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A NOTE ON ENDPOINT BOCHNER–RIESZ ESTIMATES

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ABSTRACT. We revisit an ε-removal argument of Tao to obtain sharp $L^p \to L^r(L^p)$ estimates for sums of Bochner–Riesz bumps which are conditional on non-endpoint bounds for single scale bumps. These can be used to obtain sharp conditional sparse bounds for Bochner–Riesz multipliers at the critical index, refining the conditional weak-type (p, p) estimates of Tao.

1. INTRODUCTION

Let Ω be a convex open subset of $\mathbb{R}^d$, $d \geq 2$, containing the origin. We assume that Ω has C^∞-boundary with non-vanishing Gaussian curvature. Let

$$\rho(\xi) = \inf\{t > 0 : \xi/t \in \Omega\}$$

be the Minkowski functional of Ω. Then $\rho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and ρ is homogeneous of degree 1, $\rho(\xi) > 0$ for $\xi \neq 0$ and $\rho(\xi) = 1$ on the boundary $\partial\Omega$. Given $\lambda > 0$, consider the Bochner–Riesz type operator

$$\mathcal{R}^\lambda := (1 - \rho(D))_+^\lambda.$$

The critical index for $L^p \to L^r$ boundedness is defined by

$$\lambda(r) = d\Big(\frac{1}{r} - \frac{1}{2}\Big) - \frac{1}{2}.$$

In this note we establish $L^p \to L^r(L^p)$ vector-valued inequalities for Bochner–Riesz bumps, and acting on families of functions $\{f_Q\}$ indexed by dyadic cubes $\mathfrak{D}$. We denote by $\mathfrak{D}_j$ the dyadic cubes of of sidelength 2^j.

For $M \geq 1$ define $\mathcal{Y}_M$ as the class of all C^M functions χ supported on $(\frac{1}{2}, 2)$ so that $\|\chi\|_{C^M} = \sum_{\nu=0}^M \|\chi^{(\nu)}\|_\infty \leq 1$.

Definition 1.1. *For $1 \leq p \leq r < \infty$ let $\mathrm{VBR}(p, r)$ denote the following statement. There exists $M > 0$ such that for all collections of functions χ_j in $\mathcal{Y}_M$, the inequality*

$$\Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} \chi_j(2^j(1 - \rho(D)))\big[\sum_{Q \in \mathfrak{D}_j} f_Q\big]\Big\|_{L^r(\mathbb{R}^d)} \leq C\Big(\sum_Q |Q| \|f_Q\|^r_{L^p(\mathbb{R}^d)}\Big)^{1/r} \tag{1.1}$$

holds for all families $\{f_Q\}_{Q \in \mathfrak{D}}$ of L^p functions f_Q, with $\mathrm{supp}\,(f_Q) \subseteq Q$.

The statement $\mathrm{VBR}(p, r)$ plays a significant role in [1] which deals with essentially sharp sparse domination results for the operator $\mathcal{R}^{\lambda(p)}$ in the sense that such sparse bounds follow from $\mathrm{VBR}(p, r)$. Satisfactory $\mathrm{VBR}(p, r)$ bounds are known for

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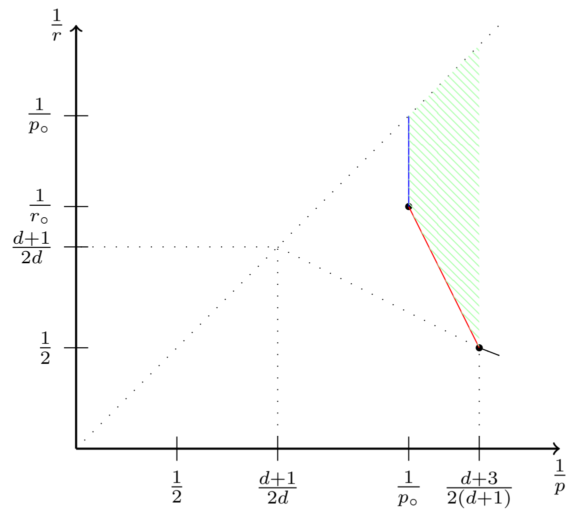

FIGURE 1. The conclusion of Theorem 1.2 holds in the interior of the green region. The red line corresponds to the line $\frac{1}{r} = \frac{1}{r_*(p,p_\circ,r_\circ)}$.

example in two dimensions for p,r in the range $1 \le p < 4/3$, $p \le r < \min\{p'/3, 2\}$ and in higher dimensions for $p \le \frac{2(d+1)}{d+3}$ and $r = 2$ (the Stein-Tomas range), see [13, 14, 16]. Familiar necessary conditions based on Knapp examples show that we need to have $r_\circ \le \frac{d-1}{d+1}p'_\circ$; and thus for $p_\circ \ge \frac{2(d+1)}{d+3}$ we must have $r_\circ \le 2$. Our result will involve the exponent $r_*(p,p_\circ,r_\circ)$ obtained by interpolation of the pairs $(p_\circ, r_\circ)$ and the Stein-Tomas pair $(\frac{2(d+1)}{d+3}, 2)$; the desired vector-valued inequalities for the latter follow from well-known arguments. It is given by

$$\frac{1}{r_*(p,p_\circ,r_\circ)} := \frac{\frac{1}{r_\circ}(\frac{d+3}{2(d+1)} - \frac{1}{p}) + \frac{1}{2}(\frac{1}{p} - \frac{1}{p_\circ})}{\frac{d+3}{2(d+1)} - \frac{1}{p_\circ}}. \tag{1.2}$$

Theorem 1.2. *Let $\frac{2(d+1)}{d+3} < p_\circ < \frac{2d}{d+1}$ and $r_\circ \in [p_\circ, \frac{d-1}{d+1}p'_\circ]$. Assume that $\mathcal{R}^\lambda$ maps $L^{p_\circ}(\mathbb{R}^d)$ to $L^{r_\circ}(\mathbb{R}^d)$ for all $\lambda > \lambda(r_\circ)$. Let $\frac{2(d+1)}{d+3} \le p < p_\circ$. Then* VBR$(p,r)$ *holds for $p \le r < r_*(p,p_\circ,r_\circ)$*

It is useful to note that $r_*(p,p_\circ,r_\circ) \to \frac{d-1}{d+1}p'$ as $r_\circ \nearrow \frac{d-1}{d+1}p'_\circ$. This implies that if we have the non-endpoint Bochner-Riesz $L^{p_\circ} \to L^{r_\circ}$ assumption for some $p_\circ \in [\frac{2(d+1)}{d+3}, \frac{2d}{d+1})$ and all $r \in [p_\circ, \frac{d-1}{d+1}p'_\circ)$ then the conclusion of VBR(p,r) holds for the full non-endpoint range $\frac{2(d+1)}{d+3} < p < p_\circ$ and $r \in [p, \frac{d-1}{d+1}p')$.

Theorem 1.2 corresponds to an off-diagonal version of a theorem of Tao [16] in which the $r_\circ = p_\circ$ version was obtained. The purpose of the resulting VBR(p,r) estimate in [16] was to prove conditional weak type (p,p) bounds for $\mathcal{R}^{\lambda(p)}$ and strong type results for a class of related multipliers such as $(1-\rho)_+^\lambda(1-\log(1-\rho))^{-\gamma}$, based on reductions in [3, 2, 13, 16]. These reductions also work in the off-diagonal case and yield the following endpoint multiplier theorems (we will not provide more details).

Corollary 1.3. *Let $p_\circ, r_\circ$ be as in Theorem 1.2. Assume $1 \le p < p_\circ$, $p \le r < r_*(p, p_\circ, r_\circ)$, $r \le 2$, $r \le \sigma \le \infty$. Then, for sequences $a = \{a_j\}_{j=1}^\infty \in \ell^\sigma$ we have the inequality*

$$\Big\| \sum_{j>0} a_j 2^{-j\lambda(q)} \chi_j(2^j(1-\rho(D)))f \Big\|_{L^{r,\sigma}} \lesssim \|a\|_{\ell^\sigma} \|f\|_{L^p} \tag{1.3}$$

The proof of Theorem 1.2 for $r > p$ is a re-elaboration of that of Tao for $r = p$. We claim no originality but provide full details of the argument in view of the applicability in [1] and also in view of Tao's question [16] concerning the possibility of ε-removal results for $L^p \to L^r$ bounds. As Tao remarks, such bounds would be especially interesting for the critical line $\frac{1}{r_{\mathrm{crit}}(p)} = \frac{d+1}{d-1}(1-\frac{1}{p})$. For applicability in [1] we only need to address the case $p < r < r_*(p, p_\circ, r_\circ)$; the latter condition becomes $p < r < r_{\mathrm{crit}}(p)$ if we assume $\mathrm{VBR}(p_\circ, r_\circ)$ for all $r_\circ = \frac{d-1}{d+1}p'_\circ - \varepsilon$ and $\varepsilon \to 0$.

1.1. *Notation.* We list some frequently used notation.

- *Families of dyadic cubes.* We let $\mathfrak{D}$ be a fixed dyadic lattice, which may or may not satisfy the assumptions in the setup by Lerner-Nazarov [10] (this requirement is only important when considering sparse bounds as in [1]). Let $\mathfrak{D}_j$ denote the subset of cubes in $\mathfrak{D}$ of sidelength 2^j. Cubes in $\mathfrak{D}_j$ are assumed to be half open, i.e. of the form $\prod_{i=1}^d [a_i, a_i+2^j)$ for suitable $a \in \mathbb{R}^d$. We use $\mathfrak{Q}$ for general subcollections of $\mathfrak{D}$, and let $\mathfrak{Q}_j$ be the cubes in $\mathfrak{Q}$ which are of sidelength 2^j. The sidelength of a dyadic cube Q is denoted by $2^{L(Q)}$ with $L(Q) \in \mathbb{Z}$.
- *Constants.* Given a list of objects L and real numbers A, $B \ge 0$, we write $A \lesssim_L B$ or $B \gtrsim_L A$ to indicate $A \le C_L B$ for some constant C_L which depends only items in the list L. We write $A \sim_L B$ to indicate $A \lesssim_L B$ and $B \lesssim_L A$.
- *Normalized bump functions.* Throughout the paper we shall fix a number $N \ge d+1$, and set $\mathcal{Y} = \mathcal{Y}_{d+1+N}$. The functions χ_j are throughout assumed to belong to $\mathcal{Y}$.
- *Multiplier notation.* For $m \in L^\infty(\mathbb{R}^d)$ we define the multiplier operator $m(D)$ which acts initially on Schwartz functions by

$$m(D)f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x, \xi\rangle} m(\xi) \widehat{f}(\xi)\, \mathrm{d}\xi.$$

Structure of this note. In §2 we provide some single scale estimates for suitable frequency or spatially localized bumps associated to Bochner–Riesz multipliers, as well as a vector-valued Stein–Tomas type estimate. In §3 we present some L^2-estimates based on a finer localization that will feature in the proof of Theorem 1.2. In §4 we formulate a discrete variant of Theorem 1.2. In §5 we recall a stopping time lemma due to Tao that features in many ε-removal arguments. In §6, in which we give an $L^p \to L^r$ variant of an argument by Tao that deduces Fourier restriction estimates from non-endpoint Bochner-Riesz assumptions. We present the proof of Proposition 4.2 in §7, which in turn implies that of Theorem 1.2.

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2. Single-scale estimates

In this section we provide estimates for suitable frequency and spatially localised Bochner–Riesz bumps. Before going into details, we make a couple of observations regarding the function ρ that will be useful in upcoming arguments. First, we can use polar coordinates for the distance function ρ and write $\xi = \varrho\xi'$ with $\xi' \in \partial\Omega = \{\xi : \rho(\xi) = 1\}$, and

$$\mathrm{d}\xi = \varrho^{d-1}\,\mathrm{d}\varrho\,\mathrm{d}\mu(\xi') \quad \text{where } \mathrm{d}\mu(\xi') = \frac{|\langle \xi', \nabla\rho(\xi')\rangle|}{|\nabla\rho(\xi')|}\,\mathrm{d}\sigma(\xi'). \tag{2.1}$$

Second, by the homogeneity and positivity of ρ there are constants $c_0 < 1$ and $C_0 > 2$ such that

$$c_0|\xi| \le \rho(\xi), \quad |\nabla\rho(\xi)| \le C_0 \tag{2.2}$$

for all $\xi \in \mathbb{R}^d$.

2.1. *Fractional derivatives and subordination formula.* Given $L^p \to L^r$ bounds for $(1-\rho(D))_+^\lambda$ one can derive analogous estimates for the Fourier multiplier operators $\chi_j(2^j(1-\rho(D)))$ and their spatially localized versions using the subordination formula [18]

$$h_j(\varrho) = \frac{1}{\Gamma(\lambda+1)}\int_0^\infty (s-\varrho)_+^\lambda h_j^{(\lambda+1)}(s)\,\mathrm{d}s. \tag{2.3}$$

Here, for smooth h compactly supported in $(0,\infty)$, $h^{(a)}$ for $a \in (0,\infty)\setminus\mathbb{N}$ refers to a fractional derivative for functions on $(0,\infty)$ introduced in [5]. More precisely, when $a \in (0,1)$ one defines

$$h^{(a)}(\varrho) = \frac{-1}{\Gamma(1-a)}\lim_{u\to\infty}\frac{d}{d\varrho}\int_\varrho^u (s-\varrho)^{-a}h(s)\,\mathrm{d}s,$$

and for $a \in (0,\infty)\setminus\mathbb{N}$, $a > 1$ one defines inductively $h^{(a)} = \frac{d}{d\rho}h^{(a-1)}$. This leads to the formula

$$\widehat{h^{(a)}}(\tau) = (-i\tau)^a\widehat{h}(\tau) = (\cos\tfrac{\pi a}{2} - i\,\mathrm{sign}(\tau)\sin\tfrac{\pi a}{2})|\tau|^a\widehat{h}(\tau).$$

Note that $h^{(a)}$ coincides with $(-1)^a$ times the ordinary derivative of order a when a is a positive integer. The following lemma will be relevant in the aforementioned transference of bounds.

