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Mathematical Logic: Proof Theory, Constructive Mathematics

Organized by
Samuel R. Buss, La Jolla
Rosalie Iemhoff, Utrecht
Ulrich Kohlenbach, Darmstadt
Michael Rathjen, Leeds

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Abstract. The Workshop 'Mathematical Logic: Proof Theory, Constructive Mathematics' focused on proof-theoretic research on the foundations of mathematics, on the extraction of explicit computational content from given proofs in core areas of ordinary mathematics using proof-theoretic methods as well as on topics in proof complexity. The workshop contributed to the following research strands:

- Interactions between foundations and applications.
- Proof mining.
- Constructive and semi-constructive reasoning.
- Proof theory and theoretical computer science.
- Structural proof theory.

Mathematics Subject Classification (2020): 03Fxx.

Introduction by the Organizers

The workshop *Mathematical Logic: Proof Theory, Constructive Mathematics* was held November 12-17, 2023 in a hybrid format due to the Corona pandemic. It had 46 participants at the Oberwolfach Institute and 4 virtual participants who were connected via ZOOM. The program consisted of 23 talks of 40 minutes (2 of which were given via ZOOM).

The purpose of the workshop was

To promote the interaction between the foundations of mathematics and applications to mathematics as done for example in the field of 'proof mining'. M. Neri,

P. Oliva and T. Powell talked on the very recent novel development of applying proof-theoretic proof mining techniques in the context of probability theory. N. Pischke extended the framework of previously existing logical metatheorems for proof mining to include concepts such as dual and bidual spaces of a Banach space, gradients of uniformly Fréchet differentiable convex functions and their Fenchel conjugates and, finally, Bregman distances which allows one to treat for the first time important algorithms in optimization which compute zeros of maximally monotone operators in Banach spaces. P. Pinto used a concrete proof mining (of a celebrated theorem of S. Reich) due to Kohlenbach and Sipoş to generalize Reich's result (together with a quantitative analysis) to a newly defined class of uniformly smooth and convex hyperbolic spaces (which covers CAT(0)-spaces as a special case). L. Leuştean gave a survey on recently extracted effective rates of asymptotic regularity in optimization with a special focus on case where linear rates can be obtained using proof-mining methods. This topic was further extended in the talk by H. Cheval who, moreover, discussed the potential use of proof assistants such as LEAN in partially automatizing parts of the mining process.

A. Sipoş gave a quantitative treatment of the class of super strongly nonexpansive mappings which was recently introduced by Liu et al. as a counterpart to maximally monotone and uniformly monotone operators. This leads to a quantitative inconsistent feasibility result which was even qualitatively new. Talks on the interplay between foundational research in the context of reverse mathematics (RM) and core mathematics where given by J. Aguilera, who spoke about recent results on the reverse mathematics of systems of determinacy provable in second-order arithmetic and on some which go beyond it, and by S. Sanders, who studied, in particular, the status of various weak forms of continuity in the context of higher order reverse mathematics. V. Brattka's talk discussed a number of uniform dichotomies for problems in the Weihrauch lattice. M. Baaz showed that a Skolemization method due to P. Andrews - and used prominently in the context of resolution - can have a non-elementary speed up over the standard Skolemization method. R. Kahle and I. Oitavem talked about a problem in the proof complexity of a Hilbert-type system for propositional logic and for combinatorial logic. S. Negri developed a natural deduction calculus for Gurevich logic and related it to a previously proposed cut-free sequent calculus to prove a normalization result.

To explore connections between proof theory, constructive formal systems and computer science. M. Fujiwara's talk investigated the formula classes U_k, E_k, introduced in 2004 by Akama et al., from the point of view of the standard transformation procedure for prenex normalization showing that they are exactly the classes of formulas induced by Σ_k and Π_k resp. via these transformations. M.E. Maietti proved that the formal system for the 'Minimalist Foundation for Constructive Mathematics', introduced in 2005 by herself and G. Sambin, is equiconsistent with its extension by the law of the excluded middle. I. van der Giessen presented an intuitionistic version of Gödel-Löb logic that includes both modalities Box and Diamond, and allows for a Gentzen-Gödel negative translation of its classical counterpart. P. Schuster (jww G. Fellin) talked about a generalization of Glivenko's theorem to an arbitrary nucleus and to an inductively generated abstract consequence relation. M. Zorzi presented extensional proof systems for modal logics, focussing on a "geometric" approach that entails a notion of position.

To investigate further the connections between logic and computational complexity. E. Jeřábek's talk addressed the question of characterizing and axiomatizing ordered rings that are existential integer parts of real-closed exponential fields, and especially the first-order theory of such rings. P. Pudlák discussed implicit proof systems for propositional logic, and the use of iterated implicit proof systems to capture self-consistency statements. N. Thapen presented first-order theories of bounded arithmetic for semi-algebraic reasoning about polynomial inequalities, such as used by the Sum-of-Squares (SoS) proof system. M. Müller presented a proof of the independence of circuit-lower bounds for nondeterministic exponential time from theories of bounded arithmetic.

Acknowledgement: The workshop organizers would like to thank the MFO for supporting the participation of graduate students and recent post docs in the workshop via the Oberwolfach Leibniz Graduate Student program.

Workshop: Mathematical Logic: Proof Theory, Constructive Mathematics

Table of Contents

Abstracts

On a shortest proof of $\varphi \to \varphi$

Reinhard Kahle, Isabel Oitavem
(joint work with Paulo Guilherme Santos)

1. The standard proof of $\varphi \to \varphi$ in a Hilbert-style calculus

Let us consider the *Pure Positive Implication Propositional Calculus* in a Hilbert-style calculus, based on Frege's axioms for implication [2]:

$$\vdash \varphi \to (\psi \to \varphi) \tag{F1}$$

$$\vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \tag{F2}$$

The only rule is Modus ponens (MP).

Although not an axiom, $\varphi \to \varphi$ is a derivable formula:

Theorem. $\varphi \to \varphi$ is derivable, for every formula φ.

Proof. Consider the derivation D_1:

1	$\vdash (\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$	(F2)
2	$\vdash \varphi \to ((\varphi \to \varphi) \to \varphi)$	(F1)
3	$\vdash (\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$	MP[1,2]
4	$\vdash \varphi \to (\varphi \to \varphi)$	(F1)
5	$\vdash \varphi \to \varphi$	MP[3,4]

$\square$

We call D_1 the *standard proof* of $\varphi \to \varphi$ (for the given axiomatization).

Some historical notes concerning the discovery of this proof can be found in [5].

Is D_1 the shortest proof of $\varphi \to \varphi$?

Of course, this question makes sense only, when the formal system is fixed, and when an appropriate measure of length is defined.

We consider the formal system described above, and we focus on the measure $\mathsf{M1}$ which counts the lines of the proof. For instance $\mathsf{M1}(D_1) = 5$.

It is an easy combinatorial exercise to see that there is no shorter proof of $\varphi \to \varphi$, for an arbitrary formula φ, in this formal system.

However, could there exist shorter proofs than D_1 for special instances of φ?

2. The special case $(\varphi \to \varphi) \to (\varphi \to \varphi)$

Consider D_2:

1	$\vdash (\varphi \to (\varphi \to \varphi)) \to ((\varphi \to \varphi) \to (\varphi \to \varphi))$	(F2)
2	$\vdash \varphi \to (\varphi \to \varphi)$	(F1)
3	$\vdash (\varphi \to \varphi) \to (\varphi \to \varphi)$	MP[1,2]

For the length we have, $\mathsf{M1}(D_2) = 3 < \mathsf{M1}(D_1)$.

Is this the only "special case"? We answer this question via *Combinatory Logic.*

3. Combinatory Logic and the Curry-Howard Correspondence

Schönfinkel [6] and Curry [1] developed the framework of Combinatory Logic which turned out to be a "computational counterpart" of the Hilbert-style calculus with Frege's axioms (F1) and (F2) for implication.

Combinatory terms are build inductively from the two constants, K and S, variables $(X, Y, \cdots)$, and closure under application: If X and Y are combinatory terms, then the application $(X \cdot Y)$ is also a combinatory term. As usual, the dot for application is often suppressed; and one uses left associativity to reduce parentheses.

Combinatory terms serve as a kind of programming language, when one considers the following equalities:

- $\mathsf{K}\, X\, Y = X$;
- $\mathsf{S}\, X\, Y\, Z = X\, Z\, (Y\, Z)$.

The combinators can be *typed* by formulas, such that the combinatory terms represent proofs of these formulas:

- $\mathsf{K}^{\varphi\to(\psi\to\varphi)}$ for the axiom (F1);
- $\mathsf{S}^{(\varphi\to(\psi\to\chi))\to((\varphi\to\psi)\to(\varphi\to\chi))}$ for the axiom (F2);
- Application relates to (an application of) *modus ponens*: $X^{\varphi\to\psi} Y^{\varphi}$ has type ψ.

In this way, the derivation D_1 can be written by the following (typed) combinatory term: $\mathsf{S}^{(\varphi\to((\varphi\to\varphi)\to\varphi))\to((\varphi\to(\varphi\to\varphi))\to(\varphi\to\varphi))}\, \mathsf{K}^{\varphi\to((\varphi\to\varphi)\to\varphi)}\, \mathsf{K}^{\varphi\to(\varphi\to\varphi)}$. This term has, indeed, type $\varphi \to \varphi$.

When taking the application dot into account, we have also a one-to-one correspondence between the number of lines of the proof and the length of the combinatorial term: $\mathsf{M1}(D_1) = 5 = lh(S \cdot K \cdot K)$.

4. Identity Combinators and Fixed Points

The identity combinator I with $\mathsf{I}\, X = X$ can be defined by $\mathsf{I} = \mathsf{S}\,\mathsf{K}\,\mathsf{K}$.

According to the Curry-Howard Correspondence, any identity combinator, i.e., a combinator M with $M\, X = X$, for all X, will give rise to a proof of (an instance of) $\varphi \to \varphi$. But it does not need to be an identity combinator.

Definition. Let M be a closed combinatory term of type $\varphi_0 \to (\cdots \to (\varphi_n \to \psi)\cdots)$, $n \geq 0$. X is a *fixed point*, if for all terms $Y_1, \ldots Y_n$:

$$M\, X\, Y_1\, \cdots\, Y_n = X\, Y_1\, \cdots\, Y_n.$$

Theorem. Let M be a closed combinatory term.

- If M has a fixed point, then M corresponds to a proof of an instance of $\varphi \to \varphi$.
- Moreover, the number of lines of that proof is $lh(M)$.

Considering only combinatory terms of length less than or equal to 5, we obtain the following *special cases*. For terms starting with K:

Comb. M	F.P.	Proof of $\varphi \to \varphi$ for φ being
K K	K	$\varphi \to (\psi \to \varphi)$
K S	S	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
K (K K)	K K	$\varphi \to (\psi \to (\chi \to \psi))$
K (K S)	K S	$\varphi \to ((\psi \to (\chi \to \tau)) \to ((\psi \to \chi) \to (\psi \to \tau)))$
K (S K)	S K	$(\varphi \to \psi) \to (\varphi \to \varphi)$
K (S S)	S S	$((\varphi \to (\psi \to \chi)) \to (\varphi \to \psi)) \to$ $((\varphi \to (\psi \to \chi)) \to (\varphi \to \chi))$

For terms starting with S:

Combinator M	Fixed point	Proof of $\varphi \to \varphi$ for φ being
S K	I	$\varphi \to \varphi$
S S	*	$(\varphi \to (\psi \to \psi)) \to (\varphi \to \psi)$
S K K	X	φ
S K S	X	$\varphi \to (\psi \to \chi)$
S (S K)	X	$(\varphi \to \psi) \to \varphi$

5. Further considerations

- S S does not has a fixed point in the sense defined above; the analysis of this case gives, indeed, reason for further considerations.
- Hindley [3] provided a typing algorithm for combinators. From this algorithm one obtains a more general type of S K K which is of interest when considering other measures (which, for instance, take the length of formulas in a proof into account).
- The present study is some ground work for more detailed investigations on *Hilbert's 24th problem* [4]. This problem, preserved in Hilbert's mathematical notebook, asks for criteria of simplicity of proofs, proposing, in particular, to take the length of proofs into account.

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Consistency, implicit proofs, and cut-elimination

Pavel Pudlák

Two computable operators have been conjectured to be jumps.

Definition 1 (consistency jump)**.** *For a proof system P, define $con(P)$,* the consistency jump, *to be the strongest proof systems Q such that $S^1_2 + Con(S^1_2 + Rfn(P))$ proves the reflection principle for Q.*

The second operator is based on implicit proofs.

