Mathematisches Forschungsinstitut Oberwolfach

Report No. 52/2023

DOI: 10.4171/OWR/2023/52

Control Methods in Hyperbolic PDEs

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05 November – 10 November 2023

Abstract. Control of hyperbolic partial differential equations (PDEs) is a truly interdisciplinary area of research in applied mathematics nurtured by challenging problems arising in most modern applications ranging from road traffic, gas pipeline management, blood circulation, to opinion dynamics and socio-economical models, as well as in environmental and biological issues, or more recently in the analysis of deep learning and machine learning methods. The topic has gained an increasing attraction of researchers in the last twenty years due to fundamental theoretical as well as numerical advances achieved in the field of nonlinear hyperbolic PDEs. The hyperbolic and the control of PDEs communities, while pursuing separate interests in their respective range of action with a different focus and, often, with a different array of technical tools, do share a substantial body of common knowledge and background. We think the time is right and the momentum is propitious to bring those communities together at a joint workshop, to mutually stimulate each other and interact with one another, for a marked advancement of this area of research on a broad spectrum of control ranging from theoretical to numerical problems and covering also the emerging challenges involving the interplay between (topics of) control and learning. In order to also attract young scientists to this striving field we focus on selected lectures, in-depth discussions, transfer of information from senior to young researchers, and vice versa, and invited plenary talks.

Mathematics Subject Classification (2020): 35Lxx, 35L65, 35Q93, 49J20, 93B05, 93C20.

Introduction by the Organizers

The workshop *Control Methods in Hyperbolic PDEs*, organized by F. Ancona (Padua), B. Anvari (London), O. Glass (Paris) and M. Herty(Aachen) was well attended with over 40 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds. The presentations fostered discussions and exchange on emerging points of interest. Young researchers had also the opportunity to give ad hoc presentations in the afternoon to present their research and to benefit from discussions with colleagues in their field of research. The intention of the workshop has been to cover three topics that are currently very relevant for the future of the field.

First, hyperbolic equations on networks have been studied recently inspired by applications in infrastructure networks. Control questions arise naturally by considering for example nodal flow regulation or capacity planing problems. Usually, the problems lead to spatially one-dimensional hyperbolic equations that allow for a broader range of applicable tools. Natural boundary controllability questions need to be addressed. In the case of regular solutions very general results were obtained in the recent past while the case of entropy weak solutions is much less clear. For example, a very important question is to understand how these equations can be controlled when the control is applied only on one side of the boundary. New ideas of methods of vanishing viscosity and relative entropy analysis in order to understand the well-posedness and control for the dynamics shall be considered. The topic is strategically located in a dimension between one and two and bridges the rather broad knowledge on one-dimensional problems with the less established theory in multiple dimensions by studying problems on highly connected graphs.

Second, the topic of closed–loop feedback controls has been intensively studied in the context of ordinary differential equations but its development for hyperbolic problems is still in early stage besides strong progress on methods and tools in the case of smooth solutions. The rigorous analysis of control strategies, the incorporation of complex flow models in the numerical simulation, as well as, error estimates for the numerical approximation are not yet fully understood. Challenges like nonlinear models, online efficient control in particular for large scale networks are also currently absent. Feedback laws using methods of suitable Lyapunov functions have been applied to control smooth and mildly nonlinear flow patterns. Questions on how to extend this to strong nonlinearities and on how for example entropy and entropy flux pairs may be applied to obtain results also for weakly differentiable functions will be considered in this part. Closed–loop systems might be studied without relying on a sensitivity calculus and therefore we placed this topic in the middle between a pure Cauchy problem and a possible optimality system arising in open loop control. Techniques developed here might therefore shed light on possible ways to tackle problems in open loop control.

Third, apart from formal approaches, very little is known about the link of the sensitivity of particle based models with corresponding sensitivity of the formal kinetic equations. Presentations on individual-based models, their rigorous coarse-graining into macroscopic models and possible applications will be given. The agent based models have the advantage that they consists of finitely many ordinary differential equations such that control concepts are readily available. The coarse graining of those concepts to derive a suitable calculus on the kinetic level will be a first step towards a simi lar calculus on the macroscopic, i.e., hyperbolic level. Therefore, this topic might open new ways to derive suitable concepts for control questions by revisiting existing hierarchies from particles to kinetic and hyperbolic equations. In a similar spirit, recent advances in the (characteristic-like) Lagrangian representation techniques developed for nonlinear conservation laws could provide powerful tools to address controllability and optimality issues in mixed PDE-ODE problems arising for example in mixed traffic flow. This may include also control problems for PDE-ODE models and nonlocal models that naturally arise in pedestrian traffic and autonomous vehicles applications, as well as in supply-chain for complex production networks. Further, there are formulations of dense neural networks where coarse graining has been proposed to develop an efficient description that is amend- able for optimization and control methods. Those in turn could be beneficial in understanding training process of complex learning tasks.

The workshop succeeded in having talks on all fields as well as intensive discussions across those. The following extended abstracts illustrate this nicely and summarize very well the successful workshop.

Workshop: Control Methods in Hyperbolic PDEs

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Abstracts

Bilinear control of PDE's by bounded or unbounded scalar-input control and constructive algorithms for concrete applications

Fatiha Alabau-Boussouira
(joint work with Piermarco Cannarsa, Cristina Urbani)

Scalar-input bilinear controlled abstract PDE's systems take the form:

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, \ t \in [0,T], \\ u(0) = u_0 \in X, \end{cases} \tag{1}$$

where X is the state space, A stands for the diffusion operator, $A : D(A) \subset X \mapsto X$ is a linear unbounded operator and the infinitesimal generator of a $\mathcal{C}^0$ semigroup of bounded linear operators on X, B is a fixed given control operator with $B : D(B) \subset X \mapsto X$ is a linear (bounded or unbounded) operator on X. Here u stands for the state depending on time and taking values in X, and the control p satisfies $p : [0,T] \mapsto \mathbb{R}$, so that the control is a real scalar-valued function that appears as a multiplicative factor acting on the state u (see [7, 9, 10] and references therein).

Such abstract formulation have applications in quantum control (Schrödinger equation) or for the evolution of a probability density (Fokker-Planck equation), on the modeling of nuclear chain reactions for the processes of production of neutrons in fission (heat-based models) or for mechanical systems (such as beams) for describing the dynamics of smart materials. We consider mainly in the sequel heat-based models.

1. Local and semi-global exact controllability to the ground state

We consider (1) under the following assumptions: $(X, \langle\cdot,\cdot\rangle, ||\cdot||)$ is a separable Hilbert space, $B \in \mathcal{L}(X)$, $p \in L^1_{loc}(0,\infty);\mathbb{R})$, $A : D(A) \subset X \mapsto X$ is a densely defined linear operator satisfying:

$$\begin{cases} A \text{ is self-adjoint} \\ \exists\ \sigma > 0 : \ \langle Ax, x\rangle \geq -\sigma||x||^2 \quad \forall\ x \in D(A) \quad (\textit{Assumptions on } A) \\ \exists\ \lambda > 0 : \ \text{such that } (\lambda I + A)^{-1} : X \mapsto X \text{ is compact} \end{cases} \tag{2}$$

so that X has an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}^*}$ formed of eigenvectors of A, the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ of A are bounded below by σ, ordered in a nondecreasing sequence converging to ∞ as k goes to infinity. Setting $\psi_j(t) = e{-}\lambda_j t\varphi_j$, we prove:

Theorem 1 (A.-B., Cannarsa, Urbani NoDEA 2022). *Let A be a densely defined linear operator on X satisfying the above assumption, and $B \in \mathcal{L}(X)$. Assume that $\{A,B\}$ is $j{-}$null controllable in any time $T > 0$ for some $j \in \mathbb{N}^*$. Suppose that the control cost $N(\cdot)$ satisfies: $N(\tau) \leq e^{\nu/\tau} \quad \forall\ \tau \in (0,T_0] \quad (CCC)$, for some constants $\nu > 0$ and $T_0 > 0$. Then for any $T > 0$, there exists a constant $R_T > 0$*

such that for any $u_0 \in B(\varphi_j, R_T)$*, there exists a control* $p \in L^2(0,T)$ *such that the solution of* (1) *satisfies* $u(T; u_0, p) = \psi_j(T)$ *(with explicit estimates on the control cost).*

Theorem 2 (A.-B., Cannarsa, Urbani NoDEA 2022, with applications to the Fokker-Planck equation)**.** *Let A satisfy (Assumptions on A) and be such that the (Gap Condition holds). Let* $B \in \mathcal{L}(X)$ *be such that there exist* $b > 0$*,* $q > 0$ *such that the following non vanishing condition (NVC) (first assumption) and asymptotic behavior (second inequality) hold:*

$$(3) \qquad \langle B\varphi_j, \varphi_j\rangle \neq 0 \ (NVC) \ \& \ |\lambda_k - \lambda_j|^q |\langle B\varphi_j, \varphi_k\rangle| \geq b \quad \forall\, k \neq j \ (AB).$$

Then the pair $\{A, B\}$ *is* $j-$*null controllable in any time* $T > 0$ *and the control cost satisfies (Control Cost Condition) with explicit estimates.*

Thus, (1) is locally controllable to the jth eigensolution ψ_j in any time $T > 0$. Furthermore when A is an accretive operator, we prove additional two semi-global controllability results, showing that every $u_0 \in X \backslash \varphi_1^{\perp}$ is controllable to the evolution (i.e. the dynamical system without control) of its orthogonal projection along the ground state.

2. Unbounded control operator and applications

Let $(X, \langle\cdot,\cdot\rangle, ||\cdot||)$ be a separable Hilbert space. Let $A : D(A) \subset X \to X$ be a densely defined linear operator with the following properties:

$$\begin{array}{ll} (a) & A \text{ is self-adjoint,} \\ (b) & \langle Ax, x\rangle \geq 0,\ \forall\, x \in D(A), \quad (New\ Assumptions\ on\ A) \\ (c) & \exists\, \lambda > 0 \,:\, (\lambda I + A)^{-1} : X \to X \text{ is compact.} \end{array}$$

Under the above assumptions A is a closed operator and $D(A)$ is also a Hilbert space with respect to the scalar product $(x, y)_{D(A)} = \langle x, y\rangle + \langle Ax, Ay\rangle, \quad \forall\, x, y \in D(A)$. Note also that $-A$ is the infinitesimal generator of a strongly continuous semigroup of contractions on X which is also analytic and will be denoted by e^{-tA}. We also assume that $B : D(B) \subset X \to X$ be a linear unbounded operator such that $D(A^{1/2}) \hookrightarrow D(B)$ with continuous embedding.

Theorem 3 (A.-B., Cannarsa, Urbani Comptes Rendus Mathématique 2023)**.** *Let* A *and* $B : D(B) \subset X \to X$ *be a linear unbounded operator satisfying the above hypotheses. Let* $\{A, B\}$ *be* 1*-null controllable in any* $T > 0$ *with control cost* $N(\cdot)$ *such that there exist* $\nu, T_0 > 0$ *for which (CCC) holds. Then, for any* $T > 0$*, there exists a constant* $R_T > 0$ *such that, for any* $u_0 \in B_{R_T,1/2}(\varphi_1)$*, there exists a control* $p \in L^2(0,T)$ *for which* (1) *with initial data* $u_0 \in D(A^{1/2})$ *is locally controllable to the ground state solution in time* T*, that is,* $u(T; u_0, p) = \psi_1(T)$*.*

Remark 1. *We show that the above assumptions can be weakened in order to include the cases when A admits negative eigenvalues. We also prove two semi-global exact controllability results to the ground state, or to its orthogonal projection.*

3. An original methodology and algorithm

We developed a general method for producing infinite classes of potential functions that fulfill the non vanishing condition (NVC) holds (see the first condition in (3)), for the above results to hold. We also provide an algorithm to derive functions μ (the dipole moment for the Schrödinger equation for instance). The asymptotic lower estimates (AB) for an appropriate q can also be deduced thanks to our results. It also allows to derive new spaces X in which several of the above theorems hold.

Theorem 4 (A.-B., Urbani 2019-2020)**.** *Let $T > 0$ be given and $\mu \in (S)_{NVC}$ (as given in A.-B.-Urbani's algorithm). Then there exist $\delta > 0$ and a $\mathcal{C}^1$ map $\Gamma : \mathcal{R}_T \longrightarrow L^2(0,T)$ such that $\Gamma\Big(\psi_1(T)\Big) = 0$, and for all $u_f \in \mathcal{R}_T$, the solution of the Schrödinger equation with initial data $u_0 = \varphi_1 = \psi_1(0)$ and control $p = \Gamma(u_f)$ satisfies $u(T) = u_f$ where $\psi_1(t,x) = e^{-i\lambda_1 t}\varphi_1(x) \quad \forall\ t \geq 0, \forall\ x \in [0,1]$ and for $T > 0$ and $\delta > 0$, $\mathcal{R}_T = \{u_f \in \mathcal{S} \cap D(A), ||u_f - \psi_1(T)||_{H^2} < \delta\}$*

4. Some perspectives and open question

Whenever the moment method is used, it reduces the scope to $1D$ or radial results. This raises the question of proving $j-$null controllability, by other strategies for $2D$ and $3D$ controllability results such that the ones relying on (3). Extension to semi-linear parabolic dynamical systems, more complex models and to build efficient numerics are also of great interest. For the Fokker-Planck equation we cannot handle the case of perfectly reflecting boundary conditions for which $D(A)$ becomes a domain that both depends on the time and the control function p. It is an open question to treat such boundary conditions.