Lemma 2.1. *For $\lambda > -1$ we have the inequality*

$$\int_0^\infty t^{d(\frac1p-\frac1r)+\lambda}|h_j^{(\lambda+1)}(t)|\,\mathrm{d}t \lesssim 2^{j\lambda}. \tag{2.4}$$

Proof. Observe that h_j is supported in $I_j := [1-2^{-j+1}, 1-2^{-j-1}]$. If $\lambda+1$ is an integer we have $|h^{(\lambda+1)}(s)| \lesssim 2^{j(\lambda+1)}\mathbb{1}_{I_j}(s)$ and the asserted inequality is immediate.

Assume that $\kappa < \lambda + 1 < \kappa + 1$ for $\kappa \in \mathbb{N}_0$. By the definition of fractional derivative

$$h_j^{(\lambda+1)}(\varrho) = 0 \quad \text{for } \varrho > 1 - 2^{j-1}. \tag{2.5}$$

We claim that

$$|h_j^{(\lambda+1)}(\varrho)| \lesssim \frac{2^{j(\lambda+1)}}{1 + (2^j(1-\varrho))^{\lambda+2}}, \quad 0 \le \varrho \le 1\,. \tag{2.6}$$

The inequality (2.4) is now immediate from (2.5) and (2.6).

To show (2.6), let $a = \lambda + 1 - \kappa \in (0,1)$ and observe that by integration by parts we have the formulas

$$\begin{aligned}
\text{(2.7a)} \qquad h_j^{(\lambda+1)}(\varrho) &= c_{\kappa,1}\Big(\frac{d}{d\varrho}\Big)^{(\kappa+1)} \int_\varrho^\infty (s-\varrho)^{-a}\chi_j(2^j(1-s))\,\mathrm{d}s \\
&= c_{\kappa,2}\Big(\frac{d}{d\varrho}\Big)^{(\kappa+1)} \int_\varrho^\infty (s-\varrho)^{\kappa+2-a} 2^{j(\kappa+2)}\chi_j^{(\kappa+2)}(2^j(1-s))\,\mathrm{d}s \\
\text{(2.7b)} \qquad &= c_{\kappa,3} 2^{j(\kappa+2)} \int_\varrho^\infty (s-\varrho)^{1-a}\chi_j^{(\kappa+2)}(2^j(1-s))\,\mathrm{d}s.
\end{aligned}$$

From (2.7b) we get that $|h_j^{(\lambda+1)}(\varrho)| \lesssim 2^{j(\kappa+a)}$ for $\varrho > 1 - 2^{-j+2}$ which gives (2.6) in this range.

Next assume $\varrho < 1 - 2^{-j+2}$. We can now differentiate under the integral sign directly in (2.7a) and use

$$h^{(\lambda+1)}(\varrho) = c_{\kappa,4} \int_\rho^\infty (s-\varrho)^{-a-\kappa-1}\chi_j(2^j(1-s))\,\mathrm{d}s.$$

We can estimate this integral by $2^{-j}(1-\varrho)^{-a-\kappa-1}$ and since $a + \kappa + 1 = \lambda + 1$ we obtain (2.6) also for $\varrho < 1 - 2^{-j+2}$. □

2.2. *Spatial localizations and single scale-estimates.* Let ϕ_0 be a $C_c^\infty(\mathbb{R}^d)$ function supported in $\{x : |x| < 1\}$ such that $\phi_0(x) = 1$ for $|x| \le 1/2$. For $j > 0$ define

$$\begin{aligned}
&\phi_{j,0}(x) := \phi_0(2^{-j}x) \\
&\phi_{j,n}(x) := \phi_0(2^{-j-n}x) - \phi_0(2^{-j-n+1}x) \quad \text{for } n \ge 1.
\end{aligned} \tag{2.8}$$

For any $j > 0$, define with $\chi_j \in \mathcal{Y}$,

$$m_j(\xi) := h_j(\rho(\xi)), \quad \text{where } h_j(\varrho) := \chi_j(2^j(1-\varrho)), \tag{2.9}$$

and let, for any $n \ge 0$,

$$m_{j,n} := m_j * \widehat{\phi_{j,n}}. \tag{2.10}$$

In forthcoming arguments, we will use the estimate

(2.11)

$$|m_{j,n}(\xi)| \lesssim_N \sum_{k=1}^N \int\int_0^1 \tfrac{(1-s)^{N-1}}{(N-1)!} 2^{jN}|\eta|^N |\widehat{\phi_{j,n}}(\eta)||\chi_j^{(k)}(2^j(1-\rho(\xi-s\eta)))|\,\mathrm{d}s\,\mathrm{d}\eta$$

for all $n > 0$. Note that this follows from the vanishing moments of $\widehat{\phi_{j,n}}$ and Taylor's formula for $m_j(\xi - \eta)$, which together with the multidimensional Faà di Bruno formula allow to write

(2.12)
$$\begin{aligned} m_{j,n}(\xi) &= \int \int_0^1 \tfrac{(1-s)^{N-1}}{(N-1)!} \langle -\eta, \nabla \rangle^N \big[\chi_j(2^j(1-\rho(\xi - s\eta)))\big] \widehat{\phi_{j,n}}(\eta)\, \mathrm{d}s\, \mathrm{d}\eta \\ &= \sum_{k=1}^N \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = N} 2^{jk} \int_0^1 \int b_k(s, \xi, \eta) \eta^\alpha \widehat{\phi_{j,n}}(\eta) \chi_j^{(k)}(2^j(1-\rho(\xi - s\eta)))\, \mathrm{d}\eta\, \mathrm{d}s \end{aligned}$$

for $b_k \in C^\infty$. From (2.11) we also obtain the pointwise estimate

$$|m_{j,n}(\xi)| \lesssim_{N_1} 2^{-nN}(1 + 2^j|1 - \rho(\xi)|)^{-N_1} \tag{2.13}$$

for all $n > 0$, where $N_1 > 0$ is arbitrary. This inequality also extends to the case $n = 0$ by a straightforward convolution inequality.

One can transfer bounds for $(1 - \rho(D))_+^\lambda$ to bounds on m_j and $m_{j,n}$ through the following lemma.

Lemma 2.2. *Let $1 \le p \le r \le \infty$ and assume that $(1 - \rho(D))_+^\lambda$ is bounded from $L^p(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$. Let m_j be as in (2.9), with $\chi_j \in \mathcal{Y}_{d+1}$. Then we have*

$$\|m_j(D)\|_{L^p \to L^r} \lesssim 2^{j\lambda}. \tag{2.14}$$

If $\chi_j \in \mathcal{Y}$ then

$$\|m_{j,n}(D)\|_{L^p \to L^r} \lesssim 2^{-nN} 2^{j\lambda}, \qquad n \ge 0. \tag{2.15}$$

Moreover, if $p \le q \le r$,

$$\|m_j(D)\|_{L^p \to L^q} \lesssim 2^{j(\lambda + d(\frac{1}{q} - \frac{1}{r}))}, \tag{2.16a}$$

$$\|m_{j,n}(D)\|_{L^p \to L^q} \lesssim 2^{-nN} 2^{j(\lambda + d(\frac{1}{q} - \frac{1}{r}))}, \qquad n \ge 0. \tag{2.16b}$$

Proof. Since ρ is homogeneous of degree 1 we get using (2.3) for $m_j = h_j \circ \rho$,

$$\begin{aligned} \|m_j(D)\|_{L^p \to L^r} &\le \int_0^\infty s^\lambda |h_j^{(\lambda+1)}(s)|\, \|(1 - \rho(D)/s)_+^\lambda\|_{L^p \to L^r}\, \mathrm{d}s \\ &= \|(1 - \rho(D))_+^\lambda\|_{L^p \to L^r} \int_0^\infty s^{d(\frac{1}{p} - \frac{1}{r}) + \lambda} |h_j^{(\lambda+1)}(s)|\, \mathrm{d}s \lesssim 2^{j\lambda} \end{aligned} \tag{2.17}$$

where in the last inequality we have used the hypothesis and Lemma 2.1. Similarly, for $n = 0$ we obtain

$$\|m_{j,0}(D)\|_{L^p \to L^r} \lesssim \int |\widehat{\phi_{j,0}}(\eta)| \|m_j(D - \eta)\|_{L^p \to L^r}\, \mathrm{d}\eta \lesssim 2^{j\lambda}$$

where we have used the modulation invariance of the operator norm.

For $n > 0$ we use (2.11) to obtain that $\|m_{j,n}(D)\|_{L^p \to L^r}$ is bounded by a constant times

$$\sum_{k=1}^N \int \int_0^1 \tfrac{(1-s)^{N-1}}{(N-1)!} 2^{jN} |\eta|^N |\widehat{\phi_{j,n}}(\eta)| \big\| \chi_j^{(k)}(2^j(1 - \rho(D - s\eta))) \big\|_{L^p \to L^r}\, \mathrm{d}s\, \mathrm{d}\eta.$$

Since the functions $\chi_j^{(k)}$ are by assumption fixed multiples of $\mathcal{Y}_{d+1}$ functions and $\int 2^{jN}|\eta|^N|\widehat{\phi_{j,n}}(\eta)|\,\mathrm{d}\eta = O(2^{-nN})$, we get (2.15) by the modulation invariance of the multiplier norms.

Finally, note that the convolution kernel of $m_{j,n}$ is supported on a set of diameter $O(2^{j+n})$. Therefore, using (2.15) and Hölder's inequality,

$$\begin{aligned}\|m_{j,n}(D)f\|_q &= \Big\| \sum_{Q\in\mathfrak{D}_{j+n}} m_{j,n}(D)[f\mathbb{1}_Q]\Big\|_q \lesssim \Big(\sum_{Q\in\mathfrak{D}_{j+n}} \|m_{j,n}(D)[f\mathbb{1}_Q]\|_q^q\Big)^{1/q}\\ &\lesssim \Big(\sum_{Q\in\mathfrak{D}_{j+n}} \big[\|m_{j,n}(D)[f\mathbb{1}_Q]\|_r 2^{(j+n)d(\frac1q-\frac1r)}\big]^q\Big)^{1/q}\\ &\lesssim \Big(\sum_{Q\in\mathfrak{D}_{j+n}} \big[2^{-nN}2^{j\lambda}\|f\mathbb{1}_Q\|_p 2^{(j+n)d(\frac1q-\frac1r)}\big]^q\Big)^{1/q}\\ &\lesssim 2^{-n(N-d(\frac1q-\frac1r))}2^{j(\lambda+d(\frac1q-\frac1r))}\|f\|_p\end{aligned}$$

which is (2.16b). Inequality (2.16a) follows after summing in $n \geq 0$. $\square$

It is well-known by the work of Fefferman and Stein [8] that $\mathcal{R}^\lambda$ maps $L^{\frac{2(d+1)}{d+3}}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ for all $\lambda > -1/2$. Taking this into account we note in the following corollary that boundedness of the Bochner–Riesz operator $\mathcal{R}^\lambda$ for a specific pair of exponents $(p_\circ, r_\circ)$ implies $L^p \to L^r$ bounds for the region in Figure 1.

Corollary 2.3. *Let $\frac{2(d+1)}{d+3} < p_\circ < \frac{2d}{d+1}$ and $r_\circ \in [p_\circ, \frac{d-1}{d+1}p_\circ']$. Assume that $\mathcal{R}^\lambda$ maps $L^{p_\circ}(\mathbb{R}^d)$ to $L^{r_\circ}(\mathbb{R}^d)$ for all $\lambda > \lambda(r_\circ)$. Let $\frac{2(d+1)}{d+3} \leq p_1 < p_\circ$. Then for all $\varepsilon > 0$, the inequalities*

$$\begin{aligned}&\|m_j(D)\|_{L^{p_1}\to L^r} \lesssim_\varepsilon 2^{j(\varepsilon+\lambda(r))}\\ &\|m_{j,n}(D)\|_{L^{p_1}\to L^r} \lesssim_\varepsilon 2^{-nN}2^{j(\varepsilon+\lambda(r))}, \qquad n \geq 0\end{aligned}$$

hold for all $p_1 \leq r \leq r_*(p_1, p_\circ, r_\circ)$. Moreover, in this range, $\mathcal{R}^\lambda$ maps L^{p_1} to L^r for $\lambda > \lambda(r)$.