Definition 2 (Krajíček [4], implicit proofs)**.** *Let P, Q be proof systems; we define a proof system $[P, Q]$ as follows. A $[P, Q]$-proof of ϕ is a pair (π, c), where*

- *c is a circuit that defines bits of a (possibly exponential size) Q-proof of ϕ,[1] and*
- *π is a P-proof of the fact above.*

We conjecture that $imp(P) := [P, P]$ is a jump.

We want to find connection between the two operators and believe that it could be proved by showing that cut-elimination produces implicit proofs in the sense of the above definition. The fact that elimination of one level of cuts produces exponential size proofs that have succinct representations has already been observed before, [1, 2]. The problem is, however, that we still do not fully understand the concept of an implicit proof. Part of the reason is that it is not a robust concept. For instance Khaniki proved under plausible complexity-theoretical assumption that there are two proof systems P and Q such that $P \equiv_p Q$, but $imp(P) \not\equiv imp(Q)$, (cf. [3]). Therefore we decided to first study a restricted version of implicit proofs.

A restricted kind of implicit proofs is defined by requiring that the circuit computes *formulas*, not single bits, see [4]. Thus the formulas must be of polynomial size (in the size of the implicit proof). Such proof systems are denoted by $[P, Q]^m$. (This operation is only defined when Q is a proof system based on formulas.) In our modification circuits compute *sequents*, because we want to use the sequent calculus. We will consider the following proof systems (cf. [5]):

- SF denotes the sequent calculus for propositional logic augmented with the *substitution rule*:
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma[x/\alpha] \longrightarrow \Delta[x/\alpha]},$$
where α is a Boolean formula.

[1] If c has n input bits, then it can define bits of a string of length 2^n.

- G denotes the *the quantified propositional sequent calculus.* E.g., the $\exists$-right rule is
$$\frac{\Gamma \longrightarrow \Delta, \phi(\alpha)}{\Gamma \longrightarrow \Delta, \exists x.\phi(x)},$$
where α is a Boolean formula.
- For $i \geq 1$, G_i denotes the Σ_i^q fragment of G.

Theorem 1.

(1) $[SF, SF]^m \equiv_p G_1$*,*

(2) $[SF, G_i]^m \equiv_p G_{i+1}$ *for* $i \geq 1$*.*

The more technical part of the proofs are polynomial simulations $[SF, SF]^m \geq_p G_1$ and $[SF, G_i]^m \geq_p G_{i+1}$. They are based on eliminating cuts with the highest quantifier complexity and showing that this produces implicit proofs.

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New applications of proof theory: Greedy algorithms, probability, and proof assistants

Thomas Powell

I will give a brief and high-level overview of some new research projects that I believe have the potential to yield exciting results over the next few years.

The first revolves around greedy approximation schemes in Hilbert and Banach spaces. This is an area replete with convergence results, proofs of which are often nonconstructive and hinge on geometric properties of the underlying space, such as uniform smoothness. I will present an initial case study and argue that the area in general may form a fertile ground for applied proof theory, with particular relevance at the moment given its connections to learning algorithms.

I will also present an overview of some ongoing work in probability theory (joint with Morenikeji Neri). My focus will be on our efforts to understand some of the basic notions of probabilistic convergence and the relationships between them from a computational perspective. Here things seem to get particularly interesting where uniform integrability plays a role, and implications between convergence statements seem computationally subtle. A much broader open question is how to formalise the underlying proofs in a suitable abstract system.

Finally, I will outline some broad goals in formalised mathematics and automated reasoning, which are relevant to both of the above themes and applied proof theory in general.

First-Order Reasoning and Efficient Semi-Algebraic Proofs

Neil Thapen

(joint work with Fedor Part, Iddo Tzameret)

Semi-algebraic proof systems such as sum-of-squares (SoS) [5] have attracted a lot of attention due to their relation to approximation algorithms: constant degree semi-algebraic proofs lead to conjecturally optimal polynomial-time approximation algorithms for important NP-hard optimization problems [1]. Motivated by the need to allow a more streamlined and uniform framework for working with SoS proofs than the restrictive propositional level, we initiate a systematic first-order logical investigation into the kinds of reasoning possible in algebraic and semi-algebraic proof systems. Specifically, we develop first-order theories that capture in a precise manner constant degree algebraic and semi-algebraic proof systems: every statement of a certain form that is provable in our theories translates into a family of constant degree polynomial calculus or SoS refutations, respectively; and using a reflection principle, the converse also holds.

This places algebraic and semi-algebraic proof systems in the established framework of bounded arithmetic, while providing theories corresponding to systems that vary quite substantially from the usual propositional-logic ones [2, 4, 6].

We give examples of how our semi-algebraic theory proves statements such as the pigeonhole principle, we provide a separation between algebraic and semi-algebraic theories, and we describe initial attempts to go beyond these theories by introducing extensions that use the inequality symbol, identifying along the way which extensions lead outside the scope of constant degree SoS. Moreover, we prove new results for propositional proofs, and specifically extend Berkholz's [3] dynamic-by-static simulation of polynomial calculus (PC) by SoS to PC with the radical rule.

An earlier version of this work appeared as [7].

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Quantitative Probability from a Logician's Perspective

Morenikeji Neri

Over the last few decades, proof mining has enjoyed many successes in numerous areas of mathematics, mostly within analysis. To date, there have only been a handful of papers that extend proof mining to probability and measure theory. On the other hand, probability theorists have been informally extracting quantitative bounds for many years, in particular, obtaining rates for probabilistic convergence theorems.

In this talk, I shall first discuss some results from quantitative probability theory obtained by logicians and probability theorists, giving an overview of the relevant notions from probability theory. I shall then present my own ongoing work in obtaining quantitative bounds from strong law of large numbers type results, that not only build on the existing body of work in the proof mining of probability theory literature but also extend work done by probability theorists obtaining quantitative results. Lastly, I shall look towards the future and introduce some questions in quantitative probability theory that one could potentially answer using ideas from the proof mining program.

On the consistency of circuit lower bounds for non-deterministic time

Moritz Müller
(joint work with Albert Atserias, Sam Buss)

We prove the first unconditional consistency result for superpolynomial circuit lower bounds with a relatively strong theory of bounded arithmetic. Namely, we show that the theory V_2^0 is consistent with the conjecture that NEXP $\not\subseteq$ P/poly, i.e., some problem that is solvable in non-deterministic exponential time does not have polynomial size circuits. We suggest this is the best currently available evidence for the truth of the conjecture. The same techniques establish the same results with NEXP replaced by the class of problems decidable in non-deterministic barely superpolynomial time such as NTIME$(n^{O(\log\log\log n)})$. Additionally, we establish a magnification result on the hardness of proving circuit lower bounds.

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Structural proof theory for logics of strong negation

Sara Negri

(joint work with Norihiro Kamide)

Gurevich logic is an extended constructive three-valued logic obtained from intuitionistic logic by adding a connective $\sim$ of *strong negation*, with the following axiom schemata, where $\neg$ is intuitionistic negation:[1]

(1) $\sim\sim A \supset\subset A$,
(2) $\sim\neg A \supset\subset A$,
(3) $\sim A \supset \neg A$,
(4) $\sim(A \wedge B) \supset\subset \sim A \vee \sim B$,
(5) $\sim(A \vee B) \supset\subset \sim A \wedge \sim B$,
(6) $\sim(A \supset B) \supset\subset A \wedge \sim B$.

Nelson logic [11], also known as Nelson's constructive three-valued logic N3, is the intuitionistic negation-less fragment of Gurevich logic.

The primary formal difficulty in developing a natural deduction system for Gurevich logic, and more generally for logics that employ strong negation, lies in the requirement of having rules for $\neg$ and $\sim$ without $\perp$. This is solved using the rules of *explosion*, of $\neg$-*introduction*, and of *excluded middle*:[2]

$$\frac{\neg A \quad A}{C}\,\text{Exp} \qquad \frac{\begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[A]\\ \vdots \\ \neg C\end{matrix}}{\neg A}\,\neg\text{I} \qquad \frac{\begin{matrix}[\neg A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C}\,\text{Em}$$

The natural deduction system for intuitionistic logic NI* is obtained replacing the rule of *ex falso quodlibet* of NI with the rule of explosion and adding rule $\neg$I, and the natural deduction system for classical logic NK* is obtained from NI* by adding the rule of excluded middle. Next, the natural deduction system for Gurevich logic NG is obtained from NI* by adding the following rules for strong negation:

$$\frac{\sim A \quad A}{C}\,{\sim}\text{Exp}$$

$$\frac{A}{\sim\sim A}\,{\sim\sim}\text{I} \qquad \frac{\sim\sim A}{A}\,{\sim\sim}\text{E} \qquad \frac{A}{\sim\neg A}\,{\sim\neg}\text{I} \qquad \frac{\sim\neg A}{A}\,{\sim\neg}\text{E}$$

$$\frac{A \quad \sim B}{\sim(A \supset B)}\,{\sim\supset}\text{I} \qquad \frac{\sim(A \supset B)}{A}\,{\sim\supset}\text{E}_1 \qquad \frac{\sim(A \supset B)}{\sim B}\,{\sim\supset}\text{E}_2$$

[1]Gurevich logic can also be obtained by adding intuitionistic negation to Nelson logic (N3) [1, 11], which, in turn, is obtained by adding the principle of explosion to Nelson's paraconsistent four-valued logic, N4 [1, 11]. In the original study by Gurevich [4], completeness with respect to three-valued Kripke semantics, embedding into intuitionistic logic, functional completeness, and duality theorems for Gurevich logic were proven using a Hilbert-style axiomatic system. Cut-free Gentzen-style sequent calculi for Gurevich logic have been introduced in [4, 6].

[2]Systems with primitive negation for intuitionistic logic have a long history, dating back to the the 1930s with the work of Heyting and Gentzen [5, 12].

$$\frac{\sim A}{\sim (A \wedge B)} \sim\wedge\mathrm{I}_1 \qquad \frac{\sim B}{\sim (A \wedge B)} \sim\wedge\mathrm{I}_2 \qquad \frac{\sim (A \wedge B) \quad \begin{array}{c}[\sim A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[\sim B]\\ \vdots \\ C\end{array}}{C} \sim\wedge\mathrm{E}$$

$$\frac{\sim A \quad \sim B}{\sim (A \vee B)} \sim\vee\mathrm{I} \qquad \frac{\sim (A \vee B)}{\sim A} \sim\vee\mathrm{E}_1 \qquad \frac{\sim (A \vee B)}{\sim B} \sim\vee\mathrm{E}_2$$

The natural deduction system NN for Nelson logic N3 is obtained from NG by deleting Exp, $\neg$I, $\sim\neg$I, and $\sim\neg$E (i.e., NN is the $\neg$-less fragment of NG).

Equivalence between these natural deduction systems and correspondence with previously proposed cut-free Gentzen-style sequent calculi are proven and used to obtain normalization of the corresponding natural deduction systems. The normalization theorem for NK* cannot be obtained using the equivalence with LK, and therefore the single-succedent sequent calculus for classical logic LC originally introduced by von Plato in [13] (see also [9]) is used. In particular, an equivalence is established between NG and the previously proposed cut-free Gentzen-style sequent calculus LG for Gurevich logic, and this result is used to prove normalization for NG, and, as a bonus, also normalization for NN and NI*.

Next, G3-style sequent calculi are introduced for these logics and Avron and De-Omori logic. G3-style sequent calculi are sequent calculi with *all* structural rules admissible, not only cut but also weakening and contraction, and with all or most of the rules invertible. They are especially suited for root-first proof search and therefore useful for automated deduction, but also for meta-theoretical purposes because of their analyticity [9, 10]

First, the G3-style intuitionistic calculus with primitive negation G3ip$^\neg$ is obtained from G3ip by admitting an empty succedent and replacing the initial sequents $\bot, \Gamma \Rightarrow C$ for the falsity constant $\bot$ with the following rules for $\neg$:

$$\frac{\neg A, \Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow} \neg\mathrm{L} \qquad \frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A} \neg\mathrm{R}$$

Then, the G3-style sequent calculus for Gurevich logic G3gv is obtained from G3ip$^\neg$ by adding the following initial sequents and rules for $\sim$, where γ represents a formula or the empty multiset:

$$\sim P, \Gamma \Rightarrow \sim P \ \ \mathrm{init}_2 \qquad\qquad \sim P, P, \Gamma \Rightarrow \ \ \mathrm{init}_3$$

$$\frac{A, \Gamma \Rightarrow \gamma}{\sim\sim A, \Gamma \Rightarrow \gamma} \sim\sim\mathrm{L} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \sim\sim A} \sim\sim\mathrm{R}$$

$$\frac{A, \sim B, \Gamma \Rightarrow \gamma}{\sim(A \supset B), \Gamma \Rightarrow \gamma} \sim\supset\mathrm{L} \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \supset B)} \sim\supset\mathrm{R}$$

$$\frac{\sim A, \Gamma \Rightarrow \gamma \quad \sim B, \Gamma \Rightarrow \gamma}{\sim(A \wedge B), \Gamma \Rightarrow \gamma} \sim\wedge\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \sim A}{\Gamma \Rightarrow \sim(A \wedge B)} \sim\wedge\mathrm{R}_1 \qquad \frac{\Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \wedge B)} \sim\wedge\mathrm{R}_2$$

$$\frac{\sim A, \sim B, \Gamma \Rightarrow \gamma}{\sim(A \vee B), \Gamma \Rightarrow \gamma} \sim\vee\mathrm{L} \qquad \frac{\Gamma \Rightarrow \sim A \quad \Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \vee B)} \sim\vee\mathrm{R}$$

$$\frac{A, \Gamma \Rightarrow \gamma}{\sim\neg A, \Gamma \Rightarrow \gamma}\sim\neg\mathrm{L} \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \sim\neg A}\sim\neg\mathrm{R}$$

The G3-style sequent calculus for Nelson N3, G3n3, is obtained from G3gv by deleting the rules $\neg$L, $\neg$R, $\sim\neg$L, and $\sim\neg$R (i.e., as the $\neg$-less part of G3gv, and the calculus for Nelson N4, G3n4, is obtained from G3n3 by deleting init3.

