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Ground state of Bose gases interacting through singular potentials

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Abstract

We consider a system of N bosons on the three-dimensional unit torus. The particles interact through repulsive pair interactions of the form $N^{3\beta-1}v(N^\beta x)$ for $\beta \in (0,1)$. We prove the next order correction to Bogoliubov theory for the ground state and the ground state energy.

1 Introduction and main results

We consider a system of N interacting bosons on the three-dimensional unit torus, i.e., the box $\Lambda = [0,1]^3$ with periodic boundary conditions. The system is described by the N-body Hamiltonian

$$H_{N,\beta} = \sum_{j=1}^{N} -\Delta_j + \frac{1}{N} \sum_{1\leq i<j\leq N} v_{N,\beta}(x_i - x_j)\,, \tag{1.1}$$

where

$$v_{N,\beta}(x) = N^{3\beta} v(N^\beta x)\,, \qquad \beta \in (0,1)\,. \tag{1.2}$$

We make the following assumption on the interaction v:

Assumption 1. *Let $v = \kappa V$ for $\kappa \in \mathbb{R}$ suffiently small and let $V : \mathbb{R}^3 \to \mathbb{R}$ be bounded, compactly supported, non-negative and spherically symmetric. Moreover, assume that v is of positive type, i.e., that it has a non-negative Fourier transform.*

The scaling parameter β interpolates between the mean-field regime ($\beta = 0$), which describes a system of many weak and long-range interactions, and the Gross–Pitaevskii regime ($\beta = 1$), featuring few and essentially on-site interactions. In this paper, we focus on the regime $\beta \in (1/2, 1)$ of singular interactions well beyond the mean-field regime.

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The Hamiltonian $H_{N,\beta}$ acts on the Hilbert space of permutation symmetric, square integrable functions on Λ^N, which we denote as $L^2_s(\Lambda^N) \subset L^2(\Lambda^N)$. It is well known that $H_{N,\beta}$ has a discrete spectrum and a unique ground state Ψ_N, which is defined as

$$H_{N,\beta}\Psi_N = E_{N,\beta}\Psi_N\,, \qquad E_{N,\beta} := \inf \sigma(H_{N,\beta})\,. \tag{1.3}$$

An important property of the potential $v_{N,\beta}$ is its scattering length, which can be defined as follows: Denote by $f : \Lambda \to \mathbb{R}$ the scattering solution on the torus, i.e., the solution of the equation

$$\left(-\Delta + \frac{1}{2N} v_{N,\beta}\right) f = 0\,. \tag{1.4}$$

The scattering length of $v_{N,\beta}$ on the torus is then given by

$$8\pi \mathfrak{a}_{N,\beta} := \sum_{p\in\Lambda^*} \widehat{v}\left(\frac{p}{N^\beta}\right) \widehat{f}_p\,, \tag{1.5}$$

where $\widehat{f}_p := \int_\Lambda f(x) \mathrm{e}^{\mathrm{i}p\cdot x}$ denotes the Fourier transform of the scattering solution. We furthermore introduce

$$\eta_p := N\left(\widehat{f}_p - \delta_{p,0}\right)\,. \tag{1.6}$$

Our main goal in this paper is to prove the first three terms in an expansion of the ground state energy $E_{N,\beta}$ for large N. The leading and next-to-leading order have first been predicted by Bogoliubov in 1947 [6]; a rigorous proof was given for our model by Boccato, Brennecke, Cenatiempo and Schlein in [4]. In our main result, we now prove the third order in the expansion:

Theorem 1. *Let Assumption 1 be satisfied and let $\beta \in (\frac{1}{2}, 1)$. Then there exists a constant $C > 0$ such that*

$$\left| E_{N,\beta} - 4\pi(N-1)\mathfrak{a}_N^\beta - E_{0,0} - E_{\mathrm{corr}} \right| \le C N^{\frac{3}{2}(\beta-1)} \tag{1.7}$$

for sufficiently large N, where

$$E_{0,0} = \frac{1}{2}\sum_{p\in\Lambda_+^*}\left(-p^2 - \widehat{v}(0) + \sqrt{|p|^4 + 2p^2\widehat{v}(0)} + \frac{\widehat{v}(0)^2}{2p^2}\right) \tag{1.8}$$

and

$$E_{\mathrm{corr}} = C_{N,\beta}\left(- \frac{1}{2N}\sum_{p\in\Lambda_+^*} \frac{\widehat{v}_N^\beta(p)^2}{2p^2}\right), \tag{1.9}$$

with

$$C_{N,\beta} = \frac{1}{2}\sum_{p\in\Lambda_+^*}(s_p c_p - \eta_p) + \widehat{v}(0)^2 \sum_{p\in\Lambda_+^*} \frac{1}{\sqrt{|p|^4 + 2p^2\widehat{v}(0)}\left(p^2 + \sqrt{|p|^4 + 2p^2\widehat{v}(0)}\right)} \tag{1.10a}$$

$$+ \sum_{p\in\Lambda_+^*} 4\Big(c_p\widetilde{s}_p + 2\widetilde{c}_p s_p\Big)^2. \tag{1.10b}$$

Here, we introduced the notation $s_p := \sinh(\eta_p)$ and $c_p := \cosh(\eta_p)$ with η_p given by (1.6), and $\widetilde{s}_p := \sinh(\tau_p)$ and $\widetilde{c}_p := \cosh(\tau_p)$ with τ_p given by (2.39).

The coefficient $E_{0,0}$, which is of order one, is the leading order of the Bogoliubov ground state energy. It has been proven rigorously for our model in [4] under the assumption of a sufficiently small interaction potential. Previously, Bogoliubov theory had been rigorously justified in the mean-field regime ($\beta = 0$) [28, 17, 20, 25, 16]. For the mathematically very difficult Gross–Pitaevskii regime ($\beta = 1$), the Bogoliubov correction to the ground state energy was obtained in [3, 26, 11, 19, 14, 1]. Note that all these works are not only restricted to the ground state energy but extend to the full low-energy excitation spectrum.

The coefficient E_{corr} is of order $N^{\beta-1}$. The contribution from line (1.10a) comes from the next order of the Bogoliubov ground state energy and can therefore also be retrieved from the analysis in [4], where it was absorbed in the error term (see Section 4). The main novelty of our result is the proof of the contribution coming from (1.10b). A comparable result beyond Bogoliubov theory has previously been obtained for $\beta = 0$ in [10], see also [27] for related results in a different setting.

After the first version of our work was published as a preprint, two new works appeared which improved the error estimate for the Bogolibov approximation in the Gross–Pitaevskii regime [13] and proved the third-order correction to the ground state energy [15].

In addition to the ground state energy, we are interested in the ground state wave function Ψ_N, and in particular in the reduced one-body density matrix of the ground state,

$$\gamma_N^{(1)} := \mathrm{Tr}_{L^2(\Lambda^{N-1})} |\Psi_N\rangle\langle\Psi_N| \,. \tag{1.11}$$

Since we are in the translation invariant setting on the torus, Ψ_N exhibits Bose–Einstein condensation (BEC) in the state $\varphi_0 \equiv 1$ (see, e.g., [21, 2, 5, 23, 12, 18] for proofs of complete BEC in various settings). In our second main theorem, we prove the next order correction to the ground state. This allows us to show that the projection onto the condensate state approximates the reduced one-body density matrix up to an error of order $N^{\frac{3}{2}(\beta-1)}$:

Theorem 2. *Let Assumption 1 be satisfied, let $\beta \in (\frac{1}{2}, 1)$ and let N be sufficiently large. Then there exists a constant $C > 0$ such that*

$$\|\Psi_N - \Psi_{N,0} - \Psi_{N,1} - \Psi_{N,2}\|_{L^2(\Lambda^N)} \leq C N^{\frac{3}{2}(\beta-1)} \,, \tag{1.12}$$

where the functions $\Psi_{N,\ell} \in L^2_s(\Lambda^N)$ are defined in (2.61)*. Moreover, it holds that*

$$\mathrm{Tr}\left|\gamma_N^{(1)} - \gamma_0\right| \leq C N^{\frac{3}{2}(\beta-1)} \tag{1.13}$$

for $\gamma_0(x, y) = 1$.

The leading order γ_0 is determined by the condensate. From our perturbative approach, we obtain a correction γ_1 to γ_0, which is, on the torus, of order N^{-1} because the leading term of γ_1 vanishes by translation invariance. Hence, γ_1 is subleading compared to the overall error, while we expect to see a nonzero contribution in the inhomogeneous setting. The proof of Theorem 2 is given in Section 9. A comparable expansion of the reduced density matrix was previously obtained for the mean-field regime $\beta = 0$ in [10, 22, 7].

Remark 1.1. (a) The bounds in (1.7) and (1.13) for the ground state energy and the reduced densities are due to the fact that we do a perturbative expansion and each step of the perturbation series gains a factor $N^{(\beta-1)/2}$. Since we truncate the expansion after two iterations, this leads to the error $N^{3(\beta-1)/2}$, which is not optimal because one can infer from parity arguments that all orders with non-integer powers of $N^{\beta-1}$ are zero. Hence, we expect the optimal errors to be of the order $N^{2(\beta-1)}$. We conjecture that our expansion could be extended to higher orders as in [10], which would also yield the optimal bounds.

(b) We assume that the interaction potential is sufficiently small. The only reason why we need this smallness assumption is that we use the result from [4] to construct the ground state projector (see Section 2.6). Given the work [3], where the corresponding result was shown for $\beta = 1$ without the smallness assumption, we expect that [4] can be extended to work without this restriction. Then our result holds without restricting to small potentials.

(c) In [3], $E_{N,\beta}$ is approximated for $\beta = 1$ by $E_{0,0}$ but with $\widehat{v}(0)$ replaced by the scattering length. Given this result, we expect that we should be able to reconstruct the scattering length in $E_{0,0}$ from the corrections E_{corr}, at least for values of β close to one. This can nicely be seen from the formula (1.9), where the next order of the Born series of the scattering length appears. Let us also remark that in [3] the formulas contain the infinite volume scattering length, while we work with the box scattering length on the torus as in [4]. The difference between both quantities is of order N^{-1} (see, e.g., [18, Lemma 9]).

(d) The restriction $\beta > 1/2$ is only for technical reasons since it allows to summarize several error terms in a more efficient way. Our proof works for the full range $\beta \in (0, 1)$, but for small β we would obtain some extra contributions to the energy which are for $\beta > 1/2$ subleading.

(e) We expect that our result can be extended to the full low-energy excitation spectrum similarly to [10].

Formally, the next order corrections in Theorems 1 and 2 can be computed by Rayleigh-Schrödinger perturbation theory. To make this rigorous, we follow the general approach introduced in [10] for the mean-field regime, which was inspired by the previous works [8, 9] in the dynamical context. The general idea in the mean-field regime is to first unitarily transform the N-body Hamiltonian to the corresponding excitation Hamiltonian $\mathbb{H}^{\mathrm{mf}}$ on the excitation Fock space, as is by now the standard procedure [20]. Subsequently, $\mathbb{H}^{\mathrm{mf}}$ is expanded in a power series in $N^{-1/2}$ around the Bogoliubov Hamiltonian $\mathbb{H}_0^{\mathrm{mf}}$. Finally, one expresses the ground state projector as

$$\mathbb{P}^{\mathrm{mf}} = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{1}{z - \mathbb{H}^{\mathrm{mf}}} \, \mathrm{d}z$$

for a suitable contour γ encircling the ground state energy and uses the expansion of $\mathbb{H}^{\mathrm{mf}}$ to construct an expansion of $\mathbb{P}^{\mathrm{mf}}$ around the Bogoliubov ground state projector

$$\mathbb{P}_0^{\mathrm{mf}} = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{1}{z - \mathbb{H}_0^{\mathrm{mf}}} \, \mathrm{d}z \, .$$

In our situation, where $\beta \in (1/2, 1)$ is far beyond the mean field regime and the interaction potentials converge to δ-interactions, any naive perturbative expansion must fail. The idea is therefore to first reduce the problem to a mean-field problem by conjugating the excitation Hamiltonian $\mathbb{H}$ with a suitable quadratic transformation which regularizes $\mathbb{H}$.

There are different suggestions for such a quadratic transformation in the literature, see e.g. [4, 26, 19]. We choose a transformation (2.25) which slightly differs from these works. It is a true Bogoliubov transformation of the form

$$\mathbb{T} = \exp\left\{ \frac{1}{2} \sum_p \eta_p (a_p^\dagger a_{-p}^\dagger - a_p a_{-p}) \right\}$$

for η_p from (1.6). In this way, conjugating $\mathbb{H}$ with $\mathbb{T}$ directly reconstructs the scattering length (1.5) in the leading order of $E_{N,\beta}$. Another advantage is that the action of $\mathbb{T}$ on creation and annihilation operators is explicit, which makes the computation of $\mathbb{T}\mathbb{H}\mathbb{T}^*$ relatively simple. On the other hand, the transformation $\mathbb{T}$ does not preserve the truncation of the excitation Fock space (in contrast to e.g. [4], where a generalized Bogoliubov transformation was used), which makes the estimates more involved.

The transformed Hamiltonian $\mathbb{T}\mathbb{H}\mathbb{T}^*$ can now be used for perturbation theory. While we follow in principle the road of [10], obtaining sufficiently strong estimates is much more difficult than in the mean-field setting because we deal with very singular potentials.

The paper is structured as follows: In Section 2 we introduce all necessary notation and explain the strategy of the proof. The two main building blocks of the proof are Proposition 2.2, which concerns the renormalization of $\mathbb{H}$ and the estimates of the errors, and Proposition 2.4, containing the perturbation theory. Section 3 collects some useful bounds on the quadratic transformations. In Section 4 we diagonalize the Bogoliubov Hamiltonian and extract the relevant contributions to the ground state energy. In Section 5, we prove Proposition 2.2. Section 6 contains estimates of the kinetic energy and the number of excitations in the ground state. Finally, we prove Proposition 2.4 in Section 7. In Section 8, we explicitly calculate the energy correction terms, and in Section 9 we prove Theorem 2.

Notation

- We denote the Fourier transform on Λ as

$$\widehat{f}(p) := \int_\Lambda f(x) \mathrm{e}^{-\mathrm{i}p\cdot x}\, \mathrm{d}x\,, \qquad \check{f}(x) := \sum_{p\in\Lambda^*} f(p)\mathrm{e}^{\mathrm{i}p\cdot x} \tag{1.14}$$

  for $\Lambda^* = 2\pi\mathbb{Z}^3$. We also use the notation $\Lambda_+^* = \Lambda^* \setminus \{0\}$.

- We abbreviate $\widehat{v}_N^\beta := \widehat{v}(\cdot/N^\beta)$.
- The notation $A \lesssim B$ indicates that there exists some constant $C > 0$ such that $A \leq CB$.

2 Method

From now on we will always assume that Assumption 1 is satisfied.

2.1 Excitation Fock space

To compute the higher order terms in the ground state energy, we need to focus on the excitations from the condensate. Since we are in the translation invariant setting on the torus, the condensate is described by the constant function $\varphi_0 : \Lambda \to \mathbb{C}$, $\varphi_0 \equiv 1$. We follow the method introduced in [20] to split the ground state Ψ_N into a condensate and excitations as

$$\Psi_N = \sum_{k=0}^{N} \varphi_0{}^{\otimes(N-k)} \otimes_s \widetilde{\chi}^{(k)}\,, \qquad \widetilde{\chi}^{(k)} \in \bigotimes_{\text{sym}}^{k} L^2_\perp(\Lambda)\,, \tag{2.1}$$

with $\otimes_s$ the symmetric tensor product and where $L^2_\perp(\Lambda)$ denotes the L^2-orthogonal complement of φ_0. The sequence

$$\widetilde{\boldsymbol{\chi}} := \big(\widetilde{\chi}^{(k)}\big)_{k=0}^{N} \oplus 0 \tag{2.2}$$

of k-particle excitations forms a vector in the (truncated) excitation Fock space over $L^2_\perp(\Lambda)$,

$$\mathcal{F}_\perp^{\leq N} = \bigoplus_{k=0}^{N} \bigotimes_{\text{sym}}^{k} L^2_\perp(\Lambda) \ \subset\ \mathcal{F}_\perp = \bigoplus_{k=0}^{\infty} \bigotimes_{\text{sym}}^{k} L^2_\perp(\Lambda)\,, \tag{2.3}$$

where the direct sum in (2.2) is with respect to the decomposition $\mathbb{1} = \mathbb{1}^{\leq N} \oplus \mathbb{1}^{>N}$. In the following, direct sums are always understood in this sense, unless otherwise specified.

The creation and annihilation operators on $\mathcal{F}_\perp$ are

$$(a^\dagger(f)\boldsymbol{\phi})^{(k)}(x_1,...,x_k) = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} f(x_j)\phi^{(k-1)}(x_1,...,x_{j-1},x_{j+1},...,x_k)\,,\ k \geq 1\,, \tag{2.4}$$

$$(a(f)\boldsymbol{\phi})^{(k)}(x_1,...,x_k) = \sqrt{k+1}\int \mathrm{d}x\overline{f(x)}\phi^{(k+1)}(x_1,...,x_k,x)\,,\ k \geq 0 \tag{2.5}$$

for $f \in L^2_\perp(\Lambda)$ and $\boldsymbol{\phi} \in \mathcal{F}_\perp$. They satisfy the canonical commutation relations

$$[a(f), a^\dagger(g)] = \langle f, g\rangle\,, \qquad [a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0\,. \tag{2.6}$$

Since we consider the translation invariant setting, we will mostly work in the momentum space representation. We denote

$$\Lambda^* := 2\pi\mathbb{Z}^3\,, \qquad \Lambda^*_+ := \Lambda^* \setminus \{0\} \tag{2.7}$$

and define for $p \in \Lambda^*$ the normalized plane waves $\varphi_p(x) = \mathrm{e}^{\mathrm{i}p\cdot x} \in L^2(\Lambda)$. The operators which create/annihilate a particle in the state φ_p are given as

$$a_p^\dagger := a^\dagger(\varphi_p)\,, \qquad a_p := a(\varphi_p)\,. \tag{2.8}$$

We denote the vacuum of $\mathcal{F}_\perp$ as $|\Omega\rangle = (1, 0, 0, \dots)$. The number operator on $\mathcal{F}_\perp$ is given by

$$\mathcal{N}_\perp := \sum_{p\in\Lambda^*_+} a_p^\dagger a_p\,, \qquad (\mathcal{N}_\perp\boldsymbol{\phi})^{(k)} = k\phi^{(k)} \ \text{ for } \boldsymbol{\phi} \in \mathcal{F}_\perp\,. \tag{2.9}$$

The N-body ground state Ψ_N is mapped onto its excitation vector $\widetilde{\boldsymbol{\chi}}$ by the excitation map

$$U_N : L^2(\Lambda^N) \to \mathcal{F}_\perp^{\leq N}\,, \quad \widetilde{\boldsymbol{\chi}} := U_N\Psi_N \oplus 0\,. \tag{2.10}$$

The map U_N is unitary and acts as

$$U_N a_0^\dagger a_0 U_N^* = N - \mathcal{N}_\perp\,, \tag{2.11a}$$
$$U_N a_p^\dagger a_0 U_N^* = a_p^\dagger\sqrt{N - \mathcal{N}_\perp}\,, \tag{2.11b}$$
$$U_N a_0^\dagger a_p U_N^* = \sqrt{N - \mathcal{N}_\perp}\,a_p\,, \tag{2.11c}$$
$$U_N a_p^\dagger a_q U_N^* = a_p^\dagger a_q \tag{2.11d}$$

for $p, q \in \Lambda^*_+$.

2.2 Excitation Hamiltonian

In this section we conjugate the N-body Hamiltonian $H_{N,\beta}$ with the excitation map U_N. The embedding of $H_{N,\beta}$ in the Fock space is given by

$$\mathcal{H}_{N,\beta} = \sum_{p\in\Lambda^*} p^2 a_p^\dagger a_p + \frac{1}{2N} \sum_{p,q,r\in\Lambda^*} \widehat{v}_N^\beta(r) a_p^\dagger a_q^\dagger a_{q-r} a_{p+r} \,, \tag{2.12}$$

where we used the abbreviation

$$\widehat{v}_N^\beta := \widehat{v}(\cdot/N^\beta)\,. \tag{2.13}$$

Using (2.11), we compute

$$\begin{aligned} U_N H_{N,\beta} U_N^* = \frac{1}{2}(N-1)\widehat{v}(0) + \mathbb{K}_0 + \left[\frac{N-\mathcal{N}_\perp}{N}\right]_+ \mathbb{K}_1 \\ + \left(\mathbb{K}_2 \frac{\sqrt{[(N-\mathcal{N}_\perp)(N-\mathcal{N}_\perp-1)]_+}}{N} + \text{h.c.}\right) \\ + \left(\mathbb{K}_3 \frac{\sqrt{[N-\mathcal{N}_\perp]_+}}{N} + \text{h.c.}\right) + \frac{1}{N}\mathbb{K}_4 \end{aligned} \tag{2.14}$$

as operator on $\mathcal{F}_\perp^{\leq N}$, where we denoted by $[\cdot]_+$ the positive part and used the shorthand notation

$$\mathbb{K}_0 := \sum_{p\in\Lambda^*} p^2 a_p^\dagger a_p \,, \tag{2.15a}$$

$$\mathbb{K}_1 := \sum_{p\in\Lambda_+^*} \widehat{v}_N^\beta(p) a_p^\dagger a_p \,, \tag{2.15b}$$

$$\mathbb{K}_2 := \tfrac{1}{2} \sum_{p\in\Lambda_+^*} \widehat{v}_N^\beta(p) a_p^\dagger a_{-p}^\dagger \,, \tag{2.15c}$$

$$\mathbb{K}_3 := \sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}} \widehat{v}_N^\beta(p) a_{p+q}^\dagger a_{-p}^\dagger a_q \,, \tag{2.15d}$$

$$\mathbb{K}_4 := \tfrac{1}{2} \sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}} \widehat{v}_N^\beta(r) a_{p+r}^\dagger a_q^\dagger a_p a_{q+r} \,. \tag{2.15e}$$

Note that we added the positive part in the prefactors of $\mathbb{K}_0$, $\mathbb{K}_2$ and $\mathbb{K}_3$ for free since the operator $U_N H_{N,\beta} U_N^*$ is defined on the truncated Fock space where $\mathcal{N}_\perp \leq N$. To be able to renormalize it in the next step, we first need to extend $U_N H_{N,\beta} U_N^*$ to an operator $\mathbb{H}$ on the full Fock space $\mathcal{F}_\perp$ in such a way that its low-energy spectrum coincides with the low-energy spectrum of $H_{N,\beta}$. In particular, we require that $\widetilde{\boldsymbol{\chi}} = U_N \Psi_N \oplus 0$ is an eigenstate of the new operator $\mathbb{H}$ satisfying

$$\mathbb{H}\widetilde{\boldsymbol{\chi}} = E_{N,\beta}\widetilde{\boldsymbol{\chi}} \tag{2.16}$$

for $E_{N,\beta}$ from (1.3). This is satisfied by the choice $\mathbb{H} : \mathcal{F}_\perp \to \mathcal{F}_\perp$,

$$\mathbb{H} := \frac{N-1}{2}\widehat{v}(0) + \mathbb{K}_0 + \frac{1}{N}\mathbb{K}_4$$

$$+\mathbb{K}_1\left[\frac{N-\mathcal{N}_\perp}{N}\right]_+\oplus 0+\left(\mathbb{K}_2\frac{\sqrt{[(N-\mathcal{N}_\perp)(N-\mathcal{N}_\perp-1)]_+}}{N}\oplus 0+\text{h.c.}\right)$$
$$+\left(\mathbb{K}_3\frac{\sqrt{[N-\mathcal{N}_\perp]_+}}{N}\oplus 0+\text{h.c.}\right). \tag{2.17}$$

Here we extended the particle number preserving operators $\mathbb{K}_0$ and $\mathbb{K}_4$ trivially to the full space, whereas all other operators are extended by zero outside $\mathcal{F}_\perp^{\leq N}$. To see that (2.16) holds true, one observes that

$$\mathbb{H}-4\pi(N-1)\mathfrak{a}_{N,\beta}=\mathbb{H}^<\oplus\mathbb{H}^>, \tag{2.18}$$

where

$$\mathbb{H}^<:=U_N H_{N,\beta}U_N^*-4\pi(N-1)\mathfrak{a}_{N,\beta}, \tag{2.19}$$
$$\mathbb{H}^>:=\mathbb{1}^{>N}\left(\mathbb{K}_0+\frac{1}{N}\mathbb{K}_4\right). \tag{2.20}$$

From [4, Theorem 1.1], we know that the two lowest eigenvalues of $\mathbb{H}^<$ are of order one. Since $\mathbb{K}_4\geq 0$ by Assumption 1 and as $|p|^2\geq 4\pi^2$ as operator on $L^2_\perp(\Lambda)$, we conclude that

$$\sigma\left(\mathbb{H}^>\right)\subset(4\pi^2 N,\infty). \tag{2.21}$$

Consequently, neither the ground state energy nor the first excited eigenvalue of $\mathbb{H}^<$ are elements $\sigma(\mathbb{H}^>)$. Since

$$\sigma(\mathbb{H})=\sigma(\mathbb{H}^<)\cup\sigma(\mathbb{H}^>), \tag{2.22}$$

this implies that the ground state of $\mathbb{H}$ is given by $\widetilde{\boldsymbol{\chi}}=U_N\Psi_N\oplus 0$; in particular, (2.16) is satisfied. Hence, we can from now on work with the operator $\mathbb{H}$, to which we refer as the excitation Hamiltonian.

Remark 2.1. We remark that there was a small mistake in [10], where the prefactor of $\mathbb{K}_1$ was defined without the positive part and the prefactors of $\mathbb{K}_2$ and $\mathbb{K}_3$ were expanded in Taylor series on $\mathcal{F}_\perp$ and not only on $\mathcal{F}_\perp^{\leq N}$. Hence, there are in fact some additional remainder terms, which were not taken into account. However, these remainders are arbitrarily small because they live only on $\mathcal{F}_\perp^{>N}$ and any power of the number operator with respect to the low-energy states of both the full Hamiltonian and the Bogoliubov Hamiltonian can be bounded uniformly in N. Hence, this does not affect the result of [10].

2.3 Quadratic transformation

In this section we construct the Bogoliubov transformation which we will use to renormalize the excitation Hamiltonian $\mathbb{H}$. Recall that f denotes the scattering solution on the torus as defined in (1.4) and

$$\eta_p=N\left(\widehat{f}_p-\delta_{p,0}\right)$$

as in (1.6). Then it follows that

$$p^2\eta_p+\frac{1}{2N}\sum_{q\in\Lambda_+^*}\widehat{v}\left(\frac{p-q}{N^\beta}\right)\eta_q=-\frac{1}{2}\widehat{v}\left(\frac{p}{N^\beta}\right) \tag{2.23}$$

for $p \in \Lambda_+^*$. As discussed in [19, Section 3], there exists a solution to (2.23), namely $\check{\eta} \in L^2(\Lambda)$ given by

$$\check{\eta} = -\frac{1}{2} q_0 \left[q_0 \left(-\Delta + \frac{1}{2N} v_{N,\beta} \right) q_0 \right]^{-1} q_0 v_{N,\beta} \,, \tag{2.24}$$

where q_0 denotes the projector onto the orthogonal complement of $\varphi_0 \equiv 1$. The Fourier coefficients of the solution $\check{\eta}$ are given through (2.23).

Now we can define the Bogoliubov transformation $\mathbb{T}$ on $\mathcal{F}_\perp$ as

$$\mathbb{T} := \exp \left\{ \frac{1}{2} \sum_{p \in \Lambda_+^*} \eta_p \left(a_p^\dagger a_{-p}^\dagger - a_p a_{-p} \right) \right\} . \tag{2.25}$$

It acts on creation and annihilation operators as

$$\mathbb{T} a_p^\dagger \mathbb{T}^* = c_p a_p^\dagger + s_p a_{-p} \,, \tag{2.26a}$$

$$\mathbb{T} a_p \mathbb{T}^* = c_p a_p + s_p a_{-p}^\dagger \,, \tag{2.26b}$$

where

$$c_p := \cosh(\eta_p) \,, \qquad s_p := \sinh(\eta_p) \,. \tag{2.27}$$

Since v is spherically symmetric and consequently $\widehat{v}(p) = \widehat{v}(-p)$, it follows that

$$s_{-p} = s_p \,, \qquad c_{-p} = c_p \,. \tag{2.28}$$

2.4 Regularized excitation Hamiltonian

The next step is to renormalize the excitation Hamiltonian $\mathbb{H}$ by conjugating it with the Bogoliubov transformation $\mathbb{T}$. This is done in the following proposition, whose proof we postpone to Section 5.

Proposition 2.2. *For $\mathbb{H}$ as in (2.17), it holds that*

$$\mathbb{G} := \mathbb{T}\mathbb{H}\mathbb{T}^* - \mathcal{C} = \mathbb{G}_0 + \mathbb{G}_1 + \mathbb{G}_2 + \mathbb{R}_2 \,, \tag{2.29}$$

where

$$\begin{aligned} \mathcal{C} = {} & \frac{1}{2}(N-1)\widehat{v}(0) + \sum_{p \in \Lambda^*} \left(p^2 + \widehat{v}_N^\beta(p) \right) s_p^2 + \sum_{p \in \Lambda^*} \widehat{v}_N^\beta(p) c_p s_p \\ & + \frac{1}{2N} \sum_{p \in \Lambda_+^*} (\widehat{v}_N^\beta * cs)_p c_p s_p - \frac{1}{N} \sum_{p \in \Lambda_+^*} \widehat{v}_N^\beta(p) \left(\frac{1}{2} c_p s_p + c_p s_p^3 \right) , \end{aligned} \tag{2.30}$$

$$\mathbb{G}_0 = \sum_{p \in \Lambda_+^*} F_p a_p^\dagger a_p + \frac{1}{2} \sum_{p \in \Lambda_+^*} G_p (a_p^\dagger a_{-p}^\dagger + a_p a_{-p}) \,, \tag{2.31}$$

$$\mathbb{G}_1 := \frac{1}{\sqrt{N}} \mathbb{T}(\mathbb{K}_3 + \mathbb{K}_3^*)\mathbb{T}^* \,, \tag{2.32}$$

$$\mathbb{G}_2 := \frac{1}{2N} \sum_{\substack{p,q,r \in \Lambda_+^* \\ p+r \neq 0, q+r \neq 0}} \widehat{v}_N^\beta(r) c_{p+r} c_q c_p c_{q+r} a_{p+r}^\dagger a_q^\dagger a_p a_{q+r} \,, \tag{2.33}$$

with coefficients F_p, G_p *given by*

$$F_p := (c_p^2 + s_p^2)p^2 + (c_p + s_p)^2 \widehat{v}_N^\beta(p) + \frac{2}{N}(\widehat{v}_N^\beta * cs)_p c_p s_p \,, \tag{2.34a}$$

$$G_p := 2c_p s_p p^2 + (c_p + s_p)^2 \widehat{v}_N^\beta(p) + \frac{1}{N}(\widehat{v}_N^\beta * cs)_p (c_p^2 + s_p^2) \,. \tag{2.34b}$$

Here, we abbreviated

$$(\widehat{v}_N^\beta * cs)_p := \sum_{\substack{q \in \Lambda_+^* \\ q \neq p}} \widehat{v}_N^\beta(p-q) c_q s_q \,. \tag{2.35}$$

The remainder satisfies for $\ell \in \mathbb{R}$ *and* $\psi, \xi \in \mathcal{F}_\perp$

$$\begin{aligned} |\langle \psi, \mathbb{R}_2 \xi \rangle| \lesssim N^{\frac{3}{2}(\beta-1)} \Big(& \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4-\ell} \psi\| \, \|(\mathcal{N}_\perp+1)^{5/4+\ell} \xi\| \\ & + \|((\mathcal{N}_\perp+1)^{5/4-\ell} \psi\| \, \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4+\ell} \xi\| \Big) \,. \end{aligned} \tag{2.36}$$

for all $\beta \in (1/2, 1)$.

