCD}(K,N)$ spaces.

For the sake of this extended abstract, I will briefly present the result in a simplified form in the smooth framework. For the general statement, see [2].

Let (M^n, g) be a smooth globally hyperbolic Lorentzian manifold. Let V, S be two Cauchy hypersurfaces, with $S \subset I^+(V)$, where

$$I^+(V) = \{y \in M \colon \exists x \in V \text{ such that } x << y\}$$

denotes the chronological future of V. Let

$$\begin{aligned} \tau_V(y) &:= \sup_{x\in V} \tau(x,y),\ \forall y \in I^+(V), \quad \text{time-separation from } V, \\ \mathrm{dist}(V,S) &:= \inf_{y\in S} \tau_V(y) \quad \text{"distance" from } V \text{ to } S, \\ C(V,S) &:= \{\gamma(t)\colon t\in[0,1],\ \text{s.t. } \gamma \text{ is a timelike maximizing geodesic with} \\ &\qquad\qquad \gamma(0)\in V,\ \gamma(1)\in S\}, \end{aligned}$$

i.e. $C(V,S)$ is region spanned by timelike maximzing geodesics from V to S.

Theorem [Cavalletti-M. 2023, [2]] Let (M^n, g) be a globally hyperbolic Lorentzian manifold satisfying Hawking-Penrose's strong energy condition (i.e. Ric ≥ 0 on timelike vectors). Let $V, S \subset M$ be Cauchy hypersurfaces with $S \subset I^+(V)$. Then

$$\mathrm{Area}(S)\ \mathrm{dist}(V,S) \leq n\ \mathrm{Vol}(C(V,S)).$$

We actually prove a more general result, holding for any timelike Ricci lower bound $K \in \mathbb{R}$, and in the higher generality of timelike non-branching Lorentzian pre-length spaces satisfying the $\mathsf{TCD}(K,N)$ condition. Also the assumptions on V

can be relaxed: it is enough to assume that V is a Borel, achronal, future timelike complete subset.

The inequality is sharp (as equality is achieved in conical regions of model spaces: de Sitter in positive curvature, Minkowski in zero curvature, anti-de Sitter in negative curvature), and seems to be new even for smooth space-times satisfying the Hawking-Penrose strong energy condition. When specialised to convenient domains, it gives apparently new information to the geometry of cosmological spacetimes and interior regions of black holes. The proof is obtained via disintegration by Lorentzian geodesics maximising τ_V and localising the $\mathsf{TCD}(K,N)$ condition to each geodesic.

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A new varifold solution concept for mean curvature flow

Tim Laux
(joint work with Sebastian Hensel)

Folklore says that mean curvature flow is the "formal" gradient flow of the area functional with respect to the standard L^2 metric tensor on normal velocities. Indeed, for a smoothly evolving surface $(\Sigma(t))_{t\in[0,T)}$, being a mean curvature flow (i.e., $V=-H$ holds on $\Sigma(t)$ for all $t\in(0,T)$, where V denotes the normal velocity and H the mean curvature of $\Sigma(t)$) is equivalent to the energy dissipation relation

$$\text{Area}(\Sigma(T)) + \frac{1}{2}\int_0^T \int_{\Sigma(t)} (V^2 + H^2)\, dS\, dt \le \text{Area}(\Sigma(0)). \tag{1}$$

In other words, formally, all information on the evolution equation is contained in a single inequality. Here, the non-trivial implication follows from inserting the change of area formula $\frac{d}{dt}\text{Area}(\Sigma(t)) = \int_{\Sigma(t)} VH\, dS$ into (1), which yields $\frac{1}{2}\int_0^T \int_{\Sigma(t)} (V+H)^2\, dS\, dt \le 0$, and hence $V = -H$ on $\Sigma(t)$ for all $t \in (0,T)$.