Structural properties including cut elimination are established for these calculi and a Glivenko theorem for embedding G3gv into G3ip$^{\neg}$ is shown, providing at the same time an indirect alternative proof of the cut-elimination theorem for G3gv.

The G3-style sequent calculus G3cp$^{\neg}_{\sim}$ is obtained from the intuitionistic calculus turning it to a multisuccedent system. In G3cp$^{\neg}_{\sim}$, $\neg$ is equivalent to $\sim$. Thus, G3cp$^{\neg}_{\sim}$ is a redundant G3-style sequent calculus for classical propositional logic, however, the interest in this calculus lies in the fact that it provides a platform to obtain G3 calculi for a wealth of logical systems, already studied in the literature, that lacked a G3-style proof system: it is used to define G3-style sequent calculi for classical versions of N3 and N4, for Avron logic [2], and for De–Omori logic (the extension of Belnap–Dunn logic with classical negation) [3].

Finally, the explicit use of $\sim$ in G3cp$^{\neg}_{\sim}$ as an auxiliary connective makes it possible to prove a Glivenko theorem for embedding G3cp$^{\neg}_{\sim}$ into G3gv.

For the details, cf. [7, 8].

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Determinacy and $\Pi^1_n{-}\mathsf{CA}_0$

Juan P. Aguilera

There is an extremely large body of work on the metamathematics of determinacy principles in the context of set theory and reverse mathematics. From the perspective of the former, it was known from work of Steel, Tanaka, Heinatsch-Möllerfeld, Montalbán-Shore, and Nemoto that most of the usual subsystems of second-order arithmetic, such as WKL_0, ACA_0, ACA_0^+, ATR_0, $\Pi^1_1{-}\mathsf{CA}_0$, $\Pi^1_2{-}\mathsf{CA}_0$, and $Z_2 = \Pi^1_\infty{-}\mathsf{CA}_0$, are equiconsistent with schemata of axioms asserting the determinacy of games with complexity at various levels of the hierarchy of continuous or Lipschitz reducibility. It was open whether the same result is true for the missing subsystems $\Pi^1_n{-}\mathsf{CA}_0$, where $2 < n < \infty$.

In this talk, we mentioned the main ingredients of the proof behind the theorem asserting that the systems $\Pi^1_n{-}\mathsf{CA}_0$ are not equiconsistent with any schema of determinacy assertions when $n \neq 1, 2, \infty$. The main tool was the representation of the Wadge classes between the levels of the difference hierarchy over the $G_{\delta,\sigma}$ sets in terms of *separated Boolean* connectives in the style of Louveau, together with an argument by transfinite induction employing an abstract determinacy transfer theorem which is provable from hypotheses asserting the existence of certain nonstandard models of Kripke-Platek set theory admitting infinitely nested sequences of elementarity gaps of various kinds. This type of determinacy transfer theorem, although provable in the weak theory RCA_0, also has applications in the context of ZFC and its extensions. The specific theorem mentioned in the talk was:

Theorem. *Suppose that every $x \in \mathbb{R}$ belongs to a nonstandard β_m-model M of Kripke-Platek set theory satisfying $V = L$ and Γ-determinacy, where Γ is a Borel Wadge class, and such that there exists a sequence $\{(\zeta_i, s_i) : i \in \mathbb{N}\}$ of M-ordinals for which the following hold for all i:*

(1) $\zeta_i < \zeta_{i+1} \in wfp(M)$,
(2) $s_{i+1} < s_i$,
(3) $M \models L_{\zeta_i} \prec_{\Sigma_{m+1}} L_{s_i}$,
(4) $M \models L_{s_{i+1}} \prec_{\Sigma_{m-1}} L_{s_i}$.

Then, all games in the class $\mathsf{LU}(\Sigma^0_2, \Gamma, m{-}\Sigma^0_3)$ are determined. This is the class of all sets of the form

$$W = \bigcup_{i\in\mathbb{N}} \Big(A_i \cap C_i\Big) \cup B \setminus \bigcup_{i\in\mathbb{N}} C_i,$$

where $A_i \in \Gamma$, $C_i \in \Sigma^0_2$, $B \in m{-}\Sigma^0_3$, and $W \cap C_i = A_i \cap C_i$ for all $i \in \mathbb{N}$.

Although quite technical, the theorem is very powerful. Relating the sets provided by the theorem to those considered in Louveau's analysis of the Wadge ranks of Borel sets and using an analog of his of the Hausdorff-Kuratowski theorem, one can then transfinitely iterate the theorem inside $\Pi^1_n{-}\mathsf{CA}_0$, leading to the following dichotomy:

Theorem. *Suppose that* $\Gamma = \bigcup_{i\in\mathbb{N}} \Gamma_i$ *is a Borel Wadge class, provably so in* $\Pi^1_{n+3}{-}\mathsf{CA}_0$*. Write* Γ*-Determinacy for the schema* $\{\Gamma_i$*-Determinacy:* $i \in \mathbb{N}\}$*. Then, one of the following holds:*

(1) $\Pi^1_{n+3}{-}\mathsf{CA}_0 \vdash \Gamma$*-Determinacy* & $\mathsf{con}(\Gamma$*-Determinacy*)*; or*
(2) $\mathsf{RCA}_0 + \Gamma$*-Determinacy* $\vdash \mathrm{con}(\Pi^1_{n+3}{-}\mathsf{CA}_0)$*.*

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Proof Mining and duality in Banach spaces

Nicholas Pischke

We present a proof-theoretically tame approach for treating the dual space of an abstract Banach space in systems amenable to proof mining metatheorems on bound extractions, unlocking a major branch of functional analysis to these methods. The approach relies on using intensional methods to deal with the high quantifier complexity of the predicate defining the dual space as well as on a novel treatment of suprema over certain bounded sets in normed spaces to deal with the norm induced on the functionals of the dual. Beyond this, we provide an overview of the many possible extensions and concrete applications to core mathematics obtainable from this (which in particular includes a theory of convex functions and corresponding Fréchet derivatives and their duality theory through Fenchel conjugates, together with the associated Bregman distances).

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Proof mining and asymptotic regularity

Laurenţiu Leuştean
(joint work with Horaţiu Cheval, Paulo Firmino, Ulrich Kohlenbach, Pedro Pinto)

Proof mining is a research program that consists in the extraction of new information from mathematical proofs by applying proof-theoretic techniques. This program was systematically developed beginning with the 1990s by Kohlenbach and collaborators, in connection with applications to approximation theory, nonlinear analysis, ergodic theory, topological dynamics, Ramsey theory, (partial) differential equations, and convex optimization. Kohlenbach's monograph [11] is the standard reference for proof mining.

Asymptotic regularity is a very useful property in the study of the asymptotic behaviour of nonlinear iterations, introduced in the 1960s by Browder and Petryshyn [3] for the Picard iteration and extended to general iterations by Borwein, Reich, and Shafrir [1]. If (x_n) is a sequence in a metric space (X, d), $\emptyset \neq C \subseteq X$, and $T : C \to C$, then (x_n) is said to be asymptotically regular if $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and T-asymptotically regular if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It turns out that in numerous results on the weak or strong convergence of a nonlinear iteration (x_n), the first step is to prove the $(T\text{-})$asymptotic regularity of (x_n). Usually one proves first that (x_n) is asymptotically regular and afterwards that (x_n) is T-asymptotically regular.

A mapping $\varphi : \mathbb{N} \to \mathbb{N}$ is said to be a rate of asymptotic regularity of (x_n) if φ is a rate of convergence of $(d(x_n, x_{n+1}))$ towards 0, that is

$$\forall k \in \mathbb{N}\, \forall n \geq \varphi(k) \left(d(x_n, x_{n+1}) \leq \frac{1}{k+1} \right).$$

One defines similarly the notion of a rate of T-asymptotic regularity of (x_n). As pointed out in [14], the notion of T-asymptotic regularity can be extended to countable families of mappings. Thus, if $(T_n : C \to C)$ is such a family, then we say that (x_n) is (T_n)-asymptotically regular with rate φ if $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$ with rate of convergence φ.

In this talk I present recent applications of proof mining consisting in quantitative asymptotic regularity results for different nonlinear iterations.

In [5] we define the Tikhonov-Mann iteration as a generalization to W-hyperbolic spaces [11] of a modified Mann iteration studied by Yao, Zho, and Liou [18] and rediscovered by Boţ, Csetnek, and Meier [2]. Applying proof mining, we compute uniform rates of $(T\text{-})$asymptotic regularity for the Tikhonov-Mann iteration. Furthermore, we prove in [4] that there is a strong relation between the Tikhonov-Mann iteration and the modified Halpern iteration introduced by Kim and Xu [10]. Thus, asymptotic regularity and strong convergence results can be translated from one iteration to the other and the translation holds also for quantitative versions of these results, providing rates of $(T\text{-})$asymptotic regularity and rates of metastability. As an application of a lemma on real sequences due to Sabach and Shtern [16] we also obtain in [4] linear rates of $(T\text{-})$asymptotic regularity for both the Tikhonov-Mann and the modified Halpern iterations for a special choice of the parameter sequences.

Dinis and Pinto introduced recently [7] the alternating Halpern-Mann iteration as an iterative scheme associated with two mappings T, U that alternates between the well-known Halpern and Mann iterations. They proved, in the setting of CAT(0) spaces, quantitative results that provide rates of $(T, U\text{-})$asymptotic regularity and rates of metastability for this iteration by using proof mining techniques developed in [8]. In [15], we show that the quantitative $(T, U\text{-})$asymptotic regularity results obtained in [7] can be extended to UCW-hyperbolic spaces [12, 13], a class of W-hyperbolic spaces that generalize both CAT(0) spaces and uniformly convex normed spaces. Moreover, we apply again Sabach and Shtern's lemma to compute for the alternating Halpern-Mann iteration linear rates of asymptotic regularity in W-hyperbolic spaces and quadratic rates of T, U-asymptotic regularity in CAT(0) spaces, for a special case of the scalars.

In [6] we show that Sabach and Shtern's lemma can be applied to compute linear rates of (T-)asymptotic regularity or ((T_n)-)asymptotic regularity for other Halpern-type iterations studied in optimization and nonlinear analysis.

The viscosity approximation method (VAM), associated to resolvents $J^A_{\lambda_n}$ ($\lambda_n \subseteq (0,\infty)$) of an accretive operator A in a Banach space X, was studied by Xu et al. in a recent paper [17], where they prove results on the convergence of VAM to a zero of the operator A. We obtain in [9] quantitative versions of the asymptotic regularity results from [17] and, as a consequence, we compute uniform rates of (($J^A_{\lambda_n}$)-)asymptotic regularity for VAM. Sabach and Shtern's lemma gives us again linear rates when we consider a particular case of the parameters.

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A New Intuitionistic Version of Gödel-Löb Logic: Box and Diamond

Iris van der Giessen
(joint work with Anupam Das, Sonia Marin)

We introduce an intuitionistic version of Gödel-Löb modal logic GL (the provability logic of Peano Arithmetic) in the style of Simpson [7]. We develop a non-wellfounded labelled proof theory and coinciding birelational semantics, and we call the resulting logic IGL. While existing intuitionistic versions of GL are typically defined over only the box (and not the diamond), IGL includes both modalities. One of its interests is that it allows for the Gödel-Gentzen negative translation into GL which is promising to recover a computational interpretation of classical GL.