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Instabilities in machine learning and in PDEs

Giovanni S. Alberti

(joint work with Rima Alaifari, Tandri Gauksson)

Adversarial examples in classification. Deep neural networks (DNN) in classification have been shown to be susceptible to adversarial perturbations [8]: small changes in the inputs can lead to misclassification. In other words, given an image x that is correctly classified by the network, it is possible to construct an adversarial example x' that is visually indistinguishable from x but misclassified by the network. The standard measure of the distance between x and x' is given by a suitable norm of $x - x'$, e.g.

$$\|x - x'\|_\infty \leq \varepsilon.$$

Here, it is required that the maximum pixel perturbation in the adversarial example is smaller than or equal to ε. For this reason, x' is an adversarial (additive) *perturbation* of x.

It is natural to wonder about the best way to measure the distance between x and x'. While the above approach is easy to interpret and straightforward to implement, it is also clear that many small transformations between two images are not captured. For example, if x' is a small translation of x, their image content is unchanged, but $\|x - x'\|_\infty$ will not be small. This observation leads us to consider adversarial *deformations*. Given an image $x \in L^2([0,1]^2)$, we consider

$$x_\tau(p) = x(p + \tau(p)),$$

where $\tau\colon [0,1]^2 \to \mathbb{R}^2$ is a vector field determining the deformation. The distance between x and x_τ is measured by a suitable norm of τ, e.g. $\|\tau\|_{L^\infty([0,1]^2)}$. In [1], we design a simple iterative algorithm that constructs adversarial deformations, and show that neural networks that are commonly used in classification (for MNIST and ImageNet) are indeed vulnerable to adversarial deformations.

Adversarial examples in inverse problems. These observations indicate that DNN tend to be unstable, meaning that given two inputs that are very close, their outputs may be far apart. Another domain where stability plays a crucial role is inverse problems, where unknown physical quantities need to be reconstructed from indirect measurements. Since these measurements are typically noisy, it is important for any reconstruction method to be stable. In recent years, machine learning, and especially deep learning, has become a popular tool for solving inverse problems. Therefore, it is crucial to understand how stable these methods are. In [4], the authors demonstrate that several deep learning based methods developed for solving the inverse problem (undersampled Fourier measurements) in accelerated magnetic resonance imaging (MRI) are indeed vulnerable to adversarial perturbations. The reconstructions may contain hallucinations, namely, undesirable features that are completely created by the network and may not be easily identified. Another conclusion of [4] is that state-of-the-art reconstruction methods based on total variation (TV) regularization and with guarantees provided by the theory of compressed sensing are more stable and do not hallucinate.

A more quantitative analysis is provided in [5], where deep learning and TV regularization are compared by using the relative L^2 error of the reconstruction as a metric. The results show a similar vulnerability of the two methods to adversarial errors and to Gaussian errors. However, as discussed above, the L^2 norm does not fully capture the meaningful features of an image. For instance, given a (discretized) image x, one can construct two noisy perturbations:

$$x_1 = x + g, \qquad x_2 = x + a\chi_A,$$

where g is white Gaussian noise with $\|g\|_2 = \varepsilon$ and χ_A is the indicator function of a small region A, so that $\|a\chi_A\|_2 = \varepsilon$. As a consequence, we have

$$\|x - x_1\|_2 = \|x - x_2\|_2.$$

Thus, if we use the L^2 norm, these two perturbations are quantitatively identical. However, in terms of visual impact, if ε is small, x_1 will be visually indistinguishable from x (or the noise can easily be identified), while the feature A added in x_2 may be more problematic. A possible strategy to construct adversarial perturbations that are visually meaningful as well as quantitatively measurable is proposed in [2], where localized adversarial artifacts are constructed for the inverse problem of undersampled MRI. We show that TV regularization is more vulnerable than DNN-based methods. Furthermore, we show how the vulnerability to this type of attacks is inherently connected to the exact recovery guarantees given by compressed sensing theory for TV regularization.

Instabilities and adversarial examples in PDEs. The stability properties of DNN can also be analyzed through the lens of the theory of partial differential equations (PDEs). It was shown in [7] that when both the layers and the space variables are considered in the continuous limit, the action of a residual convolutional neural network (CNN) on an input f may be written as

$$f \longmapsto u(T),$$

where u is the solution of a (nonlinear) dynamic PDE of the form:

$$\partial_t u(t) = F_t(u(t)), \qquad u(0) = f,$$

where F_t is a, possibly nonlinear, differential operator. Whenever energy conservation is crucial, this *parabolic CNN* may be replaced by a *hyperbolic CNN*:

$$\partial_t^2 u(t) = F_t(u(t)), \qquad u(0) = f, \qquad \partial_t u(0) = 0.$$

In both cases, under suitable assumptions on the differential operators and on the nonlinearities included in F_t, it is possible to show that

$$\|u_1(T) - u_2(T)\|_2 \lesssim \|u_1(0) - u_2(0)\|_2.$$

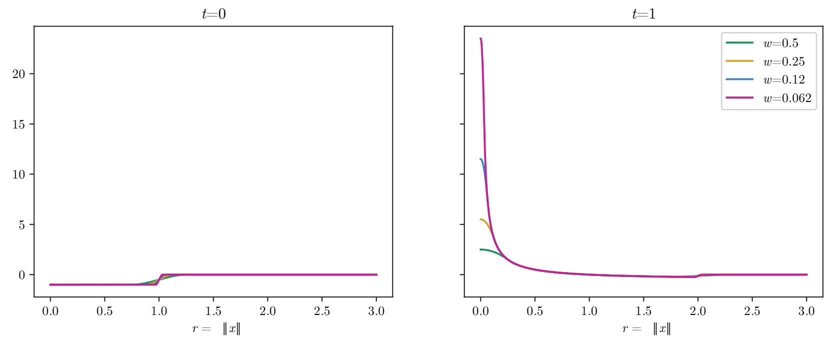

FIGURE 1. An example of a sequence as in (2). These two figures show the radial components of f_n and of $u_n(1,:)$, respectively. The initial states f_n, $n=1,2,3,4$, are continuously differentiable functions that transition from the value -1 to the value 0 in a window of width $w=2^{-n}$, centered at $r=1$.

Namely, these maps are Lipschitz stable.

These bounds are obtained by using standard energy estimates, and are specific to the L^2 norm. Because of the above discussions on the different possibile notions of perturbations, and in particular on the observations on the limited relevance of the L^2 norm in certain contexts, it is natural to wonder whether these stability estimates can be extended, for instance, to the L^∞ norm. The answer is negative, as a simple example shows. Consider the (linear) wave equation with constant coefficients in $\mathbb{R}^3$:

$$\partial_t^2 u - \Delta u = 0, \qquad u(0) = f, \qquad \partial_t u(0) = 0. \tag{1}$$

With a radial initial condition $f(x) = g(|x|)$, u is a spherical wave of the form

$$u(t,x) = \frac{1}{2|x|}\left(\varphi(|x|-t) + \varphi(|x|+t)\right), \qquad \varphi(r) = rg(r).$$

If we evaluate this expression at $x=0$, we obtain $u(t,0) = g(t) + t\cdot g'(t)$. Therefore, it is possible to find a sequence of adversarial examples f_n such that

$$\|f_n\|_\infty \leq 1, \qquad \|u_n(1,\cdot)\|_\infty \to +\infty, \tag{2}$$

see Figure 1. In particular, the map $f \mapsto u(1,\cdot)$ is unstable with respect to the L^∞ norm.

From an abstract point of view, this instability can be analysed by using the framework of Fourier multipliers [6]. The map $f \mapsto u(1,\cdot)$ may be written as

$$B\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad f \mapsto \mathcal{F}^{-1}\left[\cos\left(2\pi\,|\cdot|\right)\mathcal{F}f\right],$$

and it is easy to see that this map is not bounded with respect to the L^∞ norm. In [3], we propose a method to regularize the operator B in order to obtain a family of approximations that are bounded as operators $L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. More

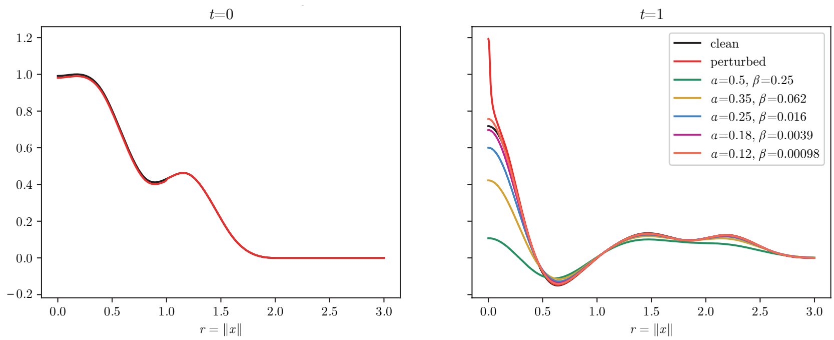

FIGURE 2. Radial component of spherical waves at times $t = 0$ and $t = 1$. Left: the clean initial state f and a perturbed initial state $f + r$, with $\|r\|_{L^\infty(\mathbb{R}^3)} = 0.01 \cdot \|f\|_{L^\infty(\mathbb{R}^3)}$. Right: the end states Bf, $B(f + r)$, and $B_{\alpha,\beta}(f + r)$.

precisely, let

$$B_{\alpha,\beta}(f) = \mathcal{F}^{-1}\left(\cos\left(2\pi\,|\cdot|\right)\cdot\kappa_\alpha\cdot\mathcal{F}\left(h_\beta f\right)\right),$$

with filters $\kappa_\alpha, h_\beta \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. The operator $B_{\alpha,\beta}\colon L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ is well-defined and bounded for every $p \in [2, +\infty]$. Moreover, if the family of filters k_α and h_β is suitably chosen, then $B_{\alpha,\beta} \to B$ in a suitable sense as $\alpha, \beta \to 0$.

An example of this behavior is shown in Figure 2. A radial input f is modified with a small (with respect to both the L^∞ and the L^2 norm) adversarial perturbation r. The quantity $B(f + r)$, namely, the solution to (1) with initial value $f + r$ calculated at time $t = 1$, presents a visible artifact in the origin. By using the regularized quantities $B_{\alpha,\beta}(f + r)$, it is possible to substantially reduce the artifact, while maintaining a good overall quality of the output. As is typical in regularization, this procedure requires a suitable choice of the parameters α and β. This regularization strategy turns out to be effective for this simple linear PDE, and it would be interesting to investigate whether similar ideas can be used also for more complicated nonlinear PDEs or for the corresponding neural networks.

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Source and subdomain control of scalar conservation laws

Boris Andreianov
(joint work with Shyam S. Ghoshal)

Given a trajectory $v : [s, T] \mapsto X$ for evolution equation of the form $\dot{u} + Au \ni g$ governed by an m-accretive densely defined operator on a Banach space X ([3]), given any initial state $u_0 \in X$, we provide avery simple construction of feedback control that permits to reach, at $t = T$, the final state $v(T)$ starting from u_0 at $t = 0$. The construction is reniniscent of Luenberger's observers (the nudging strategy) in Data Assimilation ([6]), but the exponential stabilisation to the target state at large times is replaced by exact control at final time T. In turn, this provides a distributed control in $L^1(0, T; X)$ where X is the state space. Therefore, if s a state v_T is attainable (with source) at some time τ, it is attainable - with source- from any initial/boundary states, at any time $T \geq \tau$.

The specific application we have in mind is to scalar conservation laws, possibly multi-dimensional and non-convex; moreover, replacing the abstract semigroup arguments by the standard PDE arguments, we can also handle e.g. the Cauchy-Dirichlet problem with time-dependent boundary conditions. The construction of the distributed control has a numerical counterpart, in the setting of Finite Volume approximation by a monotone scheme.

Backward constructions with source are proposed, showing that wide classes of data (including BV and many fractional Sobolev spaces) are attainable - with distributed source in L^1- at any time (cf. [5] for bounded source). In $1D$, a variant of backward front-tracking construction provides reconstructions under the form of a continuous juxtaposition of compression and rarefaction fans, for BV data.

Further, combining the idea of source control, now localized in a compact interval in space, with geometric observability-kind constructions in terms of backward characteristics ([2, 1]), we are also able to explore the framework of subdomain control (cf. [5]). This approach remains however restricted to a 1D, strictly convex scalar conservation law, in the Cauchy or the Cauchy-Dirichlet setting.