The decomposition of $\mathbb{G} = \mathbb{T}\mathbb{H}\mathbb{T}^* - \mathcal{C}$ in Proposition 2.2 allows to extract the leading-order contribution of the renormalized excitation Hamiltonian. In Section 5, we analyze $\mathbb{G}$ and show that $\mathcal{C}$ is of order N, while $\mathbb{G}_0$ is of order one, $\mathbb{G}_1$ is of order $N^{(\beta-1)/2}$, and $\mathbb{G}_2$ is of order $N^{\beta-1}$. The remainder $\mathbb{R}_2$ is of even lower order $N^{3(\beta-1)/2}$. We can further decompose $\mathbb{G}_1$ as

$$\begin{aligned} \mathbb{G}_1 = & \frac{1}{\sqrt{N}} \sum_{\substack{p,q \in \Lambda_+^* \\ p+q \neq 0}} \widehat{v}_N^\beta(p) c_{p+q} c_p c_q (a_{p+q}^\dagger a_{-p}^\dagger a_q + \text{h.c.}) \\ & + \frac{1}{\sqrt{N}} \sum_{\substack{p,q \in \Lambda_+^* \\ p+q \neq 0}} \widehat{v}_N^\beta(p) c_{p+q} c_p s_q (a_{p+q}^\dagger a_{-p}^\dagger a_{-q}^\dagger + \text{h.c.}) \\ & + \mathbb{R}_d \,, \end{aligned} \tag{2.37}$$

where $\mathbb{R}_d$ is of order $N^{-1/2}$ and thus even smaller than the remainder $\mathbb{R}_2$ in the parameter regime $\beta \in (2/3, 1)$(see Section 5.1, Lemma 5.1).

2.5 Bogoliubov-Hamiltonian

The quadratic operator $\mathbb{G}_0$ is known as the Bogoliubov Hamiltonian, and its ground state energy E_0, satisfying

$$\mathbb{G}_0 \boldsymbol{\chi}_0 = E_0 \boldsymbol{\chi}_0 \,, \tag{2.38}$$

contributes the next-to-leading order in the ground state energy of $H_{N,\beta}$. To extract it, we need to diagonalize $\mathbb{G}_0$, which can be done explicitly by means of a Bogoliubov transformation. To construct the diagonalizing transformation, we define for $p \in \Lambda_+^*$ the sequence τ_p by

$$\tanh(2\tau_p) = -\frac{G_p}{F_p} \,. \tag{2.39}$$

By Lemma 3.5 below, this is well defined. Note that τ_p is real-valued by definition. Making use of this sequence τ_p, we define the Bogoliubov transformation

$$\mathbb{U}_\tau := \exp\left[\frac{1}{2} \sum_{p \in \Lambda_+^*} \tau_p (a_p^\dagger a_{-p}^\dagger - a_p a_{-p}) \right] \,. \tag{2.40}$$

It is well known that $\mathbb{U}_\tau$ diagonalizes the quadratic Hamiltonian $\mathbb{G}_0$ as

$$\mathbb{U}_\tau \mathbb{G}_0 \mathbb{U}_\tau^* = \frac{1}{2} \sum_{p\in\Lambda_+^*} \big[-F_p + \sqrt{F_p^2 - G_p^2}\big] + \sum_{p\in\Lambda_+^*} \sqrt{F_p^2 - G_p^2} a_p^\dagger a_p \tag{2.41}$$

(see, e.g., [4, Lemma 5.1]). Consequently,

$$\boldsymbol{\chi}_0 = \mathbb{U}_\tau^* |\Omega\rangle \tag{2.42}$$

and

$$E_0 = \frac{1}{2} \sum_{p\in\Lambda_+^*} \big[-F_p + \sqrt{F_p^2 - G_p^2}\big] \,. \tag{2.43}$$

From Lemma 3.5 below, it is easy to see that $E_0 = \mathcal{O}(1)$. Adding E_0 to the constant $\mathcal{C}$ from Proposition 2.2 yields the leading and next-to-leading order contribution to the ground state energy:

Lemma 2.3. *Let $\beta \in (0,1)$. Then, for $\mathcal{C}$ as in* (2.30)*, it holds for every $\alpha < \beta$ that*

$$\mathcal{C} + E_0 = 4\pi(N-1)\mathfrak{a}_N^\beta + E_{0,0} + E_{0,1} + \mathcal{O}(N^{2(\beta-1)}) + \mathcal{O}(N^{-1}) + \mathcal{O}(N^{-\alpha}) \,, \tag{2.44}$$

where $\mathfrak{a}_N^\beta$ denotes the scattering length defined in (1.5) *and with $E_{0,0}$ and $E_{0,1}$ given in* (1.8) *and* (4.28)*. Moreover,*

$$|E_{0,0}| \lesssim 1 \,, \qquad |E_{0,1}| \lesssim N^{\beta-1} \,. \tag{2.45}$$

The proof of Lemma 2.3 is given in Section 4.

2.6 Perturbation theory

Our goal is a perturbative expansion of the ground state $\boldsymbol{\chi}$ of the operator

$$\mathbb{G} = \mathbb{T}\mathbb{H}\mathbb{T}^* - \mathcal{C} \,.$$

The ground state $\boldsymbol{\chi}$ satisfies the eigenvalue equation

$$\mathbb{G}\boldsymbol{\chi} = (E_{N,\beta} - \mathcal{C})\boldsymbol{\chi} =: E\boldsymbol{\chi} \,. \tag{2.46}$$

Equivalently, $\boldsymbol{\chi} = \mathbb{T}\widetilde{\boldsymbol{\chi}} = \mathbb{T}(U_N\Psi_N \oplus 0)$. We denote the spectral projectors of $\mathbb{G}$ and $\mathbb{G}_0$ corresponding to their ground state energies E and E_0, defined in (2.46) and (2.38), respectively, by

$$\mathbb{P} := |\boldsymbol{\chi}\rangle\langle\boldsymbol{\chi}| \,, \qquad \mathbb{Q} := \mathbb{1} - \mathbb{P} \,, \tag{2.47}$$

$$\mathbb{P}_0 := |\boldsymbol{\chi}_0\rangle\langle\boldsymbol{\chi}_0| \,, \qquad \mathbb{Q}_0 := \mathbb{1} - \mathbb{P}_0 \,. \tag{2.48}$$

In [4, Theorem 1.1], it is shown that

$$E_{N,\beta} = 4\pi(N-1)\mathfrak{a}_{N,\beta} + E_{0,0} + \mathcal{O}(N^{-\alpha}) \tag{2.49}$$

for $\alpha < \min\{\beta, \frac{1-\beta}{2}\}$. More precisely, the scattering length $\mathfrak{a}_{N,\beta}$ is constructed in [4] via its Born series, which is truncated after sufficiently many terms. Hence, we infer from Lemma 2.3 that

$$\lim_{N\to\infty} |E_{N,\beta} - \mathcal{C} - E_0| = \lim_{N\to\infty} |E - E_0| = 0 \,. \tag{2.50}$$

Moreover, [4, Theorem 1.1] together with the reasoning in Section 2.2 implies that the spectral gap of $\mathbb{G}$ above $E_{N,\beta}$ is of order one. By (2.41), the same holds true for the spectral gap of $\mathbb{G}_0$ above E_0. Hence, there exists a constant $c = \mathcal{O}(1)$ such that the closed contour

$$\gamma := \{E_0 + c\,\mathrm{e}^{\mathrm{i}t} \;:\; t \in [0, 2\pi)\} \subset \mathbb{C} \tag{2.51}$$

encloses both E and E_0 but contains no other point of the spectra of $\mathbb{G}$ and $\mathbb{G}_0$. Consequently, the projectors $\mathbb{P}$ and $\mathbb{P}_0$ can be expressed as

$$\mathbb{P} = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{1}{z - \mathbb{G}}\,, \qquad \mathbb{P}_0 = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{1}{z - \mathbb{G}_0}\,. \tag{2.52}$$

Now we follow the strategy of [10, Lemma 3.13, Proposition 3.14 and Theorem 2] to expand $\mathbb{P}$ around $\mathbb{P}_0$. The proof of this proposition is given in Section 7.

Proposition 2.4. *Let $\mathbb{A}$ be an operator on $\mathcal{F}_\perp$ such that $\|\mathbb{A}\psi\| \lesssim \|\mathcal{N}_\perp\psi\|$ for $\psi \in \mathcal{F}_\perp$. Then*

$$\left| \mathrm{Tr}\mathbb{A}\mathbb{P} - \sum_{\ell=0}^{2} \mathrm{Tr}\mathbb{A}\mathbb{P}_\ell \right| \lesssim N^{\frac{3}{2}(\beta-1)}\,, \tag{2.53}$$

where

$$\mathbb{P}_1 := \mathbb{P}_0\mathbb{G}_1\frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} + \mathrm{h.c.}\,, \tag{2.54a}$$

$$\begin{aligned}\mathbb{P}_2 := &\left(\mathbb{P}_0\mathbb{G}_2\frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} + \mathbb{P}_0\mathbb{G}_1\frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0}\mathbb{G}_1\frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} + \mathrm{h.c.}\right)\\ &- \left\langle \boldsymbol{\chi}_0, \mathbb{G}_1, \frac{\mathbb{Q}_0}{(E_0 - \mathbb{G}_0)^2}\mathbb{G}_1\boldsymbol{\chi}_0 \right\rangle \mathbb{P}_0 + \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0}\mathbb{G}_1\mathbb{P}_0\mathbb{G}_1\frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0}\,.\end{aligned} \tag{2.54b}$$

Moreover, we find that

$$|E - E_0 - E_{\mathrm{pert}}| \lesssim N^{\frac{3}{2}(\beta-1)}\,, \tag{2.55}$$

where

$$E_{\mathrm{pert}} := \langle \boldsymbol{\chi}_0, \mathbb{G}_2\boldsymbol{\chi}_0 \rangle + \left\langle \boldsymbol{\chi}_0, \mathbb{G}_1\frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0 \right\rangle\,. \tag{2.56}$$

Important ingredients for this proof are estimates of the operators $\mathbb{G}_1$ and $\mathbb{G}_2$ and of the remainders, which are given in Section 5. Moreover, it is crucial that

$$\left\langle \boldsymbol{\chi}_0, (\mathbb{K}_0 + 1)(\mathcal{N}_\perp + 1)^\ell \boldsymbol{\chi}_0 \right\rangle \lesssim 1\,, \qquad \left\langle \boldsymbol{\chi}, (\mathbb{K}_0 + 1)(\mathcal{N}_\perp + 1)^\ell \boldsymbol{\chi} \right\rangle \lesssim 1\,, \tag{2.57}$$

which we prove in Lemma 3.6 and Lemma 6.2, respectively.

2.7 Expansion of the ground state

Proposition 2.4 implies that

$$|E_{N,\beta} - \mathcal{C} - E_0 - E_{\mathrm{pert}}| \lesssim N^{\frac{3}{2}(\beta-1)} \tag{2.58}$$

by definition (2.38) of E. In Lemma 2.3, we have shown that $\mathcal{C} + E_0 = 4\pi(N-1)\mathfrak{a}_{N,\beta} + E_{0,0} + E_{0,1}$ up to small error terms. Finally, in Section 8, we compute $E_{0,1} + E_{\mathrm{pert}}$ explicitly to obtain the expression (1.9) for E_{corr}. This concludes the proof of Theorem 1.

Another consequence of Proposition 2.4 is an expansion of the ground state wave function and of its one-body reduced density matrix. Since (2.53) holds in particular for any bounded operator $\mathbb{A} \in \mathcal{L}(\mathcal{F}_\perp)$, we conclude analogously to [10, Corollary 3.4] that

$$\operatorname{Tr}\left|\mathbb{P} - \sum_{\ell=0}^{2} \mathbb{P}_\ell\right| \lesssim N^{\frac{3}{2}(\beta-1)}\,. \tag{2.59}$$

Since $\mathbb{P}$ is a rank-one projector, [10, Theorem 4] implies that

$$\|\Psi_N - \Psi_{N,0} - \Psi_{N,1} - \Psi_{N,2}\|_{L^2(\Lambda^N)} \lesssim N^{\frac{3}{2}(\beta-1)}\,, \tag{2.60}$$

where

$$\Psi_{N,\ell} = U_N^*\left(\mathbb{1}^{\leq N}\mathbb{T}^*\boldsymbol{\chi}_\ell\right) \tag{2.61}$$

for $\boldsymbol{\chi}_0$ from (2.38) and with

$$\boldsymbol{\chi}_1 = \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\,, \tag{2.62}$$

$$\boldsymbol{\chi}_2 = \left(\frac{\mathbb{Q}_0}{(E_0 - \mathbb{G}_0)^2}\mathbb{G}_2 + \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0}\mathbb{G}_1\frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0}\mathbb{G}_1 - \frac{1}{2}\left\langle \mathbb{G}_1\frac{\mathbb{Q}_0}{(E_0 - \mathbb{G}_0)^2}\mathbb{G}_1\right\rangle\right)\boldsymbol{\chi}_0\,, \tag{2.63}$$

where we used the shorthand notation

$$\langle\mathbb{B}\rangle := \langle\boldsymbol{\chi}_0, \mathbb{B}\boldsymbol{\chi}_0\rangle$$

for operators $\mathbb{B}$ on $\mathcal{F}_\perp$. Finally, following the proof of [7, Corollary 1.1], we obtain the estimate for the reduced density matrix, which is proven in Section 9.

3 Bogoliubov transformations

In this section we collect and prove useful properties of the quadratic transformation $\mathbb{T}$, which regularizes the excitation Hamiltonian $\mathbb{H}$, and of the transformation $\mathbb{U}_\tau$, which diagonalizes the Bogoliubov Hamiltonian $\mathbb{G}_0$.

3.1 Properties of $\widehat{v}_N^\beta$

As a preparation, we provide two useful estimates for the interaction potential.

Lemma 3.1. *Recall that $\widehat{v}_N^\beta = \widehat{v}(\cdot/N^\beta)$. Then*

$$\sum_{p\in\Lambda_+^*} \frac{\left(\widehat{v}_N^\beta(p)\right)^2}{p^2} \lesssim N^\beta \tag{3.1}$$

and

$$\sup_{q\in\Lambda_+^*}\left\{\sum_{r\in\Lambda_+^*, r\neq -q} \frac{\widehat{v}_N^\beta(r)}{(q+r)^2}\right\} \lesssim N^\beta. \tag{3.2}$$

Proof. Concerning the first bound, we find

$$\sum_{p\in\Lambda_+^*}\left(\widehat{v}_N^\beta(p)\right)^2/p^2 \lesssim \left(\|\widehat{v}_N^\beta\|_{\ell^\infty}^2\||\,|p|^{-2}\chi_{|p|\le N^\beta}\|_{\ell^1}+\|\widehat{v}_N^\beta\|_{\ell^\infty}\|\widehat{v}_N^\beta\|_{\ell^2}\||p|^{-2}\chi_{|p|>N^\beta}\|_{\ell^2}\right)\lesssim N^\beta \tag{3.3}$$

because $\|\widehat{v}_N^\beta\|_{\ell^\infty}\lesssim 1$ and $\|\widehat{v}_N^\beta\|_{\ell^2}\lesssim N^{3\beta/2}$. For the second estimate, we proceed similarly and find

$$\sum_{r\in\Lambda_+^*,r\neq -q}\frac{\left|\widehat{v}_N^\beta(r)\right|}{(q+r)^2}\lesssim \|\widehat{v}\|_{\ell^\infty}\|\chi_{|p|\le N^\beta}|p|^{-2}\|_{\ell^1}+\|\widehat{v}_N^\beta\|_{\ell^2}\|\chi_{|p|\ge N^\beta}|p|^{-2}\|_{\ell^2}\,, \tag{3.4}$$

which concludes the proof. □

3.2 Quadratic transformation $\mathbb{T}$

We recall from (2.25) that the Bogoliubov transformation $\mathbb{T}$ is given by

$$\mathbb{T}=\exp\left\{\frac{1}{2}\sum_{p\in\Lambda_+^*}\eta_p(a_p^\dagger a_{-p}^\dagger-a_pa_{-p})\right\},$$

for η_p as in (1.6). In Lemma 3.2 below we summarize some useful properties of the sequence η_p. In fact this Lemma is a modification of [18, Lemma 14], where the Gross-Pitaevski regime is considered (i.e. $\beta=1$). The arguments presented there easily translate to our setting $\beta\in(0,1)$.

Lemma 3.2. *It holds that $\eta_p\in\mathbb{R}$, $\eta_p=\eta_{-p}$ for all $p\in\Lambda_+^*$ and*

$$|\eta_p|\lesssim\frac{1}{p^2},\quad \|\eta\|_{\ell^\infty},\ \|\eta\|_{\ell^2}\lesssim 1\quad \textit{and}\quad \|p\eta\|_{\ell^2}\lesssim N^{\beta/2}. \tag{3.5}$$

Proof. We follow the lines of the proof of [18, Lemma 14]. We multiply (2.23) with η_p and find, summing over $p\in\Lambda_+^*$ and using $v\ge 0$,

$$2\|p\eta\|_{\ell^2}^2=-\sum_{p\in\Lambda_+^*}\widehat{v}_N^\beta(p)\eta_p-\frac{1}{N}\sum_{p,q\in\Lambda_+^*}\widehat{v}_N^\beta(p-q)\eta_p\eta_q\le-\sum_{p\in\Lambda_+^*}\widehat{v}_N^\beta(p)\eta_p\le\|p\eta\|_{\ell^2}\|\widehat{v}_N^\beta/p\|_{\ell^2}\,. \tag{3.6}$$

We conclude that

$$\|p\eta\|_{\ell^2}^2\lesssim\sum_{p\in\Lambda_+^*}\left(v_N^\beta(p)\right)^2/p^2\lesssim N^\beta \tag{3.7}$$

by Lemma 3.1. Using once more (2.23) we get the pointwise estimate

$$|p^2\eta_p|\le|\widehat{v}_N^\beta(p)|+\frac{1}{2N}\left(\sum_{q\in\Lambda_+^*}\frac{|\widehat{v}_N^\beta(p-q)|^2}{q^2}\right)^{1/2}\|q\eta\|_{\ell^2}\lesssim 1 \tag{3.8}$$

proceeding similarly with Lemma 3.1. Consequently,

$$\|\eta\|_{\ell^\infty}\le\|\eta\|_{\ell^2}\lesssim 1\,. \tag{3.9}$$

□

Lemma 3.2 immediately implies the following bounds on $c_p = \cosh(\eta_p)$ and $s_p = \sinh(\eta_p)$:

Lemma 3.3. *For $p \in \Lambda_+^*$, we have the pointwise estimates*

$$|c_p| \lesssim 1\,, \qquad |c_p - 1| \lesssim \frac{1}{|p|^4} \leq 1\,, \tag{3.10}$$

$$|s_p| \lesssim \frac{1}{|p|^2} \lesssim 1\,, \qquad |c_p s_p - \eta_p| \lesssim \frac{1}{|p|^6} \lesssim 1\,. \tag{3.11}$$

Finally, we prove that the number operator conjugated with $\mathbb{T}$ can be estimated in terms of the number operator.

Lemma 3.4. *For any $k \in \mathbb{N}_0$, $\ell \in \mathbb{R}$ and $\psi, xi \in \mathcal{F}_\perp$, it holds that*

$$\left|\left\langle \psi, \mathbb{T}(\mathcal{N}_\perp + 1)^k \mathbb{T}^* \xi \right\rangle\right| \leq C(k) \|(\mathcal{N}_\perp + 1)^{\frac{k}{2}+\ell}\psi\| \|(\mathcal{N}_\perp + 1)^{\frac{k}{2}-\ell}\xi\|, \tag{3.12}$$

$$\|(\mathcal{N}_\perp + 1)^{-k}\,\psi\| \leq C(k) \|(\mathcal{N}_\perp + 1)^{-k}\,\mathbb{T}^*\psi\|. \tag{3.13}$$

Proof. By Lemma 3.3, we compute for $k = 1$

$$\begin{aligned}
&|\langle \psi, \mathbb{T}(\mathcal{N}_\perp + 1)\mathbb{T}^*\xi\rangle| \\
&\quad \leq \sum_{p \in \Lambda_+^*} (|c_p|^2 + |s_p|^2) \left|\left\langle (\mathcal{N}_\perp + 1)^\ell\,\psi, a_p^\dagger a_p\,(\mathcal{N}_\perp + 1)^{-\ell}\,\xi \right\rangle\right| \\
&\qquad + \sum_{p \in \Lambda_+^*} |c_p||s_p| \left|\left\langle \psi, \big(\,(\mathcal{N}_\perp + 1)^\ell\, a_p^\dagger a_{-p}^\dagger\,(\mathcal{N}_\perp + 3)^{-\ell} + (\mathcal{N}_\perp + 3)^\ell\, a_p a_{-p}\,(\mathcal{N}_\perp + 1)^{-\ell}\,\big)\xi \right\rangle\right| \\
&\qquad + \left(\sum_{p \in \Lambda_+^*} |s_p|^2 + 1\right) \left|\left\langle (\mathcal{N}_\perp + 1)^\ell\,\psi, (\mathcal{N}_\perp + 1)^{-\ell}\,\xi \right\rangle\right| \\
&\quad \lesssim \|(\mathcal{N}_\perp + 1)^{1/2+\ell}\psi\| \|(\mathcal{N}_\perp + 1)^{1/2-\ell}\xi\|\,.
\end{aligned} \tag{3.14}$$

The case $k > 1$ follows from this by induction, similarly to [9, Lemma 4.4]. In order to prove (3.13), note that (3.12) with $\ell = \frac{k}{2}$ implies

$$\begin{aligned}
\|(\mathcal{N}_\perp + 1)^{-k}\,\mathbb{T}\,(\mathcal{N}_\perp + 1)^k\,\mathbb{T}^*\psi\|^2 &\leq \left\langle (\mathcal{N}_\perp + 1)^{-2k}\,\mathbb{T}\,(\mathcal{N}_\perp + 1)^k\,\mathbb{T}^*\psi, \mathbb{T}\,(\mathcal{N}_\perp + 1)^k\,\mathbb{T}^*\psi \right\rangle \\
&\lesssim \|(\mathcal{N}_\perp + 1)^{-k}\,\mathbb{T}\,(\mathcal{N}_\perp + 1)^k\,\mathbb{T}^*\psi\| \|\psi\|,
\end{aligned} \tag{3.15}$$

showing

$$\|(\mathcal{N}_\perp + 1)^{-k}\,\mathbb{T}\,(\mathcal{N}_\perp + 1)^k\,\mathbb{T}^*\|_{\mathrm{op}} \lesssim 1. \tag{3.16}$$

Hence,

$$\begin{aligned}
\|(\mathcal{N}_\perp + 1)^{-k}\,\psi\| &= \|(\mathcal{N}_\perp + 1)^{-k}\,\mathbb{T}\,(\mathcal{N}_\perp + 1)^k\,\mathbb{T}^*\mathbb{T}\,(\mathcal{N}_\perp + 1)^{-k}\,\mathbb{T}^*\psi\| \\
&\lesssim \|\mathbb{T}\,(\mathcal{N}_\perp + 1)^{-k}\,\mathbb{T}^*\psi\| \\
&= \|(\mathcal{N}_\perp + 1)^{-k}\,\mathbb{T}^*\psi\|.
\end{aligned} \tag{3.17}$$

□

3.3 Quadratic transformation $\mathbb{U}_\tau$

Recall from (2.31) that the Bogoliubov Hamiltonian is given by

$$\mathbb{G}_0 = \sum_{p\in\Lambda_+^*} F_p a_p^\dagger a_p + \frac{1}{2}\sum_{p\in\Lambda_+^*} G_p(a_p^\dagger a_{-p}^\dagger + a_p a_{-p})\,,$$

where the coefficients F_p and G_p are defined in (2.34). From similar arguments as given in [4, Lemma 5.1], it follows that the operators F and G satisfy the following properties:

Lemma 3.5. *For all $p\in\Lambda_+^*$ and N large enough,*

$$p^2/2 \le F_p \lesssim (1+p^2), \quad |G_p| \lesssim p^{-2}, \quad |G_p|/F_p \lesssim p^{-4}, \quad |G_p|/F_p \le 1/2\,. \tag{3.18}$$

Proof. The lemma can be proven analogously to [4, Lemma 5.1]. Note that in [4] the sequence η_p was chosen differently. However, as η_p defined in (2.23) satifies analogous estimates as the sequence from [4], the proof of [4, Lemma 5.1] applies here, too. We will briefly sketch the proof. First, note that

$$\Big|\frac{1}{N}\sum_{q\in\Lambda_+^*}\widehat{v}_N^\beta(p-q)s_q c_q\Big| \le \frac{C}{N}\sum_{q\in\Lambda_+^*}\frac{|\widehat{v}_N^\beta(p-q)|}{q^2} \le CN^{\beta-1} \tag{3.19}$$

by Lemmas 3.1 and 3.2. Since $\widehat{v}\ge 0$ we arrive by definition (2.34) of F_p (note that $c_p^2+s_p^2\ge 1$) at

$$F_p \ge p^2 - CN^{\beta-1} \ge \frac{1}{2}p^2 \tag{3.20}$$

for N large enough. The upper bound follows since $\widehat{v}_N^\beta, s_p, c_p$ are bounded in $\ell^\infty(\Lambda_+^*)$ uniformly in N, thus

$$F_p \le C(1+p^2)\,. \tag{3.21}$$

To prove the second bound on G_p, we write

$$G_p = 2p^2\eta_p + \widehat{v}_N^\beta(p) + \frac{1}{N}\sum_{q\in\Lambda_+^*}\widehat{v}_N^\beta(p-q)\eta_q + \widetilde{G}_p \tag{3.22}$$

with $|\widetilde{G}_p|\le Cp^{-2}$ following from $|s_pc_p-\eta_p|\lesssim p^{-6}$ by Lemma 3.2 and similarly $|(s_p+c_p)^2-1|\lesssim p^{-2}$, $|s_p^2|\lesssim p^{-4}, |c_p^2-1|\lesssim p^{-4}$. The remaining three terms of the r.h.s. of (3.22) vanish by (2.23) and thus we arrive at (3.18). □

The Bogoliubov Hamiltonian is diagonalized by the quadratic transformation

$$\mathbb{U}_\tau = \exp\left\{\frac{1}{2}\sum_{p\in\Lambda_+^*}\tau_p(a_p^\dagger a_{-p}^\dagger - a_p a_{-p})\right\}$$

defined in (2.40), where τ_p is given by $\tanh(2\tau_p) = -G_p/F_p$ for all $p\in\Lambda_+^*$ by (2.39). Note that $\tau_p=\tau_{-p}$. Equivalently, we can write

$$\tau_p = \frac{1}{4}\ln\frac{1-G_p/F_p}{1+G_p/F_p} \quad \text{for all} \quad p\in\Lambda_+^*\,, \tag{3.23}$$

hence Lemma 3.5 yields the estimate

$$|\tau_p| \lesssim |G_p|/F_p \lesssim p^{-4} \quad \text{for all} \quad p \in \Lambda_+^* \, . \tag{3.24}$$

We now use this estimate to show that $\mathbb{U}_\tau$ approximately preserves the number of particles and the kinetic energy:

Lemma 3.6. *For $\ell \in \mathbb{N}_0$ it holds that*

$$\mathbb{U}_\tau (\mathcal{N}_\perp + 1)^\ell \mathbb{U}_\tau^* \le C(\ell)(\mathcal{N}_\perp + 1)^\ell \, , \qquad \mathbb{U}_\tau^* (\mathcal{N}_\perp + 1)^\ell \mathbb{U}_\tau \le C(\ell)(\mathcal{N}_\perp + 1)^\ell \tag{3.25}$$

as well as

$$\begin{aligned} \mathbb{U}_\tau^* (\mathcal{N}_\perp + 1)^\ell (\mathbb{K}_0 + 1) \mathbb{U}_\tau &\le C(\ell)(\mathcal{N}_\perp + 1)^\ell (\mathbb{K}_0 + 1) \, , \\ \mathbb{U}_\tau (\mathcal{N}_\perp + 1)^\ell (\mathbb{K}_0 + 1) \mathbb{U}_\tau^* &\le C(\ell)(\mathcal{N}_\perp + 1)^\ell (\mathbb{K}_0 + 1). \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \mathbb{U}_\tau^* (\mathcal{N}_\perp + 1)^\ell \frac{1}{\mathbb{K}_0 + 1} \mathbb{U}_\tau &\le C(\ell)(\mathcal{N}_\perp + 1)^\ell \frac{1}{\mathbb{K}_0 + 1} \, , \\ \mathbb{U}_\tau (\mathcal{N}_\perp + 1)^\ell \frac{1}{\mathbb{K}_0 + 1} \mathbb{U}_\tau^* &\le C(\ell)(\mathcal{N}_\perp + 1)^\ell \frac{1}{\mathbb{K}_0 + 1} \, . \end{aligned} \tag{3.27}$$

Proof. **Proof of (3.25).** We compute

$$\mathbb{U}_\tau \mathcal{N}_\perp \mathbb{U}_\tau^* = \sum_{p \in \Lambda_+^*} (\sinh(\tau_p)^2 + \cosh(\tau_p)^2) a_p^\dagger a_p + \sum_{p \in \Lambda_+^*} \sinh(\tau_p)^2 \tag{3.28}$$

$$+ \sum_{p \in \Lambda_+^*} \cosh(\tau_p) \sinh(\tau_p) (a_p^\dagger a_{-p}^\dagger + a_p a_{-p}) \, . \tag{3.29}$$

By (3.24), it follows that $|\cosh(\tau_p)| \lesssim 1$ and $|\sinh(\tau_p)| \lesssim |p|^{-4}$. Consequently (3.25) follows analogously to the proof of Lemma 3.4.