Semantics for IGL

Well-known intuitionistic modal logic iGL is sound and complete with respect to *birelational models* $(W, \leq, R, V)$ such that $(\leq; R) \subseteq R$ and R is transitive and conversely wellfounded [8]. The valuation V is *persistent*, i.e., monotone in $\leq$. To interpret the $\Diamond$, the models for iGL are too restrictive. In this work we adopt the same frame conditions as [7], i.e., $(R^{-1}; \leq) \subseteq (\leq; R^{-1})$ and $(R; \leq) \subseteq (\leq; R)$, and further require R to be transitive and $(R; \leq)$ to be conversely wellfounded. We call this class of models $\mathscr{B}\mathsf{IGL}$.

One can view (this form of intuitionistic) modal logic as a fragment of (intuitionistic) predicate logic under the standard translation, cf. [7]. In this sense, we obtain another intuitionistic reading of GL, by interpreting the converse wellfoundedness of $(R; \leq)$ within a predicate Kripke models. We denote this class by $\mathscr{P}\mathsf{IGL}$.

Proof theory for IGL

To obtain intuitionistic versions of classical modal logics, it typically suffices to restrict a ‘standard’ calculus, to having one formula on the right of a sequent. For GL, restricting the sequent calculus in [1] and cyclic sequent system in [6], yields calculi for logic iGL [3, 4]. For our setting, labelled systems admitting independent treatments of $\Box$ and $\Diamond$ have been fruitful to define intuitionistic calculi [7]. We develop a labelled calculus for GL taking inspiration from *non-wellfounded proof theory*, where (co)induction principles are devolved to the proof structure rather than explicit rules or axioms. Note that, in contrast to the labelled system for GL in [5], we do not modify the usual labelled rules for $\Box$ and $\Diamond$. From this we define a single-succedent and a multi-succedent non-wellfounded labelled system for IGL, denoted $\ell\mathsf{IGL}$ and $\mathsf{m}\ell\mathsf{IGL}$, respectively.

$$\begin{tikzcd} \ell\mathsf{IGL} \arrow[r, "\text{soundness}"] & \text{birelational models } \mathscr{B}\mathsf{IGL} \arrow[d] \\ \mathsf{m}\ell\mathsf{IGL} \arrow[u, "\text{simulation + (partial) cut-elim}"] & \text{predicate models } \mathscr{P}\mathsf{IGL} \arrow[l, "\text{countermodel via proof search}"] \end{tikzcd}$$

FIGURE 1. Summary of main results. All arrows denote inclusions of modal logics, so the four characterisations coincide.

Results

Our main result is that these notions coincide as depicted in Figure 1. Soundness for both aforementioned classes of models is readily established via an infinite descent argument by contradiction that is now standard in non-wellfounded proof theory. For completeness, we provide a predicate countermodel construction from a failed proof search in the multi-succedent calculus $\mathsf{m}\ell\mathsf{IGL}$ by appealing to the (lightface) analytic determinacy result for the corresponding 'proof search game'. Simulations using cuts show the equivalence between $\ell\mathsf{IGL}$ and $\mathsf{m}\ell\mathsf{IGL}$ concluding our result. All results can be found in [2].

In future work we would like to establish an explicit axiomatisation for the logic introduced. At the same time it would also be pertinent to investigate the complexity of our logic, given our hitherto non-finitary-presentations. Finally, we would like to examine the role of our logic as a logic of provability in appropriate models of Heyting Arithmetic.

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The Biggest Five of Reverse Mathematics

Sam Sanders
(joint work with Dag Normann)

1. The Biggest Five phenomenon and its limits

The aim of the program *Reverse Mathematics* (RM for short) is to find the minimal axioms needed to prove a given theorem of ordinary mathematics. The *Big Five phenomenon* of RM is the observation that many (perhaps even 'most') theorems are equivalent to one of four logical systems, assuming a weak logical system called the *base theory*. These five systems are called the *Big Five*.

In [7, 12], the Big Five phenomenon is greatly **extended** by establishing numerous equivalences involving the **second-order** Big Five on one hand, and well-known **third-order** theorems from analysis about discontinuous functions on the other hand, working in Kohlenbach's base theory RCA_0^ω from [3, §2]. By [7, §2.8], *slight* variations/generalisations of these third-order theorems cannot be proved from the Big Five and *much* stronger systems. A basic example is as follows.

- Over RCA_0^ω, WKL_0 is equivalent to the *supremum principle* for any of the following: Baire 1, cadlag, quasi-continuity, normal bounded variation.
- Over RCA_0^ω, the Big Five (and much stronger[1] systems like Z_2^ω) cannot prove the supremum principle for any of the following: bounded variation, regulated, cliquish, semi-continuity, Baire 2.

The supremum principles and associated function classes in the first item are called **second-order ish**: although they are third-order in nature, they can be proved from second-order comprehension principles (only). While second-order RM generally deals with countable and separable constructs, quasi-continuity is much wilder[2], yet part of the RM of WKL_0, which is perhaps unexpected.

Many similar examples exist, including for the other Big Five, e.g. the supremum principle for *effectively Baire 2* functions, the Jordan decomposition, and basic properties of the Riemann integral. A full(er) list may be found in [7, 12].

Finally, Rathjen states in [8] that Π_2^1-CA_0 *dwarfs* Π_1^1-CA_0 and Martin-Löf talks of a *chasm* and *abyss* between these two in [4]. The previous examples show that small variations of second-order-ish theorems go *far* beyond the Big Five and Π_2^1-CA_0, far beyond the aforementioned abyss.

[1]The system Z_2^ω proves the same second-order sentences as Z_2 ([2]). Here, Z_2^ω is RCA_0^ω extended with, for each $k \geq 1$, the functional S_k^2 which decides Π_k^1-formulas.

[2]If $\mathfrak{c}$ is the cardinality of $\mathbb{R}$, there are $2^{\mathfrak{c}}$ non-measurable quasi-continuous $[0,1] \to \mathbb{R}$-functions and $2^{\mathfrak{c}}$ measurable quasi-continuous $[0,1] \to [0,1]$-functions (see [1]).

2. Exploring the Abyss: Kleene's quantifiers

The results in Section 1 are based on the RM of Kleene's quantifiers $(\exists^2)$ and $(\exists^3)$, which is interesting in its own right, and discussed in this section.

First of all, Kohlenbach proves the equivalence between the following in [3, §2].

- Kleene's $(\exists^2) : (\exists E : \mathbb{N}^{\mathbb{N}} \to \mathbb{N})(\forall f \in \mathbb{N}^{\mathbb{N}})(E(f) = 0 \leftrightarrow (\exists n \in \mathbb{N})(f(n) = 0)$.
- *There exists a discontinuous function $f : \mathbb{R} \to \mathbb{R}$.*

Moreover, $(\exists^2)$ is also equivalent to the following (see [7, §2] for a complete list).

- There exists a function $f : [0, 1] \to \mathbb{R}$ that is not Baire 1.

There are *many* similar equivalences, but following surprise also lies in wait: the system Z_2^ω, a conservative extension of Z_2, cannot prove that

There exists a function $f : [0, 1] \to \mathbb{R}$ that is not Baire 2.

We invite the reader to contemplate the meaning of 'a code for a Baire 3 function' in light of the previous result. Since it is consistent with Z_2^ω that all functions are Baire 2, we find there to be very little meaning in this coding construct.

Secondly, while at the far edges of the subject, the RM of $(\exists^3)$ can be surprisingly basic, as follows. Now, there are dozens (hundreds?) of **decompositions of continuity**, where continuity is shown to be equivalent to the combination of two or more weak continuity[3] notions, going back to Baire, as follows:

$$\textit{continuity} \leftrightarrow \textit{weak continuity notion } \mathsf{A} \text{ plus } \textit{weak continuity notion } \mathsf{B}. \qquad (\mathsf{D})$$

It is then a natural question whether these weak continuity notions are as *tame* as continuity, e.g. how hard is it to find the supremum of weakly continuous functions? We note that Kohlenbach in [3, §3] singles out this supremum functional as an interesting object of study.

Now, most of these weak continuity notions are rather *tame*: working in $\mathsf{RCA}_0^\omega + (\exists^2)$, one can define the supremum functional $\lambda p, q, f. \sup_{y\in[p,q]} f(y)$ restricted to f satisfying the weak continuity notion at hand.

By contrast, there are seven weak continuity notions that are rather exceptional. In particular, over RCA_0^ω, the following are equivalent.

- Kleene's $(\exists^3)$: $(\exists E)(\forall Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N})(E(Y) = 0 \leftrightarrow (\exists f \in \mathbb{N}^{\mathbb{N}})(Y(f) = 0)$,
- Kleene's quantifier $(\exists^2)$ plus the existence of a supremum functional for any of these classes: the Young condition, almost continuity (Husain), graph continuity, not of Cesàro type, peripheral, pre-, or $\mathcal{C}$-continuity.

These weak continuity notions exist in the literature, side-by-side with the tame ones, and two go back over a hundred years.

[3] We note that *weak and generalised continuity* come with its own AMS code, namely 54C08, i.e. weak continuity is not a fringe topic in mathematics.

3. New Big systems

We list four third-order theorems that boast many equivalences, similar to the original Big Five, with some hints on the kind of principles involved.

- The *uncountability of* $\mathbb{R}$ ([6, 10, 12]) is equivalent to basic properties of regulated and bounded variation functions.
- The *Jordan decomposition theorem* ([5, 12]) is equivalent to the fact that countable sets can be enumerated.
- The *Baire category theorem* ([11, 12]) is equivalent to basic properties of semi-continuous functions.
- The *pigeon-hole principle* for the Lebesgue measure ([11, 12]) is equivalent to one direction of the Vitali-Lebesgue theorem.

For the first two items, the following definition of 'countable set' is used. No elegant equivalences are known for the usual definition based on injections to $\mathbb{N}$.

Definition 1. *A set* $A \subset \mathbb{R}$ *is* height-countable *if there is a* height *function* $H : \mathbb{R} \to \mathbb{N}$ *for* A*, i.e. for all* $n \in \mathbb{N}$*,* $A_n := \{x \in A : H(x) < n\}$ *is finite.*

Definition 2 (Finite set)**.** *Any* $X \subset \mathbb{R}$ *is* finite *if there is* $N \in \mathbb{N}$ *such that for any finite sequence* $(x_0, \ldots, x_N)$ *of distinct reals, there is* $i \leq N$ *such that* $x_i \notin X$.

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On the theory of exponential integer parts

Emil Jeřábek

An *integer part (IP)* of an ordered ring R is a discretely ordered subring $I \subseteq R$ such that every $x \in R$ is within distance 1 from I. (By abuse of language, we will conflate a discretely ordered ring I with the ordered semiring $I_{\geq 0}$.) A classical result of Shepherdson [5] characterizes models of IOpen (= Robinson's arithmetic + induction for open formulas in the language $\mathcal{L}_{\mathrm{OR}} = \langle 0, 1, +, \cdot, < \rangle$):

Theorem 1. *Integer parts of real-closed fields are exactly the models of* IOpen*.*

Let an *exponential field* be an ordered field R endowed with an isomorphism $\exp\colon \langle R, 0, 1, +, < \rangle \to \langle R_{>0}, 1, 2, \cdot, < \rangle$, optionally satisfying the *growth axiom (GA)* $\exp(x) > x$. Introduced by Ressayre [4], an *exponential integer part (EIP)* of an exponential ordered field $\langle R, \exp \rangle$ is an IP $I \subseteq R$ such that $I_{\geq 0}$ is closed under exp. We are interested in the question of characterizing (non-negative parts of) ordered rings that are EIP of real-closed exponential fields (RCEF), and in particular, what is the first-order theory of such rings. This problem (and in particular, the question whether this theory properly extends IOpen) was raised by Jeřábek [2], who provided an upper bound: all countable models of the bounded arithmetical theory VTC^0 in $\mathcal{L}_{\mathrm{OR}}$ are EIP of RCEF.

Extensions of Theorem 1 to exponential ordered fields were previously studied by Boughattas and Ressayre [1] and Kovalyov [3], but they focussed on generalizing the other direction of the theorem (e.g., what additional properties of RCEF ensure that their EIP are models of open induction in a language with exponentiation?). Moreover, they were mostly concerned with EIP in a language with the binary powering operation $x^y = \exp(y \log x)$. Since $\langle I, +, \cdot, <, x^y \rangle$ can define approximations of exp on its fraction field F, we can canonically extend exp to the completion of F; but no such direct construction seems possible for EIP in $\mathcal{L}_{\mathrm{OR}}$ or $\mathcal{L}_{\mathrm{OR}} \cup \{2^x\}$, hence our arguments will be of different nature.

The main goal of this talk is to present complete axiomatizations of the first-order theories of EIP of RCEF in $\mathcal{L}_{\mathrm{OR}} \cup \{2^x\}$, $\mathcal{L}_{\mathrm{OR}} \cup \{P_2\}$ (where P_2 is a predicate for the image of 2^x), and $\mathcal{L}_{\mathrm{OR}}$, and determine some properties of these theories.