However, this computation is only formal and does not apply to weak solutions. In fact, Brakke flows exhibit a complete failure of uniqueness as they may, for example, jump to the empty set at any given instant: For any fixed $t_0 \in (0,T)$, if $(\Sigma(t))_{t\in[0,T)}$ is a Brakke flow, then so is $(\widetilde{\Sigma}(t))_{t\in[0,T)}$ with $\widetilde{\Sigma}(t) := \Sigma(t)$ for $t \le t_0$ and $\widetilde{\Sigma}(t) := \emptyset$ for $t > t_0$. Another severe obstacle in finding a sound solution concept is that certain symmetric singular configurations lead to physical non-uniqueness. Take for example the symmetric cross $\Sigma(0) = \{x_1x_2 = 0\} \subset \mathbb{R}^2$ viewed as the boundary of the union of the first and third quadrants $\Omega(0) = \{x_1x_2 > 0\}$, which has two symmetric outflows, either connecting the two components of $\Omega(0)$ or disconnecting them.

Our goal here is to rigorously interpret mean curvature flow as a gradient flow in light of De Giorgi's vision for gradient flows despite the fact that the induced distance on the space of surfaces is completely degenerate. This is a challenge as the abstract theory of gradient flows works in great generality, but requires the state space to be a metric space and therefore does not apply here.

In this talk, the new weak solution concept from [2] is presented, which characterizes solutions in terms of an energy dissipation inequality just like (1). It is phrased in terms of (i) a measure $\mu = \mu_t \otimes dt = \omega_t \otimes \lambda_{x,t} \otimes dt$ on space, directions and time describing a one-parameter family of oriented varifolds which encode the evolving surfaces $\Sigma(t)$; and (ii) a characteristic function $\chi = \chi(x,t)$ on space and time encoding the volume $\Omega(t)$ enclosed by the surface. The natural counterpart of (1) then is

$$\omega_T(\mathbb{R}^d) + \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} (V^2 + |\mathrm{H}|^2)\, d\omega_t\, dt \le \omega_0(\mathbb{R}^d). \tag{2}$$

Here, morally, the normal velocity V is encoded by a transport equation for the enclosed volume and the mean curvature vector H by the change of area formula.

The former is a modified version of Rayleigh's theorem inspired by the weak formulation of Luckhaus and Sturzenhecker [4] in the BV setting of sets of finite perimeter, in which case it reads $\partial_t\chi + V|\nabla\chi| = 0$. In our setting, the total variation measure $|\nabla\chi|$ is replaced by the base measure ω of the varifold so that the transport equation becomes

$$\chi_0\, dx \otimes \delta_{\{t=0\}} + \partial_t\chi + V\omega = 0. \tag{3}$$

Here, we also allow for test functions that do not vanish at $t = 0$ in order to encode the initial condition $\chi(0) = \chi_0$ in a weak form.

The latter is completely standard in the geometric analysis community and states in physical terms that the forces can be written as the divergence of the stress tensor, i.e., we have the distributional relation

$$\mathrm{H}\,\omega = \nabla\cdot\mathbf{T}, \quad \text{where} \quad \mathbf{T}(x,t) = \int_{\mathbb{S}^{d-1}} (I_d - p\otimes p)\mu(x,dp,t). \tag{4}$$

The only additional condition is the following compatibility between the normals of the varifold and the set of finite perimeter

$$\nabla\chi = \int_{\mathbb{S}^{d-1}} p\,\mu(x,dp,t). \tag{5}$$

Since the central part of [2] is the solution concept itself, we record the precise weak formulation here.

Definition. *A couple (μ,χ) – with $\mu = \omega_t\otimes\lambda_{x,t}\otimes dt \in \mathcal{M}(\mathbb{R}^d\times\mathbb{S}^{d-1}\times(0,\infty))$ and $\chi\in L^1_{\mathrm{loc}}(\mathbb{R}^d\times(0,\infty);\{0,1\})$ which are compatible according to* (5) *– is a De Giorgi solution with initial data (ω_0,χ_0) if* (2) *holds for a.e. $T\in(0,\infty)$, where V and* H *are given by the distributional relations* (3) *and* (4)*, respectively.*

For a smoothly evolving surface $\Sigma(t) = \partial\Omega(t)$, we may set $\omega_t = \mathcal{H}^{d-1}\llcorner\partial\Omega(t)$, $\lambda_{x,t} = \delta_{\nu_{\Sigma(t)}(x)}$, and $\chi(\cdot,t) = \chi_{\Omega(t)}(\cdot)$; hence a smooth De Giorgi solution according to our definition satisfies (1) and is therefore a classical mean curvature flow. Furthermore, we stress that the initial conditions are only encoded through the first term in (3) and the right-hand side of (2). We also mention that it is natural to work with the mean curvature vector instead of the scalar mean curvature as we allow for general varifolds for which the mean curvature vector may not point in normal direction. Finally, it is easy to see that the definiton directly implies Hölder continuity of the volumes, $\int_{\mathbb{R}^d} |\chi(x,t)-\chi(x,s)|\,dx \le \sqrt{2}\,\omega_0\sqrt{t-s}$ for all $0\le s<t<\infty$, so that the example of sudden vanishing above is ruled out.

