Mathematisches Forschungsinstitut Oberwolfach

Report No. 51/2023

DOI: 10.4171/OWR/2023/51

Mini-Workshop: Positivity and Inequalities in Convex and Complex Geometry

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29 October – 3 November 2023

Abstract. The workshop convened researchers from algebraic geometry, convex geometry, and complex geometry to explore themes arising from the Alexandrov-Fenchel and Brunn-Minkowski inequalities. It featured three introductory talks delving into the basics of Lorentzian polynomials, valuations in convex geometry, and plurisubharmonic functions, that served as a foundation for the subsequent research talks. As anticipated, significant overlap emerged among the varied perspectives within these three areas, evident in the presentations and ensuing discussions.

Mathematics Subject Classification (2020): 32J27, 52A39, 52B40, 14C17, 52A40.

Introduction by the Organizers

The workshop was organized by Andreas Bernig (Goethe-Universität Frankfurt), Julius Ross (University of Illinois at Chicago) and Thomas Wannerer (Friedrich-Schiller-Universität Jena). It was held over 5 days and included five introductory talks over three topics, and 13 research talks.

The mini-workshop revolved around a recent theme that has connected many seemingly different areas of mathematics, the so-called "Kähler package" that contains Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann bilinear relations. Originally coming from Kähler and algebraic geometry, it is now understood that this also appears in algebra, combinatorics and convex geometry. For example, each of the following admits a version of the Kähler package: McMullen's algebra generated by the Minkowski summands of a simple convex polytope, the combinatorial intersection cohomology of a convex polytope, the Chow ring of a matroid, and the ring of algebraic cycles modulo homological equivalence on a smooth projective variety via Grothendieck's standard conjectures on algebraic cycles. A powerful idea in the groundbreaking work of the Fields medalist June Huh and his collaborators is that the existence of a log-concave sequence is strong evidence for a Kähler package in the background. The celebrated Alexandrov–Fenchel inequality of convex geometry is an important example of a log-concave sequence, and therefore it is no surprise that this inequality can be deduced from at least three different incarnations of the Kähler package.

The aim of the mini-workshop was to bring together researchers interested in different aspects of the Kähler package, with an emphasis on aspects that relate most closely to complex and convex geometry.

The first introductory talk was titled *Plurisubharmonic functions and complex Brunn–Minkowski theory* and was given by Bo Berndtsson. He introduced the class of plurisubharmonic functions, sketched Bedford–Taylor theory, and discussed the complex version of Prekopa's theorem. The second introductory talk, over two hours, was titled *Valuations and convex geometry* and was given by Semyon Alesker. He introduced the algebraic structures on the space of valuations on convex bodies that are fundamental in the (partly conjectured) Kähler package for valuations. The final introductory talk, over two hours, was titled *Lorentzian polynomials* and given by Hendrik Süß. He introduced the concept of Lorentzian polynomials according to the work by Brändén and Huh and then explained various operations that map Lorentzian polynomials to Lorentzian polynomials. Thanks to the introductory talks, a common background knowledge was established at the beginning of the week on which later talks could rely.

The research talks covered various related topics such as the Alexandrov–Fenchel inequality and related inequalities for mixed volumes, the theory of valuations on convex bodies and on manifolds, the Hodge–Riemann bilinear relations on Kähler manifolds, Weighted Ehrhart theory, Gamma-positivity, Superforms and Hodge–Riemann classes coming from ample vector bundles.

The stimulating atmosphere of the mini-workshop led to many fruitful discussion that strengthened the links between different, but in fact closely related, areas of mathematics.

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Abstracts

Plurisubharmonic functions and complex Brunn-Minkowski theory

Bo Berndtsson

This was an introductory lecture. I first defined the notion of plurisubharmonic function. A function ϕ, defined in an open subset of $\mathbb{C}^n$ is plurisubharmonic if it is upper semicontinuos and its restriction to any complex line is subharmonic as a function of one complex variable. If the function is smooth, this is equivalent to saying that its complex Hessian

$$(\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k})$$

is positively semidefinite everywhere. With a plurisubharmonic function one can associate the *positive* differential form – or current –

$$i\partial\bar{\partial}\phi := i\sum\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}dz_j\wedge d\bar{z}_k.$$

If ϕ is smooth, this is a differential form of bidegree $(1,1)$ (meaning that it contains one barred differential and one unbarred); in general it is a current (meaning that the coefficients should be interpreted as distributions). That this form is positive means, in the smooth case, that the coeffcient matrix is positive semidefinite everywhere. In the general case it means that for any vector $(\lambda_1,...\lambda_n)$

$$\sum\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}\lambda_j\bar{\lambda}_k,$$

is a positive distribution, i.e. a positive measure.

If our funtion $\phi(z)=\phi(x+iy)=\phi(x)$ depends only on the real part of z, then ϕ is plurisubharmonic if and only if it is convex, and its complex Hessian coincides with the real Hessian, modulo a factor $1/4$. Moreover, one checks that

$$i\partial\bar{\partial}\phi=(1/2)\sum\phi_{jk}dx_j\wedge dy_k$$

then. So, this expression has a meaning as a current for any convex function, not necessarily smooth.

It was discovered by E. Bedford and B.A. Taylor, that the $(1,1)$ currents associated to plurisubharmonic functions can be multiplied, provided that the functions are locally bounded, so that e.g.

$$i\partial\bar{\partial}\phi\wedge i\partial\bar{\partial}\psi$$

is a well defined current. This is remarkable since distributions, or even measures, cannot be multiplied in general, but it turns out that the cancellation from the wedge product of differential forms works in our favour. When ϕ depends only on the real part of z, this gives in particular that

$$(\sum\phi_{jk}dx_j\wedge dy_k)^n/n!$$

is a well defined measure, the Monge-Ampère measure of ϕ. It can be shown that this coincides with Alexandrov's definition of Monge-Ampère measure.

The second topic of the lecture further developed the analogy between convexity and plurisubharmonicity. The analog for convex functions of the Brunn-Minkowski inequality for convex sets is Prékopa's theorem. Prékopa's theorem says that if $\phi(t,x)$ is a convex on $\mathbb{R}^{n+1}$, then

$$\tilde{\phi}(t) := -\log \int_{\mathbb{R}^n} e^{-\phi(t,x)} dx$$

is again convex. One well known proof of this, by Brascamp and Lieb, uses a certain Poincaré-type inequality: If u is a function on $\mathbb{R}^n$ sucht that

$$\int u e^{-\phi} dx = 0,$$

then

$$\int u^2 e^{-\phi} \leq \int |du|^2_{(\phi^{jk})} e^{-\phi}.$$

Here $|du|^2_{(\phi^{jk})} := \sum u_j u_k \phi^{jk}$ is the norm of du measured with the inverse of the Hessian of ϕ.

The Brascamp-Lieb inequality can be seen as the real variable counterpart of Hormander's L^2-estimates for the $\bar{\partial}$-equation. A natural question is then if there is a complex version of Prékopa's theorem? In one sense the answer is no: A counterexample by Kiselman shows that the function

$$\tilde{\phi}(\tau) := -\log \int_{\mathbb{C}^n} e^{-\phi(\tau,z)} dm(z)$$

is in general not subharmonic for plurisubharmonic $\phi(\tau, z)$. It does, however, hold under various extra assumptions, most notably if ϕ is S^1-invariant in z, for fixed τ:

$$\phi(\tau, e^{i\theta} z) = \phi(\tau, z).$$

One way to see this is via the Bergman kernel, here defined as

$$B_\tau(z) := \sup_h \frac{|h(z)|^2}{\int |h|^2 e^{-\phi(\tau,z)} dm(z)},$$

with the supremum taken over all holomorphic functions. The real variable analog of this would be to take supremum over all constant functions, leading to the function $\tilde{\phi}(\tau)$ (as $\log B_\tau$) introduced above. The first complex Prékopa theorem says that in general

$$\log B_\tau(z)$$

is plurisubharmonic in (τ, z). In the special case of S^1-invariance it is easily seen that

$$B_\tau(0) = 1/\int_{\mathbb{C}^n} e^{-\phi(\tau,z)} dm(z),$$

giving a more concrete Prekopa theorem in this case.

The complex Prékopa (or Brunn-Minkowski) theory is, however, considerably richer than this. One way to explain the general picture is to start from the observation that the Bergman kernel is the squared norm of the evaluation functional

$$h \to h(z)$$

on the Hilbert space

$$A^2_\tau := \{h \in H(\mathbb{C}^n), \int |h|^2 e^{-\phi(\tau,z)} dm(z) < \infty\}.$$

It turns out that one can replace the evaluation functional by any other family of functionals μ_τ, that depend holomorphically on τ in the sense that

$$\tau \to \mu_\tau(h)$$

is holomorphic for holomorphic h. This means, intuitively, that the bundle of Hilbert spaces $\tau \to A^2_\tau$ has positive curvature.

This is finally the most general statement along these lines , in the setting of Euclidean space. The complex case is, however, more naturally studied in the setting of complex manifolds. We then replace $\mathbb{C}^{n+1}$ by a complex manifold X, and the projection from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ by a surjective holomorphic map to another manifold. It turns out that one can develop a similar theory in this setting, under the crucial assumption that X be Kahler.

Valuations and convex geometry

Semyon Alesker

(1) I gave two introductory talks on translation invariant valuations on convex sets focusing mostly on the structures on the space of smooth translation invariant valuations (product, convolution, Fourier type transform), their relations to the recent Kotrbatý's conjectures on mixed hard Lefschetz (mHL) and mixed Hodge-Riemann (mHR) type results, to McMullen's polytope algebra, and to toric varieties. Below we briefly summarize main relevant definitions and theorems.

(2) Let V be a finite dimensional real vector space, $n = \dim V$. Let $\mathcal{K}(V)$ denote the family of all convex compact non-empty subsets of V. Its elements are also called convex bodies.

Definition 1. A valuation is a functional $\phi\colon \mathcal{K}(V) \to \mathbb{C}$ which is finitely additive:

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}(V)$.

Definition 2. A valuations ϕ is called translation invariant if

$$\phi(K + v) = \phi(K) \text{ for any } K \in \mathcal{K}(V), v \in V.$$

(3) Let us denote by $Val(V)$ the set of all continuous (in the Hausdorff metric) translation invariant valuations. It is a vector space over $\mathbb{C}$. Being equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$ it becomes a Banach space.

Definition 3. A valuation ϕ is called α-homogeneous if

$$\phi(\lambda K) = \lambda^\alpha \phi(K) \text{ for any } K \in \mathcal{K}(V), \lambda > 0.$$

Let $Val_\alpha(V) \subset Val(V)$ denote the subset of α-homogeneous valuations. Clearly it is a closed linear subspace. The following structural result is very important in the theory.

Theorem 4 (P. McMullen [6], 1977)**.** *One has the decomposition*

$$Val(V) = \oplus_{i=0}^n Val_i(V).$$

It is known that:
(1) $Val_0(V) = \mathbb{C} \cdot \chi$. This is trivial.
(2) $Val_n(V) = \mathbb{C} \cdot vol$. This is not trivial and due to Hadwiger.
The space $Val(V)$ has an important distinguished dense subspace $Val^\infty(V)$ of so called smooth valuations which carries rich algebraic structures. The definition was given in the lectures.

(4) The space $Val^\infty(V)$ carries a canonical multiplicative structure. Let us fix a positive Lebesgue measure vol_V on V.

Theorem 5 (Alesker [2], 2004)**.** *(1) $Val^\infty(V)$ has a canonical (thus $GL(V)$-equivariant) continuous (in the Garding topology) product $Val^\infty \times Val^\infty \to Val^\infty$ which is uniquely characterized by the following property: Let $\phi(K) = vol_V(K + A), \psi(K) = vol_V(K + B)$. Then*

$$(\phi \cdot \psi)(K) = vol_{V^2}(\Delta(K) + (A \times B)),$$

where $vol_{V^2} := vol_V \times vol_V$ is the product measure, and $\Delta\colon V \to V \times V$ is the diagonal imbedding, i.e. $\Delta(x) = (x, x)$.

(2) Equipped with this product $Val^\infty(V)$ is an associative commutative algebra with a unit ($= \chi$).

(3) $Val^\infty(V)$ is a graded: $Val_i^\infty \cdot Val_j^\infty \subset Val_{i+j}^\infty$.

(4) Poincaré duality: the bilinear map

$$Val_i^\infty \times Val_{n-i}^\infty \to Val_n = \mathbb{C} \cdot vol$$

is a perfect pairing, i.e. for any $0 \neq \phi \in Val_i^\infty$ there exists $\psi \in Val_{n-j}^\infty$ such that $\phi \cdot \psi \neq 0$.

Furthermore $Val^\infty(V)$ satisfies a version of the hard Lefschetz theorem which is a combination of results of Alesker [1, 3] and Bernig-Bröcker [4].

