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Homotopy Theory

Organized by
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Abstract. The workshop brought together experts in homotopy theory from many areas, including chromatic homotopy theory, algebraic K-theory, derived algebra and equivariant homotopy theory. We had a lecture series on the recent disproof of the telescope conjecture. In addition to a total of 24 research talks we had two gong-shows where participants presented their research in 10-minute talks.

Mathematics Subject Classification (2020): 55Pxx.

Introduction by the Organizers

The workshop *Homotopy Theory* was organised by Jesper Grodal (Copenhagen), Michael Hill (Los Angeles), and Birgit Richter (Hamburg) and it was well attended with 48 in-person participants and 4 online participants representing a number of countries around Europe and the world. Participants from all career stages attended, ranging from advanced graduate students to senior faculty, and the workshop also represented almost all research areas in homotopy theory. The workshop consisted of 24 talks and two gong shows. The talks ranged in length from 30 minutes to an hour. At the evening gong-shows the atmosphere was very lively. In these short 10-minute talks 17 participants presented their research and open problems. All of the talks described cutting-edge research in homotopy theory and were the starting point for many discussions.

This summer saw a surge in in-person conferences and many participants had a busy conference schedule. Still, as usual, the excellent atmosphere at Oberwolfach lead to many discussions during coffee breaks, meals, evenings and during hikes.

A focus was the recent announcement in Oxford in June of the disproof of the telescope conjecture which has a huge impact on our view of the structure of chromatic homotopy theory. We had a lecture series on that topic by Robert Burklund, Jeremy Hahn, Ishan Levy and Tomer Schlank. Other featured topics were chromatic homotopy theory, algebraic K-theory, derived algebra and equivariant homotopy theory.

Workshop: Homotopy Theory

Table of Contents

Abstracts

K-theory and the telescope conjecture

Ishan Levy
(joint work with Robert Burklund, Jeremy Hahn, Tomer Schlank)

In this talk, I sketch the simplest case of our recent disproof of Ravenel's telescope conjecture which is stated below:

Conjecture 1 (Ravenel, 1977)**.** The inclusion $\mathrm{Sp}_{K(n)} \subset \mathrm{Sp}_{T(n)}$ is an equality.

The telescope conjecture was the last remaining open conjecture of a series of influential conjectures of Ravenel [1] describing the chromatic picture of the category of spectra. The rest of the conjectures were resolved shortly after the nilpotence theorem of Devinatz–Hopkins–Smith [2].

For $n = 1$, the telescope conjecture was proven by Mahowald [3] for $p = 2$ using bo-resolutions, and by Miller [4] for $p > 2$ using the localized Adams spectral sequence.

We use algebraic K-theory to show that the telescope conjecture is false for $n \geq 2$. In the case $n = 2$, the result is as follows:

Theorem 1. $L_{T(2)}K(L_{K(1)}\mathbb{S})$ is not $K(2)$-local, and hence a counterexample to the height 2 telescope conjecture.

A consequence of the above result, which is additionally joint with Shachar Carmeli and Lior Yanovski, is the following, which is the best known lower bound on the asymptotic growth rate of the mod p stable stems:

Theorem 2. Let $f(n)$ be the average rank from $i = 1$ to n of $\pi_i(\mathbb{S})/p$. Then $f(n) \to \infty$ as $n \to \infty$.

The goal of this talk is to explain the main ideas in the simplest version of our disproof, so we assume $n = 2, p \geq 7$ from now on. The condition $p \geq 7$ guarantees that the Smith–Toda complex $V(2) = \mathbb{S}/(p, v_1, v_2) = ((\mathbb{S}/p)/v_1)/v_2$ exists and is a homotopy commutative and associative ring spectrum.

The $K(1)$-local sphere is the homotopy fixed points of the Adams summand L of p-complete complex K-theory by the $\mathbb{Z}$-action coming from the Adams operation Ψ^{1+p}. A major input in our proof is the following result, which is a consequence of cyclotomic redshift, the subject of Shachar's talk:

Theorem 3 (Ben-Moshe–Carmeli–Schlank–Yanovski)**.** There is an equivalence

$$L_{K(2)}K(L^{h\mathbb{Z}}) \cong L_{K(2)}K(L)^{h\mathbb{Z}}$$

To prove Theorem 1, we show that the map

$$L_{T(2)}K(L^{h\mathbb{Z}}) \to L_{T(2)}K(L^{h\mathbb{Z}})$$

is not an equivalence.

We first explain how the K-theory of L can be understood. Let ℓ denote the connective cover of L. Then there is a localization sequence of categories

$$\mathrm{Mod}(\ell)^{\omega, v_1-\mathrm{nil}} \to \mathrm{Mod}(\ell)^{\omega} \to \mathrm{Mod}(L)^{\omega}$$

that induces a cofibre sequence of K-theory spectra. The category on the left has a bounded t-structure, so applying Barwick's theorem of the heart [5] (which is a form of devissage) we obtain a cofibre sequence, due to Blumberg–Mandell [6]:

$$K(\mathbb{Z}_p) \to K(\ell) \to K(L)$$

According to a theorem of Mitchell [7], $K(\mathbb{Z}_p)$-vanishes $T(2)$-locally, so we get to replace L with ℓ. The advantage of ℓ over L is that its K-theory can be understood via trace methods. Namely, the Dundas–Goodwillie–McCarthy (DGM) theorem [8] gives a pullback square of commutative ring spectra

$$\begin{tikzcd} K(\ell) \arrow[r] \arrow[d] & \mathrm{TC}(\ell) \arrow[d] \\ K(\pi_0\ell) \arrow[r] & \mathrm{TC}(\pi_0\ell) \end{tikzcd}$$

Using Mitchell's theorem again, we learn that the bottom two terms vanish $T(2)$-locally. Overall, we obtain an equivalence

$$L_{T(2)}K(L) \cong L_{T(2)}\mathrm{TC}(\ell)$$

In my paper on the K-theory of the $K(1)$-local sphere [9], I worked out how to make an analog of all of the above work with L replaced by $L^{h\mathbb{Z}} = L_{K(1)}\mathbb{S}$. One difference is that the map $\mathrm{Mod}(\ell^{h\mathbb{Z}})^{\omega} \to \mathrm{Mod}(L^{h\mathbb{Z}})^{\omega}$, though still a localization map, has kernel the subcategory of modules that are both p and v_1-nilpotent. Constructing the bounded t-structure is a bit more subtle, and can be done using [10].

The other key difference is that $\ell^{h\mathbb{Z}}$ is not a connective ring, but is only -1-connective, and so the DGM theorem doesn't apply. It is observed in [9] that the DGM actually extends to certain maps of -1-connective rings including the map $\ell^{h\mathbb{Z}} \to (\pi_0\ell)^{h\mathbb{Z}}$. In this case that result can be proven using [11].

Summarizing, we are now reduced to proving that the map

$$L_{T(2)}\mathrm{TC}(\ell^{h\mathbb{Z}}) \to L_{T(2)}\mathrm{TC}(\ell)^{h\mathbb{Z}}$$

is not an equivalence.

Since the telescope $T(2)$ can be chosen to be $\mathbb{S}/(p, v_1)[v_2^{\pm 1}]$, we only need to understand everything modulo p and v_1.

The first step in understanding $\mathrm{TC}/(p, v_1)$ is to understand $\mathrm{THH}/(p, v_1)$. This was first worked out by McClure–Staffeldt [12] for ℓ, but we present a different proof.

The Adams filtration on ℓ has associated graded the polynomial algebra $\mathbb{F}_p[v_0, v_1]$. THH satisfies base change with respect to symmetric monoidal functors, so we learn that there is a filtration on $\mathrm{THH}(\ell)$ whose associated graded is $\mathrm{THH}(\mathbb{F}_p[v_0,v_1])$. This filtration in particular gives rise to a spectral sequence

$$E_1 = \pi_*\mathrm{THH}(\mathbb{F}_p[v_0,v_1])/(v_0,v_1) \implies \pi_*\mathrm{THH}(\ell)/(p,v_1)$$

Symmetric monoidality and base change properties of THH give equivalences

$$\mathrm{THH}(\mathbb{F}_p[v_0,v_1] \cong \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{HH}(\mathbb{F}_p[v_0,v_1]/\mathbb{F}_p)$$

At the level of homotopy rings, the first term in the tensor product is $\mathbb{F}_p[\sigma^2 p]$ with $|\sigma^2 p| = 2$ by Bökstedt's theorem, and the latter term is $\mathbb{F}_p[v_0,v_1]\langle dv_0, dv_1\rangle$ by the HKR theorem. Running the spectral sequence mod (v_0,v_1), there are two differentials $d_1\sigma^2 p = dv_0, d_p(\sigma^2 p)^p = dv_1$ which are propogated via the Leibniz rule, and the resulting answer is that

$$\pi_*\mathrm{THH}(\ell)/(p,v_1) \cong \mathbb{F}_p[\mu]\langle \lambda_1, \lambda_2\rangle$$

with $|\mu| = 2p^2, |\lambda_i| = 2p^i - 1$.

David Lee and I worked out in [13] how to compute $\mathrm{THH}(\ell^{h\mathbb{Z}})/(p,v_1)$, i.e add $\mathbb{Z}$-fixed points to the above computation. By taking the $\mathbb{Z}$-fixed points of the Adams filtration on ℓ, we get a filtration on $\ell^{h\mathbb{Z}}$ whose associated graded is $\mathbb{F}_p[v_0,v_1] \otimes_{\mathbb{F}_p} \mathbb{F}_p^{h\mathbb{Z}}$. This is because at the level of the associated graded of the Adams filtration, the Adams operation Ψ^{1+p} acts *trivially.* Thus we get $\mathrm{THH}(\mathbb{F}_p[v_0,v_1] \otimes_{\mathbb{F}_p} \mathbb{F}_p^{h\mathbb{Z}}) \cong \mathrm{THH}(\mathbb{F}_p[v_0,v_1]) \otimes_{\mathbb{F}_p} \mathrm{HH}(\mathbb{F}_p^{h\mathbb{Z}}/\mathbb{F}_p)$

Let us discuss the second tensor factor. We can view $\mathbb{F}_p^{h\mathbb{Z}}$ as the ring $C^*(B\mathbb{Z}_p;\mathbb{F}_p)$ of continuous cochains on $B\mathbb{Z}_p$ with values in $\mathbb{F}_p$. Since Hochschild homology takes cochain algebras on a p-profinite space X to the cochain algebra on the free loop space LX, we learn that $\mathrm{HH}(\mathbb{F}_p^{h\mathbb{Z}}/\mathbb{F}_p) \cong C^*(LB\mathbb{Z}_p;\mathbb{F}_p)$. The free loop space of $B\mathbb{Z}_p$ is $B\mathbb{Z}_p \times \mathbb{Z}_p$, so we obtain $\mathrm{HH}(\mathbb{F}_p^{h\mathbb{Z}}/\mathbb{F}_p) \cong \mathbb{F}_p^{h\mathbb{Z}} \otimes C^0(\mathbb{Z}_p;\mathbb{F}_p)$, where $C^0(\mathbb{Z}_p;\mathbb{F}_p)$ is the ring of continuous functions from $\mathbb{Z}_p$ to $\mathbb{F}_p$, which we from now on shorten to $C^0(\mathbb{Z}_p)$.

Furthermore, the map $\mathrm{HH}(\mathbb{F}_p^{h\mathbb{Z}}/\mathbb{F}_p) \to \mathrm{HH}(\mathbb{F}_p/\mathbb{F}_p)^{h\mathbb{Z}}$ at the level of π_* is the map $C^0(\mathbb{Z}_p)\langle\zeta\rangle \to \mathbb{F}_p\langle\zeta\rangle$, where $|\zeta| = 1$, which sends ζ to itself and a continuous function f to its value at 0. Our ultimate goal is to see that TC doesn't commute with $\mathbb{Z}$-fixed points, and this is the place where that phenomenon originates.

David and I showed that this $\mathrm{HH}(\mathbb{F}_p^{h\mathbb{Z}}/\mathbb{F}_p)$-tensor fact doesn't interact much with the rest of the spectral sequence, so that one learns that

$$\pi_*\mathrm{THH}(\ell^{h\mathbb{Z}})/(p,v_1) \cong \pi_*\mathrm{THH}(\ell)/(p,v_1) \otimes \mathrm{HH}_*(\mathbb{F}_p^{h\mathbb{Z}}/\mathbb{F}_p)$$

In other words, at the level of the homotopy ring, $\mathrm{THH}(\ell^{h\mathbb{Z}})/(p,v_1)$ behaves as if the $\mathbb{Z}$-action on ℓ were trivial.

A key point in our disproof of the telescope conjecture is to find a sense in which the phenomenon of behaving like a trivial $\mathbb{Z}$-action persists to the level of the $T(2)$-local TC.

The first observation is the following:

Observation 1. All proofs so far go through if we replace the group $\mathbb{Z}$ that acts with the subgroup $p^k\mathbb{Z}$ for any $k \geq 0$.

The next observation is the key to gaining control over the TC:

Observation 2. For any finite computation towards $\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$, the answer is formally determined by the one for ℓ if we make $k \gg 0$.

Let us now elaborate on the latter observation. The algebras that appear in the spectral sequences as we vary k are $\mathrm{HH}(\mathbb{F}_p^{hp^k\mathbb{Z}}/\mathbb{F}_p)$, and at the level of π_*, the natural maps between these are of the form

$$\begin{tikzcd}
\pi_*\mathrm{HH}(\mathbb{F}_p^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, equal] & \pi_*\mathrm{HH}(\mathbb{F}_p^{hp^{k+1}\mathbb{Z}}) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \pi_*\mathrm{HH}(\mathbb{F}_p/\mathbb{F}_p) \arrow[d, Rightarrow] \\
C^0(p^k\mathbb{Z}_p)\langle\zeta\rangle \arrow[r] & C^0(p^{k+1}\mathbb{Z}_p)\langle\zeta\rangle \arrow[r] & \cdots & \mathbb{F}_p
\end{tikzcd}$$

The lower horizontal maps send ζ to 0, and restrict continuous functions.

Then the following lemma is the key to making Observation 2 precise:

Lemma 1. Let $f : A \to B$ be a map of finite rank free $C^0(\mathbb{Z}_p)$-modules. Then for $k \gg 0$, there is an isomorphism $f \otimes_{C^0(\mathbb{Z}_p)} C^0(p^k\mathbb{Z}_p) \cong f \otimes_{C^0(\mathbb{Z}_p)} \mathbb{F}_p \otimes_{\mathbb{F}_p} C^0(p^k\mathbb{Z}_p)$, where the map $C^0(\mathbb{Z}_p) \to \mathbb{F}_p$ is evaluation at 0.

If we think of f as a continuous family of maps of $\mathbb{F}_p$-vector spaces parameterized by $\mathbb{Z}_p$, the conclusion of the above lemma can be written as $f|_{p^k\mathbb{Z}_p} \cong f|_0 \otimes C^0(p^k\mathbb{Z}_p)$ for $k \gg 0$. The proof of the lemma is very easy: one reduces to the case where $A = B = C^0(\mathbb{Z}_p)$, at which point f is a continuous function $\mathbb{Z}_p \to \mathbb{F}_p$. By continuity, it must by constant in some neighborhood $p^k\mathbb{Z}_p$ of 0, giving the conclusion.

By applying the above lemma to maps on π_* and differentials in spectral sequences that appear in computations towards $\mathrm{TC}(\ell^{hp^k\mathbb{Z}})/(p, v_1)$, we learn that as long as there are finitely many needed pieces of information in our computation, we can make k large enough, and the answer will behave as if the action were a trivial action, i.e it will be formally determined by analogous computation for ℓ.

The original method of computation of $\pi_*\mathrm{TC}(\ell)/(p, v_1)$ due to Ausoni–Rognes [14] is not a finite computation, but recent work of Hahn–Raksit–Wilson [15] shows that there is a way to do it making it into a finite computation. The key difference between what they do and what Ausoni–Rognes do is that they first compute $\mathrm{TC}(\ell)/(p, v_1, v_2)$, and then show that the v_2-Bockstein spectral sequence degenerates.

Nikolaus–Scholze [16] describe TC as fitting into an equalizer diagram

$$\mathrm{TC} \to \mathrm{THH}^{hS^1} \rightrightarrows \mathrm{THH}^{tS^1}$$

where the two maps are the canonical and frobenius maps. Thus to understand $\mathrm{TC}(\ell)/(p, v_1, v_2)$, it suffices to understand the homotopy fixed point spectral sequence computing $\mathrm{THH}^{hS^1}(\ell)/(p, v_1, v_2)$ (which then determines the Tate spectral sequence), and the frobenius map.

Up to grading conventions, the homotopy fixed point spectral sequence can be found in [15, Corollary 6.5.2]: it has E_2-page $\pi_*\mathrm{THH}(\ell)/(p, v_1)[t]\langle\sigma v_2\rangle$ with $|\sigma v_2| = 2p^2 - 1$ and $|t| = -2$. Up to propogation via the Leibniz rule, there are finitely many differentials: $d_2(\sigma v_2) = t\mu, d_{2p}t = t^{p+1}\lambda_1, d_{2p^2}t^p = t^{p^2+p}\lambda_2$. The spectral sequence moreover degenerates at a finite page for degree reasons. The important point is that there are only finitely many pieces of information about finitely many classes that determine the spectral sequence. The frobenius map is determined by the fact that $\phi(\lambda_i) = \lambda_i, \phi(\mu) = t^{-p^2}$ and the frobenius of any class with a t in its name is 0.

The above paragraph contains all the facts needed to compute $\mathrm{TC}(\ell)/(p, v_1, v_2)$, and a motivic associated graded version and chart at the prime 2 can be found in [15, Theorem 6.0.2]. In particular, its homotopy groups are finite dimensional and live in a bounded range of degrees. This means that the v_2-Bockstein spectral sequence, which degenerates, is also a finite amount of information. In particular we learn that $\pi_*\mathrm{TC}(\ell)/(p, v_1)[v_2^{\pm 1}]$ is finite dimensional over $\mathbb{F}_p[v_2^{\pm 1}]$. This also implies that $\pi_*\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}/(p, v_1)[v_2^{\pm 1}]$ is also a finite dimensional $\mathbb{F}_p[v_2^{\pm 1}]$-vector space for each k. In fact for $k \gg 0$ we have an isomorphism $\pi_*\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}/(p, v_1) \cong \pi_*\mathrm{TC}(\ell)/(p, v_1)\langle\zeta\rangle$.

The above discussion along with Observation 2 suggest that computing the homotopy groups of $\mathrm{TC}(\ell^{hp^k\mathbb{Z}})/(p, v_1)$ is a formal consequence of that of ℓ for $k \gg 0$. Indeed, there is a class $\partial \in \mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ such that $\pi_*\mathrm{TC}(\ell)/(p, v_1, v_2)$ as a module over $\mathbb{F}_p\langle\partial\rangle$ determines $\pi_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})/(p, v_1)$ for $k \gg 0$ as follows:

- Each summand of $\mathbb{F}_p\langle\partial\rangle$ appearing in $\pi_*\mathrm{TC}(\ell)/(p, v_1, v_2)$ contributes a copy of $\mathbb{F}_p[v_2] \oplus \partial\zeta \otimes C^0(p^{k+1}\mathbb{Z}_p^\times)[v_2]$ to $\pi_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})/(p, v_1)$.
- Each summand of $\mathbb{F}_p$ appearing in $\pi_*\mathrm{TC}(\ell)/(p, v_1, v_2)$ contributes a copy of $\mathbb{F}_p\langle\zeta\rangle \otimes C^0(p^{k+1}\mathbb{Z}_p)[v_2]$ to $\pi_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})/(p, v_1)$.

The contributions described above are all infinite dimensional $\mathbb{F}_p[v_2^{\pm}]$-vector spaces after inverting v_2.

We have in other words completely understood at the level of homotopy groups for $k \gg 0$ the map

$$\pi_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})/(p, v_1)[v_2^{\pm 1}] \to \pi_*\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}/(p, v_1)[v_2^{\pm 1}]$$

The failure of the telescope conjecture in this case is then the fact that this isn't an isomorphism. In fact it is quite far from an isomorphism: the source is infinite dimensional over $\mathbb{F}_p[v_2^{\pm 1}]$ and the target is not!

References

[1] D. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math **106** 1984, 351–414.