Extension to kinetic formulation systems of conservation laws makes sense ([4]).

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Non-admissibility of spiral-like strategies

Stefano Bianchini
(joint work with Martina Zizza)

1. Introduction

We study a dynamic blocking problem first proposed by Bressan in [3]. The problem is concerned with the model of wild fire spreading in a region of the plane $\mathbb{R}^2$ and the possibility to block it constructing some *barriers* in real time. If we denote by $R(t) \subset \mathbb{R}^2$ the region burned by the fire at time t, then we can describe it as the reachable set for a differential inclusion. More precisely, one considers the Cauchy Problem

$$\dot{x} \in F(x), \qquad x(0) \in R_0,$$

where the set $R_0 \subset \mathbb{R}^2$ represents the region burnt by the fire at the initial time $t = 0$, while the function F describes the speed of spreading of the fire. The set $R_0 \subset \mathbb{R}^2$ is assumed to be open, bounded, non-empty and connected with Lipschitz boundary, whereas the standard assumptions on $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$, which is a Lipschitz-continuous multifunction, are:

(1) there exists $r > 0$ such that $B_r(0) \subset F(x) \quad \forall x \in \mathbb{R}^2$;
(2) $F(x)$ is compact and convex $\forall x \in \mathbb{R}^2$;
(3) $x \rightrightarrows F(x)$ is Lipschitz-continuous in the Hausdorff topology.

If no barriers are present the *reachable set* for the differential inclusion is

$$R(t) = \Big\{x(t), \quad x(\cdot) \text{ abs. cont.}, x(0) \in R_0, \dot{x}(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0,t]\Big\}.$$

When the fire starts spreading, a fireman can construct some barriers, modeled by a one-dimensional rectifiable set $\zeta \subset \mathbb{R}^2$, in order to block the fire. More in detail, we consider a continuous function $\psi : \mathbb{R}^2 \rightrightarrows \mathbb{R}^+$ together with a positive constant $\psi_0 > 0$ such that $\psi \geq \psi_0$. If we denote by $\zeta(t) \subset \mathbb{R}^2$ the portion of the barrier constructed within the time $t \geq 0$, we say that ζ is an admissible barrier (or admissible strategy) if

(1) (H1) $\zeta(t_1) \subset \zeta(t_2)$, $\forall t_1 \leq t_2$;
(2) (H2) $\int_{\zeta(t)} \psi d\mathcal{H}^1 \leq t, \quad \forall t \geq 0$,

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Once we have an admissible strategy ζ, then we define the reachable set for ζ at time t the set

(1)
$$R^\zeta(t) = \Big\{ x(t) : x \text{ abs. cont.}, \dot{x}(\tau) \in F(x(\tau)) \text{ a.e. } \tau \in [0,t], x(\tau) \notin \zeta(\tau)\ \forall \tau \in [0,t] \Big\}.$$

Definition. Let $t \Rightarrow \zeta(t)$ be an admissible strategy. We say that it is a *blocking* strategy if

$$R^\zeta_\infty \doteq \bigcup_{t \geq 0} R^\zeta(t)$$

is a bounded set.

We call *isotropic* the case in which the fire is assumed to propagate with unit speed in all directions, while the barrier is constructed at a constant speed $\sigma > 0$, namely

$$F \equiv \overline{B_1(0)}, \quad R_0 = B_1(0), \quad \psi \equiv \frac{1}{\sigma}, \tag{2}$$

where $\overline{B_1(0)}$ denotes the closure of the unit ball of the plane centered at the origin. We remark that in [5] there are comparison results between more general choices of the data R_0 and F and the isotropic problem for the study of the fire problem in a more general setting.

The existence of admissible blocking (or winning) strategies for the isotropic blocking problem is a very challenging open problem and it has been addressed mainly in [3],[5].[1] In particular, the following theorems hold:

Theorem. Assume that (2) hold. Then if $\sigma > 2$ there exists an admissible blocking strategy.

Theorem. Assume that (2) hold. Then if $\sigma \leq 1$ no admissible blocking strategy exists.

The two theorems are proved in [3] and they motivate the following Fire Conjecture [4]:

Conjecture. Let (2) hold. Then if $\sigma \leq 2$ no admissible blocking strategy exists.

In this talk we study spiral-like strategies: namely, admissible barriers that are constructed putting all the effort on a single branch. The study of spiraling strategies is of key importance in the complete solution of Bressan's Fire conjecture, indeed there is a strongly belief that these strategies are the best possible barriers that can be constructed when $\sigma \leq 2$.

We start giving the definition of spiral-like strategies:

[1] One can prove that the existence of blocking strategy does not depend on the starting set R_0 but only on the speed σ [2].

Definition. Let $Z = \zeta([0, S]) \subset I\!\!R^2$ be a strategy, where ζ is a parametrization by length. We say that it is a spiral-like strategy if it satisfies:

- $\zeta(0) = (1, 0)$ and $\zeta|_{[0,S)}$ is simple;
- $s \mapsto u \circ \zeta(s)$ is increasing.

Finally, we say that Z is an admissible spiral if it is a spiral-like strategy, the curve is locally convex, in the sense of the definition above and moreover it satisfies the following assumption

$$(\mathrm{A1}) \qquad 0 \leq \angle(\mathbf{t}^+(0), \mathbf{e}_2) \leq \frac{\pi}{2},$$

where $\mathbf{e}_2$ is the vertical vector of the canonical base $\mathbf{t}(0)$ is the tangent vector of the spiral in the starting point (commonly $(1, 0)$) and $\angle$ denotes the angle between the two vectors. What really matters in the definition of spiral-like strategies is the requirement

$$s \Rightarrow u \circ \zeta(s) \qquad \text{is increasing},$$

which corresponds to the fact that either a portion of the barrier lies on the level set $\{u =\}$ (so that the previous function is constant), or the fire can not burn two portions of the barrier simultaneously. We believe instead that the hypothesis of local convexity and the assumption (A1) are automatically satisfied by *optimal* spiral-like strategies, which is an open question.

In addition to the parametrization by arc-length, it is possible to parametrize any admissible spiral by $(r(\phi), \phi)$, where ϕ denotes the angle of rotation on the spiral, while $r(\phi)$ represents the length of the final segment of the fire ray reaching the point $(r(\phi), \phi)$.

The only results known on these barriers can be found in [5] and [8]. In the two papers it is proved independently and with different techniques the following

Theorem. *Let $\sigma > 2.6144..$ (critical speed). Then there exists a spiral-like strategy which confines the fire.*

This theorem inspires therefore the following

Conjecture. If $\sigma \leq 2.6144...$ then no spiral-like strategy is admissible.

A partial answer to this conjecture has been given in [8] where the authors use a geometric argument to prove that if $\sigma \leq \frac{1+\sqrt{5}}{2}$ then no spiral-like strategy is admissible.

We proved the following

Theorem. *No admissible spiral-like strategy confines the fire if $\sigma \leq 2.3$.*

The bound 2.3 is not sharp, since it is obtained by purely numerical computations. It could be an interesting question to investigate the numerical optimization of the parameter σ accordingly to the method we will propose for the solution of this problem. But unfortunately, even if the speed σ could be optimized, the critical case $\sigma = 2.6144..$ at the present time seems out of reach and very delicate. We remark that this theorem proves Bressan's Fire Conjecture in the case of spiral strategies.

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Optimal control strategies for moving sets

Alberto Bressan

We consider a family of control problems for a moving set. In applications, this set can describe the region infested by an invasive biological population, which can grow or shrink in time, depending on the control applied along the boundary. For example, a region infested by mosquitoes can be reduced in size by spraying pesticides along its boundary. Optimal strategies are sought, which minimize the contaminated area plus a cost for implementing the control.

In mathematical terms, given an initial bounded set $\Omega(0) = \Omega_0 \subset \mathbb{R}^2$, for $t \in [0,T]$ we seek a motion $t \mapsto \Omega(t)$ which minimizes a cost functional of the form

$$J \doteq \int_0^T \phi\big(\mathcal{E}(t)\big)\, dt + \int_0^T \mathrm{meas}\big(\Omega(t)\big)\, dt + \kappa\, \mathrm{meas}\big(\Omega(T)\big). \tag{1}$$

This accounts for a control cost, and a cost depending on the area of the contaminated region at various times.

Denoting by $\beta(t,x)$ the **inward-pointing velocity** of the boundary of the set $\Omega(t)$ at the point $x \in \partial\Omega(t)$, the **total control effort** at time t is measured by

$$\mathcal{E}(t) \doteq \int_{\partial\Omega(t)} E\big(\beta(t,x)\big)\, dx. \tag{2}$$

A natural choice for the effort function E in (2) is

$$E(\beta) = \max\{1+\beta, 0\}. \tag{3}$$

In other words, if no effort is made, the contaminated region expands with unit speed ($E = 0$ corresponds to $\beta = -1$). By applying a control along the boundary, this expansion can be reduced, or even reversed.

Possible choices for the function ϕ in (1) are:

$$\phi(s) \;=\; \frac{s^2}{2} \qquad\qquad \text{or} \qquad\qquad \phi(s) = \begin{cases} 0 & \text{if } s \le M, \\ +\infty & \text{if } s > M. \end{cases} \tag{4}$$

Notice that, if the second definition applies, then the first term in (1) has the meaning of a constraint: at every time $t \in [0,T]$ the total control effort must satisfy $\mathcal{E}(t) \le M$.

Optimization problems of the form (1) were first considered in [2], and formally derived from the control of a parabolic equation, by taking a sharp interface limit. Existence of optimal solutions was proved in [3]. Necessary conditions for optimality were also determined, in the form a Pontryagin maximum principle. A basic setting is the following:

(OP) Optimization Problem. *Given a bounded initial set $\Omega(0) = \Omega_0 \subset \mathbb{R}^2$, find a motion $t \mapsto \Omega(t) \subset \mathbb{R}^2$ that minimizes the cost*

$$\mathcal{J} \;=\; meas\big(\Omega(T)\big),$$

subject to $\beta(t,x) \ge -1$ and

$$\mathcal{E}(t) \;\doteq\; \int_{\partial\Omega(t)} \big(1 + \beta(t,x)\big)\, d\sigma \;\le\; M \qquad \textit{for every } t \in [0,T]. \tag{5}$$

Notice that in this case:

(i) With no control, the contaminated set $\Omega(t)$ expands with unit speed in all directions.

(ii) Implementing a control along the boundary, we can clear a region of area M per unit time.

In turn, this implies that the increase of the area of $\Omega(t)$ is determined by

$$\frac{d}{dt}\text{meas}\big(\Omega(t)\big) \;=\; \text{length}\big(\partial\Omega(t)\big) - M.$$

This suggest that, in order to reduce the area, it is always most convenient to reduce the perimeter as quickly as possible. A rigorous proof of this fact was recently given in [1].

Theorem 1. *In connection with the optimization problem* **(OP)***, assume that the initial set $\Omega(0) = \Omega_0 \subset \mathbb{R}^2$ is convex. Then, at each time $t \in [0,T]$, the optimal set $\Omega(t)$ is convex.*

The optimal control is active precisely along the portion of the boundary $\partial\Omega(t)$ where the curvature is maximum. This is a union of arcs of circumferences, all with the same radius $r(t)$.

At the present time, several related questions remain open. In particular:

(Q1) What is the regularity of the optimal sets $\Omega(t)$? We recall that, to derive the necessary conditions in [3], one needs to construct trajectories $t \mapsto x(t,\xi) \in \partial\Omega(t)$ orthogonal to the boundary. Can these trajectories be always well defined?

(Q2) If the initial set Ω_0 is not convex, what can one say about the optimal strategy? Under what conditions is it true that the sets $\Omega(t)$ are connected, for all $t \in [0, T]$?

(Q3) More generally, all the above problems can be formulated on $\mathbb{R}^n$, or even on an n-dimensional Riemann manifold. How much of the theory remains valid in a multidimensional setting? Does the convexity result stated in Theorem 1 remain true for optimal set motion in $\mathbb{R}^3$?

The case with geographical constraints, where the pest population needs to be eradicated from an island (a bounded open set V in the plane), is also of interest. As before it is assumed that, within V, the contaminated region expands with unit speed in all directions, while the control can "clean up" an area M per unit time. In this setting, two main problems arise.

Eradication problem. *Find an admissible strategy $t \mapsto \Omega(t) \subseteq V$ that completely eradicates the contamination in finite time. This means: $\Omega(0) = V$, $\Omega(T) = \emptyset$, and the following constraint is satisfied:*

$$\mathcal{E}(t) \doteq \int_{\partial\Omega(t)\cap V} \big(1 + \beta(t, x)\big)\, d\sigma \leq M \qquad \textit{for every } t \in [0, T].$$

The existence or non-existence of such a strategy can be determined by comparing the speed M with two geometric invariants of the set V. In the positive case, one can further consider:

Minimum time problem. *Among all strategies that eradicate the contamination, find one that minimizes the time T.*

The analysis of optimality conditions indicates that, in an optimal strategy, at each time $t > 0$ the interface between free and contaminated zone should be the union of

(i) arcs of circumferences, all with the same radius $r(t)$, where the control is active, together with
(ii) additional arcs where the control is not active, and the contamination expands with unit speed.