Proof of (3.26). Let us define the symmetric operator

$$\mathbb{A} = \frac{\mathrm{i}}{2} \sum_{p \in \Lambda_+^*} \tau_p (a_p^\dagger a_{-p}^\dagger - a_p a_{-p}) \, , \qquad \mathbb{U}_\tau(\theta) = \mathrm{e}^{-\mathrm{i}\mathbb{A}\theta}$$

such that $\mathbb{U}_\tau(0) = \mathbb{1}$ and $\mathbb{U}_\tau(1) = \mathbb{U}_\tau$. Next, we compute

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \left\langle \psi, \mathbb{U}_\tau^*(\theta) (\mathcal{N}_\perp + 1)^\ell (\mathbb{K}_0 + 1) \mathbb{U}_\tau(\theta) \psi \right\rangle$$

$$= \mathrm{i} \left\langle \psi, \mathbb{U}_\tau^*(\theta) (\mathcal{N}_\perp + 1)^\ell \, [\mathbb{A}, \mathbb{K}_0] \, \mathbb{U}_\tau(\theta) \psi \right\rangle \tag{3.30a}$$

$$+ \mathrm{i} \left\langle \psi, \mathbb{U}_\tau^*(\theta) \left[\mathbb{A}, (\mathcal{N}_\perp + 1)^\ell \right] (\mathbb{K}_0 + 1) \mathbb{U}_\tau(\theta) \psi \right\rangle . \tag{3.30b}$$

Note that the second term is zero if $\ell = 0$ and that

$$[\mathbb{A}, \mathbb{K}_0] = \mathrm{i} \sum_{p \in \Lambda_+^*} p^2 \tau_p \left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p} \right) \, . \tag{3.31}$$

Together with $|\tau_p| \lesssim p^{-4}$ (see (3.24)), the shifting property of the number operator and the Cauchy–Schwarz inequality, this leads to

$$
\begin{aligned}
|(3.30\text{a})| \leq & \left| \sum_{p\in\Lambda_+^*} p^2 \tau_p \left\langle \psi, \mathbb{U}_\tau^*(\theta)(\mathcal{N}_\perp+1)^{\frac{\ell}{2}} a_p^\dagger a_{-p}^\dagger (\mathcal{N}_\perp+3)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi \right\rangle \right| \\
& + \left| \sum_{p\in\Lambda_+^*} p^2 \tau_p \left\langle \psi, \mathbb{U}_\tau^*(\theta)(\mathcal{N}_\perp+1)^{\frac{\ell}{2}} a_p a_{-p} (\mathcal{N}_\perp-1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi \right\rangle \right| \\
\leq & \sum_{p\in\Lambda_+^*} p^2 |\tau_p| \, \|a_p(\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi\| \|a_{-p}^\dagger(\mathcal{N}_\perp+3)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi\| \\
& + \sum_{p\in\Lambda_+^*} p^2 |\tau_p| \, \|a_p^\dagger(\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi\| \|a_{-p}(\mathcal{N}_\perp-1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi\| \\
\lesssim & \left(\sum_{p\in\Lambda_+^*} p^2 \|a_p(\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi\|^2 \right)^{1/2} \left(\sum_{p\in\Lambda_+^*} p^2 |\tau_p|^2 \, \|a_{-p}^\dagger(\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi\|^2 \right)^{1/2} \\
\lesssim & \, \|(\mathbb{K}_0+1)^{\frac{1}{2}} (\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi\|^2 \, .
\end{aligned}
\tag{3.32}
$$

Using the shifting property of the number operator again, we get

$$
\begin{aligned}
\left[\mathbb{A}, (\mathcal{N}_\perp+1)^\ell\right] = -\frac{\mathrm{i}}{2} \sum_{p\in\Lambda_+^*} \tau_p \Big(& \left((\mathcal{N}_\perp+1)^\ell - (\mathcal{N}_\perp-1)^\ell \right) (\mathcal{N}_\perp-1)^{-\frac{\ell}{2}} \, a_p^\dagger a_{-p}^\dagger \, (\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \\
& + \left((\mathcal{N}_\perp+3)^\ell - (\mathcal{N}_\perp+1)^\ell \right) (\mathcal{N}_\perp+3)^{-\frac{\ell}{2}} \, a_p a_{-p} \, (\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \Big) .
\end{aligned}
\tag{3.33}
$$

Hence,

$$
\begin{aligned}
& |(3.30\text{b})| \\
& \leq \left| \left\langle \psi, \mathbb{U}_\tau^*(\theta) \left[\mathbb{A}, (\mathcal{N}_\perp+1)^\ell\right] \mathbb{U}_\tau(\theta)\psi \right\rangle \right| \\
& \quad + \left| \left\langle \psi, \mathbb{U}_\tau^*(\theta) \left[\mathbb{A}, (\mathcal{N}_\perp+1)^\ell\right] (\mathcal{N}_\perp+1)^{-\frac{\ell}{2}} \, \mathbb{K}_0 \, (\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \, \mathbb{U}_\tau(\theta)\psi \right\rangle \right| \\
& \lesssim \|(\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \, \mathbb{U}_\tau(\theta)\psi\| \sum_{p\in\Lambda_+^*} |\tau_p| \, \|a_p a_{-p} \left((\mathcal{N}_\perp+1)^\ell - (\mathcal{N}_\perp-1)^\ell \right) (\mathcal{N}_\perp-1)^{-\frac{\ell}{2}} \, \mathbb{U}_\tau(\theta)\psi\| \\
& \quad + \|(\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \, \mathbb{U}_\tau(\theta)\psi\| \sum_{p\in\Lambda_+^*} |\tau_p| \, \|a_p^\dagger a_{-p}^\dagger \left((\mathcal{N}_\perp+3)^\ell - (\mathcal{N}_\perp+1)^\ell \right) (\mathcal{N}_\perp+3)^{-\frac{\ell}{2}} \, \mathbb{U}_\tau(\theta)\psi\| \\
& \quad + \sum_{k\in\Lambda_+^*} k^2 \|a_k \, (\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \, \mathbb{U}_\tau(\theta)\psi\| \\
& \qquad \times \sum_{p\in\Lambda_+^*} |\tau_p| \, \|a_k a_p a_{-p} \Big(\left((\mathcal{N}_\perp+1)^\ell - (\mathcal{N}_\perp-1)^\ell \right) (\mathcal{N}_\perp-1)^{-\frac{\ell}{2}} \, \mathbb{U}_\tau(\theta)\psi\| \\
& \quad + \sum_{k\in\Lambda_+^*} k^2 \|a_k \, (\mathcal{N}_\perp+1)^{\frac{\ell}{2}} \, \mathbb{U}_\tau(\theta)\psi\| \\
& \qquad \times \sum_{p\in\Lambda_+^*} |\tau_p| \, \|a_k a_p^\dagger a_{-p}^\dagger \Big(\left((\mathcal{N}_\perp+3)^\ell - (\mathcal{N}_\perp+1)^\ell \right) (\mathcal{N}_\perp+3)^{-\frac{\ell}{2}} \, \mathbb{U}_\tau(\theta)\psi\| \, .
\end{aligned}
\tag{3.34}
$$

Using $(\mathcal{N}_\perp + 3)^\ell - (\mathcal{N}_\perp + 1)^\ell \lesssim \ell\,(\mathcal{N}_\perp + 3)^{\ell-1}$ and $|\tau_p| \lesssim p^{-4}$, it is easily seen that the first two lines can be bounded by $\|(\mathcal{N}_\perp + 1)^{\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\|^2$. Estimating the remaining terms then by the Cauchy–Schwarz inequality and Young's inequality for products leads to

$$
\begin{aligned}
(3.30\text{b}) \lesssim\ & \|(\mathbb{K}_0 + 1)^{1/2}\,(\mathcal{N}_\perp + 1)^{\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\|^2 \\
&+ \sum_{k\in\Lambda_+^*} k^2 \left| \sum_{p\in\Lambda_+^*} |\tau_p|\, \|a_k a_p a_{-p}\Big(\Big((\mathcal{N}_\perp + 1)^\ell - (\mathcal{N}_\perp - 1)^\ell\Big)\,(\mathcal{N}_\perp - 1)^{-\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\| \right|^2 \qquad (3.35)\\
&+ \sum_{k\in\Lambda_+^*} k^2 \left| \sum_{p\in\Lambda_+^*} |\tau_p|\, \|a_k a_p^\dagger a_{-p}^\dagger\Big(\Big((\mathcal{N}_\perp + 3)^\ell - (\mathcal{N}_\perp + 1)^\ell\Big)\,(\mathcal{N}_\perp + 3)^{-\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\| \right|^2 . \qquad (3.36)
\end{aligned}
$$

Similarly to above, we obtain

$$
(3.35) \lesssim \sum_{k\in\Lambda_+^*} k^2 \|a_k\,(\mathcal{N}_\perp + 1)^{\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\|^2 = \|\mathbb{K}_0^{\frac{1}{2}}\,(\mathcal{N}_\perp + 1)^{\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\|^2, \qquad (3.37)
$$

and, using that $\left[a_k, a_p^\dagger a_{-p}^\dagger\right] = (\delta_{k,p} + \delta_{k,-p})a_k^\dagger$, we estimate

$$
\begin{aligned}
(3.36) \lesssim\ & \sum_{k\in\Lambda_+^*} k^2 \left| \sum_{p\in\Lambda_+^*} |\tau_p|\, \|a_p^\dagger a_{-p}^\dagger a_k\Big(\Big((\mathcal{N}_\perp + 3)^\ell - (\mathcal{N}_\perp + 1)^\ell\Big)\,(\mathcal{N}_\perp + 3)^{-\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\| \right|^2 \\
&+ \sum_{k\in\Lambda_+^*} k^2\, |\tau_k|^2\, \|a_k^\dagger\Big(\Big((\mathcal{N}_\perp + 3)^\ell - (\mathcal{N}_\perp + 1)^\ell\Big)\,(\mathcal{N}_\perp + 3)^{-\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\|^2 \\
\lesssim\ & \|\mathbb{K}_0^{\frac{1}{2}}\,(\mathcal{N}_\perp + 1)^{\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\|^2. \qquad (3.38)
\end{aligned}
$$

Hence,

$$
(3.30\text{b}) \lesssim \|(\mathbb{K}_0 + 1)^{1/2}\,(\mathcal{N}_\perp + 1)^{\frac{\ell}{2}}\, \mathbb{U}_\tau(\theta)\psi\|^2 . \qquad (3.39)
$$

Collecting the estimates, we find that

$$
\frac{\mathrm{d}}{\mathrm{d}\theta} \left\langle \psi, \mathbb{U}_\tau^*(\theta)(\mathcal{N}_\perp + 1)^\ell(\mathbb{K}_0 + 1)\mathbb{U}_\tau(\theta)\psi \right\rangle \le C(\ell) \left\langle \psi, \mathbb{U}_\tau^*(\theta)(\mathcal{N}_\perp + 1)^\ell(\mathbb{K}_0 + 1)\mathbb{U}_\tau(\theta)\psi \right\rangle , \qquad (3.40)
$$

hence Gronwall's lemma lead to

$$
\left\langle \psi, \mathbb{U}_\tau^*(1)(\mathcal{N}_\perp + 1)^\ell(\mathbb{K}_0 + 1)\mathbb{U}_\tau(1)\psi \right\rangle \le e^{C(\ell)} \left\langle \psi, \mathbb{U}_\tau^*(0)(\mathcal{N}_\perp + 1)^\ell(\mathbb{K}_0 + 1)\mathbb{U}_\tau(0)\psi \right\rangle . \qquad (3.41)
$$

Since $\mathbb{U}_\tau^*(1) = \mathbb{U}_\tau^*$ and $\mathbb{U}_\tau^*(0) = 1$ this shows the first inequality of (3.26). The second one follows analogously.

Proof of (3.27). We prove this statement via a similar Gronwall argument. Analogously to above, we compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\theta}\left\langle \psi, \mathbb{U}_\tau^*(\theta)(\mathcal{N}_\perp+1)^\ell \frac{1}{\mathbb{K}_0+1}\mathbb{U}_\tau(\theta)\psi\right\rangle \\
&\quad = \mathrm{i}\left\langle \psi, \mathbb{U}_\tau^*(\theta)(\mathcal{N}_\perp+1)^\ell \left[\mathbb{A}, \frac{1}{\mathbb{K}_0+1}\right]\mathbb{U}_\tau(\theta)\psi\right\rangle \qquad (3.42a)\\
&\qquad + \mathrm{i}\left\langle \psi, \mathbb{U}_\tau^*(\theta)\left[\mathbb{A}, (\mathcal{N}_\perp+1)^\ell\right]\frac{1}{\mathbb{K}_0+1}\mathbb{U}_\tau(\theta)\psi\right\rangle . \qquad (3.42b)
\end{aligned}$$

For (3.42a), we observe that

$$\left[\mathbb{A}, \frac{1}{\mathbb{K}_0+1}\right] = \mathrm{i}\sum_{p\in\Lambda_+^*}\frac{1}{\mathbb{K}_0+1}(a_p^\dagger a_{-p}^\dagger + a_p a_{-p})\frac{1}{\mathbb{K}_0+1}\,. \qquad (3.43)$$

Analogously to the estimate of (3.30a), this yields

$$|(3.42\mathrm{a})| \lesssim \left\langle \psi, \mathbb{U}_\tau^*(\theta)(\mathcal{N}_\perp+1)^\ell \frac{1}{\mathbb{K}_0+1}\mathbb{U}_\tau(\theta)\psi\right\rangle \qquad (3.44)$$

because, for example,

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*} p^2|\tau_p|\|a_p(\mathcal{N}_\perp+1)^\ell\frac{1}{\mathbb{K}_0+1}\mathbb{U}_\tau(\theta)\psi\|\|a_{-p}^\dagger(\mathcal{N}_\perp+3)^{\frac{\ell}{2}}\frac{1}{\mathbb{K}_0+1}\mathbb{U}_\tau(\theta)\psi\| \\
&\lesssim \left(\sum_{p\in\Lambda_+^*} p^2\|a_p(\mathcal{N}_\perp+1)^{\frac{\ell}{2}}\frac{1}{\mathbb{K}_0+1}\mathbb{U}_\tau(\theta)\psi\|^2\right)^{\frac12}\left(\sum_{p\in\Lambda_+^*}p^2|\tau_p|^2\right)^{\frac12}\|(\mathcal{N}_\perp+3)^{\frac{\ell+1}{2}}\frac{1}{\mathbb{K}_0+1}\mathbb{U}_\tau(\theta)\psi\| \\
&\lesssim \|(\mathcal{N}_\perp+1)^{\frac{\ell}{2}}\left(\frac{1}{\mathbb{K}_0+1}\right)^{\frac12}\mathbb{U}_\tau(\theta)\psi\|^2\,. \qquad (3.45)
\end{aligned}$$

For the second contribution (3.42b), recall that $(\mathbb{K}_0+1)^{-\frac12} = \pi^{-1}\int_0^\infty s^{-\frac12}(\mathbb{K}_0+s+1)^{-1}\,\mathrm{d}s$, which yields

$$\left[\left(\frac{1}{\mathbb{K}_0+1}\right)^{\frac12}, a_p a_{-p}\right] = -\frac{2p^2}{\pi}\int_0^\infty \frac{1}{\sqrt{s}}\frac{1}{\mathbb{K}_0+s+1}a_p a_{-p}\frac{1}{\mathbb{K}_0+s+1}\,\mathrm{d}s\,. \qquad (3.46)$$

Consequently, abbreviating

$$F_1(\mathcal{N}_\perp) := ((\mathcal{N}_\perp+1)^\ell - (\mathcal{N}_\perp-1)^\ell)(\mathcal{N}_\perp+1)^{-\frac{\ell}{2}}\,,\quad F_3(\mathcal{N}_\perp) := ((\mathcal{N}_\perp+3)^\ell - (\mathcal{N}_\perp+1)^\ell)(\mathcal{N}_\perp+3)^{-\frac{\ell}{2}}\,,$$

we find

$$\begin{aligned}
|(3.42\mathrm{b})| \lesssim &\sum_{p\in\Lambda_+^*}|\tau_p|\left\|a_p a_{-p}\left(\frac{1}{\mathbb{K}_0+1}\right)^{\frac12}F_1(\mathcal{N}_\perp)\mathbb{U}_\tau(\theta)\psi\right\| \\
&\qquad\times\left\|\left(\frac{1}{\mathbb{K}_0+1}\right)^{\frac12}(\mathcal{N}_\perp+1)^{\frac{\ell}{2}}\mathbb{U}_\tau(\theta)\psi\right\| \qquad (3.47a)\\
&+\sum_{p\in\Lambda_+^*}|\tau_p|\left\|\left(\frac{1}{\mathbb{K}_0+1}\right)^{\frac12}(\mathcal{N}_\perp+1)F_3(\mathcal{N}_\perp)\mathbb{U}_\tau(\theta)\psi\right\|
\end{aligned}$$

$$\times \left\| (\mathcal{N}_\perp + 1)^{-1} a_p a_{-p} \left(\frac{1}{\mathbb{K}_0 + 1}\right)^{\frac{1}{2}} (\mathcal{N}_\perp + 1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi \right\| \tag{3.47b}$$

$$+ \int_0^\infty \mathrm{d}s \frac{1}{\sqrt{s}} \sum_{p \in \Lambda_+^*} |\tau_p| p^2 \left\| a_p a_{-p} \frac{1}{\mathbb{K}_0 + s + 1} F_1(\mathcal{N}_\perp) \mathbb{U}_\tau(\theta)\psi \right\|$$

$$\times \left\| \frac{1}{\mathbb{K}_0 + s + 1} \left(\frac{1}{\mathbb{K}_0 + 1}\right)^{\frac{1}{2}} (\mathcal{N}_\perp + 1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi \right\| \tag{3.47c}$$

$$+ \int_0^\infty \mathrm{d}s \frac{1}{\sqrt{s}} \sum_{p \in \Lambda_+^*} |\tau_p| p^2 \left\| \frac{1}{\mathbb{K}_0 + s + 1} F_3(\mathcal{N}_\perp) \mathbb{U}_\tau(\theta)\psi \right\|$$

$$\times \left\| a_p a_{-p} \frac{1}{\mathbb{K}_0 + s + 1} \left(\frac{1}{\mathbb{K}_0 + 1}\right)^{\frac{1}{2}} (\mathcal{N}_\perp + 1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi \right\|. \tag{3.47d}$$

It is easily seen that

$$(3.47a) + (3.47b) \lesssim \left\| \left(\frac{1}{\mathbb{K}_0 + 1}\right)^{\frac{1}{2}} (\mathcal{N}_\perp + 1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi \right\|^2. \tag{3.48}$$

For the third term, we use the estimates $\mathbb{K}_0 + s + 1 \geq \max\{\mathbb{K}_0 + 1, s + 1\}$ to compute

$$\begin{aligned}
(3.47c) \lesssim & \int_0^\infty \mathrm{d}s \frac{1}{\sqrt{s}(s+1)} \left(\sum_{p \in \Lambda_+^*} \tau_p^2 p^2 \right)^{\frac{1}{2}} \left(\sum_{p \in \Lambda_+^*} p^2 \left\| a_p \frac{1}{\mathbb{K}_0 + s + 1} \mathcal{N}_\perp^{\frac{1}{2}} F_1(\mathcal{N}_\perp) \mathbb{U}_\tau(\theta)\psi \right\|^2 \right)^{\frac{1}{2}} \\
& \times \left\| \left(\frac{1}{\mathbb{K}_0 + 1}\right)^{\frac{1}{2}} (\mathcal{N}_\perp + 1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi \right\| \\
\lesssim & \int_0^\infty \mathrm{d}s \frac{1}{\sqrt{s}(s+1)} \left\| \mathbb{K}_0^{\frac{1}{2}} \frac{1}{\mathbb{K}_0 + s + 1} \mathcal{N}_\perp^{\frac{1}{2}} F_1(\mathcal{N}_\perp) \mathbb{U}_\tau(\theta)\psi \right\| \left\| \left(\frac{1}{\mathbb{K}_0 + 1}\right)^{\frac{1}{2}} (\mathcal{N}_\perp + 1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi \right\| \\
\lesssim & \left\| \left(\frac{1}{\mathbb{K}_0 + 1}\right)^{\frac{1}{2}} (\mathcal{N}_\perp + 1)^{\frac{\ell}{2}} \mathbb{U}_\tau(\theta)\psi \right\|^2,
\end{aligned} \tag{3.49}$$

and the estimate of (3.47d) works similarly. This concludes the Gronwall argument. □

4 Diagonalization of $\mathbb{G}_0$

In this section we prove Lemma 2.3.

Proof. For this proof we use ideas from [4, Proof of Lemma 5.3], where an expansion up to $o(1)$ is derived. Here we generalize ideas from [4] and provide an expansion up to higher order that is $o(N^{\beta-1})$. First we observe from (2.34) and (2.41) and the properties of the hyperbolic functions that

$$\begin{aligned}
\mathcal{C} - \frac{1}{2} \sum_{p \in \Lambda_+^*} F_p = & \frac{N-1}{2} \widehat{v}(0) - \frac{1}{2} \sum_{p \in \Lambda_+^*} \left(p^2 + \widehat{v}_N^\beta(p) + \frac{1}{N} (v_N^\beta * sc)_p c_p s_p \right) \\
& - \frac{1}{N} \sum_{p \in \Lambda_+^*} v_N^\beta(p) \left(\frac{s_p c_p}{2} + s_p^3 c_p \right)
\end{aligned} \tag{4.1}$$

for $(v_N^\beta * sc)_p$ as in (2.35). Since $|s_p^3 c_p| \lesssim |p|^{-6}$ and $|c_p s_p - \eta_p| \lesssim |p|^{-6}$ by Lemma 3.3, we get

$$\begin{aligned}\mathcal{C} - \frac{1}{2}\sum_{p\in\Lambda_+^*} F_p =& \frac{N-1}{2}\widehat{v}(0) - \frac{1}{2}\sum_{p\in\Lambda_+^*}\Big(p^2 + \widehat{v}_N^\beta(p) + \frac{1}{N}(v_N^\beta * sc)_p c_p s_p\Big) \\ & - \frac{1}{2N}\sum_{p\in\Lambda_+^*} v_N^\beta(p)\eta_p + \mathcal{O}(N^{-1}) \; . \end{aligned} \tag{4.2}$$

By definition of the scattering length on the box in (1.5), we can write the term in the last line as

$$\frac{1}{N}\sum_{p\in\Lambda_+^*} v_N^\beta(p)\eta_p = 8\pi\mathfrak{a}_N^\beta - \widehat{v}(0) \tag{4.3}$$

so that

$$\mathcal{C} - \frac{1}{2}\sum_{p\in\Lambda_+^*} F_p = \frac{N}{2}\widehat{v}(0) - 4\pi\mathfrak{a}_N^\beta - \frac{1}{2}\sum_{p\in\Lambda_+^*}\Big(p^2 + \widehat{v}_N^\beta(p) + \frac{1}{N}(v_N^\beta * sc)_p c_p s_p\Big) + \mathcal{O}(N^{-1}) \; . \tag{4.4}$$

Furthermore,

$$F_p^2 - G_p^2 = |p|^4 + 2p^2\widehat{v}_N^\beta(p) + A_p \tag{4.5}$$

with

$$A_p = -\frac{1}{N}\Big[2\widehat{v}_N^\beta(p)(v_N^\beta * sc)_p + \frac{1}{N}(v_N^\beta * sc)_p^2\Big]. \tag{4.6}$$

Note that $\|(v_N^\beta * \eta)\|_{\ell^\infty} \lesssim N^\beta$ by Lemma 3.3 and thus

$$|A_p| \lesssim N^{\beta-1} \; . \tag{4.7}$$

Note that $\widehat{v}(p) \geq 0$ for all $p \in \Lambda_+^*$. Since A_p is of lower order we obtain that $|p|^4 + 2p^2\widehat{v}_N^\beta(p) + A_p \geq 0$ and $|p|^4 + 2p^2\widehat{v}_N^\beta(p) \geq 0$ for N large enough. Therefore, recalling that we are interested in the energy expansion for $o\big(N^{(\beta-1)}\big)$, we expand the square root of $F_p^2 - G_p^2$, i.e., the r.h.s. of (4.5), using the identity $\sqrt{a+b} = \sqrt{b} + \frac{a}{2\sqrt{b}} - \frac{a^2}{2\sqrt{b}(\sqrt{a+b}+\sqrt{b})^2}$, we find

$$\begin{aligned}\sqrt{F_p^2 - G_p^2} =& \sqrt{|p|^4 + 2p^2\widehat{v}_N^\beta(p)} + \frac{A_p}{2\sqrt{|p|^4 + 2p^2\widehat{v}_N^\beta(p)}} \\ & - \frac{A_p^2}{2\Big(\sqrt{|p|^4 + 2p^2\widehat{v}_N^\beta(p)} + \sqrt{|p|^4 + 2p^2\widehat{v}_N^\beta(p) + A_p}\Big)^2\sqrt{|p|^4 + 2p^2\widehat{v}_N^\beta(p)}} \; . \end{aligned} \tag{4.8}$$

We will show that the last term is $\mathcal{O}(N^{2(\beta-1)})$. To this end we observe that

$$Cp^2 \leq \sqrt{|p|^4 + 2p^2\widehat{v}_N^\beta(p)} + \sqrt{|p|^4 + 2p^2\widehat{v}_N^\beta(p) + A_p} \leq Cp^2\Big(1 + C\Big(\frac{A_p}{|p|^4} + \frac{\widehat{v}_N^\beta(p)}{p^2}\Big)\Big) \tag{4.9}$$

for sufficiently large N and thus

$$\frac{A_p^2}{|p|^6}\Big(1 - C\Big(\frac{A_p}{|p|^4} + \frac{\widehat{v}_N^\beta(p)}{p^2}\Big)\Big)$$

$$\leq \frac{A_p^2}{\Big(\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)+A_p}\Big)^2\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}} \leq \frac{A_p^2}{|p|^6}\,. \tag{4.10}$$

With (4.7) we find

$$\sum_{p\in\Lambda_+^*}\frac{A_p^2}{|p|^6} \leq CN^{2(\beta-1)}\,, \tag{4.11}$$

which leads with (4.8) to

$$\sqrt{F_p^2-G_p^2} = \sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)} + \frac{A_p}{2\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}} + \mathcal{O}(N^{2(\beta-1)})\,. \tag{4.12}$$

Next, we further expand the second term of the r.h.s. of (4.12) and write

$$\frac{1}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}} = \frac{1}{p^2} - \frac{2\widehat{v}_N^\beta(p)}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big)}\,, \tag{4.13}$$

which leads with the definition

$$B_p := \frac{A_p\widehat{v}_N^\beta(p)}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big)} \tag{4.14}$$

to

$$\sqrt{F_p^2-G_p^2} = \sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)} + \frac{A_p}{2p^2} - B_p + \mathcal{O}(N^{2(\beta-1)})\,. \tag{4.15}$$

We use this expansion together with (4.4) to derive an expansion

$$\begin{aligned}
&\mathcal{C} + \frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-F_p+\sqrt{F_p^2-G_p^2}\big] \\
&= \frac{N\widehat{v}(0)}{2} - 4\pi\mathfrak{a}_N^\beta + \frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-p^2-v_N^\beta(p)+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\big] \\
&\quad + \sum_{p\in\Lambda_+^*}\Big[\frac{A_p}{4p^2}-\frac{1}{2N}(\widehat{v}_N^\beta * sc)_p c_p s_p\Big] - \frac{1}{2}\sum_{p\in\Lambda_+^*}B_p + \mathcal{O}(N^{2(\beta-1)}).
\end{aligned} \tag{4.16}$$

For the two terms of the first line we proceed as in [4] and replace $\widehat{v}_N^\beta(p)$ by $\widehat{v}(0)$ paying a price that is $\mathcal{O}(N^{-\beta})$. We add and substract $\sum_{p\in\Lambda_+^*}(v_N^\beta(p))^2/p^2$ and find since

$$\begin{aligned}
&\Big|\sum_{p\in\Lambda_+^*}\big[-p^2-\widehat{v}_N^\beta(p)+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}+\frac{(\widehat{v}_N^\beta(p))^2}{2p^2}\big] \\
&\qquad - \sum_{p\in\Lambda_+^*}\big[-p^2-\widehat{v}(0)+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(0)}+\frac{\widehat{v}(0)^2}{2p^2}\Big| \leq CN^{-\alpha}\,.
\end{aligned} \tag{4.17}$$

for every $\alpha < \beta$ from [4, p. 2362 (before (5.35)] that

$$\begin{aligned}
&\mathcal{C} + \frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-F_p + \sqrt{F_p^2 - G_p^2}\big] \\
=&\frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-p^2 - \widehat{v}(0) + \sqrt{|p|^4 + 2p^2\widehat{v}_N^\beta(0)} + \frac{\widehat{v}(0)^2}{2p^2}\big] - 4\pi\mathfrak{a}_N^\beta \\
&+ \frac{N\widehat{v}(0)}{2} - \sum_{p\in\Lambda_+^*}\frac{(\widehat{v}_N^\beta(p))^2}{4p^2} + \sum_{p\in\Lambda_+^*}\Big[\frac{A_p}{4p^2} - \frac{1}{2N}(\widehat{v}_N^\beta * sc)_p s_p c_p\Big] \\
&- \frac{1}{2}\sum_{p\in\Lambda_+^*} B_p + \mathcal{O}(N^{2(\beta-1)}) + \mathcal{O}(N^{-\alpha})\,. \qquad (4.18)
\end{aligned}$$

Using (2.23) and (4.6), we compute

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*}\Big[\frac{A_p}{4p^2} - \frac{1}{2N}(\widehat{v}_N^\beta * sc)_p s_p c_p\Big] \\
&= -\frac{1}{2N}\sum_{p\in\Lambda_+^*}(\widehat{v}_N^\beta * \eta)_p\frac{\widehat{v}_N^\beta(p)}{2p^2} - \frac{1}{2N}\sum_{p\in\Lambda_+^*}(\widehat{v}_N^\beta * (sc-\eta))_p\frac{\widehat{v}_N^\beta(p)}{2p^2} \\
&\quad - \frac{1}{2N}\sum_{p\in\Lambda_+^*}\frac{(\widehat{v}_N^\beta * sc)_p}{p^2}\Big[\frac{1}{2N}(\widehat{v}_N^\beta * (sc-\eta))_p + p^2(s_pc_p - \eta_p) - \frac{1}{2}\widehat{v}(0)\eta_p\Big]. \qquad (4.19)
\end{aligned}$$

Since $|s_pc_p| \lesssim |p|^{-2}$ from Lemma 3.2 we have $\|\widehat{v}_N^\beta * sc\|_{\ell^\infty} \lesssim N^\beta$ from Lemma 3.1 and thus

$$\Big|\frac{1}{4N^2}\sum_{p\in\Lambda_+^*}\frac{(\widehat{v}_N^\beta * sc)_p}{p^2}(\widehat{v}_N^\beta * (sc-\eta))_p\Big| \lesssim N^{\beta-2}\sum_{p,q\in\Lambda_+^*, p\neq q}\frac{\widehat{v}_N^\beta(p)}{p^2|p-q|^6} \lesssim N^{2(\beta-1)}\,, \qquad (4.20)$$

where we used once more Lemma 3.1. Moreover,

$$\Big|\frac{\widehat{v}(0)}{4N^2}\sum_{p\in\Lambda_+^*}(\widehat{v}_N^\beta * sc)_p\frac{\eta_p}{p^2}\Big| \lesssim N^{-2}\|\widehat{v}_N^\beta * sc\|_{\ell^\infty}\sum_{p\in\Lambda_+^*}\frac{1}{p^4} \lesssim N^{\beta-2}\,. \qquad (4.21)$$

Therefore we get

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*}\Big[\frac{A_p}{4p^2} - \frac{1}{2N}(\widehat{v}_N^\beta * sc)_p s_p c_p\Big] \\
&\quad = -\frac{1}{2N}\sum_{p\in\Lambda_+^*}(\widehat{v}_N^\beta * \eta)_p\frac{\widehat{v}_N^\beta(p)}{2p^2} - \frac{1}{2N}\sum_{p,q\in\Lambda_+^*, p\neq q}\widehat{v}_N^\beta(p-q)(s_pc_p - \eta_p)\Big[s_qc_q + \frac{\widehat{v}_N^\beta(q)}{q^2}\Big] \\
&+ \mathcal{O}(N^{2(\beta-1)}). \qquad (4.22)
\end{aligned}$$

We insert this into (4.18) and get

$$\mathcal{C} + \frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-F_p + \sqrt{F_p^2 - G_p^2}\big]$$

$$
\begin{aligned}
=&\frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-p^2-\widehat{v}(0)+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(0)}+\frac{\widehat{v}(0)^2}{2p^2}\big]-4\pi\mathfrak{a}_N^\beta\\
&+\frac{N\widehat{v}(0)}{2}-\sum_{p\in\Lambda_+^*}\frac{(\widehat{v}_N^\beta(p))^2}{4p^2}-\frac{1}{2N}\sum_{p\in\Lambda_+^*}(\widehat{v}_N^\beta*\eta)_p\frac{\widehat{v}_N^\beta(p)}{2p^2}\\
&-\frac{1}{2N}\sum_{p,q\in\Lambda_+^*,p\neq q}\widehat{v}_N^\beta(p-q)(s_pc_p-\eta_p)\left[s_qc_q+\frac{\widehat{v}_N^\beta(q)}{q^2}\right]\\
&-\frac{1}{2}\sum_{p\in\Lambda_+^*}B_p+\mathcal{O}(N^{2(\beta-1)})+\mathcal{O}(N^{-\alpha})\;. \qquad (4.23)
\end{aligned}
$$

The second line of the r.h.s. sums by (2.23) up to

$$
4\pi N\mathfrak{a}_{N,\beta}+\frac{\widehat{v}(0)}{4N}\sum_{p\in\Lambda_+^*}\frac{\widehat{v}_N^\beta(p)\eta_p}{p^2}\,, \qquad (4.24)
$$

where the second summand is at most of order N^{-1} by Lemma 3.3. Hence,

$$
\begin{aligned}
&\mathcal{C}+\frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-F_p+\sqrt{F_p^2-G_p^2}\big]\\
=&4\pi(N-1)\mathfrak{a}_N^\beta+\frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-p^2-\widehat{v}(0)+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(0)}+\frac{\widehat{v}(0)^2}{2p^2}\big]\\
&-\frac{1}{2N}\sum_{p,q\in\Lambda_+^*,p\neq q}\widehat{v}_N^\beta(p-q)(s_pc_p-\eta_p)\left[s_qc_q+\frac{\widehat{v}_N^\beta(q)}{q^2}\right]\\
&-\frac{1}{2}\sum_{p\in\Lambda_+^*}B_p+\mathcal{O}(N^{2(\beta-1)})+\mathcal{O}(N^{-\alpha})\;. \qquad (4.25)
\end{aligned}
$$