[2] E S. Devinatz, M. J. Hopkins, and J. H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2), **128** 1988, 207–241

[3] M. Mahowald, *bo-resolutions*, Pacific J. Math, **92** 1981 365–383

[4] H. R. Miller, *On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space*, J. Pure Appl. Algebra, **20** 1981 287-312

[5] C. Barwick, *On exact ∞-categories and the theorem of the heart*, Compos. Math., **151** 2015 2160–2186

[6] A. Blumberg and M. Mandell, *The localization sequence for the algebraic K-theory of topological K-theory*, Acta Math., **200** 2008 155–179
[7] S. A. Mitchell, *The Morava K-theory of algebraic K-theory spectra*, K-theory, **3** 1990 607-626
[8] B. I. Dundas, T. G. Goodwillie, and R. McCarthy *The local structure of algebraic K-theory* Algebra and Applications **18** 2013 xvi+435
[9] I. Levy, *The algebraic K-theory of the $K(1)$-local sphere via* TC, arXiv:2209.05314
[10] R. Burklund and I. Levy, *On the K-theory of regular coconnective rings*, Selecta Math. (N.S.), **29** 2023 Paper No. 28, 30
[11] M. Land and G. Tamme, *On the K-theory of pullbacks*, Ann. of Math. (2), **190**, 2019, 877–930
[12] J. E. McClure, and R. E. Staffeldt, *On the topological Hochschild homology of* bu. *I*, Amer. J. Math., **115** 1993 1–45
[13] D. J. Lee and I. Levy, *Topological Hochschild homology of the image of j* , arXiv:2307.04248
[14] C. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory*, Acta Math., **188** 2002 1–39
[15] J. Hahn, A. Raksit, and D. Wilson, *A motivic filtration on the topological cyclic homology of commutative ring spectra*, arXiv:2206.11208
[16] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Math., **221** 2018 203–409

Cyclotomic Redshift

Shachar Carmeli
(joint work with Shay Ben-Moshe, Tomer Schlank, Lior Yanovski)

Algebraic K-theory interacts deeply and subtly with the chromatic filtration. The redshift philosophy of Ausoni and Rognes suggests that it should increase the chromatic height precisely by one. An incarnation of this phenomenon has been established recently in the works of Clausen-Mathew-Naumann-Noel and Land-Mathew-Meier-Tamme, showing that the $T(n+1)$-localization of $K(R)$ only depends on the $T(n+1) \oplus T(n)$ localization of R, and of Burklund-Schlank-Yuan, showing that $L_{T(n+1)}K(R)$ is non-zero when R is commutative and has a non zero $T(n)$-localization. These results spotlight the $T(n+1)$-localization of the K-theory of $T(n)$-local ring spectra, which lies at the border of the chromatic support of the K-theory of such ring spectra.

Besides their contribution to redshift, Clausen-Mathew-Naumann-Noel also establish a descent result for this localized K-theory, $L_{T(n+1)}K$, as a functor of L_n^f-local ∞-categories (i.e., those with L_n^f-local mapping spectra). Namely, they show that this functor preserves homotopy fixed points for finite p-group actions. In my talk, I described work in preparation, joint with Ben-Moshe, Schlank, and Yanovski, generalizing the descent result above from finite p-group actions to actions of π-finite p-groups, i.e., truncated groups in spaces whose homotopy groups are finite p-groups. One of the main applications of this "higher descent" is the compatibility of $T(n+1)$-local K-theory with the formation of $T(n)$-local higher cyclotomic extensions. These are certain finite Galois extensions of the $T(n)$-local sphere previously constructed in a joint work with Schlank and Yanovski. These extensions provided an explicit telescopic lift of some well-known Galois extensions of the $K(n)$-local sphere, previously studied by Westerland. One of our main results is that $T(n+1)$-local K-theory carries $T(n)$-local cyclotomic extensions to $T(n+1)$-local cyclotomic extensions. This phenomenon, which we call "cyclotomic redshift", gives an example of a Galois descent for $T(n+1)$-local K-theory along a Galois extension of order prime to p (namely the p-th cyclotomic extension). This case is not covered by the results of Clausen-Mathew-Naumann-Noel, which only deal with p-group actions. Even for p-group actions, cyclotomic redshift is an example of "explicit Galois descent": not only that $T(n+1)$-local K-theory carries a Galois extension to a Galois extension, we know which one we get.

After $K(n+1)$-localization, the cyclotomic extensions assemble to a hypercomplete $\mathbb{Z}_p^\times$-extension of the $K(n+1)$-local sphere, and hence by base change, of every $K(n+1)$-local commutative ring spectrum. Consequently, our result shows that $K(n+1)$-local K-theory satisfies hyperdescent along the entire profinite $\mathbb{Z}_p^\times$-extension assembled from the finite cyclotomic extensions. The analogous hyperdescent result for $T(n+1)$-local K-theory is shown by Burklund-Hahn-Levi-Schlank not to hold. This shows that the $T(n+1)$-local and $K(n+1)$-local K-theories of $T(n)$-local ring spectra are generally not the same, thus disproving Ravenel's telescope conjecture.

Moduli spaces of equivariant *h*-cobordisms

Mona Merling

(joint work with Tom Gooodwillie, Kiyoshi Igusa, and Cary Malkiewich)

Classical parametrized stable h-cobordism theorem. Given a homotopy equivalence between smooth manifolds $M \simeq N$, the strategy to determine whether M and N are diffeomorphic is to try to construct an h-cobordism between them. By the classical h-cobordism (or s-cobordism) theorem, the obstructions to the h-cobordism being trivial, which would imply that $M \cong N$, are classified in terms of their Whitehead torsion $\tau \in Wh(\pi_1 M)$. The Whitehead group by definition is the quotient of $K_1(\pi_1 M)$ by $\pm g \in \pi_1 M$.

Theorem 1 ([2, 9, 17])**.** Suppose M is a manifold with $\dim(M) \geq 5$. There is an isomorphism

$$\{\text{iso classes of } h\text{-cobordisms on } M\} \cong Wh(\pi_1 M).$$

In order to study the space of all diffeomorphisms of M, it is necessary to topologize these obstructions [18, 12]. The Whitehead group $Wh(\pi_1 M)$ is the π_0 of the h-cobordism space $\mathcal{H}(M)$, whose k-simplices are h-cobordism bundles over Δ_k. The aforementioned theorem says that $\pi_0\mathcal{H}(M)$ can be computed in terms of K-theory. We can ask the same question about the higher homotopy groups of this moduli space:

Can we compute $\pi_i\mathcal{H}(M)$ in terms of algebraic K-theory?

The answer is yes, but only in a stable range, namely in the range where $\mathcal{H}(M)$ is equivalent to the stable version $\mathcal{H}^\infty(M)$ obtained by multiplying M by copies of I to increase its dimension [8]. This is the content of the celebrated "stable parametrized h-cobordism theorem."

Theorem 2 ([11])**.** There is a decomposition

$$\mathbf{A}(X) \simeq \Sigma^\infty X_+ \times \mathbf{Wh}(X), \tag{1}$$

where $\mathbf{Wh}(X)$ is a spectrum with the property that for a smooth compact manifold M, the underlying infinite loop space of $\Omega\mathbf{Wh}(M)$ is equivalent to the stable h-cobordism space $\mathcal{H}^\infty(M)$.

Weiss and Williams show that $\mathcal{H}^\infty(M)$ provides the information that accesses the diffeomorphism group of M in a stable range [12].

Equivariant h-cobordism spaces. Now suppose G is a finite group acting on a smooth manifold M with corners so that it has "trivial action on corners," namely M is a G-manifold modeled locally by $G \times_H V \times [0,\infty)^k$, for varying $H \leq G$ and H-representations V. Our goal is to understand the stable moduli space of equivariant h-cobordisms $\mathcal{H}_G^\infty(M)$ we constructed in [1] and show that it can be computed, at least in a range, by equivariant algebraic K-theory. Equivariant A-theory of a G-space $\mathbf{A}_G(X)$ was constructed in [14] using the machinery of spectral Mackey functors [6, 7, 10, 5, 4].[1] The main question/conjecture is the following.

Question 1. For a compact smooth G-manifold M, is there a splitting

$$\mathbf{A}_G(M) \simeq \Sigma_G^\infty M \times \mathbf{Wh}_G(M) \tag{2}$$

analogous to the nonequivariant one from equation (1), where $\Omega^{\infty+1}\mathbf{Wh}_G(M)^G \simeq \mathcal{H}_G^\infty(M)$?

An equivariant h-cobordism $(W; M, N)$ between compact G manifolds M and N is an h-cobordism W where the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are G-homotopy equivalences. For an equivariant parametrized stable h-cobordism theorem, we need to stabilize h-cobordisms with respect to representation disks. This is already apparent in the equivariant case on π_0: the equivariant Whitehead torsion of an equivariant h-cobordism $M \hookrightarrow W$ is the trivial element of the equivariant Whitehead group $Wh_G(M)$ if and only if there exists a G-representation V such that the equivariant h-cobordism $(W \times D(V); M \times D(V), N \times D(V))$ is trivial, where $D(V)$ is the unit disk in the representation V [13].

What underlies this result is the phenomenon that often equivariant results in manifold topology do not generalize unless there is a difference in the dimensions between fixed points ... An equivariant map is isovariant if if preserves stabilizers, or fixed point strata. An equivariant h-cobordism is isovariant if the inclusion maps of the boundaries are isovariant homotopy equivalences. If a G-manifold

[1] We note that $\mathbf{A}_G(X)$ is not the "K-theory of group actions for the G-action on the category of retractive spaces over X. That construction yields a G-spectrum that we call $\mathbf{A}_G^{\text{coarse}}(X)$ and study further in [15].

satisfies the so-called "weak gap hypotheses" where the difference between dimensions of different fixed points is at least 3, then an equivariant h-cobordism on M is an isovariant h-cobordism [3]. The idea is that stabilizing with respect to representation disks increases the gaps between dimensions of fixed points.

For a compact smooth G-manifold M, denote by $M_{[H]}$ be compactification of the subspace of points with isotropy H, by removing tubular neighborhoods of smaller fixed-point submanifolds. Note that these will be manifolds with corners. Furthermore, denote by WH the Weyl group of H with respect to G. In work in progress, we prove the following unstable splitting result for isovariant h-cobordism spaces, which on π_0 recovers a result of Browder-Quinn and Rothenberg.

Theorem 3 (Goodwillie-Igusa-Malkiewich-M.)**.** Let M be a compact smooth G-manifold. If $\dim M_{[H]}/WH \geq 5$, then the space of isovariant h-cobordisms satisfies a splitting

$$\mathcal{H}_G^{\mathrm{iso}}(M) \simeq \prod_{(H)\leq G} \mathcal{H}(M_{[H]}/WH).$$

When we stabilize, the spaces of isovariant and equivariant h-cobordisms agree, and we obtain the following stable result about equivariant h-cobordism spaces.

Theorem 4 (Goodwillie-Igusa-Malkiewich-M.)**.** Let M be a compact smooth G-manifold. Then the stable space of equivariant h-cobordisms (stabilized with respect to representation disks) satisfies a splitting

$$\mathcal{H}_G^{\infty}(M) \simeq \prod_{(H)\leq G} \mathcal{H}^{\infty}(M_{hWH}^H).$$

We can now ask the analogous question as before about this moduli space of equivariant h-cobordisms:

Can we describe $\mathcal{H}_G^{\infty}(M)$ in terms of equivariant algebraic K-theory?

Combined with the results of [16], we obtain a sequence in G-spectra

$$\mathbf{A}_G(M) \to \Sigma_G^{\infty} M \to \mathbf{Wh}_G(M)$$

where $\Omega^{\infty+1}\mathbf{Wh}_G(M)^G \simeq \mathcal{H}_G^{\infty}(M)$, a significant step toward Question 1.

References

[1] Tom Goodwillie, Kiyoshi Igusa, Cary Malkiewich, and Mona Merling. On the functoriality of the space of equivariant smooth h-cobordisms. *arXiv:2303.14892*, 2023.

[2] D. Barden. The structure of manifolds. *Ph.D. thesis, Cambridge*, 1963.

[3] Wolfgang Lück. *Transformation groups and algebraic K-theory*, volume 1408 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.

[4] Clark Barwick, Saul Glasman, and Jay Shah, *Spectral Mackey functors and equivariant algebraic K-theory (II)*, arXiv:1505.03098v2.

[5] Clark Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Advances in Mathematics **304**, 646–727.

[6] Bertrand Guillou and J. P. May, *Models of G-spectra as presheaves of spectra*, arXiv:1110.3571.

[7] Bertrand J. Guillou, J. Peter May, Mona Merling, and Angélica M. Osorno. Multiplicative equivariant K-theory and the Barratt-Priddy-Quillen theorem. *Adv. Math.*, 414:Paper No. 108865, 111, 2023.

[8] Kiyoshi Igusa. The stability theorem for smooth pseudoisotopies. *K-Theory*, 2(1-2):vi+355, 1988.

[9] Barry Mazur. Relative neighborhoods and the theorems of Smale. *Ann. of Math. (2)*, 77:232–249, 1963.

[10] Anna Marie Bohmann and Angélica Osorno, *Constructing equivariant spectra via categorical Mackey functors*, Algebraic & Geometric Topology **15** (2015), no. 1, 537–563.

[11] Friedhelm Waldhausen, Bjørn Jahren, and John Rognes, *Spaces of PL manifolds and categories of simple maps*, Annals of Mathematics Studies, vol. 186, Princeton University Press, Princeton, NJ, 2013. MR 3202834

[12] Michael Weiss and Bruce Williams, *Automorphisms of manifolds and algebraic K-theory. I*, *K*-Theory **1** (1988), no. 6, 575–626. MR 953917 (89h:57012)

[13] Shoro Araki and Katsuo Kawakubo, *Equivariant s-cobordism theorems*, J. Math. Soc. Japan **40** (1988), no. 2, 349–367.

[14] Cary Malkiewich and Mona Merling, *Equivariant A-theory*, *Doc. Math.*, 24:815–855, 2019.

[15] C. Malkiewich and M. Merling. Coassemly is a homotopy limit. *Annals of K-Theory*, 5(3):373–394, 2020.

[16] Cary Malkiewich and Mona Merling. The equivariant parametrized h-cobordism theorem, the non-manifold part. *Adv. Math.*, 399:Paper No. 108242, 42, 2022.

[17] John R. Stallings. On infinite processes leading to differentiability in the complement of a point. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 245–254. Princeton Univ. Press, Princeton, N.J., 1965.

[18] A. E. Hatcher. Concordance spaces, higher simple-homotopy theory, and applications. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 3–21. Amer. Math. Soc., Providence, R.I., 1978.

Invariant prime ideals in the equivariant Lazard ring

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(joint work with Markus Hausmann)

Chromatic homotopy theory is based on the paradigm that the structure of the stable homotopy category is predicted by the moduli stack of formal groups $\mathcal{M}_{FG}$. This correspondence is mediated by complex cobordism MU, whose coefficients π_*MU carry the universal formal group law. More precisely, every spectrum X defines a graded quasi-coherent sheaf $\mathcal{F}_*^X$ on $\mathcal{M}_{FG}$, corresponding to the graded (π_*MU, MU_*MU)-comodule $MU_*(X)$, and the properties of $\mathcal{F}_*^X$ reflect those of X.

We may look, for example, at the support $\operatorname{supp}\mathcal{F}_*^X$ of $\mathcal{F}_*^X$ in the space $|\mathcal{M}_{FG}|$ of points of $\mathcal{M}_{FG}$. The points of $\mathcal{M}_{FG}$ correspond to formal groups over fields and are thus classified by the residue characteristic and the height. By definition of the Balmer spectrum $\operatorname{Spc}(\mathrm{Sp}^{\mathrm{fin}})$, the support theory $\operatorname{supp}\mathcal{F}_*^{(-)}$ defines a continuous map $|\mathcal{M}_{FG}| \to \operatorname{Spc}(\mathrm{Sp}^{\mathrm{fin}})$. By the Hopkins–Smith thick subcategory theorem, this is a homeomorphism.

We claim that analogous statements are true for every compact abelian Lie group G, which we will fix throughout.

Tenet. Mediated by equivariant complex cobordism MU_G, the moduli stack of G-equivariant formal groups predicts the structure of the G-equivariant genuine stable homotopy category.

1. The moduli stack of G-equivariant formal groups

We warn that G-equivariant formal groups are *not* the same as formal groups with a G-action. The latter are relevant for theories like $K\mathbb{R}$ or $MU_{\mathbb{R}}$, while the former are relevant for G-equivariantly complex oriented theories like KU_G or MU_G (the universal example). We believe that the notion of equivariant formal groups is not as widely known as it should, and therefore we give both motivation and definition of this notion. The motivation we give is topological, but the notion should also be interesting from the purely algebro-geometric point of view.

If E is a non-equivariant complex-oriented ring spectrum, then we have an isomorphism $E^{2*}(\mathbb{CP}^\infty) \cong E^*[\![x]\!]$ and hence $\operatorname{Spf} E^{2*}(\mathbb{CP}^\infty) \cong \widehat{\mathbb{A}}^1_{E^{2*}}$. If E_G is a G-equivariant complex-oriented ring spectrum, we need to replace $\mathbb{CP}^\infty$ by $\mathbb{CP}^\infty_G$, the G-space of complex lines in the complex complete universe $\mathcal{U} = \bigoplus_{V \in G^*} V^\infty$; here $G^* = \operatorname{Hom}(G, U(1))$ is the set of irreducible complex representations of G. The complex orientation is a class $y \in E^2_G(\mathbb{CP}^\infty_G)$. The group G^* acts by tensoring on $\mathcal{U}$ and hence on $\operatorname{Spf} E^{2*}_G(\mathbb{CP}^\infty_G)$, and the map $\mathbb{CP}^\infty \to \mathbb{CP}^\infty_G$ defines a map $\widehat{\mathbb{A}}^1_{E^{2*}_G} \to \operatorname{Spf} E^{2*}_G(\mathbb{CP}^\infty_G)$. This motivates the notion of a G-equivariant formal group, defined in different language by Cole–Greenlees–Kriz [CGK00].

Definition 1. A *G-equivariant formal group* over a commutative ring k consists of

- a group object X in formal schemes over k,
- a G^*-action on the underlying formal scheme over k,
- a map $\widehat{\mathbb{A}}^1_k \xrightarrow{\varphi} X$,

such that

- the G^*-translates of φ cover X,
- the coordinate of $\widehat{\mathbb{A}}^1_k$ extends to a non-zero divisor y on X.

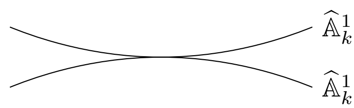

Figure 1. A schematic depiction of a C_2-equivariant formal group

Requiring y as part of the data, gives the notion of a G-equivariant formal group *law*. Every G-equivariant complex oriented theory defines such a group law in the manner sketched above. To obtain a notion of G-equivariant formal group that satisfies descent and hence defines a moduli stack $\mathcal{M}^G_{FG}$, we should weaken the definition above to asking for the existence of y (and the coordinate on $\widehat{\mathbb{A}}^1_k$) only Zariski-locally on k.

In the monograph [Str11], Strickland investigated many aspects of equivariant formal groups and showed in particular:

Theorem 1 (Strickland)**.** *The points of $\mathcal{M}_{FG}^G$ are classified by the residue characteristic p, the height n and the "subgroup of definition" $H \subseteq G$.*

The importance of the notion of equivariant formal groups to topology was cemented when, extending earlier work of Greenlees and of Hanke–Wiemeler (for $G = C_2$), Hausmann showed in seminal work an analogue of Quillen's theorem:

Theorem 2 (Hausmann, [Hau22])**.** *The coefficients $\pi_*^G MU$ carry the universal group law, and the Hopf algebroid $(\pi_*^G MU, MU_*^G MU)$ stackifies to $\mathcal{M}_{FG}^G$.*

2. The equivariant thick subcategory theorem

Relying on Hausmann's theorem, we can associate to every G-spectrum X a graded quasi-coherent sheaf $\mathcal{F}_*^X$ on $\mathcal{M}_{FG}^G$, corresponding to the $(\pi_*^G MU, MU_*^G MU)$-comodule $MU_*^G X$.

Theorem 3 ([HM23])**.** *The map*

$$\begin{aligned} \text{finite } G\text{-spectra} &\to \{\text{closed subsets of } |\mathcal{M}_{FG}^G|\} \\ X &\mapsto \text{support of } \mathcal{F}_*^X \end{aligned}$$

is the universal support theory on finite G-spectra. This induces a homeomorphism $|\mathcal{M}_{FG}^G| \to \mathrm{Spc}(\mathrm{Sp}_G^{\mathrm{fin}})$ to the Balmer spectrum of finite G-spectra.