A direction of current research aims at understanding the structure of optimal strategies in two main cases:

(i) V is a polygon.
(ii) V is a generic convex set, with smooth boundary.

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A Front Tracking Approach to an Euler-type Flocking Model

Cleopatra Christoforou
(joint work with Debora Amadori)

The study of hydrodynamic models that emerged in the area of self-organization has received alot of attention in the recent years and many new challenges in partial differential equations have arisen that yield interesting questions in the mathematical community.

We consider the system

$$(1)\qquad \begin{cases} \partial_t \rho + \partial_x(\rho \mathtt{v}) = 0, \\ \partial_t(\rho \mathtt{v}) + \partial_x\left(\rho \mathtt{v}^2 + p(\rho)\right) = K \displaystyle\int_{\mathbb{R}} \rho(x,t)\rho(x',t)\left(\mathtt{v}(x',t) - \mathtt{v}(x,t)\right)\, dx' \end{cases}$$

with $(x,t) \in \mathbb{R} \times [0,+\infty)$. Here $\rho \geq 0$ stands for the density, $\mathtt{v}$ for the velocity, p for the pressure, given by

$$(2)\qquad p(\rho) = \alpha^2 \rho\,, \qquad \alpha > 0\,,$$

and $K > 0$ is a given constant. Having set $\mathtt{m} := \rho \mathtt{v}$ as the momentum variable, we consider the Cauchy problem (1) with the initial condition

$$(3)\qquad (\rho, \mathtt{m})(x,0) = (\rho_0(x), \mathtt{m}_0(x)) \qquad x \in \mathbb{R}\,,$$

and our aim is to formulate a problem to (1)–(3) with conditions appropriate for the models of self-organized systems and then seek weak solutions that admit time-asymptotic flocking.

The pioneering work of Cucker and Smale [5] led a major part of the mathematical community to conduct research intensively on this topic. Many mathematical models have arised and most work so far is on the behavior of the particle models, the kinetic equation and the hydrodynamic formulation. However, very little is done in this area on *weak solutions* and the scope of our work is to contribute in this direction of weak solutions in the set up of the Euler-type flocking system that can be derived using a hydrodynamic formulation. We refer to the reviews [4, 7, 8] and the references therein. We stand out the work of Karper, Mellet and Trivisa in [6], in which it is shown the convergence of weak solutions to the kinetic equation Cucker-Smale flocking model to strong solutions of an Euler-type flocking system of the form (1) with pressure of the form (2). Thus, we study this system with pressure as derived in [6] although pressureless systems have received most attention so far. Also, we are interested in solutions with discontinuities and most results so far deal with regular solutions.

Our analysis goes over the following steps: we first set up the problem with initial data appropriate for flocking models, that is the initial data having finite total mass confined in a bounded interval and initial density uniformly positive therein. Next, we introduce an appropriate notion of entropy weak solutions with concentration centred along two free boundaries emanating from the endpoints of the initial support. It is shown that under this notion, solutions conserve mass and momentum and the system reduces to a local one for an all-to-all interaction kernel.

The construction of the weak solution is obtained by transforming the problem into Lagrangian variables (cf. [9]) and employing the front tracking algorithm (cf. [3]). Showing that the linear functional is non-increasing, it allows us to pass to the limit and obtain an entropy weak solutions with concentration. Additional analysis at the level of approximate solution reveals a geometric wave decay and this yields an exponential decay of the total variation in time. Having this, we conclude unconditional time-asymptotic flocking, i.e. the support of the solutions remains bounded for all times and velocity alignment occurs without any further restrictions on the data.

There are many open problems on this topic arise that would be very interesting to be studied and contribute in the better understanding of flocking phenomena. It would be important to extend our result to the non-constant case of interaction kernel K that would include especially the singular kernels. Another direction would be to study control-type problems on this set-up, i.e. given a final state at time $t = T$, to determine the initial data for which the given profile is reached at time T. An alternative interesting direction is the study of the system with pressure of the form ρ^γ for $\gamma > 1$ and its relation with the solutions to (1), we have taking the limit $\gamma \to 1$.

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Optimal Control of Propagaton Fronts

Maria Teresa Chiri
(joint work with Alberto Bressan, Najmeh Salehi)

The control of parabolic equations is by now a classical subject [8, 9, 10, 12]. More specifically, several studies have been devoted to the optimal harvesting of spatially distributed populations [5, 6, 11]. Our recent interest in the control of reaction-diffusion equations is primarily motivated by models of pest eradication [1, 2, 7, 13]. The controlled spreading of a population, in a simplest form, can be described by a semilinear parabolic equation

$$u_t \;=\; f(u) + \Delta u - \alpha u. \tag{1}$$

Here $u = u(t, x)$ denotes the population density at time t, at a location $x \in \mathbb{R}^2$. The function f describes the reproduction rate, while $\alpha = \alpha(t, x)$ is a distributed control representing the quantity of pesticides sprayed at time t at location x, and $\alpha\, u$ describes the amount of population which is eliminated by this strategy. Given an increasing, convex cost function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, we can consider the following optimization problem

(OP1) Optimization Problem for a reaction-diffusion equation. *Given an initial profile $u(0, x) = u_0(x)$ and a time interval $[0, T]$, determine a control $\alpha = \alpha(t, x) \geq 0$ so that, calling $u(t, x)$ the corresponding solution to (1), the total cost*

$$\mathcal{J} \;\doteq\; \int_0^T \phi\left(\int \alpha(t,x)\,dx\right) dt + \kappa \int_0^T \int u(t,x)\,dx\,dt \tag{2}$$

is minimized.

We think of $\int \alpha(t, x)dx$ as the *total control effort* at time t. Standard results yield the existence of an optimal strategy, and necessary conditions for optimality. However, very rarely one can find explicit formulas for the optimal solution. Assuming that $f(0) = f(1) = 0$, and observing that solutions to (1) often develop stable traveling fronts, the parabolic problem **(OP1)** can be approximated with an optimal control problem for the moving set $\Omega(t) \;=\; \big\{x \in \mathbb{R}^2\,;\; u(t, x) \approx 1\big\}$, representing the contaminated region. In this case, the control function is the speed $\beta = \beta(t, x)$ at which the boundary $\partial\Omega(t)$ is pushed in the inward normal direction. This new problem has the form

(OP2) Optimization Problem for a moving set. *Let an initial set $\Omega_0 \subset \mathbb{R}^2$ and cost functions $E : \mathbb{R} \mapsto \mathbb{R}_+$, $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \cup \{+\infty\}$ be given. Find a set-valued function $t \mapsto \Omega(t)$, with $\Omega(0) = \Omega_0$, which minimizes*

$$J \;=\; \int_0^T \phi\left(\int_{\partial\Omega(t))} E\big(\beta(t,x)\,d\sigma\right) dt + \kappa \int_0^T meas\big(\Omega(t)\big)\,dt. \tag{3}$$

In our work [3, 4] we investigated the underlying connection between the two above optimization problems. In particular, the effort $E(\beta)$, needed to achieve the inward normal speed β, can be uniquely determined by solving an optimal control problem for traveling wave profiles of (1). The cost for moving the interface at different speeds in the normal direction is determined through the analysis of traveling wave profiles for the PDE model, and justified via a sharp interface limit. More generally, the same approach remains valid for systems of (possibly degenerate) parabolic equations with spatial variable in $\mathbb{R}^n$.

A rigorous derivation of **(OP2)** would require a study of the Γ-limit of the functionals

$$\mathcal{F}_\varepsilon(u) \doteq \int_0^T \int \frac{[\varepsilon \Delta u + \varepsilon^{-1} f(u) - u_t]_+}{u}\, dx\, dt \tag{2}$$

as $\varepsilon \to 0$. Here $[s]_+ = \max\{s, 0\}$. As first step in this direction, we have proved that the cost J at (3) can be achieved as the limit of the cost (2), for a family of solutions to the rescaled parabolic equations

$$u_t^\varepsilon = \frac{1}{\varepsilon} f(u^\varepsilon) + \varepsilon\, \Delta u^\varepsilon - u^\varepsilon \alpha^\varepsilon\,, \qquad t \in [0, T],\ x \in \mathbb{R}^2.$$

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Renewal Equations: Models, Analysis and Control Problems

Rinaldo M. Colombo
(joint work with Mauro Garavello, Francesca Marcellini, Elena Rossi)

According to [13, § 3.1] by *renewal equation* the following initial boundary value problem is meant:

$$\begin{cases} \partial_t \rho + \partial_x \rho = 0 & (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ \rho(t,0) = \int_0^{+\infty} b(\xi)\, \rho(t,\xi)\, d\xi & t \in \mathbb{R}_+ \\ \rho(0,x) = \rho_o(x) & x \in \mathbb{R}_+ \,. \end{cases}$$

However, recently, this term has been used more and more also to refer to rather general problems, such as

$$(1) \qquad \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, V) = S & (t,x) \in \mathbb{R}_+ \times \Omega \\ \rho(t,\xi) = B(t,\xi) \quad \text{(if there is a boundary)} & (t,\xi) \in \mathbb{R}_+ \times \partial\Omega \\ \rho(0,x) = \rho_o(x) & x \in \Omega \end{cases}$$

that typically have in common the presence of non local terms and a motivation coming from some sort of biological model. In (1), the velocity V, the source S and the boundary term B may well depend – locally or not locally – on the unknown function ρ which, in turn, may well be both a scalar or a vector.

An attempt to obtain a rather general well posedness result for a renewal equation in the sense of (1) is presented in [10]. There, the precise form of the considered problem is

$$(2) \quad \partial_t u^h + \nabla_x \left(v^h(t,x)\, u^h\right) = p^h\left(t,x,u(t)\right) u^h + q^h\left(t,x,u,u(t)\right) \qquad h \in \{1,\ldots,k\}\,.$$

Here $t \in \mathbb{R}_+$ is time and the *"space"* variable x varies in $\mathbb{R}_+^m \times \mathbb{R}^n$. This choice allows to encompass in (2) also situations where where part of the independent variables are bounded below, while the remaining part varies in $\mathbb{R}^n$. The former variables may thus represent age or time since vaccination, for instance, see [4, 9], while the latter variables are typically space coordinates, for instance. Note also that in (2), the dependence on $u(t)$ stands for a dependence which is *non local* in the x variable. In the right hand side in (2) the choices of p^h and q^h are flexible, so that the provided estimates can be optimized for the specific model considered. Refer to [10] for specific models that fit into (2). Other well posedness results in similar settings are provided in [6, 7] and in [12], where the problem is set in the framework of evolution equations in metric spaces.

Renewal equations appear also in mixed systems devoted to some sort of predator prey dynamics, such as

$$(3) \qquad \begin{cases} \partial_t u + \nabla \cdot (u\, v(u)) = f(t,x,w)\, u + a \\ \partial_t w - \mu\, \Delta w = g(t,x,u,w)\, w + b \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \Omega$$

where u *"hunts"* w and the hunting term v is of the form

$$(4) \qquad v(u) = V\left(\nabla u * \eta\right)\,,$$

η being a smooth approximation of Dirac delta and V a Lipschitz continuous function. The case of Lotka–Volterra interaction is recovered by

$$f(t,x,w) = \alpha\, w - \beta \qquad \text{and} \qquad g(t,x,u,w) = \gamma - \delta\, u\,,$$

but further terms, for instance related to some sort of *capacity* are not excluded.

System (3)–(4) can be studied in both cases $\Omega = \mathbb{R}^n$, see [8], and Ω bounded, see [11] – in the latter case suitable boundary conditions need to be supplemented. The term $a = a(t,x)$, respectively $b = b(t,x)$, is a control parameter describing the amount of u, respectively w, that is deployed per unit time at position x and time t. Indeed, system (3)–(4) applies to cases, for instance, where the predator u is a parasitoid used against the propagation of the parasite w, see [5, 8] or also when the chemical substance w diffuses attracting and killing the pest u.

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Generalised characteristics of Hamilton-Jacobi equations, propagation of singularities, and long-time behaviour

Piermarco Cannarsa

A generalised characteristic (GC) is a solution of certain differential inclusions that play a crucial role for propagation of singularities of solutions to Hamilton-Jacobi equations $H(x,u(x),Du(x))=0$. GC's were introduced in [4], in the context of hyperbolic conservation laws and then adapted to Hamilton-Jacobi equations in [1] and [3]. In this talk, we will discuss several topics related to GC's including restricted classes of characteristics introduced in [5], uniqueness issues, continuation properties in connection with propagation of singularities [2].