(5) Another important structure is the Bernig-Fu convolution.

Theorem 6 (Bernig-Fu [5],2006)**.** *(1) $Val^\infty(V)$ has a continuous (in the Garding topology) convolution $Val^\infty \times Val^\infty \to Val^\infty$ commuting with the* *group of linear volume preserving transformaitons which is uniquely characterized by the following property: Let* $\phi(K) = vol_V(K+A), \psi(K) = vol_V(K+B)$. *Then*

$$(\phi * \psi)(K) = vol_V(K + A + B).$$

(2) Equipped with this convolution $Val^\infty(V)$ *is an associative commutative algebra with a unit (*$= vol_V$*).*

(3) $Val^\infty_{n-i} * Val^\infty_{n-j} \subset Val^\infty_{n-i-j}$.

Poincaré duality and hard Lefschetz theorem (for intrinsic volumes) are also satisfied by convolution.

(6) Alesker [1, 3] has constructed an isomorphism of topological algebras, called Fourier type transform,

$$\mathbb{F}\colon (Val^\infty(V), \cdot) \tilde{\to} (Val^\infty(V^*), *)$$

commuting with the group of linear volume preserving transformations.

(7) The Kotrbatý's conjectures are formulated in terms of convolution and they are mixed hard Lefschetz and mixed Hodge-Riemann type results for valuations. I explained a heuristic argument in favor (imho) of the conjectures. It is based on the connection of valuations to the McMullen's polytope algebra established by Bernig and Faifman. Then I indicated a relation to toric varieties.

References

[1] Semyon Alesker. Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.*, 63(1):63–95, 2003.

[2] Semyon Alesker. The multiplicative structure on continuous polynomial valuations. *Geom. Funct. Anal.*, 14(1):1–26, 2004.

[3] Semyon Alesker. A Fourier-type transform on translation-invariant valuations on convex sets. *Israel J. Math.*, 181:189–294, 2011.

[4] Andreas Bernig and Ludwig Bröcker. Valuations on manifolds and Rumin cohomology. *J. Differential Geom.*, 75(3):433–457, 2007.

[5] Andreas Bernig and Joseph H. G. Fu. Convolution of convex valuations. *Geom. Dedicata*, 123:153–169, 2006.

[6] Peter McMullen. Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc. (3)*, 35(1):113–135, 1977.

Gamma-positivity for symmetric edge polytopes

Martina Juhnke-Kubitzke

Symmetric edge polytopes are a class of lattice polytopes that has seen a surge of interest in recent years for their intrinsic combinatorial and geometric properties [MHN+11, HKM17, OT21a, OT21b, CDK23] as well as for their relations to metric space theory [Ver15, GP17, DH20], optimal transport [ÇJM+21] and physics, where they appear in the context of the Kuramoto synchronization model [CDM18, Che19] (see [DDM22] for a more detailed account of these connections).

Given a finite simple graph $G = ([n], E)$, the associated *symmetric edge polytope* $\mathcal{P}_G$ is defined as

$$\mathcal{P}_G = \operatorname{conv}(\pm(e_i - e_j) \ : \ ij \in E).$$

Symmetric edge polytope have been shown to exhibit several nice properties, independent of the underlying graph: all of these polytopes are known to admit a pulling regular unimodular triangulation [OH14, HJM19] and to be centrally symmetric, terminal and reflexive [Hig15]. In particular, by this latter property, it follows from work of Hibi [Hib92] that their h^*-vectors are palindromic. Thus, given the h^*-vector $h^*(\mathcal{P}_G) = (h^*_0, \ldots, h^*_d)$ of a symmetric edge polytope, one can define the γ-vector of $\mathcal{P}_G$ by applying the following change of basis:

$$\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (t+1)^{d-2i} = \sum_{j=0}^{d} h^*_j t^j. \tag{1}$$

Obviously, $\gamma(\mathcal{P}_G) = (\gamma_0, \ldots, \gamma_{\lfloor \frac{d}{2} \rfloor})$ stores the same information as $h^*(\mathcal{P}_G)$ in a more compact form. More generally, in the same way, one can associate a γ-vector with any symmetric vector and this has been done and studied extensively in a lot of cases. One of the most prominent examples in topological combinatorics, which is strongly related to the just mentioned example of h^*-vectors of reflexive polytopes, are h-vectors of simplicial spheres. For *flag* spheres, Gal's conjecture [Gal05] states that their γ-vectors are nonnegative. Several related conjectures exist, including the Charney–Davis conjecture [CD95], claiming nonnegativity only for the last entry of the γ-vector, and the Nevo–Petersen conjecture [NP11] which even conjectures the γ-vector of a flag sphere to be the f-vector of a balanced simplicial complex.

If a polytope $\mathcal{P}$ admits a regular unimodular triangulation Δ, which is the case for symmetric edge polytopes, then the restriction of Δ yields a unimodular triangulation of the boundary complex of $\mathcal{P}$, as well. If, in addition, $\mathcal{P}$ is reflexive, it is well-known that the h^*-vector of $\mathcal{P}$ equals the h-vector of any unimodular triangulation Δ of its boundary, which in particular is a simplicial sphere. This provides a link between the study of the γ-vector of $\mathcal{P}_G$ and the rich world of conjectures on the γ-nonnegativity of simplicial spheres; however, note that the objects we are interested in will *not* be flag in general. Despite the lack of flagness, in all the cases known so far the γ-vector of $\mathcal{P}_G$ is nonnegative, and this brought Ohsugi and Tsuchiya to formulate the following conjecture, which is the starting point of this paper:

Conjecture 1. [OT21a, Conjecture 5.11] Let G be a graph. Then $\gamma_i(\mathcal{P}_G) \geq 0$ for every $i \geq 0$.

On the one hand, it is already known and follows e.g. from [BR07] that a weaker property, namely, unimodality of the h^*-vector holds. On the other hand, though it is tempting to hope that even the stronger property of the h^*-polynomial being real-rooted is true, this is not the case in general, as shown by the 5-cycle. The γ-nonnegativity conjecture above has been verified for special classes of graphs, mostly by direct computation: as shown in [OT21a, Section 5.3], such classes encompass cycles, suspensions of graphs (which include both complete graphs and wheels), outerplanar bipartite graphs and complete bipartite graphs. This last instance was originally proved in [HJM19] but was generalized in [OT21a] to bipartite graphs $\widetilde{H}$ obtained from another bipartite graph H as in [OT21a, p. 708].

The main goal of this work is to provide some supporting evidence to the γ-nonnegativity conjecture, independent of the graph. We take two different approaches: a *deterministic* and a *probabilistic* one.

In the deterministic part, we focus on the coefficient γ_2. Through some delicate combinatorial analysis, we are able to prove that γ_2 is always nonnegative. Moreover, we provide a characterization of those graphs for which $\gamma_2(\mathcal{P}_G) = 0$:

Theorem 2. *] Let $G = ([n], E)$ be a graph. Then $\gamma_2(\mathcal{P}_G) \geq 0$. Moreover, if G is 2-connected, then $\gamma_2(\mathcal{P}_G) = 0$ if and only if either $n < 5$, or $n \geq 5$ and G is isomorphic to one of the following two graphs:*

- *the graph G_n with edge set $\{12\} \cup \{1k, 2k \ : \ k \in \{3, \ldots, n\}\}$; or*
- *the complete bipartite graph $K_{2,n-2}$.*

The "if" part of the equality statement can be deduced from the results in [HJM19] and [OT21a], where the authors compute explicitly the γ-vector of the families of graphs appearing in 2.

For the probabilistic approach, we consider the Erdős-Rényi model $G(n, p(n))$, which is one of the most popular and well-studied ways to generate a graph on the vertex set $[n]$ via a random process: for a graph $G \in G(n, p)$, the probability of ij with $1 \leq i < j \leq n$ being an edge of G equals $p(n)$, and all of these events are mutually independent. Our question is then: for an Erdős-Rényi graph $G \in G(n, p)$, how likely is it that the entries of the γ-vector of $\mathcal{P}_G$ are nonnegative? As an extension, we pose the question of how *big* those entries will most likely be. Our main result, answering both questions, is the following:

Theorem 3. *Let k be a positive integer. For the Erdős-Rényi model $G(n, p(n))$, where $p(n) = n^{-\beta}$ for some $\beta > 0$, $\beta \neq 1$, the following statements hold:*

- (subcritical regime) *if $\beta > 1$, then asymptotically almost surely $\gamma_\ell = 0$ for all $\ell \geq 1$;*
- (supercritical regime) *if $0 < \beta < 1$, then asymptotically almost surely $\gamma_\ell \in \Theta(n^{(2-\beta)\ell})$ for every $0 < \ell \leq k$.*

In particular, this shows that $\gamma_\ell \geq 0$ for $1 \leq \ell \leq k$ with high probability, thereby proving that (up to a fixed entry of the γ-vector) Gal's conjecture holds with high probability. To prove this result, we need to distinguish two regimes: subcritical ($\beta > 1$) and supercritical ($0 < \beta < 1$), the subcritical one being the easier one. Along the proof, we derive concentration inequalities for the number of non-faces and faces of the triangulation of $\mathcal{P}_G$ studied in [HJM19, Proposition 3.8]. Unfortunately, our approach does not give results for the critical regime.

This is joint work with Alessio D'Alí, Daniel Köhne and Lorenzo Venturello.

References

[BR07] Winfried Bruns and Tim Römer. h-vectors of Gorenstein polytopes. *J. Combin. Theory Ser. A*, 114(1):65–76, 2007.

[CD95] Ruth Charney and Michael Davis. The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold. *Pacific J. Math.*, 171(1):117–137, 1995.

[CDK23] Tianran Chen, Robert Davis, and Evgeniia Korchevskaia. Facets and facet subgraphs of symmetric edge polytopes. *Discrete Appl. Math.*, 328:139–153, 2023.

[CDM18] Tianran Chen, Robert Davis, and Dhagash Mehta. Counting equilibria of the Kuramoto model using birationally invariant intersection index. *SIAM J. Appl. Algebra Geom.*, 2(4):489–507, 2018.

[Che19] Tianran Chen. Directed acyclic decomposition of Kuramoto equations. *Chaos*, 29(9):093101, 12, 2019.

[ÇJM+21] Türkü Özlüm Çelik, Asgar Jamneshan, Guido Montúfar, Bernd Sturmfels, and Lorenzo Venturello. Wasserstein distance to independence models. *J. Symbolic Comput.*, 104:855–873, 2021.

[DDM22] Alessio D'Alì, Emanuele Delucchi, and Mateusz Michałek. Many Faces of Symmetric Edge Polytopes. *Electron. J. Combin.*, 29(3):Paper No. 3.24–, 2022.

[DH20] Emanuele Delucchi and Linard Hoessly. Fundamental polytopes of metric trees via parallel connections of matroids. *European J. Combin.*, 87:103098, 18, 2020.

[Gal05] Światosław R. Gal. Real root conjecture fails for five- and higher-dimensional spheres. *Discrete Comput. Geom.*, 34(2):269–284, 2005.

[GP17] Joseph Gordon and Fedor Petrov. Combinatorics of the Lipschitz polytope. *Arnold Math. J.*, 3(2):205–218, 2017.

[Hib92] Takayuki Hibi. Dual polytopes of rational convex polytopes. *Combinatorica*, 12(2):237–240, 1992.

[Hig15] Akihiro Higashitani. Smooth Fano polytopes arising from finite directed graphs. *Kyoto J. Math.*, 55(3):579–592, 2015.

[HJM19] Akihiro Higashitani, Katharina Jochemko, and Mateusz Michałek. Arithmetic aspects of symmetric edge polytopes. *Mathematika*, 65(3):763–784, 2019.

[HKM17] Akihiro Higashitani, Mario Kummer, and Mateusz Michałek. Interlacing Ehrhart polynomials of reflexive polytopes. *Selecta Math. (N.S.)*, 23(4):2977–2998, 2017.

[MHN+11] Tetsushi Matsui, Akihiro Higashitani, Yuuki Nagazawa, Hidefumi Ohsugi, and Takayuki Hibi. Roots of Ehrhart polynomials arising from graphs. *J. Algebraic Combin.*, 34(4):721–749, 2011.

[NP11] Eran Nevo and T. Kyle Petersen. On γ-vectors satisfying the Kruskal-Katona inequalities. *Discrete Comput. Geom.*, 45(3):503–521, 2011.

[OH14] Hidefumi Ohsugi and Takayuki Hibi. Centrally symmetric configurations of integer matrices. *Nagoya Math. J.*, 216:153–170, 2014.

[OT21a] Hidefumi Ohsugi and Akiyoshi Tsuchiya. The h^*-polynomials of locally anti-blocking lattice polytopes and their γ-positivity. *Discrete Comput. Geom.*, 66(2):701–722, 2021.