This theorem has a curious history, as the topological side, namely the Balmer spectrum $\mathrm{Spc}(\mathrm{Sp}_G^{\mathrm{fin}})$, was calculated first, in work of Strickland, Balmer–Sanders, Barthel–Hausmann–Naumann–Noel–Nikolaus–Stapleton and Barthel–Greenlees–Hausmann [BS17], [BHNNNS19], [BGH20]. In our work, we calculate the algebraic side, namely the topology on $|\mathcal{M}_{FG}^G|$, and establish that the map above is a support theory; this induces the required map $|\mathcal{M}_{FG}^G| \to \mathrm{Spc}(\mathrm{Sp}_G^{\mathrm{fin}})$. Establishing this support theory is harder than in the non-equivariant case: $\pi_* MU$ is known to be coherent, but the analogous result is not known for $\pi_*^G MU$. We conjecture:

Conjecture 1. *The stacks $\mathcal{M}_{FG}^G$ are coherent in the sense that coherent sheaves on them (corresponding to comodules whose underlying module is finitely presented) form an abelian category.*

3. Further results and the road ahead

The points in $\mathcal{M}_{FG}$ corresponds to the invariant prime ideals $I_{p,n} \subseteq \pi_* MU$. These are generated by the elements $p = v_0, v_1, \dots, v_{n-1}$, where each v_k is canonically defined modulo $I_{p,k}$. We show that the v_n naturally refine to elements $\mathbf{v}_n$ in $\pi_*^{C_p^n \times U(1)} MU$, canonically defined modulo a certain invariant prime ideal. These allow us to write down generators for many of the invariant prime ideals of $\pi_*^G MU$; these do not form regular sequences in general, however.

The $\mathbf{v}_n$ are crucial in determining the topology on $|\mathcal{M}_{FG}|$ and form, in some sense, the algebraic replacements of the partition complexes used for the determination of the topology on $\mathrm{Spc}(\mathrm{Sp}_G^{\mathrm{fin}})$.

We expect that the $\mathbf{v}_n$ will play a fundamental role in the chromatic picture for G-spectra, especially for equivariant analogues of the periodicity theorem.

References

[BGH20] Tobias Barthel, J. P. C. Greenlees, and Markus Hausmann. On the Balmer spectrum for compact Lie groups. *Compos. Math.*, 156(1):39–76, 2020.

[BHNNNS19] Tobias Barthel, Markus Hausmann, Niko Naumann, Thomas Nikolaus, Justin Noel, and Nathaniel Stapleton. The Balmer spectrum of the equivariant homotopy category of a finite abelian group. *Invent. Math.*, 216(1):215–240, 2019.

[BS17] Paul Balmer and Beren Sanders. The spectrum of the equivariant stable homotopy category of a finite group. *Invent. Math.*, 208(1):283–326, 2017.

[CGK00] Michael Cole, J. P. C. Greenlees, and I. Kriz. Equivariant formal group laws. *Proc. London Math. Soc. (3)*, 81(2):355–386, 2000.

[Hau22] Markus Hausmann. Global group laws and equivariant bordism rings. *Ann. of Math. (2)*, 195(3):841–910, 2022.

[HM23] Markus Hausmann, and Lennart Meier. Invariant prime ideals in equivariant Lazard rings. arXiv:2309.00850

[Str11] N. P. Strickland. Multicurves and equivariant cohomology. *Mem. Amer. Math. Soc.*, 213(1001):vi+117, 2011.

Quillen stratification in equivariant homotopy theory

Natàlia Castellana Vila

(joint work with Tobias Barthel, Drew Heard, Niko Naumann, Luca Pol)

In the 1970's Quillen [8] published a celebrated theorem known as strong Quillen stratification theorem. The strong stratification theorem provides a decomposition of the Zariski spectrum of the cohomology of any finite group G with coefficients in a field k

$$\mathrm{Spec}\, H^\bullet(G,k) = \bigsqcup_{(E)\subseteq G} \mathcal{V}^+_{G,E}.$$

in terms of locally closed subsets indexed on the conjugacy classes of elementary abelian subgroups and the strata $\mathcal{V}^+_{G,E}$ are orbits of the Weyl group action on an open subset of the Zariski spectrum $\mathrm{Spec}\, H^\bullet(E,k)$ of the cohomology of E, which is well-known. The weak version describes the spectrum as a colimit of $\mathrm{Spec}\, H^\bullet(E,k)$ on the orbit category on elementary abelian subgroups.

Previous generalizations of Quillen's weak result had been obtained. On the one hand, Mathew–Naumann–Noel [7] produce a generalization for coefficients in an arbitrary commutative equivariant ring spectrum; on the other hand, the weak statement has found a tt-geometric incarnation for the spectrum of $\mathrm{D}^b(\mathbb{F}_pG)$ in Balmer's [1].

In this project (see [5]) we prove a generalization of this result in the context of equivariant homotopy theory formulated in the language of tensor-triangular geometry. So it conceptualizes these results in a uniform point of view getting together equivariant tensor-triangular geometry, Quillen's stratification of group cohomology, and stratifications in modular representation theory. In particular, we establish a Quillen-type decomposition of the Balmer spectrum of equivariant tensor-triangulated category and study the extent to which it is reflected in a stratification of the category defined over it.

Let $\mathcal{K}$ be an essentially small tt-category. Balmer [2] constructed a topological space $\mathrm{Spc}(\mathcal{K})$, called the *spectrum* of $\mathcal{K}$ whose points are the prime $\otimes$-ideals of $\mathcal{K}$, and the topology of $\mathrm{Spc}(\mathcal{K})$ is the one having $\{\mathrm{supp}(a)\}_{a\in\mathcal{K}}$ as a basis of closed subsets, where $\mathrm{supp}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) | a \notin \mathcal{P}\}$. Balmer shows that there is an order-preserving bijection between radical thick $\otimes$-ideals of $\mathcal{K}$ and Thomason subsets of $\mathrm{Spc}(\mathcal{K})$.

More generally one would like to understand when the Balmer spectrum of compact objects also parameterizes the localizing $\otimes$-ideals. Suppose that $\mathcal{T}$ is a rigidly-compactly generated tt-category whose Balmer spectrum of compact objects is Noetherian. Then, Balmer–Favi [4] and Stevenson [9] have extended the notion of Balmer support from compact objects,$\mathcal{T}^c$, to all of $\mathcal{T}$.

$$\{\text{Localizing } \otimes\text{-ideals of } \mathcal{T}\} \xrightarrow{\text{Supp}} \{\text{Subsets of } \mathrm{Spc}(\mathcal{T}^c)\} \tag{1}$$

If the map Supp from (1) is a bijection, then we say that $\mathcal{T}$ *is stratified*. So, the first question is when this happens. Techniques to approach this question are developed in [6].

Another question then is to identify $\mathrm{Spc}(\mathcal{T}^c)$ as a set or as a topological space. If R is the graded endomorphism ring of the unit, Balmer [3] defines a natural continuous comparison map $\rho\colon \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}^h(R)$ which combined with $\mathrm{Spec}^h(R) \to \mathrm{Spec}(R_0)$ (that sends a homogeneous prime ideal $\mathfrak{p}$ to $\mathfrak{p} \cap R_0$), gives rise to an *ungraded comparison map*

$$\rho_0\colon \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}(R_0).$$

We specialize to the following context. Let G be a finite group. We let Sp_G denote the stable ∞-category of G-spectra. Given a commutative equivariant ring spectrum R, let $\mathrm{Mod}_G(R)$ denote the ∞-category of R-modules internal to Sp_G, and write $\mathrm{Perf}_G(R)$ for its full subcategory of compact (or perfect) modules. Given a subgroup $H \subseteq G$, the geometric fixed point functor is denoted by Φ^H.

In the following main result we establish an analogue of Quillen stratification for an arbitrary commutative equivariant ring spectrum R and we show that the category is stratified in terms of geometric fixed points.

Theorem 1. Let R be a commutative equivariant ring spectrum and write $\mathrm{Mod}_G(R)$ for the category of G-equivariant modules over R. Then:

(1) The spectrum of perfect R-modules admits a locally-closed decomposition

$$\mathrm{Spc}(\mathrm{Perf}_G(R)) \simeq \bigsqcup_{(H)\subseteq G} \mathrm{Spc}(\mathrm{Perf}(\Phi^H R))/W_G(H),$$

with the set-theoretic disjoint union being indexed on conjugacy classes of subgroups of G;

(2) $\mathrm{Mod}_G(R)$ is stratified if the categories $\mathrm{Mod}(\Phi^H R)$ are stratified with Noetherian spectrum for all subgroups H in G.

In both (a) and (b) it suffices to index on a family $\mathcal{F}$ of subgroups $H \subseteq G$ such that R is $\mathcal{F}$-nilpotent.

Specializing to the Borel-equivariant theory for $H\mathbb{F}_p$, the Eilenberg-MacLane spectrum for $\mathbb{F}_p$, one recovers a version of Quillen's theorem. We apply our methods to the case of the Borel-equivariant Lubin-Tate E-theory.

Theorem 2. Let $\underline{E} = \underline{E_n}$ be a G-Borel-equivariant Lubin–Tate E-theory of height n and at the prime p. The category $\mathrm{Mod}_G(\underline{E})$ is cohomologically stratified, and there is a decomposition into locally-closed subsets.

$$\mathrm{Spc}(\mathrm{Perf}_G(\underline{E})) \cong \mathrm{Spec}(E^0(BG)) \simeq \bigsqcup_A \mathrm{Spec}(\pi_0 \Phi^A \underline{E})/W_G^Q(A),$$

where the disjoint union is indexed on abelian p-subgroups A of G generated by at most n elements. In particular, the generalized telescope conjecture holds for $\mathrm{Mod}_G(\underline{E})$ and there are explicit bijections

$$\left\{\begin{matrix}\text{Thick } \otimes\text{-ideals of}\\ \mathrm{Perf}_G(\underline{E})\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{Specialization closed}\\ \text{subsets of } \mathrm{Spec}(E^0(BG))\end{matrix}\right\}$$

and

$$\left\{\begin{matrix}\text{Localizing } \otimes\text{-ideals of}\\ \mathrm{Mod}_G(\underline{E})\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{Subsets of}\\ \mathrm{Spec}(E^0(BG))\end{matrix}\right\}.$$

A key input for the proof of the previous theorem is the following statement.

Theorem 3. The commutative ring $\pi_0 \Phi^A \underline{E}$ is regular Noetherian for any finite abelian p-group A.

Finally, the techniques developed in the general context can be applied to the following examples obtaining stratification theorems in each case:

(1) The integral constant Green functor $R = H\underline{\mathbb{Z}}$ for any cyclic p-group G.
(2) Equivariant K-theory $R = KU_G$ for any finite group G. In this case, $\mathrm{Spc}(\mathrm{Perf}_G(KU_G)) \cong \mathrm{Spec}(\pi_0 KU_G)$, where $\pi_0 KU_G \cong R(G)$ is the complex representation ring of G.

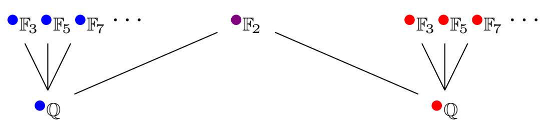

FIGURE 1. Here $\mathrm{Spc}(\mathrm{Perf}_{C_2}(KU_{C_2})) \cong \mathrm{Spec}(R(C_2))$. The closure goes upwards, and the primes are labeled by their residue fields.

(3) Atiyah's K-theory with reality $R = K\mathbb{R}$ for $G = C_2$. In this case, $\mathrm{Spc}(\mathrm{Perf}_{C_2}(K\mathbb{R})) \cong \mathrm{Spec}(\mathbb{Z})$.

References

[1] P. Balmer, *Separable extensions in tensor-triangular geometry and generalized Quillen stratification*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), 907–925.

[2] P. Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005), 149–168.

[3] P. Balmer, *Spectra, spectra, spectra – Tensor triangular spectra versus Zariski spectra of endomorphism rings*, Algebr. Geom. Topol. **10** (2010), 1521–1563.

[4] P. Balmer, G. Giordano, *Generalized tensor idempotents and the telescope conjecture*, Proc. Lond. Math. Soc. (3) **6** (2011), 1161–1185.

[5] T. Barthel, N. Castellana, D. Heard, N. Naumann, L. Pol, *Quillen stratification in equivariant homotopy theory*, arXiV:2301.02212 (2023).

[6] T. Barthel, D. Heard, B. Sanders, *Stratification in tensor triangular geometry with applications to spectral Mackey functors*, to appear in Cambridge Journal of Mathematics.

[7] A. Mathew, N. Naumann, J. Noel, *Derived induction and restriction theory*, Geom. Topol. **23** (2019), 541–636.

[8] D. Quillen, *The spectrum of an equivariant cohomology ring. I, II* , Ann. of Math. (2) **94** (1971), 549–572, ibid. (2) **94** (1971), 573–602.

[9] G. Stevenson, *Support theory via actions of tensor triangulated categories*, J. Reine Angew. Math. **681** (2013), 219–254.

N_∞ operads, transfer systems, and the combinatorics of bi-incomplete Tambara functors

Kyle Ormsby

(joint work with Linus Bao, Christy Hazel, Tia Karkos, Alice Kessler, Austin Nicolas, Jeremie Park, Cait Schleff, Scotty Tilton)

Let G be a finite group. The theory of N_∞ operads was created by Blumberg–Hill [4] to parametrize homotopy coherent normed multiplicative structures on G-equivariant ring spectra. The homotopy category of G-N_∞ operads is equivalent to the lattice $\mathrm{Tr}\, G$ of G-transfer systems. The combinatorial nature of $\mathrm{Tr}\, G$ makes it amenable to study by elementary means. In this talk, I report on work by the 2023 Electronic Computational Homotopy Theory REU to determine the structure of $\mathrm{Tr}\, G$ when $G = C_p \times C_p$ is an elementary Abelian p-group of rank two. This leads to an application in equivariant algebra: a quick derivation of the number of compatible pairs of transfer systems underlying bi-incomplete Tambara functors on $C_p \times C_p$.

Eschewing the standard homotopical conceit of writing Σ_n, let $\mathfrak{S}_n$ denote the symmetric group on n letters.

Definition 1. A *G-N_∞ operad* $\mathscr{O}$ is an operad in G-spaces such that (1) $\mathscr{O}(0)$ is G-contractible, (2) the action of $\mathfrak{S}_n = e \times \mathfrak{S}_n$ on $\mathscr{O}(n)$ is free, (3) for all $\Gamma \le G \times \mathfrak{S}_n$, $\mathscr{O}(n)^\Gamma$ is either contractible or empty, and (4) $\mathscr{F}_{\mathscr{O}} := \{\Gamma \le G \times \mathfrak{S}_n \mid \mathscr{O}(n)^\Gamma \simeq *\}$ is a $G \times \mathfrak{S}_n$-family[1] containing all subgroups of the form $H \times e$.

[1] A collection of subgroups forms a *family* when it is closed under conjugation and taking subgroups.

Let $H \leq G$ and let T be a finite H-set. Let $\Gamma(T)$ denote the graph of a permutation representation $H \to \mathfrak{S}_{|T|}$ of T. We say that $\mathscr{O}$ *admits T-norms* when $\mathscr{O}(n)^{\Gamma(T)} \simeq *$.

Note that when an $\mathscr{O}$-algebra X admits H/K-norms, we get a "wrong way" map

$$X^K \to X^H.$$

These are what practicitioners typically think of as norms (or transfers in an additive setting).

We write N_∞-$\mathbf{Op}^G$ for the category of G-N_∞ operads and G-equivariant maps of G-operads. A weak equivalence of G-N_∞ operads is map $\varphi\colon \mathscr{O}_1 \to \mathscr{O}_2$ such that the induced map $\mathscr{O}_1(n)^\Gamma \to \mathscr{O}_2(n)^\Gamma$ is a weak equivalence for all $n \geq 0$ and $\Gamma \leq G \times \mathfrak{S}_n$. We write $\mathrm{Ho}(N_\infty\text{-}\mathbf{Op}^G)$ for the associated homotopy category. Since we wish to classify N_∞ operads up to homotopy, it is desirable to have a tractable model for $\mathrm{Ho}(N_\infty\text{-}\mathbf{Op}^G)$, and this is provided by transfer systems.

Definition 2. Let $(P, \leq)$ be a partially ordered set (poset). A *transfer system* R on P is a partial order on the set P refining $\leq$ (so $x\ R\ y \implies x \leq y$) such that

$$x\ R\ y,\ z \leq y,\ \text{and } w \text{ maximal among } w' \leq x \implies w\ R\ z. \tag{1}$$

If $(\mathrm{Sub}\,G, \leq)$ denotes the subgroup lattice of a finite group G ordered by inclusion, then a *G-transfer system* is a transfer system R on $\mathrm{Sub}\,G$ that is further closed under conjugation: $K\ R\ H \implies {}^gK\ R\ {}^gH$ where ${}^gH := gHg^{-1}$.

Note that when P is a lattice (like $\mathrm{Sub}\,G$), condition (1) reduces to

$$x\ R\ y \text{ and } z \leq y \implies x \wedge z\ R\ z, \tag{2}$$

which we refer to as the *restriction condition.* Categorically speaking, a transfer system on a lattice P is a wide subcategory of P closed under pullbacks.

Let $\mathrm{Tr}\,P$ denote the collection of transfer systems on P, and let $\mathrm{Tr}\,G$ denote the set of G-transfer systems. There is a canonical refinement partial order $\leq$ on $\mathrm{Tr}\,P$ given by

$$R \leq R' \iff (x\ R\ y \implies x\ R'\ y),$$

and when P is a finite lattice, $\mathrm{Tr}\,P$ is a finite lattice; the same is true of $\mathrm{Tr}\,G$.

Work of many authors [4, 8, 6, 9, 10, 1] establishes that $\mathrm{Tr}\,G$ models the homotopy category of G-N_∞ operads. Given an N_∞ operad $\mathscr{O}$, write $R_{\mathscr{O}} \in \mathrm{Tr}\,G$ for the transfer system given by

$$K\ R_{\mathscr{O}}\ H \iff \mathscr{O} \text{ admits } H/K \text{ norms}.$$

Theorem 1. The assignment $R \mapsto R_{\mathscr{O}}$ is a functor N_∞-$\mathbf{Op}^G \to \mathrm{Tr}\,G$ and descends to an equivalence of categories

$$\mathrm{Ho}(N_\infty\text{-}\mathbf{Op}^G) \simeq \mathrm{Tr}\,G.$$

Transfer systems are elementary but subtle, and enumerations of $\mathrm{Tr}\,G$ have only slowly appeared. Prior to our work, the only infinite family of transfer system lattices completely understood was for $G = C_{p^n}$, the cyclic group of order p^n, p

prime.[2] Indeed, Balchin–Barnes–Roitzheim [1] prove that $\operatorname{Tr} C_{p^n}$ is isomorphic to the Tamari lattice $\mathcal{A}_{n+1}$ of planar binary rooted trees with $n+2$ leaves partially ordered by tree rotation. It follows that

$$|\operatorname{Tr} C_{p^n}| = \mathrm{Cat}_{n+1} = \frac{1}{2n+3}\binom{2n+3}{n+1},$$

the $(n+1)$-th Catalan number.

In our work [3], we completely determine and enumerate the lattice of transfer systems for $C_p \times C_p$, p prime. In order to state the theorem, set $[n] := \{0 < 1 < \cdots < n\}$ and note that $[1]^k$ is isomorphic to the lattice of subsets of a k-element set partially ordered by inclusion.

Theorem 2 (Bao, Hazel, Karkos, Kessler, Nicolas, O., Park, Schleff, Tilton [3, Theorem 5.4])**.** For p prime there are exactly

$$2^{p+2} + p + 1$$

transfer systems on $C_p \times C_p$, and the lattice of transfer systems consists of three disjoint induced subposets: $B, T \cong [1]^{p+1}$ and M consisting of $p+1$ incomparable points. The only covering relations in $\operatorname{Tr}(C_p \times C_p)$ not internal to B or T are of the following forms:

(i) each element of B covered by $\max B$ is also covered by exactly one element of M,
(ii) each element of T covering $\min T$ also covers exactly one element of M,
(iii) $\min T$ covers $\max B$.