Our first result can be proved by an easy application of Robinson's joint consistency theorem:

Theorem 2. *The theory* TEIP_{2^x} *of EIP of RCEF in* $\mathcal{L}_{\mathrm{OR}} \cup \{2^x\}$ *is axiomatized over* IOpen *by*

$$\begin{aligned}&x > 0 \to \exists y\, x < 2^y \leq 2x,\\ &2^{x+y} = 2^x 2^y,\\ &2^x \neq 0.\end{aligned}$$

The theory of EIP of RCEF satisfying GA is $\mathsf{TEIP}_{2^x} + \mathsf{GA}$.

Next, we treat the language with a predicate for powers of 2:

Theorem 3. *The theory* TEIP_{P_2} *of EIP of RCEF, with or without GA, in* $\mathcal{L}_{\mathrm{OR}} \cup \{P_2\}$ *is axiomatized over* IOpen *by*

$$\begin{aligned}&x > 0 \to \exists u\,(P_2(u) \wedge u \leq x < 2u),\\ &P_2(u) \wedge P_2(v) \wedge u \leq v \to \exists w\,(P_2(w) \wedge uw = v).\end{aligned}$$

The conservativity of TEIP_{2^x} over TEIP_{P_2} is, again, proved by a simple application of joint consistency; for $\mathsf{TEIP}_{2^x} + \mathsf{GA}$, we need a rather more complex back-and-forth argument on a countable recursively saturated model of TEIP_{P_2}.

We mention here that Shepherdson's [5] model of IOpen expands to a model of TEIP_{P_2}, but not to a model of TEIP_{2^x}.

For any $\mathfrak{M} \vDash \mathsf{IOpen}$ and $n \in \mathbb{N}$, the *power-of-two game* $\mathrm{PowG}_n(\mathfrak{M})$ is played between two players, Challenger (C) and Powerator (P), in n rounds: in each round $0 \leq i < n$, C picks $x_i \in M_{>0}$, and P responds with $u_i \in M_{>0}$ such that $u_i \leq x_i < 2u_i$. C wins if $u_i u_j < u_k < 2u_i u_j$ for some $i, j, k < n$, otherwise P wins. (While not part of the official rules, we may note that if $u_i < u_j$ but $u_i \nmid u_j$ for some i, j, C can force a win in the next round by playing $\lfloor u_j / u_i \rfloor$.)

The motivation for the game is that if $\langle \mathfrak{M}, P_2 \rangle \vDash \mathsf{TEIP}_{P_2}$, then "play $u_i \in P_2$" is a winning strategy for P. The theory of EIP in the basic language $\mathcal{L}_{\mathrm{OR}}$ is now axiomatized by a schema asserting that Powerator has a winning strategy in PowG for an arbitrary number of rounds:

Theorem 4. *The theory* TEIP *of EIP of RCEF (with or without GA) in* $\mathcal{L}_{\mathrm{OR}}$ *is axiomatized over* IOpen *by the sentences*

$$\forall x_0\, \exists u_0 \dots \forall x_{n-1}\, \exists u_{n-1} \Big(\bigwedge_{i<n} (x_i > 0 \to u_i \leq x_i < 2u_i) \wedge \bigwedge_{i,j,k<n} \neg (u_i u_j < u_k < 2u_i u_j) \Big)$$

for all $n \in \mathbb{N}$.

The idea of the proof is that if $\mathfrak{M} \vDash \mathsf{TEIP}$ is countable and recursively saturated, then P has a winning strategy in "$\mathrm{PowG}_\omega(\mathfrak{M})$", and if we let C enumerate all elements of M, the responses of P form a set P_2 such that $\langle \mathfrak{M}, P_2 \rangle \vDash \mathsf{TEIP}_{P_2}$.

We mention that Svenonius [7] gave a general construction of an axiomatization of a reduct of a given theory by means of sentences expressing the existence of winning strategies in a certain game, mimicking a Henkin completion procedure. However, this axiomatization is rather opaque; in contrast, our game is explicit enough that we are able to derive useful properties of TEIP from it.

First, using the existence of a nonstandard model of IOpen that is a UFD (Smith [6]), we can show that TEIP properly extends IOpen:

Theorem 5. *The following consequence of* TEIP *is not provable in* IOpen*:*

$$\forall x\, \exists u > x\, \forall y\, \bigl(0 < y < x \to \exists v\, (v \le y < 2v \wedge v \mid u)\bigr).$$

We also make partial progress on the main remaining problem about TEIP:

Question 6. *Is* TEIP *finitely axiomatizable over* IOpen*?*

Let us write $\mathsf{TEIP} = \mathsf{IOpen} + \{\forall x_0 > 0\, \exists u_0\, \bigl(u_0 \le x_0 < 2u_0 \wedge \theta^1_n(u_0)\bigr) : n \in \mathbb{N}\}$, where $\theta^1_n(u_0)$ denotes

$$\forall x_1\, \exists u_1 \dots \forall x_{n-1}\, \exists u_{n-1} \Bigl(\bigwedge_{1\le i<n} (x_i > 0 \to u_i \le x_i < 2u_i) \wedge \bigwedge_{i,j,k<n} \neg(u_i u_j < u_k < 2u_i u_j)\Bigr).$$

If $\{\theta^1_n : n \in \mathbb{N}\}$ contained only finitely many inequivalent formulas, then TEIP would be finitely axiomatizable over IOpen, but this is not the case:

Theorem 7. *The formulas* θ^1_n *form an infinite hierarchy over* $\mathrm{Th}(\mathbb{N})$.

We show this by analysis of the power-of-two game. Let $\mathrm{PowG}^1_n(u)$ denote the game $\mathrm{PowG}_n(\mathbb{N})$ where the first round is fixed such that P plays $u_0 = u$ (x_0 does not matter). If u is not a power of 2, then C has a winning strategy in $\mathrm{PowG}^1_{n+1}(u)$ for sufficiently large n; let the smallest such n be denoted $c(u)$. Then Theorem 7 amounts to $\sup\{c(u) : u \text{ not a power of } 2\} = +\infty$, which follows from:

Theorem 8. *Let* $u = 2^{\nu_2(u)} v^r$*, where* $v > 1$ *is not a perfect power. Then*

$$c(u) \le \log\log\log \min\{\nu_2(u), r\} + O(1) \le \log\log\log\log u + O(1);$$

more precisely, $c(u) \le \log\log d + O(1)$ *for any* $d \nmid r$*. On the other hand,*

$$c(u) \ge \min\left\{\log\log\log \frac{\nu_2(u)}{\log v}, \log\log d : d \nmid r\right\} + O(1).$$

For example, this shows that $c\bigl(6^{2^{2^k}!}\bigr) = k + O(1)$.

Another consequence of Theorem 8 is that there are models $\langle \mathfrak{M}, P_2\rangle \vDash \mathrm{Th}(\mathbb{N}) + \mathsf{TEIP}_{P_2}$ such that P_2 is distinct from the set of "oddless numbers" (i.e., whose all nontrivial divisors are even); indeed, $u \in P_2$ may even be divisible by 3.

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Quantitative Analysis of Stochastic Approximation Methods

Paulo Oliva
(joint work with Rob Arthan)

In an ongoing case study in Stochastic Approximation Theory, Rob Arthan and I have been working on a quantitative version of Derman-Sachs' proof [3] of Dvoretzky's theorem [4], a vast generalisation of the well-known Robins-Monro seminal stochastic approximation method [5]. Our current proof mining builds on our recent quantitative analysis of the Borel-Cantelli lemmas [1] – one of the ingredients in Derman-Sachs proof. This case study has been proven to be extremely interesting for several reasons.

Firstly, arguments in Probability Theory (and also Measure Theory) look a priori extremely ineffective and non-computational. Most arguments rely on uses of set comprehension to form increasing or decreasing sequences of events, or the axiom of *countable additivity*, which does not seem to have a clear constructive interpretation. Examples of these are the Continuity from Above/Below Lemma and Egorov's Theorem. We rely on recent work of Avigad et. al. [2] and interpret almost sure convergence statements about sequences of random variables

$$\mathbb{P}[\{\omega \mid \forall \varepsilon > 0 \exists N \forall i, j \geq N(|X_i(\omega) - X_j(\omega)| \leq \varepsilon)\}] = 1$$

via a λ-uniform ε-convergent modulus Φ, i.e.

$$\forall \varepsilon, \delta > 0\, (\mathbb{P}[\{\omega \mid \forall i, j \geq \Phi(\varepsilon, \delta)(|X_i(\omega) - X_j(\omega)| \leq \varepsilon)\}] \geq 1 - \lambda)\,.$$

Egorov's Theorem states that a certain sequence of random variables converges, and our proof mining extracts an explicit λ-uniform ε-convergent modulus Φ for the sequence.

Secondly, Derman-Sachs' proof is also extremely interesting in that it uses several subtle lemmas about sequences of real numbers, which as far as we know have not been proof mined yet. The main ones are:

(1) If $\sum a_n$ is a convergent series and $\{b_n\}$ is monotone and bounded then $\sum a_n b_n$ is also a convergent series (Abel's test),
(2) If $\{b_n\}$ is a sequence of non-negative reals such that the series $\sum b_n$ converges then the sequence $\{1/B_n\}$ also converges, where $B_n = \prod_{i \leq n}(1+b_i)$,

(3) If $\{b_n\}$ is a sequence of non-negative reals such that the series $\sum b_n$ converges then there exists a sequence a_n which converges to 0 such that $\sum b_n/a_n^2$ still converges.

It seems to us that a shared repository of results about converging or diverging sequences and series of real numbers which have already been "mined" would be a very useful resource.

Finally, Derman-Sachs relies on a form a "Transfer Principle", whereby the almost sure convergence of a sequence of random variables $\{X_n\}$ is proven by finding a suitable event E where for $\omega \in E$ the convergence of the sequence of real numbers $x_n = X_n(\omega)$ can be derived. Ensuring that the rates on the convergence of the sequences of reals are uniform enough for this transfer to be possible is part of the the challenge in this proof mining case study.

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Double negation and conservation

Peter Schuster
(joint work with Giulio Fellin)

1. Heuristics

Recall that from derivability in minimal logic $\vdash_m$ one obtains derivability first in intuitionistic logic $\vdash_i$ and then in classical logic $\vdash_c$ by allowing as additional axioms finitely many instances of (in first-order logic: the universal closures of) *ex falso quodlibet* $\bot \to B$ and *tertium non datur* $B \vee \neg B$, respectively: that is,

$$\Gamma \vdash_i A \equiv \Gamma, \mathrm{EFQ} \vdash_m A\,, \qquad \Gamma \vdash_c A \equiv \Gamma, \mathrm{TND} \vdash_i A\,.$$

With double negation, Glivenko's theorem [4] for propositional logic can be put as

$$\Gamma \vdash_c A \Longrightarrow \Gamma \vdash_i \neg\neg A\,.$$

In view of $\vdash_m \neg\neg(B \vee \neg B)$, Glivenko's theorem follows from Brouwer's lemma:

$$\Delta, D \vdash_* \neg C \Longrightarrow \Delta, \neg\neg D \vdash_* \neg C \quad (* \in \{m, i\})\,.$$

This and Odintsov's [6] have brought us to analyse the conclusion of Glivenko's theorem in terms of $\vdash_m$:

$$\Gamma \vdash_i \neg\neg A \iff \Gamma, \mathrm{EFQ} \vdash_m \neg\neg A \iff \Gamma, \neg\neg\mathrm{EFQ} \vdash_m \neg\neg A\,.$$

Since $\nvdash_m \neg\neg(\bot \to B)$ in general, if $\vdash_i$ were replaced by $\vdash_m$, then Glivenko's theorem would fail already for $A \equiv \bot \to B$.