We will then focus on classical mechanical systems on the torus with the Hamiltonian

$$H(x,p)=\frac{1}{2}|p|^2+V(x),\qquad x\in\mathbb{T}^d, p\in\mathbb{R}^d$$

and consider the stationary Hamilton-Jacobi equation

$$H(x,Du(x))=\frac{1}{2}|Du(x)|^2+V(x)=\alpha[0],\qquad x\in\mathbb{T}^d, \tag{1}$$

where $\alpha(0)$ is the Mañé's critical value. For any semiconcave solution (or viscosity solution) u of (1), the set of the points of non-differentiability of u, $\mathrm{Sing}(u)$, is called the singular set of u. The singularities of a semiconcave solution u of (1) propagate along the generalized gradient flow defined by

$$\begin{cases}\dot{\mathbf{x}}(t,x)\in D^+u(\mathbf{x}(t,x)), & t\geq 0 \text{ a.e.}\\ \mathbf{x}(0,x)=x.\end{cases} \tag{2}$$

Denoting by $\mathbf{x}_u(t,x)$ the associated semi-flow for a semiconcave solution u of (2), for any $x\in\mathbb{T}^d$ and any $T>0$ we introduce the (individual) invariant occupational measure for $\mathbf{x}_u(\cdot,x)$ as the Borel probability measure μ_x^T defined by

$$\int_{\mathbb{T}^d} f(y)\ d\mu_x^T(y)=\frac{1}{T}\int_0^T f\big(\mathbf{x}_u(t,x)\big)\ dt\quad \forall f\in C(\mathbb{T}^d).$$

Then, we call any weak limit of $\mu_x^{T_n}$, as $T_n\to\infty$, a limiting occupational measure of $\mathbf{x}_u(\cdot,x)$. We denote by $\mathcal{W}_u(x)$ be the family of all limiting occupational measures of $\mathbf{x}_u(\cdot,x)$. As we shall see, there are interesting connections among the critical set of u, the set of limiting occupational measures $\mathcal{W}_u(x)$ of $\mathbf{x}_u(\cdot,x)$, and $\mathrm{Sing}(u)$. We first show that $\mathcal{W}_u(x)\neq\emptyset$ and, by the Krylov-Bogoliubov argument, that each measure in $\mathcal{W}_u(x)$ is invariant under $\mathbf{x}_u(\cdot,x)$. Then we show that the critical set of u is an attractor for the semi-flow $\mathbf{x}_u(t,x)$ in the sense that, for any $\varepsilon>0$, the probability for $\mathbf{x}_u(t,x)$ to be ε-close to the critical set of u, with t picked at random in $[0,T]$, tends to 1 as $T\to\infty$.

Given (a nonzero cohomology class) $c\in\mathbb{R}^d$, to extend the above results to the cell problem

$$\frac{1}{2}|Du(x)+c|^2+V(x)=\alpha[c],\qquad x\in\mathbb{T}^d, \tag{3}$$

is a challenging open question that will definitely require new ides. Even more so, it would be extremely interesting to adapt the current approach to a general Hamilton-Jacobi equation

$$H(x, Du(x) + c) = \alpha[c], \qquad x \in \mathbb{T}^d \tag{4}$$

with H of Tonelli type.

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Some inverse problems for shock wave control

Gui-Qiang G. Chen

Controlling shock waves is crucial in applications in various fields of science and engineering, including aerodynamics, aerospace engineering, explosion mitigation, and shock tube experiments. In this talk, we present some inverse problems for controlling multi-dimensional steady shock waves and discuss some recent progress in controlling both the leading shock waves generated by wedges/conical bodies and associated fluid flows by designing the boundary geometries of the wedges/conical bodies with desired pressure distributions and/or leading shock locations through the inverse problems. Some further perspectives and open problems in this direction will also be addressed.

Hysteresis and string stability in traffic flows

Andrea Corli
(joint work with Haitao Fan)

In this talk we discuss some new macroscopic models of traffic flow [2, 4], whose aim is to model stop-and-go waves and related phenomena by means of a hysteretic term. The approach is inspired by the modeling of fluid flows in porous media [1], where hysteresis plays an important role. We also comment on the so-called string stability of the model, a notion of stability that is widely used in microscopic models but that is here extended to macroscopic models [3].

Stop-and-go waves are observed in real traffic flows but cannot be produced by the classical Lighthill–Whitham–Richards (LWR) model. To capture stop-and-go waves, we add hysteresis to the LWR model; we call HLWR such a model [2].

The model HLWR consists of two equations for the unknown functions ρ, the vehicle density, and h, the hysteresis variable. It is hyperbolic but it is not in conservation form, because there is no reason for the hysteresis to be conserved; moreover, it involves functions that are possibly discontinuous, and solutions are to be meant in the sense of [5]. For the model under consideration, we find all possible "viscous" waves as well as necessary and sufficient conditions for their existence. In particular, deceleration and acceleration shocks appear and stop-and-go waves are produced by pairs of deceleration and acceleration shocks completing a hysteresis cycle. We solve the Riemann problem for every Riemann data and show that, where hysteresis loops exist, a deviation in speed of a few vehicles in a uniform car platoon can generate stop-and-go waves. This analysis could be possibly useful for the control of traffic flows. We also discuss how the previous approach can be extended to the Aw-Rascle-Zhang (ARZ) model [4].

In microscopic models, string stability or instability is concerned with the propagation of oscillations in a car platoon caused by the leading vehicle. The issue is whether and when such oscillations are amplified or damped; in the former case, traffic jams occur. We propose a suitable notion of string stability for continuum models and show that the LWR and AWR models model are string stable for wide classes of perturbations. In the case of the previous hysteretic model, we show that string instability can occur for large perturbations, while, under small perturbations, examples as well as approximate solution analysis suggest that the hysteretic traffic flow modeled for instance by the HLWR model is string stable.

In this framework there are many open problems. First of all, the Riemann solvers are not unique, as it was already the case in [1]; is it possible to select a "rational" Riemann solver? Second, the analysis of the initial-value problem for general data with bounded variation is missing; we conjecture that such a solution exists, at least for small initial data. Third, it would be interesting to validate the model by analyzing real traffic flows. Indeed, the evidence of hysteresis loops in traffic flows is known in the applied literature since several decades, but the modeling of hysteresis as a variable of the models has never been checked with such real flows.

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Inverse design for the chromatography system

Carlotta Donadello
(joint work with Giuseppe M. Coclite, Nicola De Nitti and Florian Peru)

We present some results on the inverse design problem for the $n \times n$ chromatography system. The particular structure of the system, which can be written as a triangular system consisting of one autonomous strictly concave conservation law and $n-1$ linear continuity equations, allows to combine the backward reconstruction method introduced in [2] with recent results obtained by Colombo and Perrollaz, [3], and by Liard and Zuazua, [5, 4], on the inverse design problem for a strictly convex scalar conservation law. More precisely, in the case of two components, the original system

$$\begin{cases} \partial_t u_1 + \partial_x \left(\dfrac{u_1}{1+u_1+u_2} \right) = 0, & t > 0,\ x \in \mathbb{R}, \\ \partial_t u_2 + \partial_x \left(\dfrac{u_2}{1+u_1+u_2} \right) = 0, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

rewrites as

$$\begin{cases} \partial_t v + \partial_x \left(\dfrac{v}{1+v} \right) = 0, & t > 0,\ x \in \mathbb{R}, \\ \partial_t w + \partial_x \left(\dfrac{w}{1+v} \right) = 0, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

thanks to the change of variables

$$v := u_1 + u_2, \qquad w := u_1 - u_2.$$

For any $v_0 \in L^\infty(\mathbb{R})$, the Cauchy problem

$$\begin{cases} \partial_t v + \partial_x \left(\dfrac{v}{1+v} \right) = 0, & t > 0,\ x \in \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases}$$

admits a unique entropy solution in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$. From it we can define the vector field $\left(A(t,x) = v(t,x), B(t,x) = \dfrac{v(t,x)}{1+v(t,x)} \right)$, satisfying the hypothesis which, in [6], are necessary to prove the existence of a unique weak renormalized solution z of

$$\begin{cases} \partial_t (vz) + \partial_x \left(\dfrac{vz}{1+v} \right) = 0, & t > 0,\ x \in \mathbb{R}, \\ z(0,x) = z_0(x), & x \in \mathbb{R}, \end{cases}$$

for any $z_0 \in L^\infty(\mathbb{R})$. In particular, z is time-reversible, in the sense that if $A(T,x)z(T,x) = A(T,x)z_T(x)$, then $t \mapsto z(T-t)$ is a generalized solution of

$$\begin{cases} \partial_t (A\rho) - \partial_x (B\rho) = 0, & t > 0,\ x \in \mathbb{R}, \\ A(0,x)\rho(0,x) = A(0,x)z_T(x), & x \in \mathbb{R}. \end{cases}$$

Using this approach to implement a backward reconstruction, it was proven in [2] that the physically relevant attainable profiles for the chromatography system in the (u_1, u_2) variables are

$$\mathsf{A}_T(\mathbb{R}) = \Big\{(v_T, w_T)\ :\ v \in \mathcal{A}_T\left(\mathbb{R}, v \mapsto \frac{v}{1+v}\right) \text{ and there exists}$$
$$z \in L^\infty(\mathbb{R}; [-1,1]) \text{ such that } w_T = z v_T\Big\},$$

where the set $\mathcal{A}_T$ contains the attainable profiles at time T for the first equation, as described in [1]. For a given positive time T and an attainable profile $V_T = (v_T = u_1^T + u_2^T, w_T = u_1^T - u_2^T) \in \mathsf{A}_T(\mathbb{R})$, the set of initial conditions leading to V_T can be easily characterized, and its topological properties inferred, thanks to a 1–to–1 correspondence with the set $I(v_T)$ of inverse designs for the scalar conservation law with strictly concave flux, [3, 5].

For a target profile V which is not attainable in time T, we recover the initial condition which would steer the system as close as possible to V in the L^2 norm thank to a minimization procedure analogous to the one in [4].

Building on the numerical scheme in [2] and results in [4], we implemented a finite volume numerical scheme which, for any attainable profile U_T and positive integer r, provides an initial condition leading to U_T and which first component (the initial condition for the nonlinear conservation law) suffers of exactly r discontinuities.

The final part of the presentation explains why these results are limited to a system with a very specific structure and cannot easily be generalized to the class of triangular systems considered in [2].

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Nonlocal macroscopic models of multi-population pedestrian flows for walking facilities optimization

Paola Goatin

(joint work with R. Bürger, D. Inzunza, E. Rossi, L. M. Villada)

We consider a class of nonlocal crowd dynamics models for N populations, $N \geq 1$, with different destinations trying to avoid each other in a confined walking domain $\Omega \subset \mathbb{R}^2$ and described by their densities $\boldsymbol{\rho} = (\rho^1, \ldots, \rho^N)^T$. This can be formalized in the following initial-boundary value problem:

$$
(1) \quad \begin{cases} \partial_t \rho^k + \operatorname{div}_{\mathbf{x}} \left[f_k \left(\rho^k \right) \boldsymbol{\nu}^k \left(t, \mathbf{x}, \mathcal{J}^k[\boldsymbol{\rho}] \right) \right] = 0, & \mathbf{x} \in \Omega, t \geq 0, \, k = 1, \ldots, N, \\ \boldsymbol{\rho}(0, \mathbf{x}) = \boldsymbol{\rho}_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \boldsymbol{\rho}(t, \mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega, \end{cases}
$$

where $\boldsymbol{\nu}^k = (\nu_1^k, \nu_2^k)$ is the velocity vector field of the k-th population, $\mathcal{J}^k$ is a nonlocal operator, i.e. $\mathcal{J}^k[\boldsymbol{\rho}] = \left(\mathcal{J}^k[\boldsymbol{\rho}(t)] \right)(\mathbf{x})$, and $\boldsymbol{\rho}_0$ is a given initial datum. Usually, the vector fields $\boldsymbol{\nu}^k$ consist of a fixed smooth vector field of preferred directions (e.g. given by the regularized solution of an eikonal equation) together with nonlocal correction terms depending on the current density distribution.