Proof idea. The subgroup lattice of $C_p \times C_p$ consists of the trivial subgroup e, $p+1$ rank 1 subgroups (isomorphic to C_p), and the full group. As such, $\operatorname{Sub}(C_p \times C_p) \cong [2]^{*(p+1)}$, the $(p+1)$-fold *fusion* of $[2]$ with itself. Here the fusion of two lattices is their disjoint union with minimal elements identified and maximal elements identified. We provide a general recursion for $|\operatorname{Tr}(P * Q)|$ in [3, Theorem 4.11], and leverage this to enumerate $\operatorname{Tr}(C_p \times C_p) \cong \operatorname{Tr}([2]^{*(p+1)})$.

With the enumeration in hand, it is easy to construct all transfer systems and check the covering relations between them. The subposet B consists of transfer systems that only have non-identity relations between e and some subset of rank 1 subgroups. The subposet T consists of transfer systems that have all relations from e to other subgroups and some subset of relations between rank 1 subgroups and $C_p \times C_p$. The transfer systems in M have all but one of the relations from e to rank 1 subgroups, along with one relation from the excluded rank 1 subgroup to $C_p \times C_p$. □

Such an explicit enumeration of transfer systems allows us to study other structures related to transfer systems on $C_p \times C_p$. By work of Chan [7], we know that

[2] Balchin–MacBrough–Ormsby [2] have also determined an elaborate set of interleaving recursions which determine the cardinalities of $\operatorname{Tr} D_{p^n}$ and $\operatorname{Tr} C_{qp^n}$

special pairs of transfer systems enumerate compatible choices of transfers and norms for bi-incomplete Tambara functors in the sense of Blumberg–Hill [5].[3]

Definition 3. Let G be a finite group. A pair $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$ of G-transfer systems is called *compatible* when $\xrightarrow{m} \leq \overset{a}{\dashrightarrow}$ and the following condition holds:

$$K, L \leq H \leq G,\ K \xrightarrow{m} H, \text{ and } K \cap L \overset{a}{\dashrightarrow} K \implies L \overset{a}{\dashrightarrow} H. \tag{3}$$

We write $\operatorname{Comp} G$ for the collection of compatible pairs of G-transfer systems.

We may encode (3) diagramatically as

$$\begin{tikzcd}
& H & \\
L \arrow[ur, dashed, "a"] & & K \arrow[ul, "m"'] \arrow[ll, phantom, "\Leftarrow"] \\
& K \cap L \arrow[ul, "m"] \arrow[ur, dashed, "a"'] &
\end{tikzcd}$$

where the double arrow indicates logical implication. Note that $K \cap L \xrightarrow{m} L$ is already guaranteed by (2).

Based on Theorem 2, we may enumerate the compatible pairs of transfer systems for $C_p \times C_p$ with relatively little pain.

Theorem 3 (O.)**.** For p prime, there are exactly

$$2^p(2^{p+2} + p + 3) + 3^{p+1} + 2p + 2$$

compatible pairs of $(C_p \times C_p)$-transfer systems.

To give the reader a sense for these numbers, we record the first few values:

p	2	3	5	7	11	13
$\lvert \operatorname{Comp}(C_p \times C_p) \rvert$	117	393	5 093	73 393	17 337 353	273 349 525

Proof sketch. Set $n = p + 1$. For each $\xrightarrow{m} \in \operatorname{Tr}(C_p \times C_p)$ we determine which $\overset{a}{\dashrightarrow} \geq \xrightarrow{m}$ satisfy (3). First focus on the 2^n transfer systems in B. Since no relations in these transfer systems are restrictions of other relations, no conditions are imposed by (3) and we only need to count the size of the up-set of each $\xrightarrow{m}$ in B. If $\xrightarrow{m}$ has rank k, then there are 2^{n-k} elements of B at least as large as it, along with $n - k$ elements of M and all 2^n elements of T. Since there are $\binom{n}{k}$ elements of B of rank k, we find that there are exactly

$$\sum_{k=0}^{n} \binom{n}{k} (2^{n-k} + n - k + 2^n)$$

[3]Bi-incomplete Tambara functors arise in the context of equivariant ring spectra R defined over G-universes that might not be complete. In this scenario, $\underline{\pi}_* R$ is a bi-incomplete Tambara functor with additive transfers encoded by the G-universe and multiplicative norms encoded by an N_∞ operad over which R is an algebra.

compatible pairs $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$ with $\xrightarrow{m}$ in B. Standard combinatorial identities reduce this expression to

$$3^n + 2^{n-1} \cdot n + 2^{2n}.$$

Now let $\xrightarrow{m}$ be one of the n transfer systems in M. While there are $1 + 2^{n-1}$ transfer systems at least as large as $\xrightarrow{m}$, only $\xrightarrow{m}$ and the complete transfer system $\leq$ pair with $\xrightarrow{m}$ to satisfy (3). Thus there are $2n$ compatible pairs $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$ with $\xrightarrow{m}$ in M.

Finally, if $\xrightarrow{m}$ is in T, then it can only pair with the complete transfer system to satisfy (3), so there are 2^n compatible pairs $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$ with $\xrightarrow{m}$ in T. Adding things up, we see that there are exactly

$$3^n + 2^{n-1} \cdot n + 2^{2n} + 2n + 2^n = 2^{n-1}(2^{n+1} + n + 2) + 3^n + 2n$$

compatible pairs for $C_p \times C_p$. Substituting $n = p + 1$ gives the expression from the theorem statement. $\square$

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References

[1] S. Balchin, D. Barnes, and C. Roitzheim, *N_∞-operads and associahedra*, Pacific J. Math. **315** (2021), no. 2, 285–304.

[2] S. Balchin, E. MacBrough, and K. Ormsby, *The combinatorics of N_∞ operads for C_{qp^n} and D_{p^n}*, preprint (2023), arXiv:2209.06992.

[3] L. Bao, C. Hazel, T. Karkos, A. Kessler, A. Nicolas, K. Ormsby, J. Park, C. Schleff, and S. Tilton, *Transfer systems for rank two elementary Abelian groups: characteristic functions and matchstick games*, preprint (2023), arXiv:2310.13835.

[4] A.J. Blumberg and M.A. Hill, *Operadic multiplications in equivariant spectra, norms, and transfers*, Adv. Math. **285** (2015), 658–708.

[5] A.J. Blumberg and M.A. Hill, *Bi-incomplete Tambara functors*, preprint (2021), arXiv:2104.10521.

[6] P. Bonventre and L.A. Pereira, *Genuine equivariant operads*, Adv. Math. **381** (2021), Paper No. 107502, 133.

[7] D. Chan, *Bi-incomplete Tambara functors as $\mathscr{O}$-commutative monoids*, preprint (2022), arXiv:2208.05555.

[8] J.J. Gutiérrez and D. White, *Encoding equivariant commutativity via operads*, Algebr. Geom. Topol. **18** (2018), no. 5, 2919–2962.

[9] J. Rubin, *Combinatorial N_∞ operads*, Algebr. Geom. Topol. **21** (2021), no. 7, 3513–3568.

[10] J. Rubin, *Detecting Steiner and linear isometries operads*, Glasg. Math. J. **63** (2021), no. 2, 307–342.

Equivariant orientations of bundles over disconnected G-spaces

Foling Zou
(joint work with Prasit Bhattacharya)

Classically, orientations of vector bundles can be described via several equivalent ways, among which the existence of a Thom isomorphism. Equivariantly, different approaches of orientations bifurcate when the base space of the bundle is not G-connected. The issue is that fibers over different components could be different representations. With Prasit Bhattacharya, we develop a two-stage obstruction theory that is computable, and used it to prove the existence of Thom classes and Thom isomorphisms of some Thom space that appears in equivariant cohomology operations.

Transchromatic phenomena in the equivariant slice spectral sequence

XiaoLin Danny Shi
(joint work with Lennart Meier, Mingcong Zeng)

In this talk, we will construct a stratification for the equivariant slice spectral sequence. This stratification is achieved through the localized slice spectral sequences, which compute the geometric fixed points equipped with residual quotient group actions. As an application, we will utilize this stratification to investigate norms of Real bordism theories and their quotients. These quotients hold significant importance in Hill-Hopkins-Ravenel's resolution of the Kervaire invariant one problem, as well as in the study of fixed points of Lubin-Tate theories by finite subgroups of the Morava stabilizer group. For these theories, the stratification exhibits a transchromatic phenomenon: the slice spectral sequence of a higher height theory is stratified into distinct regions, each isomorphic to the slice spectral sequences of the lower height theories. This provides an inductive approach and various structural insights when computing the fixed points of Lubin-Tate theories.

The telescope conjecture

Robert Burklund
(joint work with Jeremy Hahn, Ishan Levy, Tomer Schlank)

In this talk I elaborated on joint work with J. Hahn, I. Levy and T. Schlank discussed in previous talks this week. The first part of the talk focused on calculating $\pi_*\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$. Using this I explained a heuristic for understanding why the map $\mathrm{TC}(\mathbb{S}^{B\mathbb{Z}}) \to \mathrm{TC}(\mathbb{S})^{B\mathbb{Z}}$ is unlikely to be an equivalence. Following this I began discussing why the Adams operations on ℓ are sufficiently close to trivial for $\mathrm{TC}(-) \otimes V(2)$ to be calculable by reduction to the case of a trivial action.

Separability in homotopical algebra

Maxime Ramzi

In classical algebra, separable algebras are a generalization of both étale algebras in the commutative setting, and Azumaya algebras in the noncommutative setting. Introduced by Auslander and Goldman in [1], they have been re-introduced in homotopical algebra by Balmer in [2], where he studies separable algebras in the context of tensor-triangulated categories. Their definition can be given in any symmetric monoidal (∞-)category:

Definition 1. Let C be a symmetric monoidal ∞-category, and A an algebra in C. It is said to be *separable* if its multiplication map $A \otimes A \to A$ admits a bimodule section.

One of the key features of separable algebras in the context of tensor-triangulated categories is that they have a good module theory, without having to remember higher homotopical structure; and furthermore this module theory agrees with the homotopically coherent version. For example, we have:

Theorem 1 ([2, Corollary 6.6])**.** Let R be a commutative ring and S a commutativeétale R-algebra. The category of S-modules in the derived (1-)category of R, $D(R)$, is equivalent to the derived (1-)category of S, $D(S)$.

In this talk, I reported on my paper [3] where I explain this phenomenon from the perspective of symmetric monoidal stable ∞-categories. The goal of the talk was to state and explain the following pair of theorems:

Theorem 2. Let C be a stably symmetric monoidal ∞-category, and A a homotopy algebra in C, i.e. an algebra in $\mathrm{ho}(C)$, and assume A is separable in $\mathrm{ho}(C)$. In this case, we have:

1. The moduli space of lifts of A to an algebra in C, $\mathcal{M}_A^{\mathbb{E}_1} := \mathrm{Alg}(C) \times_{\mathrm{Alg}(\mathrm{ho}(C))} \{A\}$ is simply-connected, and in particular non-empty;
2. Given any lift $\tilde{A} \in \mathcal{M}_A^{\mathbb{E}_1}$, $\tilde{A}$ is itself separable (rather than only separable in $\mathrm{ho}(C)$);
3. Given any lift $\tilde{A}$ as above, the canonical functor $\mathrm{ho}(\mathrm{LMod}_{\tilde{A}}(C)) \to \mathrm{LMod}_A(\mathrm{ho}(C))$ is an equivalence.

Further, given a lift $\tilde{A}$ as above, and given any algebra R in C, the canonical map $\pi_0 \mathrm{Map}_{\mathrm{Alg}(C)}(\tilde{A}, R) \to \hom_{\mathrm{Alg}(\mathrm{ho}(C))}(A, R)$ is an isomorphism.

Warning 1. For a general A, the moduli space $\mathcal{M}_A^{\mathbb{E}_1}$ is *not* contractible.

Similarly, for a general A and R, the mapping space $\mathrm{Map}_{\mathrm{Alg}(C)}(\tilde{A}, R)$ is not discrete.

In a sense, this theorem explains the results of [2], and explains that most of the results proved about separable algebras in tensor-triangulated categories extend readily to their coherent analogue in stably symmetric monoidal ∞-categories. There are still some subtleties related to Warning 1, and the following theorem shows that they disappear in the commutative case:

Theorem 3. Let C be a stably symmetric monoidal ∞-category, and A a homotopy algebra in C, i.e. an algebra in $\mathrm{ho}(C)$, and assume A is separable in $\mathrm{ho}(C)$. Assume further that A is homotopy commutative, i.e. A is a commutative algeba in $\mathrm{ho}(C)$. In this case, we have:

(1) For any $d \geq 1$ (including $d = 1$ and $d = \infty$), the moduli space of lifts of A to an $\mathbb{E}_d$-algebra in C, $\mathcal{M}_A^{\mathbb{E}_d} := \mathrm{Alg}_{\mathbb{E}_d}(C) \times_{\mathrm{Alg}(\mathrm{ho}(C))} \{A\}$ is contractible;
(2) The 1-category $\mathrm{LMod}_A(\mathrm{ho}(C))$ acquires a canonical tensor product for which the equivalence from Theorem 2(3) is an equivalence.

Further, given any $d \geq 1$, any lift $\tilde{A}$ of A to an $\mathbb{E}_d$-algebra and any homotopy commutative $\mathbb{E}_d$-algeba R in C[1], the mapping space $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_d}(C)}(\tilde{A}, R)$ is discrete and equivalent to $\hom_{\mathrm{Alg}(\mathrm{ho}(C))}(A, R)$.

A corollary of the study of the commutative case is the fact that, unlike general separable algebras, the category of commutative separable algebras satisfies a strong form of descent, namely the functor that assigns to C its category of commutative separable algebras preserves limits.

We conclude with a few words about proofs: a key point in these theorems is that except for one statement, they are entirely elementary. In particular, we only use obstruction theory to "get off the ground", namely to go from an algebra up to homotopy to a coherent algebra. In other words, all the results except for the fact that $\mathcal{M}_A^{\mathbb{E}_1}$ is non-empty (with the notation from Theorem 2) can be proved with no obstruction theory. The obstruction theory that we do use to get this non-emptiness is a version of Goerss–Hopkins obstruction theory, as developed by Pstragowski–VanKoughnett [4].

References

[1] M. Auslander, O. Goldman, *The Brauer group of a commutative ring*, Transactions of the American Mathematical Society, **97(3)** (1960), 367–409
[2] P. Balmer, *Separability and triangulated categories*, Advances in Mathematics, **226(5)** (2011), 4352–4372
[3] M. Ramzi, *Separability in homotopical algebra*, arXiv preprint, arXiv:2305.17236 (2023)
[4] P. Pstragowski, P. VanKoughnett, *Abstract Goerss–Hopkins theory*, Advances in Mathematics, **395** (2022), 108098.

Separable commutative algebras and Galois theory in stable homotopy theories

Niko Naumann
(joint work with Luca Pol)

We report on results of the joint paper available at arxiv.org/abs/2305.01259.

We relate two different proposals to extend the étale topology into homotopy theory, namely via the notion of finite cover introduced by Mathew and via the notion of separable commutative algebra introduced by Balmer. We show that

[1] For $d \geq 2$, the homotopy commutative condition is vacuous, but it is important for $d = 1$.

finite covers are precisely those separable commutative algebras with underlying dualizable module, which have a locally constant and finite degree function. We then use Galois theory to classify separable commutative algebras in numerous categories of interest. Examples include the category of modules over an $\mathbb{E}_\infty$-ring R which is either connective or even periodic with $\pi_0(R)$ regular Noetherian, the stable module category of a finite group of p-rank one and the derived category of a qcqs scheme.

Scissors congruence *K*-theory and traces

Anna Marie Bohmann
(joint work with Teena Gerhardt, Cary Malkiewich, Mona Merling, Inna Zakharevich)

The scissors congruence problem asks the following question: given two polytopes P and Q, when is it possible to cut one up using finitely many straight cuts and reassemble the pieces to get the other? It is immediately apparent that a necessary condition is that P and Q have the same volume. When we consider polygons—i.e., polytopes in two-dimensional Euclidean space—this necessary condition is also sufficient. That is, any two polygons in the plane with the same area are "scissors congruent."

The more general question dates back to Hilbert's Third Problem, which asked if volume was correspondingly the only scissors congruence invariant for polytopes in Euclidean 3-space E^3. This question was quickly answered in the negative by Dehn [2], who built a second scissors congruence invariant of three-dimensional polytopes and showed it took different values on tetrahedra and cubes of the same volumes. In the 1960s, Sydler [4] showed that, together, volume and the Dehn invariant form a complete scissors congruence invariant for polytopes in E^3; the same is true for Euclidean 4-space, but finding complete scissors congruence invariants remains an open question in dimensions 5 and higher.

Nowadays, the usual approach to scissors congruence is to algebraicize the question by building scissors congruence groups. For a nice geometry X, such as Euclidean, hyperbolic or spherical geometry of some fixed dimension, and a subgroup of G of the isometry group of X, we form the scissors congruence group $P(X, G)$. This is the free abelian group on polytopes in X modulo the relations

- $P + Q = P \amalg Q$
- $gP = P$ for $g \in G$.

By construction, polytopes are identified in $P(X, G)$ if they are scissors congruent with "moves" from G allowed. Typical choices of G are things like the subgroup of all translations or the subgroup of orientation-preserving isometries. When G is the trivial group, this simply measures whether two polytopes can be cut up into the same pieces with no moving allowed at all.

We can think of $P(X, G)$ as an example of a "group complete, break stuff up, and impose relations" construction, and from such a perspective, it's no surprise that it arises from algebraic K-theory. Making good on this promise does require a new approach to K-theory, however. In [6], Zakharevich builds a spectrum $K(\mathcal{P}_G^X)$ with $\pi_0(K(\mathcal{P}_G^X)) = P(X,G)$ as an example of her construction of K-theory of an assembler. This allows us to consider not only the scissors congruence groups $P(X,G)$, but also the "higher scissors congruence groups" defined as $\pi_n(K(\mathcal{P}_G^X))$ for $n > 0$. These groups encode scissors congruence automorphisms, for example.

In [1], we generalize the notion of K-theory of an assembler to K-theory of a structure we call a "category with covering families." The motivating example $\mathcal{P}_G^X$ comes from polytopes in a geometry X, with "covering families" given by cutting polytopes up and applying moves from $G < \mathrm{Isom}(X)$. We show that the K-theory here agrees with Zakharevich's scissors congruence K-theory, so that the notational collision is harmless. Another key example comes from an abelian group A where "covering families" encode addition in A. The K-theory in this example recovers the Eilenberg–MacLane spectrum HA. A "measure" on polytopes, valued in A, assigns each polytope an element of A in way that induces a functor between the corresponding categories with covering families, and thus a map of K-theory spectra. An essential example is volume, which is a measure valued in $\mathbb{R}$. The fact that assigning a polytope to its volume induces a functor of categories with covering families boils down to the observation that cutting polytopes into pieces preserves the total volume.

These ideas allow us to construct trace maps from (higher) scissors congruence groups to group homology as follows. We first make an important identification.

Theorem 1 (BGMMZ)**.** For any group of isometries G, the scissors congruence K-theory $K(\mathcal{P}_G^X)$ is actually the homotopy orbit spectrum $K(\mathcal{P}_1^X)_{hG}$ of the scissors congruence K-theory with trivial isometry group.

This result is closely related to Thomason's work on commuting homotopy colimits with algebraic K-theory [5].

We next see that a G-equivariant measure on polytopes, now valued in a $\mathbb{Z}[G]$-module A, induces a map of K-theory spectra

$$K(\mathcal{P}_1^X) \to HA.$$

By passing to homotopy orbits for the G-action on both sides and using the previous theorem, we obtain a trace map.

Theorem 2 (BGMMZ)**.** If A is a $\mathbb{Z}[G]$-module and $\mu\colon P(X,1) \to A$ is a $\mathbb{Z}[G]$-module map, there is a trace map

$$K(\mathcal{P}_G^X) \to HA_{hG}$$

which on homotopy groups produces maps

$$\mathrm{tr}\colon \pi_n(K(\mathcal{P}_G^X)) \to H_n(G;A)$$

Note that the right-hand side is the homology of G as a discrete group.

These trace maps are surprisingly computable. In [1], we give several accessible examples and Malkiewich uses them in [3] to identify scissors congruence K-theory in terms of Tits buildings. In this talk, we elaborate on the framework for these trace maps and some of these computable examples.