[OT21b] Hidefumi Ohsugi and Akiyoshi Tsuchiya. Symmetric edge polytopes and matching generating polynomials. *Comb. Theory*, 1:Paper No. 9, 19, 2021.

[Ver15] A. M. Vershik. Classification of finite metric spaces and combinatorics of convex polytopes. *Arnold Math. J.*, 1(1):75–81, 2015.

Lorentzian polynomials

Hendrik Süss

In [1] Brändén and Huh introduced the notion of Lorentzian polynomials. This is a class of polynomials which generalizes the log-concavity properties of volume polynomials appearing in convex and algebraic geometry and behaves well with respect to many natural operations, such as multiplication, specialization and (positive) linear transformation. The theory of Lorentzian polynomials has been used to prove and reprove important conjectures in matroid theory, see [1, 2]. Moreover, many polynomials arising from representation theory are conjectured to be (denormalized) Lorentzian [3].

In my introductory talk I gave an overview of the definitions and basic theory of Lorentzian polynomials as presented in [1] and discussed the proof of the Strong Mason Conjecture also given in [1] as an exemplary application to matroid theory.

References

[1] Petter Brändén and June Huh. Lorentzian polynomials. *Ann. Math. (2)*, 192(3):821–891, 2020.

[2] Petter Brändén and Jonathan Leake. Lorentzian polynomials on cones, 2023.

[3] June Huh, Jacob P. Matherne, Karola Mészáros, and Avery St. Dizier. Logarithmic concavity of Schur and related polynomials. *Trans. Am. Math. Soc.*, 375(6):4411–4427, 2022.

Some analogies between valuation on convex bodies and algebraic cycles on varieties

Nguyen-Bac Dang

Fix $X = \mathbb{P}^d(\mathbb{C})$ the complex projective space of dimension $d \geq 1$ and a rational map $f\colon X \dashrightarrow X$ on X whose image is not contained in a hypersurface (i.e f is called dominant). Fix a Kähler form ω, for each $k \leq d$, a general problem is to estimate the growth of the sequence $(\deg_k(f^n))_{n\in\mathbb{N}}$ where

$$\deg_k(f^n) = \int_X (f^n)^*(\omega^k) \wedge \omega^{d-k}.$$

When the map f is holomorphic, these sequences can be understood on the cohomology of X. Namely, the element ω^k represents a class in the Dolbeaut cohomology $H^{k,k}(X)$ and f induces a pullback action this vector space by multiplication by $\deg_1(f)^k$. To tackle this problem for general meromorphic maps, the general strategy is to consider the pullback action of f on an infinite vector space:

$$\text{n-C}^k(\mathcal{X}) = \varinjlim H^{k,k}(Y),$$

where the inductive limit is taken over all birational models Y with a birational morphism $\pi\colon Y \to X$. More generally, one sees that the group of bimeromorphic transformations $Cr(d)$ of $\mathbb{P}^d(\mathbb{C})$ (the Cremona group) induces an action on the graded algebra:

$$Cr(d) \hookrightarrow \oplus_{k=0}^{d} \text{n-C}^k(\mathcal{X}). \tag{1}$$

This viewpoint was very fruitful and allowed for example Cantat [Can11] to study group theoretic properties of the Cremona group of dimension 2.

One can then read the growth of the sequence $\deg_k(f^n)$ on the growth of the sequence of vectors $(f^n)^*\omega^k \in$ n-$\mathrm{C}^k(\mathcal{X})$. In [BFJ08, DF21] a purely exponential growth of the sequence $\deg_1(f^n)$ was obtained under some conditions. The method was to complete the space n-$\mathrm{C}^1(\mathcal{X})$ with a suitable Banach norm so that the sequence of classes $(f^n)^*\omega$ converges to a unique eigenvector for the operator f^*.

The situation is very well-understood when $k \geq 2$ if the map f is defined by monomials. Fix a matrix $A = (a_{ij}) \in GL_d(\mathbb{Z})$, the monomial map associated to A is:

$$f_A\colon (x_1, \dots, x_d) \mapsto \left(y_1 = \prod_{j=1}^{d} x_j^{a_{1j}}, \dots, y_d = \prod_{j=1}^{d} x_j^{a_{dj}}\right).$$

The map $A \in GL_d(\mathbb{Z}) \to f_A \in Cr(d)$ induces an injection of $GL_d(\mathbb{Z})$ in the Cremona group. In that case, this subgroup acts on the subspace:

$$\oplus_{k=0}^{d} \text{n-C}^k(\mathcal{X}_{tor}) = \oplus_k \varinjlim_{Y \text{toric}} H^{k,k}(Y),$$

where the injective limit is taken over all toric compactifications Y of $(\mathbb{C}^*)^d$. On one hand, elements of n-$\mathrm{C}^k(\mathcal{X}_{tor})$ correspond to collection of classes of algebraic cycles living on a toric variety, but on the other hand, the theory of toric varieties allows one to view those as valuations on convex bodies. Namely, if P is the fundamental polytope of $\mathbb{R}^d$, then

$$\deg_k(f_A) = MV(A(P)[k], P[d-k]),$$

where $MV(A(P)[k], P[d-k])$ denotes the mixed volume of $A(P)$ taken k times and P taken $d-k$ times. Precisely, the class of ω^k is associated to the translation invariant valuation ϕ_{ω^k} homogeneous of degree $d-k$ such that

$$\phi_{\omega^k}(K) = MV(P[k], K[d-k]),$$

for all K convex body in $\mathbb{R}^d$. The action by $GL_d(\mathbb{Z})$ is then given by $A \cdot \phi(K) = \phi(A^{-1}(K))$. Denote by $Val_k(\mathbb{R}^d)$ vector space of translation invariant valuations of given degree k, one recovers an action on the graded vector space

$$GL_d(\mathbb{Z}) \hookrightarrow \oplus_k Val_k(\mathbb{R}^d). \tag{2}$$

Comparing (1) with (2), one sees that the previous space had a structure of graded algebra while in the second, it is only a graded vector space since the convolution between two valuations is not always well-defined. When $d = 2$, the analog in convex geometry of the norm defined by Boucksom-Favre-Jonsson is given by:

$$||\phi||^2 = 2\phi(B)^2 - \phi \star \phi \tag{3}$$

where B is a ball of volume 1 and where ϕ is a smooth valuation of degree 1. Taking the completion of smooth valuations for this norm yields a smaller space on which the convolution extends continuously. The fact that the above formula yields a norm is a consequence of Hodge-index theorem in algebraic geometry, and in convex geometry is the Legendre-Fenchel inequality or the Hodge-Riemann property for degree 1 valuations.

References

[BFJ08] Sébastien Boucksom, Charles Favre, and Mattias Jonsson, *Degree growth of meromorphic surface maps*, Duke Math. J. **141** (2008), no. 3, 519–538. MR 2387430

[Can11] Serge Cantat, *Sur les groupes de transformations birationnelles des surfaces*, Ann. of Math. (2) **174** (2011), no. 1, 299–340. MR 2811600

[DF21] Nguyen-Bac Dang and Charles Favre, *Spectral interpretations of dynamical degrees and applications*, Ann. of Math. (2) **194** (2021), no. 1, 299–359. MR 4276288

Schur polynomials, positivity and the Hodge-Riemann property

Matei Toma

(joint work with Julius Ross)

We present recent joint work with Julius Ross, [1], [2], [3], showing that Schur polynomials evaluated on "positive" forms or cycle classes exhibit strong positivity properties themselves, such as the Hard Lefschetz property and the Hodge-Riemann property. Our work was motivated by the need to understand intersection properties of algebraic cycles on complex projective manifolds and was inspired by two parallel developments. On one hand, in algebraic geometry the extension of the classical Hard Lefschetz Theorem proved by Bloch and Gieseker, [4], paved the way towards the work of Fulton and Lazarsfeld, [5], on positivity of Schur classes of ample vector bundles. On the other hand, in Kähler geometry it was suggested by Gromov in [6] and proved by Dinh and Nguyen, [7], that the Hard Lefschetz Theorem and the Hodge-Riemann bilinear relations may be extended to a mixed situation, meaning by this that both work with a product of Kähler classes replacing the power of a single Kähler class in the classical statements. A natural question arises, whether other combinations of positive classes, besides those exhibited by the Bloch-Gieseker and Dinh-Nguyen theorems, have similar Hard Lefschetz and Hodge-Riemann properties. Our results, which we next describe, say that this is the case for two-codimensional Schur classes.

We will denote by $c_0, c_1, \dots, c_e \in k[x_1, \dots, x_e]$ the elementary symmetric polynomials, by $\Lambda(d,e)$ the set of partitions $\lambda = (\lambda_1, \dots, \lambda_N)$ of d with

$$0 \le \lambda_N \le \dots \lambda_1 \le e, \quad \text{and} \quad \sum_{i=1}^{N} \lambda_i = d$$

and we will set

$$s_\lambda := \det \begin{pmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+N-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+N-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{\lambda_N-N+1} & c_{\lambda_N-N+2} & \cdots & c_{\lambda_N} \end{pmatrix}.$$

The s_λ are called *Schur polynomials* and build a basis of the space $k[x_1, \dots, x_e]_d^{sym}$ of degree d homogeneous symmetric polynomials in e variables, when λ runs through $\Lambda(d, e)$.

Then we can prove the following three instances of Hard Lefschetz and Hodge-Riemann properties for "Schur classes" of degree $d = n - 2$.

Theorem (linear algebra case). Let $\omega_1, \dots, \omega_e$ be strictly positive $(1,1)$-forms on $V = \mathbb{C}^n$, λ be a partition in $\Lambda(n-2, e)$ and vol be the standard volume form on V. Then the linear map

$$\bigwedge_{\mathbb{R}}^{2} V^* \to \bigwedge_{\mathbb{R}}^{2n-2} V^*, \ \alpha \mapsto \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e)$$

is an isomorphism and the quadratic form

$$Q_{s_\lambda(\omega_1,\dots,\omega_e)} : (\bigwedge^{1,1} V^*)_{\mathbb{R}} \to \mathbb{R}, \ \alpha \mapsto (\alpha \wedge s_\lambda(\omega_1, \dots, \omega_e) \wedge \alpha)/\mathrm{vol},$$

is non-degenerate of signature $(1, n^2 - 1)$.

Theorem (Kähler case). If $\omega_1, \dots, \omega_e$ are Kähler classes on a compact Kähler manifold X of dimension n and λ is a partition in $\Lambda(n-2, e)$, then the linear map

$$H^2(X, \mathbb{R}) \to H^{2n-2}(X, \mathbb{R}), \ \alpha \mapsto \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e)$$

is an isomorphism and the quadratic form

$$Q_{s_\lambda(\omega_1,\dots,\omega_e)} : H^{1,1}(X)_{\mathbb{R}} \to \mathbb{R}, \ \alpha \mapsto \int_X \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e) \wedge \alpha,$$

is non-degenerate of signature $(1, h^{1,1} - 1)$.

Theorem (ample vector bundle case). If E is a rank e ample vector bundle on a complex projective manifold X of dimension n and $\lambda \in \Lambda(n-2, e)$, then the linear map

$$H^2(X, \mathbb{R}) \to H^{n-2}(X, \mathbb{R}), \ \alpha \mapsto \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e)$$

is an isomorphism and the quadratic form

$$Q_{s_\lambda(\omega_1,\dots,\omega_e)} : H^{1,1}(X)_{\mathbb{R}} \to \mathbb{R}, \ \alpha \mapsto \int_X \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e) \wedge \alpha,$$

is non-degenerate of signature $(1, h^{1,1} - 1)$.

We start by proving the "ample vector bundle case" and in doing so we make use of the Bloch-Gieseker Theorem and of the Fulton-Lazarsfeld cone construction. We then successively deduce the "linear algebra case" and the "Kähler case". Let us note that a different, more algebraic, approach to prove the "linear algebra case" has appeared in the meantime in [8]. It seems however that for the "ample vector bundle case" a geometric proof is needed.

References

[1] J. Ross, M. Toma, *Hodge-Riemann bilinear relations for Schur classes of ample vector bundles*, Ann. Sci. Éc. Norm. Supér. (4) **56** (2023), 197–241.

[2] J. Ross, M. Toma, *On Hodge-Riemann Cohomology Classes*, Birational Geometry, Kaehler-Einstein Metrics and Degenerations, Springer Proceedings in Mathematics & Statistics, **Volume 409**, Springer 2023, edited by Ivan Cheltsov, Xiuxiong Chen, Ludmil Katzarkov and Jihun Park, 763-793.

[3] J. Ross, M. Toma, *Hodge-Riemann Relations for Schur Classes in the Linear and Kähler Cases*, Int. Math. Res. Not. IMRN **16** (2023), 13780–13816.