Proof. By Lemma 2.2 it suffices to prove this for $r = r_*(p_1, p_\circ, r_\circ)$. By the same lemma and the boundedness assumption on the Bochner-Riesz operator we have

$$\|m_{j,n}(D)\|_{L^{p_\circ}\to L^{r_\circ}} \lesssim_\varepsilon 2^{-nN}2^{j(\varepsilon+\lambda(r_\circ))}.$$

On the other hand, by the same lemma and the aforementioned $L^{\frac{2(d+1)}{d+3}}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ boundedness,

$$\|m_{j,n}(D)\|_{L^{\frac{2(d+1)}{d+3}}\to L^2} \lesssim_\varepsilon 2^{-nN}2^{-j/2+j\varepsilon}.$$

Interpolating these two inequalities yields the assertion on the $L^{p_1} \to L^{r_*(p_1,p_\circ,r_\circ)}$ operator norm of $m_{j,n}(D)$ and summing in n yields the corresponding assertion for $m_j(D)$. The implication on $\mathcal{R}^\lambda$ follows by the standard decomposition of $(1-\rho(D))_+^\lambda$ as a sum of operators of type $m_j(D)$ and an L^1 bounded operator. $\square$

Remark 2.4. A standard argument using the Stein–Tomas theorem [7, 8] reveals that that the $L^p \to L^2$ operator norm of $m_j(D)$ is $O(2^{-j/2})$ for $1 \le p \le \frac{2(d+1)}{d+3}$, without any ε-loss. Using the orthogonality of the m_j, then running the decomposition $m_j = \sum_{n=0}^\infty m_{j,n}$, and using the support assumptions of $m_{j,n}$ we can use this to upgrade the bounds for $m_j(D)$ when $r = 2$ in Corollary 2.3 to the estimates

$$\Big\| \sum_{j>0} \sum_{Q\in\mathfrak{D}_j} 2^{j/2} m_j(D)[f_Q \mathbb{1}_Q] \Big\|_2 \lesssim \Big(\sum_{Q\in\mathfrak{D}} \|f_Q\|_p^2 \Big)^{\frac12} \tag{2.18}$$

which correspond to VBR$(p,2)$ for $1 \le p \le \frac{2(d+1)}{d+3}$.

2.3. *A kernel estimate.* We finish this section with a pointwise bound for the kernels associated to the multipliers $|m_{j,n}|^2$. This will be used in T^*T arguments in Section 3, and the proof is based on stationary phase calculations.

Lemma 2.5. *For $j > 0$, $n \ge 0$ let $\kappa_{j,n} = \mathcal{F}^{-1}[|m_{j,n}|^2]$. Then*

$$\sup_{x\in\mathbb{R}^d} (1 + |x|)^{\frac{d-1}{2}} |\kappa_{j,n}(x)| \lesssim 2^{-2nN} 2^{-j}. \tag{2.19}$$

Proof. We give the proof assuming $n > 0$; a small modification yields the cases $n = 0$. Using (2.13) it is straightforward to see that

$$|\kappa_{j,n}(x)| \lesssim 2^{-j} 2^{-2nN}; \tag{2.20}$$

we use this for $|x| \lesssim C_0$ (where C_0 is as in (2.2)). Note that $\kappa_{j,n}(x) = 0$ if $|x| \ge 2^{j+n+2}$.

Now assume $C_0 \le |x| \le 2^{j+n+2}$. We use formula (2.12) for $m_{j,n}$ and its complex conjugate. We then write

$$\kappa_{j,n} = \kappa_{j,n,0} + \kappa_{j,n,\complement}$$

where

$$\kappa_{j,n,0}(x) = (2\pi)^{-d} \int e^{i\langle x,\xi\rangle} \iiiint\limits_{[0,1]^2\times\mathcal{U}} \frac{((1-s_1)(1-s_2))^{N-1}}{((N-1)!)^2} \widehat{\phi_{j,n}}(v) \overline{\widehat{\phi_{j,n}}(w)} \times$$

$$\langle -v, \nabla\rangle^N \big[\chi_j(2^j(1-\rho(\xi - s_1 v)))\big] \overline{\langle -w, \nabla\rangle^N \big[\chi_j(2^j(1-\rho(\xi - s_2 w)))\big]} \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}\xi$$

where the set $\mathcal{U} \equiv \mathcal{U}(x,j,n)$ is defined by

$$\mathcal{U} = \{(v,w) : |v| < \tfrac{1}{8C_0} 2^{-j-n/2} |x|^{1/4}, |w| < \tfrac{1}{8C_0} 2^{-j-n/2} |x|^{1/4}\}.$$

The term $\kappa_{j,n,\complement}(x)$ is the analogous expression where the region $\mathcal{U}$ is replaced by $\mathcal{U}^\complement = \mathbb{R}^{2d} \setminus \mathcal{U}$. We first analyze the terms $\kappa_{j,n,\complement}(x)$. We first note that for all $v, w \in \mathbb{R}^d$, $(s_1, s_2) \in [0,1]^2$,

$$\mathrm{meas}(\{\xi : \max\{|\rho(\xi - s_1 v) - 1|, |\rho(\xi - s_2 v) - 1|\} \le 2^{-j+1}\}) \lesssim 2^{-j}$$

and interchange the order of integration to apply the integral in ξ first. For $|x| \gg 1$

$$\int_{|v|\ge 2^{-j-n/2}|x|^{1/4}} |v|^N |\widehat{\phi_{j+n}}(v)| \,\mathrm{d}\eta \lesssim_{N_2} 2^{(j+n)(d-N_2)} \int_{|\eta|\ge 2^{-j-n/2}|x|^{1/4}} |v|^{N-N_2} \,\mathrm{d}v$$

$$\lesssim 2^{-jN} 2^{-n\frac{N_2-d}{2}} |x|^{\frac{N+d-N_2}{4}}$$

provided we take $N_2 > N + d$. The same consideration applies to the w-integral. We use this in conjunction with the second part of the formula (2.12) we get for all $N_1 \in \mathbb{N}$,

$$|\kappa_{j,n,\complement}(x)| \lesssim_{N_1} 2^{-j}2^{-nN_1}|x|^{-N_1}, \quad |x| \geq C_0. \tag{2.21}$$

We now turn to the main term $\kappa_{j,n,0}$. By (2.12), $\kappa_{j,n,0}$ is a linear combination of terms of the form

$$\iiiint_{\mathcal{U}} 2^{j(k_1+k_2)} v^\alpha w^\beta \widehat{\phi_{j,n}}(v)\overline{\widehat{\phi_{j,n}}(w)}\mathcal{J}_{k_1,k_2}(x, s_1, s_2, v, w)\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}s_1\, \mathrm{d}s_2$$

where $\alpha, \beta \in \mathbb{N}_0^d$, $|\alpha| = |\beta| = N$, $1 \leq k_1, k_2 \leq N$, and $\mathcal{J}_{k_1,k_2}$ is given by

$$\begin{aligned} &\mathcal{J}_{k_1,k_2}(x, s_1, s_2, v, w) = \\ &\int e^{i\langle x,\xi\rangle} b_{k_1,k_2}(\xi, s_1, s_2, v, w)\chi_j^{(k_1)}(2^j(1 - \rho(\xi - s_1 v))\overline{\chi_j^{(k_2)}(2^j(1 - \rho(\xi - s_2 w))}\, \mathrm{d}\xi\,, \end{aligned} \tag{2.22}$$

with $b_{k_1,k_2} \in C^\infty$.

We now use polar coordinates $\xi = \rho(\xi)\Xi^\xi$ where $\Xi^\xi \in \partial\Omega$, with the intent to apply the method of stationary phase in the variables parametrizing $\partial\Omega$. Some care is needed since these variables show up in the rough terms $\chi_j^{(k_i)}(2^j(1 - \rho(\xi - s_i v)))$, and for an application of the method of stationary phase we need that the amplitude behaves reasonably well under differentiation.

Let $|v| \leq (4C_0)^{-1}$. Then $|\rho(\xi - sv) - \rho(\xi)| \leq C_0|v| \leq 1/4$ and we have

$$\begin{aligned} \xi - sv &= \rho(\xi - sv)\Xi^{\xi - sv} \\ &= \rho_{s,v}(\rho(\xi), \Xi^\xi)\, \Xi_{s,v}(\rho(\xi), \Xi^\xi) \end{aligned} \tag{2.23}$$

where

$$\rho_{s,0}(\varrho, \Xi) = \varrho, \quad \Xi_{s,0}(\varrho, \Xi) = \Xi \tag{2.24}$$

and

$$(\varrho, \Xi) \mapsto \big(\rho_{s,v}(\varrho, \Xi), \Xi_{s,v}(\varrho, \Xi)\big)$$

is a diffeomorphism that maps $(\frac{1}{2}, 2) \times \partial\Omega$ into an open set containing $(\frac{3}{4}, \frac{7}{4}) \times \partial\Omega$ and contained in $(\frac{1}{4}, \frac{9}{4}) \times \partial\Omega$. For $(v, w) \in \mathcal{U}$ we have $|v| < (8C_0)^{-1}2^{-j-n/2}|x|^{1/4} \leq (8C_0)^{-1}2^{-j3/4-n/4+1/2} \leq (4C_0)^{-1}$ and if $1/4 < \varrho < 3$ then also $\rho_{s_1,v}(\varrho, \Xi) \approx 1$ and $\rho_{s_2,w}(\varrho, \Xi) \approx 1$.

Using ρ-polar coordinates we write $\mathcal{J}_{k_1,k_2}$ in (2.22) as

$$\begin{aligned} \mathcal{J}_{k_1,k_2}(x, s_1, s_2, v, w) \;=\; &\int_\varrho \int_{\partial\Omega} e^{i\langle x,\varrho\Xi\rangle} \beta_{s_1,s_2,v,w}(\varrho, \Xi) \times \\ &\chi_j^{(k_1)}(2^j(1 - \rho_{s_1,v}(\varrho, \Xi)))\overline{\chi_j^{(k_2)}(2^j(1 - \rho_{s_2,w}(\varrho, \Xi)))}\, \mathrm{d}\mu(\Xi)\, \mathrm{d}\varrho. \end{aligned}$$

For a coordinate patch on $\partial\Omega$ with parametrization $y \mapsto \Xi(y)$ we observe that the y-derivatives of

$$y \mapsto \chi_j^{(k)}(2^j(1 - \rho_{s_1,v}(\varrho, \Xi(y)))) \tag{2.25}$$

vanish for $v = 0$, by (2.24). Hence the y-derivatives of order L of the function in (2.25) are $O(1 + (2^j|v|)^L)$. By the assumption $|v| \lesssim 2^{-j-n/2}|x|^{1/4}$, this is $O(|x|^{L/4})$.

The same applies to the entire amplitude of the y integral. By the inequalities $|v|,|w|\lesssim 2^{-j-n/2}|x|^{1/4}$ we see that the derivatives of total order L in Ξ are $O(|x|^{L/4})$. Since the oscillation parameter is $|x|$, we are still able to use the method of stationary phase to see that the inner Ξ-integral is $O(|x|^{-\frac{d-1}{2}})$, uniformly in ϱ and $(s_1,s_2)\in[0,1]$, $(v,w)\in\mathcal{U}$. For each $(s_1,s_2,v,w)\in[0,1]^2\times\mathcal{U}$, the ρ integration is over a set of measure $O(2^{-j})$ and we obtain for $C_0\le|x|\le 2^{j+n+2}$

$$|\mathcal{J}_{k_1,k_2}(x,s_1,s_2,v,w)|\lesssim 2^{-j}|x|^{-(d-1)/2}.$$

Finally the v,w integrations give a bound of $O(2^{-nN})$ each and we arrive at the estimate

$$|\kappa_{j,n,0}(x)|\lesssim 2^{-2Nn}2^{-j}|x|^{-\frac{d-1}{2}} \text{ for } C_0\le|x|\le 2^{j+n+2}. \tag{2.26}$$

We finish by combining (2.20), (2.21) and (2.26). □

3. Auxiliary L^2 bounds and finer localizations

Let η be a real valued non-negative Schwartz function such that $\widehat{\eta}$ has compact support in $\{\xi:|\xi|\le 2\}$ and such that $\eta(x)\ge 1$ for $\max_{1\le i\le d}|x_i|\le 2$. Let B be a cube of sidelength $R=R_B>1$ and center x_B and define

$$\eta_B(x)=\eta\Big(\frac{x-x_B}{R_B}\Big). \tag{3.1}$$

Let $j>0$ and m_j be as in (2.9). For $Q\in\mathfrak{D}_j$, let $\mathcal{B}_Q$ a family of pairwise disjoint subcubes of Q of sidelength R. From Plancherel's theorem and the decay properties of η, it is easy to see that for $R\le 2^j$ and functions $\{f_{Q,B}\}_{Q\in\mathfrak{D}_j,B\in\mathcal{B}_Q}$

$$\Big\|m_j(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_B f_{Q,B}\big]\Big\|_2\lesssim\Big(\sum_{Q,B}\|f_{Q,B}\|_2^2\Big)^{1/2}. \tag{3.2}$$

A key insight used in Tao's work [16] is that certain standard L^2 bounds can be improved under the assumptions that the families $\mathcal{B}_Q$ are sufficiently separated. We start with a definition.

Definition 3.1. *Let $R\ge 1$ and $S\ge 3R$. A family $\mathcal{B}$ of axis-parallel cubes of sidelength R is S-separated if* $\operatorname{dist}(x_B,x_{B'})>S$ *for all $B,B'\in\mathcal{B}$, $B\ne B'$, where x_B denotes the center of B.*

Proposition 3.2. *Let $j>0$, $R\le 2^j$. For each $Q\in\mathfrak{D}_j$ let $S_Q\ge 3R$ and $\mathcal{B}_Q$ be a finite family of S_Q-separated cubes of sidelength R intersecting Q. Then*

$$\Big\|m_j(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_B f_{Q,B}\big]\Big\|_2 \lesssim (2^{-j}R)^{1/2}\sup_Q\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{1/2}\Big(\sum_{Q,B}\|f_{Q,B}\|_2^2\Big)^{1/2} \tag{3.3}$$

and

$$\Big\| m_j(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_B f_{Q,B}\big]\Big\|_1 \lesssim 2^{j(d-1)/2}R^{1/2}\sup_Q\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{1/2}\sum_Q\Big(\sum_B\|f_{Q,B}\|_2^2\Big)^{1/2} \tag{3.4}$$

for all families of functions $\{f_{Q,B}\}$*.*

Clearly (3.3) is a significant improvement over (3.2) if S_Q is large enough for all Q, specifically if $S_Q > (R^{d-1}\#\mathcal{B}_Q)^{2/(d-1)}$.