We assume that $\Omega^c = \mathbb{R}^2 \setminus \Omega$ is a compact set consisting of a finite number $M \in \mathbb{N}$ of connected components $\Omega^c = \Omega_1^c \cup \ldots \cup \Omega_M^c$. To account for the presence of these obstacles, in [3] we proposed to evaluate the nonlocal operators on the extended density $\boldsymbol{\rho}_\Omega : \mathbb{R}^2 \to \mathbb{R}_+^N$ including the presence of obstacles:

$$
\rho_\Omega^k(t, \mathbf{x}) := \begin{cases} \rho^k(t, \mathbf{x}) & \text{if } \mathbf{x} \in \Omega, \\ R_\ell & \text{if } \mathbf{x} \in \Omega_\ell^c, \end{cases}
$$

with $R_\ell \geq R > 0$, $\ell = 1, \ldots, M$, big enough so that $\boldsymbol{\nu}^k \left(t, \mathbf{x}, \mathcal{J}^k[\boldsymbol{\rho}_\Omega] \right) \cdot \mathbf{n}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \partial\Omega$, $t \geq 0$, $\mathbf{n}$ being the outward normal to Ω. In this way, (1) can be rewritten as

$$
(2) \quad \begin{cases} \partial_t \rho^k + \operatorname{div}_{\mathbf{x}} \left[f_k \left(\rho^k \right) \boldsymbol{\nu}^k \left(t, \mathbf{x}, \mathcal{J}^k[\boldsymbol{\rho}_\Omega] \right) \right] = 0, & \mathbf{x} \in \mathbb{R}^2, t \geq 0, \, k = 1, \ldots, N, \\ \boldsymbol{\rho}(0, \mathbf{x}) = \boldsymbol{\rho}_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2. \end{cases}
$$

Under suitable regularity assumptions, well-posedness results for weak entropy solutions of (2) can be proved relying on [1, 2], see [3, 5].

The trick of incorporating the obstacles in the nonlocal operator allows to avoid including them in the vector field of preferred directions. In particular, we can address shape optimization problems aiming at finding the optimal position of the obstacles to minimize the total travel time, rewriting them as standard PDE-constrained optimization [4]. In addition, to accelerate the numerical optimization procedure, we propose to address the computational bottleneck represented by the convolution products by a Finite Difference scheme that couples high-order WENO approximations for spatial discretization, a multi-step TVD method for temporal discretization, and a high-order numerical derivative formula to approximate the derivatives of nonlocal terms, and in this way avoid excessive calculations.

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Boundary feedback control problems for systems of balance laws

Simone Göttlich

(joint work with Sonja Groffmann)

The main objective of this work is to investigate the exponential and Input-to-state (ISS) stability of solutions to linear hyperbolic balance laws that are paired with boundary feedback conditions. In particular, we consider the following $k \times k$ system of linear hyperbolic balance laws

$$\partial_t W(x,t) + \Lambda(x)\partial_x W(x,t) + \Pi(x)W(x,t) = 0, \qquad (x,t) \in [0,l] \times [0,+\infty) \tag{1}$$

for the state vector $W := W(x,t) : [0,l] \times [0,+\infty) \to \mathbb{R}^k$ over some space interval of length l and an open time horizon. The variable coefficients are chosen such that $\Pi(x)$ is a non-zero matrix in $\mathbb{R}^{k\times k}$ and $\Lambda(x)$ is a diagonal matrix with non-zero eigenvalues in $\mathbb{R}^{k\times k}$.

For an analysis of the hyperbolic system, characteristic variables are applied and $\Lambda(x)$ is split into a positive and a negative part according to the sign of its eigenvalues. Therefore, $\Lambda(x) = \operatorname{diag}\{\Lambda^+(x), -\Lambda^-(x)\}$, where $\Lambda^+(x) = \operatorname{diag}\{\lambda_i(x) > 0 : i = 1,...,m\}$ and $-\Lambda^-(x) = \operatorname{diag}\{\lambda_i(x) < 0 : i = m+1,...,k\}$ for some $m \in \mathbb{N}$ with $m \leq k$. Analogously, the state vector is split into the parts that correspond to the positive and negative eigenvalues of $\Lambda(x)$, respectively. Hence, $W = [W^+, W^-]^T$ with $W^+ \in \mathbb{R}^m$ and $W^- \in \mathbb{R}^{k-m}$.

The hyperbolic system of balance laws will be paired with an initial condition

$$W(x,0) = W_0(x), \qquad x \in (0,l), \tag{2}$$

for a function $W_0 : (0,l) \to \mathbb{R}^k$, as well as boundary conditions consisting of a disturbed linear feedback law

$$\begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix} = K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} + Mb(t), \qquad t \in [0,+\infty), \tag{3}$$

where $K, M \in \mathbb{R}^{k\times k}$ are constant matrices and $b : [0,+\infty) \to \mathbb{R}^k$ is a vector-valued function describing the disturbances. This model has been also studied in [4].

In case of exponential stability an undisturbed feedback boundary condition is required. Therefore, the following special case of (3) with $Mb(t) \equiv 0$ for all $t \in [0, +\infty)$ will sometimes be considered as well:

$$\begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix} = K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}, \qquad t \in [0, +\infty). \tag{4}$$

Both of the above stated boundary conditions are called feedback laws, as the inflow into the system $W^+(0,t)$ and $W^-(l,t)$ at the spacial boundaries $x = 0$ and $x = l$ is a function of the outflow out of the system at the boundaries given by $W^+(l,t)$ and $W^-(0,t)$. The information about the outflow is returned via feedback to influence the inflow. Similar types of feedback boundary conditions are often employed for boundary control problems or boundary stabilization as presented for example in [2, 3].

The problem is now completed by the following assumptions which shall hold for all $x \in [0, l]$ and $t \in [0, +\infty)$.

- **A1** Λ is a real diagonal matrix of class $C^1([0, l])$, i.e., $\Lambda(x)$ is a function that is once continuously differentiable.
- **A2** Π is a real matrix of class $C^0([0, l])$, i.e., $\Pi(x)$ is a continuous function.
- **A3** b is a vector of boundary disturbances of class $C^0([0, +\infty))$, i.e., $b(t)$ is a continuous function.

Based on these assumptions, a detailed Input-to-state stability analysis in the L^2-norm has been provided in [4]. More precisely, the steady state $W \equiv 0$ of the system (1) with the boundary conditions (3) is Input-to-state stable in L^2-norm with respect to the disturbance function b if there exist positive real constants $\eta > 0$, $\xi > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every initial condition $W_0(x) \in L^2((0, l); \mathbb{R}^k)$, the L^2-solution to the system (1) with initial condition (2) and boundary conditions (3) satisfies for all $t \in \mathbb{R}^+$

(5)
$$\|W(\cdot,t)\|^2_{L^2((0,l);\mathbb{R}^k)} \le C_1 \exp(-\eta t) \|W_0\|^2_{L^2((0,l);\mathbb{R}^k)} + \frac{C_2}{\eta}\left(1 + \frac{1}{\xi}\right) \sup_{s \in [0,t]} (|b(s)|^2).$$

If the disturbance function disappears, i.e., $b(t) \equiv 0$ for all $t \in [0, +\infty)$, the definition of exponential stability can be retrieved from the notion on Input-to-state stability. The latter notion is of course weaker than its exponential counterpart, as the disturbance term in inequality (5) counteracts the exponential decay of the first term and is highly dependent on the given disturbances.

A possible Lyapunov function candidate to study the exponential stability of hyperbolic balance laws has been originally introduced in [1], i.e.,

$$\mathcal{L}(W(\cdot,t)) = \int_0^l W^T P(x) W dx, \quad t \in [0, +\infty), \tag{6}$$

for continuously differentiable positive definite matrices $P(x)$. This Lyapunov function is then said to be an ISS-Lyapunov function for the system (1) with the boundary conditions (3) if there exist positive real constants $\eta > 0$, $\xi > 0$ and

$\nu > 0$ such that, for all functions $b(t) \in C^0([0,+\infty))$, for L^2-solutions of the system (1) satisfying the boundary conditions (3), and for all $t \in [0,+\infty)$,

$$\frac{d\mathcal{L}(W(\cdot,t))}{dt} \leq -\eta\mathcal{L}(W(\cdot,t)) + \nu\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}(|b(s)|^2). \tag{7}$$

It turns out that if the matrix

$$-\Lambda(x)P'(x) - \Lambda'(x)P(x) + \Pi^T(x)P(x) + P(x)\Pi(x) \tag{8}$$

is positive definite for all $x \in [0,l]$ and the matrix

$$\begin{bmatrix}\Lambda^+(l)P^+(l) & 0\\ 0 & \Lambda^-(0)P^-(0)\end{bmatrix} - (1+\xi)K^T\begin{bmatrix}\Lambda^+(0)P^+(0) & 0\\ 0 & \Lambda^-(l)P^-(l)\end{bmatrix}K \tag{9}$$

is positive semi-definite, the steady state $W(x,t) \equiv 0$ of the system (1) with boundary conditions (3) is Input-to-state stable in the L^2-norm with respect to the disturbance function b.

A further approach intends to answer the question how the results from the continuous setting in terms of Input-to-state stability can be transferred to a discretized version of (1)-(3). Following the the numerical discretization presented in [3] based on the operator splitting technique

$$\partial_t W(x,t) + \Lambda(x)\partial_x W(x,t) = 0 \qquad (x,t) \in [0,l]\times[0,+\infty) \tag{10}$$

$$\partial_t W(\cdot,t) + \Pi(\cdot)W(\cdot,t) = 0 \qquad t \in [0,+\infty). \tag{11}$$

then allows for a rigorous numerical analysis of Input-to-state stability. In fact, a notable advantage of the discretized approach is the explicit computation of decay rates η that appear in (5). For an illustration of the results obtained so far, we make use of the Telegrapher's equations which is a 2×2 system of linear hyperbolic balance laws of the form

$$\partial_t W(x,t) + \Lambda\partial_x W(x,t) + \Pi W(x,t) = 0 \tag{12}$$

for $x \in [0,l]$, where Π and Λ are independent of x and defined as

$$\Pi := \frac{1}{2}\begin{bmatrix}RL^{-1}+ZC^{-1} & RL^{-1}-ZC^{-1}\\ RL^{-1}-ZC^{-1} & RL^{-1}+ZC^{-1}\end{bmatrix}, \tag{13}$$

$$\Lambda := \begin{bmatrix}\lambda^+ & 0\\ 0 & \lambda^-\end{bmatrix}, \text{ with } \lambda^\pm = \pm(\sqrt{LC})^{-1} \tag{14}$$

and positive constants R, L, Z, C. Applying the weighted matrices

$$P_j = \mathrm{diag}\{p_1\exp(-\mu_1 x_j), p_2\exp(\mu_2 x_j)\} \tag{15}$$

with $\mu_1, \mu_2 > 0$ and $p_1, p_2 > 0$ to the disctrization of (6), we are able to show that for the Telegrapher's equations given by (12) - (14) the decay rate η is defined by

$$\eta = \frac{1}{\Delta x\sqrt{LC}}(1-\exp(-\mu\Delta x)).$$

Future considerations will investigate the notion of Input-to-state stability for balance laws of type (1) on networks. This requires in particular a discussion on coupling conditions and their influence on the continuous and discretized problem, respectively.

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Isothermal flow in gas networks: Synchronization of observer systems

Martin Gugat

We study systems that are governed by neworked hyperbolic partial differential equations. As an example, think of gas pipeline networks.

We discuss the necessity to design observers for this type of system. They can be applied to produce an approximation of the full system state that can be used as input in a feedback law and also provide initial data for optimal control problems. We present an observer system that is also defined as a networked system of hyperbolic partial differential equations and is fed with pointwise measurements from the original system. These measurements are taken at a finite number of locations in the network. In our model, we consider the measurements as continuous in time. This is not completely realistic, and in future studies we will also consider time-discrete measurements and also an observer system that is defined on discrete times and is implementable on a computer.

We show that for a sufficiently large number of measurement locations, the observer system synchronizes exponentially fast with the original system, that is the error decays exponentially fast to zero. This is a local result that we can only proof under a number of smallness assumption for the state of the original system and also for the error of the estimate of the initial state that is used in the observer system. Since the semi-global classical solutions of quasilinear hyperbolic systems are a suitable framework to prove the synchronization results, we use an existence result for semi-global classical solutions in our analysis of the well-posedness of the system. These results have been investigated thoroughly in the group of Ta-Tsien Li.

We discuss a Lyapunov function with exponential weights that can be used for the proof of the exponential synchronization in the L^2–sense. Such exponential weights have been used very successfully by Jean-Michel Coron and his group.

As an outlook, we discuss the extension of the observer to the case where the gas is a mixture of hydrogen and natural gas. We consider a model of the following type that is similar as in [2]:

$$(1) \qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t c + v\, \partial_x(c) = 0 \\ \partial_t(\rho v) + \partial_x \left(\rho v^2 + \hat{p}(\rho, c)\right) = -\dfrac{\lambda^F}{2D}\rho|v|v \end{cases}$$

where ρ_1 and ρ_2 denote the densities of the two components, $\rho := \rho_1 + \rho_2$, v is the velocity of the gas mixture and $c := \rho_1/\rho$. Note that in order to close the system, the knowledge of the pressure law $\hat{p}(\rho, c)$ for the mixture is mandatory. The number λ^F is a friction parameter and D denotes the diameter of the pipe.