References

[1] Anna Marie Bohmann, Teena Gerhardt, Cary Malkiewich, Mona Merling and Inna Zakharevich. "A trace map on higher scissors congruence groups." 2023. arXiv:2303.13183.
[2] M. Dehn. "Ueber den Rauminhalt." *Math. Ann.*, 55(3):465–478, 1901.
[3] Cary Malkiewich. "Scissors congruence K-theory is a Thom spectrum," 2022. arXiv:2210.08082.
[4] J.-P. Sydler. "Conditions nécessaires et suffisantes pour l'équivalence des polyèdres de l'espace euclidien à trois dimensions." *Comment. Math. Helv.*, 40:43–80, 1965.
[5] Robert W. Thomason. "First quadrant spectral sequences in algebraic K-theory via homotopy colimits." *Comm. Algebra*, 10(15):1589–1668, 1982.
[6] Inna Zakharevich. The K-theory of assemblers. *Adv. Math.*, 304:1176–1218, 2017.

Integral endotrivial modules

Achim Krause
(joint work with Jesper Grodal)

For a ring k and finite group G, the stable module category is defined as

$$\mathrm{stmod}(k[G]) := \mathrm{Fun}(BG, \mathrm{D}(k)^\omega)/\langle k[G]\rangle,$$

i.e. the Verdier quotient of the full subcategory of $\mathrm{D}(k[G])$ consisting of all complexes perfect over k by the thick subcategory generated by the free module $k[G]$. Trading cells, one sees that the homotopy category of $\mathrm{stmod}(k[G])$ admits an elementary description with

- Objects given by $k[G]$-modules M which are finitely generated projective as k-modules, and
- Morphisms between objects represented by such modules M and N given by ordinary $k[G]$-module morphisms $M \to N$, modulo those which factor through projective $k[G]$-modules.

In fact, mapping spectra in $\mathrm{stmod}(k[G])$ are described by $\mathrm{map}_{\mathrm{stmod}(k[G])}(M,N) = \mathrm{Hom}_k(M,N)^{tG}$, the Tate construction.

The k-linear tensor product induces a symmetric-monoidal structure on $\mathrm{stmod}(k[G])$, and every object is dualizable, with duals computed as k-linear dual on representatives as above. An object M is invertible if and only if it is inverse to its dual, i.e. if the map $k \to M \otimes M^\vee \cong \mathrm{End}_k(M)$ is an equivalence. This happens if and only if its cokernel is projective, i.e. if $\mathrm{End}_k(M) \cong k \oplus P$ for P a projective $k[G]$-module. Such M are called *endotrivial modules.*

Example 1. The shift $k[1]$ always provides an invertible object of $\mathrm{stmod}(k[G])$ (which can be represented also by the cokernel of $k \to k[G]$). If $G = C_n$, $k[1]^{\otimes 2} \simeq k$, since Tate cohomology of C_n is 2-periodic. In general, $k[1]$ has infinite order, since Tate cohomology is typically not periodic (for example for $G = C_p \times C_p$ and p not invertible in k).

With coefficients k a field of characteristic p, the Picard group $\mathrm{Pic}(\mathrm{stmod}(k[G]))$ has been well-studied in the literature (for a more precise discussion, see the introduction of [2]). In this talk, we ask: What about integer coefficients? These are best understood in two steps, passing from finite field coefficients to $\mathbb{Z}_p$-coefficients, and then from $\mathbb{Z}_p$-coefficients for all p to $\mathbb{Z}$-coefficients.

Lemma 1 ("Alperin lifting")**.** For $\mathcal{O}_K$ the ring of integers in a p-adic number field, with maximal ideal $\mathfrak{m}$ and residue field $k = \mathcal{O}_K/\mathfrak{m}$, we have an exact sequence

$$0 \to \mathrm{Hom}(G, (1+\mathfrak{m})^\times) \to \mathrm{Pic}(\mathrm{stmod}(\mathcal{O}_K[G])) \to \mathrm{Pic}(\mathrm{stmod}(k[G])) \to 0$$

Proof. The surjectivity of the right-hand map is due to Alperin [1]. For determining the kernel, we assume that M is a $\mathcal{O}_K[G]$-module which is finitely generated projective over $\mathcal{O}_K$ which is $P \oplus k$ after base-change to k, with P a projective $k[G]$-module. We may lift P to a projective $\mathcal{O}_K[G]$-module $\widetilde{P}$, and the inclusion map to a map $\widetilde{P} \to M$. The cokernel is $\mathcal{O}_K$, with some G-action which reduces to 1 modulo $\mathfrak{m}$, hence is determined by a homomorphism $G \to (1+\mathfrak{m})^\times$. □

In particular, we have $\mathrm{Pic}(\mathrm{stmod}(\mathbb{Z}_p[G])) \cong \mathrm{Pic}(\mathrm{stmod}(\mathbb{F}_p[G]))$ for odd p, and an exact sequence

$$0 \to \mathrm{Hom}(G, \{\pm 1\}) \to \mathrm{Pic}(\mathrm{stmod}(\mathbb{Z}_2[G])) \to \mathrm{Pic}(\mathrm{stmod}(\mathbb{F}_2[G])) \to 0,$$

so the $\mathbb{Z}_p$-coefficients version is quite close to the $\mathbb{F}_p$-coefficients version. For $\mathbb{Z}$-coefficients, something more interesting happens. Motivated by fracture squares for ordinary modules, one might expect that the $\mathbb{Z}$-coefficient case is determined by $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{Q}$-coefficients. Since $\mathrm{stmod}(k[G]) \simeq 0$ if $|G|$ is invertible in k, $\mathbb{Q}_p$ and $\mathbb{Q}$ do not contribute and one might hope that the canonical map

$$\mathrm{stmod}(\mathbb{Z}[G]) \to \prod_{p||G|} \mathrm{stmod}(\mathbb{Z}_p[G])$$

is an equivalence. It is in fact fully faithful, but it is not essentially surjective: Already for $G = C_6$, the element given by $\mathbb{Z}_2 \in \mathrm{stmod}(\mathbb{Z}_2[C_6])$ and $0 \in \mathrm{stmod}(\mathbb{Z}_3[C_6])$ is not in the image, as explained below. The problem lies in the fact that $\mathrm{stmod}(\mathbb{Z}[G])$ is not idempotent complete.

Lemma 2 ([4, Prop. 5.2])**.** The functor

$$\mathrm{stmod}(\mathbb{Z}[G]) \to \prod_{p||G|} \mathrm{stmod}(\mathbb{Z}_p[G])$$

exhibits the target as idempotent completion of the source.

Proof. We have $\pi_0 \,\mathrm{end}_{\mathrm{stmod}(\mathbb{Z}[G])}(\mathbb{Z}) = \mathbb{Z}/|G|$, and so every object X is a retract of $X/|G|$. Since torsion complexes split canonically into their p-completions, this shows that the functor is fully faithful, and that every object in the target is a retract of an object in the essential image. It remains to check that the target is idempotent complete. Given some object of $\mathrm{stmod}(\mathbb{Z}_p[G])$ with an idempotent, we may represent it by a $\mathbb{Z}_p[G]$-module M which is finitely generated free over $\mathbb{Z}_p$, and some endomorphism $\varepsilon : M \to M$. This is not necessarily idempotent, but all powers ε^n represent the same idempotent in $\mathrm{stmod}(\mathbb{Z}_p[G])$. By compactness, some subsequence of the ε^n converges to a true idempotent $\widetilde{\varepsilon}$, whose image splits the original idempotent in $\mathrm{stmod}(\mathbb{Z}_p[G])$. □

In particular, fully faithfulness shows that the canonical map

$$\mathrm{Pic}(\mathrm{stmod}(\mathbb{Z}[G])) \to \prod_{p||G|} \mathrm{Pic}(\mathrm{stmod}(\mathbb{Z}_p[G]))$$

is injective. What remains is the question of how to characterize its image.

Remark 1. By an observation of Thomason [5, Theorem 2.1], to see whether an object of the idempotent completion lies in the image of the original category, it suffices to check that its K_0-class does. In our case, the relevant localisation sequence reads

$$K_0(\mathrm{Fun}(BG, \mathrm{D}(\mathbb{Z})^\omega)) \to K_0(\mathrm{stmod}(\mathbb{Z}[G])) \to K_{-1}(\mathbb{Z}[G]) \to 0,$$

so the relevant obstruction lies in $K_{-1}(\mathbb{Z}[G])$. In particular, the subsequent discussion can be interpreted as an explicit character-theoretic description of $K_{-1}(\mathbb{Z}[G])$. This can be seen as a more explicit version of the description given in [6].

Lemma 3. An object $X \in \mathrm{stmod}(\mathbb{Z}_p[G])$ has a well-defined *partial character*

$$\chi_X(g) : \mathrm{Sing}_p(G) \to \mathbb{Q}_p,$$

where $\mathrm{Sing}_p(G)$ is the subset of elements of G of order divisible by p.

Proof. For any $\mathbb{Z}_p[G]$-module M, we may consider $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$ as $\mathbb{Q}_p$-linear G-representation, which has a character $\chi_M : G \to \mathbb{Q}_p$. We may extend this to $\mathrm{Fun}(BG, \mathrm{D}(\mathbb{Z}_p)^\omega)$ by alternating sums. For a projective $\mathbb{Z}_p[G]$-module P, $\chi_P : G \to \mathbb{Q}_p$ vanishes on elements of order divisible by p (see e.g. [3, Theorem 36]), so the restriction of χ_M to $\mathrm{Sing}_p(G)$ remains well-defined after passing to $\mathrm{stmod}(\mathbb{Z}_p[G])$. □

Theorem 1. An element $(X_p) \in \prod_{p||G|} \mathrm{stmod}(\mathbb{Z}_p[G])$ lies in the image of $\mathrm{stmod}(\mathbb{Z}[G])$ if and only if the following conditions are satisfied:

(1) Each of the partial characters $\chi_{X_p} : \mathrm{Sing}_p(G) \to \mathbb{Q}_p$ takes values in $\mathbb{Q}$.
(2) For each pair of primes, χ_{X_p} and χ_{X_ℓ} agree on $\mathrm{Sing}_p(G) \cap \mathrm{Sing}_\ell(G)$.
(3) The resulting map $\bigcup_p \chi_{X_p} : G \setminus \{e\} \to \mathbb{Q}$ is the restriction of a character of a (virtual) rational representation of G.

This provides a very computable criterion to determine the image. For example, in the aforementioned case $G = C_6$, the partial character of the trivial representation $\mathbb{Z}_2$ is constant 1 on $\mathrm{Sing}_2(G)$, the partial character of 0 is constant 0 on $\mathrm{Sing}_3(G)$, and so condition 2 is violated since $\mathrm{Sing}_2(C_6) \cap \mathrm{Sing}_3(C_6) \neq \emptyset$. On the other hand, if $G = \Sigma_3$, there are no elements of composite order, and one checks that all three conditions are trivially satisfied, so

$$\mathrm{stmod}(\mathbb{Z}[\Sigma_3]) \simeq \mathrm{stmod}(\mathbb{Z}_2[\Sigma_3]) \times \mathrm{stmod}(\mathbb{Z}_3[\Sigma_3]).$$

Remark 2. A more general version of Theorem 1 exists, where $\mathbb{Z}$ is replaced by an arbitrary localisation of a number ring, and the $\mathbb{Z}_p$ by the completions at all finite places. Conditions 1,2,3 are replaced by their obvious analogues, paying careful attention to the fact that condition 2 for two prime ideals $\mathfrak{p}, \mathfrak{p}'$ dividing the same integral prime p is not vacuous!

For invertible objects, the situation simplifies, since the partial characters are then required to take values in $\{\pm 1\}$. On $\mathrm{Sing}_p(G)$ for an odd prime p, one can even see that the partial characters are constant $+1$ or -1, which leads to clean descriptions of $\mathrm{Pic}(\mathrm{stmod}(\mathbb{Z}[G]))$ for odd $|G|$. For $p = 2$, more complicated partial characters can occur, and we are currently investigating uniform descriptions of $\mathrm{Pic}(\mathrm{stmod}(\mathbb{Z}[G]))$ for even $|G|$.

References

[1] J. L. Alperin, *Lifting endo-trivial modules*, J. Group Theory 4 (2001).

[2] J. Grodal, *Endotrivial modules for finite groups via homotopy theory*, Journal of the American Mathematical Society 36.1 (2023): 177–250.

[3] J. P. Serre, *Linear representations of finite groups*, Vol. 42. Springer, 1977.

[4] A. Krause *The Picard group in equivariant homotopy theory via stable module categories*, arXiv preprint arXiv:2008.05551 (2020).

[5] R. W. Thomason *The classification of triangulated subcategories*, Compositio Mathematica, 105(1), 1–27. doi:10.1023/A:1017932514274

[6] D. W. Carter *Localization in lower algebraic K-theory*, Communications in Algebra, 8:7, 603–622, doi:10.1080/00927878008822478

Classifying modules of equivariant Eilenberg–MacLane spectra

Clover May

(joint work with Jacob Grevastad)

Classically, any module over the Eilenberg–MacLane spectrum $H\mathbb{F}_p$ splits as a wedge of suspensions of $H\mathbb{F}_p$ itself. Equivariantly, the module theory of G-equivariant Eilenberg–MacLane spectra is much more complicated. For the group $G = C_p$ and the constant Mackey functor $\underline{\mathbb{F}}_p$, there are infinitely many indecomposable $H\underline{\mathbb{F}}_p$-modules. Previous work joint with Dugger and Hazel classified all indecomposable $H\underline{\mathbb{F}}_2$-modules for the group $G = C_2$. The isomorphism classes of indecomposables fit into just three families. By contrast, we show for $G = C_p$ with p an odd prime, the classification of indecomposable $H\underline{\mathbb{F}}_p$-modules is wild. That is, any complete description would necessarily include a simultaneous classification of indecomposable modules for every finite-dimensional $\mathbb{F}_p$-algebra. This is joint work in progress with Grevstad.

This talk began with recalling a theorem of Hopkins and Smith about graded fields and spectra. A graded field is a graded ring in which every nonzero homogeneous element is a unit.

Theorem 1 (Hopkins–Smith 1998 [3])**.** If R is a ring spectrum with $\pi_*(R)$ a graded field, then every $X \in R$–Mod splits as a wedge of suspensions of R. That is

$$X \simeq \bigvee_{i \in I} \Sigma^{n_i} R.$$

As a first example we have the Eilenberg–MacLane spectrum $H\mathbb{F}_p$. Another example is a spectrum that has played a central role at this workshop, and the one Hopkins and Smith were concerned with, Morava K-theory $K(n)$ at a prime p. Recall $\pi_*(K(n)) \cong \mathbb{F}_p[v_n^{\pm 1}]$, where $|v_n| = 2(p^n - 1)$.

For the group $G = C_2$, there is a genuine equivariant C_2-spectrum $H\underline{\mathbb{F}}_2$ that represents $RO(C_2)$-graded cohomology with an analogue of $\mathbb{F}_2$-coefficients. This is an equivariant cohomology theory graded on representations. Previous work joint with Dugger and Hazel classified compact $H\underline{\mathbb{F}}_2$-modules.

Theorem 2 (Dugger–Hazel–M. 2023 [2])**.** If Z is a compact $H\underline{\mathbb{F}}_2$-module then Z splits as a wedge of $RO(G)$-suspensions of

$$H\underline{\mathbb{F}}_2, \qquad (S^n_a)_+ \wedge H\underline{\mathbb{F}}_2, \qquad \text{and} \qquad \mathrm{cof}\,(\tau^m)\,,$$

where $n \geq 0$ and $m \geq 1$.

For the moment, let us focus not on these particular objects, but emphasize that there are three families of (isomorphism classes of) indecomposable $H\underline{\mathbb{F}}_2$-modules Observe that the $RO(C_2)$-graded homotopy $\pi_{*,*}H\underline{\mathbb{F}}_2 = \mathbb{M}_2$ is not a graded field. It is a non-Noetherian ring with two cones of infinitely-many elements. One cone is polynomial in elements ρ and τ, and the other has an element θ that is infinitely divisible by ρ and τ. (This is the same τ that appears in the decomposition theorem, and one could rename $(S^n_a)_+ \wedge H\underline{\mathbb{F}}_2$ as $\mathrm{cof}(\rho^{n+1})$). In fact, this ring $\mathbb{M}_2$ has a complicated module theory, so it is no surprise we need more indecomposables beyond $H\underline{\mathbb{F}}_2$. On the other hand, $H\underline{\mathbb{F}}_2$-modules look a bit like finitely-generated modules over a PID, even though $\mathbb{M}_2$ is not a graded PID. That is, there are infinitely-many indecomposables, but they fit into families that are easy enough to describe.

Our proof of the decomposition of $H\underline{\mathbb{F}}_2$-modules used a result by Schwede and Shipley to translate the problem to the algebra of Mackey functors.

Theorem 3 (Schwede–Shipley 2003 [4])**.** There is a Quillen equivalence

$$H\underline{\mathbb{F}}_2\mathrm{-Mod} \simeq Ch(\underline{\mathbb{F}}_2).$$

In [2], we classified perfect complexes of $\underline{\mathbb{F}}_2$-modules up to isomorphism via a change of basis algorithm. Then we translated the result to $H\underline{\mathbb{F}}_2$-modules.

Perhaps I ought to mention, we saw $\mathbb{M}_2 = \pi_{*,*}H\underline{\mathbb{F}}_2$ was not a graded field, but there are graded fields in the equivariant context. For example, still taking $G = C_2$, we can consider $H\underline{\mathbb{F}}_p$ where p is an odd prime. Now $\pi_{*,*}H\underline{\mathbb{F}}_p$ is a graded field, so indeed any $X \in H\underline{\mathbb{F}}_p$–Mod splits as a wedge of $RO(G)$-suspensions of $H\underline{\mathbb{F}}_p$ as we would expect.

Let us shift to the group $G = C_p$ for p an odd prime and consider $H\underline{\mathbb{F}}_p$. There are many indecomposables one can find. The first example of course is $H\underline{\mathbb{F}}_p$. There is also a free C_p action on any odd-dimensional sphere, and there are indecomposables of the form $\left(S^{2n+1}_{\text{free}}\right)_+ \wedge H\underline{\mathbb{F}}_p$. There is another C_p-space I call the eggbeater and an indecomposable $EB \wedge H\underline{\mathbb{F}}_p$, etc. I have tried and failed several times to prove a complete classification. It turns out to be impossible.

Theorem 4 (Grevstad–M. in progress)**.** For $G = C_p$ the classification of compact $H\underline{\mathbb{F}}_p$-modules is wild.

Morally, this means it is impossible to classify all indecomposables. A bit more precisely, this means any such classification would include every finite-dimensional indecomposable module of every finite-dimensional $\mathbb{F}_p$-algebra. For example, it is well known that $\mathbb{F}_p[C_p \times C_p]$ is wild unless $p = 2$. See for example [1].

Definition 1. We say a k-algebra R has **wild representation type** if there is a functor $k\langle a, b\rangle$–Mod $\to R$–Mod that reflects isomorphisms and preserves indecomposables.

The conditions here tell us every isomorphism class of indecomposable in the module category for the free k-algebra on two generators $k\langle a, b\rangle$ contributes to an isomorphism class of indecomposable in R–Mod. Furthermore, this implies R–Mod contains every finite-dimensional indecomposable module of every finite-dimensional k-algebra by the following result.

Proposition 1. Let S be a finite-dimensional k-algebra with generators and relations with $S \cong k\langle x_1, x_2, \ldots, x_n\rangle / \sim$. There is a functor S–Mod $\to k\langle a, b\rangle$–Mod that reflects isomorphisms and preserves indecomposables.

We prove that $\underline{\mathbb{F}}_p$ is representation finite, so has finitely-many isomorphism classes of indecomposables, but derived wild, meaning $\mathcal{D}_{\mathrm{perf}}(\underline{\mathbb{F}}_p)$ is wild. The proof uses quiver representations. Analogously to Theorem 3, Schwede–Shipley showed $Ho(H\underline{\mathbb{F}}_p\text{–Mod})^{\omega} \simeq \mathcal{D}_{\mathrm{perf}}(\underline{\mathbb{F}}_p)$, from which we conclude our main result, Theorem 4.

References

[1] Bondarenko, V. & Drozd, J. The representation type of finite groups. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI).* **71** pp. 24-41, 282 (1977), Modules and representations

[2] Dugger, D., Hazel, C. & May, C. Equivariant $\underline{\mathbb{Z}}/\ell$-modules for the cyclic group C_2. *J. Pure Appl. Algebra.* **228**, Paper No. 107473, 48 (2024), https://doi.org/10.1016/j.jpaa.2023.107473

[3] Hopkins, M. & Smith, J. Nilpotence and stable homotopy theory. II. *Ann. Of Math. (2).* **148**, 1-49 (1998), https://doi.org/10.2307/120991

[4] S. Schwede and B. Shipley, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. **3** (2003), no. 1, 287–334.