Lemma 1. $\neg\neg$EFQ *is equivalent, over* $\vdash_m$*, to the* double negation shift for $\to$*:*

$$\mathrm{DNS}_{\to}: \quad (B \to \neg\neg C) \to \neg\neg(B \to C).$$

So Glivenko's theorem can alternatively be put with $\vdash_m$ as follows:

$$\Gamma \vdash_c A \Longrightarrow \Gamma, \mathrm{DNS}_{\to} \vdash_m \neg\neg A.$$

While $\mathrm{DNS}_{\to}$ is provable with $\vdash_i$ and thus has hitherto remained invisible in Glivenko's theorem, it is in analogy to

(1) the double negation shift for $\forall$, viz.

$$\mathrm{DNS}_{\forall}: \quad \forall x \neg\neg Cx \to \neg\neg\forall x Cx\,,$$

in Kuroda's [5] generalisation of Glivenko's theorem to first-order logic:

$$\Gamma \vdash_c A \Longrightarrow \Gamma, \mathrm{DNS}_{\forall} \vdash_i \neg\neg A\,;$$

(2) the double negation shift for $\bigwedge_{\mathbb{N}}$, viz.

$$\mathrm{DNS}_{\bigwedge_{\mathbb{N}}}: \quad \bigwedge_{n\in\mathbb{N}} \neg\neg Cn \to \neg\neg \bigwedge_{n\in\mathbb{N}} Cn\,,$$

in Tesi's [7] counterpart of Kuroda's theorem for infinitary logic:

$$\Gamma \vdash_c A \Longrightarrow \Gamma, \mathrm{DNS}_{\bigwedge_{\mathbb{N}}} \vdash_i \neg\neg A\,.$$

2. Conservation for nuclei

Let S be a set and $\rhd$ an *inductively generated* single-succedent *entailment relation*: that is, $\rhd \subseteq \mathcal{P}_{<\omega}(S) \times S$ is the least such relation which satisfies certain generating axioms and rules on top of the following three *structural rules*:[1]

$$\text{reflexivity: } \frac{}{U, a \rhd a} \qquad \text{monotonicity: } \frac{U \rhd a}{U, V \rhd a} \qquad \text{transitivity: } \frac{U \rhd a \quad V, a \rhd b}{U, V \rhd b}$$

By a *nucleus* over $\rhd$ we understand a map $j\colon S \to S$ satisfying

$$U, a \rhd jb \iff U, ja \rhd jb.$$

We consider two entailment relations which contain $\rhd$:

— the *weak* or *Kleisli extension* is defined by $U \rhd_j a \equiv U \rhd ja$;

— the *strong* or *stable extension* $\rhd^j$ is inductively generated by the same axioms and rules as $\rhd$ plus the axiom of *stability* $ja \rhd a$.

[1]By an axiom we understand a premissless rule; for instance, reflexivity is an axiom. Unless one needs to distinguish axioms from rules, one may subsume the former under the latter.

Note that always $\rhd_j \subseteq \rhd^j$. If $\rhd \equiv \vdash_i$ and $j \equiv \neg\neg$, then $\rhd^j \equiv \vdash_c$, and Glivenko's theorem means *conservation*: that is, $\rhd_j \supseteq \rhd^j$.

While the stable extension $\rhd^j$ by its very inductive definition satisfies all axioms and rules of $\rhd$, the Kleisli extension $\rhd_j$ *a priori* satisfies—in addition to the structural rules—only all axioms of $\rhd$.

Theorem 1. *$\rhd_j \supseteq \rhd^j$ if and only if $\rhd_j$ satisfies all (non-axiom) rules of $\rhd$.*

In fact, stability is automatic for $\rhd_j$, because $ja \rhd_j a \equiv ja \rhd ja$.

Corollary 1. *$\rhd_j = \rhd^j$ whenever $\rhd$ is inductively generated by axioms only.*

We hasten to add that for applying Theorem 1 and Corollary 1 it is irrelevant which axioms and rules we take for the inductive generation of $\rhd$. In fact, collections $\mathcal{R}$ and $\mathcal{R}'$ of axioms and rules generate the same $\rhd$ precisely when every member of $\mathcal{R}$ is the composition of members of $\mathcal{R}'$ and vice versa; and "to hold for the Kleisli extension $\rhd_j$" is closed under composition of rules.

3. Applications to logic

Let $\rhd$ be $\vdash_m$. For propositional logic this is generated by the axioms and rules

$$\frac{}{A \wedge B \rhd A}\mathrm{L}\wedge_1 \qquad \frac{}{A \wedge B \rhd B}\mathrm{L}\wedge_2 \qquad \frac{}{A, B \rhd A \wedge B}\mathrm{R}\wedge$$

$$\frac{}{A \vee B, A \to C, B \to C \rhd C}\mathrm{L}\vee \qquad \frac{}{A \rhd A \vee B}\mathrm{R}\vee_1 \qquad \frac{}{B \rhd A \vee B}\mathrm{R}\vee_2$$

$$\frac{}{A \to B, A \rhd B}\mathrm{L}\to \qquad \frac{\Gamma, A \rhd B}{\Gamma \rhd A \to B}\mathrm{R}\to \qquad \frac{}{\rhd \top}\mathrm{R}\top$$

From this variant of minimal propositional logic one obtains

(1) minimal first-order logic by adding the axioms and rules

$$\frac{}{\forall x\, A \rhd A[t/x]}\mathrm{L}\forall \qquad \frac{\Gamma \rhd A[y/x]}{\Gamma \rhd \forall x\, A}\mathrm{R}\forall \quad (y \text{ fresh})$$

$$\frac{}{\exists x\, A, \forall x(A \to B) \rhd B}\mathrm{L}\exists \quad (x \notin \mathrm{FV}(B)) \qquad \frac{}{A[t/x] \rhd \exists x\, A}\mathrm{R}\exists$$

(2) minimal infinitary logic by adding the axioms and rules

$$\frac{}{\bigwedge_{i\in\mathbb{N}} A_i \rhd A_n}\mathrm{L}\bigwedge\nolimits_n \ (n \in \mathbb{N}) \qquad \frac{\{\Gamma \rhd A_n : n \in \mathbb{N}\}}{\Gamma \rhd \bigwedge_{i\in\mathbb{N}} A_i}\mathrm{R}\bigwedge$$

$$\frac{}{\bigvee_{i\in\mathbb{N}} A_i, \bigwedge_{i\in\mathbb{N}}(A_i \to B) \rhd B}\mathrm{L}\bigvee \qquad \frac{}{A_n \rhd \bigvee_{i\in\mathbb{N}} A_i}\mathrm{R}\bigvee\nolimits_n \ (n \in \mathbb{N})$$

The only non-axiom rules of the calculi above are R$\to$, R$\forall$ and R$\bigwedge$.

Now let j be a nucleus—compatible with substitution for first-order logic [8]:

$$j(A[t/x]) = (jA)[t/x]\,.$$

Lemma 2. *Each of* R$\to$, R$\forall$ *and* R$\bigwedge$ *holds for $\rhd_j$ if any only if the variant of* $\mathrm{DNS}_\to$, $\mathrm{DNS}_\forall$ *and* $\mathrm{DNS}_\bigwedge$, *respectively, obtains in which $\neg\neg$ is replaced by j.*

logic	**non-axiom rule**	R_ **holds for** $\rhd_j$ **iff**	**case** $j \equiv \neg\neg$
propositional	R$\to$	$B \to jC \rhd j(B \to C)$	DNS$_\to$
first-order	R$\forall$	$\forall x\, jB \rhd j\forall x\, B$	DNS$_\forall$
infinitary	R$\bigwedge$	$\bigwedge_{n\in\mathbb{N}} jB_n \rhd j \bigwedge_{n\in\mathbb{N}} B_n$	DNS$_\bigwedge$

As for the true DNS, the converse $\lhd$ is automatic in the third column. E.g. R$\to$ holds for $\rhd_j$ if and only if j commutes with every open nucleus [1, 8].

Generalisations include Glivenko-style conservation theorems for the translations ascribed to Kolmogorov, Gentzen and Kuroda in place of double negation.

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Prenex normalization and the hierarchical classification of formulas

Makoto Fujiwara

(joint work with Taishi Kurahashi)

In this workshop, I gave a talk about my recent work [3] on the prenex normalization of first-order formulas by the standard reduction procedure without any reference to the notion of derivability, as well as some ongoing attempt after the work.

The prenex normal form theorem states that for any first-order theory based on classical logic, every formula is equivalent (over the theory in question) to some formula in prenex normal form. This theorem is verified by using the fact that several transformations of formulas moving quantifiers in the formula from inside to outside in a suitable way are admissible in first-order classical logic. For example, if x is not contained in δ, then $\forall x \xi(x) \to \delta$ is transformed into $\exists x(\xi(x) \to \delta)$ with preserving classical validity because they are classically equivalent. For each first-order formula, one can obtain an equivalent formula in prenex normal form by the following procedure:

(1) Apply the above mentioned transformations finitely many times to the subformulas of the form $A \circ B$ with A and B in prenex normal form where $\circ \in \{\wedge, \vee, \to\}$, and transform the subformulas into equivalent formulas in prenex normal form;
(2) Repeating this procedure until when all subformulas become to be in prenex normal form.

Akama, Berardi, Hayashi and Kohlenbach [1] introduced the classes E_k and U_k of formulas defined by counting the number of the alternations of quantifiers in a given formula (the formal definitions are given in [2]). The class E_k (resp. U_k) is intended to form the class of formulas which are classically equivalent to some Σ_k-formula (resp. Π_k-formula). In addition, as mentioned in [1], the class P_k is intended to represent the set of Δ_{k+1}-formulas, namely, formulas which is equivalent to some Σ_{k+1}-formula and also to some Π_{k+1}-formulas. Note that every formula with quantifier occurrences is classified into exactly one of E_{k+1}, U_{k+1} and P_{k+1}.

In [3], Kurahashi and the author gave a proper justification for the hierarchical classes. They formalized the above mentioned procedure for prenex normalization and investigated the relation between the classes of prenex formulas and the hierarchical classes in [1, 2] modulo the transformation procedure in a general language of a first-order theory. In particular, they showed that a formula is in E_k^+ (resp. U_k^+) if and only if it can be transformed into a formula in Σ_k^+ (resp. Π_k^+) by the transformation procedure, where $\mathrm{E}_k^+, \mathrm{U}_k^+, \Sigma_k^+$ and Π_k^+ are cumulative variants of $\mathrm{E}_k, \mathrm{U}_k, \Sigma_k$ and Π_k, respectively. By the results for the cumulative classes, it also follows that non-cumulative classes E_k, U_k and P_k (except P_0) are the cumulative counterparts of Σ_k, Π_k and Δ_{k+1} respectively modulo the transformation procedure.

In addition, I have presented an ongoing attempt about a classification of first-order formulas based on hierarchical prenex normalization procedures restricted to those which are admissible in intuitionistic and semi-classical theories. For that purpose, we introduce new hierarchical classes of first-order formulas and characterize the classes by the hierarchical prenex normalization procedures restricted to some of those classes.

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Proof mining, applications to optimization, and interactive theorem proving

Horaţiu Cheval

Let H be a Hilbert space, $(T_n : H \to H)$ be a family of nonexpansive mappings and consider the problem of finding a common fixed point $x \in \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(T_n)$. Boţ, and Meier [1] introduced an iterative method for finding such a point, which proceeds by constructing the sequence (x_n) via

$$x_{n+1} = (1-\lambda_n)\beta_n x_n + \lambda_n T_n(\beta_n x_n), \tag{1}$$

where $(\lambda_n), (\beta_n)$ are sequences in $[0,1]$, and $x_0 \in H$ is arbitrary. The main results of [1] show that, under certain conditions on (λ_n), (β_n) and (T_n), it holds that

- $\lim\limits_{n\to\infty} \|x_n - x_{n+1}\| = 0$;
- $\lim\limits_{n\to\infty} \|x_n - T_n x_n\| = 0$;
- $\lim\limits_{n\to\infty} x_n = x$, for some $x \in \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(T_n)$.

The first two results are also known as the asymptotic (resp. (T_n)-asymptotic) of (x_n).

We present [3] an extension of this iteration from the setting of Hilbert spaces to the nonlinear case of W-hyperbolic spaces, in the sense of [7]. For X a W-hyperbolic space and $(T_n : X \to X)$ a family of nonexpansive self-mappings thereof, we define its associated *Tikhonov-Mann iteration* (x_n) by

$$x_{n+1} = (1-\lambda_n)u_n + \lambda_n T_n u_n, \quad \text{where} \tag{2}$$

$$u_n = (1-\beta_n)u + \beta_n x_n, \tag{3}$$

with $(\lambda_n), (\beta_n)$. This simultaneously generalizes (1), as well as the single mapping case studied in W-hyperbolic spaces in [2]. As the main results of [3], we show that the asymptotic regularity of (x_n) still holds in this setting, i.e. that, under certain conditions on (λ_n), (β_n), (T_n) we have that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$, $\lim\limits_{n\to\infty} d(x_n, T_n x_n) = 0$ and that for any $m \in \mathbb{N}$, $\lim\limits_{n\to\infty} d(x_n, T_m x_n) = 0$.

Furthermore, the convergence theorems obtained are enriched with quantitative information, in the form of rates of ((T_n)-, T_m-)asymptotic regularity, which display a high degree of uniformity with respect to the space X and the mappings (T_n). We also present work in progress regarding the generalization of the strong convergence of (x_n) from Hilbert to CAT(0) spaces. This can be carried out effectively, adapting arguments from [4–6, 9]. These results are part of the program of *proof mining* [8].

Finally, in the second part, we discuss work in progress and research ideas for using the Lean interactive theorem prover in proof mining, which can be found at `https://github.com/hcheval/`. This includes the formalization of mathematical results used and obtained in the context proof mining (for example quantitative versions of lemmas widely used in optimization about sequences of real numbers), ideally in the form of a unified library. See also `https://github.com/Kejineri` for such formalizations.