Recalling the definition of A_p in (4.6), the estimates $\|\widehat{v}_N^\beta*sc\|_{\ell^\infty}\lesssim N^\beta$, $\|\widehat{v}_N^\beta*(sc-\eta)\|_{\ell^\infty}\lesssim 1$ and the definition of B_p in (4.14), we can extract the leading order contribution of $-\frac{1}{2}\sum_{p\in\Lambda_+^*}B_p$ and find

$$
\begin{aligned}
&\mathcal{C}+\frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-F_p+\sqrt{F_p^2-G_p^2}\big]\\
&\quad=4\pi(N-1)\mathfrak{a}_N^\beta+E_{0,0}+E_{0,1}\mathcal{O}(N^{2(\beta-1)})+\mathcal{O}(N^{-1})+\mathcal{O}(N^{-\alpha})\,, \qquad (4.26)
\end{aligned}
$$

for every $\alpha<\beta$, where

$$
E_{0,0}=\frac{1}{2}\sum_{p\in\Lambda_+^*}\big[-p^2-\widehat{v}(0)+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(0)}+\frac{\widehat{v}(0)^2}{2p^2}\big] \qquad (4.27)
$$

as defined in (1.8) and

$$
\begin{aligned}
E_{0,1}:=&-\frac{1}{2N}\sum_{p,q\in\Lambda_+^*,p\neq q}\widehat{v}_N^\beta(p-q)(s_pc_p-\eta_p)\left[s_qc_q+\frac{\widehat{v}_N^\beta(q)}{q^2}\right]\\
&+\frac{1}{N}\sum_{p,q\in\Lambda_+^*,p\neq q}\frac{\widehat{v}_N^\beta(p)^2\;\;\widehat{v}_N^\beta(p-q)s_qc_q}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\left(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\right)}\;. \qquad (4.28)
\end{aligned}
$$

Proof of (2.45)**.** Recalling the definition of $E_{0,0}$, we find by expanding the square root that $\big| -p^2 - \widehat{v}(0) + \sqrt{|p|^4 + 2p^2 \widehat{v}_N^\beta(0)} + \frac{\widehat{v}(0)^2}{2p^2} \big| \lesssim |p|^{-4}$, yielding the desired estimate

$$|E_{0,0}| \lesssim 1 \,. \tag{4.29}$$

Furthermore, we conclude from the definition of $E_{0,1}$ with the estimates $|s_p c_p - \eta_p| \lesssim |p|^{-6}$, $\sum_{q\in\Lambda_+^*} \frac{\widehat{v}_N^\beta(q)}{q^2} \lesssim N^\beta$ and $\|\widehat{v}_N^\beta * sc\|_{\ell^\infty} \lesssim N^\beta$ that $|E_{0,1}| \lesssim N^{\beta-1}$. □

5 Analysis of $\mathbb{G}$

In this section we study properties of

$$\mathbb{G} = \mathbb{T}\mathbb{H}\mathbb{T}^* - \mathcal{C} \tag{5.1}$$

defined in (2.29), which will in the end yield Proposition 2.2. The idea is to compute $\mathbb{G}$ using the explicit action of $\mathbb{T}$ on creation and annihilation operators (2.26). For this we decompose $\mathbb{H}$ from (2.17) as

$$\mathbb{H} = \mathbb{H}_\beta + \widetilde{\mathbb{R}}_{\surd} \,, \tag{5.2}$$

where we defined

$$\mathbb{H}_\beta := \frac{1}{2}(N-1)\widehat{v}(0) + \mathbb{K}_0 + \mathbb{K}_1 + \mathbb{K}_2 + \mathbb{K}_2^* + \frac{1}{\sqrt{N}}\left(\mathbb{K}_3 + \mathbb{K}_3^*\right) + \frac{1}{N}\mathbb{K}_4 \tag{5.3}$$

with $\mathbb{K}_j$ given by (2.15) and

$$\begin{aligned}\widetilde{\mathbb{R}}_{\surd} := &\left(\left[\frac{N-\mathcal{N}_\perp}{N}\right]_+ - 1\right)\mathbb{K}_1 \\ &+ \left(\mathbb{K}_2 \frac{\sqrt{[(N-\mathcal{N}_\perp)(N-\mathcal{N}_\perp - 1)]_+} - N}{N} + \text{h.c.}\right) \\ &+ \left(\mathbb{K}_3 \frac{\sqrt{[N-\mathcal{N}_\perp]_+} - \sqrt{N}}{N} + \text{h.c.}\right) \,.\end{aligned} \tag{5.4}$$

We prove Proposition 2.2 by computing the action of the Bogoliubov transformation $\mathbb{T}$ on the individual contributions of $\mathbb{H}$ in (5.2) and decompose the result w.r.t. to the order of the terms in N. The first observation is that the conjugation of $\widetilde{\mathbb{R}}_{\surd}$ with $\mathbb{T}$ contributes only to sub-leading order. To be more precise, we define

$$\mathbb{R}_c := \mathbb{T}\widetilde{\mathbb{R}}_{\surd}\mathbb{T}^* + \frac{1}{N}\sum_{p\in\Lambda_+^*} \widehat{v}_N^\beta(p) c_p s_p \mathbb{T}(\mathcal{N}_\perp + \frac{1}{2})\mathbb{T}^* \,. \tag{5.5}$$

In Section 5.5 (Lemma 5.5), we prove that the remainder $\mathbb{R}_c$ is of order $N^{3(\beta-1)/2}$. Next, we observe that the explicit action of the Bogoliubov transformation on creation and annihilation operators (2.26) yields

$$\mathbb{T}\left(\mathbb{K}_0 + \mathbb{K}_1 + \mathbb{K}_2 + \mathbb{K}_2^*\right)\mathbb{T}^* = \mathbb{G}_0 + \sum_{p\in\Lambda^*}\left(p^2 s_p^2 + \widehat{v}_N^\beta (s_p^2 + c_p s_p)\right)$$

$$
\begin{aligned}
&- \frac{2}{N} \sum_{p\in\Lambda_+^*} (\widehat{v}_N^\beta * sc)_p c_p s_p a_p^\dagger a_p \\
&- \frac{1}{2N} \sum_{p\in\Lambda_+^*} (\widehat{v}_N^\beta * cs)_p (c_p^2 + s_p^2)(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}) \qquad (5.6)
\end{aligned}
$$

for $\mathbb{G}_0$ as defined in (2.31). The last two terms of the r.h.s. will be compensated for by $N^{-1}\mathbb{T}\mathbb{K}_4\mathbb{T}^*$. In fact, a lengthy computation shows that

$$
\begin{aligned}
&\mathbb{T}\left(\mathbb{K}_0 + \mathbb{K}_1 + \mathbb{K}_2 + \mathbb{K}_2^*\right)\mathbb{T}^* + \frac{1}{N}\mathbb{T}\mathbb{K}_4\mathbb{T}^* \\
=&\mathbb{G}_0 + \mathbb{G}_2 + \mathbb{R}_a + \mathbb{R}_b \\
&+ \frac{1}{2N}\sum_{p\in\Lambda_+^*} (\widehat{v}_N^\beta * cs)_p c_p s_p + \sum_{p\in\Lambda^*} \left(p^2 s_p^2 + \widehat{v}_N^\beta (s_p^2 + c_p s_p)\right) - \frac{1}{N}\sum_{p\in\Lambda_+^*} \widehat{v}_N^\beta(p) c_p s_p \Big(\frac{1}{2} + s_p^2\Big) \\
&+ \frac{1}{N}\sum_{p\in\Lambda_+^*} \widehat{v}_N^\beta(p) c_p s_p \mathbb{T}(\mathcal{N}_\perp + \frac{1}{2})\mathbb{T}^* \\
=&\mathbb{G}_0 + \mathbb{G}_2 + \mathbb{R}_a + \mathbb{R}_b + \frac{1}{N}\sum_{p\in\Lambda_+^*} \widehat{v}_N^\beta(p) c_p s_p \mathbb{T}(\mathcal{N}_\perp + \frac{1}{2})\mathbb{T}^* + \mathcal{C} - \frac{N-1}{2}\widehat{v}(0)\,, \qquad (5.7)
\end{aligned}
$$

where $\mathbb{R}_a$ and $\mathbb{R}_b$ are given in (5.29) and (5.33) below. In Section 5.3 (Lemma 5.3), we prove that $\mathbb{R}_a$ is of order $N^{-1+\beta/2}$. In Section 5.4 (Lemma 5.4), we show that $\mathbb{R}_b$ is of order N^{-1}. Recalling the definition of $\mathbb{G}_1$ in (2.32) and combining this with (5.3) and (5.5), we arrive at

$$
\mathbb{G} = \mathbb{G}_0 + \mathbb{G}_1 + \mathbb{G}_2 + \mathbb{R}_a + \mathbb{R}_b + \mathbb{R}_c =: \mathbb{G}_0 + \mathbb{G}_1 + \mathbb{G}_2 + \mathbb{R}_2\,, \qquad (5.8)
$$

where we set

$$
\mathbb{R}_2 := \mathbb{R}_a + \mathbb{R}_b + \mathbb{R}_c\,. \qquad (5.9)
$$

Then Proposition 2.2 follows by Lemmas 5.3 – 5.5 proven in Sections 5.3 – 5.5 below.

5.1 Analysis of $\mathbb{G}_1$

With (2.26) we find

$$
\mathbb{G}_1 := \widetilde{\mathbb{G}}_1 + \mathbb{R}_d\,, \qquad (5.10)
$$

where the leading order contributions of $\mathbb{G}_1$, which we will show to be of order $N^{(\beta-1)/2}$, is given by

$$
\begin{aligned}
\widetilde{\mathbb{G}}_1 :=&\frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \widehat{v}_N^\beta(p) c_{p+q} c_p c_q (a_{p+q}^\dagger a_{-p}^\dagger a_q + \text{h.c.}) \\
&+ \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \widehat{v}_N^\beta(p) c_{p+q} c_p s_q (a_{p+q}^\dagger a_{-p}^\dagger a_{-q}^\dagger + \text{h.c.}) \\
=&: \left(A_{d_1} + A_{d_2} + \text{h.c.}\right)\ . \qquad (5.11)
\end{aligned}
$$

The remaining terms are of order $N^{-1/2}$ and given by

$$\begin{aligned}
\mathbb{R}_d :=& \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \widehat{v}_N^\beta(p) c_{p+q} s_p c_q (a_{p+q}^\dagger a_p a_q + \text{h.c.}) \\
&+ \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \widehat{v}_N^\beta(p) s_{p+q} c_p c_q (a_{-(p+q)} a_{-p}^\dagger a_q + \text{h.c.}) \\
&+ \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \widehat{v}_N^\beta(p) s_{p+q} s_p c_q (a_{-(p+q)} a_p a_q + \text{h.c.}) \\
&+ \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \widehat{v}_N^\beta(p) c_{p+q} s_p s_q (a_{p+q}^\dagger a_p a_{-q}^\dagger + \text{h.c.}) \\
&+ \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \widehat{v}_N^\beta(p) s_{p+q} c_p s_q (a_{-(p+q)} a_{-p}^\dagger a_{-q}^\dagger + \text{h.c.}) \\
&+ \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \widehat{v}_N^\beta(p) s_{p+q} s_p s_q (a_{-(p+q)} a_p a_{-q}^\dagger + \text{h.c.}) \\
=:& \,(A_{d_3} + A_{d_4} + A_{d_5} + A_{d_6} + A_{d_7} + A_{d_8} + \text{h.c.})\;. \qquad (5.12)
\end{aligned}$$

Lemma 5.1. *For $\psi, \xi \in \mathcal{F}_\perp$ and $\ell_1, \ell_2 \in \mathbb{R}$, we have*

$$\begin{aligned}
\left|\left\langle \psi, \widetilde{\mathbb{G}}_1 \xi \right\rangle\right| \lesssim & N^{\frac{\beta-1}{2}} \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{\ell_1} \psi\| \|(\mathcal{N}_\perp+1)^{-\ell_1+1} \xi\| \\
&+ N^{\frac{\beta-1}{2}} \|(\mathcal{N}_\perp+1)^{-\ell_2+1} \psi\| \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{\ell_2} \xi\| \qquad (5.13)
\end{aligned}$$

and, moreover,

$$|\langle \psi, \mathbb{R}_d \xi\rangle| \lesssim N^{-\frac{1}{2}} \|(\mathcal{N}_\perp+1)^{\ell_1} \psi\| \|(\mathcal{N}_\perp+1)^{-\ell_1+\frac{3}{2}} \xi\|\;. \qquad (5.14)$$

Proof. We start with the first bound for $\widetilde{\mathbb{G}}_1$ defined in (5.10) and consider both its contributions separately.

Term $\widetilde{\mathbb{G}}_1$. Since $\|c\|_{\ell^\infty} \lesssim 1$ by Lemma 3.3,

$$\begin{aligned}
&|\langle \psi, A_{d_1} \xi\rangle| \\
&\lesssim N^{-1/2} \sum_{p,q\in\Lambda_+^*} \widehat{v}_N^\beta(p) \left|\left\langle \psi, (\mathcal{N}_\perp+1)^\ell a_{p+q}^\dagger a_{-p}^\dagger a_q (\mathcal{N}_\perp+2)^{-\ell} \xi \right\rangle\right| \\
&\lesssim N^{-1/2} \left(\sum_{p,q\in\Lambda_+^*} |p|^2 \|a_{p+q} a_{-p} (\mathcal{N}_\perp+1)^\ell \psi\|^2 \right)^{\frac{1}{2}} \||p|^{-1} \widehat{v}_N^\beta\|_{\ell^2(\Lambda_+^*)} \left(\sum_{q\in\Lambda_+^*} \|a_q (\mathcal{N}_\perp+3)^{-\ell} \xi\|^2 \right)^{\frac{1}{2}}. \qquad (5.15)
\end{aligned}$$

With Lemma 3.1 we arrive at the desired bound

$$|\langle \psi, A_{d_1} \xi\rangle| \lesssim N^{\frac{\beta-1}{2}} \|\mathbb{K}_0 (\mathcal{N}_\perp+1)^{\ell+\frac{1}{2}} \psi\| \|(\mathcal{N}_\perp+1)^{-\ell+\frac{1}{2}} \xi\|. \qquad (5.16)$$

Using similar ideas we continue with

$$\begin{aligned}&|\langle\psi, A_{d_2}\xi\rangle|\\ &\quad\lesssim N^{-1/2}\|\mathbb{K}_0^{1/2}\,(\mathcal{N}_\perp+1)^{\ell+\frac{1}{2}}\,\psi\|\,\||p|^{-1}\,\widehat{v}_N^{\beta}\|_{\ell^2}\Big(\sum_{q\in\Lambda_+^*}|s_q|^2\,\|a^\dagger_{-q}\,(\mathcal{N}_\perp+4)^{-\ell}\,\xi\|^2\Big)^{1/2}\end{aligned}\tag{5.17}$$

Using once more Lemmas 3.1 and 3.3, we obtain

$$|\langle\psi, A_{d_2}\xi\rangle|\lesssim N^{\frac{\beta-1}{2}}\|\mathbb{K}_0^{1/2}\,(\mathcal{N}_\perp+1)^{\ell+\frac{1}{2}}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+\frac{1}{2}}\,\xi\|.$$

The hermitian conjugates can be estimated simiarly.

Term $\mathbb{R}_d$. Next we prove the bound on $\mathbb{R}_d$. For this we consider all its contributions separately. We start with

$$\begin{aligned}&|\langle\psi, A_{d_3}\xi\rangle|\\ &\quad\lesssim N^{-1/2}\sum_{p,q\in\Lambda_+^*}\widehat{v}_N^{\beta}\,|s_p|\left|\left\langle\psi,(\mathcal{N}_\perp+2)^{\ell}\,a^\dagger_{p+q}a_pa_q\,(\mathcal{N}_\perp+1)^{-\ell}\,\xi\right\rangle\right|\\ &\quad\lesssim N^{-1/2}\|\widehat{v}\|_{\ell^\infty}\left(\sum_{p,q\in\Lambda_+^*}|s_p|^2\,\|a_{p+q}\,(\mathcal{N}_\perp+2)^{\ell}\,\psi\|^2\right)^{1/2}\left(\sum_{p,q\in\Lambda_+^*}\|a_pa_q\,(\mathcal{N}_\perp+1)^{-\ell}\,\xi\|^2\right)^{1/2}\\ &\quad\lesssim N^{-1/2}\|(\mathcal{N}_\perp+1)^{\ell+\frac{1}{2}}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+1}\,\xi\|\,,\end{aligned}\tag{5.18}$$

by Lemma 3.3. In order to estimate the next term, note that

$$\begin{aligned}A_{d_4}&=N^{-1/2}\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}\widehat{v}_N^{\beta}(p)s_{p+q}c_pc_qa_{-(p+q)}a^\dagger_{-p}a_q\\ &=N^{-1/2}\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}\widehat{v}_N^{\beta}(p)s_{p+q}c_pc_qa^\dagger_{-p}a_{-(p+q)}a_q\end{aligned}\tag{5.19}$$

because $q\in\Lambda_+^*$. This allows us to estimate with Lemma 3.3

$$\begin{aligned}&|\langle\psi, A_{d_4}\xi\rangle|\\ &\quad\lesssim N^{-1/2}\|\widehat{v}\|_{\ell^\infty}\Bigg(\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}|s_{p+q}|^2\,\|a_{-p}\,(\mathcal{N}_\perp+2)^{\ell}\,\psi\|^2\Bigg)^{1/2}\Bigg(\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}\|a_{-(p+q)}a_q\,(\mathcal{N}_\perp+1)^{-\ell}\,\xi\|^2\Bigg)^{1/2}\\ &\quad\lesssim N^{-1/2}\|(\mathcal{N}_\perp+1)^{\ell+\frac{1}{2}}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+1}\,\xi\|.\end{aligned}\tag{5.20}$$

For the next term, we find with similar arguments

$$\begin{aligned}&|\langle\psi, A_{d_5}\xi\rangle|\\ &\quad\lesssim N^{-1/2}\|\widehat{v}\|_{\ell^\infty}\Bigg(\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}|s_{p+q}|^2\,|s_p|^2\,\|a^\dagger_{-(p+q)}\,(\mathcal{N}_\perp+4)^{\ell}\,\psi\|^2\Bigg)^{1/2}\Bigg(\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}\|a_pa_q\,(\mathcal{N}_\perp+1)^{-\ell}\,\xi\|^2\Bigg)^{1/2}\\ &\quad\lesssim N^{-1/2}\|(\mathcal{N}_\perp+1)^{\ell+\frac{1}{2}}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+1}\,\xi\|\end{aligned}\tag{5.21}$$

and, moreover, by Lemma 3.3

$$
\begin{aligned}
&|\langle \psi, A_{d_6}\xi\rangle| \\
&\quad\lesssim N^{-1/2}\|\widehat{v}\|_{\ell^\infty(\Lambda_+^*)}\Bigg(\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}|s_p|^2\,\|a_{p+q}\,(\mathcal{N}_\perp+1)^\ell\,\psi\|^2\Bigg)^{1/2}\Bigg(\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}|s_q|^2\,\|a_p a^\dagger_{-q}\,(\mathcal{N}_\perp+2)^{-\ell}\,\xi\|^2\Bigg)^{1/2} \\
&\quad\lesssim N^{-1/2}\|(\mathcal{N}_\perp+1)^{\ell+\frac{1}{2}}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+1}\,\xi\|. && (5.22)
\end{aligned}
$$

For the next term, we put the creation and annihilation operators in normal order using that $\delta_{-(p+q),-p}=0$ for all $q\neq 0$ and $\delta_{-(p+q),-q}=0$ for all $p\neq 0$ so that

$$
A_{d_7} = N^{-1/2}\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}\widehat{v}_N^\beta(p)s_{p+q}c_p s_q a^\dagger_{-p}a^\dagger_{-q}a_{-(p+q)}. \tag{5.23}
$$

Then we can proceed analogously as for A_{d_3} and thus get

$$
|\langle \psi, A_{d_7}\xi\rangle| \lesssim N^{-1/2}\|(\mathcal{N}_\perp+1)^{\ell+1}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+\frac{1}{2}}\,\xi\|. \tag{5.24}
$$

For the last term, we find, using that $[a_{-(p+q)},a^\dagger_{-q}]=0$,

$$
\begin{aligned}
&|\langle \psi, A_{d_8}\xi\rangle| \\
&\quad\lesssim N^{-1/2}\|\widehat{v}\|_{\ell^\infty}\|s\|_{\ell^\infty}\Bigg(\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}|s_p|^2\,\|a_{-q}a^\dagger_p\,(\mathcal{N}_\perp+2)^\ell\,\psi\|^2\Bigg)^{1/2} \\
&\quad\times\Bigg(\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}}|s_q|^2\,\|a_{-(p+q)}\,(\mathcal{N}_\perp+1)^{-\ell}\,\xi\|^2\Bigg)^{1/2} \\
&\quad\lesssim N^{-1/2}\|(\mathcal{N}_\perp+1)^{\ell+1}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+\frac{1}{2}}\,\xi\|. && (5.25)
\end{aligned}
$$

The hermitian conjugates can be estimated similarly. □

5.2 Analysis of $\mathbb{G}_2$

Here we show that the operator $\mathbb{G}_2$, given by

$$
\mathbb{G}_2 = \frac{1}{2N}\sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}}\widehat{v}_N^\beta(r)c_{p+r}c_q c_p c_{q+r}a^\dagger_{p+r}a^\dagger_q a_p a_{q+r}\,, \tag{5.26}
$$

is of order $N^{\beta-1}$. The proof of the following Lemma closely follows ideas of [4, Lemma 7.3]. For completeness, we nevertheless sketch the proof here.

Lemma 5.2. *Let $\psi,\xi\in\mathcal{F}_\perp$ and $\ell\in\mathbb{R}$. Then,*

$$
|\langle\psi,\mathbb{G}_2\xi\rangle| \lesssim N^{\beta-1}\|\mathbb{K}_0^{\frac{1}{2}}\,(\mathcal{N}_\perp+1)^{\frac{1}{2}+\ell}\,\psi\|\|\mathbb{K}_0^{\frac{1}{2}}\,(\mathcal{N}_\perp+1)^{\frac{1}{2}-\ell}\,\xi\|\,. \tag{5.27}
$$

Proof. Since $|c_p| \lesssim 1$ by Lemma 3.3, we find with Lemma 3.1

$$
\begin{aligned}
|\langle \psi, \mathbb{G}_2 \xi \rangle| &\leq \frac{1}{2N} \Bigg(\sum_{\substack{p,q,r \in \Lambda_+^* \\ p+r \neq 0, q+r \neq 0}} \frac{\left| \widehat{v}_N^\beta(r) \right|}{(q+r)^2} (p+r)^2 \| a_q a_{p+r} \left(\mathcal{N}_\perp + 1 \right)^\ell \psi \|^2 \Bigg)^{1/2} \\
&\qquad \times \Bigg(\sum_{\substack{p,q,r \in \Lambda_+^* \\ p+r \neq 0, q+r \neq 0}} \frac{\left| \widehat{v}_N^\beta(r) \right|}{(p+r)^2} (q+r)^2 \| a_p a_{q+r} \left(\mathcal{N}_\perp + 1 \right)^{-\ell} \xi \|^2 \Bigg)^{1/2} \\
&\lesssim N^{\beta - 1} \| \mathbb{K}_0^{1/2} \left(\mathcal{N}_\perp + 1 \right)^{\ell + \frac{1}{2}} \psi \| \| \mathbb{K}_0^{1/2} \left(\mathcal{N}_\perp + 1 \right)^{-\ell + \frac{1}{2}} \xi \|.
\end{aligned} \tag{5.28}
$$

□

5.3 Analysis of $\mathbb{R}_a$

In this section, we show that the remainder term $\mathbb{R}_a$, given by

$$
\begin{aligned}
\mathbb{R}_a &:= \frac{1}{N} \sum_{\substack{p,q,r \in \Lambda_+^* \\ p+r \neq 0, q+r \neq 0}} \widehat{v}_N^\beta(r) c_p c_q c_{q+r} s_{p+r} \left(a_{q+r}^\dagger a_p^\dagger a_{-(p+r)}^\dagger a_q + \text{h.c.} \right) \\
&\quad + \frac{1}{2N} \sum_{\substack{p,q,r \in \Lambda_+^* \\ p+r \neq 0, q+r \neq 0}} \widehat{v}_N^\beta(r) c_{p+r} c_q s_p s_{q+r} \left(a_{p+r}^\dagger a_q^\dagger a_{-p}^\dagger a_{-(q+r)}^\dagger + \text{h.c.} \right) \\
&=: \left(A_{a1} + A_{a2} + \text{h.c.} \right),
\end{aligned} \tag{5.29}
$$

is of order $N^{\beta/2-1}$.

Lemma 5.3. *Let $\psi, \xi \in \mathcal{F}_\perp$ and $\ell_1, \ell_2 \in \mathbb{R}$. Then*

$$
\begin{aligned}
|\langle \psi, \mathbb{R}_a \xi \rangle| &\lesssim N^{\frac{\beta}{2} - 1} \| \mathbb{K}_0^{1/2} \left(\mathcal{N}_\perp + 1 \right)^{\frac{1}{2} - \ell_1} \psi \| \| \left(\mathcal{N}_\perp + 1 \right)^{1 + \ell_1} \xi \| \\
&\quad + N^{\frac{\beta}{2} - 1} \| \left(\mathcal{N}_\perp + 1 \right)^{1 + \ell_2} \psi \| \| \mathbb{K}_0^{1/2} \left(\mathcal{N}_\perp + 1 \right)^{\frac{1}{2} - \ell_2} \xi \|.
\end{aligned} \tag{5.30}
$$

Proof. For A_{a1}, we compute with Lemma 3.3 and (3.2)

$$
\begin{aligned}
&|\langle \psi, A_{a1} \xi \rangle| \\
&\lesssim \frac{\| \widehat{v} \|_{\ell^\infty}^{\frac{1}{2}}}{N} \Bigg(\sum_{p,q,r \in \Lambda_+^*} |q+r|^2 \| a_p a_{-(p+r)} a_{q+r} \left(\mathcal{N}_\perp + 1 \right)^{-\frac{1}{2} + \ell} \psi \| \Bigg)^{\frac{1}{2}} \\
&\qquad \times \Bigg(\sum_{\substack{p,q,r \in \Lambda_+^* \\ p+r \neq 0, q+r \neq 0}} \frac{\widehat{v}_N^\beta(r)}{|q+r|^2} |s_{p+r}|^2 \| a_q (\mathcal{N}_\perp + 3)^{\frac{1}{2} - \ell} \xi \|^2 \Bigg)^{\frac{1}{2}} \\
&\lesssim N^{\frac{\beta}{2} - 1} \| \mathbb{K}_0^{1/2} \left(\mathcal{N}_\perp + 1 \right)^{\frac{1}{2} + \ell} \psi \| \| \left(\mathcal{N}_\perp + 1 \right)^{1 - \ell} \xi \|
\end{aligned} \tag{5.31}
$$

and similarly

$$
|\langle \psi, A_{a2} \xi \rangle|
$$

$$\lesssim \frac{\|\widehat{v}\|_{\ell^\infty}^{\frac{1}{2}}}{N}\Bigg(\sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}} \frac{\widehat{v}_N^\beta(r)}{|p+r|^2}|s_p|^2|s_{q+r}|^2\|a^\dagger_{-p}(\mathcal{N}_\perp+1)^{\frac{1}{2}-\ell}\xi\|^2\Bigg)^{\frac{1}{2}}$$

$$\times\Bigg(\sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}} |p+r|^2\|a_q a_{-(q+r)}a_{p+r}\,(\mathcal{N}_\perp+5)^{-\frac{1}{2}+\ell}\,\psi\|^2\Bigg)^{\frac{1}{2}}$$

$$\lesssim N^{\frac{\beta}{2}-1}\|(\mathcal{N}_\perp+1)^{1-\ell}\,\psi\|\|\mathbb{K}_0^{1/2}\,(\mathcal{N}_\perp+1)^{\frac{1}{2}+\ell}\,\xi\|\,. \tag{5.32}$$

The hermitian conjugates can be estimated similarly and we arrive at the desired bounds. □

5.4 Analysis of $\mathbb{R}_b$

In this section we show that the remainder term $\mathbb{R}_b$

$$\begin{aligned}
\mathbb{R}_b = &\frac{1}{2N}\sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}} \widehat{v}_N^\beta(r)c_q s_p s_{p+r}s_{q+r}\left(a^\dagger_{-p}a^\dagger_q a^\dagger_{-(q+r)}a_{-(p+r)}+\text{h.c.}\right)\\
&+\frac{1}{2N}\sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}} \widehat{v}_N^\beta(r)c_{p+r} s_p s_q s_{q+r}\left(a^\dagger_{p+r}a_{-q}a^\dagger_{-p}a^\dagger_{-(q+r)}+\text{h.c.}\right)\\
&+\frac{1}{2N}\sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}} \widehat{v}_N^\beta(r)c_p c_q s_{p+r}s_{q+r}a^\dagger_q a^\dagger_{-(q+r)}a_{-(p+r)}a_p\\
&+\frac{1}{2N}\sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}} \widehat{v}_N^\beta(r)c_p c_{p+r} s_q s_{q+r}\left(a^\dagger_{p+r}a^\dagger_{-(q+r)}a_p a_{-q}+\text{h.c.}\right)\\
&+\frac{1}{2N}\sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}} \widehat{v}_N^\beta(r)c_{p+r} c_{q+r} s_p s_q a^\dagger_{p+r}a_{-q}a^\dagger_{-p}a_{q+r}\\
&+\frac{1}{2N}\sum_{\substack{p,q,r\in\Lambda_+^*\\ p+r\neq 0, q+r\neq 0}} \widehat{v}_N^\beta(r)s_p s_{p+r} s_q s_{q+r} a_{-(p+r)}a_{-q}a^\dagger_{-p}a^\dagger_{-(q+r)}\\
&+\frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq q}} \widehat{v}_N^\beta(p-q)c_p^2 s_q^2 a^\dagger_p a_p\\
&+\frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq q}} \widehat{v}_N^\beta(p-q)c_p s_p s_q^2\left(a^\dagger_{-p}a^\dagger_p+\text{h.c.}\right)\\
&-\frac{1}{N}\sum_{p\in\Lambda_+^*}\widehat{v}_N^\beta c_p s_p\left((c_p^2+s_p^2)a^\dagger_p a_p + c_p s_p(a^\dagger_p a^\dagger_{-p}+\text{h.c.})\right)\\
&=: A_{b1}+A_{b1}^*+A_{b2}+A_{b2}^*+A_{b3}+A_{b4}+A_{b4}^*+A_{b5}+A_{b6}+A_{b7}+A_{b8}+A_{b8}^*+A_{b9}+A_{b9}^*\,.
\end{aligned}\tag{5.33}$$

is of order N^{-1} and thus subleading for our discussion.