Chern classes in equivariant bordism

Stefan Schwede

This talk was a report on my paper `arxiv:2303.12366` with the same title. Complex cobordism **MU** is arguably the most important cohomology theory in algebraic topology. It represents the bordism theory of stably almost complex manifolds, and it is the universal complex oriented cohomology theory; via Quillen's celebrated theorem [7], **MU** is the entry gate for the theory of formal group laws into stable homotopy theory, and thus the cornerstone of chromatic stable homotopy theory.

Tom Dieck's homotopical equivariant bordism $\mathbf{MU}_G$ [10], defined with the help of equivariant Thom spaces, strives to be the legitimate equivariant refinement of complex cobordism, for compact Lie groups G. The theory $\mathbf{MU}_G$ is the universal equivariantly complex oriented theory; and for abelian compact Lie groups, the coefficient ring $\mathbf{MU}_G^*$ carries the universal G-equivariant formal group law [5]. Homotopical equivariant bordism receives a homomorphism from the geometrically defined equivariant bordism theory; due to the lack of equivariant transversality, this homomorphism is *not* an isomorphism for non-trivial groups. In general, the equivariant bordism ring $\mathbf{MU}_G^*$ is still largely mysterious; we elucidate its structure for unitary groups, and for products of unitary groups.

Chern classes are important characteristic classes for complex vector bundles that were originally introduced in singular cohomology. Conner and Floyd [3, Corollary 8.3] constructed Chern classes for complex vector bundles in complex cobordism; in the universal cases, these yield classes $c_k \in \mathbf{MU}^{2k}(BU(m))$ that are nowadays referred to as Conner-Floyd-Chern classes. Conner and Floyd's construction works in much the same way for any complex oriented cohomology theory; in singular cohomology, it reduces to the classical Chern classes.

We define Chern classes in $U(m)$-equivariant homotopical bordism that map to the Conner-Floyd-Chern classes under tom Dieck's bundling homomorphism [10, Proposition 1.2]. Since $\mathbf{MU}$ comes with the structure of a global ring spectrum, it supports graded-commutative multiplications on $\mathbf{MU}_G^*$, as well as external multiplication pairings

$$\times \ : \ \mathbf{MU}_G^k \times \mathbf{MU}_K^l \ \to \ \mathbf{MU}_{G\times K}^{k+l}$$

for all pairs of compact Lie groups G and K. We write

$$e_k \ \in \ \mathbf{MU}_{U(k)}^{2k}$$

for the Euler class of the tautological representation of the unitary group $U(k)$ on $\mathbb{C}^k$, and $\mathrm{tr}_{U(k,m-k)}^{U(m)} : \mathbf{MU}_{U(k,m-k)}^* \to \mathbf{MU}_{U(m)}^*$ for the transfer from the block subgroup $U(k, m-k)$ to $U(m)$.

For $0 \leq k \leq m$, the *k-th Chern class* in homotopical equivariant bordism is

$$c_k^{(m)} \ = \ \mathrm{tr}_{U(k,m-k)}^{U(m)}(e_k \times 1_{m-k}) \ \in \ \mathbf{MU}_{U(m)}^{2k} \ ,$$

where $1_{m-k} \in \mathbf{MU}_{U(m-k)}^0$ is the multiplicative unit.

For example, the class $c_0^{(m)} = 1_m$ is the multiplicative unit, and $c_m^{(m)} = e_m$ is the Euler class of the tautological $U(m)$-representation. The familiar structural properties of the Conner-Floyd-Chern classes already hold for the Chern classes in $U(m)$-equivariant $\mathbf{MU}$-theory:

(i) For all $0 \leq k \leq m = i + j$, the relation

$$\mathrm{res}_{U(i,j)}^{U(m)}(c_k^{(m)}) \ = \ \sum_{d=0,\ldots,k} c_d^{(i)} \times c_{k-d}^{(j)}$$

holds in the group $\mathbf{MU}_{U(i,j)}^{2k}$.

(ii) The relation

$$\mathrm{res}^{U(m)}_{U(m-1)}(c_k^{(m)}) \;=\; \begin{cases} c_k^{(m-1)} & \text{for } 0\le k\le m-1, \text{ and}\\ 0 & \text{for } k=m \end{cases}$$

holds in the group $\mathbf{MU}^{2k}_{U(m-1)}$.

(iii) Let T^m denote the diagonal maximal torus of $U(m)$. Then the restriction homomorphism

$$\mathrm{res}^{U(m)}_{T^m} \;:\; \mathbf{MU}^{2k}_{U(m)} \;\to\; \mathbf{MU}^{2k}_{T^m}$$

takes the class $c_k^{(m)}$ to the k-th elementary symmetric polynomial in the classes $p_1^*(e_1), \dots, p_m^*(e_1)$, where $p_i : T^m \to T = U(1)$ is the projection to the i-th factor.

(iv) The bundling homomorphism

$$\mathbf{MU}^*_{U(m)} \;\to\; \mathbf{MU}^*(BU(m))$$

takes $c_k^{(m)}$ to the k-th Conner-Floyd-Chern class.

Since the Chern classes are defined as transfers, the main ingredient in proving part (i) above is to work out the double coset formula for $\mathrm{res}^{U(m)}_{U(i,j)} \circ \mathrm{tr}^{U(m)}_{U(k,m-k)}$.

Despite many formal similarities, there are crucial qualitative differences compared to Chern classes in complex oriented cohomology theories: our Chern classes are *not* characterized by their restriction to the maximal torus – in contrast to the non-equivariant situation for complex oriented cohomology theories. Indeed, there is no 'splitting principle' in homotopical equivariant bordism, as the restriction homomorphism

$$\mathrm{res}^{U(m)}_{T^m} \;:\; \mathbf{MU}^{*}_{U(m)} \;\to\; \mathbf{MU}^{*}_{T^m}$$

to the maximal torus T^m is *not* injective for $m \ge 2$. For example, the class

$$1 - \mathrm{tr}_N^{U(m)}(1) \;\in\; \mathbf{MU}^0_{U(m)}$$

is nonzero and in the kernel of restriction from $U(m)$ to T^m, where $N = N_{U(m)}T^m$ is the maximal torus normalizer. Moreover,

$$c_1^{(2)} \cdot (1 - \mathrm{tr}_N^{U(2)}(1)) \;=\; 0 \;,$$

so the Chern class $c_1^{(2)}$ is a zero-divisor in the ring $\mathbf{MU}^*_{U(2)}$, also in stark contrast to Chern classes in complex oriented cohomology theories.

The Chern classes feature in new structure results about the equivariant bordism rings $\mathbf{MU}^*_{U(m)}$ for unitary groups. To put this into context, we recall that in the special case when G is an *abelian* compact Lie group, the graded ring $\mathbf{MU}^*_G$ is concentrated in even degrees and free as a module over the non-equivariant cobordism ring $\mathbf{MU}^*$ [2, Theorem 5.3], and the bundling homomorphism $\mathbf{MU}^*_G \to \mathbf{MU}^*(BG)$ is completion at the augmentation ideal of $\mathbf{MU}^*_G$ [1, Theorem 1.1]. For non-abelian compact Lie groups G, however, the equivariant bordism rings $\mathbf{MU}^*_G$ are still largely mysterious.

Theorem. Let $m \ge 1$ be a natural number.

(i) The sequence of Chern classes $c_m^{(m)}, c_{m-1}^{(m)}, \dots, c_1^{(m)}$ is a regular sequence that generates the augmentation ideal of $\mathbf{MU}^*_{U(m)}$.
(ii) The completion of $\mathbf{MU}^*_{U(m)}$ at the augmentation ideal is a graded $\mathbf{MU}^*$-power series algebra in the above Chern classes.
(iii) The bundling homomorphism $\mathbf{MU}^*_{U(m)} \to \mathbf{MU}^*(BU(m))$ extends to an isomorphism

$$(\mathbf{MU}^*_{U(m)})^\wedge_I \ \to \ \mathbf{MU}^*(BU(m))$$

from the completion at the augmentation ideal.

The proof of the theorem makes crucial use of the splitting theorem for global functors established in [9]. Therefore, it is highly relevant that the theories $\mathbf{MU}_G$ for varying compact Lie groups G assemble into a global stable homotopy type, see [8, Example 6.1.53].

Greenlees and May [4, Corollary 1.6] construct a local homology spectral sequence

$$E_2^{p,q} \ = \ H^I_{-p,-p}(\mathbf{MU}^*_G) \ \Longrightarrow \ \mathbf{MU}^{p+q}(BG) \ .$$

The regularity of the Chern classes implies that for $U(m)$ the $E_2^{p,q}$-term vanishes for all $p \neq 0$, and the spectral sequence degenerates into the previous isomorphism

$$E_2^{0,*} \ \cong \ (\mathbf{MU}^*_{U(m)})^\wedge_I \ \cong \ \mathbf{MU}^*(BU(m)) \ .$$

We also use the Chern classes to reformulate the $\mathbf{MU}_G$-completion theorem of Greenlees-May [4] and La Vecchia [6], for any compact Lie group G. The new insight is that the ideal generated by the Chern classes of any faithful G-representation is 'sufficiently large' in the sense of [4, Definition 2.4].

References

[1] G Comezaña, J P May, *A completion theorem in complex bordism.* Chapter XXVII in: *Equivariant homotopy and cohomology theory.* CBMS Regional Conference Series in Mathematics, 91. Amer. Math. Soc., 1996, xiv+366 pp.

[2] G Comezaña, *Calculations in complex equivariant bordism.* Chapter XXVIII in: *Equivariant homotopy and cohomology theory.* CBMS Regional Conference Series in Mathematics, 91. Amer. Math. Soc., 1996. xiv+366 pp.

[3] P E Conner, E E Floyd, *The relation of cobordism to K-theories.* Lecture Notes in Mathematics, No. 28 Springer-Verlag, Berlin-New York 1966, v+112 pp.

[4] J P C Greenlees, J P May, *Localization and completion theorems for MU-module spectra.* Ann. of Math. (2) 146 (1997), no. 3, 509–544.

[5] M Hausmann, *Global group laws and equivariant bordism rings.* Annals of Math. (2) 195 (2022), no. 3, 841–910.

[6] M La Vecchia, *The local (co)homology theorems for equivariant bordism.* To appear in Geom. Topol.

[7] D Quillen, *On the formal group laws of unoriented and complex cobordism theory.* Bull. Amer. Math. Soc. 75 (1969), 1293–1298.

[8] S Schwede, *Global homotopy theory.* New Mathematical Monographs 34, Cambridge University Press, 2018, xviii+828 pp.

[9] S Schwede, *Splittings of global Mackey functors and regularity of equivariant Euler classes.* Proc. London Math. Soc. (3) 125 (2022), 258–276.

[10] T tom Dieck, *Bordism of G-manifolds and integrality theorems.* Topology 9 (1970), 345–358.

Hecke operators on topological modular forms

Jack Morgan Davies

Let us write TMF for the cohomology theory of *topological modular forms*, originally constructed by Goerss–Hopkins–Miller. Similar to how topological K-theory enjoys connections to operator algebras, physics, commutative algebra, and algebraic geometry, TMF also has connections to other areas of mathematics. These include geometry and physics through the sigma orientation MString $\to$ TMF, and number theory through the comparison map of rings $\pi_*\mathrm{TMF} \to \mathrm{MF}_{*/2}$ into the ring of *meromorphic modular forms*; much of this background is described in the book [4].

In [3], we construct *stable Hecke operators* on TMF, so morphisms of spectra $\mathrm{T}_n \colon \mathrm{TMF}[\frac{1}{n}] \to \mathrm{TMF}[\frac{1}{n}]$, and show that the comparison map above commutes with these operations and n-times the classical Hecke operators on MF_*. The construction of these operators, as well as *Adams operations* ψ^k and *Atkin–Lehner involutions* w, and the proofs of there various properties and relations, are all a consequence of the following theorem. Let us write Isog for the category whose objects are those in the small étale site of the moduli stack of elliptic curves, and whose morphisms are given by morphisms of stacks and isogenies of elliptic curves of invertible degree.

Theorem 1 ([3, Th.A])**.** There is a functor $\mathcal{O}^{\mathrm{top}} \colon \mathrm{Isog}^{\mathrm{op}} \to \mathrm{CAlg}$ into the ∞-category of $\mathbf{E}_\infty$-rings, whose restriction to the small étale site of the moduli stack of elliptic curves agrees with the sheaf $\mathcal{O}^{\mathrm{top}}_{\mathrm{GHM}}$ used originally to construct TMF. There is also a further extension of $\mathcal{O}^{\mathrm{top}}$ to a category of spans.

This theorem allows us to define the above mentioned operations and also to prove various relations between them. For example, one can show that Hecke operators and Adams operations commute with one another and that the homotopies witnessing this commutativity are coherent up to all higher homotopies.

There are some curious consequences of these operations. For example, armed with the knowledge of some of the homotopy groups of TMF and their relation to the homotopy groups of $\mathbf{S}$, one obtains some simple number-theoretic congruences. Applying T_n to the element $\beta^3 \in \pi_{30}\mathrm{TMF}_{(3)} \simeq \mathbf{Z}/3\mathbf{Z}$, which lies in the image of the unit map $\mathbf{S} \to \mathrm{TMF}$, we obtain the formula

$$\mathrm{T}_n(\beta^3) = \beta^3\mathrm{T}_n(1) = \sigma(n)\beta^3, \qquad \sigma(n) = \sum_{d|n} d.$$

Here we choose n to not be divisible by 3, which is necessary to define the stable Hecke operators above; more on this point shortly. Similarly, using the expression of $\beta^3 = \alpha[\alpha\Delta]$ from the *descent spectral sequence* for TMF, we see that

$$\mathrm{T}_n(\beta^3) = \mathrm{T}_n(\alpha[\alpha\Delta]) = \alpha[n\alpha\mathrm{T}^{\mathrm{cl}}_n(\Delta)] = n\tau(n)\beta^3$$

where $\mathrm{T}^{\mathrm{cl}}_n$ is the classical Hecke operator on modular forms and $\tau(n)$ is the *Ramanujan τ-function.* These two calculations yield the congruence $\sigma(n) \equiv_3 n\tau(n)$ for all n not divisible by 3.

This small calculation recovers simple number-theoretic relations from purely topological sources, and the stream of information also flows in the opposite direction. Suppose there were an endomorphism T_3 on TMF, **without** inverting 3, that agreed with the classical Hecke operators through the comparison map. Then after localising at 3, the previous computation would go through without a hitch and we would obtain $3\tau(3) \equiv_3 \sigma(3) = 4$. The left-hand side vanishes modulo 3 and the right-hand side clearly does not. Hence a contradiction is reached. It is worth noting that $\tau(3)$ also vanishes modulo 3. This shows the necessity of inverting n in our construction of the nth Hecke operator on TMF, at least for $n = 3$.

The constructions of the operations mentioned above, the relations between them, the consequential congruences between Hecke eigenvalues on modular forms, and nonexistence statements for ψ^p and T_p at the prime p are all generalised and thoroughly explored in [3].

Let us add two more points, which we discussed both before, during, and after the presentation of this topic at Oberwolfach.

Firstly, there is a variant of TMF, denoted as Tmf and hereby called *projective* (or *proper* or *dualisable*) topological modular forms, such that its connective cover tmf gives a good model for holomorphic modular forms. All of the operations discussed above can be lifted to Tmf and hence also tmf; see [2] for a lift of ψ^k and upcoming work of the author for a lift of the Hecke operators. Unfortunately, the current constructions using Goerss–Hopkins obstruction theory do not produce a “compactified” version of the main theorem above. We ask: can the theorem above can be extended to moduli stack of generalised elliptic curves? Is there an approach to Tmf similar to Lurie’s approach to TMF in [5]?

Secondly, just as the definition of TMF is as the limit of $\mathcal{O}^{\mathrm{top}}_{\mathrm{GHM}}$, one might ask the question: what is the limit of $\mathcal{O}^{\mathrm{top}}$ over Isog? Notice that this limit has a copy of $\mathbf{Z}$ in π_0 as it factors $\mathbf{S} \to \mathrm{TMF}$, so no integers are inverted. Moreover, by projecting this diagram onto various subdiagrams, one notices that this limit also maps to all of Behrens’ $Q(N)$ spectra [1], so it in particular detects all of the divided α and β families (at least for primes greater than 3). The sections of $\mathcal{O}^{\mathrm{top}}$ are all TMF-local, so this limit recieves a map from the TMF-local sphere $\mathbf{S}_{\mathrm{TMF}}$. Is this map a good approximation of the TMF-local sphere? Can this limit diagram produce integral (or near integral) resolutions of the TMF-local sphere? We will come back to these questions another day.

References

[1] M. Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology (2006).
[2] J. M. Davies, *Constructing and calculating Adams operations on dualisable topological modular forms*, arXiv preprint (2021).
[3] J. M. Davies, *Hecke operators on topological modular forms*, arXiv preprint (2022).

[4] C. L. Douglas, J. Francis, A. G. Henriques, M. A. Hill, *Topological modular forms. Based on the Talbot workshop, North Conway, NH, USA, March 25–31, 2007*, Mathematical Surveys and Monographs (2014).
[5] J. Lurie, *Elliptic Cohomology II: Orientations*, available at `https://www.math.ias.edu/~lurie/` (2018).

Postnikov Towers of Logarithmic Ring Spectra

Tommy Lundemo

Logarithmic geometry [7] is a variant of algebraic geometry in which the notions of étaleness and smoothness are less rigid than usual. For example, tamely ramified extensions of (complete) discrete valuation rings (in mixed characteristic with perfect residue fields) participate in log étale morphisms, despite not being étale.

When advertised to homotopy theorists, log structures are often described as "intermediate localizations." By definition, a *(pre-)log ring* (R, P, β) consists of a commutative ring R, a commutative monoid P, and a map $\beta\colon P \to (R, \cdot)$ of commutative monoids. If R is a discrete valuation ring, a choice of uniformizer π_R gives rise to a log structure $\langle \pi_R \rangle \to (R, \cdot)$ on R, simply by including the multiplicative monoid $\langle \pi_R \rangle := \{\pi_R^0, \pi_R^1, \dots\}$ in R. We think of the log ring $(R, \langle \pi_R \rangle)$ as an intermediate localization in-between R and the fraction field $F := R[1/\pi_R]$.

Logarithmic THH**.** The perspective of log structures as "intermediate localizations" is reinforced by THH-cofiber sequences constructed by Rognes–Sagave–Schlichtkrull [13]. For example, dévissage implies that there is a fiber sequence

$$K(\mathbb{F}_p) \to K(\mathbb{Z}) \to K(\mathbb{Z}[1/p])$$

in algebraic K-theory. This does not work for THH: One cannot identify the fiber of $\mathrm{THH}(\mathbb{Z}) \to \mathrm{THH}(\mathbb{Z}[1/p])$ with $\mathrm{THH}(\mathbb{F}_p)$. The introduction of [4] highlights this point very eloquently.

One can associate to any log ring (R, P) a commutative R-algebra in spectra $\mathrm{THH}(R, P)$ [12, Definition 8.11]. It is shown in [13, Theorem 5.5, Example 5.7] that this construction participates in a cofiber sequence

$$\mathrm{THH}(R) \to \mathrm{THH}(R, \langle x \rangle) \to \mathrm{THH}(R/x)[1]$$

for any non-zero divisor x in R. The available constructions of the cofiber sequence are *not* an instance of dévissage but rather come to life by a direct analysis of the map $\mathrm{THH}(R) \to \mathrm{THH}(R, \langle x \rangle)$. In particular, the construction of the cofiber sequences makes no reference to Morita-invariance type properties of logarithmic THH (at the time of writing, no such property is known to the author).

Consequently, the relationship between logarithmic THH and algebraic K-theory is not at all clear. Nonetheless, the more flexible notion of étaleness in log geometry is useful in this context. For example, there is a base-change formula

$$A \otimes_R \mathrm{THH}(R) \xrightarrow{\simeq} \mathrm{THH}(A)$$

for étale morphisms of commutative rings $R \to A$ (by e.g. [11, Theorem 1.3]). Many examples of log étale morphisms $(R, P) \to (A, M)$ give rise to a base-change formula

$$A \otimes_R \mathrm{THH}(R, P) \xrightarrow{\simeq} \mathrm{THH}(A, M)$$

by (the proof of) [8, Theorem 1.11]; this covers the example of tamely ramified extensions of discrete valuation rings. The analogous property for *log ring spectra* has proven useful for both THH and K-theory computations, as we explain below.