In the proof of Proposition 3.2, we will work with variants of m_j and η_B which are localized in space. We may decompose $m_j = \sum_{n_1=0}^\infty m_{j,n_1}$ with m_{j,n_1} as in (2.10). We also decompose η_B using a decomposition analogous to (2.8). Define

$$\begin{aligned} &\eta_{B,0}(x) := \phi_0(\tfrac{x-x_B}{R_B})\eta_B(x)\\ &\eta_{B,n}(x) := \big(\phi_0(\tfrac{x-x_B}{2^nR_B})-\phi_0(\tfrac{x-x_B}{2^{n-1}R_B})\big)\eta_B(x) \quad \text{for } n\geq 1, \end{aligned} \tag{3.5}$$

so that $\eta_B=\sum_{n_2=0}^\infty \eta_{B,n_2}$. The key estimate towards establishing Proposition 3.2 is the following lemma (in which the constant $N\geq d+1$ is as in the notation section).

Lemma 3.3. *Let $j>0$, $R\leq 2^j$, $Q\in\mathfrak{D}_j$, $S_Q\geq 3R$ and $\mathcal{B}_Q$ be a finite family of S_Q-separated cubes of sidelength R. Then*

$$\Big\| m_{j,n_1}(D)\big[\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_B\big]\Big\|_2 \lesssim 2^{-n_1N}2^{-n_2N_2}(2^{-j}R)^{1/2}\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{1/2}\Big(\sum_B\|f_B\|_2^2\Big)^{1/2} \tag{3.6}$$

for all $N_2>0$ and all families of functions $\{f_B\}$ indexed in $\mathcal{B}_Q$.

Proof. We first give the proof under the stronger separation condition

$$B,B'\in\mathcal{B}_Q,\ B\neq B' \implies \mathrm{dist}(2^{n_2+2}B, 2^{n_2+2}B') > S_Q\,. \tag{3.7}$$

We write

$$\Big\| m_{j,n_1}(D)\big[\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_B\big]\Big\|_2^2 = \sum_{B,B'\in\mathcal{B}_Q}\langle T_{B,B'}f_B, f_{B'}\rangle$$

where $T_{B,B'}$ is defined by

$$T_{B,B'}f(x) = \overline{\eta_{B',n_2}(x)}|m_{j,n_1}|^2(D)[\eta_{B,n_2}f](x)$$

and has Schwartz kernel

$$K_{B,B'}(x,y) = \overline{\eta_{B',n_2}(x)}\mathcal{F}^{-1}[|m_{j,n_1}|^2](x-y)\eta_{B,n_2}(y).$$

Note that if $x,y\in\mathrm{supp}\,(\eta_{B,n_2})$ then $\phi_0(\frac{x-y}{2^{n_2+2}R})=1$ and thus for the case $B'=B$

$$K_{B,B}(x,y) = \overline{\eta_{B,n_2}(x)}\mathcal{F}^{-1}[a_{j,B,n_1,n_2}](x-y)\eta_{B,n_2}(y)$$

where

$$a_{j,B,n_1,n_2}(\xi) = \int |m_{j,n_1}(\xi-v)|^2(2^{n_2+2}R)^d\widehat{\phi_0}(2^{n_2+2}Rv)\,\mathrm{d}v.$$

Since the operator $f \mapsto \eta_{B,n} f$ is bounded on L^2 with operator norm $O(2^{-n(N_2+d)})$ for all $N_2 \in \mathbb{N}$, then $\|T_{B,B}\|_{L^2 \to L^2} \lesssim_{N_2} 2^{-2n_2(N_2+d)} \|a_{j,B,n_1,n_2}\|_\infty$. By (2.13)

$$|a_{j,B,n_1,n_2}(\xi)| \lesssim_{N_2,N_3} 2^{-2n_1 N} \sup_\xi \int \frac{(2^{n_2+2}R)^d |\widehat{\phi}_0(2^{n_2+2}Rv)|}{(1+2^j|1-\rho(\xi-v)|)^{N_3}} \,\mathrm{d}v$$

for any $N_3 \geq 0$. A computation shows that the integral is $\lesssim \min\{1, 2^{-j+n_2}R\}$. Hence we obtain

$$\|T_{B,B}\|_{L^2 \to L^2} \lesssim_{N,N_2} 2^{-2n_1 N - 2n_2(N_2+d)} \min\{1, 2^{-j+n_2}R\}. \tag{3.8}$$

We now consider the case $B \neq B'$; recall from (3.7) that $\operatorname{dist}(B, B') \geq 2^{n_2+10} S_Q$. We use Lemma 2.5 and (3.7) together with the pointwise bound for η_{B,n_2} to get

$$|K_{B,B'}(x,y)| \lesssim 2^{-2n_1 N} 2^{-2n_2(N_2+d)} 2^{-j} |S_Q|^{-\frac{d-1}{2}} \mathbb{1}_{2^{n_2+2}B}(x) \mathbb{1}_{2^{n_2+2}B'}(y)$$

for all $N_2 > 0$. By Schur's test, for $B \neq B'$

$$\|T_{B,B'}\|_2 \lesssim 2^{-2n_1 N} 2^{-2n_2 N_2 - n_2 d} R^d 2^{-j} |S_Q|^{-\frac{d-1}{2}}. \tag{3.9}$$

Now we estimate the left-hand side of (3.6) by

$$\sum_{B \in \mathcal{B}_Q} |\langle T_{B,B} f_B, f_B \rangle| + \sum_{\substack{B,B' \in \mathcal{B}_Q \\ B \neq B}} |\langle T_{B,B'} f_B, f_{B'} \rangle| = I + II.$$

The diagonal terms are estimated using Cauchy-Schwarz and (3.8), and we get

$$I \lesssim_{N,N_2} 2^{-2n_1 N - 2n_2 N_2} \min\{1, 2^{-j+n_2}R\} \sum_{B \in \mathcal{B}_Q} \|f_B\|_2^2.$$

Moreover, using instead (3.9) we obtain for the off-diagonal terms that

$$\begin{aligned} II &\lesssim 2^{-n_2(2N_2+d)} 2^{-n_1 N} R^d 2^{-j} |S_Q|^{-\frac{d-1}{2}} \sum_{B \in \mathcal{B}_Q} \|f_B\|_2 \sum_{B' \in \mathcal{B}_Q} \|f_{B'}\|_2 \\ &\lesssim 2^{-n_2(2N_2+d)} 2^{-n_1 N} R^d 2^{-j} |S_Q|^{-\frac{d-1}{2}} \#\mathcal{B}_Q \sum_{B \in \mathcal{B}_Q} \|f_B\|_2^2. \end{aligned}$$

Thus, under the additional assumption (3.7), we have estimated the left-hand side of (3.6) by $2^{-n_2 d}$ times the right-hand side of (3.6). For the general case note that each S_Q-separated family of cubes of sidelength $R < 3S_Q$ can be split into $O(2^{n_2 d})$ sub-families, each of them $S_Q 2^{n_2+10}$-separated. Applying Minkowski's inequality we lose a factor of $O(2^{n_2 d})$ which leads to (3.6). □

Now we are in position of proving Proposition 3.2.

Proof of Proposition 3.2. We first consider (3.3) and estimate the left-hand side by

$$\sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \Big\| m_{j,n_1}(D) \big[\sum_{Q \in \mathfrak{D}_j} \sum_{B \in \mathcal{B}_Q} \eta_{B,n_2} f_{Q,B} \big] \Big\|_2.$$

Now fix j, n_1, n_2, R and set

$$U \equiv U_{j,n_1,n_2,R} = \max\{2^{j+n_1+10}, 2^{n_2+10}R\}. \tag{3.10}$$

Let $\mathfrak{Q}_j \subset \mathfrak{D}_j$ be a family of U-separated 2^j-cubes. We use the localization properties of $\mathcal{F}^{-1}[m_{j,n_1}]$ and η_{B,n_2} followed by Lemma 3.3 to obtain

$$\Big\| m_{j,n_1}(D)\big[\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_2 \lesssim \Big(\sum_{Q\in\mathfrak{Q}_j}\Big\| m_{j,n_1}(D)\big[\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_2^2\Big)^{1/2}$$
$$\lesssim_{N_2} \Big(\sum_{Q\in\mathfrak{Q}_j} 2^{-2n_1N}2^{-2n_2N_2}2^{-j}R\,\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)\sum_{B\in\mathcal{B}_Q}\|f_{Q,B}\|_2^2\Big)^{1/2}$$
$$\lesssim 2^{-n_1N}2^{-n_2N_2}(2^{-j}R)^{1/2}\sup_{Q\in\mathfrak{Q}_j}\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{1/2}\Big(\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q}\|f_{Q,B}\|_2^2\Big)^{1/2}.$$

We can write $\mathfrak{D}_j$ as a union of $O((2^{-j}U)^d)$ U-separated families of 2^j-cubes. By an application of the Minkowski and Cauchy-Schwarz inequalities we lose a factor of $O((2^{-j}U)^{d/2}) = O(2^{(n_1+n_2)d/2})$ and obtain

$$\Big\| m_{j,n_1}(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_2 \lesssim 2^{-n_1(N-\frac{d}{2})-n_2(N_2-\frac{d}{2})}(2^{-j}R)^{\frac12}\quad\times$$
$$\sup_{Q\in\mathfrak{D}_j}\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{\frac12}\Big(\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\|f_{Q,B}\|_2^2\Big)^{\frac12}. \tag{3.11}$$

Since $N_1 > d/2$, $N_2 > d/2$, we may sum in n_1, n_2 and obtain (3.3).

We now turn to (3.4) and estimate the left-hand side by

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Big\| m_{j,n_1}(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_1.$$

Fix j, n_1, n_2, R and let U be as in (3.10). We now tile $\mathbb{R}^d$ by dyadic cubes of sidelength $\approx U$. Then

$$\Big\| m_{j,n_1}(D)\big[\sum_{Q\in\mathfrak{D}_j}\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_1 \lesssim \sum_{\square}\sum_{\substack{Q\in\mathfrak{D}_j\\ Q\cap\square\neq\emptyset}} U^{d/2}\Big\| m_{j,n_1}(D)\big[\sum_{B\in\mathcal{B}_Q}\eta_{B,n_2}f_{Q,B}\big]\Big\|_2$$
$$\lesssim 2^{-n_1(N-\frac{d}{2})-n_2(N_2-\frac{d}{2})}2^{j\frac{d-1}{2}}R^{\frac12}\sup_{Q\in\mathfrak{D}_j}\big(1+S_Q^{-\frac{d-1}{2}}R^{d-1}\#\mathcal{B}_Q\big)^{\frac12}\sum_{Q\in\mathfrak{D}_j}\Big(\sum_{B\in\mathcal{B}_Q}\|f_{Q,B}\|_2^2\Big)^{\frac12}.$$

Here we have used the Cauchy-Schwarz inequality and the localization properties in the first inequality, and an application of the estimate (3.3) in the last inequality. Summing in n_1, n_2 yields (3.4). $\square$

4. Discretization

Given $\mathfrak{z} \in \mathbb{Z}^d$, let $q_{\mathfrak{z}}$ denote the unique dyadic cube in $\mathfrak{Q}_0$ containing $\mathfrak{z}$. Let $\{F_{Q,\mathfrak{z}}\}$ be a collection of C^∞ functions parametrized by $(Q,\mathfrak{z}) \in \mathfrak{D}\times\mathbb{Z}^d$ satisfying

$$\operatorname{supp} F_{Q,\mathfrak{z}} \subseteq 2q_{\mathfrak{z}}\cap Q \qquad \text{and} \qquad \sup_{Q\in\mathfrak{D}}\sup_{\mathfrak{z}\in\mathbb{Z}^d}\|F_{Q,\mathfrak{z}}\|_\infty \le 1. \tag{4.1}$$

We start with a reformulation of Theorem 1.2.

Theorem 4.1. *Let $\frac{2(d+1)}{d+3} < p_\circ < \frac{2d}{d+1}$ and $r_\circ \in [p_\circ, \frac{d-1}{d+1}p'_\circ]$. Assume that $\mathcal{R}^\lambda$ maps $L^{p_\circ}(\mathbb{R}^d)$ to $L^{r_\circ}(\mathbb{R}^d)$ for all $\lambda > \lambda(r_\circ)$. Let $\frac{2(d+1)}{d+3} < p < p_\circ$, $p \le r < r_*(p, p_\circ, r_\circ)$. Then the inequality*

$$\Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D) \big[\sum_{Q\in\mathfrak{D}_j} \sum_{\mathfrak{z}\in Q\cap\mathbb{Z}^d} \gamma(Q,\mathfrak{z}) F_{Q,\mathfrak{z}} \big] \Big\|_r \lesssim_{p,r} \Big(\sum_{Q\in\mathfrak{D}} |Q| \Big(\sum_{\mathfrak{z}} |\gamma(Q,\mathfrak{z})|^p \Big)^{r/p} \Big)^{1/r} \tag{4.2}$$

holds for all functions $\gamma : \mathfrak{D} \times \mathbb{Z}^d \to \mathbb{C}$ and all $F_{Q,\mathfrak{z}}$ satisfying (4.1).