[4] S. Bloch, D. Gieseker, *The positivity of the Chern classes of an ample vector bundle*, Invent. Math. **12** (1971), 112–117.

[5] W. Fulton, R. Lazarsfeld, *Positive polynomials for ample vector bundles*, Ann. of Math. (2) **118** (1983), no. 1, 35–60.

[6] M. Gromov, *Convex sets and Kähler manifolds*, Advances in differential geometry and topology, World Sci. Publ., Teaneck, NJ, 1990, pp. 1–38.

[7] T.-C. Dinh, V.-A. Nguyên, *The mixed Hodge-Riemann bilinear relations for compact Kähler manifolds*, Geom. Funct. Anal. **16** (2006), no. 4, 838–849.

[8] J. Ross, H. Süß, Th. Wannerer, *Dually Lorentzian Polynomials*, arxiv2304.08399.

Bezout inequalities for mixed volumes

Maud Szusterman

Bezout inequality (in $\mathbb{P}^n$) and the Bernstein-Khovanskii-Kushnirenko (BKK) theorem allows to derive inequalities of mixed volumes

$$V(A_1, ..., A_r, \Delta)V(\Delta)^{r-1} \leq \prod_{i\leq r} V(A_i, \Delta[n-1]),$$

where A_i are arbitrary convex bodies in $\mathbb{R}^n$, and Δ is an n-simplex. Another consequence of the BKK theorem is

$$V(A_1, ..., A_n)V(\Delta) \leq V(A_2, ..., A_n, \Delta)V(A_1, \Delta[n-1]).$$

We introduce the affine invariant quantities $b_r(K)$ and $b(K)$ as the least $b_r, b \geq 1$ such that

$$V(A_1, ..., A_r, K)V(K)^{r-1} \leq b_r \prod_{i\leq r} V(A_i, K[n-1]), \quad \text{respectively}$$

$$V(A_1, ..., A_n)V(K) \leq bV(A_2, ..., A_n, K)V(A_1, K[n-1]),$$

holds true for any (A_i). In particular note that $1 \leq b_2(K) \leq b_r(K) \leq b(K)^{r-1}$ for any $n \geq 2$, and for any K.

In [1], C. Saroglou, I. Soprunov and A. Zvavitch have proven that $b(K) = 1$ characterizes the simplex among all convex bodies, and that $b_2(K) = 1$ characterizes the simplex among all n-polytopes: we shall review the proof of this latter characterization, and explain where it fails to generalize to the setting of convex bodies (if one uses Wulff-shape perturbations of K rather than “perturbated polytopes”). Moreover it follows from Fenchel’s inequality, respectively from Diskant’s inequality (see also [3]) that $b_2(K) \leq 2$ and $b(K) \leq n$ for all K (both constants are sharp, as shown by the cross-polytope for b_2, and by the unit cube for b). While the characterization of all K such that $b_2(K) = 2$ is known, that of all K such that $b(K) = n$ remains open.

This study of Bezout inequalities for mixed volumes was initiated by Soprunov and Zvavitch in [2], where they conjectured that the n-simplex is the only minimizer of b_2. Though this conjecture remains open, we will discuss recent progress on restricting the set of potential minimizers; namely we will present a necessary condition on the support of the surface area measure of K. In dimension 3, this necessary condition, together with previously established restrictions, is enough to answer positively Soprunov-Zvavitch's conjecture.

References

[1] Christos Saroglou, Ivan Soprunov, and Artem Zvavitch. Wulff shapes and a characterization of simplices via a Bezout type inequality. *Adv. Math.*, 357:106789, 24, 2019.

[2] Ivan Soprunov and Artem Zvavitch. Bezout inequality for mixed volumes. *Int. Math. Res. Not. IMRN*, (23):7230–7252, 2016.

[3] Jian Xiao. Bézout-type inequality in convex geometry. *Int. Math. Res. Not. IMRN*, (16):4950–4965, 2019.

Applications of Legendre transforms in Kähler geometry

Xu Wang

The Legendre transform of a generalized function $\phi : \mathbb{R}^n \to [-\infty, \infty]$ is defined by

$$\phi^*(y) := \sup_{x \in \mathbb{R}^n} x \cdot y - \phi(x).$$

It is one of the most important concepts in convex geometry. For instance, it can be used to define the interpolating family between two convex functions and prove that the mixed volume function is a polynomial for convex bodies (see formula (3.2) and Corollary 3.8 in [1]). It also plays an crucial role in the intersection theory in algebraic geometry. For example, it can be used to prove compactness of a Delzant toric manifold and the Bernstein-Kushnirenko inequality (see section 2 in [6]). In this talk, we will introduce a few recent applications of Legendre transforms in Kähler geometry. The first result is the following generalization [3] of McDuff–Polterovich's result [4] (for $\beta = (1, \cdots, 1)$, in which case $\epsilon_x(\omega; \beta)$ is called the Seshadri constant).

Theorem A. *Let (X, ω) be a compact Kähler manifold. Fix $x \in X$, we have*

$$\epsilon_x(\omega; \beta) = c_x(\omega; \beta), \ \beta = (\beta_1, \cdots, \beta_n), \beta_j > 0, \ 1 \leq j \leq n,$$

where the β-Seshadri constant of (X, ω) at $x \in X$ is defined by

$$\begin{aligned}\epsilon_x(\omega; \beta) := \sup\{\gamma \geq 0 : & \textit{ there exists } \psi \in \mathrm{PSH}(X, \omega) \textit{ such that} \\ & \psi = \gamma \log(|z_1|^{2/\beta_1} + \cdots + |z_n|^{2/\beta_n}) \textit{ near } x\},\end{aligned}$$

and "$\psi \in \mathrm{PSH}(X,\omega)$" means that ψ is upper semi continuous on X and $\omega + dd^c\psi \geq 0$, $d^c := (\partial - \bar\partial)/(4\pi i)$, in the sense of currents on X. The β-Kähler width

$$c_x(\omega;\beta) := \sup\left\{\pi r^2 : B_r^\beta \hookrightarrow hol_x(X,\tilde\omega),\ \exists\, \tilde\omega \in \mathcal{K}_\omega\right\},$$

$$B_r^\beta := \left\{ z \in \mathbb{C}^n : \sum_{j=1}^n \beta_j |z_j|^2 < r^2 \right\},$$

where "$B_r^\beta \hookrightarrow hol_x(X,\tilde\omega)$" means that there exists a holomorphic injection $f : B_r^\beta \to X$ such that $f(0) = x$ and $f^*(\tilde\omega) = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar z_j$, $\mathcal{K}_\omega$ denotes the space of Kähler metrics in $[\omega]$.

The main ingredient of our proof is the following Legendre transform result.

Theorem B ([3, Theorem 3.7]). *Let ϕ be smooth strictly convex on $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ be closed. Put $\phi_A(x) = \sup_{y\in A} y\cdot x - \phi^*(y)$. If x satisfies $\phi_A(x) < \phi(x)$ then $\phi_A(x) = \sup_{y\in\partial A} y \cdot x - \phi^*(y)$, where ∂A denotes the boundary of A.*

Another application of Theorem B is the following Ross-Witt Nyström theorem [5].

Theorem C. *Let ϕ be a smooth strictly convex function on $\mathbb{R}^n$. Assume that $A := \nabla\phi(\mathbb{R}^n)$ is bounded. Fix a concave function u on A and assume that $u \in C^\infty(\mathbb{R}^n)$. Then for every $t > 0$,*

$$(\phi^* - tu)^* = \sup_{\alpha\in\mathbb{R}}\{\phi_\alpha + t\alpha\}, \quad \phi_\alpha(x) := \sup_{u(y)\geq\alpha} y\cdot x - \phi^*(y).$$

The compact Kähler version of the above theorem is known as the Ross-Witt Nyström correspondence between the maximal test curves and geodesic rays.

Definition D. *Let $(L, e^{-\phi})$ be a positive line bundle over a compact complex manifold X. A map $\alpha \mapsto v_\alpha$ from $\mathbb{R}$ to $\mathrm{PSH}(X,\omega)$, $\omega := dd^c\phi$, is called a bounded test curve if*

(1) $\lambda_v := \inf\{\alpha \in \mathbb{R} : v_\alpha \equiv -\infty\} < \infty$;
(2) $\alpha \mapsto v_\alpha(x)$ is concave, decreasing and usc for any $x \in X$;
(3) $v_\alpha \equiv 0$ for $\alpha \leq 0$ and $\sup\{\alpha \in \mathbb{R} : v_\alpha \equiv 0\} = 0$.

A bounded test curve is said to be maximal if $P[v_\alpha] = v_\alpha$ for every $\alpha \in \mathbb{R}$, where

$$P[v_\alpha] := \sup{}^{*}\{v \in \mathrm{PSH}(X,\phi) : v \leq 0 \text{ and } v - v_\alpha \text{ is bounded on } X\},$$

is called the maximal envelope of v_α.

Definition E. *Let $(L, e^{-\phi})$ be a positive line bundle over a compact complex manifold X. A map $t \mapsto u_t$ from $(0,\infty)$ to $\mathrm{PSH}(X,\omega)$, $\omega := dd^c\phi$, is called a sub-linear sub-geodesic ray if*

(1) $\phi(x) + u_{-\log|\xi|^2}(x)$ is psh on $X \times \{\xi \in \mathbb{C} : |\xi| < 1\}$;
(2) $u_t \geq \lim_{t\to 0} u_t = 0$ and $\lambda_u := \lim_{t\to\infty} \sup_X u_t/t < \infty$.

A sub-linear sub-geodesic ray u_t is called a geodesic ray if for every $0 < a < t < b$ we have $u_t = \sup\{v_t\}$ where the supremum is taken over all sub-geodesics v_t with $\limsup_{t\to a,b} v_t \leq u_{a,b}$.

The main theorem in the Ross-Witt Nyström correspondence theory is the following result.

Theorem F ([5, Theorem 1.1])**.** *The α-Legendre transform $\hat{v}_t := \sup_{\alpha\in\mathbb{R}}\{v_\alpha + t\alpha\}$, $t > 0$, gives a bijective map, say $\mathcal{L}$, between*

(1) bounded test curves and sub-linear sub-geodesic rays;
(2) maximal bounded test curves and geodesic rays.

Moreover, we have $\lambda_v = \lambda_{\hat{v}}$ and $\mathcal{L}^{-1}(u_t)$ is given by the t-Legendre transform

$$\check{u}_\alpha := \inf_{t>0}\{u_t - t\alpha\}, \quad \alpha \in \mathbb{R}.$$

The above theorem implies the following Bergman kernel estimate in [2].

Theorem G. *Let $(L, e^{-\phi})$ be a positive line bundle over an n-dimentional compact complex manifold X. Assume that the Seshadri constant of L is $> n$ on X. Then $B_\phi \geq \mathrm{HS}_\phi$, where*

$$B_\phi(x) := \sup_{f\in H^0(X,\mathcal{O}(K_X+L))} \frac{i^{n^2} f(x)\wedge\overline{f(x)}e^{-\phi(x)}}{\int_X i^{n^2} f\wedge\bar{f}e^{-\phi}}, \quad \forall\, x\in X,$$

denotes the ϕ-weighted Bergman kernel form on X and

$$\mathrm{HS}_\phi(x) := \frac{(dd^c\phi)^n(x)}{\int_{T_xX} e^{-\phi_{L,x}}(dd^c\phi)^n(x)}, \quad \forall\, x\in X,$$

is called the Hele-Shaw form on X, where

$$\phi_{L,x} := \sup\left\{G_{hom,x} : G\in \mathrm{PSH}(X,\omega) \text{ with } \sup_X G = 0\right\}$$

is called the canonical growth condition [7] *of $\omega := dd^c\phi$ at x, here*

$$G_{hom,x}(w) := \limsup_{t\to 0}\{G(\exp_x(tw)) - \nu_x(G)\log(|t|^2)\}, \quad w\in T_xX,$$

$\exp_x$ *denotes the exponential map from T_xX to X with respect to ω and*

$$\nu_x(G) := \liminf_{z\to 0}\frac{G(z)}{\log(|z|^2)}$$

denotes the Lelong number of G at x.

The proof of Theorem G is to use an Ohsawa-Takegoshi extension theorem (see Theorem A in [2]) behind Theorem F.

References

[1] B. Berndtsson, Notes on complex and convex geometry, `http://www.math.chalmers.se/~bob/3notes.pdf`.

[2] Yan He, Johannes Testorf and Xu Wang, *Ross-Witt Nyström correspondence and Ohsawa-Takegoshi extension*, arXiv:2311.03840.

[3] F. Luef and X. Wang *Gaussian Gabor frames, Seshadri constants and generalized Buser-Sarnak invariants*, Geom. Funct. Anal. **33** (2023), 778–823.

[4] D. McDuff and L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. **115** (1994), 405–429.