Our weak formulation of mean curvature flow allows soft existence and convergence results as one only needs to prove lower bounds for the terms on the left-hand side of (2). As an example, in [2] we show that in the setting of Ilmanen's fundamental work [3], solutions of the Allen–Cahn equation converge to these solutions. More precisely, denoting by $u_\varepsilon = u_\varepsilon(x,t)$ the solution to the Allen–Cahn equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \tfrac{1}{\varepsilon^2}W'(u_\varepsilon), \quad \text{where } W(u) := \tfrac{1}{18}(1-u)^2u^2, \tag{6}$$

we have the following convergence result.

Theorem. *For well-prepared initial data $u_\varepsilon(0)$ in the sense of Ilmanen* [3]*, the solutions u_ε are precompact in $L^1_{\mathrm{loc}}(\mathbb{R}^d\times(0,\infty))$ and the associated oriented varifolds μ^ε are precompact as measures. Furthermore, any limit point (μ,χ) is a De Giorgi solution with $\omega_0 = \lim_\varepsilon\big(\frac{\varepsilon}{2}|\nabla u_\varepsilon(\cdot,0)|^2 + \frac{1}{\varepsilon}W(u_\varepsilon(\cdot,0))\big)dx$ and $\chi_0 = \lim_\varepsilon u_\varepsilon(0)$.*

The varifolds $\mu^\varepsilon = \mu_t^\varepsilon \otimes dt$ above are naturally related to the energy measure $e_t^\varepsilon = \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon(\cdot,t)|^2 + \frac{1}{\varepsilon}W(u_\varepsilon(\cdot,t))\big)dx$. It is convenient to work with the base measure $\omega_t^\varepsilon := \sqrt{2W(u_\varepsilon(\cdot,t))}|\nabla u_\varepsilon(\cdot,t)|\,dx \le e_t^\varepsilon$ which is suggested by the Modica–Mortola/Bogomol'nyi-trick, and to set $\mu_t^\varepsilon := \omega_t^\varepsilon \otimes \delta_{\nu_\varepsilon(x,t)}$, where $\nu_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$.

Strikingly and in stark contrast to Brakke's formulation, these evolving varifolds satisfy a weak-strong uniqueness principle – even without rectifiablity (let alone integrality) assumptions. Precisely, we show that the following relative energy

$$E(t) := E\big[(\mu,\chi),(\Sigma,\Omega)\big](t) := \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (1-\xi(x,t)\cdot p)\,d\mu_t(x,p)$$

satisfies an integral version of the estimate $\frac{d}{dt}E(t) \le CE(t)$. Here, the vector field $\xi(\cdot,t)$ is a suitable extension of the normal vector field of $\Sigma(t)$, reminiscent of calibrations in the static setting of minimal surface theory.

Theorem. *Let $\Sigma(t) = \partial\Omega(t)$, $t \in [0,T]$ be a smooth mean curvature flow and (μ,χ) be a De Giorgi solution. Then there exists $C = C((\Sigma(t))_{t\in[0,T]}) < \infty$ such that*

$$E\big[(\mu,\chi),(\Sigma,\Omega)\big](t) \le e^{Ct}E\big[(\mu,\chi),(\Sigma,\Omega)\big](0) \quad \textit{for a.e. } t \in (0,T).$$

Moreover, if $\chi_0 = \chi_{\Omega(0)}$ and $\omega_0 = \mathcal{H}^{d-1}\llcorner\Sigma(0)$, then we have $\chi(\cdot,t) = \chi_{\Omega(t)}$, $\omega_t = \mathcal{H}^{d-1}\llcorner\Sigma(t)$, and $\lambda_{\cdot,t} = \delta_{\nu_{\Sigma(t)}(\cdot)}$ for a.e. $t \in (0,T)$.