Proof of Theorem 1.2 (assuming Theorem 4.1). Let u be a Schwartz function such that $\widehat{u}$ is compactly supported and $\widehat{u}(\xi) = 1$ if $\rho(\xi) < 2$; then clearly $m_j = m_j \widehat{u}$. Let $\Phi \in C_c^\infty(\mathbb{R}^d)$ be supported in $\{x : |x| < 1/2\}$ and such that $\widehat{\Phi}(\xi) \ge 1/2$ on the support of $\widehat{u}$. Observe that $\widehat{u}/\widehat{\Phi}$ is a Schwartz function and therefore convolution with its inverse Fourier transform is bounded on L^r. Thus it suffices to prove

$$\Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D) \big[\sum_{Q\in\mathfrak{D}_j} \Phi * (f_Q \mathbb{1}_Q) \big] \Big\|_r \lesssim \Big(\sum_{Q\in\mathfrak{D}} |Q| \|f_Q\|_p^r \Big)^{1/r}. \tag{4.3}$$

Now, given $Q \in \mathfrak{D}$, $\mathfrak{z} \in \mathbb{Z}^d$, let

$$F_{Q,\mathfrak{z}}(x) = \frac{\Phi * [f_Q \mathbb{1}_{q_\mathfrak{z}\cap Q}](x)}{\|\Phi\|_{p'} \|f_Q \mathbb{1}_{q_\mathfrak{z}\cap Q}\|_p} \quad \text{if } \|f_Q \mathbb{1}_{q_\mathfrak{z}\cap Q}\|_p \neq 0 \tag{4.4}$$

and $F_{Q,\mathfrak{z}} = 0$ otherwise. Notice that $F_{Q,\mathfrak{z}}$ is supported in the double of $q_\mathfrak{z}$ and that $\|F_{Q,\mathfrak{z}}\|_\infty \le 1$ by Hölder's inequality. Let

$$\gamma(Q,\mathfrak{z}) = \|f_Q \mathbb{1}_{q_\mathfrak{z}\cap Q}\|_p.$$

Then the left-hand side of (4.3) becomes

$$\|\Phi\|_{p'} \Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D) \big[\sum_{Q\in\mathfrak{D}_j} \sum_{\mathfrak{z}\in\mathbb{Z}^d} \gamma(Q,\mathfrak{z}) F_{Q,\mathfrak{z}} \big] \Big\|_r$$

which by assumption is $\lesssim \big(\sum_{Q\in\mathfrak{Q}} |Q| \big(\sum_\mathfrak{z} |\gamma(Q,\mathfrak{z})|^p \big)^{r/p} \big)^{1/r}$. By definition of γ this gives the right-hand side of (4.3) and thus the estimate $\mathrm{VBR}(p,r)$ claimed in Theorem 1.2. □

We next reduce the main estimate for the proof of Theorem 4.1 to the situation where for each Q the function $\mathfrak{z} \to \gamma(Q,\mathfrak{z})$ is replaced by the characteristic function of a finite set $\mathcal{E}_Q \subset \mathbb{Z}^d \cap Q$.

Proposition 4.2. *Let $\frac{2(d+1)}{d+3} < p_\circ < \frac{2d}{d+1}$ and $r_\circ \in [p_\circ, \frac{d-1}{d+1}p'_\circ]$. Assume that $\mathcal{R}^\lambda$ maps $L^{p_\circ}(\mathbb{R}^d)$ to $L^{r_\circ}(\mathbb{R}^d)$ for all $\lambda > \lambda(r_\circ)$. Fix $\mathcal{M} \ge 2$ and $\delta > 0$. Let $\frac{2(d+1)}{d+3} < p < p_\circ$ and $p \le r < r_*(p, p_\circ, r_\circ)$. Then the inequality*

$$\Big\| \sum_{j>0} \sum_{\substack{Q\in\mathfrak{D}_j \\ \#\mathcal{E}_Q \le \mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} 2^{j\frac{d+1}{2}} \beta_Q m_j(D) F_{Q,\mathfrak{z}} \Big\|_{L^r(\mathbb{R}^d)} \lesssim_\delta \mathcal{M}^\delta \Big(\sum_{Q\in\mathfrak{D}} |Q| |\beta_Q|^r (\#\mathcal{E}_Q)^{\frac{r}{p_1}} \Big)^{\frac{1}{r}} \tag{4.5}$$

holds for all $p \le p_1 < p_\circ$, all subsets $\mathcal{E}_Q \subseteq \mathbb{Z}^d$, all real-valued coefficients β_Q and all families of functions $F_{Q,\mathfrak{z}}$ satisfying (4.1)*.*

Proof of Theorem 4.1 (assuming Proposition 4.2). Fix $p, p_\circ, r, r_\circ$ as in the assumptions. If $p_1 > p$ observe that $r_*(p_1, p_\circ, r_\circ) < r_*(p, p_\circ, r_\circ)$ and $r_*(p_1, p_\circ, r_\circ) \to r_*(p, p_\circ, r_\circ)$ as $p_1 \to p$. Thus we can choose p_1 with $p < p_1 < p_\circ$ such that $r < r_*(p_1, p_\circ, r_\circ)$.

Let $F_{Q,\mathfrak{z}}$ be functions satisfying (4.1) and consider a function $\gamma : \mathfrak{D} \times \mathbb{Z}^d \to \mathbb{C}$. Define $\gamma_Q(\mathfrak{z}) := \gamma(Q, \mathfrak{z})$. Without loss of generality we may assume that $\|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}$ is finite (otherwise there is nothing to prove). Clearly $|\gamma(Q,\mathfrak{z})| \le \|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}$ and therefore we can decompose $\gamma_Q \mathbb{1}_Q = \sum_{k\ge 0} \gamma_Q \mathbb{1}_{\mathcal{E}_Q^k}$ where

$$\mathcal{E}_Q^k := \{\mathfrak{z} \in \mathbb{Z}^d \cap Q : 2^{-(k+1)/p}\|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)} < |\gamma(Q,\mathfrak{z})| \le 2^{-k/p}\|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}\}.$$

For each Q we apply Chebyshev's inequality to get $\#\mathcal{E}_Q^k \le 2^{k+1}$. Let

$$\beta_Q^k := 2^{-k/p}\|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}, \qquad F_{Q,\mathfrak{z}}^k := \frac{\gamma(Q,\mathfrak{z})}{\beta_Q^k} F_{Q,\mathfrak{z}} \mathbb{1}_{\mathcal{E}_Q^k}(\mathfrak{z}).$$

Then for each k the family of functions $F_{Q,\mathfrak{z}}^k$ continues to satisfy (4.1). Hence, by (4.5) with exponents (p_1, r), with $\mathcal{M} = 2^{k+1}$ and $\delta < \frac{1}{2}(\frac{1}{p} - \frac{1}{p_1})$ we obtain

$$\begin{aligned}
&\Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D)\big[\sum_{Q\in\mathfrak{D}_j} \sum_{\mathfrak{z}\in Q\cap\mathbb{Z}^d} \gamma(Q,\mathfrak{z}) F_{Q,\mathfrak{z}}\big]\Big\|_r \\
&\lesssim \sum_{k\ge 0} \Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D)\big[\sum_{Q\in\mathfrak{D}_j} \beta_Q^k \sum_{\mathfrak{z}\in\mathcal{E}_Q^k} F_{Q,\mathfrak{z}}^k\big]\Big\|_r \\
&\lesssim C(\delta, p_1) \sum_{k\ge 0} 2^{(k+1)\delta} \Big(\sum_{Q\in\mathfrak{D}} |Q| |\beta_Q^k|^r (\#\mathcal{E}_Q^k)^{\frac{r}{p_1}}\Big)^{\frac{1}{r}} \\
&\lesssim C(\delta, p_1) \sum_{k\ge 0} 2^{(k+1)\delta} \Big(\sum_{Q\in\mathfrak{D}} |Q| \big[2^{-\frac{k}{p}}\|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}\big]^r 2^{(k+1)\frac{r}{p_1}}\Big)^{\frac{1}{r}}
\end{aligned}$$

and thus we get

$$\begin{aligned}
&\Big\| \sum_{j>0} 2^{j\frac{d+1}{2}} m_j(D)\big[\sum_{Q\in\mathfrak{D}_j} \sum_{\mathfrak{z}\in Q\cap\mathbb{Z}^d} \gamma(Q,\mathfrak{z}) F_{Q,\mathfrak{z}}\big]\Big\|_r \\
&\lesssim C(\delta, p_1) \sum_{k\ge 0} 2^{k\delta} 2^{-k(\frac{1}{p}-\frac{1}{p_1})} \Big(\sum_{Q\in\mathfrak{D}} |Q| \|\gamma_Q\|_{\ell^p(\mathbb{Z}^d)}^r\Big)^{\frac{1}{r}} \lesssim \tfrac{C(\delta,p_1)}{p_1 - p} \Big(\sum_{Q\in\mathfrak{D}} |Q| \|\gamma_Q\|_p^r\Big)^{\frac{1}{r}}.
\end{aligned}$$

Hence (4.2) is established. $\square$

5. A decomposition lemma

The following lemma is a discretized version of the stopping time argument by Tao [16, Lemma 4.3].

Lemma 5.1. *Let $\mathcal{E}$ be a finite subset of $\mathbb{Z}^d$ and $V \in \mathbb{N}$. Let $\{L_k\}_{k=0}^V$ be a sequence of integers such that $L_0 = 0$ and $L_k > L_{k-1}$ for $k = 1, \dots, V$. Then for each*

$0 \le k \le V-1$ *there exist indexing sets* A_k *satisfying*

$$\#A_k \le 2^d (\#\mathcal{E})^{1/V} \tag{5.1}$$

and families $\{\mathcal{B}_{k,\alpha}\}_{\alpha\in A_k}$ *with the following properties:*

(i) $\mathcal{B}_{k,\alpha}$ *is a collection of dyadic cubes in* $\mathfrak{D}_{L_k}$.

(ii) Any two different cubes in $\mathcal{B}_{k,\alpha}$ *have mutual distance at least* $2^{L_{k+1}}$.

(iii) For each k, $\alpha \in A_k$, $B \in \mathcal{B}_{k,\alpha}$ *there exists non-empty subsets* $\mathcal{E}_{k,B} \subset \mathcal{E} \cap B$ *such that*

$$\mathcal{E} = \bigcup_{k=0}^{V-1} \bigcup_{\alpha\in A_k} \bigcup_{B\in\mathcal{B}_{k,\alpha}} \mathcal{E}_{k,B}\,.$$

Proof. For each $\nu \in \{0,1\}^d$ and each nonnegative integer L we denote by $\mathfrak{D}_{L,\nu}$ the collection of dyadic cubes $\prod_{i=1}^d [n_i 2^L, (n_i{+}1)2^L)$ where $n_i = \nu_i \mod 2$ for $i = 1,\dots d$. Notice that for fixed ν two different cubes in $\mathfrak{D}_{L,\nu}$ have mutual distance at least 2^L. For each $\mathfrak{z} \in \mathbb{Z}^d$ and each nonnegative integer L, let $B(\mathfrak{z}, L)$ be the unique cube in $\mathfrak{D}_L$ containing $\mathfrak{z}$.

For each $\mathfrak{z} \in \mathcal{E}$, let $\kappa(\mathfrak{z}) \in [1,N] \cap \mathbb{N}$ denote the least positive integer such that

$$\#(\mathcal{E} \cap B(\mathfrak{z}, L_{\kappa(\mathfrak{z})})) \le (\#\mathcal{E})^{\kappa(\mathfrak{z})/V}.$$

For $k = 0,\dots,V-1$, let $\mathcal{E}_k := \{\mathfrak{z} \in \mathcal{E} : \kappa(\mathfrak{z}) = k+1\}$. Clearly $\mathcal{E} = \bigcup_{k=0}^{V-1} \mathcal{E}_k$.

Let $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ be the collection of cubes in $\mathfrak{D}_{L_k}$ that contain a point in $\mathcal{E}_k$. Each cube in $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ is contained in a unique cube in $\mathfrak{D}_{L_{k+1}}$. For each $\nu \in \{0,1\}^d$ denote by $\mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k)$ be the family of dyadic cubes B' in $\mathfrak{D}_{L_{k+1},\nu}$ which contain a point in $\mathcal{E}_k$; hence each such B' also contains a cube $B \in \mathfrak{D}_{L_k}(\mathcal{E}_k)$.

For $B' \in \mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k)$, enumerate the cubes in $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ that are contained in B' by

$$B_\ell(B'), \quad \text{with} \quad \ell = 1,\dots,n(B').$$

By the definition of the stopping time $\kappa(\mathfrak{z}) = k+1$ for $\mathfrak{z} \in \mathcal{E}_k$ we have

$$\#(\mathcal{E} \cap B_\ell(B')) \ge (\#\mathcal{E})^{k/V}, \quad \#(\mathcal{E} \cap B') \le (\#\mathcal{E})^{(k+1)/V},$$

and since the cubes $B_\ell(B')$ are disjoint this implies $n(B') \le (\#\mathcal{E})^{1/V}$ for all cubes $B' \in \mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k)$.

Now for fixed k, $\nu \in \{0,1\}^d$ and $1 \le \ell \le (\#\mathcal{E})^{1/V}$ let

$$\mathcal{B}_{k,(\nu,\ell)} = \{B_\ell(B') :\, B' \in \mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k),\, n(B') \ge \ell\}$$

and

$$A_k = \{\alpha \equiv (\nu,\ell) : \nu \in \{0,1\}^d,\, 1 \le \ell \le (\#\mathcal{E})^{1/V},\, \mathcal{B}_{k,(\nu,\ell)} \ne \emptyset\}.$$

Then clearly $\#A_k \le 2^d(\#\mathcal{E})^{1/V}$. If $\mathcal{B}_{k,(\nu,\ell)}$ is not empty then it consists of cubes in $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ which are $2^{L_{k+1}}$-separated, and we get properties (i) and (ii). Moreover for every cube B in $\mathfrak{D}_{L_k}(\mathcal{E}_k)$ there is a unique ℓ,ν and $B' \in \mathfrak{D}_{L_{k+1},\nu}(\mathcal{E}_k)$ such that $B = B_\ell(B')$. This implies

$$\mathcal{E}_k = \bigcup_{(\nu,\ell)\in A_k} \bigcup_{B\in\mathcal{B}_{k,(\nu,\ell)}} (\mathcal{E}_k \cap B)\,.$$

Recall that $\mathcal{E} = \bigcup_{k=0}^{V-1} \mathcal{E}_k$ is a disjoint union, and setting $\mathcal{E}_{k,B} = \mathcal{E}_k \cap B$ property (iii) follows. $\square$

6. A Fourier restriction bound

For the proof of Proposition 4.5 we shall use a Fourier restriction bound when j in the sum in (4.5) is very large. We show that such a Fourier restriction bound is implied by the non-endpoint Bochner-Riesz assumption in Theorem 1.2.