[5] J. Ross and D. Witt Nyström, *Analytic test configuration and geodesic rays*, Journal of Symplectic Geometry **12** (2014), 125–169.
[6] X. Wang, Convexity in complex geometry, `https://wiki.math.ntnu.no/_media/ma8108/2023h/ccg_1.pdf`.
[7] D. Witt Nyström, *Canonical growth conditions associated to ample line bundles*, Duke Math. J. **167** (2018), 449–495.

Towards Hodge theory for smooth translation-invariant valuations

Jan Kotrbaty

Let $A = \bigoplus_{k=0}^{n} A_k$ be a commutative, associative, graded algebra over $\mathbb{R}$ with $A_n \cong \mathbb{R}$ and a fixed cone $K \subset A_1$. Let k be any integer between 0 and $\frac{n}{2}$. We say that A satisfies

(1) *Poincaré duality* if for each $x \in A_k$ with $x \neq 0$ there exists $y \in A_{n-k}$ such that $x \cdot y \neq 0$;
(2) *hard Lefschetz theorem* if for each $x_1, \ldots, x_{n-2k} \in K$, the map $A_k \to A_{n-k}$ given by $y \mapsto y \cdot x_1 \cdots x_{n-2k}$ is an isomorphism;
(3) *Hodge–Riemann relations* if for each $x_1, \ldots, x_{n-2k+1} \in K$ and $y \in A_k$ such that $y \neq 0$ and $y \cdot x_1 \cdots x_{n-2k+1} = 0$ one has $(-1)^k\, y \cdot y \cdot x_1 \cdots x_{2n-k} > 0$.

A prototypical example of such an algebra—from which the terminology was inherited—is the subring $\bigoplus_k H^{k,k}$ of the Dolbeault cohomology of a compact Kähler manifold (M, ω). The statement is classical for the one-dimensional cone $K = \mathbb{R}_{>0}\omega$. However, the case when K is the full Kähler cone was proved only recently by Dinh–Nguyên [9]. A more elementary example is the linear counterpart $\bigoplus_k \bigwedge^{k,k}(\mathbb{C}^n)^*$ proven by Timorin [15]. Further examples are the McMullen's algebra $\Pi(P)$ generated by polytopes strongly isomorphic to a fixed simple polytope P [14] or the Chow ring of a matroid, as proven by Adiprasito–Huh–Katz [1]. Many more examples along with remarkable applications of these properties to combinatorics are listed in the excellent account of Huh [10].

Let $\mathcal{K}$ denote the space of convex bodies, i.e., compact convex subsets in $\mathbb{R}^n$. We call $\phi : \mathcal{K} \to \mathbb{R}$ a valuation if

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}$. The space Val of translation-invariant, continuous valuations is a Banach space. It carries a natural left $GL(n)$ action given by $g \cdot \phi = \phi \circ g^{-1}$. $GL(n)$-smooth vectors in Val are called smooth valuations. It follows from a classical result of McMullen [14] that the space of smooth valuations is graded by the degree of homogeneity of a valuation: $\mathrm{Val}^\infty = \bigoplus_{k=0}^{n} \mathrm{Val}_k^\infty$. Moreover, by a combination of results of Alesker and Bernig–Fu [2, 4, 8], Val^∞ is in fact a graded algebra satisfying Poincaré duality with respect to a natural product given as follows: Denoting the mixed volume on $\mathbb{R}^n$ by V and k copies of a convex body by $[k]$, one has $A \mapsto V(B_1, \ldots, B_k, A[n-k]) \in \mathrm{Val}_{n-k}^\infty$ provided the convex bodies B_i are from $\mathcal{K}_+^\infty$, i.e., have smooth boundaries with positive curvature. Then we define

$$V(B_1,\dots,B_k,\bullet[n-k]) * V(C_1,\dots,C_l,\bullet[n-l])$$
$$= c^n_{k,l} V(B_1,\dots,B_k,C_1,\dots,C_l,\bullet[n-k-l])$$

where $c^n_{k,l} = \frac{(n-k)!(n-l)!}{n!(n-k-l)!}$.

Motivated by the aforementioned results in other contexts and by known special cases listed below, the following conjecture was formulated in [11]:

Conjecture 1. The algebra Val^∞ satisfies the hard Lefschetz theorem and the Hodge–Riemann relations with respect to $K = \{V(C,\bullet[n-1]) \mid C \in \mathcal{K}^\infty_+\}$.

The conjecture is now known to hold for the one-dimensional cone

$$\{V(D,\bullet[n-1]) \mid D \in \mathcal{K}^\infty_+ \text{ is a Euclidean ball}\}.$$

In this case, the hard Lefschetz theorem was first showed by Alesker [3] for the subalebra of even valuations. Later on, Bernig–Bröcker [7] removed the evenness assumption and prove the statement for Val^∞. Similarly, the Hodge–Riemann relations were first proved in the even case by Kotrbatý [11]. Somewhat later, Kotrbatý–Wannerer [13] proved the Hodge–Riemann relations for Val^∞ and also gave a new proof of the hard Lefschetz theorem. The point of working with the Euclidean cone is that the Lefschetz map then commutes with the group $SO(n)$. This makes it possible to use representation theory, in particular the known decomposition of Val^∞ into $SO(n)$-types established by Alesker–Bernig–Schuster [6].

For the full cone K, Conjecture 1 is proven in general only for $k = 0, 1$. The former case is easily seen to be equivalent to non-negativity of the mixed volume. The latter was proved by Kotrbatý–Wannerer [12] (and observed independently by Alesker) by generalizing the Alexandrov's second proof of the Alexandrov–Fenchel inequality

$$V(A,B,C_1,\dots,C_{n-2})^2 \geq V(B,B,C_1,\dots,C_{n-2})V(A,A,C_1,\dots,C_{n-2}).$$

Conversely, it was first observed by Alesker that the Hodge–Riemann relations for valuations subsume geometric inequalities: Taking $n = 2$, $k = 1$, and $y = V(K,\bullet) - \frac{V(K,D)}{V(D,D)}V(D,\bullet)$, where $K \in \mathcal{K}^\infty_+$ is arbitrary and $D \in \mathcal{K}^\infty_+$ is a Euclidean ball, the Hodge–Riemann relations together with the definition of the product $*$ of valuations yield at once the isoperimetric inequality on the plane. More generally, the case $k = 1$ of Conjecture 1 implies for general n the Alexandrov–Fenchel inequality [11]. Moreover, Alesker [5] and Kotrbatý–Wannerer [13] deduced in this way from the Hodge–Riemann relations new inequalities for mixed volumes, apparently beyond the previously known geometric methods.

References

[1] K. Adiprasito, J. Huh, and E. Katz. Hodge theory for combinatorial geometries. *Ann. of Math. (2)*, 188(2):381–452, 2018.

[2] S. Alesker. Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geom. Funct. Anal.*, 11(2):244–272, 2001.

[3] S. Alesker. Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.*, 63(1):63–95, 2003.
[4] S. Alesker. The multiplicative structure on continuous polynomial valuations. *Geom. Funct. Anal.*, 14(1):1–26, 2004.
[5] S. Alesker. Kotrbatý's theorem on valuations and geometric inequalities for convex bodies. *Israel J. Math.*, 247(1):361–378, 2022.
[6] S. Alesker, A. Bernig, and F. E. Schuster. Harmonic analysis of translation invariant valuations. *Geom. Funct. Anal.*, 21(4):751–773, 2011.
[7] A. Bernig and L. Bröcker. Valuations on manifolds and Rumin cohomology. *J. Differential Geom.*, 75(3):433–457, 2007.
[8] A. Bernig and J. H. G. Fu. Convolution of convex valuations. *Geom. Dedicata*, 123:153–169, 2006.
[9] T.-C. Dinh and V.-A. Nguyên. The mixed Hodge-Riemann bilinear relations for compact Kähler manifolds. *Geom. Funct. Anal.*, 16(4):838–849, 2006.
[10] J. Huh. Combinatorial applications of the Hodge-Riemann relations. pages 3093–3111, 2018.
[11] J. Kotrbatý. On Hodge-Riemann relations for translation-invariant valuations. *Adv. Math.*, 390:Paper No. 107914, 28, 2021.
[12] J. Kotrbatý and T. Wannerer. On mixed Hodge-Riemann relations for translation-invariant valuations and Aleksandrov-Fenchel inequalities. *Commun. Contemp. Math.*, 24(7):Paper No. 2150049, 24, 2022.
[13] J. Kotrbatý and T. Wannerer. From harmonic analysis of translation-invariant valuations to geometric inequalities for convex bodies. *Geom. Funct. Anal.*, 33(2):541–592, 2023.
[14] P. McMullen. On simple polytopes. *Invent. Math.*, 113(2):419–444, 1993.
[15] V. A. Timorin. Mixed Hodge-Riemann bilinear relations in a linear context. *Funktsional. Anal. i Prilozhen.*, 32(4):63–68, 96, 1998.

Octonionic Monge-Ampère operator and its applications to valuations theory and PDE

Semyon Alesker

(1) In this talk I introduce an octonionic Monge-Ampère (MA) operator for 2 octonionic variables, apply it to a construction of translation invariant continuous valuations on $\mathbb{R}^{16}$, in particular a $Spin(9)$-invariant example. Then I introduce (jointly with Peter Gordon) an octonionic analogue of Kähler metrics on 16-torii and prove a Calabi-Yau type theorem for them. The latter states solvability of certain non-linear elliptic second order PDE.

(2) Let $\mathbb{O}$ be the (non-commutative, non-associative) field of octonions. Recall that any octonion $q \in \mathbb{O}$ can be written uniquely

$$q = \sum_{i=0}^{7} x_p e^p,$$

where $x_p \in \mathbb{R}$, and e^p are octonionic units such that $e^0 = 1$ and $(e^p)^2 = -1$ for $p > 0$. The conjugate is defined by

$$\bar{q} = x_0 - \sum_{i=1}^{7} x_p e^p.$$

(3) Let F be a smooth $\mathbb{O}$-valued function on $\mathbb{O} \simeq \mathbb{R}^8$. Define two operators

$$\frac{\partial F}{\partial \bar{q}} := \sum_{i=0}^{7} e^p \frac{\partial F}{\partial x_p}, \quad \frac{\partial F}{\partial q} := \sum_{i=0}^{7} \frac{\partial F}{\partial x_p} \bar{e}^p.$$

Such operator can be defined in the case of several octonionic variables for each variable.

(4) For a smooth function $f\colon \mathbb{O}^n \to \mathbb{R}$ define its octonionic Hessian

$$Hess_{\mathbb{O}}(f) = \left(\frac{\partial^2 f}{\partial \bar{q}_i \partial q_j} \right).$$

This $n \times n$ matrix is Hermitian, i.e. $a_{ij} = \bar{a}_{ji}$.

(5) In order to define the MA operator we need a notion of determinant. There is such a notion for 2×2 octonionic Hermitian matrices. A general such a matrix has the form

$$\begin{bmatrix} a & q \\ \bar{q} & b \end{bmatrix}, \quad a, b \in \mathbb{R}, q \in \mathbb{O}.$$

Its determinant is defined by the usual formula $ab - q\bar{q} = ab - \bar{q}q$.

Finally we define the octonionic MA operator for a C^2-smooth real valued function f by

$$MA_{\mathbb{O}}(f) := \det Hess_{\mathbb{O}}(f).$$

(6) We show that $MA_{\mathbb{O}}(f)$ can be defined by continuity (with respect to the uniform convergence) for arbitrary continuous plurisubharmonic (in particular for convex) functions on $^2 \simeq \mathbb{R}^{16}$ which is not necessarily C^0-smooth as a non-negative measure.

Theorem 1 (Alesker [1], 2008)**.** *Fix $\psi \in C_c^0(\mathbb{R}^{16}, \mathbb{R})$. Define the functional on the family of all convex compact subsets of $\mathbb{O}^2 \simeq \mathbb{R}^{16}$ by*

$$K \mapsto \int_{\mathbb{O}^2} \psi \cdot MA_{\mathbb{O}}(h_K),$$

where h_K is the supporting functional of K. This is a continuous translation invariant 2-homogeneous valuation.

Note that the valuation property follows from a version of the Blocki's formula saying that if u, v are continuous octonionic psh functions and $\min\{u, v\}$ is also psh then

$$MA_{\mathbb{O}}(\min\{u, v\}) + MA_{\mathbb{O}}(\max\{u, v\}) = MA_{\mathbb{O}}(u) + MA_{\mathbb{O}}(v).$$

If the function ψ is $O(16)$-invariant then the corresponding valuation is $Spin(9)$-invariant. For different such ψ's the corresponding valuations are proportional.

The same argument works to construct continuous valuations on the class of continuous octonionic psh (in particular, on convex) functions on $\mathbb{O}^2 \simeq \mathbb{R}^{16}$.