Lemma 5.4. *Let $\psi, \xi \in \mathcal{F}_\perp$ and $\ell \in \mathbb{R}$. Then,*

$$|\langle \psi, \mathbb{R}_b \xi \rangle| \lesssim \frac{1}{N} \|(\mathcal{N}_\perp + 1)^{1+\ell} \psi\| \|(\mathcal{N}_\perp + 1)^{1-\ell} \xi\| \tag{5.34}$$

Proof. We estimate all the contributions of $\mathbb{R}_b$ separately. Using Lemma 3.3, we start with

$$\begin{aligned}|\langle \psi, A_{b1} \xi \rangle| &\leq \frac{\|\widehat{v}\|_{\ell^\infty} \|s\|_{\ell^\infty}}{2N} \Bigg(\sum_{p,q,r \in \Lambda_+^*} \|a_{-p} a_q a_{-(q+r)} (\mathcal{N}_\perp + 1)^{\ell - \frac{1}{2}} \psi\|^2 \Bigg)^{\frac{1}{2}} \\ &\quad \times \Bigg(\sum_{\substack{p,q,r \in \Lambda_+^* \\ q+r \neq 0}} |s_p|^2 |s_{q+r}|^2 \|a_{-(p+r)} (\mathcal{N}_\perp + 3)^{-\ell + \frac{1}{2}} \xi\|^2 \Bigg)^{\frac{1}{2}} \\ &\lesssim \frac{1}{N} \|(\mathcal{N}_\perp + 1)^{\ell+1} \psi\| \|(\mathcal{N}_\perp + 1)^{-\ell+1} \xi\| .\end{aligned} \tag{5.35}$$

Since $[a_q, a^\dagger_{-(q+r)}] = 0$ for $r \neq 0$, normal ordering yields

$$\begin{aligned}|\langle \psi, A_{b2} \xi \rangle| &\leq \frac{\|\widehat{v}\|_{\ell^\infty}}{2N} \sum_{\substack{p,q,r \in \Lambda_+^* \\ q+r \neq 0}} |s_p| |s_q| |s_{q+r}| \left| \left\langle \psi, (\mathcal{N}_\perp + 1)^\ell a^\dagger_{p+r} a^\dagger_{-p} a^\dagger_{-(q+r)} a_{-q} (\mathcal{N}_\perp + 3)^{-\ell} \xi \right\rangle \right| \\ &\quad + \frac{\|\widehat{v}\|_{\ell^\infty} \|s\|_{\ell^\infty}}{2N} \sum_{p,r \in \Lambda_+^*} |s_p|^2 \left| \left\langle \psi, (\mathcal{N}_\perp + 1)^\ell a^\dagger_{p+r} a^\dagger_{-(p+r)} (\mathcal{N}_\perp + 3)^{-\ell} \xi \right\rangle \right| \\ &\lesssim \frac{1}{N} \Bigg(\sum_{p,q,r \in \Lambda_+^*} |s_q|^2 \|a_{-p} a_r (\mathcal{N}_\perp + 1)^\ell \psi\|^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{p,q,r \in \Lambda_+^*} |s_p|^2 |s_r|^2 \|(\mathcal{N}_\perp + 1)^{-\ell + \frac{1}{2}} a_q \xi\|^2 \Bigg)^{\frac{1}{2}} \\ &\quad + \frac{1}{N} \sum_{p,q \in \Lambda_+^*} \|a_q (\mathcal{N}_\perp + 1)^\ell \psi\| \|(\mathcal{N}_\perp + 1)^{-\ell + \frac{1}{2}} \xi\| \\ &\lesssim \frac{1}{N} \|(\mathcal{N}_\perp + 1)^{\ell+1} \psi\| \|(\mathcal{N}_\perp + 1)^{-\ell+1} \xi\| .\end{aligned} \tag{5.36}$$

Moreover,

$$\begin{aligned}&|\langle \psi, A_{b3} \xi \rangle| \\ &\quad \lesssim \frac{\|\widehat{v}\|_{\ell^\infty}}{N} \sum_{\substack{p,q,r \in \Lambda_+^* \\ p+r \neq 0, q+r \neq 0}} |s_{q+r}| |s_{p+r}| \|a_{-(q+r)} a_q (\mathcal{N}_\perp + 1)^\ell \psi\| \|a_{-(p+r)} a_p (\mathcal{N}_\perp + 1)^{-\ell} \xi\| \\ &\quad \lesssim \frac{1}{N} \Bigg(\sum_{\substack{p,q,r \in \Lambda_+^* \\ p+r \neq 0}} |s_{p+r}|^2 \|a_{-(q+r)} a_q (\mathcal{N}_\perp + 1)^\ell \psi\|^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{p,q,r \in \Lambda_+^* \\ q+r \neq 0}} |s_{q+r}|^2 \|a_{-(p+r)} a_p (\mathcal{N}_\perp + 1)^{-\ell} \xi\|^2 \Bigg)^{\frac{1}{2}} \\ &\quad \lesssim \frac{1}{N} \|(\mathcal{N}_\perp + 1)^{\ell+1} \psi\| \|(\mathcal{N}_\perp + 1)^{-\ell+1} \xi\| .\end{aligned} \tag{5.37}$$

The next term can be estimated as

$$\begin{aligned}&|\langle \psi, A_{b4} \xi \rangle| \\ &\quad \lesssim \frac{\|\widehat{v}\|_{\ell^\infty}}{2N} \Bigg(\sum_{p,q,r \in \Lambda_+^*} |s_q|^2 \|a_{p+r} a_{-(q+r)} (\mathcal{N}_\perp + 1)^\ell \psi\|^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{p,q,r \in \Lambda_+^* \\ q+r \neq 0}} |s_{q+r}|^2 \|a_p a_{-q} (\mathcal{N}_\perp + 1)^{-\ell} \xi\|^2 \Bigg)^{\frac{1}{2}}\end{aligned}$$

$$\lesssim \frac{1}{N} \|(\mathcal{N}_\perp + 1)^{\ell+1} \psi\| \|(\mathcal{N}_\perp + 1)^{-\ell+1} \xi\| \,. \tag{5.38}$$

Normal ordering yields

$$\begin{aligned}
&|\langle \psi, A_{b5} \xi \rangle| \\
&\quad \lesssim \frac{\|\widehat{v}\|_{\ell^\infty}}{2N} \Big(\sum_{p,q,r \in \Lambda_+^*} |s_q|^2 \|a_{p+r} a_{-p} \, (\mathcal{N}_\perp + 1)^{\ell} \, \psi\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{p,q,r \in \Lambda_+^*} |s_p|^2 \|a_{-q} a_{q+r} \, (\mathcal{N}_\perp + 1)^{-\ell} \, \xi\|^2 \Big)^{\frac{1}{2}} \\
&\qquad + \frac{\|\widehat{v}\|_{\ell^\infty}}{2N} \sum_{p \in \Lambda_+^*} |s_p|^2 \sum_{r \in \Lambda_+^*} \|a_{p+r} \, (\mathcal{N}_\perp + 1)^{\ell} \, \psi\| \|a_{p+r} \, (\mathcal{N}_\perp + 1)^{-\ell} \, \xi\| \\
&\quad \lesssim \frac{1}{N} \|(\mathcal{N}_\perp + 1)^{\ell+1} \, \psi\| \|(\mathcal{N}_\perp + 1)^{-\ell+1} \, \xi\|
\end{aligned} \tag{5.39}$$

and

$$\begin{aligned}
&|\langle \psi, A_{b6} \xi \rangle| \\
&\quad \lesssim \frac{\|\widehat{v}\|_{\ell^\infty} \|s\|_{\ell^\infty}^2}{2N} \Big(\sum_{p,q,r \in \Lambda_+^*} |s_q|^2 \|a_{-p} a_{-(q+r)} \, (\mathcal{N}_\perp + 1)^{\ell} \, \psi\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{p,q,r \in \Lambda_+^*} |s_p|^2 \|a_{-q} a_{-(p+r)} \, (\mathcal{N}_\perp + 1)^{-\ell} \, \xi\|^2 \Big)^{\frac{1}{2}} \\
&\quad + \frac{\|\widehat{v}\|_{\ell^\infty}}{2N} \sum_{p,r \in \Lambda_+^*} |s_p|^2 |s_{p+r}|^2 \|(\mathcal{N}_\perp + 1)^{\ell + \frac{1}{2}} \psi\| \|(\mathcal{N}_\perp + 1)^{-\ell + \frac{1}{2}} \xi\| \\
&\quad \lesssim \frac{1}{N} \|(\mathcal{N}_\perp + 1)^{\ell+1} \psi\| \|(\mathcal{N}_\perp + 1)^{-\ell+1} \xi\| \,.
\end{aligned} \tag{5.40}$$

The remaining contributions of $\mathbb{R}_b$ can be estimated similarly. □

5.5 Analysis of $\mathbb{R}_c$

We will show that the remainder $\mathbb{R}_c$, given by

$$\mathbb{R}_c = \mathbb{T} \widetilde{\mathbb{R}}_{\surd} \mathbb{T}^* + \frac{1}{N} \sum_{p \in \Lambda_+^*} \widehat{v}_N^\beta(p) c_p s_p \mathbb{T}(\mathcal{N}_\perp + \frac{1}{2}) \mathbb{T}^*$$

with

$$\begin{aligned}
\widetilde{\mathbb{R}}_{\surd} = &\left(\left[\frac{N - \mathcal{N}_\perp}{N} \right]_+ - 1 \right) \mathbb{K}_1 \\
&+ \left(\mathbb{K}_2 \frac{\sqrt{[(N - \mathcal{N}_\perp)(N - \mathcal{N}_\perp - 1)]_+} - N}{N} + \text{h.c.} \right) \\
&+ \left(\mathbb{K}_3 \frac{\sqrt{[N - \mathcal{N}_\perp]_+} - \sqrt{N}}{N} + \text{h.c.} \right) \\
=: &A_{c_1} + A_{c_2} + A_{c_2}^* + A_{c_3} + A_{c_3}^*
\end{aligned}$$

is of order $N^{3(\beta-1)/2}$ for all $\beta > 1/2$.

Lemma 5.5. *Let $\psi, \xi \in \mathcal{F}_\perp$ and $\ell_1, \ell_2, \ell_3 \in \mathbb{R}$. Then*

$$\begin{aligned} |\langle \psi, \mathbb{R}_c \xi \rangle| \quad &\lesssim N^{\beta/2-1} \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{\ell_1+1/2} \psi\| \|(\mathcal{N}_\perp+1)^{-\ell_1+1} \xi\| \\ &+ N^{\beta/2-1} \|(\mathcal{N}_\perp+1)^{-\ell_2+1} \psi\| \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{\ell_2+1/2} \xi\| \\ &+ \left(N^{-1} + N^{\frac{3}{2}(\beta-1)}\right) \|(\mathcal{N}_\perp+1)^{\ell_3+\frac{5}{4}} \psi\| \|(\mathcal{N}_\perp+1)^{-\ell_3+\frac{5}{4}} \xi\| , \end{aligned} \tag{5.41}$$

which, for $\beta \in (1/2, 1)$, simplifies to

$$|\langle \psi, \mathbb{R}_c \xi \rangle| \lesssim N^{\frac{3}{2}(\beta-1)} \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4} \psi\| \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4} \xi\|. \tag{5.42}$$

Proof. In order to estimate the first contribution $\mathbb{T} A_{c_1} \mathbb{T}^*$, we first observe that the function $g_0(\mathcal{N}_\perp) := \left[\frac{N-\mathcal{N}_\perp}{N}\right]_+ - 1$ satisfies

$$g_0^2(\mathcal{N}_\perp) \lesssim \mathcal{N}_\perp / N \tag{5.43}$$

as operator inequality on $\mathcal{F}_\perp$. Thus,

$$\begin{aligned} |\langle \psi, \mathbb{T} A_{c_1} \mathbb{T}^* \xi \rangle| &\leq \sum_{p \in \Lambda_+^*} \widehat{v}_N^\beta(p) \left| \left\langle \psi, \mathbb{T} (\mathcal{N}_\perp+1)^{\ell-\frac{1}{2}} g_0(\mathcal{N}_\perp) a_p^\dagger a_p (\mathcal{N}_\perp+1)^{-\ell+\frac{1}{2}} \mathbb{T}^* \xi \right\rangle \right| \\ &\leq \|\widehat{v}\|_{\ell^\infty(\Lambda_+^*)} \sum_{p \in \Lambda_+^*} \|a_p g_0(\mathcal{N}_\perp) (\mathcal{N}_\perp+1)^{\ell-\frac{1}{2}} \mathbb{T}^* \psi\| \|a_p (\mathcal{N}_\perp+1)^{-\ell+\frac{1}{2}} \xi\| \\ &\lesssim \|(\mathcal{N}_\perp+1)^{\ell+1/2} \psi\| \|(\mathcal{N}_\perp+1)^{-\ell+1} \xi\| , \end{aligned} \tag{5.44}$$

where we used Lemma 3.4 for the last line. For the next term A_{c_2}, we compute the action of the Bogoliubov transformation on the annihilation and creation operators

$$\begin{aligned} \mathbb{T} \mathbb{K}_2 \mathbb{T}^* &= \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{v}_N^\beta(p) \mathbb{T} a_p^\dagger a_{-p}^\dagger \mathbb{T}^* \\ &= \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{v}_N^\beta(p) \left(c_p s_p + c_p^2 a_p^\dagger a_{-p}^\dagger + 2 c_p s_p a_p^\dagger a_p + s_p^2 a_p a_{-p} \right) . \end{aligned} \tag{5.45}$$

We denote $g_1(\mathcal{N}_\perp) := \frac{\sqrt{[(N-\mathcal{N}_\perp)(N-\mathcal{N}_\perp-1)]_+}}{N} - 1$ and observe that $g_1^2(\mathcal{N}_\perp) \lesssim N^{-1} \mathcal{N}_\perp$. Since $\sum_{p \in \Lambda_+^*} \widehat{v}_N^\beta(p) c_p s_p \lesssim N^\beta$ by Lemma 3.1, we obtain the bound

$$\begin{aligned} &\left| \left\langle \psi, \left(\mathbb{T} A_{c_2} \mathbb{T}^* - \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{v}_N^\beta(p) c_p s_p \mathbb{T} g_1(\mathcal{N}_\perp) \mathbb{T}^* \right) \xi \right\rangle \right| \\ &= \left| \left\langle \psi, \left(\mathbb{T} \mathbb{K}_2 \mathbb{T}^* - \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{v}_N^\beta(p) c_p s_p \right) \mathbb{T} g_1(\mathcal{N}_\perp) \mathbb{T}^* \xi \right\rangle \right| \\ &\lesssim \sum_{p \in \Lambda_+^*} \widehat{v}_N^\beta(p) \left| \left\langle a_p a_{-p} (\mathcal{N}_\perp+1)^\ell \psi, (\mathcal{N}_\perp+3)^{-\ell} \mathbb{T} g_1(\mathcal{N}_\perp) \mathbb{T}^* \xi \right\rangle \right| \\ &\quad + \sum_{p \in \Lambda_+^*} |s_p| \widehat{v}_N^\beta(p) \left| \left\langle a_p^\dagger a_p (\mathcal{N}_\perp+1)^\ell \psi, (\mathcal{N}_\perp+1)^{-\ell} \mathbb{T} g_1(\mathcal{N}_\perp) \mathbb{T}^* \xi \right\rangle \right| \end{aligned}$$

$$
\begin{aligned}
&+\sum_{p\in\Lambda_+^*}|s_p|^2\,\widehat{v}_N^{\beta}(p)\left|\left\langle a_p^\dagger a_{-p}^\dagger\,(\mathcal{N}_\perp+3)^{\ell}\,\psi,(\mathcal{N}_\perp+1)^{-\ell}\,\mathbb{T}g_1(\mathcal{N}_\perp)\mathbb{T}^*\xi\right\rangle\right|\\
&\lesssim\sum_{p\in\Lambda_+^*}|p|^{-1}\,\widehat{v}_N^{\beta}(p)\||p|\,a_p\,(\mathcal{N}_\perp+1)^{\ell+\frac12}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell}\,\mathbb{T}g_1(\mathcal{N}_\perp)\mathbb{T}^*\xi\|\\
&\quad+\|(\mathcal{N}_\perp+1)^{\ell+1}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell}\,\mathbb{T}g_1(\mathcal{N}_\perp)\mathbb{T}^*\xi\|\\
&\lesssim N^{\beta/2}\|\mathbb{K}_0^{1/2}\,(\mathcal{N}_\perp+1)^{\ell+\frac12}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell}\,\mathbb{T}g_1(\mathcal{N}_\perp)\mathbb{T}^*\xi\|\\
&\lesssim N^{\beta/2-1}\|\mathbb{K}_0^{1/2}\,(\mathcal{N}_\perp+1)^{\ell+\frac12}\,\psi\|\|(\mathcal{N}_\perp+1)^{1-\ell}\,\xi\| \qquad (5.46)
\end{aligned}
$$

by Lemma 3.4. Next we show that the additional term subtracted from A_{c_2} above together with the last term of the r.h.s. of (5.5) is small. For this we observe that

$$
g_1(\mathcal{N}_\perp)=-\frac{\mathcal{N}_\perp+\frac12}{N}+N^{-2}\widetilde{R}_1^{(2)}\quad\text{with}\quad\|\widetilde{R}_1^{(2)}\psi\|\le C\|(\mathcal{N}_\perp+1)^2\,\psi\| \tag{5.47}
$$

and by Lemma 3.4

$$
\begin{aligned}
\left|\left\langle\psi,\left(\mathbb{T}g_1(\mathcal{N}_\perp)\mathbb{T}^*+N^{-1}\mathbb{T}\left(\mathcal{N}_\perp+1/2\right)\mathbb{T}^*\right)\xi\right\rangle\right|&=\left|\left\langle\psi,\mathbb{T}\left(g_1(\mathcal{N}_\perp)+N^{-1}\left(\mathcal{N}_\perp+1/2\right)\right)\mathbb{T}^*\xi\right\rangle\right|\\
&\lesssim N^{-2}\|(\mathcal{N}_\perp+1)^{\ell+1}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+1}\,\xi\|\,. \qquad (5.48)
\end{aligned}
$$

In combination with (5.46) this leads to

$$
\begin{aligned}
&\frac12\left|\left\langle\psi,\sum_{p\in\Lambda_+^*}\widehat{v}_N^{\beta}(p)c_ps_p\left(\mathbb{T}g_1(\mathcal{N}_\perp)\mathbb{T}^*+N^{-1}\mathbb{T}\left(\mathcal{N}_\perp+1/2\right)\mathbb{T}^*\right)\xi\right\rangle\right|\\
&\quad\lesssim N^{-2+\beta}\|(\mathcal{N}_\perp+1)^{\ell+1}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+1}\,\xi\| \qquad (5.49)
\end{aligned}
$$

and thus altogether

$$
\begin{aligned}
&\left|\left\langle\psi,\left(\mathbb{T}A_{c_2}\mathbb{T}^*+\frac12N^{-1}\sum_{p\in\Lambda_+^*}\widehat{v}_N^{\beta}(p)c_ps_p\mathbb{T}\left(\mathcal{N}_\perp+1/2\right)\mathbb{T}^*\right)\xi\right\rangle\right|\\
&\lesssim N^{-1+\beta/2}\|\mathbb{K}_0^{1/2}\,(\mathcal{N}_\perp+1)^{\ell+1/2}\,\psi\|\|(\mathcal{N}_\perp+1)^{-\ell+1}\,\xi\|. \qquad (5.50)
\end{aligned}
$$

The hermitian conjugate can be bounded similarly.

Finally, observing that $g_2(\mathcal{N}_\perp)^2\lesssim N^{-3/2}\mathcal{N}_\perp$ for $g_2(\mathcal{N}_\perp):=\frac{\sqrt{[N-\mathcal{N}_\perp]_+}-\sqrt{N}}{N}$ and using Lemma 3.4 again, we obtain

$$
\begin{aligned}
&|\langle\psi,\mathbb{T}A_{c_3}\mathbb{T}^*\xi\rangle|\\
&\quad\lesssim\sum_{p,q\in\Lambda_+^*}\widehat{v}_N^{\beta}(p)\left|\left\langle\psi,\mathbb{T}\left(\mathcal{N}_\perp+1\right)^{\ell+1/4}a_{p+q}^\dagger a_{-p}^\dagger a_qg_2(\mathcal{N}_\perp)(\mathcal{N}_\perp+2)^{-\ell-1/4}\mathbb{T}^*\xi\right\rangle\right|\\
&\quad\lesssim\left(\sum_{p,q\in\Lambda_+^*}\|a_{-p}a_{p+q}\left(\mathcal{N}_\perp+1\right)^{\ell+1/4}\mathbb{T}^*\psi\|^2\right)^{1/2}\\
&\quad\times\left(\sum_{p,q\in\Lambda_+^*}\widehat{v}_N^{\beta}(p)\|a_qg_2(\mathcal{N}_\perp)(\mathcal{N}_\perp+2)^{-\ell-1/4}\mathbb{T}^*\xi\|^2\right)^{1/2}
\end{aligned}
$$

$$\lesssim CN^{3(-1+\beta)/2} \|(\mathcal{N}_\perp + 1)^{\ell+5/4} \psi\| \|(\mathcal{N}_\perp + 1)^{-\ell+5/4} \xi\| . \tag{5.51}$$

As the hermitian conjugate can be estimated similarly, this concludes the proof of the lemma. □

6 Estimates for the excitation vector

The goal of this section is to prove that the quantity

$$\left\langle \boldsymbol{\chi}, (\mathbb{K}_0 + 1)(\mathcal{N}_\perp + 1)^\ell \boldsymbol{\chi} \right\rangle$$

is bounded uniformly in N. To show this, we first need a bound on the kinetic energy in terms of the generator $\mathbb{G}$.

6.1 Estimate of the kinetic energy in terms of $\mathbb{G}$

Lemma 6.1.

$$\mathbb{K}_0 \lesssim \mathbb{G} + (\mathcal{N}_\perp + 1)^4 . \tag{6.1}$$

Proof. Let $\psi \in \mathcal{F}_\perp$. Since $a_0\psi = 0$, a straightforward computation yields

$$\begin{aligned}
\langle \psi, \mathbb{G}_2 \psi \rangle &= \left\langle \psi, \left(\sum_{p,q,r \in \Lambda} \frac{\widehat{v}_N^\beta(p)}{2N} c_{p+r} c_q c_p c_{q+r} a_{p+r}^\dagger a_q^\dagger a_p a_{q+r} - frac\widehat{v}(0)2N \sum_{p,q \in \Lambda_+^*} c_p^2 c_q^2 a_p^\dagger a_q^\dagger a_p a_q \right) \psi \right\rangle \\
&= \left\langle \psi, \int \mathrm{d}x \, \mathrm{d}y v(x-y) a^\dagger(\check{c}_x) a^\dagger(\check{c}_y) a(\check{c}_x) a(\check{c}_y) \psi \right\rangle \\
&\quad - \frac{\widehat{v}(0)}{2N} \sum_{p,q \in \Lambda_+^*} c_p^2 c_q^2 \left\langle \psi, a_p^\dagger a_q^\dagger a_p a_q \psi \right\rangle \\
&\geq - \frac{\widehat{v}(0)}{2N} \sum_{p,q \in \Lambda_+^*} c_p^2 c_q^2 \left\langle \psi, a_p^\dagger a_q^\dagger a_p a_q \psi \right\rangle .
\end{aligned} \tag{6.2}$$

We consequently have that

$$\mathbb{G} = \mathbb{G}_0 + \mathbb{G}_1 + \mathbb{G}_2 + \mathbb{R}_2 \geq \mathbb{G}_0 + \mathbb{G}_1 + \mathbb{R}_2 - \frac{\widehat{v}(0)}{2N} \sum_{p,q \in \Lambda_+^*} c_p^2 c_q^2 a_p^\dagger a_q^\dagger a_p a_q \tag{6.3}$$

in the sense of operators on $\mathcal{F}_\perp$.

Estimate for $\mathbb{G}_0$. Using Lemma 3.5, we obtain

$$\begin{aligned}
\frac{1}{2} \langle \psi, \mathbb{K}_0 \psi \rangle &\leq \left\langle \psi, \sum_{p \in \Lambda_+^*} F_p a_p^\dagger a_p \psi \right\rangle \\
&\leq \langle \psi, \mathbb{G}_0 \psi \rangle + \left| \left\langle \psi, \sum_{p \in \Lambda_+^*} G_p (a_p^\dagger a_{-p}^\dagger + a_p a_{-p}) \psi \right\rangle \right| \\
&\leq \langle \psi, \mathbb{G}_0 \psi \rangle + C \|(\mathcal{N}_\perp + 1)^{1/2} \psi\|^2 .
\end{aligned} \tag{6.4}$$

Estimate for the other terms. From Lemma 5.1 and Proposition 2.2, we obtain

$$|\langle \psi, \mathbb{G}_1 \psi \rangle| \leq C N^{(\beta-1)/2} \|(\mathbb{K}_0+1)^{1/2}\psi\| \|(\mathcal{N}_\perp + 1)\,\psi\|\,, \tag{6.5}$$
$$|\langle \psi, \mathbb{R}_2 \psi \rangle| \leq C N^{\frac{3}{2}(\beta-1)} \|(\mathbb{K}_0+1)^{1/2}\,\psi\| \|(\mathcal{N}_\perp + 1)^2\,\psi\|. \tag{6.6}$$

Moreover,

$$\left| \frac{\widehat{v}(0)}{2N} \sum_{p,q\in\Lambda_+^*} c_p^2 c_q^2 \left\langle \psi, a_p^\dagger a_q^\dagger a_p a_q \psi \right\rangle \right| \leq C N^{-1} \|(\mathcal{N}_\perp+1)\psi\|^2\,. \tag{6.7}$$

Final estimate. Combining these estimates with (6.3) yields

$$\begin{aligned}
\frac{1}{2}\langle \psi, \mathbb{K}_0 \psi \rangle &\leq \langle \psi, \mathbb{G}\psi\rangle + |\langle \psi, (\mathbb{G}_1 + \mathbb{R}_2)\,\psi\rangle| + \left| \frac{\widehat{v}(0)}{2N} \sum_{p,q\in\Lambda_+^*} c_p^2 c_q^2 \left\langle \psi, a_p^\dagger a_q^\dagger a_p a_q \psi \right\rangle \right| \\
&\quad + C\|(\mathcal{N}_\perp+1)^{1/2}\,\psi\|^2 \\
&\leq \langle \psi, \mathbb{G}\psi\rangle + C\|(\mathcal{N}_\perp+1)^2\,\psi\|^2 + C\|\mathbb{K}_0^{1/2}\psi\|\|(\mathcal{N}_\perp+1)^2\,\psi\| \\
&\leq \langle \psi, \mathbb{G}\psi\rangle + C\|(\mathcal{N}_\perp+1)^2\,\psi\|^2 + \frac{1}{4}\langle \psi, \mathbb{K}_0\psi\rangle\,.
\end{aligned} \tag{6.8}$$

Subtracting the third summand on the right hand side from both sides of the inequality then shows the claim. □

6.2 Kinetic energy of the excitations

With this tool, we can now prove the main result of this section.

Lemma 6.2. *Recall that* $\boldsymbol{\chi} = \mathbb{T}(U_N\Psi_N \oplus 0)$ *and let* $\ell \geq 0$*. Then*

$$\left\langle \boldsymbol{\chi}, (\mathcal{N}_\perp+1)^\ell\,(\mathbb{K}_0+1)\,\boldsymbol{\chi} \right\rangle \lesssim 1\,. \tag{6.9}$$

Proof. We first show that any moment of the number of particles operator is bounded. We then use this bound to show that the kinetic energy is bounded, and as a last step estimate products of the kinetic energy and any power of the number of particles operator.

Powers of the number of particles. We recall an a priori bound on powers of the number of excitations in the ground state Ψ_N that was recently proven in [24, Remark 1.2]. More precisely, we have

$$\langle \Psi_N, \mathcal{N}_\perp^\ell \Psi_N\rangle_{L^2(\Lambda^N)} \leq C(\ell) \tag{6.10}$$

for any power $\ell \in \mathbb{N}$. Since $U_N \mathcal{N}_\perp U_N^* = \mathcal{N}_\perp$ and

$$\mathbb{T}\mathcal{N}_\perp^\ell \mathbb{T}^* \lesssim (\mathcal{N}_\perp+1)^\ell \tag{6.11}$$

from Lemma 3.4, we consequently get

$$\langle \boldsymbol{\chi}, (\mathcal{N}_\perp+1)^\ell \boldsymbol{\chi}\rangle \lesssim 1\;. \tag{6.12}$$

Kinetic energy. From (6.12), we now derive a bound on the kinetic energy of $\boldsymbol{\chi}$. For this we recall that $\boldsymbol{\chi}$ is the ground state of $\mathbb{G}$, i.e. it satisfies the eigenvalue equation $\mathbb{G}\boldsymbol{\chi} = E\boldsymbol{\chi}$ for $E = E_{N,\beta} - \mathcal{C}$ as defined in (2.46). From Lemma 6.1 we have the upper bound

$$\mathbb{G} \gtrsim \mathbb{K}_0 - (\mathcal{N}_\perp + 1)^4 \,, \tag{6.13}$$

which we use to estimate

$$\langle \boldsymbol{\chi}, \mathbb{K}_0 \chi \rangle \lesssim \langle \boldsymbol{\chi}, \big(\mathbb{G} + (\mathcal{N}_\perp + 1)^4\big)\boldsymbol{\chi} \rangle = \langle \boldsymbol{\chi}, (E + (\mathcal{N}_\perp + 1)^4)\boldsymbol{\chi} \rangle \,. \tag{6.14}$$

Since $|E| \lesssim 1$ by (2.50), we conclude with (6.12) that

$$\langle \boldsymbol{\chi}, \mathbb{K}_0 \boldsymbol{\chi} \rangle \le 1 \,. \tag{6.15}$$

Moments. Next we prove (6.9). For this we find with (6.13) and (6.12)

$$\begin{aligned}
\langle \boldsymbol{\chi}, (\mathcal{N}_\perp + 1)^{2(\ell+1)} \mathbb{K}_0 \boldsymbol{\chi} \rangle \lesssim & \langle \boldsymbol{\chi}, (\mathcal{N}_\perp + 1)^{\ell+1} \big(\mathbb{G} + (\mathcal{N}_\perp + 1)^4\big) (\mathcal{N}_\perp + 1)^{\ell+1} \boldsymbol{\chi} \rangle \\
\lesssim & \langle \boldsymbol{\chi}, (\mathcal{N}_\perp + 1)^{\ell+1} \mathbb{G} (\mathcal{N}_\perp + 1)^{\ell+1} \boldsymbol{\chi} \rangle + 1 \\
\lesssim & \langle \boldsymbol{\chi}, (\mathcal{N}_\perp + 1)^{\ell+1} \big[\mathbb{G}, (\mathcal{N}_\perp + 1)^{\ell+1}\big] \boldsymbol{\chi} \rangle + E \langle \boldsymbol{\chi}, (\mathcal{N}_\perp + 1)^{2(\ell+1)} \boldsymbol{\chi} \rangle + 1 \\
\lesssim & \langle \boldsymbol{\chi}, (\mathcal{N}_\perp + 1)^{\ell+1} \big[\mathbb{G}, (\mathcal{N}_\perp + 1)^{\ell+1}\big] \boldsymbol{\chi} \rangle + 1 \,.
\end{aligned} \tag{6.16}$$

Since

$$\begin{aligned}
\big[\mathbb{G}, (\mathcal{N}_\perp + 1)^{\ell+1}\big] &= \sum_{j=0}^{\ell+1} (\mathcal{N}_\perp + 1)^j \big[\mathbb{G}, \mathcal{N}_\perp\big] (\mathcal{N}_\perp + 1)^{\ell-j} \\
&= \sum_{j=1}^{\ell+1} (\mathcal{N}_\perp + 1)^j \left[\mathbb{G}_1 + \tfrac{1}{2} \sum_{p \in \Lambda_+^*} G_p (a_p^\dagger a_{-p}^\dagger + \text{h.c.}) + \mathbb{R}_2, \mathcal{N}_\perp \right] (\mathcal{N}_\perp + 1)^{\ell-j} \,,
\end{aligned} \tag{6.17}$$

we find from Lemma 5.1, Proposition 2.2 and 3.5

$$\Big\langle \boldsymbol{\chi}, (\mathcal{N}_\perp + 1)^j \left[\mathbb{G}_1, \mathcal{N}_\perp\right] (\mathcal{N}_\perp + 1)^{\ell+1-j} \boldsymbol{\chi} \Big\rangle \lesssim N^{\frac{\beta-1}{2}} \| (\mathbb{K}_0 + 1)^{\frac{1}{2}} \boldsymbol{\chi} \| \| (\mathcal{N}_\perp + 1)^{\ell+2} \boldsymbol{\chi} \| \,, \tag{6.18}$$

$$\Big\langle \boldsymbol{\chi}, (\mathcal{N}_\perp + 1)^j \left[\mathbb{R}_2, \mathcal{N}_\perp\right] (\mathcal{N}_\perp + 1)^{\ell+1-j} \boldsymbol{\chi} \Big\rangle \lesssim N^{\frac{3}{2}(\beta-1)} \| (\mathbb{K}_0 + 1)^{\frac{1}{2}} \boldsymbol{\chi} \| \| (\mathcal{N}_\perp + 1)^{3+\ell} \boldsymbol{\chi} \| \,, \tag{6.19}$$

and

$$\tfrac{1}{2} \sum_{p \in \Lambda_+^*} |G_p| \Big\langle \boldsymbol{\chi}, (\mathcal{N}_\perp + 1)^j \Big[(a_p^\dagger a_{-p}^\dagger + \text{h.c.}), \mathcal{N}_\perp \Big] (\mathcal{N}_\perp + 1)^{\ell+1-j} \boldsymbol{\chi} \Big\rangle \tag{6.20}$$

$$\lesssim \| (\mathcal{N}_\perp + 1)^{\ell+1} \boldsymbol{\chi} \| \| (\mathcal{N}_\perp + 1) \boldsymbol{\chi} \| \,. \tag{6.21}$$

Hence, by (6.15) and (6.12), we conclude (6.9).