Finally we mention the challenge to deal with hydrogen embrittlement in steel pipelines. Since pressure fluctuations promote the damage of the pipes by hydrogen embrittlement, one aim of the operation control must be to mitigate these fluctuations. In the mathematical model this leads to new state constraints that have to be compatible with a realistic damage model, for example the rainflow-counting algorithm used for calculations of the fatigue. Problems of optimal boundary control for gas pipeline networks have been investigated in [3].

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Stabilization methods for evolution equations

Amaury Hayat

The topic of this talk is the stabilization of systems of (partial) differential equations. We focus on three parts: *Fredholm backstepping*, also known as *F-equivalence*, a powerful approach that allows the derivation of explicit feedback laws for a large class of systems of PDEs; *stabilization of traffic flow*, modeled by hyperbolic systems of PDEs, and the use of *Artificial Intelligence (AI) in mathematics*. The latter extends beyond the sole area of stabilization.

1. The F-equivalence method

We study the general stabilization problem for linear systems. Consider

$$(1) \qquad \partial_t y(t) = \mathcal{A}y(t) + Bu(t),$$

where $y(t) \in X$ describes the state of the system at time t, X is a Banach space, $\mathcal{A}$ is a differential operator with discrete spectrum and defined on a Banach space X with the domain $D(\mathcal{A})$, $B : H \to D(\mathcal{A}^*)'$ is a so-called control operator and $u(t) = Ky(t)$ is the control chosen as a feedback law, and K is consequently a feedback operator with values in H. The goal is to find K such that the system (1) is exponentially stable. Ideally, we would even want to ensure that, for any $\lambda > 0$, there exists K_λ such that the system (1) is exponentially stable with decay rate λ. When B is not bounded in X and H is finite dimensional, this becomes a complex problem. The F-equivalence approach is as follows: instead of trying to directly find a feedback operator K, one attempts to simultaneously find a mapping $\mathcal{T}$ and a feedback operator K such that $\mathcal{T}$ is an isomorphism and maps the system (1) to an exponentially stable target system

$$\partial_t z = \mathcal{A}' z, \tag{2}$$

where $\mathcal{A}'$ is a carefully chosen exponentially stable operator. If this can be done then, as a consequence, the original system (1) is exponentially stable (with the same decay rate).

The *backstepping method*, introduced in 2003 by Krstic and several collaborators, relied on this paradigm and suggested looking for $\mathcal{T}$ in the form of a Volterra transform, which led to many proven results. Another method, sometimes called *Fredholm backstepping*, and relying on more general transformations, was introduced in 2013 by Coron and Lu for a particular case. We present a new method [3] to generalize the F-equivalence approach to generic systems where $\mathcal{A}$ is skew adjoint with eigenvalues scaling more than linearly, B is admissible, and the system is exactly null controllable. In particular, this allow to solve an open question presented at the College de France in 2017. The hope is that the F-equivalence approach can provide quantitative estimates of T and K with respect to λ and can extend locally to nonlinear systems, even for systems where classical perturbation arguments do not apply.

2. Traffic flow stabilization

We examine a particular problem: the stabilization of traffic flow. In traffic, when the density of cars is high, the uniform flow steady-state becomes unstable and stop-and-go waves appear, resulting in a phenomenon commonly known as a traffic jam. Our goal is to stabilize these uniform steady-states. Our approach is to use autonomous vehicles (AVs) as controls on the traffic. The resulting mathematical system is modeled by one or several PDEs coupled with an ODE: the PDE(s) represent the bulk of traffic, while the ODE represents the location of the autonomous vehicle. The PDE model could be, for instance, the LWR model

$$\partial_t \rho + \partial_x(\rho V(\rho)) = 0, \tag{3}$$

where V is a function that is concave and C^2. A more realistic approach would be to consider a second-order model, for instance, the Generalized-ARZ (GARZ) equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho V(\rho, \omega)) = 0, \\ \partial_t(\rho\omega) + \partial_x(\rho\omega V(\rho, \omega)) = 0. \end{cases} \tag{4}$$

The dynamics of the AV's location $y(t)$ can be modeled by

$$\dot{y}(t) = \min\{u(t), V(\rho(t, y(t)+), w(t, y(t)+))\}, \tag{5}$$

where $u(t)$ refers to the control, and for a BV function f, $f(y(t)+)$ refers to the right limit at $y(t)$. The min comes from the fact that the AV cannot move faster than the traffic flow in front of it (to avoid a crash). This system has both practical and mathematical interests. From a practical perspective, reducing stop-and-go waves could lead to a significant reduction in fuel consumption and CO_2 emissions, as well as safer traffic. From a mathematical perspective, the PDE/ODE system modeling this situation is interesting in that the relevant physical solutions are not the entropic solutions, usually considered as the natural solutions for hyperbolic systems and studied for decades. More specifically, the physical solutions are not necessarily entropic at the AV's location. For this reason, we need a new condition to replace the entropy condition at the AV's location. We use a flux condition similar to the *Delle Monache-Goatin* flux condition

$$\rho(t, y(t))(V(\rho(t, y(t)), \omega(t, y(t))) - \dot{y}) \leq \alpha \max_{\rho,\omega}(\rho(V(\rho, \omega) - u(t))) \tag{6}$$

where $\alpha \in (0, 1)$. We present an existence result (in the class of BV solutions, entropic outside of the AV's location) for a solution to the Cauchy problem (4)–(6) for any initial condition (in the same class), provided that the control is constant [5]. The question of a time-varying control and, *a fortiori*, of the stabilization problem is widely open.

We also present field experiment results obtained in Nashville, TN in November 2022 as part of the CIRCLES project, where 100 autonomous cars were sent on a highway in dense traffic during peak hours, showing that our candidate controller drastically reduces the speed variance of the stop-and-go waves [4].

3. AI for mathematics

AI has seen many successes in the last decade, especially in natural language processing. A natural question we would like to ask is:

Can an AI learn mathematics in some sense?

We focus on two interpretations of this question:

- Can an AI predict the solution to an advanced mathematical problem?
- Can an AI prove a mathematical statement and provide a proof?

The answer to the first question seems to be yes. In several works, a trained AI model (a Transformer) managed to predict the solution to several mathematical problems, for instance, predicting explicit solutions to ODEs, the controllability of the linearized system given a nonlinear system, or a suitable feedback law [2]. We illustrate with two examples how this approach can assist mathematicians in solving open problems. The first one is a preliminary neural network trained to find Lyapunov functions for a nonlinear system, which is a general open question. The second one is a mix of deep reinforcement learning and mathematical analysis

[1] that allowed the finding of a control feedback law and the stabilization of a system for which finding such a feedback law was an open question until now.

Regarding the second question, we present a work [6] inspired by AlphaZero, where we train a neural network to demonstrate (small) mathematical statements and provide proof. This neural network is capable of demonstrating high school or undergraduate exercises and some exercises from the International Mathematical Olympiads.

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Convex Optimisation Methods for Variational Mean Field Games

Dante Kalise

(joint work with Luis Briceño-Arias and Francisco J. Silva)

We discuss the numerical approximation of stationary mean field games [1]

$$(MFG_\infty)\begin{cases} -\sigma^2\Delta v(x) + H(x,\nabla v(x)) - \lambda = f(x,m(x)) & \text{in } \mathbb{T}^d\,, \\ -\sigma^2\Delta m(x) - \nabla\cdot\big(\partial_p H(x,\nabla v(x))m(x)\big) = 0 & \text{in } \mathbb{T}^d\,, \\ \qquad m\geq 0, \quad \displaystyle\int_{\mathbb{T}^d} m\,dx = 1, \quad \int_{\mathbb{T}^d} v\,dx = 0\,, \end{cases}$$

by convex optimisation methods. Under suitable hypotheses on the coupling term f and the Hamiltonian H, we begin by defining the momentum variable $w(x) = -\partial_p H(x,\nabla v(x))m(x)$ and reformulating the MFG system as a PDE-constrained optimisation problem

$$(OC_\infty) : \begin{cases} \min\limits_{(m,w)} \int\limits_{\mathbb{T}^d} b(x, m(x), w(x)) + F(x, m(x))\, dx \\ \text{s.t.} \quad -\sigma^2 \Delta m(x) + \nabla \cdot (w(x)) = 0\,, \quad \text{and} \quad \int\limits_{\mathbb{T}^d} m\, dx = 1\,, \end{cases},$$

where

$$F(x, m) := \begin{cases} \int\limits_0^m f(x, \mu)\, d\mu & \text{if } m \geq 0\,, \\ +\infty & \text{otherwise} \end{cases}$$

and

$$b(x, m, w) := \begin{cases} mH^*(x, -\frac{w}{m}) & \text{if } m > 0\,, \\ 0 & \text{if } (m, w) = (0, 0)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Here, $H^*(x, p^*)$ denotes the convex conjugate $H^*(x, p^*) = -\inf\limits_p \{H(x, p) - \langle p^*, p\rangle\}$. Assuming $f(x, m(x))$ increasing in m, (OC_∞) is a convex optimisation problem [10, 11]. This variational formulation is related to other problems of interest, such as:

- the **Schrödinger Bridge problem** [4, 5]

$$(SB) \begin{cases} \min\limits_{(m,u)} \dfrac{1}{2} \int_0^1 \int_{\mathbb{R}^d} m(x, t) |u(x, t)|^2 dx dt \\ \text{s.t.} \\ \partial_t m(x, t) + \nabla \cdot (m(x, t) u(x, t)) - \dfrac{\epsilon}{2} \Delta m(x, t) = 0\,, \\ m(x, 0) = m_0(x)\,, \quad m(x, 1) = m_1(x)\,, \end{cases}$$

  which in the deterministic limit ($\sigma = 0$) corresponds to the Benamou-Brenier formulation of the optimal transport problem [2, 3].
- the **JKO scheme for gradient flows** [6]

$$\partial_t m = \nabla \cdot \left[\rho \nabla \left(U'(m) + V + W * m\right)\right]$$

  where, at every discrete time step, the mass is recovered as the solution of

$$m_{\Delta t}^{n+1} \in \operatorname*{arginf}_m \left\{ \frac{1}{\Delta t} \mathcal{W}_2(m_{\Delta t}^n, m) + \mathcal{F}(m; U, V, W) \right\}\,,$$

  and the Wasserstein distance $\mathcal{W}_2$ can be computed using a similar formulation as in (SB) above [7].

The construction of a numerical scheme for these problems begins with a suitable discretization of the transport equation and the cost, leading to

$$(OC_\infty^h): \begin{cases} \min_{(m_h,w_h)} \sum_{i,j} \hat{b}(x_{i,j}, m_{i,j}, w_{i,j}) + F(x_{i,j}, m_{i,j}) \\ \text{s.t} \quad A_h m_h + B_h w_h = 0\,, \quad \forall i,j = 1,\dots,N\,, \\ \qquad \sum_{i,j} h^2 m_{i,j} = 1\,, \end{cases}$$

where $\hat{b}(m,w) = b(m,w) + \iota_{\mathbb{R}\times K}(m,w)$, with $K := \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R}^-$. Problem (OC_∞^h) belongs to a class of convex optimization problems of the form

$$\min_y \; \Phi(y) + \Psi \circ C(y)$$

where Φ, Ψ are convex, l.s.c, proper functions, and C is a linear operator. Algorithms for this class of problems rely on duality and on the proximal operator

$$\operatorname{prox}_{\lambda f}(v) := \underset{x}{argmin} \left(f(x) + \frac{1}{2\lambda} ||x - v||^2 \right).$$

Within a wide class of suitable convex optimisation algorithms, we focus on the application of the primal-dual algorithm proposed by Chambolle and Pock [8, 9]. Given $\gamma, \tau \geq 0$, such that $\gamma\tau < \|C\||^{-2}$, the iteration reads:

$$\begin{aligned} \sigma^{k+1} &= \operatorname{prox}_{\gamma\Psi^*}(\sigma^k + \gamma C \bar{y}^k) \\ y^{k+1} &= \operatorname{prox}_{\tau\Phi}(y^k - \tau C^* \sigma^{k+1}) \\ \bar{y}^{k+1} &= 2y^{k+1} - y^k \end{aligned}$$

We discuss convergence to the solution of (MFG_∞), assignments for Φ and Ψ, and the effective computation of proximal operators.

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Structure of non entropy solutions to scalar conservation laws

Elio Marconi

We consider bounded weak solutions of conservation laws of the form

$$\partial_t u + \mathrm{div}_x(F(u)) = 0, \qquad u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}, \quad F : \mathbb{R} \to \mathbb{R}^d. \tag{1}$$

In particular we are interested in solutions with finite entropy production, namely solutions for which for every smooth entropy-entropy flux pair (η, Q) the distribution

$$\mu_\eta := \partial_t \eta(u) + \mathrm{div}_x(Q(u))$$

is a locally finite Radon measure. Compared to the more usual notion of entropy solution we do *not* require that if η is convex, then μ_η is a negative measure.