(7) P. Gordon and me introduced a class of metrics on $\mathbb{O}^2$ which are octonionic analogues of Kähler metrics and proved a Calabi-Yau type theorem for an octonionic MA equation on 16-torii for such metrics.

References

[1] Semyon Alesker. Plurisubharmonic functions on the octonionic plane and Spin(9)-invariant valuations on convex sets. *J. Geom. Anal.*, 18(3):651–686, 2008.

Singularities of plurisubharmonic functions

Dano Kim

This talk was a survey on recent results in the study of singularities of plurisubharmonic functions, a topic which has seen active interactions among complex analysis, algebraic geometry and convex geometry.

A plurisubharmonic (psh for short) function φ on a complex manifold X is said to have *analytic singularities* (of type $\mathfrak{a}^c$) if it is locally of the form $\varphi = c \log \sum_{i=1}^m |g_i| + u$ where $c \geq 0$ is real, u bounded and $g_1, \ldots, g_m$ are local holomorphic functions generating a (global) coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$. Informally, let us say that such φ is *algebraic psh* in that its singularities are encoded in $\mathfrak{a}^c$ which is algebro-geometric data. Otherwise, let us say φ is *general psh*, which is a transcendental object.

General psh functions emerge in several different contexts in algebraic geometry: for example, from the study of graded sequence of ideal sheaves (cf. [5], [11]) or from local weight functions of singular hermitian metrics for a pseudoeffective line bundle (cf. [4]). In many concrete statements/results, one can observe two patterns. 1) A general psh function behaves very differently from algebraic ones. 2) A general psh function behaves similarly to algebraic ones.

An instance of 1) is a recent result [12, Thm. 5.7] joint with Hoseob Seo on psh functions with accumulation points of jumping numbers, which generalized a single initial example due to [8] to infinitely many examples, in fact characterizing them among all toric psh functions in dimension 2. Seo generalized this result to arbitrary dimension in [14]. Connection with convex analysis and geometry played an important role in these works. In this regard, another recent paper of Seo with An [1] developed further methods of using convex analysis to study equisingular approximation of psh functions.

On the other hand, as an instance of 2), the following result (joint with J. Kollár, in preparation) was announced. ($\mathcal{J}(\varphi)$ is the multiplier ideal sheaf of φ, cf. [4].)

Theorem 1. *Let X be a complex manifold and φ a quasi-plurisubharmonic function on X such that (X, φ) is log canonical. Then every point of X has a Stein open neighborhood $U \subset X$ with holomorphic functions g_i on U and real $c_i > 0$, such that $\psi := \sum_{i=1}^m c_i \log |g_i|$ is log canonical at every point of U, and $\mathcal{J}(\varphi) = \mathcal{J}(\psi)$.*

In the second part of this talk, we consider psh functions with isolated singularities at a point, say $0 \in \mathbb{C}^n$. For such psh functions $u_1, \ldots, u_n$, we denote their mixed Monge-Ampère mass at $0 \in \mathbb{C}^n$ by

$$m(u_1, \ldots, u_n) = \int_{\{0\}} (dd^c u_1) \wedge \ldots \wedge (dd^c u_n)$$

which is defined due to work of Demailly [3]. In the case when $u_k = \log|\mathfrak{a}_k|, k = 1,\dots,n$, for zero-dimensional ideals $\mathfrak{a}_k$ at 0, $m(u_1,\dots,u_n)$ is equal to the mixed multiplicity $\mu(\mathfrak{a}_1,\dots,\mathfrak{a}_n)$ of the ideals. In the joint work with Alexander Rashkovskii [10], we have the following Alexandrov-Fenchel inequality for mixed Monge-Ampère masses (generalizing a result of [5]).

Theorem 2. *Let $u_1,\dots,u_n$ be psh functions with isolated singularities at $0 \in \mathbb{C}^n$. Then we have the inequality*

$$m(u_1,u_1,u_3,\dots,u_n)\, m(u_2,u_2,u_3,\dots,u_n) \geq m(u_1,u_2,u_3,\dots,u_n)^2.$$

As a consequence for convex geometry, we derive an Alexandrov–Fenchel inequality for mixed covolumes [10, Cor. 1.5] using the case when $u_1,\dots,u_n$ are appropriate toric psh functions.

An important special case of mixed MA masses is 'higher Lelong numbers' of u defined as

$$e_k(u) = \int_{\{0\}} (dd^c u)^k \wedge (dd^c \log|z|)^{n-k} \tag{1}$$

for $k = 1,\dots,n$, generalizing the usual Lelong number $e_1(u)$. The main result of [5] (cf. [2]) is the following lower bound for the log canonical threshold $\mathrm{lct}(u)$ at 0,

$$\mathrm{lct}(u) \geq \frac{e_{n-1}(u)}{e_n(u)} + \frac{e_{n-2}(u)}{e_{n-1}(u)} + \dots + \frac{1}{e_1(u)}. \tag{2}$$

When u is algebraic psh associated to a zero-dimensional ideal $\mathfrak{a}$, this improves an earlier result of [7], $\mathrm{lct}(u) \geq n(\frac{1}{e_n(u)})^{\frac{1}{n}}$ which was applied in the topic of birational rigidity from birational geometry. The author does not yet know of an instance where (2) itself was used in birational geometry so far.

On the other hand, in a recent paper [9], we discovered an application of (2) to a completely different topic in algebraic geometry, namely hypersurface singularities. Let $(f = 0)$ be a germ of an isolated hypersurface singularity at $0 \in \mathbb{C}^n$. In [15], Teissier defined the *polar invariant* $\theta(f)$ which measures the rate of vanishing of the Jacobian ideal J_f of f with respect to that of the maximal ideal $\mathfrak{m}$ of $0 \in \mathbb{C}^n$. Also consider $\theta(f_1),\dots,\theta(f_{n-1})$ where f_j denotes the restriction of f to general j-codimensional planes containing $0 \in \mathbb{C}^n$. We have the following upper bound for the particular combination of these polar invariants from a question in [15].

Theorem 3. [9] *Let $\mathrm{lct}(\mathfrak{m}\cdot J_f)$ be the log canonical threshold at $0 \in \mathbb{C}^n$ of the product ideal $\mathfrak{m}\cdot J_f$. We have*

$$\frac{1}{1+\theta(f)} + \frac{1}{1+\theta(f_1)} + \dots + \frac{1}{1+\theta(f_{n-1})} \leq \mathrm{lct}(\mathfrak{m}\cdot J_f). \tag{3}$$

In fact, in his question [15, p.7], Teissier conjectured that one can put the *Arnold exponent* $\sigma(f)$ of f at 0, in the place of $\mathrm{lct}(\mathfrak{m}\cdot J_f)$ in (3). The Arnold exponent $\sigma(f)$ is related to log canonical thresholds by $\mathrm{lct}(f) = \min\{\sigma(f), 1\}$. The conjectured upper bound was proved by [6] whose methods are very different from [9] and based on more algebraic theories such as Saito's theory of mixed Hodge modules and the theory of Hodge ideals (cf. [13]).

References

[1] J. An and H. Seo, *On equisingular approximation of plurisubharmonic functions*, J. Math. Anal. Appl. 521 (2023), no. 2, Paper No. 126987, 16 pp.

[2] C. Bivià-Ausina, *Log canonical threshold and diagonal ideals*, Proc. Amer. Math. Soc. 145 (2017), no. 5, 1905–1916.

[3] J.-P. Demailly, *Monge-Ampère operators, Lelong numbers and intersection theory*, Complex analysis and geometry, 115-193, Univ. Ser. Math., Plenum, New York, 1993.

[4] J.-P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, 1. International Press, Somerville, MA; Higher Education Press, Beijing, 2012.

[5] J.-P. Demailly and H. H. Pham, *A sharp lower bound for the log canonical threshold*, Acta Math. 212 (2014), no. 1, 1-9.

[6] B. Dirks and M. Mustaţă, *Minimal exponents of hyperplane sections: a conjecture of Teissier*, J. Eur. Math. Soc. (JEMS) 25 (2023), no. 12, 4813–4840.

[7] T. de Fernex, L. Ein and M. Mustaţă, *Multiplicities and log canonical threshold*, J. Algebraic Geom. 13 (2004), no. 3, 603–615.

[8] Q. Guan and Z. Li, *Cluster points of jumping coefficients and equisingularties of plurisubharmonic functions*, Asian J. Math. 24 (2020), no. 4, 611–620.

[9] D. Kim, *On a question of Teissier*, Ann. Mat. Pura Appl. (4) 200 (2021), no. 5, 2305–2311.

[10] D. Kim and A. Rashkovskii, *Higher Lelong numbers and convex geometry*, J. Geom. Anal. 31 (2021), no. 3, 2525–2539.

[11] D. Kim and A. Rashkovskii, *Asymptotic multiplicities and Monge-Ampère masses (with an appendix by S. Boucksom)*, Math. Ann. 385 (2023), no. 3-4, 1947–1972.

[12] D. Kim and H. Seo, *Jumping numbers of analytic multiplier ideals*, Ann. Polon. Math. 124 (2020), no. 3, 257–280.

[13] M. Mustaţă and M. Popa, *Hodge ideals for* **Q***-divisors, V-filtration, and minimal exponent*, Forum Math. Sigma 8 (2020), Paper No. e19, 41 pp.

[14] H. Seo, *Cluster points of jumping numbers of toric plurisubharmonic functions*, J. Geom. Anal. 31 (2021), no. 12, 12624–12632.

[15] B. Teissier, *Polyèdre de Newton jacobien et équisingularité*, Seminar on Singularities (Paris, 1976/1977), pp. 193–221, Publ. Math. Univ. Paris VII, 7, Univ. Paris VII, Paris, 1980.

Weighted Ehrhart theory

Katharina Jochemko

The convex hull of finitely many points in the integer lattice $\mathbb{Z}^d$ is called a lattice polytope. Ehrhart [2] showed that for any lattice polytope $P \subset \mathbb{R}^d$, there is a polynomial $\mathrm{E}_P(n)$ such that $\mathrm{E}_P(n) = |nP \cap \mathbb{Z}^d|$ for all integers $n \geq 0$. The polynomial $\mathrm{E}_P(n)$ is called the Ehrhart polynomial and is the central object of study in Ehrhart theory. At the heart of Ehrhart theory are questions about the interpretation and characterization of the coefficients of the Ehrhart polynomial. A standard technique is to consider the h^*-polynomial $h^*_P(t)$, a linear transform of the Ehrhart polynomial, with many desirable properties. For a d-dimensional polytope P it is a polynomial of degree at most d given by the numerator of the generating series

$$\sum_{n\geq 0} \mathrm{E}_P(n)t^n \;=\; \frac{h_P^*(t)}{(1-t)^{d+1}}$$

A fundamental result by Stanley [4] states that the coefficients of the h^*-polynomial are always nonnegative and integers, in contrast to the coefficients of the Ehrhart polynomial which can be negative and rational in general. Another desirable property due to Stanley [5] is monotonicity of the coefficients, that is, for lattice polytopes $P, Q \subset \mathbb{R}^d$ with h^*-polynomials $h_P^*(t) = \sum_{i\geq 0} h_i^*(P)t^i$ and $h_Q^*(t) = \sum_{i\geq 0} h_i^*(Q)t^i$, if $P \subseteq Q$ then

$$h_i^*(P) \;\leq\; h_i^*(Q) \quad \text{for all } i \geq 0\,.$$

In this talk we present extension of Stanley's nonnegativity and monotonicity results [4, 5] to weighted lattice point enumeration. These results were obtained in joint collaboration with Esme Bajo, Robert Davis, Jesús A. De Loera, Alexey Garber, Sofía Garzón Mora and Josephine Yu [1].

A naive way to express the number of lattice points in a polytope P is $\sum_{\mathbf{x}\in P\cap\mathbb{Z}^d} 1$. We consider more general expressions of the form

$$\operatorname{ehr}(P, w) \;=\; \sum_{\mathbf{x}\in P\cap\mathbb{Z}^d} w(\mathbf{x})$$

where $w\colon \mathbb{R}^d \to \mathbb{R}$ is a polynomial function. Weighted sums of that type appear in various different areas, in particular, in enumerative combinatorics, optimization, convex geometry and statistics, see [1] and references therein. By results of Khovanskiĭ and Puklikov [3], Ehrhart's polynomiality result [2] extends to this weighted setup. More precisely, if w is a polynomial function of degree at most m and P a lattice polytope of dimension d then $\operatorname{ehr}(nP, w)$ is given by a polynomial of degree at most $d+m$ in the dilation factor $n \geq 0$. It follows that the corresponding generating series is again a rational function and we define the weighted h^*-polynomial of P, denoted $h^*_{P,w}(t)$, to be its numerator:

$$\sum_{n\geq 0} \operatorname{ehr}(nP, w)t^n \;=\; \frac{h^*_{P,w}(t)}{(1-t)^{d+m+1}}\,.$$

A natural question to ask is for which classes of polynomial functions w the weighted h^*-polynomial $h^*_{P,w}(t)$ satisfies nonnegativity and monotonicity of its coefficients. We consider two families of weights: sums of products of linear forms that are nonnegative on P, denoted R_P, and nonnegative sums of products of linear forms, denoted S_P. Clearly, $R_P \subset S_P$. In general, this inclusion is strict. For example, if $P = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2) \subset \mathbb{R}^2$ then $w(x_1, x_2) = (x_1 - x_2)^2$ is in S_P but not in R_P. It is rather easy to find examples of non-homogeneous weight functions for which the weighted h^*-polynomial has negative coefficients, even if the value of the weight function at every point in the polytope is nonnegative [1]. We thus restrict to homogeneous polynomial weight functions.