□

7 Perturbation theory

In this final section we complete the proof of the perturbative expansion of the ground state projector $\mathbb{P} = |\boldsymbol{\chi}\rangle\langle\boldsymbol{\chi}|$. As a first step, we expand $\mathbb{P}$ around $\mathbb{P}_0$; subsequently, we prove estimates on the remainders of this expansion (Proposition 2.4).

7.1 Proof of Proposition 2.4

Recall that Proposition 2.4 states that

$$\left| \mathrm{Tr}\mathbb{A}\mathbb{P} - \sum_{\ell=0}^{2} \mathrm{Tr}\mathbb{A}\mathbb{P}_\ell \right| \leq N^{\frac{3}{2}(\beta-1)}$$

for $\mathbb{P}_1$ and $\mathbb{P}_2$ as in (2.54). Our goal is now to compute these quantities and also the remainder term explicitly. In Sections 7.4– 7.5, we show that the remainder satisfies the bound above.

Since the strategy is very close to [10, Proposition 3.13, Proposition 3.14 and Theorem 2], we only sketch the main ideas.

Proof of Proposition 2.4. The first step is to express $\mathbb{G}$ as

$$\mathbb{G} = \mathbb{G}_0 + \mathbb{G}_1 + \mathbb{G}_2 + \mathbb{R}_2 =: \mathbb{G}_0 + \mathbb{G}_1 + \mathbb{R}_1 =: \mathbb{G}_0 + \mathbb{R}_0 \tag{7.1}$$

(see Proposition 2.2). The remainder terms $\mathbb{R}_i$ correspond to errors of of different order in N. Lemmas 5.3-5.5 show that $\mathbb{R}_2 = \mathbb{R}_a + \mathbb{R}_b + \mathbb{R}_c$ satisfies for $\ell, \ell_1, \ell_2 \in \mathbb{R}$

$$\begin{aligned}|\langle \psi, \mathbb{R}_2 \xi \rangle| \lesssim N^{\frac{3}{2}(\beta-1)} \Big(&\|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4-\ell_1} \psi\| \, \|(\mathcal{N}_\perp+1)^{5/4+\ell_1} \xi\| \\ &+ \|((\mathcal{N}_\perp+1)^{5/4-\ell_2} \psi\| \, \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4+\ell_2} \xi\| \Big) .\end{aligned} \tag{7.2}$$

From Lemma 5.2-5.5, it follows that $\mathbb{R}_1 = \mathbb{G}_2 + \mathbb{R}_2$ scales as

$$|\langle \psi, \mathbb{R}_1 \xi \rangle| \lesssim N^{\beta-1} \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4-\ell} \psi\| \, \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4+\ell} \xi\| . \tag{7.3}$$

The remainder term $\mathbb{R}_0 = \mathbb{G}_1 + \mathbb{G}_2 + \mathbb{R}_2$ is by Lemmas 5.1-5.5 of order

$$|\langle \psi, \mathbb{R}_0 \xi \rangle| \lesssim N^{(\beta-1)/2} \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4-\ell} \psi\| \, \|(\mathbb{K}_0+1)^{1/2} (\mathcal{N}_\perp+1)^{3/4+\ell} \xi\| . \tag{7.4}$$

We now expand $(z-\mathbb{G})^{-1}$ around $(z-\mathbb{G}_0)^{-1}$ as

$$\begin{aligned}\frac{1}{z-\mathbb{G}} = &\frac{1}{z-\mathbb{G}_0} + \frac{1}{z-\mathbb{G}_0}\mathbb{G}_1\frac{1}{z-\mathbb{G}_0} + \frac{1}{z-\mathbb{G}_0}\left(\mathbb{G}_2 + \mathbb{G}_1\frac{1}{z-\mathbb{G}_0}\mathbb{G}_1\right)\frac{1}{z-\mathbb{G}_0} \\ &+ \frac{1}{z-\mathbb{G}}\left(\mathbb{R}_0\frac{1}{z-\mathbb{G}_0}(\mathbb{G}_2 + \mathbb{G}_1\frac{1}{z-\mathbb{G}_0}\mathbb{G}_1) + \mathbb{R}_1\frac{1}{z-\mathbb{G}_0}\mathbb{G}_1 + \mathbb{R}_2\right)\frac{1}{z-\mathbb{G}_0} .\end{aligned} \tag{7.5}$$

Then (2.54) follows from an application of the residue theorem analogously to [10, Proposition 3.14]. Note that $\mathbb{G}_1$ is cubic in the number of creation/annihilation operators and $\boldsymbol{\chi}_0$ is a quasi-free state, hence $\langle \boldsymbol{\chi}_0, \mathbb{G}_1 \boldsymbol{\chi}_0 \rangle = 0$ by Wick's rule. Analogously to [10, Proposition 3.14], we find that

$$\mathrm{Tr}\mathbb{A}\mathbb{P} = \sum_{\ell=0}^{2} \mathrm{Tr}\mathbb{A}\mathbb{P}_\ell + \mathrm{Tr}\mathbb{A}\mathbb{B}_P + \mathrm{Tr}\mathbb{A}\mathbb{B}_Q , \tag{7.6}$$

where the remainder $\mathrm{Tr}\mathbb{A}\mathbb{B}_P$ is given as

$$\mathrm{Tr}\mathbb{A}\mathbb{B}_P = \oint_\gamma' \frac{1}{z-E}\Bigg(\bigg\langle \boldsymbol{\chi}, \mathbb{R}_0\frac{1}{z-\mathbb{G}_0}\left(\mathbb{G}_2 + \mathbb{G}_1\frac{1}{z-\mathbb{G}_0}\mathbb{G}_1\right)\frac{1}{z-\mathbb{G}_0}\mathbb{A}\boldsymbol{\chi} \bigg\rangle$$

$$+\left\langle \boldsymbol{\chi}, \mathbb{R}_1 \frac{1}{z-\mathbb{G}_0}\mathbb{G}_1\frac{1}{z-\mathbb{G}_0}\mathbb{A}\boldsymbol{\chi}\right\rangle + \left\langle \boldsymbol{\chi}, \mathbb{R}_2\frac{1}{z-\mathbb{G}_0}\mathbb{A}\boldsymbol{\chi}\right\rangle\Bigg)\,\mathrm{d}z\,, \tag{7.7}$$

where we used the abbreviation $\oint_\gamma' := \frac{1}{2\pi\mathrm{i}}\oint_\gamma$. For $\mathrm{Tr}\mathbb{A}\mathbb{B}_Q$, we find analogously to the computation in [10] that

$$\begin{aligned}
\mathrm{Tr}\mathbb{A}\mathbb{B}_Q = \oint_\gamma' \frac{\mathrm{d}z}{z-E_0}\Bigg(&\left\langle \mathbb{G}_2\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\right\rangle + \left\langle \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_2\right\rangle \\
&+ \left\langle \mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\right\rangle + \left\langle \mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\right\rangle \\
&+ \left\langle \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\right\rangle + \left\langle \mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_1\right\rangle \\
&+ \left\langle \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\right\rangle + \left\langle \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_2\right\rangle\Bigg) \\
+ \oint_\gamma' \frac{\mathrm{d}z}{(z-E_0)^2}\Bigg(&\langle\mathbb{G}_2\rangle\left\langle \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\right\rangle + \langle\mathbb{G}_1\rangle\left\langle \mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\right\rangle \\
&+ \left\langle \mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\right\rangle\left\langle \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\right\rangle + \langle\mathbb{G}_1\rangle\left\langle \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\right\rangle \\
&+ \langle\mathbb{G}_1\rangle\left\langle \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_1\right\rangle\Bigg) \\
+ \oint_\gamma' \frac{\mathrm{d}z}{(z-E_0)^3}&\langle\mathbb{G}_1\rangle^2\left\langle \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\right\rangle\,,
\end{aligned} \tag{7.8}$$

where we used the notation

$$\langle\mathbb{B}\rangle := \langle\boldsymbol{\chi}_0, \mathbb{B}\boldsymbol{\chi}_0\rangle\,. \tag{7.9}$$

Recall that $\langle\mathbb{G}_1\rangle = 0$ by Wick's rule, hence some of the terms in (7.8) vanish. Since $z \in \gamma$ with γ as defined in (2.51), it follows that $|z-E_0| = c$ and $|z-E| \geq |z-E_0| - |E-E_0| \geq c/2$ for sufficiently large N, hence

$$\sup_{z\in\gamma}\left|\frac{1}{z-E_0}\right| \lesssim 1\,, \qquad \sup_{z\in\gamma}\left|\frac{1}{z-E}\right| \lesssim 1\,. \tag{7.10}$$

We use this argument to estimate $\mathrm{Tr}\mathbb{A}\mathbb{B}_P$ by (7.43) in Section 7.4 and $\mathrm{Tr}\mathbb{A}\mathbb{B}_Q$ by (7.49) in Section 7.5. The final estimates of Section 7.4 and Section 7.5 (see (7.48) and (7.67)) show that

$$|\mathrm{Tr}\mathbb{A}\mathbb{B}_P| + |\mathrm{Tr}\mathbb{A}\mathbb{B}_Q| \lesssim N^{\frac{3}{2}(\beta-1)}\,, \tag{7.11}$$

which proves the first statement (2.53) of Proposition 2.4.

Ground state energy. We observe analogously to [10, Theorem 2] that

$$E = \mathrm{Tr}\,\mathbb{G}\,\mathbb{P} = \frac{1}{2\pi\mathrm{i}}\mathrm{Tr}\oint_\gamma \frac{\mathbb{G}}{z-\mathbb{G}}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\mathrm{Tr}\oint_\gamma \frac{z}{z-\mathbb{G}}\,\mathrm{d}z = E_0 + \frac{1}{2\pi\mathrm{i}}\mathrm{Tr}\oint_\gamma\frac{z-E_0}{z-\mathbb{G}}\,\mathrm{d}z\,. \tag{7.12}$$

Consequently, expanding $(z-\mathbb{G})^{-1}$ around $(z-\mathbb{G}_0)^{-1}$ as above yields

$$E - E_0 = \mathrm{Tr}\oint_\gamma' \frac{1}{z-\mathbb{G}_0}\mathbb{G}_1\frac{z-E_0}{z-\mathbb{G}_0}\,\mathrm{d}z$$

$$+ \operatorname{Tr} \oint_\gamma' \frac{1}{z - \mathbb{G}_0} \mathbb{G}_2 \frac{z - E_0}{z - \mathbb{G}_0} + \operatorname{Tr} \oint_\gamma' \frac{1}{z - \mathbb{G}_0} \mathbb{G}_1 \frac{1}{z - \mathbb{G}_0} \mathbb{G}_1 \frac{z - E_0}{z - \mathbb{G}_0} \, \mathrm{d}z$$

$$+ \sum_{\nu=0}^{2} \sum_{m=1}^{2-\nu} \sum_{\substack{\boldsymbol{j} \in \mathbb{N}^m \\ |\boldsymbol{j}|=2-\nu}} \operatorname{Tr} \oint_\gamma' \frac{1}{z - \mathbb{G}} \mathbb{R}_\nu \frac{1}{z - \mathbb{G}_0} \mathbb{G}_{j_1} \frac{1}{z - \mathbb{G}_0} \mathbb{G}_{j_2} \cdots \mathbb{G}_{j_m} \frac{z - E_0}{z - \mathbb{G}_0} \, \mathrm{d}z \tag{7.13}$$

For the first term, we observe that

$$\operatorname{Tr} \oint_\gamma' \frac{1}{z - \mathbb{G}_0} \mathbb{G}_1 \frac{z - E_0}{z - \mathbb{G}_0} \, \mathrm{d}z = \oint_\gamma' \frac{1}{z - E_0} \langle \boldsymbol{\chi}_0, \mathbb{G}_1 \boldsymbol{\chi}_0 \rangle \, \mathrm{d}z = 0 \tag{7.14}$$

by Wick's rule. Applying the residue theorem, we find that the second and third term give E_{pert} as in (2.56). This is of the same form as $\mathrm{Tr}\mathbb{A}\mathbb{B}_P + \mathrm{Tr}\mathbb{A}\mathbb{B}_Q$ but with $\mathbb{A}$ replaced by $(z - E_0)$ in each term in (7.7) and (7.8). Together with the estimate (7.10), this implies that the absolute value of the third line in (7.13) is bounded by

$$|\mathrm{Tr}\mathbb{1}\mathbb{B}_P| + |\mathrm{Tr}\mathbb{1}\mathbb{B}_Q| \leq N^{\frac{3}{2}(\beta-1)} \tag{7.15}$$

following from (7.11). This proves the second statement (2.55) of Proposition 2.4. □

7.2 Bounds on the resolvent of $\mathbb{G}_0$

In this section we prove estimates on the resolvent of the quadratic Hamiltonian $\mathbb{G}_0$.

Lemma 7.1. *Let $\ell \in \mathbb{R}$, $\psi \in \mathcal{F}_\perp$ and $z \in \gamma$ for γ as in* (2.51) *and let*

$$\mathbb{O} \in \left\{ \frac{1}{z - \mathbb{G}_0}, \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \right\}.$$

Then

$$\|(\mathbb{K}_0 + 1)^{\frac{1}{2}} (\mathcal{N}_\perp + 1)^\ell \mathbb{O} (\mathbb{K}_0 + 1)^{\frac{1}{2}} \psi\| \lesssim \|(\mathcal{N}_\perp + 1)^\ell \psi\| . \tag{7.16}$$

Proof. We consider first the case $\mathbb{O} = (z - \mathbb{G}_0)^{-1}$. First note that by diagonalizing $\mathbb{G}_0$ using $\mathbb{U}_\tau$ we get

$$\mathbb{U}_\tau \mathbb{G}_0 \mathbb{U}_\tau^* \geq c\mathcal{N}_\perp - C . \tag{7.17}$$

We multiply (7.17) with $\mathbb{U}_\tau^*$ from the left and $\mathbb{U}_\tau$ from the right and arrive with $\mathbb{U}_\tau^*(\mathcal{N}_\perp + 1)\mathbb{U}_\tau \geq \frac{1}{c}\mathcal{N}_\perp$ from Lemma 3.6 at

$$\mathbb{G}_0 \geq c_1 \mathcal{N}_\perp - c_2 \tag{7.18}$$

for some constants $c_1, c_2 > 0$. Moreover, Lemma 3.5 shows that

$$\mathbb{G}_0 \geq c_3 \mathbb{K}_0 - c_4(\mathcal{N}_\perp + 1) . \tag{7.19}$$

Interpolating both from bounds (7.18), (7.19) we arrive at

$$\mathbb{G}_0 \geq c_5(\mathbb{K}_0 + \mathcal{N}_\perp) - c_6 , \tag{7.20}$$

which implies (7.16) for $\ell = 0$ and $\mathrm{Im}z = 0$. For $\ell = 0$ and $\mathrm{Im}z \neq 0$ we find

$$\|(\mathbb{K}_0 + 1)^{\frac{1}{2}} \frac{1}{z - \mathbb{G}_0} (\mathbb{K}_0 + 1)^{\frac{1}{2}} \psi\|^2$$

$$= \left\langle (\mathbb{K}_0+1)^{\frac12}\psi, \frac{\mathbb{G}_0 - \mathrm{Re}z + \mathrm{iIm}z}{(\mathbb{G}_0-\mathrm{Re}z)^2+(\mathrm{Im}z)^2}(\mathbb{K}_0+1)\frac{\mathbb{G}_0 - \mathrm{Re}z + \mathrm{iIm}z}{(\mathbb{G}_0-\mathrm{Re}z)^2+(\mathrm{Im}z)^2}(\mathbb{K}_0+1)^{\frac12}\psi \right\rangle . \quad (7.21)$$

From (7.20) we get

$$\begin{aligned} &\left\| (\mathbb{K}_0+1)^{\frac12} \frac{\mathbb{G}_0 - \mathrm{Re}z}{(\mathbb{G}_0-\mathrm{Re}z)^2+(\mathrm{Im}z)^2}(\mathbb{K}_0+1)^{\frac12}\psi \right\| \\ &\quad \lesssim \left\| \frac{(\mathbb{G}_0-\mathrm{Re}z)^{3/2}+1}{(\mathbb{G}_0-\mathrm{Re}z)^2}(\mathbb{K}_0+1)^{\frac12}\psi \right\| \lesssim \|\psi\| \end{aligned} \quad (7.22)$$

and

$$\begin{aligned} &\left\| (\mathbb{K}_0+1)^{\frac12} \frac{1}{(\mathbb{G}_0-\mathrm{Re}z)^2+(\mathrm{Im}z)^2}(\mathbb{K}_0+1)^{\frac12}\psi \right\| \\ &\quad \lesssim \left\| \frac{(\mathbb{G}_0-\mathrm{Re}z)^{1/2}+1}{(\mathbb{G}_0-\mathrm{Re}z)^2}(\mathbb{K}_0+1)^{\frac12}\psi \right\| \lesssim \|\psi\| \end{aligned} \quad (7.23)$$

which implies (7.16) for $\ell = 0$ and $z \in \gamma$. To prove (7.16) for general $\ell \in \mathbb{R}$, we use once more that $\mathbb{U}_\tau$ diagonalizes $\mathbb{G}_0$ and compute with Lemma 3.6

$$\begin{aligned} \|\mathbb{K}_0^{\frac12}\mathcal{N}_\perp^\ell \frac{1}{z-\mathbb{G}_0}(\mathbb{K}_0+1)^{\frac12}\psi\| &= \|\mathbb{K}_0^{\frac12}\mathcal{N}_\perp^\ell \mathbb{U}_\tau^* \frac{1}{z-\mathbb{U}_\tau\mathbb{G}_0\mathbb{U}_\tau^*}(\mathbb{U}_\tau\mathbb{K}_0\mathbb{U}_\tau^*+1)^{\frac12}\mathbb{U}_\tau\psi\| \\ &\lesssim \|\mathbb{K}_0^{\frac12}\mathcal{N}_\perp^\ell \frac{1}{z-\mathbb{U}_\tau\mathbb{G}_0\mathbb{U}_\tau^*}(\mathbb{U}_\tau\mathbb{K}_0\mathbb{U}_\tau^*+1)^{\frac12}\mathbb{U}_\tau\psi\| . \end{aligned} \quad (7.24)$$

As the number of particles operator commutes with the diagonal Hamiltonian $\mathbb{U}_\tau\mathbb{G}_0\mathbb{U}_\tau^*$, we thus get, by (7.20) and using Lemma 3.6 several times, that

$$\begin{aligned} \|\mathbb{K}_0^{\frac12}\mathcal{N}_\perp^\ell \frac{1}{z-\mathbb{G}_0}(\mathbb{K}_0+1)^{\frac12}\psi\| &\lesssim \|\mathbb{K}_0^{\frac12} \frac{1}{z-\mathbb{U}_\tau\mathbb{G}_0\mathbb{U}_\tau^*}\mathcal{N}_\perp^\ell\mathbb{U}_\tau(\mathbb{K}_0+1)^{\frac12}\psi\| \\ &\lesssim \|\mathbb{K}_0^{\frac12}\frac{1}{z-\mathbb{G}_0}(\mathbb{K}_0+1)^{\frac12}\left(\frac{1}{\mathbb{K}_0+1}\right)^{\frac12}\mathbb{U}_\tau^*\mathcal{N}_\perp^\ell\mathbb{U}_\tau(\mathbb{K}_0+1)^{\frac12}\psi\| \\ &\lesssim \left\| \left(\frac{1}{\mathbb{K}_0+1}\right)^{\frac12}\mathbb{U}_\tau^*\mathcal{N}_\perp^\ell\mathbb{U}_\tau(\mathbb{K}_0+1)^{\frac12}\psi \right\| \\ &\lesssim \|\mathcal{N}_\perp^\ell\mathbb{U}_\tau\psi\| . \end{aligned} \quad (7.25)$$

For $\mathbb{O} = \mathbb{Q}_0(E_0-\mathbb{G}_0)^{-1}$ and $\ell = 0$, we find similarly to above that

$$\begin{aligned} \|(\mathbb{K}_0+1)^{\frac12}\frac{\mathbb{Q}_0}{E_0-\mathbb{G}_0}(\mathbb{K}_0+1)^{\frac12}\psi\|^2 &\lesssim \left\langle \psi, (\mathbb{K}_0+1)^{\frac12}\frac{\mathbb{Q}_0}{E_0-\mathbb{G}_0}(\mathbb{K}_0+1)^{\frac12}\psi \right\rangle \\ &\lesssim \left\langle \psi, (\mathbb{K}_0+1)^{\frac12}\mathbb{Q}_0\frac{1}{\mathbb{K}_0-1}\mathbb{Q}_0(\mathbb{K}_0+1)^{\frac12}\psi \right\rangle \\ &\lesssim \|\psi\|^2 \end{aligned} \quad (7.26)$$

where the last step follows from decomposing $\mathbb{Q}_0 = \mathbb{1} - \mathbb{P}_0$ and using that $\|(\mathbb{K}_0+1)^{1/2}\boldsymbol{\chi}_0\| \lesssim 1$ by Lemma 3.6. The case $\ell \neq 0$ is analogous to the computation for $\mathbb{O} = (z-\mathbb{G}_0)^{-1}$. □

Next, we combine Lemma 7.1 and the estimates on the remainder terms $\mathbb{R}_i$ and $\mathbb{G}_1$ im (7.4)-(7.2) resp. Lemma 5.1 to derive the following weigthed norm estimates.

Lemma 7.2. *Let $\ell \in \mathbb{Z}/2$, $\psi \in \mathcal{F}_\perp$ and $z \in \gamma$ for γ as in* (2.51) *and let*

$$\mathbb{O} \in \left\{ \frac{1}{z - \mathbb{G}_0}\,,\ \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \right\}.$$

Then

$$\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^\ell \mathbb{O}\,\mathbb{R}_0\psi\| \lesssim N^{\frac{\beta-1}{2}}\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^{\ell+\frac{3}{2}}\psi\|\,, \tag{7.27}$$

$$\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^\ell \mathbb{O}\,\mathbb{G}_1\psi\| \lesssim N^{\frac{\beta-1}{2}}\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^{\ell+1}\psi\|\,, \tag{7.28}$$

$$\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^\ell \mathbb{O}\,\mathbb{R}_1\psi\| \lesssim N^{\beta-1}\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^{\ell+\frac{3}{2}}\psi\|\,. \tag{7.29}$$

$$\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^\ell \mathbb{O}\,\mathbb{G}_2\psi\| \lesssim N^{\beta-1}\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^{\ell+\frac{3}{2}}\psi\|\,. \tag{7.30}$$

Proof. We prove the statement for $\mathbb{O} = (z-\mathbb{G}_0)^{-1}$, the case $\mathbb{O} = \mathbb{Q}_0(E_0 - \mathbb{G}_0)^{-1}$ works analogously. We start with the proof of (7.27). By (7.4) we find that

$$\begin{aligned}
&\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^\ell \frac{1}{z-\mathbb{G}_0}\mathbb{R}_0\psi\|^2 \\
&= \left\langle \frac{1}{z-\mathbb{G}_0}(\mathbb{K}_0+1)(\mathcal{N}_\perp+1)^{2\ell}\frac{1}{z-\mathbb{G}_0}\mathbb{R}_0\psi, \mathbb{R}_0\psi \right\rangle \\
&\lesssim N^{\frac{\beta-1}{2}}\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^{\frac{3}{2}+\ell}\psi\| \\
&\quad \times \|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^{-\ell}\frac{1}{z-\mathbb{G}_0}(\mathbb{K}_0+1)(\mathcal{N}_\perp+1)^{2\ell}\frac{1}{z-\mathbb{G}_0}\mathbb{R}_0\psi\|\,.
\end{aligned} \tag{7.31}$$

From Lemma 7.1 we get

$$\begin{aligned}
&\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^{-\ell}\frac{1}{z-\mathbb{G}_0}(\mathbb{K}_0+1)(\mathcal{N}_\perp+1)^{2\ell}\frac{1}{z-\mathbb{G}_0}\mathbb{R}_0\psi\| \\
&\quad\lesssim \|(\mathcal{N}_\perp+1)^\ell(\mathbb{K}_0+1)^{\frac{1}{2}}\frac{1}{z-\mathbb{G}_0}\mathbb{R}_0\psi\|
\end{aligned} \tag{7.32}$$

and thus arrive at

$$\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^\ell\frac{1}{z-\mathbb{G}_0}\mathbb{R}_0\psi\| \lesssim N^{\frac{\beta-1}{2}}\|(\mathbb{K}_0+1)^{\frac{1}{2}}(\mathcal{N}_\perp+1)^{\frac{3}{2}+\ell}\psi\|\,. \tag{7.33}$$

The other two estimates follows same with similar ideas using Lemma 5.1 and (7.2) instead of (7.4). □

7.3 Bounds on the resolvent of $\mathbb{G}$

Finally, we also need to control expressions involving a resolvent of the full generator $\mathbb{G}$.

Lemma 7.3. *Let $a \geq 5/2$, $\psi \in \mathcal{F}_\perp$ and $z \in \gamma$ for γ as in* (2.51). *Then*

$$\|(\mathbb{K}_0+1)^{\frac{1}{2}}\frac{1}{(\mathcal{N}_\perp+1)^a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\| \lesssim \|\psi\|. \tag{7.34}$$

Proof. Let $a \geq 5/2$. From (7.20) and $\mathbb{G}_2 \geq -\widetilde{\mathbb{R}}$ with $\widetilde{\mathbb{R}} := \frac{\widehat{v}(0)}{2N}\sum_{p,q\in\Lambda_+^*} c_p^2 c_q^2 a_p^\dagger a_q^\dagger a_p a_q$ as in (6.2), we get

$$\|\mathbb{K}_0^{1/2}\frac{1}{(\mathcal{N}_\perp+1)^a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|^2$$

$$\lesssim \left\langle \frac{1}{(\mathcal{N}_\perp+1)^a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi, \left(\mathbb{G}_0+\mathbb{G}_2+\widetilde{\mathbb{R}}\right)\frac{1}{(\mathcal{N}_\perp+1)^a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\right\rangle$$
$$= \left\langle \frac{1}{(\mathcal{N}_\perp+1)^{2a}}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi, \left(\mathbb{G}_0+\mathbb{G}_2+\widetilde{\mathbb{R}}\right)\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\right\rangle$$
$$+ \left|\left\langle \frac{1}{(\mathcal{N}_\perp+1)^a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi, \left(\sum_{p\in\Lambda_+^*} G_p(\left[(\mathcal{N}_++3)^{-a}-(\mathcal{N}_++1)^{-a}\right]a_pa_{-p}+\text{h.c.})\right)\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\right\rangle\right|. \tag{7.35}$$

Since $G_p \in \ell^2(\Lambda_+^*)$ from Lemma 3.5, the last term of the r.h.s. is bounded by a constant times $\|\psi\|^2$. Moreover, since $\mathbb{G}_0+\mathbb{G}_2=\mathbb{G}-\mathbb{G}_1-\mathbb{R}_2=\mathbb{G}-z+z-\mathbb{G}_1-\mathbb{R}_2$, we get

$$\|\mathbb{K}_0^{1/2}\frac{1}{(\mathcal{N}_\perp+1)^a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|^2 \lesssim \|\psi\|^2$$
$$+\left|\left\langle \frac{1}{(\mathcal{N}_\perp+1)^{2a}}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi, \mathbb{G}_1\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\right\rangle\right| \tag{7.36}$$
$$+\left|\left\langle \frac{1}{(\mathcal{N}_\perp+1)^{2a}}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi, \mathbb{R}_2\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\right\rangle\right| \tag{7.37}$$
$$+\left|\left\langle \frac{1}{(\mathcal{N}_\perp+1)^{2a}}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi, \widetilde{\mathbb{R}}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\right\rangle\right|. \tag{7.38}$$

We estimate the three terms separately and start with Lemma 5.1 by

$$\begin{aligned}(7.36) &\lesssim N^{\frac{\beta-1}{2}}\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{1-2a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|\|\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\| \\ &\quad + N^{\frac{\beta-1}{2}}\|(\mathcal{N}_\perp+1)^{1+a-2a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\| \\ &\lesssim N^{\frac{\beta-1}{2}}\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|\|\psi\|. \end{aligned} \tag{7.39}$$

From (7.2) we furthermore get

$$\begin{aligned}(7.37) &\lesssim N^{\frac{3}{2}(\beta-1)}\Big[\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{2-2a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|\|\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\| \\ &\qquad + \|(\mathcal{N}_\perp+1)^{2+a-2a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|\Big] \\ &\lesssim N^{\frac{3}{2}(\beta-1)}\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|\|\psi\|. \end{aligned} \tag{7.40}$$

Finally,

$$\begin{aligned}(7.38) &\lesssim N^{-1}\sum_{p,q\in\Lambda_+^*}\|(\mathcal{N}_\perp+1)a_pa_q(\mathcal{N}_\perp+1)^{-a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|\|(\mathcal{N}_\perp+1)^{-1}a_pa_q\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\| \\ &\lesssim N^{-1}\|\psi\|^2. \end{aligned} \tag{7.41}$$

In total, this shows

$$\begin{aligned}\|\mathbb{K}_0^{1/2}\frac{1}{(\mathcal{N}_\perp+1)^a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|^2 &\lesssim \|\psi\|^2 + N^{\frac{\beta-1}{2}}\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|\|\psi\| \\ &\lesssim \|\psi\|^2 + \frac{1}{2}\|\mathbb{K}_0^{1/2}(\mathcal{N}_\perp+1)^{-a}\frac{\mathbb{Q}}{z-\mathbb{G}}\psi\|^2. \end{aligned} \tag{7.42}$$

□

7.4 Estimate of $\mathrm{Tr}\mathbb{A}\mathbb{B}_P$

By definition of $\mathrm{Tr}\mathbb{A}\mathbb{B}_P$ in (7.7), it follows from (7.10) that

$$\begin{aligned} |\mathrm{Tr}\mathbb{A}\mathbb{B}_P| \lesssim \sup_{z\in\gamma} \Bigg(\left| \left\langle \boldsymbol{\chi}, \mathbb{R}_0 \frac{1}{z-\mathbb{G}_0} \mathbb{G}_2 \frac{1}{z-\mathbb{G}_0} \mathbb{A}\boldsymbol{\chi} \right\rangle \right| + \left| \left\langle \boldsymbol{\chi}, \mathbb{R}_0 \frac{1}{z-\mathbb{G}_0} \mathbb{G}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{G}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{A}\boldsymbol{\chi} \right\rangle \right| \\ + \left| \left\langle \boldsymbol{\chi}, \mathbb{R}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{G}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{A}\boldsymbol{\chi} \right\rangle \right| + \left| \left\langle \boldsymbol{\chi}, \mathbb{R}_2 \frac{1}{z-\mathbb{G}_0} \mathbb{A}\boldsymbol{\chi} \right\rangle \right| \Bigg) . \\ =: \mathcal{R}_{P1} + \mathcal{R}_{P2} + \mathcal{R}_{P3} + \mathcal{R}_{P4} . \end{aligned} \tag{7.43}$$