Proposition 6.1. *Let $1 < p \le r \le 2$, $\alpha > 0$. Suppose that $\mathcal{R}^{\lambda(r)+\alpha}$ maps L^p to L^r. Then for all $R \ge 2$, the inequality*

$$\|\widehat{f}|_{\partial\Omega}\|_{L^r(\partial\Omega)} \lesssim R^{2\alpha}\|f\|_{L^p(\mathbb{R}^d)} \tag{6.1}$$

holds for all f supported in a cube of sidelength R.

Remark 6.2. Under the assumption that $\|\widehat{f}|_{\partial\Omega}\|_{L^{r_\circ}(\partial\Omega)} \lesssim R^{2\alpha}\|f\|_{L^{p_\circ}(\mathbb{R}^d)}$ holds for some $p_\circ > \frac{2(d+1)}{d+3}$, $r_\circ \le \frac{d-1}{d+1}p_\circ'$ and all $\alpha > 0$ one can also show a full Fourier restriction result, i.e. for $p < p_\circ$, $r < r_*(p, p_\circ, r_\circ)$ the operator $f \mapsto \widehat{f}|_{\partial\Omega}$ maps $L^p(\mathbb{R}^d)$ to $L^r(\partial\Omega)$. This is accomplished by an adaptation of Tao's ε-removal argument in [17] (also based on the previous stopping time argument). This upgrade is not needed here.

Proof of Proposition 6.1. The proof is an adaptation of Tao's argument in [17, Thm. 1.1]. For the sake of completeness we provide the details. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ supported in $\{x : 1/4 < |x| < 4\}$ and define for large A

$$T_A f(x) = \int e^{iA\rho^*(x-y)}\eta(x-y)f(y)\,\mathrm{d}y. \tag{6.2}$$

We shall first show that under the $L^p \to L^r$ boundedness assumption on $\mathcal{R}^{\lambda(r)+a}$ we have for large A

$$\|T_A\|_{L^p\to L^r} \lesssim A^{\alpha-d/p'}. \tag{6.3}$$

Let $\varepsilon > 0$ so that for every $\xi_0 \in \partial\Omega$ the portion of the boundary in $B_{8\varepsilon}(\xi_0)$ can be parametrized by a regular parametrization $y \mapsto \Xi(y)$ (with y in an open set in $\mathbb{R}^{d-1}$). Note that by the assumption of convexity with nonvanishing curvature there is for every $x \ne 0$ a unique $\xi(x) \in \partial\Omega$ such that $x/|x|$ is the outer unit normal to $\partial\Omega$ at $\xi(x)$; moreover we may choose a $\delta > 0$ such that $|\xi(x) - \xi(\widetilde{x})| < \varepsilon$ for all $x, \widetilde{x}$ with $1/4 \le |x|, |\tilde{x}| \le 4$ and $|x - \tilde{x}| \le 4\delta$. We may now construct a finite family of C_c^∞ functions η_ν such that $\sum_\nu \eta_\nu = \eta$, where η_ν is supported in a ball $B_\delta(x_\nu)$. Let $T_{\nu,a}$ be defined as in (6.2) but with η_ν in place of η; it then suffices to prove $\|T_{\nu,A}\|_{L^p\to L^r} = O(A^{\alpha-d/p'})$.

Let $w_\nu \in C_c^\infty$ be supported in $B_{2\varepsilon}(\xi_\nu)$ such that $w_\nu(\xi) = 1$ for $\xi \in B_\varepsilon(\xi_\nu)$. Let $\mathfrak{S}_\nu = \{x \ne 0 : \xi(\frac{x}{|x|}) \in B_\varepsilon(\xi_\nu)\}$ and let $y \mapsto \Xi(y)$ be a regular parametrization of $\partial\Omega \cap B_{8\varepsilon}(\xi_\nu)$. By the assumption on $\mathcal{R}^{\lambda(r)+\alpha}$, the operator $w_\nu(D)\mathcal{R}^{\lambda(r)+\alpha}$ is $L^p \to L^r$ bounded. Let K_ν be its convolution kernel. For $x \in \mathfrak{S}_\nu$ we can express $K_\nu(x)$ using ρ-polar coordinates (see the proof of Lemma 2.5) by

$$K_\nu(x) = (2\pi)^{-d}\int_0^1 (1-\varrho)^\lambda \varrho^{d-1}\int w_\nu(\varrho\Xi(y))e^{i\varrho\langle x,\Xi(y)\rangle}\langle \Xi(y), \mathfrak{n}(\Xi(y))\rangle\,\mathrm{d}y\,\mathrm{d}\varrho.$$

We use the assumption that $\partial\Omega$ has positive Gaussian curvature. By a standard asymptotic expansion based on stationary phase calculations in the y-variable and of asymptotic expansions involving Fourier transforms of $\chi(\varrho)(1-\varrho)_+^\lambda$ ([6, §2.8], [12], [15, §VIII]), with $\lambda = \lambda(r)+\alpha$, we see that there is a constant $A_\nu \gg 1$ such that

$$K_\nu(x) = b(x)|x|^{-\lambda(r)-a-1-\frac{d-1}{2}} e^{i\rho^*(x)} \quad \text{for } x \in \mathfrak{S}_\nu^\infty := \mathfrak{S}_\nu \cap \{x : |x| \geq A_\nu\} \tag{6.4}$$

where b is a standard symbol of order 0, with

$$0 < c_1 \leq |b(x)| \leq C_1 \quad \text{if } x \in \mathfrak{S}_\nu^\infty .$$

See e.g. [12]. Let $Y(x)$ be the unique critical point for which $\nabla_y \langle x, \Xi(y)\rangle = 0$. Note that x is perpendicular to the tangent space $T_{\Xi(Y(x))}$. Then the phase function ρ^* is given by

$$\rho^*(x) = \langle x, \Xi(Y(x))\rangle = \sup_{\xi:\rho(\xi)\leq 1} \langle x, \xi\rangle. \tag{6.5}$$

It turns out that ρ^* is smooth, homogeneous of degree 1 and that the level sets of ρ^* are strictly convex hypersurfaces with nonvanishing curvature (see [11], [9, §5.1] for these calculations and more background on convex bodies). By Euler's homogeneity relation we have $\rho^*(x) = \langle x, \nabla\rho^*(x)\rangle$ and thus $\langle x, \Xi(Y(x))\rangle = \langle x, \nabla\rho^*(x)\rangle$. From the strict convexity property we get

$$\nabla\rho^*(x) = \Xi(Y(x)). \tag{6.6}$$

We now choose $A \geq 8A_\nu$ and define

$$u_{\nu,A}(x) := \frac{\eta_\nu(x)|x|^{d+1+\lambda(r)+a}}{b(Ax)};$$

we verify that in view of the symbol and nonvanishing properties of b the functions $u_{\nu,A}$ form a bounded family of C_c^∞-functions. Note that the functions $\eta_\nu(A^{-1}\cdot)$ are supported in $\mathfrak{S}_\nu^\infty$ and that from (6.4) we get for all $x \in \mathbb{R}^d$

$$u_{\nu,A}(A^{-1}x)K_\nu(x) = A^{-\frac{d+1}{2}-\lambda(r)-\alpha}\eta_\nu(A^{-1}x)e^{i\rho^*(x)}.$$

Clearly $\|\widehat{u_{\nu,A}(A^{-1}\cdot)}\|_1 = O(1)$ and therefore the convolution operator with convolution kernel $u_{\nu,A}(A^{-1}x)K_\nu(x)$ is $L^p \to L^r$ bounded with operator norm uniform in A. Denote the operator with convolution kernel $A^{-\frac{d+1}{2}-\lambda(r)-\alpha}A^d\eta_\nu(x)e^{i\rho^*(Ax)}$ by $O_{\nu,A}$; then by scaling we see that $O_{\nu,A}$ has $L^p \to L^r$ operator norm $O(A^{d/p-d/r})$. Since $\rho^*(x)$ is homogeneous of degree 1 we get

$$\|T_{\nu,A}\|_{L^p\to L^r} = A^{\frac{d+1}{2}+\lambda(r)+\alpha}A^{-d}\|O_{\nu,A}\|_{L^p\to L^r} \lesssim A^{\alpha-\frac{d}{p'}}$$

and (6.3) follows by summing in ν, provided that $A \geq \max_\nu 8A_\nu$.

We now turn to the Fourier restriction operator. By (6.6) $\partial\Omega$ is described by $\theta \mapsto \nabla\rho^*(\theta)$ for $x \in S^{d-1}$, and we have

$$\Big(\int_{\partial\Omega} |\widehat{f}(\xi)|^r \,\mathrm{d}\sigma(\xi)\Big)^{\frac1r} \lesssim \Big(\int_{S^{d-1}} |\widehat{f}(\nabla\rho^*(\theta))|^r \,\mathrm{d}\theta\Big)^{\frac1r} \lesssim \Big(\int_{1\leq|x|\leq 2} |\widehat{f}(\nabla\rho^*(x))|^r \,\mathrm{d}x\Big)^{\frac1r}$$

as $\nabla\rho^*$ is homogeneous of degree 0. Our goal is therefore to show, for $R \gg 2\sqrt{d}$, the estimate

$$\Big(\int_{1\le|x|\le 2} |\widehat{f}(\nabla\rho^*(x))|^r \,\mathrm{d}x\Big)^{1/r} \lesssim R^{2\alpha}\|f\|_p \quad \text{if } \operatorname{supp}(f) \subset Q_R; \tag{6.7}$$

here Q_R is the cube of sidelength R centered at the origin.

We may choose η in the definition of T_A above so that $\eta(w) = 1$ if $1/2 \le |w| \le 3$, in particular $\eta(x-y) = 1$ for $|y| \le 1/2$ and $1 \le |x| \le 2$. Then (6.3) yields

$$\Big(\int_{1\le|x|\le 2} |T_A g(x)|^r\Big)^{1/r} \lesssim A^{\alpha - d/p'}\|g\|_p \quad \text{if } \operatorname{supp}(g) \subseteq \{y : |y| \le \tfrac{1}{2}\}. \tag{6.8}$$

Changing variables in the oscillatory integral, we get

$$\begin{aligned}&\Big(\int_{1\le|x|\le 2} |\widehat{f}(\nabla\rho^*(x))|^r \,\mathrm{d}x\Big)^{1/r} \\ &\lesssim \Big(\int_{1\le|x|\le 2}\Big|\int_{|y|_\infty\le R^{-1}} e^{-i\langle R^2 y, \nabla\rho^*(x)\rangle} R^{2d} f(R^2 y)\,\mathrm{d}y\Big|^r \mathrm{d}x\Big)^{1/r}\end{aligned} \tag{6.9}$$

where $|y|_\infty = \max_{1\le i\le d}|y_i|$. We rewrite the phase function using Taylor's formula:

$$\langle y, \nabla\rho^*(x)\rangle = \rho^*(x-y) - \rho^*(x) - \langle y, \mathcal{H}(x,y)y\rangle$$

with $\mathcal{H}(x,y) = \int_0^1 (1-s)\nabla^2\rho^*(x-sy)\,\mathrm{d}s$ where $\nabla^2\rho^*$ denotes the matrix of second derivatives of ρ^*. We then see that the right-hand side of (6.9) is estimated by

$$\Big(\int_{1\le|x|\le 2}\Big|\int_{|y|_\infty\le R^{-1}} e^{iR^2\rho^*(x-y)} e^{i\langle Ry, \mathcal{H}(x,y)Ry\rangle} R^{2d} f(R^2 y)\,\mathrm{d}y\Big|^r \mathrm{d}x\Big)^{1/r}.$$

For $|x| \le 2$, $|w|_\infty \le 1$ expand

$$e^{i\langle w, \mathcal{H}(x,\frac{w}{R})w\rangle} = \sum_{(n_1,n_2)\in\mathbb{Z}^d\times\mathbb{Z}^d} c_{n_1,n_2}(R) e^{i\langle n_1, x\rangle} e^{i\langle n_2, w\rangle}$$

with $|c_{n_1,n_2}(R)| \le C(1+|n_1|+|n_2|)^{-10d}$ and C independent of R; this is applied for $w = Ry$. After taking out the sum in (n_1,n_2) by Minkowski's inequality, we can apply (6.8) with $A = R^2$, since $f(R^2\cdot)$ is supported in $\{|y|_\infty \le R^{-1}\}$ and $R > 2\sqrt{d}$. We obtain

$$\Big(\int_{1\le|x|\le 2} |\widehat{f}\circ\nabla\rho^*(x)|^r dx\Big)^{1/r} \lesssim \sum_{(n_1,n_2)\in\mathbb{Z}^d\times\mathbb{Z}^d} |c_{n_1,n_2}|(R^2)^{\alpha-d/p'}\|R^{2d}f(R^2\cdot)e^{i\langle n_2, R\cdot\rangle}\|_p$$

which is $\lesssim R^{2\alpha}\|f\|_p$, yielding (6.7). □

7. Proof of Proposition 4.2

Let p, r be as in the statement of Proposition 4.2, i.e. $\frac{2(d+1)}{d+3} < p < p_\circ$, $p \le r < r_*(p, , p_\circ, r_\circ)$. Let $p \le p_1 < p_\circ$ and $r_1 := \max\{r, p_1\}$.