We have the following nonnegativity results.

Theorem 1 ([1]). *Let $P \subset \mathbb{R}^d$ be a lattice polytope and let $w\colon \mathbb{R}^d \to \mathbb{R}$ be a polynomial function.*

(i) If the weight w is a homogeneous element in R_P, then the coefficients of $h^*_{P,w}(t)$ are nonnegative.
(ii) If the weight w is a homogeneous element in S_P, then $h^*_{P,w}(t) \geq 0$ for all $t \geq 0$.

We remark that Theorem 1 (i) is rather sharp in the sense that it does in general not even extend to the case when w is the square of a single linear form, except for if P is a lattice polygon in $\mathbb{R}^2$ [1].

Further, we have the following monotonicity results.

Theorem 2 ([1]). *Let $P, Q \subset \mathbb{R}^d$ be a lattice polytopes, $P \subseteq Q$, and let $w\colon \mathbb{R}^d \to \mathbb{R}$ be a polynomial function.*

(i) If the weight w is a homogeneous element in R_P, then $h^*_{P,w}(t) \preceq h^*_{Q,w}(t)$ coefficient-wise.
(ii) If the weight w is a homogeneous element in S_P and $\dim P = \dim Q$, then $h^*_{P,w}(t) \geq 0$ for all $t \geq 0$.

Observe that in Theorem 2 (ii) the assumption $\dim P = \dim Q$ is necessary [1], in contrast to the classical monotonicity result by Stanley [5] and Theorem 2 (i) where P and Q may have different dimensions.

While the talk focusses on weighted Ehrhart polynomials of lattice polytopes, the theory and result hold more general for rational polytopes after suitable adjustments. See [1] for detailed statements.

References

[1] Esme Bajo, Robert Davis, Jesús A De Loera, Alexey Garber, Sofía Garzón Mora, Katharina Jochemko, and Josephine Yu. Weighted Ehrhart theory: Extending Stanley's nonnegativity theorem. *arXiv preprint arXiv:2303.09614*, 2023.

[2] Eugène Ehrhart. Sur les polyèdres rationnels homothétiques à n dimensions. *C. R. Acad. Sci. Paris*, 254:616–618, 1962.

[3] Aleksandr V. Pukhlikov and Askold Georgievich Khovanskiĭ. Finitely additive measures of virtual polyhedra. *Rossiĭskaya Akademiya Nauk. Algebra i Analiz*, 4(2):161–185, 1992.

[4] Richard P. Stanley. Decompositions of rational convex polytopes. *Annals of Discrete Mathematics*, 6:333–342, 1980. Combinatorial mathematics, optimal designs and their applications (Proc. Sympos. Combin. Math. and Optimal Design, Colorado State Univ., Fort Collins, Colo., 1978).

[5] Richard P. Stanley. A monotonicity property of h-vectors and h^*-vectors. *European Journal of Combinatorics*, 14(3):251–258, 1993.

Uncertainty and quasianalyticity on higher grassmannians

Dmitry Faifman

1. Introduction

Recall two integral transforms playing important roles in integral geometry.

The Radon transform is defined for $p < k$ by

$$\mathcal{R}_{p,k} : C(Gr_p(\mathbb{R}^n)) \to C(Gr_k(\mathbb{R}^n)), \quad \mathcal{R}_{p,k}f(F) = \int_{Gr_p(F)} f(E)dE;$$

The cosine transform is given by

$$\mathcal{C}_k : C(Gr_k(\mathbb{R}^n)) \to C(Gr_k(\mathbb{R}^n)), \quad \mathcal{C}_k f(F) = \int_{Gr_k(\mathbb{R}^n)} |\cos(E,F)| f(E)dE.$$

Both admit natural extensions to the space of distributions.

For a geometric application, let $\mathcal{K}_s(\mathbb{R}^n)$ the centrally symmetric convex bodies, and by $\mathcal{S}_s(\mathbb{R}^n)$ the centrally-symmetric star-shaped sets. Let $A_k(E;S) := vol_k(E\cap S) \in C(Gr_k(\mathbb{R}^n))$ denote the k-section function of $S \in \mathcal{S}_s(\mathbb{R}^n)$, and $P_k(E;K)) := vol_k(\mathrm{Pr}_E(K)) \in C(Gr_k(\mathbb{R}^n))$ the k-projection function of $K \in \mathcal{K}(\mathbb{R}^n)$.

It then holds that $A_k(E;S) = \mathcal{R}_{1,k}(\frac{1}{k}\rho_S^k)(E)$, where ρ_S is the radial function of S. Far less obviously, it holds also that $P_k(E;K)$ lies in the closure of $\mathrm{Image}(\mathcal{C}_k)$.

Two foundational results in geometric tomography are as follows.

Theorem 1 (Funk 1916)**.** *Fix $1 \le k \le n-1$. If $S, S' \in \mathcal{S}_s(\mathbb{R}^n)$ satisfy $A_k(E;S) = A_k(E;S')$ for all $E \in Gr_k(\mathbb{R}^n)$, then $S = S'$.*

Theorem 2 (Aleksandrov 1937 [1])**.** *Fix $1 \le k \le n-1$. If $K, L \in \mathcal{K}_s(\mathbb{R}^n)$ satisfy $P_k(E;K) = A_k(E;L)$ for all $P \in Gr_k(\mathbb{R}^n)$, then $K = L$.*

The former result is based on the injectivity of the Radon transform $\mathcal{R}_{1,k}$. The latter makes use of the injectivity of the cosine transform $\mathcal{C}_1$.

2. Results

In the following, T denotes either the Radon transform $\mathcal{R}_{p,k}$ with $\dim Gr_p < \dim Gr_k$, or alternatively $\mathcal{C}_k$ with $2 \le k \le n-2$. We will write $\mathrm{Image}_{C^N}(T) := \mathrm{Image}_{C^{-\infty}}(T) \cap C^N$, where $\mathrm{Image}_{C^{-\infty}}(T)$ is the image of T on distributions. In either cases, $\mathrm{Image}_C^N(T)$ is not dense in $C^N(Gr_k(\mathbb{R}^N))$ $N \in \{-\infty, 0, \infty\}$. A representation-theoretic description of the image is available, which we now recall.

Denote $\kappa = \min(k, n-k)$, and $\Lambda_\kappa = \{\lambda_1 \ge \cdots \ge \lambda\kappa : \lambda_i \in 2\mathbb{Z}_+\}$. One has the multiplicity-free decomposition [8]

$$L^2(Gr_k(\mathbb{R}^n)) = \oplus_{\lambda\in\Lambda_\kappa} V_\lambda,$$

where V_λ are certain pairwise distinct irreducible representations of $\mathrm{O}(n)$.

We then have

Theorem 3 (Gelfand-Graev-Rosu [5])**.** $\mathrm{Image}(\mathcal{R}_{p,k})$ *consists of those V_λ with $\lambda_{p+1} = 0$.*

Theorem 4 (Alesker-Bernstein [2]). Image$(\mathcal{C}_k)$ *consists of those* V_λ *with* $\lambda_2 \leq 2$.

It follows that any $f \in \text{Image}(T)$ has rather stringent restrictions on its spectrum. Our goal is to find a geometric rigidity manifestation of this spectral restriction. It will be realized through a quasianalyticity phenomenon. Generally speaking, a class of functions is called quasianalytic if it has a unique continuation property understood in broad terms: the values of a function from the class in an appropriate small set must determine the function uniquely.

Definition 5. Fix $F \in Gr_{n-k}(\mathbb{R}^n)$. The open Schubert cell $\Sigma_F^k \subset Gr_k(\mathbb{R}^n)$ is $\Sigma_F^k = \{E : E \cap F = \{0\}\}$. The Schubert equator $\mathcal{X}i_F^k$ is its complement, $\mathcal{X}i_F^k = \{E : E \cap F \neq \{0\}\}$.

Definition 6. A class of functions $\mathcal{A} \subset C(Gr_k(\mathbb{R}^n))$ is exp-$\mathcal{X}i$-quasianalytic if, whenever $f, g \in \mathcal{A}$ coincide exponentially on $\mathcal{X}i_F^k$, namely if for some $C, c > 0$ it holds for all E that

$$|f(E) - g(E)| \leq C \exp\left(-\frac{c}{d_{Gr}(E, \mathcal{X}i_F^j)}\right),$$

then $f = g$.
A class of distributions $\mathcal{A} \subset C(Gr_k(\mathbb{R}^n))$ is Bernstein-$\mathcal{X}i$-quasianalytic if, whenever $f, g \in \mathcal{A}$ coincide in a neighborhood of $\mathcal{X}i_F^k$, then $f = g$.

Our main result is as follows.

Theorem 7 (F [4]). *Let T denote either $\mathcal{R}_{p,k}$ with* $\dim Gr_p < \dim Gr_k$, *or* $\mathcal{C}_k$ *with* $2 \leq k \leq n-2$. *Then* $\text{Image}_{C^0}(T)$ *is exp-$\mathcal{X}i$-quasianalytic, and* $\text{Image}_{C^{-\infty}}(T)$ *is Bernstein-$\mathcal{X}i$-quasianalytic.*

This immediately implies sharper version of Funk's and Aleksandrov's theorems:

Theorem 8 (Sharper Funk, F [4]). *Fix $1 \leq k \leq n-1$. If for $S, S' \in \mathcal{S}_s(\mathbb{R}^n)$ it holds that $A_k(E; S)$ and $A_k(E; S')$ coincide on any single Schubert equator, then $S = S'$.*

Theorem 9 (Sharper Aleksandrov, F [4]). *Fix $1 \leq k \leq n-1$. If for $K, L \in \mathcal{K}_s(\mathbb{R}^n)$ it holds that $P_k(E; K)$ and $P_k(E; L)$ coincide on any single Schubert equator, then $K = L$.*

One similarly obtains a sharper version of the Klain injectivity theorem [9] in convex valuation theory, which can then be used to prove also a sharper version of the Schneider injectivity theorem [10].

3. Sketch of proof of Theorem 7

The idea of the proof is quite simple. Let us work with $T = \mathcal{C}_k$.

Assume by contradiction that a counterexample f exists, which for simplicity we assume to be supported inside Σ_F^k. The first step is producing a counterexample supported at a point. We use a trick going back to Gelfand-Graev-Rosu [5], writing $\mathcal{C}_k$ as a $\mathrm{GL}_n(\mathbb{R})$-equivariant transform between spaces of sections of certain line bundles. Considering f as such a section, this allows to take $g_\epsilon = \mathrm{Pr}_{F^\perp} + \epsilon \mathrm{Pr}_F \in \mathrm{GL}_n(\mathbb{R})$, and define $f_\epsilon = g_\epsilon^*(f)$. Evidently f_ϵ has support shrinking to $\{F^\perp\}$, and one can show that a sequence $c_\epsilon \to 0$ exists such that $c_\epsilon f_\epsilon$ converges to a distribution f_0 supported at $\{F^\perp\}$ which is still inside Image($\mathcal{C}_k$).

The second step consists of proving an uncertainty principle. Namely, we prove

Theorem 10 (F [4]). *Assume* $1 \leq k \leq n-1$. *Assume* $f_0 \in C^{-\infty}(Gr_k(\mathbb{R}^n))$ *is supported at one point. Consider* $supp(\widehat{f_0}) = \{\lambda \in \Lambda_\kappa : \widehat{f_0}(\lambda) \neq 0\}$. *Then*

$$\lim_{m\to\infty} \frac{\#\{\lambda \in supp(\widehat{f_0}) : \sum \lambda_i \leq 2m\}}{\#\{\lambda \in \Lambda_\kappa : \sum \lambda_i \leq 2m\}} = 1.$$

Moreover for $k \in \{1, n-1\}$, $supp(\widehat{f_0})$ *must be co-finite.*

However by the theorem of Alesker-Bernstein, that limit above must vanish, leading to a contradiction.