A different direction is the implementation of the general logical metatheorems from proof mining which guarantee the possibility of extracting quantitative content from certain classes of formal proofs. Given the constructive character of these metatheorems, they could be built into automatic program extraction tools, which could then be applied to proofs already formalized in Lean, in order to obtain strengthened variants thereof.

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Equiconsistency of the Minimalist Foundation with its classical version

Maria Emilia Maietti

In our Oberwolfach talk we showed that the Minimalist Foundation, which is a foundation for constructive mathematics, is equiconsistent with its classical version, obtained by extending the underlying logic with the law of excluded middle.

The Minimalist Foundation, for short **MF**, was initially conceived in 2005 in collaboration with Giovanni Sambin in [MS05] and further developed into a comprehensive two-level system in 2009 in [Mai09].

This two-level structure comprises an intensional level, referred to as **mtt**, which is envisioned as a theory possessing sufficiently decidable properties to serve as a foundation for a proof assistant which at the same time might be enriched with a mechanism of program extraction from its proofs. Additionally, there is an extensional level, named **emtt**, which is formulated in a language closely aligned with that of traditional mathematics. Then, **emtt** is interpreted within the intensional level, **mtt**, through the utilization of a quotient model.

One of the main novelties of **MF** is that of serving as a shared core among significant foundations for mathematics. Notably, its estensional level **emtt** is compatible with several prominent mathematical foundations, including the standard axiomatic set theory ZFC, Aczel's Constructive Zermelo-Fraenkel set theory, the general theory of elementary toposes, as shown in [Mai09] (see also [MS22]), and more recently, Homotopy Type Theory and Voedvosky's Univalent Foundations as shown in [CM23]. Instead, its intensional level **mtt** is compatible with Martin-Löf's intensional type theory, Coquand-Huet-Paulin's Calculus of Inductive Constructions, as shown in [Mai09], and again Homotopy Type Theory as shown in [CM23].

When we say that a theory is "compatible" with another theory, we mean that there exists a translation preserving the meaning of logical and set-theoretic operators from the first theory to the latter (and, for example, this implies that the translation commutes with the embedding of Heyting arithmetics with finite types in each of the mentioned theories if the first theory includes it).

As a byproduct **MF** is both constructive and predicative. In particular, the computational contents of proofs developed within **MF** and further extensions with inductive and coinductive topological definitions has been made explicit through realizability models described in [IMMS18, MMR21, MMR22].

In our Oberwolfach talk we showed the remarkable property that both levels of **MF** are still predicative and equiconsistent with the addition of the law of the excluded middle and are all mutually equiconsistent.

It is worth mentioning two key steps to prove our claim.

The first key step of our proof is that we can smoothly extends Goedel-Gentzen's double negation translation of classical Peano arithmetics into the intuitionistic one (for example in [Tv88]), to show that the intensional level **mtt** with the addition of proof-irrelevance for propositions is equiconsistent with its classical version obtained with the further addition of the law of excluded middle. This works because the elimination rule of the propositional identity of **mtt** is equivalent to the usual replacement rules of first-order equality. Proof-irrelevance is then needed to interpret the universe of small classical propositions as the subtype of the **mtt**-universe of small propositions that are $\neg\neg$-stable.

The second key step to show our claim is that **mtt** (with or without the addition of proof-irrelevance for propositions) is equiconsistent with the extensional level **emtt** of **MF** through the use of the quotient model and of canonical isomorphisms in [Mai09]. The proof of this fact extends smoothly to show the equiconsistency of **mtt** (with or without proof-irrelevance) with **emtt** when the law of excluded middle is added to each of them.

We also mention that Pietro Sabelli in his forthcoming PhD's thesis shows that Goedel-Gentzen's double negation translation can be extended to provide a direct interpretation of the classical version of **emtt** into **emtt** (whilst the propositional equality of **emtt** has stronger rules that the usual first-order equality) thanks to the fact that the propositional equality of **emtt** ground types is $\neg\neg$-stable and that

emtt-type-theoretic constructors preserve the $\neg\neg$-stability of their propositional equality.

We conclude by underlying that the predicativity and equiconsistency of the classical version of **MF** with **MF** itself is a peculiarity of **MF** since the other well known constructive and predicative foundations mentioned above, namely Martin-Löf's intensional type theory, Aczel's Constructive Zermelo-Fraenkel set theory and Homotopy Type Theory, do not satisfy this property because they become impredicative when the law of excluded middle is added to their underlying logic.

We leave to future research to investigate whether the extensions of **MF** with inductive and coinductive topological definitions in [MMR21] and [MMR22] are still equiconsistent with their classical version or, at least, are still predicative.

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The computational content of super strongly nonexpansive mappings

Andrei Sipoş

Strongly nonexpansive mappings are a core concept in convex optimization. Recently, they have begun to be studied from a quantitative viewpoint: U. Kohlenbach has identified in [2] the notion of a 'modulus' of strong nonexpansiveness, which leads to computational interpretations of the main results involving this class of mappings (e.g. rates of convergence, rates of metastability). This forms part of the greater research program of 'proof mining', initiated by G. Kreisel and highly developed by U. Kohlenbach and his collaborators, which aims to apply proof-theoretic tools to extract computational content from ordinary proofs in mainstream mathematics (for more information on the current state of proof mining, see the book [1] and the recent survey [3]). The quantitative study of strongly nonexpansive mappings has later led to finding rates of asymptotic regularity for the problem of 'inconsistent feasibility' [4, 7], where one essential ingredient has been a computational counterpart of the concept of rectangularity, recently identified in [5] as a 'modulus of uniform rectangularity'.

Last year, Liu, Moursi and Vanderwerff [6] have introduced the class of 'super strongly nonexpansive mappings', and have shown that this class is tightly linked to that of uniformly monotone operators. What we do is to provide a modulus of super strong nonexpansiveness, give examples of it in the cases e.g. averaged mappings and contractions for large distances and connect it to the modulus of uniform monotonicity. In the case where the modulus is supercoercive, we give a refined analysis, identifying a second modulus for supercoercivity, specifying the necessary computational connections and generalizing quantitative inconsistent feasibility.

The results in this talk may be found in the paper [8].

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Sunny nonexpansive retractions in nonlinear spaces

Pedro Pinto

Undoubtedly, one of the most complicated instances of proof mining to date is the proof-theoretical analysis of Reich's theorem, one of the most pivotal results in functional analysis, carried out in [2].

In this talk, we introduce the notion of a nonlinear smooth space generalizing both CAT(0) spaces as well as smooth Banach spaces [3]. Concretely, we say that a hyperbolic space (X, d, W) (in the sense of [1]) is a *smooth hyperbolic space* if there exists a function $\pi : X^2 \times X^2 \to \mathbb{R}$ satisfying for all $x, y, z, u, v \in X$

(P1) $\pi(\overrightarrow{xy}, \overrightarrow{xy}) = d^2(x, y)$
(P2) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) = -\pi(\overrightarrow{yx}, \overrightarrow{uv}) = -\pi(\overrightarrow{xy}, \overrightarrow{vu})$
(P3) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) + \pi(\overrightarrow{yz}, \overrightarrow{uv}) = \pi(\overrightarrow{xz}, \overrightarrow{uv})$
(P4) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) \leq d(x, y)d(u, v)$

and for any $\lambda \in [0, 1]$

$$\text{(P5)}\ \ d^2(W(x, y, \lambda), z) \leq (1 - \lambda)^2 d^2(x, z) + 2\lambda\pi\Big(\overrightarrow{yz}, \overrightarrow{W(x, y, \lambda)z}\Big).$$

Moreover, we say that (X, d, W, π) is a *uniformly smooth hyperbolic space* if it satisfies additionally

$$\text{(P6)}\ \ \begin{cases} \forall \varepsilon > 0\ \forall r > 0\ \exists \delta > 0\ \forall a \in X\ \forall u, v \in \overline{B}_r(a) \\ \quad d(u, v) \leq \delta \to \forall x, y \in X\ (|\pi(\overrightarrow{xy}, \overrightarrow{ua}) - \pi(\overrightarrow{xy}, \overrightarrow{va})| \leq \varepsilon \cdot d(x, y)) \,. \end{cases}$$

Formally we can consider these spaces in a extension of the system $\mathcal{A}^\omega[X, d, W]$ from [1] where we have a further constant π of type $1(X)(X)(X)(X)$ governed by the axioms (P1)–(P6). Clearly by (P4) π is majorizable, and using (P6) the system proves the extensionality of π. Thus, if we include a modulus of uniform continuity ω_X in the sense of providing a witness for δ in (P6), we have a metatheorem for the extraction of bounds from (formalizable) proofs in this new class of nonlinear spaces.

We discuss that this notion allows for a unified treatment of several mathematical proofs in functional analysis. In particular, we show that Kohlenbach's and Sipoş's treatment of Reich's result can be appropriately discussed in this nonlinear setting.

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Extensional Proof-Systems for Modal Logics

Margherita Zorzi
(joint work with S. Guerrini, S. Martini, A. Masini)

1. Introduction

Designing a robust proof theory for modal logics is a subtle task. The difficulty lies not merely in establishing deductive systems; rather, the real challenge is in formulating a concrete structural proof theory, in which the objects of study are (not only) modal formulas, but also modal proofs. A well defined systems satisfies some desirable properties, such as the the syntactical study of cut elimination/normalization theorem and its consequences (the sub-formula property and the consistency theorem, see [1])). Furthermore, if feasible, one could attempt to define an *extensible* system – a system capable of capturing not only a single logic but an entire family.

In the literature, several deductive styles and approaches to modal proof theory have been introduced. We recall *multidimensional systems*, where the primary concept involves equipping formulas with an index or position, (offering a kind of "spatial coordinate") and Labeled Deductive Systems, where the rules that model the accessibility relationship are explicitly integrated into the syntactical deductive instruments. In this abstract we will focus on natural deduction and on a family of multidimentional systems based on the notion of *position*. The main ideas of our frameworks are the following: formulas are marked with a spatial coordinate; only one introduction rule and one elimination rule per connective; no additional structural rules; no explicit reference to the accessibility relation; only modal operators can "change" the spatial position of the formulas and are treated in analogy of first-order quantifiers. Refer to [2–4] for complete technical details, comprehensive references to related work, and a thorough comparison with the state of the art.

2. From K to S4: the system $\mathcal{N}_{pos}$

A *position-formula* is an expression of the form A^α, where A is a modal formula and α is a position. Positions are constructed based on *tokens*, which are essentially uninterpreted symbols. According to the definition of position we adopt (a sequence, a set, a singleton set) we are able to characterize different modal systems.

The classical natural deduction system $\mathcal{N}_{pos}$ captures the normal extension of the logic K by incorporating the basic axiom $K \equiv \Box(A \to B) \to (\Box A \to \Box B)$ and one or more of the following axioms: $D \equiv \Box A \to \Diamond A$, $T \equiv \Box A \to A$, $4 \equiv \Box A \to \Box\Box A$. This results in K (containing only the basic axiom K) D (K+D), T (K+T), $\mathsf{K4}$ (K+4), $\mathsf{D4}$ (K+D+4), and **S4** (K+T+4).

In $\mathcal{N}_{pos}$ positions are interpreted as *sequences* of tokens (with related operations such as concatenation).

Rules for modal operator are designed as much as possible in analogy with fist-order logic quantifiers:

$$\dfrac{\begin{matrix}\vdots\\ A^{\alpha x}\end{matrix}}{\Box A^{\alpha}}\ (\Box I)\qquad \dfrac{\begin{matrix}\vdots\\ \Box A^{\alpha}\end{matrix}}{A^{\alpha\beta}}\ (\Box E)$$

$$\dfrac{\begin{matrix}\vdots\\ A^{\alpha\beta}\end{matrix}}{\Diamond A^{\alpha}}\ (\Diamond I)\qquad \dfrac{\begin{matrix}\vdots\\ \Diamond A^{\alpha}\end{matrix}\qquad \begin{matrix}[A^{\alpha x}]\\ \vdots\\ C^{\beta}\end{matrix}}{C^{\beta}}\ (\Diamond E)$$

In the rule $\Box I$, one has $\alpha x \notin \mathfrak{Init}[\Gamma]$, where Γ is the set of (open) assumptions on which $A^{\alpha x}$ depends and $\mathfrak{Init}[\Gamma] = \{\beta : \exists A^{\alpha} \in \Gamma.\ \beta \sqsubseteq \alpha\}$. In the rule $\Diamond E$, one has $\alpha x \notin \mathfrak{Init}[\beta]$ and $\alpha x \notin \mathfrak{Init}[\Gamma]$, where Γ is the set of (open) assumptions on which C^{β} depends, with the exception of the discharged assumptions $A^{\alpha x}$. The system K and $\mathsf{K4}$ are partial logics we use existence predicates (à la Scott) for formulating sound deduction rules to deal with partial systems.