For the first term, Lemmas 5.1 and7.1 lead to the estimate

$$\begin{aligned} |\mathcal{R}_{P1}| &\lesssim \sup_{z\in\gamma} \left| \left\langle \boldsymbol{\chi}, \mathbb{G}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{G}_2 \frac{1}{z-\mathbb{G}_0} \mathbb{A}\boldsymbol{\chi} \right\rangle \right| \\ &\lesssim N^{\beta-1} \sup_{z\in\gamma} \|\mathcal{N}_\perp^{\frac{3}{2}} \mathbb{K}_0^{\frac{1}{2}} \frac{1}{z-\mathbb{G}_0} \boldsymbol{\chi}\| \|\mathcal{N}_\perp^{-\frac{1}{2}} \mathbb{K}_0^{\frac{1}{2}} \frac{1}{z-\mathbb{G}_0} \mathbb{G}_1 \mathbb{A}\boldsymbol{\chi}\| \\ &\lesssim N^{\beta-1} \sup_{z\in\gamma} \|(\mathcal{N}_\perp+1)^2 (\mathbb{K}_0+1)^{-\frac{1}{2}} \boldsymbol{\chi}\| \|(\mathbb{K}_0+1)^{-\frac{1}{2}} (\mathcal{N}_\perp+1)^{-\frac{1}{2}} \mathbb{G}_1 \mathbb{A}\boldsymbol{\chi}\| \\ &\lesssim N^{\frac{3}{2}(\beta-1)} \|(\mathbb{K}_0+1)^{\frac{1}{2}} (\mathcal{N}_\perp+1)^2 \boldsymbol{\chi}\|^2 , \end{aligned} \tag{7.44}$$

where we used for the last two estimates Lemma 7.2, the bound $\|\mathbb{A}\psi\| \lesssim \|A\| \|\mathcal{N}_\perp \psi\|$ and Lemma 6.2. For the second term, we find similarly with Lemmas 5.1, 7.1 and 7.2

$$\begin{aligned} |\mathcal{R}_{P2}| &\lesssim \sup_{z\in\gamma} \left| \left\langle \boldsymbol{\chi}, \mathbb{R}_0 \frac{1}{z-\mathbb{G}_0} \mathbb{G}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{G}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{A}\boldsymbol{\chi} \right\rangle \right| \\ &\lesssim N^{\frac{\beta-1}{2}} \sup_{z\in\gamma} \|(\mathcal{N}_\perp+1)^{\frac{1}{2}} (\mathbb{K}_0+1)^{\frac{1}{2}} \frac{1}{z-\mathbb{G}_0} \mathbb{R}_0 \boldsymbol{\chi}\| \\ &\qquad \times \|(\mathcal{N}_\perp+1)^{\frac{1}{2}} (\mathbb{K}_0+1)^{\frac{1}{2}} \frac{1}{z-\mathbb{G}_0} \mathbb{G}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{A}\boldsymbol{\chi}\| \\ &\lesssim N^{\frac{3}{2}(\beta-1)} \|(\mathbb{K}_0+1)^{\frac{1}{2}} (\mathcal{N}_\perp+1)^3 \boldsymbol{\chi}\|^2 \lesssim N^{3(\beta-1)/2} , \end{aligned} \tag{7.45}$$

where the last estimate is again a consequence of Lemma 6.2. For the third term we proceed analogously and find

$$\begin{aligned} |\mathcal{R}_{P3}| &\lesssim \sup_{z\in\gamma} \left| \left\langle \boldsymbol{\chi}, \mathbb{R}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{G}_1 \frac{1}{z-\mathbb{G}_0} \mathbb{A}\boldsymbol{\chi} \right\rangle \right| \\ &\lesssim N^{\frac{\beta-1}{2}} \sup_{z\in\gamma} \|(\mathcal{N}_\perp+1)(\mathbb{K}_0+1)^{\frac{1}{2}} \frac{1}{z-\mathbb{G}_0} \boldsymbol{\chi}\| \|(\mathbb{K}_0+1)^{\frac{1}{2}} \frac{1}{z-\mathbb{G}_0} \mathbb{R}_1 \mathbb{A}\boldsymbol{\chi}\| \\ &\lesssim N^{\frac{3}{2}(\beta-1)} \|(\mathbb{K}_0+1)^{\frac{1}{2}} (\mathcal{N}_\perp+1)^3 \boldsymbol{\chi}\|^2 \lesssim N^{3(\beta-1)/2} . \end{aligned} \tag{7.46}$$

Finally, the last term is given by

$$|\mathcal{R}_{P3}| \lesssim \sup_{z\in\gamma} \left| \left\langle \boldsymbol{\chi}, \mathbb{R}_2 \frac{1}{z-\mathbb{G}_0} \mathbb{A}\boldsymbol{\chi} \right\rangle \right| \lesssim N^{\frac{3}{2}(\beta-1)} \|(\mathbb{K}_0+1)^{\frac{1}{2}} (\mathcal{N}_\perp+1)^{\frac{5}{2}} \boldsymbol{\chi}\|^2 . \tag{7.47}$$

In summary, we find that

$$|\mathrm{Tr}\mathbb{A}\mathbb{B}_P| \lesssim N^{3(\beta-1)/2} . \tag{7.48}$$

7.5 Estimate of $\mathrm{Tr}\mathbb{A}\mathbb{B}_Q$

In this section we estimate $\mathrm{Tr}\mathbb{A}\mathbb{B}_P$ defined in (7.8). It follows from (7.10) that

$$|\mathrm{Tr}\mathbb{A}\mathbb{B}_Q| \lesssim \sup_{z\in\gamma}\Bigg(\left|\left\langle \boldsymbol{\chi}_0, \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\boldsymbol{\chi}_0\right\rangle \langle\boldsymbol{\chi}_0, \mathbb{G}_2\boldsymbol{\chi}_0\rangle\right| \tag{7.49a}$$

$$+\left|\left\langle \boldsymbol{\chi}_0, \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_2\boldsymbol{\chi}_0\right\rangle\right| \tag{7.49b}$$

$$+\left|\left\langle \boldsymbol{\chi}_0, \mathbb{G}_2\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\boldsymbol{\chi}_0\right\rangle\right| \tag{7.49c}$$

$$+\left|\left\langle \boldsymbol{\chi}_0, \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\boldsymbol{\chi}_0\right\rangle \left\langle \boldsymbol{\chi}_0\mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\right\rangle\right| \tag{7.49d}$$

$$+\left|\left\langle \boldsymbol{\chi}_0, \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\right\rangle\right| \tag{7.49e}$$

$$+\left|\left\langle \boldsymbol{\chi}_0, \mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\right\rangle\right| \tag{7.49f}$$

$$+\left|\left\langle \boldsymbol{\chi}_0, \mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_0\boldsymbol{\chi}_0\right\rangle\right| \tag{7.49g}$$

$$+\left|\left\langle \boldsymbol{\chi}_0, \mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_1\boldsymbol{\chi}_0\right\rangle\right| \tag{7.49h}$$

$$+\left|\left\langle \boldsymbol{\chi}_0, \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\right\rangle\right| \tag{7.49i}$$

$$+\left|\left\langle \boldsymbol{\chi}_0, \mathbb{A}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{R}_2\boldsymbol{\chi}_0\right\rangle\right|\Bigg). \tag{7.49j}$$

We estimate the terms separately:

Term (7.49a). For the first line of the r.h.s. of (7.49) we find with (7.4) and Lemma 5.2

$$\begin{aligned}|(7.49\mathrm{a})| \lesssim &\sup_{z\in\gamma} N^{(\beta-1)/2}\left\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-5/2}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{A}\boldsymbol{\chi}_0\right\| \left\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^4\boldsymbol{\chi}_0\right\| \\ &\times N^{\beta-1}\left\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{1/2}\boldsymbol{\chi}_0\right\|^2 .\end{aligned} \tag{7.50}$$

With Lemma 7.3 and using that $|z|\lesssim 1$ for $z\in\gamma$, we get for the first term of the r.h.s.

$$|(7.49\mathrm{a})| \lesssim N^{3(\beta-1)/2}\ \|\mathbb{A}\boldsymbol{\chi}_0\|\|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^4\boldsymbol{\chi}_0\|\ \|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{1/2}\boldsymbol{\chi}_0\|\,, \tag{7.51}$$

which we can finally bound since

$$\|\mathbb{A}\psi\| \lesssim \|\mathcal{N}_\perp\psi\| \tag{7.52}$$

for any $\psi\in\mathcal{F}$. With Lemma 3.6, recalling that $\boldsymbol{\chi}_0 = \mathbb{U}_\tau\Omega$, we find that

$$|(7.49\mathrm{a})| \lesssim N^{3(\beta-1)/2}\ . \tag{7.53}$$

Term (7.49b)**.** We proceed similarly and find with (7.4)

$$|(7.49\text{b})| \lesssim \sup_{z\in\gamma} N^{(\beta-1)/2} \left\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-5/2} \frac{\mathbb{Q}}{z-\mathbb{G}} \mathbb{A}\boldsymbol{\chi}_0 \right\| \times \left\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{4} \frac{\mathbb{Q}_0}{z-\mathbb{G}_0} \mathbb{G}_2\boldsymbol{\chi}_0 \right\| . \tag{7.54}$$

For the first term of the r.h.s. we use Lemma 7.3 and (7.52) and for the second term Lemma 7.1. This yields

$$|(7.49\text{b})| \lesssim N^{(\beta-1)/2} \, \|\mathcal{N}_\perp\boldsymbol{\chi}_0\| \left\| (\mathcal{N}_\perp+1)^{4}(\mathbb{K}_0+1)^{\frac{1}{2}} \frac{\mathbb{Q}_0}{z-\mathbb{G}_0} \mathbb{G}_2\boldsymbol{\chi}_0 \right\| . \tag{7.55}$$

With Lemmas 7.2 and 3.6, we find that

$$|(7.49\text{b})| \lesssim N^{3(\beta-1)/2} \, \|\mathcal{N}_\perp\boldsymbol{\chi}_0\| \, \|(\mathcal{N}_\perp+1)^{11/2}(\mathbb{K}_0+1)^{1/2}\boldsymbol{\chi}_0\| \lesssim N^{3(\beta-1)/2} . \tag{7.56}$$

Term (7.49c) From (7.4) we find

$$|(7.49\text{c})| \lesssim \sup_{z\in\gamma} N^{(\beta-1)/2} \left\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-5/2} \frac{\mathbb{Q}}{z-\mathbb{G}} \mathbb{A} \frac{\mathbb{Q}_0}{z-\mathbb{G}_0} \mathbb{G}_2\boldsymbol{\chi}_0 \right\| \times \left\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{4}\boldsymbol{\chi}_0 \right\| . \tag{7.57}$$

We use Lemmas 7.3, 7.1 and (7.52) for the first and Lemma 3.6 for the second term and find (similarly as before)

$$|(7.49\text{c})| \lesssim \sup_{z\in\gamma} N^{(\beta-1)/2} \left\| \mathcal{N}_\perp \frac{\mathbb{Q}_0}{z-\mathbb{G}_0} \mathbb{G}_2\boldsymbol{\chi}_0 \right\| . \tag{7.58}$$

The remaining term can be estimated with Lemma 7.2 and thus we finally get

$$|(7.49\text{c})| \lesssim N^{3(\beta-1)/2} \left\| (\mathcal{N}_\perp+1)^{2}(\mathbb{K}_0+1)^{1/2}\boldsymbol{\chi}_0 \right\| \lesssim N^{3(\beta-1)/2} . \tag{7.59}$$

Term (7.49d) It follows immediately from the estimate in (7.49a) and Lemma 7.2 that

$$|(7.49\text{d})| \lesssim N^{\frac{3}{2}(\beta-1)} . \tag{7.60}$$

Term (7.49e)**.** We use (7.4) first and afterwards as above Lemmas 7.3, 7.1, 7.2, 3.6 and (7.52) to obtain

$$\begin{aligned} |(7.49\text{e})| &\lesssim \sup_{z\in\gamma} N^{(\beta-1)/2} \left\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-5/2} \frac{\mathbb{Q}}{z-\mathbb{G}} \mathbb{A}\boldsymbol{\chi}_0 \right\| \\ &\quad \times \left\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{4} \frac{\mathbb{Q}_0}{z-\mathbb{G}_0} \mathbb{G}_1 \frac{\mathbb{Q}_0}{z-\mathbb{G}_0} \mathbb{G}_1\boldsymbol{\chi}_0 \right\| \\ &\lesssim \sup_{z\in\gamma} N^{\beta-1} \left\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{5} \frac{\mathbb{Q}_0}{z-\mathbb{G}_0} \mathbb{G}_1\boldsymbol{\chi}_0 \right\| \, \|\mathcal{N}_\perp\boldsymbol{\chi}_0\| \lesssim N^{3(\beta-1)/2} . \end{aligned} \tag{7.61}$$

Term (7.49f). Formula (7.4) with Lemmas 7.3, 7.1, 7.2, 3.6 and well as (7.52) lead to

$$\begin{aligned}|(7.49\text{f})| \lesssim & \sup_{z\in\gamma} N^{(\beta-1)/2} \Big\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-5/2}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{A}\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\Big\| \\ & \qquad \times \|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^4\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\ \boldsymbol{\chi}_0\| \\ \lesssim & N^{(\beta-1)/2}\|\mathcal{N}_\perp\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\|\ \|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^4\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\ \boldsymbol{\chi}_0\| \\ \lesssim & N^{3(\beta-1)/2}\,. \end{aligned} \tag{7.62}$$

Term (7.49g). Similarly as before we use (7.4) first, and then Lemmas 7.3, 7.1, 7.2 and 3.6. We obtain

$$\begin{aligned}|(7.49\text{g})| \lesssim & \sup_{z\in\gamma} N^{(\beta-1)/2} \Big\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-5/2}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{A}\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\Big\| \\ \lesssim & N^{3(\beta-1)/2}\,. \end{aligned} \tag{7.63}$$

Term (7.49h). As before we use (7.3), Lemmas 7.3, 7.1, 7.2 and 3.6 as well as the estimate (7.52) to compute

$$\begin{aligned}|(7.49\text{h})| \lesssim & \sup_{z\in\gamma} N^{\beta-1} \Big\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-5/2}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{A}\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\Big\| \\ & \qquad \times \|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^4\boldsymbol{\chi}_0\| \\ \lesssim & N^{3(\beta-1)/2}\,. \end{aligned} \tag{7.64}$$

Term (7.49i). We combine (7.3), 7.3, 7.1, 7.2 and (7.52), which yields

$$\begin{aligned}|(7.49\text{i})| \lesssim & \sup_{z\in\gamma} N^{\beta-1} \Big\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-5/2}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{A}\boldsymbol{\chi}_0\Big\| \\ & \qquad \times \|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^4\frac{\mathbb{Q}_0}{z-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\| \\ \lesssim & N^{3(\beta-1)/2}\,. \end{aligned} \tag{7.65}$$

Term (7.49j). For the last term we proceed similarly as before. With (7.2), Lemma 7.3 and 3.6, we conclude that

$$\begin{aligned}|(7.49\text{j})| \lesssim & \sup_{z\in\gamma} N^{3(\beta-1)/2} \Big\| (\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^{-5/2}\frac{\mathbb{Q}}{z-\mathbb{G}}\mathbb{A}\boldsymbol{\chi}_0\Big\| \\ & \qquad \times \|(\mathbb{K}_0+1)^{1/2}(\mathcal{N}_\perp+1)^4\boldsymbol{\chi}_0\| \\ \lesssim & N^{\frac{3}{2}(\beta-1)}\,. \end{aligned} \tag{7.66}$$

Summary. Thus, in summary, we have proven that

$$|\mathrm{Tr}\mathbb{A}\mathbb{B}_Q| \lesssim N^{3(\beta-1)/2}\,. \tag{7.67}$$

8 Explicit calculation of energy correction

8.1 Calculation for $E_{0,1}$

Recall that $E_{0,1}$ is defined in (4.28). Here, we prove that for $\beta \in \left(\frac{1}{2}, 1\right)$ we have

$$E_{0,1} = C^{(1)}_{N,\beta}\left(-\frac{1}{2N}\sum_{q\in\Lambda_+^*} \frac{\widehat{v}^\beta_N(q)^2}{2q^2}\right) + O(N^{2(\beta-1)}), \tag{8.1}$$

with

$$C^{(1)}_{N,\beta} = \frac{1}{2}\sum_{p\in\Lambda_+^*}(s_p c_p - \eta_p) + \widehat{v}(0)^2 \sum_{p\in\Lambda_+^*} \frac{1}{\sqrt{|p|^4 + 2p^2\widehat{v}(0)}\left(p^2 + \sqrt{|p|^4+2p^2\widehat{v}(0)}\right)}. \tag{8.2}$$

Note that $|C^{(1)}_{N,\beta}| \lesssim 1$. We define

$$E^{(1)}_{0,1} := -\frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq q}} \widehat{v}^\beta_N(p-q)(s_pc_p - \eta_p)\left[s_qc_q + \frac{\widehat{v}^\beta_N(q)}{q^2}\right], \tag{8.3}$$

and

$$E^{(2)}_{0,1} := \frac{1}{N}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq q}} \frac{\widehat{v}^\beta_N(p)^2 \ \ \widehat{v}^\beta_N(p-q)s_qc_q}{\sqrt{|p|^4+2p^2\widehat{v}^\beta_N(p)}\left(p^2 + \sqrt{|p|^4+2p^2\widehat{v}^\beta_N(p)}\right)}, \tag{8.4}$$

then it follows that $E_{0,1} = E^{(1)}_{0,1} + E^{(2)}_{0,1}$. Note that the scattering equation (2.23) implies that

$$\left|\eta_p + \frac{\widehat{v}^\beta_N(p)}{2p^2}\right| = \frac{1}{2N}|p|^{-2}\left|\sum_{q\in\Lambda_+^*}\widehat{v}^\beta_N(p-q)\eta_q\right| \lesssim N^{\beta-1}|p|^{-2} \tag{8.5}$$

by Lemmas 3.1 and 3.2. Also, since v has compact support, we have the estimate

$$\left|\widehat{v}^\beta_N(p-q) - \widehat{v}^\beta_N(q)\right| \le N^{-\beta}|p|\int_0^1 |\nabla \widehat{v}^\beta_N(tp-q)|\,\mathrm{d}t \lesssim N^{-\beta}|p|. \tag{8.6}$$

By an expansion of the square root as in [4, Sec. 5], this implies the pointwise estimate

$$\left|\sqrt{|p|^4+2p^2\widehat{v}^\beta_N(p)} - \sqrt{|p|^4+2p^2\widehat{v}(0)}\right| \lesssim N^{-\beta}|p|. \tag{8.7}$$

Simplification of $E^{(1)}_{0,1}$. We write

$$E^{(1)}_{0,1} = \left(\frac{1}{2}\sum_{p\in\Lambda_+^*}(s_pc_p-\eta_p)\right)\left(-\frac{1}{2N}\sum_{q\in\Lambda_+^*}\frac{\widehat{v}^\beta_N(q)^2}{q^2}\right) \tag{8.8a}$$

$$-\frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq q}} \widehat{v}^\beta_N(p-q)(s_pc_p-\eta_p)\left[s_qc_q - \eta_q + \eta_q + \frac{\widehat{v}^\beta_N(q)}{2q^2}\right] \tag{8.8b}$$

$$+\frac{1}{2N}\sum_{p\in\Lambda_+^*}(s_pc_p-\eta_p)\frac{\widehat{v}_N^\beta(p)^2}{2p^2} \tag{8.8c}$$

$$-\frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq q}}\Big(\widehat{v}_N^\beta(p-q)-\widehat{v}_N^\beta(q)\Big)(s_pc_p-\eta_p)\frac{\widehat{v}_N^\beta(q)}{2q^2}. \tag{8.8d}$$

Then, using Lemma 3.1, Lemma 3.3, and (8.5), we find that

$$|(8.8\text{b})|\lesssim N^{-1}\sum_{p,q\in\Lambda_+^*,p\neq q}|\widehat{v}_N^\beta(p-q)|\,|p|^{-6}\Big(|q|^{-6}+N^{\beta-1}|q|^{-2}\Big)\lesssim N^{-1}+N^{2(\beta-1)}, \tag{8.9}$$

and

$$|(8.8\text{c})|\lesssim N^{-1}\sum_{p\in\Lambda_+^*}\frac{\widehat{v}_N^\beta(p)^2}{2p^2}|p|^{-6}\lesssim N^{-1}. \tag{8.10}$$

Furthermore, by Lemma 3.1, Lemma 3.3, and (8.6) we find

$$|(8.8\text{d})|\lesssim N^{-1}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq q}}N^{-\beta}|p||p|^{-6}\frac{|\widehat{v}_N^\beta(q)|}{q^2}\lesssim N^{-1}. \tag{8.11}$$

In conclusion, for $\beta\in\big(\frac{1}{2},1\big)$ we have

$$E_{0,1}^{(1)}=\Bigg(\frac{1}{2}\sum_{p\in\Lambda_+^*}(s_pc_p-\eta_p)\Bigg)\Bigg(-\frac{1}{2N}\sum_{q\in\Lambda_+^*}\frac{\widehat{v}_N^\beta(q)^2}{2q^2}\Bigg)+\mathcal{O}(N^{2(\beta-1)}). \tag{8.12}$$

Simplification of $E_{0,1}^{(2)}$. We start by writing

$$E_{0,1}^{(2)}=-\frac{1}{N}\sum_{p,q\in\Lambda_+^*}\frac{\widehat{v}_N^\beta(0)^2\;\;\widehat{v}_N^\beta(q)}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big)}\Bigg(\frac{\widehat{v}_N^\beta(q)}{2q^2}\Bigg) \tag{8.13a}$$

$$+\frac{1}{N}\sum_{p\in\Lambda_+^*}\frac{\widehat{v}_N^\beta(0)^2\;\;\widehat{v}_N^\beta(p)}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big)}\Bigg(\frac{\widehat{v}_N^\beta(p)}{2p^2}\Bigg) \tag{8.13b}$$

$$-\frac{1}{N}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq q}}\frac{\Big(\widehat{v}_N^\beta(p)^2\;\;\widehat{v}_N^\beta(p-q)-\widehat{v}_N^\beta(0)^2\;\;\widehat{v}_N^\beta(q)\Big)}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big)}\Bigg(\frac{\widehat{v}_N^\beta(q)}{2q^2}\Bigg) \tag{8.13c}$$

$$+\frac{1}{N}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq q}}\frac{\widehat{v}_N^\beta(p)^2\;\;\widehat{v}_N^\beta(p-q)}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big)}\Bigg(s_qc_q-\eta_q+\eta_q+\frac{\widehat{v}_N^\beta(q)}{2q^2}\Bigg). \tag{8.13d}$$

We directly find $|(8.13\text{b})|\lesssim N^{-1}$. Next, using (8.6) for momenta $|p|\le N^\beta$ and $\sup_{p\in\Lambda_+^*}\big|\widehat{v}_N^\beta(p)\big|\lesssim 1$ for momenta $|p|\ge N^\beta$, together with Lemma 3.1, we find

$$|(8.13\text{c})|\lesssim\frac{1}{N}\sum_{\substack{p,q\in\Lambda_+^*,p\neq q\\ |p|\le N^\beta}}\frac{N^{-\beta}|p|}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\Big)}\Bigg(\frac{\widehat{v}_N^\beta(q)^2}{2q^2}\Bigg)$$

$$+\frac{1}{N}\sum_{\substack{p,q\in\Lambda_+^*,p\neq q\\|p|\geq N^\beta}}\frac{1}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\left(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\right)}\left(\frac{\left|\widehat{v}_N^\beta(q)\right|}{2q^2}\right)$$
$$\lesssim N^{-1}\ln(N). \tag{8.14}$$

Furthermore, by Lemma 3.3 and (8.5),

$$|(8.13\mathrm{d})|\lesssim N^{-1}+N^{2(\beta-1)}. \tag{8.15}$$

Finally, we use (8.7) to deduce that

$$\sum_{p\in\Lambda_+^*}\left|\frac{1}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\left(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\right)}-\frac{1}{\sqrt{|p|^4+2p^2\widehat{v}(0)}\left(p^2+\sqrt{|p|^4+2p^2\widehat{v}(0)}\right)}\right|$$
$$=\sum_{p\in\Lambda_+^*}\left|\frac{p^2\left(\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}-\sqrt{|p|^4+2p^2\widehat{v}(0)}\right)+2p^2\left(\widehat{v}_N^\beta(p)-\widehat{v}(0)\right)}{\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\left(p^2+\sqrt{|p|^4+2p^2\widehat{v}_N^\beta(p)}\right)\sqrt{|p|^4+2p^2\widehat{v}(0)}\left(p^2+\sqrt{|p|^4+2p^2\widehat{v}(0)}\right)}\right|$$
$$\lesssim\sum_{p\in\Lambda_+^*}\frac{|p|^3N^{-\beta}}{p^8}\lesssim N^{-\beta}, \tag{8.16}$$

which proves, for $\beta\in\left(\frac{1}{2},1\right)$, that

$$E_{0,1}^{(2)}=\left(\widehat{v}(0)^2\sum_{p\in\Lambda_+^*}\frac{1}{\sqrt{|p|^4+2p^2\widehat{v}(0)}\left(p^2+\sqrt{|p|^4+2p^2\widehat{v}(0)}\right)}\right)\left(-\frac{1}{2N}\sum_{q\in\Lambda_+^*}\frac{\widehat{v}_N^\beta(q)^2}{2q^2}\right)$$
$$+\mathcal{O}(N^{2(\beta-1)}). \tag{8.17}$$

8.2 Calculation for E_{pert}

The goal of this section is to estimate and explicitly calculate the leading order contribution of

$$E_{\mathrm{pert}}:=\langle\boldsymbol{\chi}_0,\mathbb{G}_2\boldsymbol{\chi}_0\rangle+\left\langle\boldsymbol{\chi}_0,\mathbb{G}_1\frac{\mathbb{Q}_0}{E_0-\mathbb{G}_0}\mathbb{G}_1\boldsymbol{\chi}_0\right\rangle \tag{8.18}$$

with $\boldsymbol{\chi}_0=\mathbb{U}_\tau^*|\Omega\rangle$. To this end, we introduce the shorthand notation

$$\widetilde{c}_p=\cosh(\tau_p)\quad\text{and}\quad\widetilde{s}_p=\sinh(\tau_p). \tag{8.19}$$

By the definition of $\mathbb{G}_2$ we get

$$\langle\boldsymbol{\chi}_0,\mathbb{G}_2\boldsymbol{\chi}_0\rangle=\frac{1}{2N}\sum_{\substack{p,q,r\in\Lambda_+^*\\p+r\neq0,q+r\neq0}}\widehat{v}_N^\beta(r)c_{p+r}c_qc_pc_{q+r}\left\langle\Omega,\mathbb{U}_\tau a_{p+r}^\dagger a_q^\dagger a_pa_{q+r}\mathbb{U}_\tau^*\Omega\right\rangle. \tag{8.20}$$

Note that

$$\left\langle\Omega,\mathbb{U}_\tau a_{p+r}^\dagger a_q^\dagger a_pa_{q+r}\mathbb{U}_\tau^*\Omega\right\rangle$$

$$
\begin{aligned}
&= \widetilde{s}_{p+r}\widetilde{s}_{q+r} \left\langle \Omega, a_{-(p+r)} \mathbb{U}_\tau a_q^\dagger a_p \mathbb{U}_\tau^* a_{-(q+r)}^\dagger \Omega \right\rangle \\
&= \widetilde{s}_{p+r}\widetilde{s}_{q+r} \left\langle \Omega, a_{-(p+r)} \left(\widetilde{c}_q a_q^\dagger + \widetilde{s}_q a_{-q} \right) \left(\widetilde{c}_p a_p + \widetilde{s}_p a_{-p}^\dagger \right) a_{-(q+r)}^\dagger \Omega \right\rangle \\
&= \widetilde{s}_{p+r}\widetilde{s}_{q+r}\widetilde{c}_q\widetilde{c}_p \left\langle \Omega, a_{-(p+r)} a_q^\dagger a_p a_{-(q+r)}^\dagger \Omega \right\rangle + \widetilde{s}_{p+r}\widetilde{s}_{q+r}\widetilde{s}_q\widetilde{s}_p \left\langle \Omega, a_{-(p+r)} a_{-q} a_{-p}^\dagger a_{-(q+r)}^\dagger \Omega \right\rangle \\
&= \widetilde{s}_{p+r}\widetilde{s}_{q+r}\widetilde{c}_q\widetilde{c}_p \delta_{-(p+r),q}\delta_{-(q+r),p} + \widetilde{s}_{p+r}\widetilde{s}_{q+r}\widetilde{s}_q\widetilde{s}_p \left(\delta_{p,q}\delta_{r,0} + \delta_{r,0} \right). \qquad (8.21)
\end{aligned}
$$

Since $\delta_{-(q+r),p} = \delta_{-(p+r),q}, \delta_{r,0} = 0$ for all $r \in \Lambda_+^*$, $\widetilde{s}_{-p} = \widetilde{s}_p$ and $\widetilde{c}_{-p} = \widetilde{c}_p$ we obtain

$$
\begin{aligned}
\langle \boldsymbol{\chi}_0, \mathbb{G}_2 \boldsymbol{\chi}_0 \rangle &= \frac{1}{2N} \sum_{\substack{p,q,r \in \Lambda_+^* \\ p+r \neq 0, q+r \neq 0}} \widehat{v}_N^\beta(r) c_{p+r} c_q c_p c_{q+r} \widetilde{s}_{p+r} \widetilde{s}_{q+r} \widetilde{c}_q \widetilde{c}_p \delta_{-(p+r),q} \\
&= \frac{1}{2N} \sum_{\substack{p,r \in \Lambda_+^* \\ p+r \neq 0}} \widehat{v}_N^\beta(r) c_{p+r}^2 c_p^2 \widetilde{s}_{p+r} \widetilde{s}_p \widetilde{c}_{p+r} \widetilde{c}_p. \qquad (8.22)
\end{aligned}
$$

Using (3.24) we estimate

$$
|\langle \boldsymbol{\chi}_0, \mathbb{G}_2 \boldsymbol{\chi}_0 \rangle| \leq C N^{-1} \|\widehat{v}\|_\infty \sum_{\substack{p,r \in \Lambda_+^* \\ p+r \neq 0}} |\tau_{p+r}|\,|\tau_p| \leq C N^{-1} \|\widehat{v}\|_\infty \sum_{\substack{p,r \in \Lambda_+^* \\ p+r \neq 0}} p^{-4}(p+r)^{-4} \leq C N^{-1}, \qquad (8.23)
$$

showing that the contribution of $\langle \boldsymbol{\chi}_0, \mathbb{G}_2 \boldsymbol{\chi}_0 \rangle$ is subleading. Next, we use the splitting (5.10) and consider

$$
\begin{aligned}
\left\langle \boldsymbol{\chi}_0, \mathbb{G}_1 \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \mathbb{G}_1 \boldsymbol{\chi}_0 \right\rangle &= \left\langle \boldsymbol{\chi}_0, \widetilde{\mathbb{G}}_1 \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \widetilde{\mathbb{G}}_1 \boldsymbol{\chi}_0 \right\rangle \qquad (8.24) \\
&\quad + \left\langle \boldsymbol{\chi}_0, \mathbb{R}_d \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \mathbb{R}_d \boldsymbol{\chi}_0 \right\rangle \qquad (8.25) \\
&\quad + 2\mathrm{Re} \left\langle \boldsymbol{\chi}_0, \widetilde{\mathbb{G}}_1 \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \mathbb{R}_d \boldsymbol{\chi}_0 \right\rangle. \qquad (8.26)
\end{aligned}
$$