Logarithmic ring spectra. Rognes [12] initiated the study of log structures in the context of structured ring spectra. The role of commutative rings is now played by $\mathbb{E}_\infty$-rings, while that of the monoid is often played by the "QS^0-graded $\mathbb{E}_\infty$-spaces" (or *commutative $\mathcal{J}$-space monoids*) of Sagave–Schlichtkrull [16]. These categories participate in an adjunction which we will denote by

$$\mathbb{S}^{\mathcal{J}}[-] \colon \mathbb{E}_\infty\text{-Spaces}_{/QS^0} \leftrightarrows \mathrm{CAlg}(\mathrm{Sp}) \colon \Omega^{\mathcal{J}}(-),$$

and a *(pre-)log ring spectrum* (R, P, β) is thus an $\mathbb{E}_\infty$-ring R, a QS^0-graded $\mathbb{E}_\infty$-space P, and a map $\beta \colon P \to \Omega^{\mathcal{J}}(P)$. There are important variations of this definition: Replacing QS^0 by $\mathrm{BO} \times \mathbb{Z}$ plays a role in Sagave–Schlichtkrull's [17].

In the present setup, well-behaved log structures $(R, \langle x \rangle)$ arise from homotopy classes $x \in \pi_d(R)$ that are "strict" in a certain sense: We refer to [15, Construction 4.2] for the concrete construction. Examples include connective covers of periodic ring spectra with log structures generated by their periodicity classes; e.g. the connective Adams summand $(\ell_p, \langle v_1 \rangle)$, connective complex K-theory $(\mathrm{ku}, \langle u \rangle)$, and connective real K-theory $(\mathrm{ko}, \langle \beta \rangle)$.

To a log ring spectrum (R, P), Rognes [12, Definition 8.11] and Rognes–Sagave–Schlichtkrull [13, Definition 4.6] associate a commutative R-algebra in spectra $\mathrm{THH}(R, P)$. In the examples $(R, \langle x \rangle)$ of interest, this construction participates in cofiber sequences

$$\mathrm{THH}(R) \to \mathrm{THH}(R, \langle x \rangle) \to \mathrm{THH}(R /\!\!/ x)[1]$$

by [13, Theorem 1.1]. While $\ell_p /\!\!/ v_1 \simeq \mathbb{Z}_p$ and $\mathrm{ku} /\!\!/ u \simeq \mathbb{Z}$, we have $\mathrm{ko} /\!\!/ \beta \simeq \tau_{\leq 7}\mathrm{ko}$, which highlights the lack of reliance on dévissage in the construction of the cofiber sequences in logarithmic THH. Related to this point are the cofiber sequences

$$\mathrm{THH}(\mathrm{BP}\langle n \rangle) \to \mathrm{THH}(\mathrm{BP}\langle n \rangle, \langle v_n \rangle) \to \mathrm{THH}(\mathrm{BP}\langle n-1 \rangle)[1]$$

obtained from a corresponding sequence for MUP ([17, Example 8.6]) by using the $\mathrm{MU}[x]$-algebra stuctures on $\mathrm{BP}\langle n \rangle$ of Hahn–Wilson [6], as sketched in e.g. [5, Remark 9.8]. Results of Barwick–Lawson [2] and Antieau–Barthel–Gepner [1] suggest that this is an apparent mismatch with the corresponding sequences in algebraic K-theory, which adds to the difficulty of giving K-theoretic interpretations of the cofiber sequences in logarithmic THH.

Logarithmic deformation theory. To any map of log ring spectra one can associate a *log cotangent complex* $\mathbb{L}_{(A,M)/(R,P)}$ [12, 15, 8]. Analogously to the situation for ordinary THH (cf. the argument of [11]), its vanishing implies base-change

$$A \otimes_R \mathrm{THH}(R,P) \xrightarrow{\simeq} \mathrm{THH}(A,M)$$

in logarithmic THH in the connective case [8, Theorem 1.7]. By [15, Theorem 1.6], the log cotangent complex associated to the inclusion of the Adams summand vanishes, and so we obtain that

$$\mathrm{ku}_p \otimes_{\ell_p} \mathrm{THH}(\ell_p, \langle v_1 \rangle) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{ku}_p, \langle u \rangle);$$

this is also the content of [14, Theorem 1.5]. This is computationally useful in conjunction with the cofiber sequences in logarithmic THH: In [14], this is used to recover Ausoni's computation of $V(1)_*\mathrm{THH}(\mathrm{ku}_p)$, while Bayındır [3] has used these methods to recover Ausoni's computation of $T(2)_*K(\mathrm{ku}_p)$ in terms of $T(2)_*K(\ell_p)$.

The presence of a cotangent complex and the more flexible notion of étaleness in log geometry naturally begs the question of an obstruction theory with vanishing obstruction groups for log étale extensions. As a first step, we would like to understand the logarithmic analog of the tower of square-zero extensions

$$\cdots \to \tau_{\leq 2}(R) \to \tau_{\leq 1}(R) \to \tau_{\leq 0}(R) \simeq \pi_0(R)$$

for a connective ring spectrum R, and how to set up an inductive lifting procedure starting from a formally étale map out of its bottom-most stage.

One should first understand the analog of square-zero extensions for log ring spectra. At this point, there is some tension between

(1) the natural guess from a log geometric perspective, where a square-zero extension $(\widetilde{R}, \widetilde{P}) \to (R,P)$ is one of underlying commutative rings that is *strict*; for the purposes of this exposition, one may read this as $\widetilde{P} \simeq P$.[1]
(2) the natural guess from the perspective of derived/higher algebra, where one would ask that a square-zero extension $(\widetilde{R}, \widetilde{P}) \to (R,P)$ is pulled back from a "log derivation" $(d, d^\flat)\colon (R,P) \to (R \oplus J[1], P \oplus J[1])$; these are corepresented by the log cotangent complex.

Theorem. *These two notions of log square-zero extensions agree.*

That (2) implies (1) appears in [9, Chapter 4], while the converse is currently being written up. For a log ring spectrum (R,P), this gives rise to an essentially unique tower

$$\cdots \to (\tau_{\leq 2}(R), P) \to (\tau_{\leq 1}(R), P) \to (\pi_0(R), P)$$

of log square-zero extensions compatible with the Postinkov tower. This is quite natural from a log geometric perspective: For instance, the "residue field" associated to the log ring $(A, \langle \pi_A \rangle)$ for a discrete valuation ring A is $(A/\pi_A, \langle \pi_A \rangle)$,

[1] Making this precise would require making the distinction and passage between pre-log and log ring spectra explicit. The definition appears in e.g. [12, Definition 7.25].

where all positive powers of π_A map to zero. This is the *standard log point*, of which we consider the log ring spectrum $(\pi_0(R), P)$ to be an analog.

For example, if $(R, P) = (\ell_p, \langle v_1 \rangle)$, let us write $\langle \sqrt[p-1]{v_1} \rangle$ for the object called E in [15, Proof of Prop 4.15]. Then $(\mathbb{Z}_p, \langle v_1 \rangle) \to (\mathbb{Z}_p \otimes_{\mathbb{S}^{\mathcal{J}}[\langle v_1 \rangle]} \mathbb{S}^{\mathcal{J}}[\langle \sqrt[p-1]{v_1} \rangle], \langle \sqrt[p-1]{v_1} \rangle)$ is formally log étale, and the underlying ring spectrum of the target is equivalent to $\mathbb{Z}_p \otimes_{\ell_p} \mathrm{ku}_p$. Formally log étale maps lift uniquely along log square-zero extensions [9, Theorem 4.1.0.3]. We are currently pursuing more structured statements relating the categories formally log étale of (R, P)- and $(\pi_0(R), P)$-algebras. For this, we extend Lurie's cotangent complex formalism [10, Section 7.3] to the context of log geometry: The expected identification of the fibers of the resulting *replete tangent bundle* $T^{\mathrm{rep}}_{\mathrm{Log}}$ is available in [9, Proposition 5.1.0.1].

References

[1] B. Antieau, T. Barthel, D. Gepner, *On localization sequences in the algebraic K-theory of ring spectra*, J. Eur. Math. Soc. (JEMS) **20** (2018), no.2, 459–487.

[2] C. Barwick, T. Lawson, *Regularity of structured ring spectra and localization in K-theory*, arXiv:1402.6038

[3] H. Bayındır, *Algebraic K-theory of the two-periodic first Morava K-theory*, arXiv:2305.14308

[4] R. Burklund, I. Levy, *On the K-theory of regular coconnective rings*, Selecta Math. (N.S.) **29** (2023), no.2, Paper No. 28, 30 pp.

[5] F. Binda, T. Lundemo, D. Park, P.A. Østvær, *A Hochschild-Kostant-Rosenberg Theorem and Residue Sequences for Logarithmic Hochschild Homology*, arXiv:2209.14182

[6] J. Hahn, D. Wilson, *Redshift and multiplication for truncated Brown-Peterson spectra*, Ann. of Math. (2) **196** (2022), no.3, 1277–1351.

[7] K. Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224.

[8] T. Lundemo, *On the relationship between logarithmic TAQ and logarithmic THH*, Doc. Math. **26** (2021), 1187–1236.

[9] T. Lundemo, *On Formally Étale Morphisms in Derived and Higher Logarithmic Geometry*, PhD Thesis, Radboud University Nijmegen (2022)

[10] J. Lurie, *Higher Algebra*, available on the website of the author.

[11] A. Mathew, *THH and base-change for Galois extensions of ring spectra*, Algebr. Geom. Topol. **17** (2017), no.2, 693–704.

[12] J. Rognes, *Topological Logarithmic Structures*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), 401–544. Geom. Topol. Monogr., 16.

[13] J. Rognes, S. Sagave, C. Schlichtkrull, *Localization sequences for logarithmic topological Hochschild homology*, Math. Ann. **363** (2015), no.3-4, 1349–1398.

[14] J. Rognes, S. Sagave, C. Schlichtkrull, *Logarithmic topological Hochschild homology of topological K-theory spectra*, J. Eur. Math. Soc. (JEMS) **20** (2018), no.2, 489–527.

[15] S. Sagave, *Logarithmic structures on topological K-theory spectra*, Geom. Topol. **18** (2014), no.1, 447–490.

[16] S. Sagave, C. Schlichtkrull, *Diagram spaces and symmetric spectra*, Adv. Math. **231** (2012), no.3-4, 2116–2193.

[17] S. Sagave, C. Schlichtkrull, *Virtual vector bundles and graded Thom spectra*, Math. Z. **292** (2019), no.3-4, 975–1016.

The even filtration and prismatic cohomology

Piotr Pstragowski

The even filtration, first introduced by Hahn-Raksit-Wilson, associates to a commutative algebra in spectra R a canonical filtration which informally measures the failure of $\pi_*(R)$ to be even. Applied to ring spectra of arithmetic interest, this recovers a variety of important filtrations, such as the Adams-Novikov filtration of the sphere or the various motivic filtrations on THH of rings and its variants. More generally, applying the even filtration to variants of THH of reasonably nice commutative ring spectra allows one to define prismatic and syntomic cohomology in this more general case.

In this talk, I describe a variant of the even filtration which is naturally defined on $\mathbf{E}_1$-ring spectra and their modules, eschewing the commutativity condition. This variant is induced by Postnikov towers on sheaves of spectra on the ∞-category $\mathrm{Perf}_{ev}(R)$ of perfect even modules; that is, those which can be built using extensions and retracts from $\Sigma^{2n}R$. I will describe how the resulting even filtration agrees in good cases with the Hahn-Raksit-Wilson filtration but, unlike the latter, can also be effectively computed through resolutions of modules, which is an essentially linear process. I will describe how this implies that the resulting even cohomology groups have excellent formal properties, especially in the connective case, having a strict vanishing line and being explicitly calculable in low degrees.

Time permitting, I will describe joint work in progress with Raksit on applying this filtration to the construction of prismatic cohomology of $\mathbf{E}_2$-rings.

Telescopes under the stars

Jeremy Hahn

(joint work with Robert Burklund, Ishan Levy, Tomer Schlank)

This talk concludes a series in which Burklund, Levy, and Schlank previously spoke. The main focus is the diagram

$$\begin{tikzcd}
T(2)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}}) \arrow[rr, "\not\simeq"] \arrow[d, "\not\simeq"] & & T(2)_*\mathrm{TC}(\ell)^{hp^k\mathbb{Z}} \arrow[d, "\simeq"] \\
T(2)_*\left(L_{K(2)}\mathrm{TC}(\ell^{hp^k\mathbb{Z}})\right) \arrow[rr, "\simeq"] & & T(2)_*\left(L_{K(2)}\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}\right).
\end{tikzcd}$$

Here, ℓ is the connective Adams summand at a fixed prime $p > 5$. The restriction on the prime guarantees the existence of homotopy commutative and associative ring spectra $V(0) = \mathbb{S}/p$, $V(1) = \mathbb{S}/(p, v_1)$, and $V(2) = \mathbb{S}/(p, v_1, v_2)$. We let $T(2)$ denote $v_2^{-1}V(1)$. The action of $\mathbb{Z}$ on ℓ is by an Adams operation ψ^m, where m is a topological generator of $1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$. The integer $k \geq 0$ we think of as a parameter to be tuned, and mostly we will be interested in what happens when $k \gg 0$.

That the bottom horizontal map is an equivalence is forthcoming work of Ben Moshe–Carmeli–Schlank–Yanovski, on cyclotomic redshift. That the right vertical map is an equivalence is joint work with Raksit and Wilson [3]. That the upper horizontal map is *not* an equivalence is forthcoming joint work with Burklund, Levy, and Schlank, and is the main result of the talk. It follows that the left vertical map is not an equivalence.

To prove the non-equivalence, we first observe that $T(2)_*\mathrm{TC}(\ell)^{hp^k\mathbb{Z}}$ is a finite-dimensional $\mathbb{F}_p[v_2^{\pm}]$-vector space, by calculations of Ausoni–Rognes [1]. We explain here why, at least for $k \gg 0$, $T(2)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ is an infinite-dimensional $\mathbb{F}_p[v_2^{\pm}]$-vector space. The key point is the following:

Theorem 1 (Burklund–H–Levy–Schlank)**.** For all sufficiently large $k \gg 0$, there is an equivalence of cyclotomic spectra

$$V(2) \otimes \mathrm{THH}(\ell^{p^k\mathbb{Z}}) \cong V(2) \otimes \mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}}).$$

Here, $\ell^{\mathrm{B}\mathbb{Z}}$ refers to the fixed points of ℓ under a trivial $\mathbb{Z}$-action. To give some intuition for the result, consider the simpler claim that $V(1) \otimes \ell^{hp^k\mathbb{Z}}$ is equivalent as a spectrum to $V(1) \otimes \ell^{\mathrm{B}\mathbb{Z}}$. Indeed, $V(1) \otimes \ell \simeq \mathbb{F}_p$, and there are not many possible different $\mathbb{Z}$ actions on $\mathbb{F}_p$. It is similarly the case that $V(1) \otimes \mathrm{THH}(\ell^{p^k\mathbb{Z}}) \simeq V(1) \otimes \mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}})$ as spectra, when $k \gg 0$, but one must mod out by v_2 before obtaining an equivalence of cyclotomic spectra. In the remainder of the talk, we discuss the simpler claim that

$$V(2)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}}) \cong V(2)_*\mathrm{TC}(\ell^{\mathrm{B}\mathbb{Z}}),$$

which will be enough to disprove the telescope conjecture.

The starting point is joint work of Lee and Levy [2], which computes

$$V(2)_*\mathrm{THH}(\ell^{hp^k\mathbb{Z}}) \cong V(2)_*\mathrm{THH}(\ell^{\mathrm{B}\mathbb{Z}}) \cong C^0(\mathbb{Z}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\mu] \otimes_{\mathbb{F}_p} \Lambda(\lambda_1, \lambda_2, \epsilon, \zeta),$$

where $C^0(\mathbb{Z}_p)$ is the ring of locally constant, $\mathbb{F}_p$-valued functions on $\mathbb{Z}_p$. There is then a homotopy fixed point spectral sequence for $V(2)_*\mathrm{TC}^-(\ell^{hp^k\mathbb{Z}})$, which begins with

$$C^0(\mathbb{Z}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\mu, t] \otimes_{\mathbb{F}_p} \Lambda(\lambda_1, \lambda_2, \epsilon, \zeta).$$

The sequence of S^1-equivariant maps

$$V(2) \otimes \mathrm{THH}(\mathbb{S}^{\mathrm{B}\mathbb{Z}}) \to V(2) \otimes \mathrm{THH}(\ell^{hp^k\mathbb{Z}}) \to V(2) \otimes \mathrm{THH}(\ell)^{hp^k\mathbb{Z}}$$

gives rise to maps of S^1-homotopy fixed point spectral sequences, the final of which may be deduced from the seminal work of Ausoni–Rognes [1]. From this, one can deduce differentials

$$d_2(\zeta) = ft,$$

where $f : \mathbb{Z}_p \to \mathbb{F}_p$ is a non-zero function supported on $\mathbb{Z}_p^\times$,

$$d_2(\epsilon) = \mu t,$$

$$d_{2p}(t) = \lambda_1 t^{p+1}, \text{ and}$$

$$d_{2p^2}(t^p) = \lambda_2 t^{p^2+p}.$$

For $k \gg 0$ one proves that there are no non-zero differentials other than those implied by the above and multiplicative structure. The idea here is that, if any other differential did occur, one could increase k until it does not. Any fixed differential can be ruled out by such methods, and after accounting for multiplicative structure there are only finitely many differentials to rule out in the entire spectral sequence.

Remark 1. In contrast to the S^1-homotopy fixed point spectral sequence computing $V(2)_*\mathrm{TC}^-(\ell^{hp^k\mathbb{Z}})$, the S^1-homotopy fixed point spectral sequence computing $V(1)_*\mathrm{TC}^-(\ell^{hp^k\mathbb{Z}})$ has infinitely many potential differentials, and so cannot be controlled by setting $k \gg 0$.

From the S^1-homotopy fixed point spectral sequence for $V(2)_*\mathrm{TC}^-(\ell^{hp^k\mathbb{Z}})$, one formally inverts t to deduce the S^1-Tate spectral sequence for $V(2)_*\mathrm{TP}(\ell^{hp^k\mathbb{Z}})$. The map

$$\left(S^1 - \text{hfp sseq}\right) \to t^{-1}\left(S^1 - \text{hfp sseq}\right)$$

converges to the map

$$\mathrm{can} : V(2)_*\mathrm{TC}^-(\ell^{hp^k\mathbb{Z}}) \to V(2)_*\mathrm{TP}(\ell^{hp^k\mathbb{Z}}).$$

One can similarly compute the map

$$\varphi : V(2)_*\mathrm{TC}^-(\ell^{hp^k\mathbb{Z}}) \to V(2)_*\mathrm{TP}(\ell^{hp^k\mathbb{Z}})$$

by studying the map of spectral sequences

$$\left(S^1 - \text{hfp sseq}\right) \to \mu^{-1}\left(S^1 - \text{hfp sseq}\right),$$

though there is a non-trivial isomorphism

$$t^{-1}V(2)_*\mathrm{TC}^-(\ell^{hp^k\mathbb{Z}}) \cong \mu^{-1}V(2)_*\mathrm{TC}^-(\ell^{hp^k\mathbb{Z}})$$

to work out. This isomorphism is fairly straightforwardly determined using the sequence of cyclotomic spectra

$$V(2) \otimes \mathrm{THH}(\mathbb{S}^{\mathrm{B}\mathbb{Z}}) \to V(2) \otimes \mathrm{THH}(\ell^{hp^k\mathbb{Z}}) \to V(2) \otimes \mathrm{THH}(\ell)^{hp^k\mathbb{Z}}.$$

The upshot of all of this is that $V(2)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ is an infinite-dimensional graded $\mathbb{F}_p$-vector space concentrated in a finite range of degrees. For degree reasons it then follows, using a v_2-Bockstein spectral sequence, that the $\mathbb{F}_p[v_2]$-module $V(1)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ contains an infinitely generated free $\mathbb{F}_p[v_2]$-module as a summand. Thus, for $k \gg 0$, $T(2)_*\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ is an infinite-dimensional $\mathbb{F}_p[v_2^{\pm}]$-vector space.

References

[1] C. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory*, Acta Math. **188** (2002), 1–39.

[2] D. Lee and I. Levy, *Topological Hochschild homology of the image of j*, arXiv eprint 2307.04248 (2023).

[3] J. Hahn and A. Raksit and D. Wilson, *A A motivic filtration on the topological cyclic homology of commutative ring spectra*, arXiv eprint 2206.11208 (2022).