Let $\delta > 0$, $\mathcal{M} \ge 2$. For the proof we will make the choice

$$V \equiv V(\delta) = \lceil 2/\delta \rceil, \qquad H \equiv H(\delta) = V2^{V+1}. \tag{7.1}$$

We estimate the L^r norm of

$$\sum_{j>0}\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q}2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}.$$

We will separately consider the terms with $2^j\le(2\mathcal{M})^H$ and $2^j>(2\mathcal{M})^H$. The estimate for the terms with $2^j\le(2\mathcal{M})^H$ will rely on the Bochner-Riesz hypothesis in Theorem 1.2, which combined with Corollary 2.3 yields

$$\|2^{j\frac{d+1}{2}}m_{j,n}(D)\|_{L^{p_1}\to L^{r_1}}\lesssim_\varepsilon 2^{-nN}2^{j(\frac{d}{r_1}+\varepsilon)} \tag{7.2}$$

for all $n\ge0$ and $\varepsilon>0$ and, moreover, that $\mathcal{R}^\lambda$ maps L^{p_1} to L^{r_1} for all $\lambda>\lambda(r_1)$. The estimate for the terms with $2^j>(2\mathcal{M})^H$ will rely on the $L^{p_1}\to L^{r_1}$ Fourier restriction estimate implied by the just mentioned Bochner–Riesz bound via Proposition 6.1.

We first estimate the terms with $2^j\le(2\mathcal{M})^H$ and prove

$$\Big\|\sum_{\substack{j>0:\\ 2^j\le(2\mathcal{M})^H}}\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q}2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)} \lesssim_\delta \mathcal{M}^\delta\Big(\sum_{Q\in\mathfrak{D}}|Q||\beta_Q|^r(\#\mathcal{E}_Q)^{r/p_1}\Big)^{1/r}. \tag{7.3}$$

To prove (7.3) it suffices to establish that for fixed $j>0$ with $2^j\le(2\mathcal{M})^H$

$$\Big\|\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q}2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)}\lesssim_\delta \mathcal{M}^{\delta/2}\Big(\sum_{Q\in\mathfrak{D}}|Q||\beta_Q|^r(\#\mathcal{E}_Q)^{r/p_1}\Big)^{1/r}; \tag{7.4}$$

then (7.3) follows from the triangle inequality summing over all $j>0$ with $2^j\le(2\mathcal{M})^H$. Write $m_j=\sum_{n=0}^\infty m_{j,n}$ where $m_{j,n}$ are as in (2.10). By the support properties of $\mathcal{F}^{-1}[m_{j,n}]$ and the triangle inequality, we have for fixed $j>0$ (with $2^j\le(2\mathcal{M})^H$)

$$\begin{aligned}&\Big\|2^{j(d+1)/2}m_j(D)\big[\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q}\beta_Q F_{Q,\mathfrak{z}}\big]\Big\|_r\\ &\lesssim\sum_{n=0}^\infty\Big(\sum_{Q'\in\mathfrak{D}_{j+n}}\Big\|2^{j(d+1)/2}m_{j,n}(D)\big[\sum_{\substack{Q\in\mathfrak{Q}_j:Q\subset Q'\\ \#\mathcal{E}_Q\le\mathcal{M}}}\sum_{\mathfrak{z}\in\mathcal{E}_Q}\beta_Q F_{Q,\mathfrak{z}}\big]\Big\|_r^r\Big)^{1/r}.\end{aligned}$$

By the support properties, Hölder's inequality, (7.2) with $\varepsilon = \frac{\delta}{2H}$ and (4.1), we have we then have that for each $n \geq 0$ and $Q' \in \mathfrak{D}_{j+n}$

$$\begin{aligned}
&\Big\| 2^{j(d+1)/2} m_{j,n}(D) \big[\sum_{\substack{Q\in\mathfrak{D}_j: Q\subset Q' \\ \#\mathcal{E}_Q \leq \mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} \beta_Q F_{Q,\mathfrak{z}} \big] \Big\|_r \\
&\lesssim 2^{(j+n)d(\frac{1}{r}-\frac{1}{r_1})} \Big\| 2^{j(d+1)/2} m_{j,n}(D) \big[\sum_{\substack{Q\in\mathfrak{D}_j: Q\subset Q' \\ \#\mathcal{E}_Q \leq \mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} \beta_Q F_{Q,\mathfrak{z}} \big] \Big\|_{r_1} \\
&\lesssim_\delta 2^{(j+n)d(\frac{1}{r}-\frac{1}{r_1})} 2^{-nN} 2^{j(\frac{d}{r_1}+\frac{\delta}{2H})} \Big\| \sum_{\substack{Q\in\mathfrak{D}_j: Q\subset Q' \\ \#\mathcal{E}_Q \leq \mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} \beta_Q F_{Q,\mathfrak{z}} \Big\|_{p_1} \\
&\lesssim 2^{-n(N-d(\frac{1}{r}-\frac{1}{r_1}))} 2^{j(\frac{d}{r}+\frac{\delta}{2H})} \Big(\sum_{\substack{Q\in\mathfrak{D}_j: Q\subset Q' \\ \#\mathcal{E}_Q \leq \mathcal{M}}} |\beta_Q|^{p_1} (\#\mathcal{E}_Q) \Big)^{1/p_1}.
\end{aligned}$$

Furthermore, observe that

$$\Big(\sum_{\substack{Q\in\mathfrak{D}_j: Q\subset Q' \\ \#\mathcal{E}_Q \leq \mathcal{M}}} |\beta_Q|^{p_1} (\#\mathcal{E}_Q) \Big)^{1/p_1} \lesssim \max\{1, 2^{nd(\frac{1}{p_1}-\frac{1}{r})}\} \Big(\sum_{\substack{Q\in\mathfrak{D}_j: Q\subset Q' \\ \#\mathcal{E}_Q \leq \mathcal{M}}} |\beta_Q|^{r} (\#\mathcal{E}_Q)^{r/p_1} \Big)^{1/r};$$

this follows for $r \geq p_1$ by Hölder's inequality and for $r < p_1$ by the embedding $\ell^r \subset \ell^{p_1}$. We can combine the above observations to obtain

$$\begin{aligned}
&\Big\| 2^{j(d+1)/2} m_j(D) \big[\sum_{\substack{Q\in\mathfrak{D}_j \\ \#\mathcal{E}_Q \leq \mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} \beta_Q F_{Q,\mathfrak{z}} \big] \Big\|_r \\
&\quad \lesssim_\delta \sum_{n=0}^{\infty} 2^{-n(N+\frac{d}{r_1})} \max\{2^{n\frac{d}{p_1}}, 2^{n\frac{d}{r}}\} \mathcal{M}^{\delta/2} \Big(\sum_{\substack{Q\in\mathfrak{D}_j \\ \#\mathcal{E}_Q \leq \mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} |Q| |\beta_Q|^r (\#\mathcal{E}_Q)^{r/p_1} \Big)^{1/r}
\end{aligned}$$

where we have used that $2^j \leq (2\mathcal{M})^H$. Since $N > \max\{d/p_1, d/r\} - d/r_1$, we immediately obtain (7.4). Thus (7.3) is established.

We now address the terms with $2^j > (2\mathcal{M})^H$ and prove

$$\begin{aligned}
(7.5) \quad & \Big\| \sum_{\substack{j>0: \\ 2^j > (2M)^H}} \sum_{\substack{Q\in\mathfrak{D}_j \\ \#\mathcal{E}_Q \leq \mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} 2^{j\frac{d+1}{2}} \beta_Q m_j(D) F_{Q,\mathfrak{z}} \Big\|_{L^r(\mathbb{R}^d)} \\
&\qquad\qquad \lesssim_\delta \mathcal{M}^\delta \Big(\sum_{Q\in\mathfrak{D}} |Q| |\beta_Q|^r (\#\mathcal{E}_Q)^{r/p_1} \Big)^{1/r}.
\end{aligned}$$

To show (7.5) we use Lemma 5.1 for each non-empty set $\mathcal{E}_Q$ with $Q \in \mathfrak{D}_j$ satisfying

$$\#\mathcal{E}_Q \leq \mathcal{M} < 2^{j/H-1};$$

specifically we apply it with $V = \lceil 2/\delta \rceil$, and the integer sequence $L_0 < \cdots < L_V$ defined by

$$L_0 = 0, \quad 2L_k + \tfrac{2}{d-1} \log_2(\#\mathcal{E}_Q) < L_{k+1} \leq 2L_k + \tfrac{2}{d-1} \log_2(\#\mathcal{E}_Q) + 1\,.$$

We then write

$$\mathcal{E}_Q = \bigcup_{k=0}^{V-1} \bigcup_{\alpha\in A_Q^k} \bigcup_{B\in\mathcal{B}_Q^{k,\alpha}} \mathcal{E}_{Q,B}^k \tag{7.6}$$

where A_Q^k is an indexing set of cardinality

$$\#A_Q^k \le 2^d(\#\mathcal{E}_Q)^{1/V} \le 2^d\mathcal{M}^{\delta/2} \tag{7.7}$$

and each $\mathcal{B}_Q^{k,\alpha}$ is a family of cubes of sidelength 2^{L_k}, with each pair of them having distance at least $2^{L_{k+1}}$. It will be crucial to bound 2^{L_k} by a suitable power of $\mathcal{M}$; note that for $k\ge 1$

$$L_k \le \sum_{\ell=0}^{k} 2^\ell + \frac{\log_2(\#\mathcal{E}_Q)}{d-1}\sum_{\ell=1}^{k} 2^\ell \tag{7.8}$$

as one may check by induction from the definition. Hence, for $k=1,\dots,V-1$

$$2^{L_k} \le 2^{2^{k+1}-1}\big(\#\mathcal{E}_Q\big)^{\frac{2^{k+1}-2}{d-1}} \le (2\mathcal{M})^{2^{k+1}} \le (2\mathcal{M})^{2^V} = (2\mathcal{M})^{2^{\lceil\frac{2}{\delta}\rceil}} \tag{7.9}$$

and thus, we have

$$2^{L_k} \le (2\mathcal{M})^{2^V} \le 2^{j2^V/H} \le 2^{\frac{j}{2V}} \le 2^{j\delta/4} \quad \text{provided that } 2^j \ge (2\mathcal{M})^H. \tag{7.10}$$

Also, for each α, k and $Q\in\mathfrak{D}_j$, the cubes in $\mathcal{B}_Q^{k,\alpha}$ are $2^{L_{k+1}}$-separated, in particular, $2^{2L_k}(\#\mathcal{E}_Q)^{\frac{2}{d-1}}$-separated.

We will show that

$$\begin{aligned}\Big\| \sum_{\substack{j\ge 0\\ 2^j>(2\mathcal{M})^H}} \sum_{\substack{Q\in\mathfrak{Q}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{B\in\mathcal{B}_Q^{k,\alpha(Q)}} \sum_{\mathfrak{z}\in\mathcal{E}_{k,Q,B}} 2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)} \\ \lesssim \mathcal{M}^{\delta/2}\Big(\sum_{Q\in\mathfrak{Q}} |Q||\beta_Q|^r\Big(\sum_{B\in\mathcal{B}_Q^{k,\alpha(Q)}} \#\mathcal{E}_{Q,B}^k\Big)^{r/p_1}\Big)^{1/r}\end{aligned} \tag{7.11}$$

uniformly in $0\le k<V$, in all subcollections $\mathfrak{Q}\subset\mathfrak{D}$, and all mappings $Q\mapsto\alpha(Q)$ where $\alpha(Q)$ is an index in A_Q^k. We may then obtain (7.5) by the triangle inequality. Indeed, enumerate $A_Q^k=\{\alpha_1,\dots,\alpha_{\mathfrak{n}(k,Q)}\}$ where $\mathfrak{n}(k,Q)\le 2^d\mathcal{M}^{\delta/2}$. Then from (7.6)

and (7.11)

$$\Big\| \sum_{\substack{j>0:\\ 2^j>(2\mathcal{M})^H}} \sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{\mathfrak{z}\in\mathcal{E}_Q} 2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)}$$

$$\le \sum_{k=0}^{V-1} \sum_{1\le i\le 2^d\mathcal{M}^{\delta/2}} \Big\| \sum_{\substack{j>0:\\ 2^j>(2\mathcal{M})^H}} \sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}\\ \mathfrak{n}(k,Q)\ge i}} \sum_{B\in\mathcal{B}_Q^{k,\alpha_i}} \sum_{\mathfrak{z}\in\mathcal{E}^k_{Q,B}} 2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)}$$

$$\lesssim \mathcal{M}^{\delta/2}\sum_{k=0}^{V-1} \sum_{1\le i\le 2^d\mathcal{M}^{\delta/2}} \Big(\sum_{\substack{Q\in\mathfrak{D}\\ \mathfrak{n}(k,Q)\ge i}} |Q||\beta_Q|^r\Big(\sum_{B\in\mathcal{B}_Q^{k,\alpha_i}} \#\mathcal{E}^k_{Q,B}\Big)^{r/p_1}\Big)^{1/r}$$

$$\lesssim \mathcal{M}^{\delta/2}(V2^d\mathcal{M}^{\delta/2})\Big(\sum_{Q\in\mathfrak{D}}|Q||\beta_Q|^r(\#\mathcal{E}_Q)^{r/p_1}\Big)^{1/r}$$

and since $V2^d\mathcal{M}^{\delta/2}\lesssim_\delta \mathcal{M}^{\delta/2}$ we get (7.5).