The first term. A direct computation leads to

$$
\mathbb{U}_\tau \widetilde{\mathbb{G}}_1 \mathbb{U}_\tau^* |\Omega\rangle = \frac{1}{\sqrt{N}} \sum_{\substack{p,q \in \Lambda_+^* \\ p+q \neq 0}} f(p,q) a_q^\dagger a_p^\dagger a_{-p-q}^\dagger |\Omega\rangle, \qquad (8.27)
$$

with

$$
\begin{aligned}
f(p,q) = \frac{1}{6} c_{p+q} c_p c_q \Big(& \widehat{v}_N^\beta(p)(\widetilde{c}_p + \widetilde{s}_p)(\widetilde{c}_{p+q}\widetilde{s}_q + \widetilde{c}_q \widetilde{s}_{p+q}) + \widehat{v}_N^\beta(q)(\widetilde{c}_q + \widetilde{s}_q)(\widetilde{c}_{p+q}\widetilde{s}_p + \widetilde{c}_p \widetilde{s}_{p+q}) \\
& + \widehat{v}_N^\beta(p+q)(\widetilde{c}_{p+q} + \widetilde{s}_{p+q})(\widetilde{c}_p \widetilde{s}_q + \widetilde{c}_q \widetilde{s}_p) \Big) \\
+ \frac{1}{3} \Big(\widetilde{c}_p \widetilde{c}_q \widetilde{c}_{p+q} + \widetilde{s}_p \widetilde{s}_q \widetilde{s}_{p+q} \Big) \Big(& \widehat{v}_N^\beta(p) c_p (c_{p+q} s_q + c_q s_{p+q}) + \widehat{v}_N^\beta(q) c_q (c_{p+q} s_p + c_p s_{p+q}) \\
& + \widehat{v}_N^\beta(p+q) c_{p+q} (c_p s_q + c_q s_p) \Big). \qquad (8.28)
\end{aligned}
$$

Note that this function is written in a symmetric way, i.e., $f(p,q) = f(q,p) = f(-p-q,q) = f(p,-p-q)$. Together with

$$\mathbb{U}_\tau \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \mathbb{U}_\tau^* a_{p_1}^\dagger \dots a_{p_n}^\dagger |\Omega\rangle = -\frac{1}{\mathfrak{e}(p_1) + \dots + \mathfrak{e}(p_n)} a_{p_1}^\dagger \dots a_{p_n}^\dagger |\Omega\rangle, \tag{8.29}$$

where $\mathfrak{e}(p) = \sqrt{F_p^2 - G_p^2}$, this leads to

$$\begin{aligned}
&\left\langle \boldsymbol{\chi}_0, \widetilde{\mathbb{G}}_1 \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \widetilde{\mathbb{G}}_1 \boldsymbol{\chi}_0 \right\rangle \\
&\quad = \left\langle \mathbb{U}_\tau \widetilde{\mathbb{G}}_1 \mathbb{U}_\tau^* \Omega, \mathbb{U}_\tau \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \mathbb{U}_\tau^* \mathbb{U}_\tau \widetilde{\mathbb{G}}_1 \mathbb{U}_\tau^* \Omega \right\rangle \\
&\quad = -\frac{1}{N} \sum_{\substack{p,q \in \Lambda_+^* \\ p+q \neq 0}} \sum_{\substack{r,s \in \Lambda_+^* \\ r+s \neq 0}} \frac{f(p,q) f(r,s)}{\mathfrak{e}(p+q) + \mathfrak{e}(-p) + \mathfrak{e}(-q)} \left\langle a_{p+q}^\dagger a_{-p}^\dagger a_{-q}^\dagger \Omega, a_{r+s}^\dagger a_{-r}^\dagger a_{-s}^\dagger \Omega \right\rangle \\
&\quad = -\frac{6}{N} \sum_{\substack{p,q \in \Lambda_+^* \\ p+q \neq 0}} \frac{f(p,q)^2}{\mathfrak{e}(p+q) + \mathfrak{e}(-p) + \mathfrak{e}(-q)}.
\end{aligned} \tag{8.30}$$

From Lemma 3.5 we find that $\mathfrak{e}(p) \gtrsim |p|^2$, and thus $(\mathfrak{e}(p+q) + \mathfrak{e}(-p) + \mathfrak{e}(-q))^{-1} \lesssim |p|^{-2}$. Using this, Cauchy–Schwarz, Lemma 3.3, $|\widetilde{c}_p| \lesssim 1$, $|\widetilde{s}_p| \lesssim |p|^{-4}$, and the symmetry of $f(p,q)$, we can estimate

$$\left| \left\langle \boldsymbol{\chi}_0, \widetilde{\mathbb{G}}_1 \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \widetilde{\mathbb{G}}_1 \boldsymbol{\chi}_0 \right\rangle \right| \lesssim N^{-1} \sum_{\substack{p,q \in \Lambda_+^* \\ p+q \neq 0}} |p|^{-2} \left(\widehat{v}_N^\beta(p) \right)^2 |q|^{-4} \lesssim N^{\beta - 1}, \tag{8.31}$$

where the last step follows from Lemma 3.1.

The second term. Note that Lemma 5.1 implies

$$\|\mathbb{R}_d \boldsymbol{\chi}_0\|^2 = \langle \mathbb{R}_d \boldsymbol{\chi}_0, \mathbb{R}_d \boldsymbol{\chi}_0 \rangle \lesssim N^{-\frac{1}{2}} \|\mathbb{R}_d \boldsymbol{\chi}_0\| \|(\mathcal{N}_\perp + 1)^{\frac{3}{2}} \boldsymbol{\chi}_0\| \tag{8.32}$$

and therefore

$$\|\mathbb{R}_d \boldsymbol{\chi}_0\| \lesssim N^{-\frac{1}{2}} \|(\mathcal{N}_\perp + 1)^{\frac{3}{2}} \boldsymbol{\chi}_0\|. \tag{8.33}$$

Together with Lemma 3.6 we get

$$\left| \left\langle \boldsymbol{\chi}_0, \mathbb{R}_d \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \mathbb{R}_d \boldsymbol{\chi}_0 \right\rangle \right| \lesssim N^{-1} \|(\mathcal{N}_\perp + 1)^{\frac{3}{2}} \boldsymbol{\chi}_0\|^2 \lesssim N^{-1}, \tag{8.34}$$

showing that the left hand side only has a subleading contribution which can be absorbed in the overall error in our main theorem.

The third term. Having estimated the first and second term, the third term can immediately be bound by Cauchy–Schwarz, i.e.,

$$\left| \left\langle \boldsymbol{\chi}_0, \widetilde{\mathbb{G}}_1 \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \mathbb{R}_d \boldsymbol{\chi}_0 \right\rangle \right| \lesssim \sqrt{N^{-1}} \sqrt{N^{\beta - 1}} = N^{\beta/2 - 1} \lesssim N^{\frac{3}{2}(\beta - 1)} \tag{8.35}$$

for $\beta \in (\frac{1}{2}, 1)$.

Summary. We have shown that for $\beta \in (\frac{1}{2}, 1)$,

$$E_{\text{pert}} = \widetilde{E}_{\text{pert}} + \mathcal{O}\big(N^{\frac{3}{2}(\beta-1)}\big), \quad \text{with } \widetilde{E}_{\text{pert}} := -\frac{6}{N} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \frac{f(p,q)^2}{\mathfrak{e}(p+q)+\mathfrak{e}(p)+\mathfrak{e}(q)} \,. \tag{8.36}$$

Next, we simplify $\widetilde{E}_{\text{pert}}$. Recall that $f(p,q)$ is defined in (8.28), and $\mathfrak{e}(p) = \sqrt{F_p^2 - G_p^2}$, with F_p and G_p defined in (2.34). Our goal is to prove that

$$\widetilde{E}_{\text{pert}} = C_{N,\beta}^{(2)} \left(-\frac{1}{2N} \sum_{p\in\Lambda_+^*} \frac{\widehat{v}_N^\beta(p)^2}{2p^2} \right) + O(N^{-1}(\ln N)^2), \tag{8.37}$$

with

$$C_{N,\beta}^{(2)} = \sum_{q\in\Lambda_+^*} 4\Big(c_q\widetilde{s}_q + 2\widetilde{c}_q s_q\Big)^2. \tag{8.38}$$

Note that $|C_{N,\beta}^{(2)}| \lesssim 1$. We first collect a few preparatory estimates. Lemma 3.5 implies that

$$p^2 \lesssim \mathfrak{e}(p) \lesssim p^2 \tag{8.39}$$

for $p \in \Lambda_+^*$, and thus we have

$$\frac{1}{\mathfrak{e}(p+q)+\mathfrak{e}(p)+\mathfrak{e}(q)} \lesssim \frac{1}{p^2}, \frac{1}{q^2}, \frac{1}{|p|\,|q|}. \tag{8.40}$$

Furthermore, (4.5) implies that

$$\begin{aligned} \big|p^2 - \mathfrak{e}(p)\big| &= p^2 \left| 1 - \sqrt{1 + \frac{2\widehat{v}_N^\beta(p)}{p^2} + \frac{A_p}{p^4}} \right| \\ &= p^2 \left| \frac{1}{2} \left(\frac{2\widehat{v}_N^\beta(p)}{p^2} + \frac{A_p}{p^4} \right) \int_0^1 \mathrm{d}s \left(1 + s \left(\frac{\widehat{v}_N^\beta(p)}{p^2} + \frac{A_p}{p^4} \right) \right)^{-1/2} \right| \\ &\lesssim 1 \,, \end{aligned} \tag{8.41}$$

for A_p as in (4.6) with $|A_p| \lesssim N^{\beta-1}$ by (4.7). We start to estimate (8.36) by using Lemma 3.3 and $|\widetilde{s}_p| \lesssim |p|^{-4}$, $|\widetilde{c}_p - 1| \lesssim |p|^{-8}$. Then many contributions of $f(p,q)$ can be summed up, and renaming summation indices if necessary leads to

$$\widetilde{E}_{\text{pert}} = -\frac{1}{4N} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \frac{\widehat{v}_N^\beta(p)\Big(\widehat{v}_N^\beta(p) + \widehat{v}_N^\beta(p+q)\Big) f(q)}{\mathfrak{e}(p+q)+\mathfrak{e}(p)+\mathfrak{e}(q)} + \mathcal{O}((\ln N)^2 N^{-1}), \tag{8.42}$$

with

$$f(q) := 4\Big(c_q\widetilde{s}_q + 2\widetilde{c}_q s_q\Big)^2. \tag{8.43}$$

Note that here for the terms which decay only like $|p|^{-3}|q|^{-3}$ we have used that for any $r \in \Lambda_+^*$,

$$\sum_{p\in\Lambda_+^*} \frac{|\widehat{v}_N^\beta(p+r)|}{|p|^3} = \sum_{\substack{p\in\Lambda_+^* \\ |p|\leq N^\beta}} \frac{|\widehat{v}_N^\beta(p+r)|}{|p|^3} + \sum_{\substack{p\in\Lambda_+^* \\ |p|>N^\beta}} \frac{|\widehat{v}_N^\beta(p+r)|}{|p|^3}$$

$$\le \Big(\sup_{p\in\Lambda_+^*} |\widehat{v}_N^\beta(p)|\Big) \sum_{\substack{p\in\Lambda_+^*\\ |p|\le N^\beta}} \frac{1}{|p|^3} + \sqrt{\sum_{p\in\Lambda_+^*} |\widehat{v}_N^\beta(p)|^2} \sqrt{\sum_{\substack{p\in\Lambda_+^*\\ |p|>N^\beta}} \frac{1}{|p|^6}}$$

$$\lesssim \ln(N) + \sqrt{N^{3\beta}}\sqrt{N^{-3\beta}}. \tag{8.44}$$

Note that $|f(q)| \lesssim |q|^{-4}$. We then split the remaining double sum in (8.42) as

$$-\frac{1}{4N}\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}} \frac{\widehat{v}_N^\beta(p)\Big(\widehat{v}_N^\beta(p)+\widehat{v}_N^\beta(p+q)\Big)f(q)}{\mathfrak{e}(p+q)+\mathfrak{e}(p)+\mathfrak{e}(q)}$$

$$= -\frac{1}{4N}\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}} \frac{\widehat{v}_N^\beta(p)\Big(\widehat{v}_N^\beta(p+q)-\widehat{v}_N^\beta(p)\Big)f(q)}{\mathfrak{e}(p+q)+\mathfrak{e}(p)+\mathfrak{e}(q)} \tag{8.45a}$$

$$-\frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}} \widehat{v}_N^\beta(p)^2 f(q)\left(\frac{1}{\mathfrak{e}(p+q)+\mathfrak{e}(p)+\mathfrak{e}(q)} - \frac{1}{2p^2}\right) \tag{8.45b}$$

$$-\frac{1}{2N}\sum_{p\in\Lambda_+^*} \frac{\widehat{v}_N^\beta(p)^2}{2p^2} \sum_{q\in\Lambda_+^*} f(q) \tag{8.45c}$$

$$+\frac{1}{2N}\sum_{p\in\Lambda_+^*} \frac{\widehat{v}_N^\beta(p)^2}{2p^2} f(-p)\,. \tag{8.45d}$$

We will now prove that (8.45c) is the leading contribution. Clearly $|(8.45\text{d})| \lesssim N^{-1}$, and similarly to (8.14) we estimate

$$|(8.45\text{a})| \lesssim N^{-1}\sum_{\substack{p,q\in\Lambda_+^*\\ |q|\le N^\beta}} \frac{|\widehat{v}_N^\beta(p)|\,|q|N^{-\beta}|q|^{-4}}{p^2} + N^{-1}\sum_{\substack{p,q\in\Lambda_+^*\\ |q|>N^\beta}} \frac{|\widehat{v}_N^\beta(p)||q|^{-4}}{p^2} \lesssim N^{-1}\ln(N), \tag{8.46}$$

where we used (8.6) for momenta $|q| \le N^\beta$ and $\|\widehat{v}\|_{\ell^\infty} \lesssim 1$ for momenta $|q| \ge N^\beta$, together with Lemma 3.1. Finally, in order to estimate (8.45b), we first split into momenta $|q| \le N^\beta$ and $|q| > N^\beta$, and then use (8.39), (8.40), and (8.41). We find

$$\begin{aligned}
&|(8.45\text{b})| \\
&= \Bigg| \frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0\\ |q|\le N^\beta}} \widehat{v}_N^\beta(p)^2 f(q)\left(\frac{1}{\mathfrak{e}(p+q)+\mathfrak{e}(p)+\mathfrak{e}(q)} - \frac{1}{2p^2}\right) \\
&\quad + \frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0\\ |q|> N^\beta}} \widehat{v}_N^\beta(p)^2 f(q)\left(\frac{1}{\mathfrak{e}(p+q)+\mathfrak{e}(p)+\mathfrak{e}(q)} - \frac{1}{2p^2}\right)\Bigg| \\
&\lesssim N^{-1}\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0\\ |q|\le N^\beta}} \widehat{v}_N^\beta(p)^2 |f(q)| \left(\frac{|p^2-\mathfrak{e}(p)|+|\mathfrak{e}(q)|+|p^2-(p+q)^2|+|(p+q)^2-\mathfrak{e}(p+q)|}{p^2\big(\mathfrak{e}(p+q)+\mathfrak{e}(p)+\mathfrak{e}(q)\big)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{O}(N^{-1}) \\
&\lesssim N^{-1} \sum_{\substack{p,q\in\Lambda_+^* \\ |q|\le N^\beta}} \widehat{v}_N^\beta(p)^2 |q|^{-4} \left(\frac{1}{|p|^4} + \frac{|q|^2}{p^2|p||q|} + \frac{|p||q|}{p^2p^2} + \frac{|q|^2}{p^2|p||q|} \right) + \mathcal{O}(N^{-1}) \\
&\lesssim N^{-1} (\ln N)^2. \qquad (8.47)
\end{aligned}
$$

Note that in the last step we have additionally used (8.44).

9 Proof of Theorem 2

Let $p_0 = |\varphi_0\rangle\langle\varphi_0|$, $q_0 = 1 - p_0 = \sum_{p\in\Lambda_+^*} |\varphi_p\rangle\langle\varphi_p|$ and $A \in \mathcal{L}(L^2(\Lambda))$ with $\|A\|_{\mathcal{L}(L^2(\Lambda))} = 1$. Then,

$$
\begin{aligned}
\mathrm{Tr}\left(A\gamma^{(1)}\right) &= \mathrm{Tr}\left(A\left(p_0\gamma^{(1)}p_0 + q_0\gamma^{(1)}q_0 + p_0\gamma^{(1)}q_0 + q_0\gamma^{(1)}p_0\right)\right) \\
&= \frac{1}{N}\mathrm{Tr}\left(Ap_0\right)\left\langle \Psi_N, a^\dagger(\varphi_0)a(\varphi_0)\Psi_N \right\rangle \\
&\quad + \frac{1}{N}\sum_{p,q\in\Lambda_+^*} \mathrm{Tr}\left(A\,|\varphi_p\rangle\langle\varphi_q|\right)\left\langle \Psi_N, a^\dagger(\varphi_q)a(\varphi_p)\Psi_N \right\rangle \\
&\quad + \frac{1}{N}\sum_{p\in\Lambda_+^*} \Big(\mathrm{Tr}\left(A\,|\varphi_0\rangle\langle\varphi_p|\right)\left\langle \Psi_N, a^\dagger(\varphi_p)a(\varphi_0)\Psi_N \right\rangle \\
&\qquad\qquad + \mathrm{Tr}\left(A\,|\varphi_p\rangle\langle\varphi_0|\right)\left\langle \Psi_N, a^\dagger(\varphi_0)a(\varphi_p)\Psi_N \right\rangle \Big). \qquad (9.1)
\end{aligned}
$$

By means of (2.11) this can be written as

$$
\begin{aligned}
\mathrm{Tr}\left(A\left(\gamma^{(1)} - p_0\right)\right) &= -\frac{1}{N}\mathrm{Tr}\left(Ap_0\right)\langle \boldsymbol{\chi}, \mathbb{T}\mathcal{N}_\perp\mathbb{T}^*\boldsymbol{\chi}\rangle &(9.2a)\\
&\quad + \frac{1}{N}\sum_{p,q\in\Lambda_+^*} \mathrm{Tr}\left(A\,|\varphi_p\rangle\langle\varphi_q|\right)\left\langle \boldsymbol{\chi}, \mathbb{T}a_q^\dagger a_p\mathbb{T}^*\boldsymbol{\chi} \right\rangle &(9.2b)\\
&\quad + \frac{1}{N}\sum_{p\in\Lambda_+^*} \mathrm{Tr}\left(A\,|\varphi_0\rangle\langle\varphi_p|\right)\left\langle \boldsymbol{\chi}, \mathbb{T}a_p^\dagger\sqrt{N-\mathcal{N}_\perp}\,\mathbb{T}^*\boldsymbol{\chi} \right\rangle &(9.2c)\\
&\quad + \mathrm{Tr}\left(A\,|\varphi_p\rangle\langle\varphi_0|\right)\left\langle \boldsymbol{\chi}, \mathbb{T}\sqrt{N-\mathcal{N}_\perp}\,a_p\mathbb{T}^*\boldsymbol{\chi} \right\rangle. &(9.2d)
\end{aligned}
$$

Note that

$$
\langle \boldsymbol{\chi}, \mathbb{T}\mathcal{N}_\perp\mathbb{T}^*\boldsymbol{\chi}\rangle \lesssim 1 \qquad (9.3)
$$

because of Lemma 3.4 and Lemma 6.2. With (9.3), we obtain

$$
\begin{aligned}
\left| \sum_{p,q\in\Lambda_+^*} \mathrm{Tr}\left(A\,|\varphi_p\rangle\langle\varphi_q|\right)\left\langle \boldsymbol{\chi}, \mathbb{T}a_q^\dagger a_p\mathbb{T}^*\boldsymbol{\chi} \right\rangle \right| &= \langle \boldsymbol{\chi}, \mathbb{T}\,\mathrm{d}\Gamma(q_0Aq_0)\mathbb{T}^*\boldsymbol{\chi}\rangle &(9.4)\\
&\lesssim \|A\|_{\mathcal{L}(\ell^2(\Lambda))}\|(\mathcal{N}_\perp+1)^{1/2}\,\mathbb{T}^*\boldsymbol{\chi}\|^2 \lesssim 1. &(9.5)
\end{aligned}
$$

Hence,

$$
|(9.2a) + (9.2b)| \lesssim \frac{1}{N}. \qquad (9.6)
$$

Using

$$\sum_{p\in\Lambda_+^*} |\mathrm{Tr}\,(A\,|\varphi_0\rangle\,\langle\varphi_p|)|^2 \leq \|A\varphi_0\|^2 \leq \|A\|^2_{\mathcal{L}(L^2(\Lambda))} \leq 1, \tag{9.7}$$

Lemma 3.4, Lemma 6.2 and (2.59), we estimate

$$\begin{aligned}
|(9.2\mathrm{c})| &= \frac{1}{N}\Big| \sum_{p\in\Lambda_+^*} \mathrm{Tr}\,(A\,|\varphi_0\rangle\,\langle\varphi_p|) \Big\langle \boldsymbol{\chi}, \mathbb{T} a_p^\dagger \left(\sqrt{N-\mathcal{N}_\perp} - N^{1/2} + N^{1/2}\right) \mathbb{T}^* \boldsymbol{\chi} \Big\rangle \Big| \\
&\lesssim N^{-3/2} \|(\mathcal{N}_\perp + 1)^{3/4}\, \mathbb{T}^* \boldsymbol{\chi}\|^2 + N^{-1/2} \Big| \sum_{p\in\Lambda_+^*} \mathrm{Tr}\,(A\,|\varphi_0\rangle\,\langle\varphi_p|) \Big\langle \boldsymbol{\chi}, \mathbb{T} a_p^\dagger \mathbb{T}^* \boldsymbol{\chi} \Big\rangle \Big| \\
&\lesssim N^{-3/2} + N^{-1/2} \|\boldsymbol{\chi} - (\boldsymbol{\chi}_0 + \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2)\| \Big[\|(\mathcal{N}_\perp + 1)^{1/2}\, \mathbb{T}^* \boldsymbol{\chi}\| \\
&\qquad\qquad + \|(\mathcal{N}_\perp + 1)^{1/2}\, \mathbb{T}^* \left(\boldsymbol{\chi}_0 + \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2\right)\| \Big] \\
&\quad + N^{-1/2} \Big| \sum_{p\in\Lambda_+^*} \mathrm{Tr}\,(A\,|\varphi_0\rangle\,\langle\varphi_p|) \Big\langle \left(\boldsymbol{\chi}_0 + \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2\right), \mathbb{T} a_p^\dagger \mathbb{T}^* \left(\boldsymbol{\chi}_0 + \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2\right) \Big\rangle \Big| \\
&\lesssim N^{-3/2} + N^{\frac{3}{2}\beta - 2} \Big[1 + \|(\mathcal{N}_\perp + 1)^{1/2}\, \mathbb{T}^* \left(\boldsymbol{\chi}_0 + \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2\right)\| \Big] \\
&\quad + N^{-1/2} \Big| \sum_{p\in\Lambda_+^*} \mathrm{Tr}\,(A\,|\varphi_0\rangle\,\langle\varphi_p|) \Big\langle \left(\boldsymbol{\chi}_0 + \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2\right), \mathbb{T} a_p^\dagger \mathbb{T}^* \left(\boldsymbol{\chi}_0 + \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2\right) \Big\rangle \Big|.
\end{aligned} \tag{9.8}$$

Note that

$$\Big\langle \boldsymbol{\chi}_0, \mathbb{T} a_p^\dagger \mathbb{T}^* \boldsymbol{\chi}_0 \Big\rangle = \Big\langle \boldsymbol{\chi}_0, \mathbb{T} a_p^\dagger \mathbb{T}^* \boldsymbol{\chi}_2 \Big\rangle = \Big\langle \boldsymbol{\chi}_1, \mathbb{T} a_p^\dagger \mathbb{T}^* \boldsymbol{\chi}_1 \Big\rangle = \Big\langle \boldsymbol{\chi}_2, \mathbb{T} a_p^\dagger \mathbb{T}^* \boldsymbol{\chi}_2 \Big\rangle = 0 \tag{9.9}$$

because all these expressions are vacuum expectation values of operators with an odd number of creation/annihilation operators. Moreover, recall that

$$\begin{aligned}
\boldsymbol{\chi}_1 &= \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \mathbb{G}_1 \boldsymbol{\chi}_0 \\
&= \mathbb{U}_\tau^* \mathbb{U}_\tau \frac{\mathbb{Q}_0}{E_0 - \mathbb{G}_0} \mathbb{U}_\tau^* \mathbb{U}_\tau \mathbb{G}_1 \mathbb{U}_\tau^* |\Omega\rangle \\
&= -\frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \frac{f(p,q)}{\mathfrak{e}(p+q) + \mathfrak{e}(p) + \mathfrak{e}(q)} \mathbb{U}_\tau^* a_{p+q}^\dagger a_{-p}^\dagger a_{-q}^\dagger |\Omega\rangle
\end{aligned} \tag{9.10}$$

with $f(p,q)$ being defined as in Section 8. Consequently,

$$\Big\langle \boldsymbol{\chi}_0, \mathbb{T} a_p^\dagger \mathbb{T}^* \boldsymbol{\chi}_1 \Big\rangle = -\frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} \frac{f(p,q)}{\mathfrak{e}(p+q) + \mathfrak{e}(p) + \mathfrak{e}(q)} \Big\langle \Omega, \mathbb{U}_\tau \mathbb{T} a_p^\dagger \mathbb{T}^* \mathbb{U}_\tau^* a_{p+q}^\dagger a_{-p}^\dagger a_{-q}^\dagger \Omega \Big\rangle = 0 \tag{9.11}$$

because $\mathbb{U}_\tau \mathbb{T} a_p^\dagger \mathbb{T}^* \mathbb{U}_\tau^*$ contributes only an annihilation operator, hence the inner product vanishes. Finally,

$$\|(\mathcal{N}_\perp + 1)^{1/2}\, \mathbb{T}^* \boldsymbol{\chi}_0\| \lesssim 1, \quad \|(\mathcal{N}_\perp + 1)^{1/2}\, \mathbb{T}^* \boldsymbol{\chi}_1\| \lesssim N^{\frac{\beta-1}{2}}, \quad \|(\mathcal{N}_\perp + 1)^{1/2}\, \mathbb{T}^* \boldsymbol{\chi}_2\| \lesssim N^{\beta-1} \tag{9.12}$$

by Lemma 3.4, Lemma 3.6 and similar estimates as used in the proofs of Lemma 7.1 and Lemma 7.2. Using (9.7) again and the Cauchy–Schwarz inequality leads to

$$|(9.2c)| \lesssim N^{\frac{3}{2}(\beta-1)}. \tag{9.13}$$

By similar means we get

$$|(9.2d)| \lesssim N^{\frac{3}{2}(\beta-1)}. \tag{9.14}$$

In total this proves the claim because the space of compact operators is the dual of the space of trace-class operators. □

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References

[1] G. Basti, S. Cenatiempo, A. Olgiati, G. Pasqualetti, and B. Schlein. A second order upper bound for the ground state energy of a hard-sphere gas in the Gross–Pitaevskii regime. Commun. Math. Phys., 399(1):1–55, 2023.

[2] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein. Complete Bose–Einstein condensation in the Gross–Pitaevskii regime. Commun. Math. Phys., 359(3):975–1026, 2018.

[3] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein. Bogoliubov theory in the Gross–Pitaevskii limit. Acta Math., 222(2):219–335, 2019.

[4] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein. The excitation spectrum of Bose gases interacting through singular potentials. J. Eur. Math. Soc., 22(7):2331–2403, 2020.

[5] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein. Optimal rate for Bose–Einstein condensation in the Gross-Pitaevskii regime. Commun. Math. Phys., 376:1311–1395, 2020.

[6] N. N. Bogoliubov. On the theory of superfluidity. Izv. Akad. Nauk Ser. Fiz., 11:23–32, 1947.

[7] L. Boßmann. Low-energy spectrum and dynamics of the weakly interacting bose gas. J. Math. Phys., 2022.

[8] L. Boßmann, N. Pavlović, P. Pickl, and A. Soffer. Higher order corrections to the mean-field description of the dynamics of interacting bosons. J. Stat. Phys., 178(6):1362–1396, 2020.

[9] L. Boßmann, S. Petrat, P. Pickl, and A. Soffer. Beyond Bogoliubov dynamics. Pure Appl. Anal., 3(4):677–726, 2021.

[10] L. Boßmann, S. Petrat, and R. Seiringer. Asymptotic expansion of low-energy excitations for weakly interacting bosons. Forum Math. Sigma, 9:e28, 2021.

[11] C. Brennecke, B. Schlein, and S. Schraven. Bogoliubov theory for trapped bosons in the Gross–Pitaevskii regime. Ann. H. Poincaré, 23(5):1583–1658, 2022.

[12] C. Brennecke, B. Schlein, and S. Schraven. Bose–Einstein condensation with optimal rate for trapped bosons in the Gross–Pitaevskii regime. Mathematical Physics, Analysis and Geometry, 25(2):12, 2022.

[13] M. Brooks. Diagonalizing Bose gases in the Gross-Pitaevskii regime and beyond. arXiv:2310.11347, 2023.

[14] C. Caraci, S. Cenatiempo, and B. Schlein. The excitation spectrum of two-dimensional Bose gases in the Gross–Pitaevskii regime. Ann. Henri Poincaré, pages 1–52, 2023.

[15] C. Caraci, A. Olgiati, D. Saint Aubin, and B. Schlein. Third order corrections to the ground state energy of a Bose gas in the Gross–Pitaevskii regime. arXiv:2311.07433, 2023.

[16] J. Dereziński and M. Napiórkowski. Excitation spectrum of interacting bosons in the mean-field infinite-volume limit. Ann. Henri Poincaré, 15(12):2409–2439, 2014.

[17] P. Grech and R. Seiringer. The excitation spectrum for weakly interacting bosons in a trap. Commun. Math. Phys., 322(2):559–591, 2013.

[18] C. Hainzl. Another proof of BEC in the GP-limit. J. Math. Phys., 62(5):051901, 2021.

[19] C. Hainzl, B. Schlein, and A. Triay. Bogoliubov theory in the Gross-Pitaevskii limit: a simplified approach. Forum Math. Sigma, 10:e90, 2022.

[20] M. Lewin, P. T. Nam, S. Serfaty, and J. P. Solovej. Bogoliubov spectrum of interacting Bose gases. Commun. Pure Appl. Math., 68(3):413–471, 2015.

[21] E. H. Lieb and R. Seiringer. Proof of Bose–Einstein condensation for dilute trapped gases. Phys. Rev. Lett., 88(17):170409, 2002.

[22] P. T. Nam and M. Napiórkowski. Two-term expansion of the ground state one-body density matrix of a mean-field Bose gas. Calc. Var. Partial Differential Equations, 60(3):1–30, 2021.

[23] P. T. Nam, M. Napiórkowski, J. Ricaud, and A. Triay. Optimal rate of condensation for trapped bosons in the Gross–Pitaevskii regime. Analysis & PDE, 15(6):1585–1616, 2022.

[24] P. T. Nam and S. Rademacher. Exponential bounds of the condensation for dilute Bose gases. arXiv:2307.10622, 2023.

[25] P. T. Nam and R. Seiringer. Collective excitations of Bose gases in the mean-field regime. Arch. Ration. Mech. Anal., 215:381—417, 2015.

[26] P. T. Nam and A. Triay. Bogoliubov excitation spectrum of trapped Bose gases in the Gross–Pitaevskii regime. J. Math. Pures et Appliquées, 2023.

[27] A. Pizzo. Bose particles in a box I. A convergent expansion of the ground state of a three-modes Bogoliubov Hamiltonian. arXiv:1511.07022, 2015.

[28] R. Seiringer. The excitation spectrum for weakly interacting bosons. Commun. Math. Phys., 306(2):565–578, 2011.