Hence, our De Giorgi solutions do not exhibit the unphysical non-uniqueness of Brakke flows although they are evolving varifolds merely satisfying a single inequality. This second result builds on and extends recent work together with Fischer and Simon [1], which at the time was only valid for BV solutions as in [4].

One of the motivations for new solution concepts is the multiphase setting to which our results can be partially generalized: A suitable solution concept and the corresponding weak-strong uniqueness result is given in the same work [2] presented here, and – under an energy convergence assumption – Steinke [5] showed the convergence of the vectorial Allen–Cahn equation to a De Giorgi solution. Unconditional convergence, however, remains a challenging open problem.

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Hypersurfaces with mean curvature prescribed by an ambient function

Costante Bellettini
(joint work with Kobe Marshall-Stevens, Neshan Wickramasekera, Myles Workman)

In [6], joint work with N. Wickramasekera (Cambridge), we establish largely by PDE methods the following existence result for prescribed-mean-curvature hypersurfaces:

Theorem 1. *Let (N, h) be a compact Riemannian manifold of dimension $n+1$, $n \geq 2$, and let $g \geq 0$ be a Lipschitz function on N. There exists an immersed hypersurface $M \subset N$ of class C^2 such that (i) M is two sided, i.e. there is a global choice of unit normal ν, (ii) $dim_{\mathcal{H}}(\overline{M} \setminus M) \leq n-7$, where $dim_{\mathcal{H}}$ denotes the Hausdorff dimension, (iii) the mean curvature vector of M is given by $g\nu$. More precisely, M is quasi-embedded, i.e. for every $p \in M$ around which M fails to be embedded, there exists a neighbourhood $\mathcal{N}_\rho(p) \subset N$ such that $M \cap \mathcal{N}_\rho(p) = D_1 \cup D_2$, where D_1 and D_2 are embedded C^2 disks lying on one side of each other and intersecting tangentially (p is contained in the intersection).*

Remark 1. *When $n \leq 6$, M is closed. The failure of M to be closed can only arise for $n \geq 7$, due to the possible presence of 'singular points'. A point p is singular if $\overline{M}$ fails to be immersed in any neighbourhood of p. (It follows from the proof that $\overline{M}$ admits only non-planar tangent cones at singular points.)*

When $n = 7$ the singularities are, more precisely, isolated points.

A hypersurface M with mean curvature prescribed by g is naturally a critical point for an area-type functional. Easy examples show that it is in general a saddle-type critical point, not a minimiser, therefore minmax methods lend themselves to the construction.

The Almgren–Pitts minmax method was originally developed to solve the case $g \equiv 0$ of the above theorem, see [1], [12]. The relevant functional is defined on a non-linear space and does not satisfy a Palais–Smale condition; substantial technical machinery is required to compensate for these drawbacks.

A new, more direct proof for the case $g \equiv 0$ of Theorem 1 has been given in recent years, by means of what we will refer to as 'Allen–Cahn minmax'. This was carried out by Guaraco [10] relying on works by Hutchinson, Tonegawa, Wickramasekera [11] [16] [17]. The idea is to replace the area functional with a regularised version $\mathcal{E}_\epsilon$ of it, defined for $\epsilon > 0$ on the Hilbert space $W^{1,2}(N)$: $\mathcal{E}_\epsilon(u) = \frac{1}{2\sigma}\left(\int_N \epsilon\frac{|\nabla u|^2}{2} + \int_N \frac{W(u)}{\epsilon}\right)$, where $W : \mathbb{R} \to [0, \infty)$ is a C^2 'double well' potential, i.e. with two nondegenerate global minima at -1 and $+1$, with $W(\pm 1) = 0$, and $\sigma > 0$ is a normalising constant (determined by W). In [11] it is shown that "critical points of $\mathcal{E}_\epsilon$ converge to critical points of area (stationary integral varifolds) as $\epsilon \to 0$". In [10] a minmax construction is carried out for $\mathcal{E}_\epsilon$, for each $\epsilon > 0$, in $W^{1,2}(N)$, capitalising on the validity of the Palais–Smale condition (so a standard mountain pass lemma from classical PDE theory can be brought to bear); then a suitable limit as $\epsilon \to 0$ yields a stationary integral varifold ([11]); its smooth embeddedness away from a set of dimension $\leq n-7$ is obtained thanks to the regularity theory in [16], [17].