Let us conclude by remarking that by results of Grinberg [7], Gonzalez and Kakehi [6], the image of the Radon transform admits a description as the kernel of $\mathrm{SO}(n)$-invariant differential operator, while the image of the cosine transform lies in the kernel of another such operator by results of Alesker-Gourevitch-Sahi [3]. However, this quasianalyticity property does not appear to be a consequence of the PDEs. Furthermore, the methods and results above apply in greater generality to various $\mathrm{GL}_n(\mathbb{R})$-modules realized as spaces of sections of equivariant line bundles over the grassmannians, where no PDE description is known to exist.

References

[1] A. D. Aleksandrov. On the theory of mixed volumes. II. New inequalities between mixed volumes and their application. *Mat. Sbornik N.S.*, 2:1205–1238, 1937.

[2] S. Alesker and J. Bernstein. Range characterization of the cosine transform on higher Grassmannians. *Adv. Math.*, 184(2):367–379, 2004.

[3] S. Alesker, D. Gourevitch, and S. Sahi. On an analytic description of the α-cosine transform on real Grassmannians. *Commun. Contemp. Math.*, 18(2):1550025, 43, 2016.

[4] D. Faifman. Quasianalyticity, uncertainty, and integral transforms on higher grassmannians. *Adv. Math.*, 435(part A):Paper No. 109348, 2023.

[5] I. M. Gel'fand, M. I. Graev, and R. Roşu. The problem of integral geometry and intertwining operators for a pair of real Grassmannian manifolds. *J. Operator Theory*, 12(2):359–383, 1984.

[6] F. B. Gonzalez and T. Kakehi. Pfaffian systems and Radon transforms on affine Grassmann manifolds. *Math. Ann.*, 326(2):237–273, 2003.

[7] E. L. Grinberg. On images of Radon transforms. *Duke Math. J.*, 52(4):939–972, 1985.

[8] A. T. James and A. G. Constantine. Generalized Jacobi polynomials as spherical functions of the Grassmann manifold. *Proc. London Math. Soc. (3)*, 29:174–192, 1974.

[9] D. A. Klain. Even valuations on convex bodies. *Trans. Amer. Math. Soc.*, 352(1):71–93, 2000.

[10] R. Schneider. Simple valuations on convex bodies. *Mathematika*, 43(1):32–39, 1996.

Valuations on Kähler manifolds

Gil Solanes

(joint work with Andreas Bernig, Joseph H.G. Fu, Thomas Wannerer)

Let M^n be a smooth manifold, and let $\mathcal{P}(M)$ be the class of compact submanifolds with boundary of M. A *smooth valuation* on M (cf. [1]) is a functional $\phi\colon \mathcal{P}(M) \to \mathbb{R}$ of the form

$$\phi(A) = \int_{N(A)} \omega + \int_A \eta$$

where $N(A)$ is the so-called *conormal cycle* of A, and $\omega \in \Omega^{n-1}(S^*M)$ is a differential form on the cosphere bundle S^*M of M, while $\eta \in \Omega^n(M)$.

Remarkably (cf. [2]), the space $\mathcal{V}(M)$ of smooth valuations on M has an algebra structure fulfilling $e^*(\phi \cdot \varphi) = e^*(\phi) \cdot e^*(\varphi)$ for all $\phi, \varphi \in \mathcal{V}(N)$ and any smooth embedding $e\colon M \to N$.

As a consequence of H. Weyl's tube theorem, every riemannian manifold M^n has a canonical subalgebra $\mathcal{LK}(M) \subset \mathcal{V}(M)$, called the *Lipschitz-Killing algebra*, characterized by the following facts:

i) if $e\colon M \to N$ is an isometric embedding between riemannian manifolds, then $e^*(\mathcal{LK}(N)) = \mathcal{LK}(M)$
ii) if M is euclidean space $\mathbb{R}^n$, then $\mathcal{LK}(M)$ is the full algebra $\mathrm{Val}^{O(n)}$ of isometry invariant valuations.

It follows that the algebra structure of $\mathcal{LK}(M)$ is universal: it depends only on the dimension of M. Another simple consequence is that the algebras of invariant valuations of euclidean space $\mathbb{R}^n$ and the round sphere S^n are isomorphic to each other.

It was realized in [3] that also the algebra of isometry invariant valuations of $\mathbb{C}P^n$ is isomorphic to the algebra $\mathrm{Val}^{U(n)}$ of valuations of $\mathbb{C}^n$ invariant under hermitian isometries. This suggested the possibility that an extened version of the Lipschitz-Killing algebra might be present on any Kähler manifold. This is precisely the content of the following theorem.

Theorem 1 ([4])**.** *To every Kähler manifold M^n there is an associated subalgebra $\mathcal{KLK}(M) \subset \mathcal{V}(M)$ isomorphic to $\mathrm{Val}^{U(n)}$ in such a way that $e^*(\mathcal{KLK}(N)) = \mathcal{KLK}(M)$ for every isometric holomorphic embedding $e\colon M \to N$ of Kähler manifolds.*

References

[1] S. Alesker *Theory of valuations on manifolds II* Adv. Math. 207 (2006), no. 1, 420–454.
[2] S. Alesker, J. H. G. Fu *Theory of valuations on manifolds III. Multiplicative structure in the general case*, Trans. Amer. Math. Soc. **360** (2008), 1951–1981.
[3] A. Bernig, J.H.G. Fu, G. Solanes, *Integral Geometry of Complex Space Forms*, Geom. Funct. Anal. **24** (2014), no. 2, 403–492.
[4] A. Bernig, J.H.G. Fu, G. Solanes, T. Wannerer, *The Weyl tube theorem for Kähler manifolds*, arXiv:2209.05806.

Superforms and convex geometry

Bo Berndtsson

This lecture continues the first part of my introductory lecture. A superform on $\mathbb{R}^n$ is defined as a differential form on $\mathbb{C}^n$ whose coefficients depend only on $x = \mathrm{Re}\, z$;

$$\alpha = \sum \alpha_{JK}(x) dx_J \wedge d\xi_K.$$

The usual exterior differentiation operator acts on superforms, and we define

$$d^{\#} = \sum \partial/\partial x_j \, d\xi_j \wedge .$$

This coincides with the operator $d^c = i(\bar\partial - \partial)$ from complex analysis (the d-operator twisted with the complex structure), but we write $d^{\#}$ to emphazise that we consider only its action on superforms. If $\phi(x)$ is a function on $\mathbb{R}^n$ we have

$$d^{\#}\phi = \sum \phi_{jk} dx_j \wedge d\xi_k,$$

which makes sense as a current with measure coefficients for any convex (finite valued) function, and also for functions that can be written locally as the difference of two convex functions. By the theory of Bedford and Taylor (see the (abstract of) the introductory lecture), wedge products of such currents

$$\Omega_k = d^{\#}\phi_1 \wedge ... d^{\#}\phi_k$$

are also well defined. Taking $k = n$ and $\phi = \phi_1 = ...\phi_n$ we get the Monge-Ampère measure of ϕ, $MA(\phi)$.

The integral of a superform of maximal degree is defined as

$$\int a_0(x) dx_1 \wedge d\xi_1 \wedge ... dx_n \wedge d\xi_n := \int \alpha_0(x) dx,$$

meaning essentially that we define

$$\int d\xi_1 \wedge ... d\xi_n = \pm 1$$

(Berezin integration).

The main point of the lecture was to advocate the use of superforms for calculations involving the volume of convex bodies (this is partly based on previous work of my students A. Lagerberg and S. Larsson) , and we tried to illustrate that with a possibly new proof of the Alexandrov-Fenchel theorem. If ϕ is the support function of a convex body K, one finds that $MA(\phi)$ is a a Dirac measure at the origin of size $|K|$, the volume of K. More generally, if ϕ_j are support functions of convex bodies K_j;

$$d^{\#}\phi_1 \wedge ... d^{\#}\phi_n = V(K_1, ...K_n)\delta_0,$$

where $V(K_1, ...K_n)$ is the mixed volume of the K_j. More generally we can then define in the same way $V(u_1, ...u_n)$, where u_j are 1-homogeneous functions that can be written as differences of support functions of convex bodies. Fixing convex bodies $K_3, ...K_n$ and their support functions, we then get a quadratic form

$$Q(u, u) := V(u, u, \phi_3, ...\phi_n).$$

The essence of the Alexandrov-Fenchel theorem is that this form has Lorentzian signature, i. e. that it is positive somewhere, and seminegative on a subspace of codimension 1.

One approach to proving this (essentially Alexandrov's approach) is to note that

$$Q(u,v) = V(u,v,\phi_3...) = \int_{\partial U} d^{\#}u \wedge d^{\#}v \wedge \Omega_{n-2},$$

where U is any convex neighbourhood of the origin. After a rewrite this can be written as

$$\int_{\partial U} uA(v)dm$$

where dm is, say, surface measure and $A(v)$ is an elliptic second order differential operator. Alexandrov's proof proceeds by studying the eigenvalues of A. We sketched an alternative way, based on a study of Dirichlet problem for A on domains in ∂U of the form

$$D = \{x \in \partial U, x_1 > 0\}.$$

The main points were that the only function in D with zero boundary values, solving $A(u) = 0$, is $u = x_1$, and that this statement implies the Alexandrov-Fenchel theorem.

On the Adler-Taylor Gaussian kinematic formula

Joseph Fu

The statisticians R. Adler and J. Taylor have introduced a new type of kinematic formula based on the behavior of *Gaussian random fields* on a Riemannian manifold M^n: that is, smooth random functions $f = f_\omega, \omega \in \Omega$, on M whose value $f(x)$ is an $N(0,1)$ random variables for each $x \in M$, and which give an isometric embedding $M \to \mathbb{R}^\Omega$. Specifically, given d i.i.d. random functions of this type we obtain a random mapping $F : M \to \mathbb{R}^d$. For convex bodies $D \subset \mathbb{R}^d$, Adler-Taylor show that for nice objects $A \subset M$ that

$$\mathbb{E}[\chi(A \cap F^{-1}D)] = \sum c_i \mu_i(A)\gamma_i(D) \tag{1}$$

where the μ_i are the intrinsic volumes and the γ_i are the "Gaussian intrinsic volumes" on $\mathbb{R}^d$, viz. the Gaussian measure γ_0 and its derivatives with respect to metric expansion.

We propose that Adler-Taylor theory should be viewed as a chapter in the theory of valuations initiated by Alesker. To do so requires the resolution of two technical issues.

First, we recall that Faifman and Hofstätter [3] have shown that any π belonging to the algebra $\mathcal{V}(M)$ of smooth valuations on M may be expressed as

$$\pi = \int_{\mathcal{S}} \chi(\cdot \cap S)\, dS, \tag{2}$$

and subject to the multiplication formula

$$\nu \cdot \pi = \int_{\mathcal{S}} \nu(\cdot \cap S)\, dS \tag{3}$$

for any $\nu \in \mathcal{V}(M)$, where $(\mathcal{S}, dS)$ is a measured family of subsets of M. The left hand side of (1) is an expression of the type (2), and we expect that it is also subject to (3), but we do not yet have a full understanding of the conditions on the family $(\mathcal{S}, dS)$ that would ensure this. This is surely true of the (1), in view of the the higher Adler-Taylor formulas

$$\mu_j \cdot \mathbb{E}[\chi(\cdot \cap F^{-1}D)] = \mathbb{E}[\mu_j(\cdot \cap F^{-1}D)] = \sum c_{i,j}\mu_{i+j}(\cdot)\gamma_i(D) \tag{4}$$

Second, [2] proposes a proof of (1) via embeddings of M into spheres $\Sigma_N \subset \mathbb{R}^{N+1}$ of radius $\sqrt{N}$ and dimension N, obtaining the Gaussian projection of M into $\mathbb{R}^d$ by precomposing a given orthogonal projection $\mathbb{R}^{N+1} \to \mathbb{R}^d$ with a random rotation of Σ_N. This brings the spherical kinematic formula into play, which we understand thoroughly using the methods of algebraic integral geometry. To confirm (1) by this means requires a careful study of the resulting tube formulas in Σ_N.

Beyond these technical questions, Adler-Taylor theory also suggests an avenue towards resolving a central puzzle of integral geometry: the extension of the Weyl Tube Theorem (that the $\mu_i(M)$ are Riemannian invariants) to singular spaces such as convex hypersurfaces or sets with positive reach.

References

[1] Adler, Robert J. & Taylor, Jonathan E. *The Geometry of Random Fields.* Springer 2007

[2] Adler, Robert J. & Taylor, Jonathan E. Gaussian processes, kinematic formulae and Poincaré's limit. Ann. Probab. 37 (2009), 1459?1482.

[3] Dmitry Faifman & Georg C. Hofstätter. Convex valuations, from Whitney to Nash. arXiv:2306.07390

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