It remains to show (7.11). For this we need an auxiliary lemma. For a dyadic cube B with sidelength R_B recall the definition of η_B in (3.1) and note that $\widehat{\eta}_B$ is supported in $\{\xi : |\xi|\le 2/R_B\}$.

Lemma 7.1. *Let $L\in\mathbb{N}$ such that $2^L>8C_0$. For every $j\ge 2L$ let $\mathfrak{Q}_j$ be a collection of dyadic cubes of sidelength 2^j. For every $Q\in\mathfrak{Q}_j$ let $\mathcal{B}_Q$ be a family of dyadic subcubes Q, of sidelength 2^L. Let $S_Q\ge 2^{2L}(\#\mathcal{B}_Q)^{\frac{2}{d-1}}$ and assume that $\mathcal{B}_Q$ is S_Q-separated for all $Q\in\mathfrak{Q}_j$, $j\ge 2L$. Then for all $1\le q_2\le q_1\le 2$, the inequality*

$$\Big\|\sum_{j\ge 2L}\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}}\mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_{q_2} \tag{7.12}$$
$$\lesssim 2^{L/q_1}\Big(\sum_j\sum_{Q\in\mathfrak{Q}_j}|Q|\Big(\sum_{B\in\mathfrak{D}_L}\|G_{Q,B}\|_{q_1}^{q_1}\Big)^{q_2/q_1}\Big)^{1/q_2}$$

holds for all functions $G_{Q,B}$ indexed by $\mathfrak{Q}\times\mathfrak{D}_L$.

We first show how the proof of (7.11) is concluded, assuming the lemma. Define, for $B\in\mathcal{B}_Q^{k,\alpha(Q)}$

$$f_{Q,B}(x)=\frac{\beta_Q}{\eta_B(x)}\sum_{\mathfrak{z}\in\mathcal{E}^k_{Q,B}}F_{Q,\mathfrak{z}}(x). \tag{7.13}$$

Define the ρ-annulus

$$\mathcal{A}_{L_k}=\{\xi : 1-4C_0 2^{-L_k}\le\rho(\xi)\le 1+4C_0 2^{-L_k}\}.$$

We claim that

$$m_j(\widehat{\eta_B}*\widehat{f_{Q,B}})=m_j\big(\widehat{\eta_B}*(\mathbb{1}_{\mathcal{A}_{L_k}}\widehat{f_{Q,B}})\big) \tag{7.14}$$

whenever $j\ge L_k$; this condition is certainly guaranteed in our situation by (7.10).

To see this, first note that by the mean value theorem and by (2.2) we have $|\rho(\xi+h)-\rho(\xi)|\le C_0|h|$. Hence if $\rho(\xi)\le 1-4C_0 2^{-L_k}$ and $|h|\le 2^{1-L_k}$ then

$\rho(\xi+h)\le 1-2C_0 2^{-L_k}$. Likewise if $\rho(\xi)\ge 1+4C_0 2^{-L_k}$ and $|h|\le 2^{1-L_k}$ then $\rho(\xi+h)\ge 1+2C_0 2^{-L_k}$. The support of $\widehat{\eta}_B$ is in $\{h:|h|\le 2^{1-L_k}\}$ and the support of m_j is contained in $\{\xi: 1-2^{-j+1}<\rho(\xi)<1-2^{-j-1}\}$. Since $C_0\ge 2$ we see that for $j\ge L_k$ the supports of m_j and $\widehat{\eta}_B * (\mathbb{1}_{\mathcal{A}^{\complement}_{L_k}}\widehat{f_{Q,B}})$ are disjoint, from which (7.14) follows.

Hence

$$\sum_{\mathfrak{z}\in\mathcal{E}^k_{Q,B}} \beta_Q m_j(D)F_{Q,\mathfrak{z}} = \mathcal{F}^{-1}\big[m_j\widehat{\eta_B f_{Q,B}}\big] = \mathcal{F}^{-1}\big[m_j\big(\widehat{\eta_B} * (\mathbb{1}_{\mathcal{A}_{L_k}}\widehat{f_{Q,B}})\big)\big]$$
$$= \mathcal{F}^{-1}\big[m_j\big(\widehat{\eta_B} * G_{Q,B}\big)\big] \quad \text{where } G_{Q,B}=\mathbb{1}_{\mathcal{A}_{L_k}}\widehat{f_{Q,B}}$$

provided that $Q\in\mathfrak{D}_j$, $j\ge L_k$, $B\in\mathcal{B}^{k,\alpha(Q)}_Q$; otherwise $G_{Q,B}=0$. Apply Lemma 7.1 with $(q_1,q_2)=(r_1,r)$. This yields

$$\Big\| \sum_{\substack{j\ge 0\\ 2^j>(2\mathcal{M})^H}} \sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}} \sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q} \sum_{\mathfrak{z}\in\mathcal{E}^k_{Q,B}} 2^{j\frac{d+1}{2}}\beta_Q m_j(D)F_{Q,\mathfrak{z}}\Big\|_{L^r(\mathbb{R}^d)}$$

$$\lesssim 2^{L_k/r_1}\Big(\sum_{j\ge L_k}\sum_{\substack{Q\in\mathfrak{D}_j\\ \#\mathcal{E}_Q\le\mathcal{M}}} |Q|\Big(\sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q}\int_{\mathcal{A}_{L_k}} |\widehat{f_{Q,B}}(\xi)|^{r_1}\,\mathrm{d}\xi\Big)^{\frac{r}{r_1}}\Big)^{1/r}. \tag{7.15}$$

Using ρ-polar coordinates as in (2.1) we get

$$\int_{\mathcal{A}_{L_k}} |\widehat{f}_{Q,B}(\xi)|^{r_1}\,\mathrm{d}\xi \;\le\; \int_{1-C_0 2^{2-L_k}}^{1+C_0 2^{2-L_k}}\int_{\partial\Omega} |\widehat{f_{Q,B}}(\varrho\xi')|^{r_1}\,\mathrm{d}\mu(\xi')\varrho^{d-1}\,\mathrm{d}\varrho.$$

By Proposition 6.1 with parameters (p_1,r_1), and since $f_{Q,B}$ is supported in B we have for every $\varepsilon_1>0$,

$$\lesssim_{\varepsilon_1} 2^{L_k\varepsilon_1}\int_{1-C_0 2^{2-L_k}}^{1+C_0 2^{2-L_k}} \big\|\varrho^{-d} f_{Q,B}(\varrho^{-1}\cdot)\big\|_{p_1}^{r_1}\varrho^{d-1}\,\mathrm{d}\varrho$$

$$\lesssim_{\varepsilon_1} 2^{L_k\varepsilon_1}\|f_{Q,B}\|_{p_1}^{r_1}\int_{1-C_0 2^{2-L_k}}^{1+C_0 2^{2-L_k}} \varrho^{(\frac{d}{r_1}-d)r_1+d-1}\,\mathrm{d}\varrho \lesssim 2^{L_k\varepsilon_1}2^{-L_k}\|f_{Q,B}\|_{p_1}^{r_1}. \tag{7.16}$$

Since $\eta_B(x)\ge 1$ on B we can bound $\|f_{Q,B}\|_{p_1}\lesssim \beta_Q(\#\mathcal{E}^k_{Q,B})^{1/p_1}$ using the properties (4.1). We apply this with

$$\varepsilon_1 < \delta 2^{-\lceil\frac{2}{\delta}\rceil-1}$$

which implies $2^{L_k\varepsilon_1/r_1}\le 2^{L_k\varepsilon_1}\le\mathcal{M}^{\delta/2}$ by (7.9). Use this in (7.16) and plug it into (7.15) to obtain that the left-hand side of (7.15) is dominated by

$$\mathcal{M}^{\delta/2}\Big(\sum_{j\ge L_k}\sum_{Q\in\mathfrak{D}_j}|Q||\beta_Q|^r\Big(\sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q}(\#\mathcal{E}^k_{Q,B})^{r_1/p_1}\Big)^{r/r_1}\Big)^{1/r}.$$

Since $r_1\ge p_1$ we can use the embedding $\ell^1\hookrightarrow\ell^{r_1/p_1}$ so that

$$\sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q}(\#\mathcal{E}^k_{Q,B})^{r_1/p_1} \le \Big(\sum_{B\in\mathcal{B}^{k,\alpha(Q)}_Q}\#\mathcal{E}^k_{Q,B}\Big)^{r_1/p_1} \le (\#\mathcal{E}_Q)^{r_1/p_1}$$

and hence we get (7.11). □

Finally, we give the proof of Lemma 7.1.

Proof of Lemma 7.1. The proof follows by interpolation between the cases

(i) $q_1 = q_2 = 2$;
(ii) $q_1 = 2$, $q_2 = 1$;
(iii) $q_1 = q_2 = 1$.

As before, we use $m_j(D)\big(\eta_B \mathcal{F}^{-1}[G_{Q,B}]\big) = \mathcal{F}^{-1}\big[m_j(\widehat{\eta_B} * G_{Q,B})\big]$.

We start with (i). From (3.3) in Proposition 3.2 with $f_{Q,B} = \mathcal{F}^{-1}[G_{Q,B}]$ we get using the condition on the S_Q and Plancherel's theorem

$$2^{j/2}\Big\| \sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q} \mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_2 \lesssim 2^{L/2}\Big(\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q}\|G_{Q,B}\|_2^2\Big)^{1/2}. \tag{7.17}$$

Since the supports of m_j have bounded overlap we get

$$\begin{aligned}
&\Big\|\sum_{j\ge 2L}\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}}\mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_2\\
&\lesssim \Big(\sum_{j\ge 2L}\Big\|\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}}\mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_2^2\Big)^{1/2}\\
&\lesssim 2^{L/2}\Big(\sum_j\sum_{Q\in\mathfrak{Q}_j}|Q|\sum_{B\in\mathfrak{D}_L}\|G_{Q,B}\|_2^2\Big)^{1/2}
\end{aligned}$$

which is the case $q_1 = q_2 = 2$ of Lemma 7.1.

The case (ii) follows in a similar way, using (3.4) in Proposition 3.2 and the triangle inequality. Indeed, one has

$$\begin{aligned}
&\Big\|\sum_{j\ge 2L}\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}}\mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_1\\
&\lesssim \sum_{j\ge 2L}\Big\|\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}}\mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_2\\
&\lesssim 2^{L/2}\sum_{j\ge 2L}\sum_{Q\in\mathfrak{Q}_j}\Big(\sum_{B\in\mathfrak{D}_L}\|G_{Q,B}\|_2^2\Big)^{1/2},
\end{aligned}$$

as desired.

Finally, we prove (iii), that is,

$$\Big\|\sum_{j\ge 2L}\sum_{Q\in\mathfrak{Q}_j}\sum_{B\in\mathcal{B}_Q} 2^{j\frac{d+1}{2}}\mathcal{F}^{-1}[m_j(\widehat{\eta}_B * G_{Q,B})]\Big\|_1 \lesssim 2^L\sum_j\sum_{Q\in\mathfrak{Q}_j}|Q|\sum_{B\in\mathfrak{D}_L}\|G_{Q,B}\|_1. \tag{7.18}$$

We decompose in a familiar way (see e.g. [4])

$$m_j = \sum_{\nu\in\mathfrak{A}_j} m_j^\nu$$

where m_j^ν is supported in a $C2^{-j/2}\times\cdots\times C2^{-j/2}\times C2^{-j}$ box $R_{j,\nu}$ with long sides tangential to $\partial\Omega$ at some point, and the boxes have bounded overlap. We have

$\|\mathcal{F}^{-1}[m_j^\nu]\|_1 = O(1)$ and $\#\mathfrak{A}_j \lesssim 2^{j(d-1)/2}$. Moreover, denote by $D(\omega, 2^{1-L})$ the ball of radius 2^{1-L} centered at ω and define

$$\mathfrak{A}_j(\omega) := \{\nu \in \mathfrak{A}_j : R_{j,\nu} \cap D(\omega, 2^{1-L}) \neq \emptyset\}.$$

Then for $L \le j/2$ we have $\#\mathfrak{A}_j(\omega) \lesssim 2^{(-L+\frac{j}{2})(d-1)}$ and this bound is uniform in ω. We estimate

$$\|\mathcal{F}^{-1}[m_j(\widehat{\eta_B} * G_{Q,B})]\|_1 \le \int |G_{Q,B}(\omega)| \int \big|\mathcal{F}^{-1}\big[m_j(\cdot)\widehat{\eta_B}(\cdot - \omega)\big](x)\big| \, \mathrm{d}x \, \mathrm{d}\omega$$

and get for fixed ω

$$\begin{aligned}\big\|\mathcal{F}^{-1}\big[m_j(\cdot)\widehat{\eta_B}(\cdot - \omega)\big]\big\|_1 &\lesssim 2^{Ld} \sum_{\nu \in \mathfrak{A}_j(\omega)} \|\mathcal{F}^{-1}[m_j^\nu]\|_1 \|\mathcal{F}^{-1}[\widehat{\eta}(\tfrac{\cdot - \omega}{2^{-L}})]\|_1 \\ &\lesssim 2^{Ld} \#\mathfrak{A}_j(\omega) \lesssim 2^L 2^{j(d-1)/2}.\end{aligned}$$

Consequently,

$$\big\|2^{j(d+1)/2}\mathcal{F}^{-1}[m_j(\widehat{\eta_B} * G_{Q,B})]\big\|_1 \lesssim 2^L 2^{jd} \|G_{Q,B}\|_1$$

and (7.18) follows. This concludes the proof. □

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