Strict units of commutative ring spectra

Allen Yuan

(joint work with Shachar Carmeli, Thomas Nikolaus)

1. Introduction

A canonical and important source of commutative ring spectra is the spherical group ring construction

$$\mathbb{S}[-] : \mathrm{CMon}(\mathrm{Spaces}) \to \mathrm{CAlg}(\mathrm{Sp}).$$

For example, one can use this functor to build $\mathbb{S}$-analogues of important classical objects:

(1) $\mathbb{S}[t] := \mathbb{S}[\mathbb{N}]$, representing $\mathbb{A}^1$.
(2) $\mathbb{S}[t^{\pm 1}] := \mathbb{S}[\mathbb{Z}]$, representing $\mathbb{G}_m$.
(3) $\mathbb{S}[C_p]$, representing μ_p.

However, it is generally difficult to compute maps between such objects. For instance, the space $\mathrm{Hom}_{\mathrm{CAlg}}(\mathbb{S}[t], \mathbb{S}[t])$ is yet unknown. Nevertheless, in the case of group rings for finitely generated discrete abelian groups, we have the following result.

Theorem 1 (Carmeli–Nikolaus–Y.)**.** The functor

$$\mathbb{S}[-] : \mathrm{Ab}^{\mathrm{fg}} \to \mathrm{CAlg}(\mathrm{Sp})$$

is fully faithful.

In other words, for $A, B \in \mathrm{Ab}^{\mathrm{fg}}$, the mapping space $\mathrm{Hom}_{\mathrm{CAlg}}(\mathbb{S}[A], \mathbb{S}[B])$ is discrete and naturally equivalent to the set of abelian group maps from A to B. In particular, taking $A = \mathbb{Z}$, we have $\mathbb{G}_m(\mathbb{S}[B]) := \mathrm{Hom}_{\mathrm{CAlg}}(\mathbb{S}[t^{\pm 1}], \mathbb{S}[B]) \simeq B$. In fact, we will focus on this case moving forward, as it is the core content of the theorem.

Remark 1. The commutative ring spectrum $\mathbb{S}[t]$ is *not* free. If $\mathbb{S}\{t\} = \mathbb{S}[\coprod B\Sigma_n]$ denotes the free commutative algebra on a single generator, the natural map $\mathbb{S}\{t\} \to \mathbb{S}[t]$ sending $t \mapsto t$ is given by collapsing each of the $B\Sigma_n$ to a point. This can be thought of as giving t the structure of a *strict* element. Analogous remarks apply to $\mathbb{S}[t^{\pm 1}]$, and so we refer to $\mathbb{G}_m(-)$ as the space of *strict units*.

2. Past work

The strict units $\mathbb{G}_m(-)$ are an important and basic invariant of a commutative ring spectrum R, and have been studied extensively [4, 5, 7, 3]. One of the reasons is that these units behave more algebraically, allowing for the following well-behaved constructions given a strict unit $\alpha \in \pi_0(R)$:

- $R/(\alpha - 1) := R \otimes_{\mathbb{S}[t^{\pm 1}]} \mathbb{S}$, with the right map given by applying group ring to $\mathbb{Z} \to *$. As a module, this is simply the cofiber of $\alpha - 1$.
- $R[\sqrt[n]{\alpha}] := R \otimes_{\mathbb{S}[t^{\pm 1}]} \mathbb{S}[t^{\pm 1/n}]$, which is a rank n free module over R.

Nevertheless, there have not been many full computations of strict units. We sample two of the cases in which the answer is known:

Theorem 2 (Hopkins–Lurie [2])**.** Let $k = \overline{k}$ be an algebraically closed field of characteristic $p > 0$, and let $E_n = E_n(k)$ denote the Lubin–Tate theory of height $n > 0$. Then we have an isomorphism

$$\mathbb{G}_m(E_n) = k^\times \times K(\mathbb{Z}_p, n+1).$$

This theorem was also proved by Rezk at height 2.

Theorem 3 (Carmeli [3])**.** We have $\mathbb{G}_m(\mathbb{S}) = \{1\}$.

In fact, our theorem builds on this result of Carmeli (this is the case $A = \mathbb{Z}$ and $B = *$, in the previous notation).

3. Flat commutative ring spectra and δ_p-rings

Theorem 1 asserts a rigidity property of spherical group rings. It turns out that this rigidity is already visible in the power operation structure of these group rings, which we now discuss.

Recollection 1. Let $R \in \mathrm{CAlg}(\mathrm{Sp})$. Then there is a natural ring map $R \to R^{tC_p}$, known as the *Tate-valued Frobenius*, due to Nikolaus–Scholze [6].

Construction 1. Suppose $R \in \mathrm{CAlg}(\mathrm{Sp})$ has underlying spectrum isomorphic to the p-completion of a free $\mathbb{S}$-module. Then the Segal conjecture (for C_p, due to Lin and Gunawardena) yields an isomorphism $R^{tC_p} \simeq R$. Via this identification, the Tate-valued Frobenius can be regarded as a self-map $\varphi : R \to R$. One can check that φ induces a lift of Frobenius on π_0, and equips $\pi_0(R)$ with the structure of a δ_p-ring.

It follows that maps between spherical group algebras must induce δ_p-ring maps at the level of π_0. It is not difficult to see that $\varphi : \mathbb{S}[t^{\pm 1}]_p^\wedge \to \mathbb{S}[t^{\pm 1}]_p^\wedge$ sends $t \mapsto t^p$, so we deduce that any strict unit $f(t) \in \pi_0(\mathbb{S}[t^{\pm 1}]_p^\wedge) \simeq \mathbb{Z}[t^{\pm 1}]_p^\wedge$ must satisfy $f(t^p) = f(t)^p$. It is easy to check that this forces $f(t)$ to be a monomial.

This discussion suggests that maps between spherical group algebras are better approximated by *δ_p-ring maps* at the level of π_0, rather than just ordinary ring maps. In fact, we have the following p-complete version of our main theorem:

Theorem 4 (Carmeli–Nikolaus–Y.)**.** Let $A, B \in \mathrm{Ab}^{\mathrm{fg}}$. Then π_0 induces a natural isomorphism

$$\mathrm{Hom}_{\mathrm{CAlg}}(\mathbb{S}[A]_p^\wedge, \mathbb{S}[B]_p^\wedge) \simeq \mathrm{Hom}_{\mathrm{Ring}_{\delta_p}}(\mathbb{Z}[A]_p^\wedge, \mathbb{Z}[B]_p^\wedge).$$

4. Outline of methods

One interesting feature of our proof is that it makes extensive use of chromatic techniques, despite the statement not being chromatic. To explain one of the main ingredients, we contemplate the case of computing $\mathbb{G}_m(\mathbb{S}[B]_p^\wedge)$ in the case that B is a finite abelian p-group.

Even over $\mathbb{Z}$, it is not immediately obvious whether an element $\gamma \in \mathbb{Z}_p[B]$ will be invertible. One way to check this is to consider the embedding

$$\mathbb{Z}_p[B] \hookrightarrow \mathbb{C}[B].$$

By the representation theory of finite groups, this latter ring splits as $\mathbb{C}^{B^*}$, where B^* denotes the group of characters of B. Here, it is easier to check that γ is a unit, as one just has to see γ is non-zero component-wise.

It turns out that there is an analogous trick that one can do upon restriction to a single chromatic height, using the "chromatic Fourier transform":

Theorem 5 (Barthel–Carmeli–Schlank–Yanovski [1])**.** Let M be a p-finite spectrum and E_n as before. Then there is a natural equivalence in $\mathrm{CAlg}_{E_n}(\mathrm{Sp}_{K(n)})$

$$E_n[\Omega^\infty M] \simeq E_n^{\Omega^\infty \Sigma^n I_{\mathbb{Q}_p/\mathbb{Z}_p} M}.$$

In particular, letting $M = B$, we find that $E_n[B] \simeq E_n^{K(B^*,n)}$. Thus, instead of splitting as a B^*-indexed product, the group ring over E_n is a constant limit diagram indexed by $K(B^*, n)$. Since $\mathbb{G}_m(-)$ is corepresentable, this allows us to get a handle on $\mathbb{G}_m(E_n[B])$, from which we can understand $\mathbb{G}_m(R[B])$ for any $R \in \mathrm{CAlg}(\mathrm{Sp}_{K(n)})$. The remainder of the proof (in the special case at hand) proceeds by assembling these statements across heights and using chromatic convergence to access $\mathbb{S}_p$.

References

[1] Tobias Barthel, Shachar Carmeli, Tomer Schlank, and Lior Yanovski. *The Chromatic Fourier Transform*. ArXiv preprint arXiv:2210.12822, 2022.

[2] Burklund, Tomer M Schlank, and Allen Yuan. *The Chromatic Nullstellensatz*. ArXiv preprint arXiv:2207.09929, 2022.

[3] Shachar Carmeli. *On the strict Picard spectrum of commutative ring spectra*. Compos. Math., **159(9)** (2023), 1872–1897.

[4] Jun Hou Fung. *Strict Units of Commutative Ring Spectra*. ProQuest LLC, Ann Arbor, MI, 2020. Thesis (Ph.D.) – Harvard University.

[5] Tyler Lawson. *Adjoining roots in homotopy theory*. ArXiv preprint arXiv:2002:01997, 2020.

[6] Thomas Nikolaus and Peter Scholze. *On topological cyclic homology*. Acta Math., **221(2)** (2018), 203–409.

[7] Hisham Sati and Craig Westerland. *Twisted Morava K-theory and E-theory*. J. Topol., **8(4)** (2015), 887–916.

Periodic unstable homotopy theory and Hopf algebras

Gijs Heuts

(joint work with S. Barkan, L. Brantner, J. Hahn, Y. Shi, A. Yuan)

There are at least two ways to interpret the phrase 'monochromatic unstable homotopy theory', namely:

(1) The localization of the ∞-category of pointed spaces at the maps inducing isomorphisms on v_n-periodic homotopy groups. Let us denote the resulting ∞-category by $\mathcal{S}_{v_n}$.

(2) The localization of the ∞-category of (pointed) spaces at the maps inducing isomorphism in $T(n)$-homology, with $T(n)$ the mapping telescope of a v_n self-map of a finite type n spectrum. Let us denote the resulting ∞-category by $L_{T(n)}\mathcal{S}$.

The first goal of this talk was to discuss the relation between (1) and (2), i.e. between periodic homotopy and homology groups.

In the *stable* setting, localizing at v_n-periodic homotopy equivalences and $T(n)$-homology equivalences is the same thing. Also, for $n = 0$, the two localizations described above agree (at least for simply-connected spaces) by Serre class theory: a map of simply-connected spaces is a rational homotopy equivalence if and only if it is a rational homology equivalence. However, for $n > 0$ the localizations $\mathcal{S}_{v_n}$ and $L_{T(n)}\mathcal{S}$ behave very differently. The following are some examples of phenomena that occur:

(1) A v_n-periodic equivalence need not be a $T(n)$-homology isomorphism.

(1a) Any truncated space X (i.e., $\pi_k X = 0$ for $k \gg 0$) has vanishing v_n-periodic homotopy groups. However, $T(n)_* X$ need not be trivial. For example, $T(n)_* K(\mathbb{F}_p, k)$ is nontrivial for $k \leq n$ by the results of Ravenel–Wilson.

(1b) Fix a finite type n space V_n and write P_{V_n} for the nullification functor with respect to its suspension, i.e., left Bousfield localization with respect to the map $V_n \to *$. Then any space of the form $Y = P_{V_n} X$ has vanishing v_n-periodic homotopy groups, but generally nontrivial $T(n)$-homology. Indeed, a result of Bousfield says that a $K(n)$-equivalence of spaces is a $K(i)$-equivalence for $1 \leq i \leq n$. Hence if Y has vanishing $T(n)$-homology, it has vanishing $K(n-1)$-homology. But $K(n-1)_* Y = K(n-1)_* X$, which clearly need not be trivial.

(2) A $T(n)$-homology isomorphism need not be a v_n-periodic equivalence. For example, Langsetmo–Stanley construct 'perturbations of the Adams map' that are still $T(1)$-homology isomorphisms but not v_1-periodic equivalences.

The examples above should make clear that the relation between homotopy and homology equivalences is subtle in the periodic world. However, things improve if we consider maps $f \colon X \to Y$ that induce $T(n)$-homology isomorphisms on loop spaces. In fact, we find the following:

Theorem 1 (Barkan, H, Shi)**.** Let $f \colon X \to Y$ be a map of pointed connected spaces whose homotopy groups are p^∞-torsion in dimensions 2 and above. Then the following conditions are equivalent:

(1) The map Ωf is a $T(i)$-isomorphism for $1 \leq i \leq n$.

(2) The truncated map $\tau_{\leq n+1} f$ is an equivalence and f is a v_i-periodic equivalence for each $1 \leq i \leq n$.

Remark 1. If Bousfield's result about $K(n)$-equivalences mentioned above has an analog for $T(n)$-equivalences, then one may replace condition (1) with the following apparently weaker one:

(1') The map Ωf is a $T(n)$-isomorphism.

Provided this holds, the theorem above shows that $T(n)$-isomorphisms in the unstable case are really more of a transchromatic notion than one that is 'concentrated at height n'.

The second goal of this talk was to discuss ongoing joint work with Brantner, Hahn, and Yuan on describing an 'algebraic model' for the homotopy theory obtained from pointed spaces by inverting the maps f for which Ωf is a $T(n)$-equivalence. Let us denote it by $L^{\Omega}_{T(n)}\mathcal{S}_*$. We begin by observing that the construction of this localization guarantees the existence of the following functor:

$$L_{T(n)}\Sigma^\infty_+ \colon L^{\Omega}_{T(n)}\mathcal{S}_* \to \mathrm{Hopf}(\mathrm{Sp}_{T(n)}).$$

Here $\mathrm{Hopf}(\mathrm{Sp}_{T(n)})$ denotes the ∞-category of $T(n)$-local spectral Hopf algebras, which is defined to be the ∞-category of group objects in cocommutative $T(n)$-local coalgebra spectra. It turns out that the structure of the ∞-category $L^{\Omega}_{T(n)}\mathcal{S}_*$ is reflected rather well by the structure theory of such $T(n)$-local Hopf algebras. For now let me comment on one aspect of this, namely the 'primitively generated' Hopf algebras, which are closely related to v_n-periodic spaces. There wasn't much time left in the talk to spend on 'grouplike' Hopf algebras, which are related to truncated spaces. These results will be discussed elsewhere.

The v_n-periodic part.

The ∞-category $\mathcal{S}_{v_n}$ is known to be equivalent to that of $T(n)$-local spectral Lie algebras and bears a relation to $\mathrm{Hopf}(\mathrm{Sp}_{T(n)})$ that is reminiscent of the work of Milnor–Moore in the classical case (over $\mathbb{Q}$). To be precise, there is a commutative square

$$\begin{tikzcd}
\mathcal{S}_{v_n} \arrow[r] \arrow[d, "\simeq"] & L^{\Omega}_{T(n)}\mathcal{S}_* \arrow[d] \\
\mathrm{Lie}(\mathrm{Sp}_{T(n)}) \arrow[r, "U"] & \mathrm{Hopf}(\mathrm{Sp}_{T(n)})
\end{tikzcd}$$

in which U is the universal enveloping algebra functor. We conjecture the following parallel of the Milnor–Moore theorem in this setting:

Conjecture A. The functor U is fully faithful.

We can offer the following:

Theorem 2 (Brantner, Hahn, H, Yuan)**.** Conjecture A holds on the full subcategory of abelian $T(n)$-local spectral Lie algebras (or, equivalently, infinite loop spaces in $\mathcal{S}_{v_n}$).

Corollary 1. The functor

$$\mathrm{Sp}_{T(n)} \to \mathrm{CHopf}(\mathrm{Sp}_{T(n)}) \colon V \mapsto L_{T(n)}\mathrm{Sym}(V)$$

is a fully faithful left adjoint, where $\mathrm{CHopf}(\mathrm{Sp}_{T(n)})$ denotes the ∞-category of bicommutative $T(n)$-local spectral Hopf algebras.

Conjecture A would imply that primitively generated $T(n)$-local spectral Hopf algebras provide an alternative algebraic model for v_n-periodic spaces. What is more interesting is the following: given a space X, can its v_n-periodic homotopy type be extracted from the Hopf algebra $L_{T(n)}\Sigma^\infty_+ X$? We expect the following:

Conjecture B. The following square commutes up to natural isomorphism:

$$\begin{tikzcd} \mathcal{S}_{v_n} \arrow[d, "\simeq"] & L^{\Omega}_{T(n)}\mathcal{S}_* \arrow[l] \arrow[d] \\ \mathrm{Lie}(\mathrm{Sp}_{T(n)}) & \mathrm{Hopf}(\mathrm{Sp}_{T(n)}). \arrow[l, "\mathrm{prim}"'] \end{tikzcd}$$

Here the top horizontal arrow is localization at v_n-periodic equivalences, the bottom horizontal arrow takes primitives of a Hopf algebra.

Again we have the following special case:

Theorem 3 (Brantner, Hahn, H, Yuan)**.** Conjecture B holds on the full subcategory of $L^{\Omega}_{T(n)}\mathcal{S}_*$ on infinite loop spaces.

More concretely, for a spectrum V the primitives of the spectral Hopf algebra $L_{T(n)}\Sigma^\infty_+\Omega^\infty V$ are precisely $L_{T(n)}V$.

General inputs for the $S_\bullet$-construction

Julie Bergner

The notion of a *2-Segal space*, developed by Dyckerhoff and Kapranov [3], is a generalization of a Segal space; while a Segal space encodes the structure of an up-to-homotopy topological category, a 2-Segal space encodes a weaker structure in which composition need not be defined or unique, but is associative. More specifically, a 2-Segal space is a simplicial space $X\colon \Delta^{\mathrm{op}} \to \mathcal{SS}ets$ for which certain maps

$$X_n \to \underbrace{X_2 \times_{X_1} \cdots \times_{X_1} X_2}_{n-1}$$

are weak equivalences of simplicial spaces. Here, $n \geq 3$, and the maps are determined by the triangulations of cyclically-labeled $(n+1)$-gons by their vertices. In particular, for each $n \geq 3$ there are multiple Segal maps (indexed by the n-th Catalan number) that need to be weak equivalences, in contrast to the case of Segal spaces, for which there is one map

$$X_n \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}$$

that needs to be a weak equivalence for each $n \geq 2$.

The same structure was developed under the name of *decomposition space* by Gálvez-Carrillo, Kock and Tonks [4]. Although the two approaches differ in description and motivation, for both a key example is the output of Waldhausen's $S_\bullet$-construction when applied to an exact category. In collaboration with Osorno, Ozornova, Rovelli, and Scheimbauer, we prove that every 2-Segal space can be obtained as $S_\bullet$ of a a suitable generalization of an exact category, called an *augmented stable double Segal space* [1], [2]. Such input is characterized by objects, vertical morphisms, horizontal morphisms, and squares behave sufficiently like pullback and pushout squares.

A natural question is what happens for Waldhausen categories, where one has cofibrations (horizontal morphisms) and suitable pushouts, but lacks another distinguished vertical class of morphisms and pullback data. My student Tanner Carawan has shown that the output of a category with cofibrations under the $S_\bullet$-construction has the structure of what one could call a *left 2-Segal space*, where for each $n \geq 3$ a single map of of the above kind is a weak equivalence, namely, the map corresponding to the triangulation given by taking diagonals out of the vertex labeled by 0. For a category with fibrations, one could dually obtain a notion of *right 2-Segal space*.

Although in this situation we are only guaranteed the structure of a left (or right) 2-Segal space, Carawan shows that in fact many common examples of Waldhausen categories, such as Waldhausen's category $\mathcal{R}_f$ of retractive spaces, actually do produce (fully) 2-Segal spaces. Such examples carry sufficient additional structure so that they can be regarded as augmented stable double Segal spaces. Indeed, so far the only known examples of left 2-Segal spaces that are not 2-Segal spaces are rather artificially constructed.

References

[1] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer, 2-Segal sets and the Waldhausen construction, *Topology Appl.*, 235 (2018) 445–484.

[2] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer, 2-Segal objects and the Waldhausen construction, *Alg. Geom. Topol.* 21 (2021) 1267–1326.

[3] T. Dyckerhoff and M. Kapranov, Higher Segal spaces, *Lecture Notes in Mathematics*, 2244. Springer, Cham, 2019.

[4] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks, Decomposition spaces, incidence algebras and Möbius inversion I: Basic theory, *Adv. Math.* 331 (2018) 952–1015.

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