All the logical systems share the rules above. To obtain a specific logic, one can "tune" some syntactic constraints, described in the following tables:

name of the calculus	**constraints on the rules $\Box E$ and $\Diamond I$**
$\mathcal{N}_{\mathsf{S4}}$	no constraints
$\mathcal{N}_{\mathsf{T}}$	$\beta = \langle\ \rangle$
$\mathcal{N}_{\mathsf{D}}$	β is a singleton sequence $\langle z\rangle$
$\mathcal{N}_{D4}$	β is non empty
name of the calculus	**constraints on the rules $\Box E$ and $\Diamond I$**
$\mathcal{N}_{\mathsf{K4}}$	β is a non empty sequence
$\mathcal{N}_{\mathsf{K}}$	β is a singleton sequence $\langle z\rangle$

Following Prawitz's original proof for first-order logic, one proves a *Normalization Theorem*: for each derivation Π there exists a derivation Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and Π' is in normal form. As a Corollary, one obtains the *Consistency* of the system(s) (by syntactical arguments): for each position α, $\nvdash_{\mathcal{N}_{Pos}} \bot^{\alpha}$.

The formal definition of semantics of $\mathcal{N}_{Pos}$ is very technical but intuitive. Positions are mapped into nodes of a tree-like Kripke structure (and hence sublists of a position will range on paths of nodes). Each system captured by $\mathcal{N}_{Pos}$ is sound and complete w.r.t. its standard Hilbert-style axiomatization.

3. Beyond S4: the logics S4.2 and S5

What's happen to if we relax the "structure" of the positions?

If we release the ordering and the multiplicity of tokens, then we move from sequences to *sets*, we obtain a (classical) natural deduction system for the logic **S4**.2. The logic **S4**.2 is employed in different settings, ranging from epistemology to the metamathematics of set theory and algebraic structures. It can be derived by adding to **S4** the axiom $2 \equiv \Diamond\Box A \to \Box\Diamond A$. We do not add any additional constraints to the rules except for the usual ones on $\Box I$ and $\Diamond E$. The resulting system $\mathcal{N}_{S4.2}$ is sound and complete w.r.t. the standard Hilbert-style axiomatization of **S4**.2. Moreover, we can prove a Normalization theorem and, as a consequence, a Consistency theorem (again, by means of purely syntactical arguments).

Regarding semantics, the interpretation of position formulas requires an interesting observation. It is well-know that **S4**.2 is characterized (at the level of the accessibility relation) by direct partial preorders. If we were to consider this characterization, in attempting to assign semantics to the positions, we would encounter a non-trivial problem. We have to decide which point in the Kripke model could be uniquely associated with a set of tokens $\{x_1, \ldots, x_n\}$. The standard, naive choice would be to take one of the upper bounds of the worlds associated with each token, but this choice would not be unique, and in a direct pre-order the supremum of a finite set of elements might not exist. However, we can use some results of Goldblatt and Shetmann, which imply that **S4.2** is also characterized by a class of ordered structures different than direct pre-orders, that of semilattices with a minimum element, where the problem disappears. We can now interpret a position (i.e. a set) $\{x_1, \ldots, x_k\}$ as the *least upper bound* of the points (in a space) $x_1, \ldots, x_k$.

We have interpreted positions as general sets. Now, let's restrict the definition of positions to *singleton sets*. What we obtain is $\mathcal{N}_{S5}$ a *indexed* natural deduction for **S**5 logic. The logic **S**5 is obtained by adding the axiom $B \equiv \phi \to \Box\Diamond\phi$ to **S4** and is characterized by a universal semantics (this means that the accessibility relation is an equivalence relation). We do not add any additional constraints to the rules except for an adaptation of the usual ones on $\Box I$ and $\Diamond E$. Both the classical and intuitionistic version $\mathcal{N}_{S5}$ enjoys expected good properties such as the soundness and completeness w.r.t. their Hilbert-style axiomatization, the Normalization Theorem and its consequences. In the case of intuitionistic logic, instead of the universal semantics, it is more interesting to explore a BHK interpretation, that interpret modal operators as follows: a proof of $\Box A^x$ is a construction that for each y gives a proof $f(y)$ of A^y and a proof of $\Diamond A^x$ is a pair (y, a) such that a is a proof of A^y. The BHK interpretation, via the natural deduction system, induces a Curry–Howard Isomorphism in the usual sense. The resulting calculus is similar to λ-P, i.e. the typed lambda-calculus for the negative fragment of first order intuitionistic logic in the so called Barendregt-cube.

The described research is open to various investigations, including the study of the intuitionistic version of the $\mathcal{N}_{Pos}$ system from a Curry-Howard perspective and the extension of techniques to infinitary logics.

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Dichotomies in Weihrauch Complexity

Vasco Brattka

We discuss a number of uniform dichotomies for problems in the Weihrauch lattice. Such dichotomies have the common form that a problem is either quite well-behaved (continuous, measurable of some form, etc.) or already relatively badly behaved. We show that often such dichotomies also have non-uniform versions in terms of computable reducibility and we indicate how computability concepts such as Turing jumps, Weak Kőnig's Lemma, diagonal non-computability, etc., occur naturally in these non-uniform versions. This leads, for instance, to first-order characterizations of continuity in terms of Turing degrees. We also discuss how some known dichotomies from descriptive set theory, such as Solecki's dichotomy, can be seen in this context. The talk is based on ongoing research, but some of the discussed results are published in [1].

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Reporter: Nicholas Pischke

Participants

Dr. Juan Aguilera
Institut f. Algebra
& Diskrete Mathematik
Technische Universität Wien
Wiedner Hauptstraße 8 - 10
1040 Wien
AUSTRIA

Dr. Amir Akbar Tabatabai
Bernoulli Institute
University of Groningen
Nijenborgh 9
9747 AG Groningen
NETHERLANDS

Prof. Dr. Toshiyasu Arai
Graduate School of
Mathematical Sciences
University of Tokyo
Tokyo 153-8914
JAPAN

Prof. Dr. Matthias Baaz
Institut für Diskrete Mathematik
und Geometrie
Technische Universität Wien
Karlsplatz 13
1040 Wien
AUSTRIA

Prof. Dr. Arnold Beckmann
Department of Computer Science
Swansea University
Bay Campus
Swansea SA1 8EN
UNITED KINGDOM

Prof. Dr. Lev D. Beklemishev
Steklov Mathematical Institute of
Russian Academy of Sciences
8, Gubkina St.
119 991 Moscow GSP-1
RUSSIAN FEDERATION

Dr. Ulrich Berger
Department of Computer Science
Swansea University
Bay Campus
Crymlyn Burrows, Skewen
Swansea SA1 8EN
UNITED KINGDOM

Prof. Dr. Vasco Brattka
Institut für Theoretische Informatik,
Mathematik und Operations Research
Universität der Bundeswehr München
Werner-Heisenberg-Weg 39
85577 Neubiberg
GERMANY

Prof. Dr. Samuel Buss
Department of Mathematics
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
UNITED STATES

Prof. Dr. Yong Cheng
School of Philosophy
Wuhan University
Hubei
Wuhan 430072
CHINA

Horatiu Cheval
Dept. of Mathematics
University of Bucharest
Str. Academiei 14
70109 Bucharest 1
ROMANIA

Prof. Dr. Fernando Ferreira
Departamento de Matemática
FCUL – Universidade de Lisboa
Campo Grande, Ed. C6, Piso 2
1749-016 Lisboa
PORTUGAL

Prof. Dr. Anton Freund
Universität Würzburg
Institut für Mathematik
Emil-Fischer-Straße 40
97074 Würzburg
GERMANY

Dr. Emanuele Frittaion
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstraße 7
64289 Darmstadt
GERMANY

Dr. Makoto Fujiwara
Department of Applied Mathematics,
Faculty of Science Division I, Tokyo
University of Science
1-3 Kagurazaka, Shinjuku-ku
Tokyo 214-8571
JAPAN

Prof. em. Dr. Gerhard Jäger
Institut für Informatik
Universität Bern
Neubrückstrasse 10
3012 Bern
SWITZERLAND

Dr. Raheleh Jalali
Department of Philosophy
Utrecht University
Janskerkhof 13
3512 BL Utrecht
NETHERLANDS

Dr. Emil Jeřábek
Institute of Mathematics of the Czech
Academy of Sciences
Žitná 25
115 67 Praha 1
CZECH REPUBLIC

Dr. Joost J. Joosten
Departamento de Filosofia
Universitat de Barcelona
Montalegre, 6
08001 Barcelona
SPAIN

Prof. Dr. Reinhard Kahle
Carl Friedrich von Weizsäcker-Zentrum
Universität Tübingen
Doblerstr. 33
72074 Tübingen
GERMANY

Erfan Khaniki
Institute of Mathematics
Czech Academy of Sciences
Žitná 25
115 67 Praha
CZECH REPUBLIC

Prof. Dr. Ulrich Kohlenbach
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstraße 7
64289 Darmstadt
GERMANY

Dr. Leszek Kolodziejczyk
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warszawa
POLAND

Dr. Antonina Kolokolova
Department of Computer Science
Memorial University of Newfoundland
St. John's NL A1B 3X5
CANADA

Dr. Graham E. Leigh
Department of Philosophy, Linguistics and Theory of Science
Göteborgs Universitet
P.O. Box 200
40530 Göteborg
SWEDEN

Prof. Dr. Laurenţiu Leuştean
University of Bucharest & IMAR & ILDS
050107 Bucharest
ROMANIA

Prof. Dr. Maria Emilia Maietti
Dipartimento di Matematica
Università di Padova
Via Trieste, 63
35121 Padova
ITALY

Prof. Dr. Moritz Müller
Universität Passau
Lehrstuhl Mathematische Logik
Dr.-Hans-Kapfinger Straße 30
94032 Passau
GERMANY

Prof. Dr. Sara Negri
Dipartimento di Matematica
Università degli Studi di Genova
Via Dodecaneso 35
16146 Genova
ITALY

Dr. Takako Nemoto
Hiroshima Institute of Technology
7315193 Hiroshima
JAPAN

Morenikeji Neri
Dept. of Computer Science
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Dr. Adriana Nicolae
Faculty of Mathematics and Computer Science
Babes-Bolyai University
str. Mihail Kogalniceanu, nr. 1
400084 Cluj-Napoca
ROMANIA

Prof. Dr. Dag Normann
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
0316 Oslo
NORWAY

Prof. Dr. Isabel Oitavem
Departemento de Matematica
Universidade Nova de Lisboa
2829-516 Caparica
PORTUGAL

Dr. Paulo Oliva
School of Electronic Engineering and Computer Science
Queen Mary University of London
Mile End Road
London E1 4NS
UNITED KINGDOM

Dr. Pedro Pinto
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstraße 7
64289 Darmstadt
GERMANY

Nicholas Pischke
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstraße 7
64289 Darmstadt
GERMANY

Dr. Thomas Powell
Department of Computer Science
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Prof. Dr. Pavel Pudlák
Institute of Mathematics
Czech Academy of Sciences
Žitná 25
115 67 Praha
CZECH REPUBLIC

Prof. Dr. Michael Rathjen
School of Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Dr. Sam Sanders
Department of Philosophy II
RUB Bochum
Universitätsstraße 150
44780 Bochum
GERMANY

Prof. Dr. Peter M. Schuster
Dipartimento di Informatica
Università di Verona
Strada le Grazie 15
37134 Verona
ITALY

Dr. Andrei Sipoş
Department of Computer Science
University of Bucharest
Academiei 14
010014 Bucharest
ROMANIA

Dr. Neil Thapen
Institute of Mathematics
Czech Academy of Sciences
Žitná 25
115 67 Praha
CZECH REPUBLIC

Prof. Dr. Henry Towsner
Department of Mathematics
University of Pennsylvania
209 South 33rd Street
P.O. Box 19104
Philadelphia, PA 19104-6395
UNITED STATES

Dr. Iris van der Giessen
School of Computer Science
University of Birmingham
Birmingham B15 2TT
UNITED KINGDOM

Prof. Dr. Albert Visser
Department of Philosophy
and Religious Studies
Utrecht University
Janskerkhof 13
3512 BL Utrecht
NETHERLANDS

Prof. Dr. Andreas Weiermann
Vakgroep Wiskunde
Universiteit Gent
Krijgslaan 281
9000 Gent
BELGIUM

Dr. Keita Yokoyama
Mathematical Insitute
Tohoku University
6-3 Aza-Aoba, Aramaki, Aoba-ku
Sendai 980-8578
JAPAN

Dr. Margherita Zorzi
Dipartimento di Informatica
Università di Verona
Ca'Vignal 2, Strada Le Grazie 15
37134 Verona
ITALY