We employ an Allen–Cahn minmax in establishing Theorem 1. (We refer to [18], [19] for Almgren–Pitts minmax constructions of prescribed-mean-curvature hypersurfaces.) The starting point in [6] is again a classical mountain pass construction in $W^{1,2}(N)$, for each $\epsilon > 0$, for the energy $\mathcal{F}_{\epsilon,g}(u) = \mathcal{E}_\epsilon(u) - \frac{1}{2}\int gu$. Unlike in the case $g \equiv 0$, however, it is not necessarily true that "critical points of $\mathcal{F}_{\epsilon,g}$ converge to hypersurfaces with mean curvature g as $\epsilon \to 0$" (even in the case $g \equiv \text{cnst} > 0$). This (major) difficulty (already present in the case in which g is a positive constant) is caused by an underlying phenomenon of *cancellation of first variation* paired with the creation of *high (even) multiplicity.* This phenomenon arises, in the Allen–Cahn setting (see [11]), even for solutions u_ϵ that depend on one fixed variable (in which case, the level sets of the u_ϵ are hyperplanes, so that the level set geometry is as easy a it could be).

The strategy in [6] does not require to address the question of whether it may be possible to rule out the formation of minimal portions for interfaces arising from stable (or, more generally, with bounds on the index) Allen–Cahn solution with $g > 0$ (possibly under some metric assumption).

In recent joint work with M. Workman (UCL) [7] we consider a more special instance of Theorem 1 and obtain the following refinement:

Theorem 2. *Assume, further to the hypotheses of Theorem 1, that h has positive Ricci curvature and $g \equiv \lambda \in (0,\infty)$. Then the interface M arising from the minmax sequence is smoothly embedded with constant-mean-curvature λ (and multiplicity 1).*

In particular, no cancellation phenomenon happens in this case, giving some (albeit small) evidence that a more general result in this direction may be valid.

Exploiting the embeddedness (rather than quasi-embeddedness) obtained in Theorem 2, in recent joint work with K. Marshall–Stevens (UCL) [3], we show that any isolated singularity that appears in M of Theorem 2 must admit a neighbourhood in which M fulfils a a minimising property (for the natural area-type functional). This minimisation property is further exploited in [3] to obtain (via standard surgery) the following result (we recall that for $n = 7$ only isolated singularities can arise in Theorem 1). (The case $\lambda \equiv 0$ of this result is obtained in [8], working within the Almgren–Pitts framework.)

Theorem 3. *With hypotheses as in Theorem 2, except for $n = 7$ and $\lambda \in [0,\infty)$: for any λ, there exists a generic set of Ricci-positive metrics on N for which there exists a smoothly embedded closed (i.e. without any singular points) CMC hypersurface with mean curvature λ.*

The cancellation and multiplicity phenomenon encountered above also arises in a related and purely geometric setting. One can construct a sequence of CMC boundaries in $\mathbb{R}^3$ with scalar mean curvature constantly equal to 1 (with mean curvature vector pointing inward) in such a way that the varifold limit of this sequence is a double plane (see [9] [13]).

From [14], [15], or [4], [5], we know that uniform Morse index bounds (in addition to custumary mass bounds) would prevent the cancellation phenomenon in this geometric setting. In [2] we identify a weaker assumption that rules it out. We obtain (the following is a special instance of the result in [2]):

Theorem 4. *Assume that ∂E_ℓ are smooth boundaries in N^{n+1}, with uniformly bounded n-area, and mean curvature λ (that is $\vec{H}_\ell = \lambda \nu_{E_\ell}$). Assume a uniform bound on $\int_{\partial E_\ell} |A_{\partial E_\ell}|^q$ for some $q > 1$. Then (subsequentially) ∂E_ℓ converge to the boundary (of a Caccioppoli set E) with (generalised) mean curvature λ (that is, $\vec{H} = \lambda \nu_E$ on the reduced boundary of E).*

The proof relies on treating ∂E_ℓ as 'oriented varifolds with curvature'.

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