These solutions arise in the study of large deviations for stochastic particle scheme approximations of entropy solutions of (1) (see [11] and the recent results in [10]). A variational point of view investigated in [3] relates this problem to the following control problem for conservation laws in dimension 1: given $\varepsilon > 0$ and u_ε, let E_ε be such that

$$\partial_t u_\varepsilon + \partial_x(F(u_\varepsilon)) = \varepsilon \partial_{xx} u_\varepsilon + \partial_x E_\varepsilon \tag{2}$$

and set $I_\varepsilon(u_\varepsilon) = \min \int_{\mathbb{R}^2} E_\varepsilon^2 dxdt$, where the minimum is taken among the functions E_ε satisfying (2).

It is well known that if u_ε solves (2) with $E_\varepsilon \equiv 0$, the family u_ε converges to the entropy solution of (1) as $\varepsilon \to 0$.

An interesting regime is when $\varepsilon^{-1} I_\varepsilon(u_\varepsilon)$ remains bounded: in this case it is shown in [3] that u_ε converges up to subsequences to a solution with finite entropy production. Moreover the Γ-convergence of $\varepsilon^{-1} I_\varepsilon$ is investigated: a natural candidate H is proposed as well as a proof of

$$H \le \Gamma - \liminf_{\varepsilon\to 0} \varepsilon^{-1} I_\varepsilon.$$

The functional H can be described in terms of the entropy production measures μ_η: in the case of the Burgers equation $F(u) = u^2$ we expect that it coincides with

$$H(u) = \frac{1}{6}\int_{J_u^+} |u^+ - u^-|^3 d\mathcal{H}^1,$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure and J_u^+ is the subset of the jump set of u where the right trace u^+ is larger than the left trace u^-, namely the non-entropic shocks.

In order to complete the analysis we need to build a recovery sequence for this functional: it can be done for solutions with some additional structure (see the notion of entropy-splittable solution in [3]). It would be sufficient to check that these solutions are dense (in energy) in the class of solutions with finite entropy production, alternatively we should provide some different procedure to produce a recovery sequence directly for a general solution with finite entropy production.

In both cases it seems that a better understanding of the structure of solutions with finite entropy production is needed.

The general picture is that solutions with finite entropy production share several fine properties with solutions with bounded variation, even if in general they do not belong to BV. It is shown in [7] (in dimension 1) and in [5] in several space dimensions that under mild nonlinearity assumption on F for any solution of (1) with finite entropy production we can define a co-dimension 1 rectifiable jump set J with the following properties:

(1) every $(t,x) \notin J$ is a point of vanishing mean oscillation;
(2) for $\mathcal{H}^d$-a.e. $(t,x) \in J$ there are strong traces $u^{\pm}$ in L^1.

One may wonder if vanishing mean oscillation points are Lebesgue points (at least $\mathcal{H}^d$-a.e. as it happens for BV functions). A partial result in this direction has been obtained in [6] for $d = 1$ and in [8] in several space dimension: in both cases it is shown that the singular points has co-dimension (at least) 1.

Since the candidate Γ-limit functional H introduced in [3] can be written in terms of the measures μ_η, a desirable property that BV solutions enjoy and we would like to prove for general solutions with finite entropy production is that for every η the measure μ_η is concentrated on J_u. This is proven for Burgers equation in [9], and it is possible to extend this result to general conservation laws in the case $d = 1$ with similar techniques [2].

The results in [2, 8, 9] relies on a 'Lagrangian' description of the solutions of (1) developed in [4, 8]: inspired by the Lagrangian techniques introduced in [1] to study the linear continuity equation, we consider the kinetic formulation of (1) and we provide a decomposition along characteristics of the kinetic function associated to the solution of (1). A useful feature of this technique is that it allows to exploit some geometric constraint on the underlying characteristics structure of these solutions, despite they are a priori merely bounded. Since the geometry if $d = 1$ is simpler compared to the case of several space dimensions, this may explain why, using these techniques, we obtain better results when reducing to one space dimension.

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Optimization by kinetic equations

Lorenzo Pareschi

Since its introduction n the late 1980s [14], simulated annealing has become a popular optimization algorithm, and its applications have expanded to many fields, including artificial intelligence, machine learning, and operations research. The algorithm was developed as an extension of the Metropolis-Hastings algorithm, a Monte Carlo method used for simulating complex systems in physics [15], and adapts the concept of annealing to optimization problems by viewing the process of slowly cooling a material as a search for the lowest-energy state of the system.

Simulated annealing is similar to other metaheuristic algorithms such as genetic algorithms, ant colony optimization, particle swarm optimization and consensus based optimization in that it is based on the idea of exploring a large search space to find a global optimum without using gradient-based information. However, simulated annealing is distinct in that it uses a single solution-based probabilistic approach to accept worse positions in the hope of finding a better one, whereas other metaheuristic algorithms often use a population-based approach and other stochastic strategies.

By applying the above concepts, metaheuristic algorithms have been able to make significant advances in the search for valuable solutions to challenging optimization problems out of reach of traditional (gradient-based) methods. However, proving the rigorous convergence of metaheuristic optimization algorithms to the global minimum for non convex functionals, or to some reasonable approximation of it, remains a challenge. Indeed, metaheuristics involves the creative use of available resources to find efficient solutions without necessarily relying on a rigorous mathematical foundation that provides an analytical setting.

On the other hand, metaheuristics share similarities with statistical physics since they both deal with the complexities of large systems. The principles of statistical physics are versatile and powerful, providing insight into the behavior of large systems in a wide range of fields, from materials science to biophysics. By drawing upon the principles of statistical physics, it may be possible to provide a solid mathematical foundation to these class of optimization methods and develop more effective and efficient optimization algorithms that can handle increasingly complex problems and larger search spaces.

Mean field equations and kinetic equations are among the concepts in statistical physics that are most relevant to optimization. Mean field equations describe how each particle in a system interacts with a theoretical "average" field created by all the other particles in the system. This provides insight into the behavior of large systems, making it possible to predict macroscopic properties such as temperature or pressure. Kinetic equations, on the other hand, describe the evolution of a particle system, considering the interactions between particles as instantaneous, microscopic collisions [18]. Recently, these ideas have led to a new view of methaeuristic optimization by considering the corresponding continuous dynamics described by appropriate kinetic equations of Boltzmann type [3] and mean-field type [4, 6, 16, 13] even in constrained contexts [1, 11, 9, 10], multiobjective situations [2], or in generalizations to sampling [5]. See [17] for a recent survey.

In this talk we will focus our attention on one of the most notable examples of metaheuristics, namely the simulated annealing algorithm. This algorithm was inspired by the Monte Carlo algorithm developed by Metropolis et al. in the middle of last century. We show how classical tools of kinetic theory can be used to describe the Markov process which characterizes the method and show how its convergence to the global minimum is related to classical functional inequalities based on the so-called entropy method [8].

Furthermore, the continuous setting based on kinetic PDEs permits to investigate the relationships with Fokker-Planck equations describing the so-called mean-field Langevin dynamic [7, 12]. In particular, we illustrate how to formally derive the corresponding mean-field model taking a suitable scaling limit of the linear kinetic model describing the simulated annealing. This has been generalized to other types of kinetic equations describing variations of the simulated annealing method that avoid the acceptance-rejection process. Numerical evidence of such asymptotic behavior has also been discussed through simulation examples. From a mathematical viewpoint let us finally mention that several challenging questions remains open, like estimating the rate of convergence to equilibrium in the different functional spaces and analyzing the convergence properties of the Maxwellian variant here introduced. We leave these aspects to future researches.

Acknowledgment

This work was conducted within the activities of GNCS group of INdAM (Italian National Institute of High Mathematics). The research has been supported by the Royal Society under the Wolfson Fellowship at Heriot-Watt University, Edinburgh UK. The partial support by ICSC – Centro Nazionale di Ricerca in High Performance Computing, Big Data and Quantum Computing, funded by European Union – NextGenerationEU and by MIUR-PRIN Project 2022, No. 2022KKJP4X "Advanced numerical methods for time dependent parametric partial differential equations with applications" is also acknowledged.

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Peculiarities of non-homogeneous conservation laws

Vincent Perrollaz
(joint work with Rinaldo M. Colombo, Abraham Sylla)

It might be surprising to realize that in the context of

$$\begin{cases} \partial_t u + \partial_x f(x,u) = 0, \\ u(0,\cdot) = u_0. \end{cases} \tag{1}$$

the hypotheses of Kruzkov seminal existence result [4] may appear too restrictive. More specifically in the context of (1) — one space dimension and no source term — Kruzkov hypotheses include:

$$\sup_{(x,u)\in\mathbb{R}^2} -\partial^2_{xu} f(x,u) < +\infty. \tag{K}$$

Even within the context of regular flux functions f, some reasonable models are not directly covered. For instance, when considering a LWR model for traffic flow where the number of lanes and the speed limit may vary:

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) v_{\max}(x) \left(1 - \frac{\rho(t,x)}{\rho_{\max}(x)}\right)\right) = 0,$$

reasonable assumptions on the functions $v_{\max}$ and $\rho_{\max}$ could be

$$\forall x \in \mathbb{R}, \quad 0 < \underline{v} \le v_{\max}(x) \le \bar{v}, \quad 0 < \underline{\rho} \le \rho_{\max}(x) \le \bar{\rho} \tag{2}$$

with $\rho'_{\max} \in \mathcal{C}^2_c(\mathbb{R})$ and $v'_{\max} \in \mathcal{C}^2_c(\mathbb{R})$. This puts the model out of the scope of a direct application of Kruzkov existence result.

In [2], we provide an alternative framework of hypotheses on the flux f which allows us to still get a semigroup generated by the entropy solutions. More precisely we replace hypothesis (K) by

$$\forall \bar{f} \in \mathbb{R},\ \exists \bar{U} \in \mathbb{R},\ \forall (x,u) \in \mathbb{R}^2, \quad |f(x,u)| \le \bar{f} \implies |u| \le \bar{U}. \tag{C}$$

Let us briefly explain the role that this coercivity hypothesis play in our construction. The family of constant functions plays a central role in most techniques in the case of an x-independent flux. For instance, when combined with a locally contractive semigroup in L^1, it provides a priori L^∞ bounds. We detail, using our hypothesis (C), the construction of an alternative family of — possibly discontinuous — stationary solutions in the context of equation (1). We first make use of tools of differential topology to build such families for a special class of fluxes with nice geometric properties. We then use the method of compensated compactness to extend the result to a class of fluxes satisfying (C) (and some technical hypothesis which we expect could be relaxed).

In addition to the *a priori* L^∞ bounds granted by this family of stationary solutions, we use the method of vanishing viscosity and again compensated compactness to obtain the existence of a semigroup $(S^{CL}_t)_{t\ge 0}$ whose orbits are the unique maximal entropy solutions of (1). As a byproduct of this construction we also obtain the existence of a semigroup $(S^{HB}_t)_{t\ge 0}$ whose orbits are the unique maximal viscosity solutions of the following Hamilton-Jacobi equation

$$(3) \qquad \begin{cases} \partial_t U + f(x, \partial_x U) = 0, \\ U(0, \cdot) = U_0. \end{cases}$$

Of course, since they are both obtained as singular limits of regular viscous approximations, those semigroups are shown to be related according to the following commutative diagram

$$\begin{array}{ccc} U_o & \longrightarrow & S_t^{HJ} U_o \\ \partial_x \downarrow & & \downarrow \;\partial_x \\ u_o & \longrightarrow & S_t^{CL} u_o \end{array}$$

where the derivation ∂_x is taken in the distributional sense. A somewhat surprising fact is that the continuity properties and stability properties of the semigroups are not in full correspondence.

Finally, following [3], we describe why — even in the simplest case where the flux f is convex in u — the entropy semigroup is actually more singular than in the x-independent case. This phenomenon appears naturally when investigating the inverse design sets associated to (1)

$$I_T(w) := \{u_0 \in \mathrm{L}^\infty(\mathbb{R}) \,:\, S_T^{CL} u_0 = w\}.$$

The connection between entropy solutions of equation (1) and viscosity solutions of the Hamilton-Jacobi equation (3) is key for this analysis.

To be specific, we show in [1] for the case where the flux f does not depend on x that whenever $w \in \mathrm{L}^\infty(\mathbb{R})$ is such that $I_T(w) \neq \emptyset$ — i.e. w is reachable — then there is at least one initial data u_0 such that the solution $t \mapsto S_t^{CL} u_0$ is isentropic between $t = 0$ and $t = T$. Since the isentropic solutions are in some sense the topological closure of the classical solutions, this means that to understand the range of the semigroup one needs only to consider the classical solutions even though the blowup in time is generic with respect to initial data. In contrast, we show in [3] that when the flux f does depend on x then some states are reachable but a minimal entropy price has to be paid to reach them.

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Understanding Consensus-Based Optimization: Two Analytical Perspectives

Konstantin Riedl
(joint work with Massimo Fornasier and Timo Klock)

Consensus-based optimization (CBO) [1] is a multi-particle derivative-free optimization method capable of globally minimizing high-dimensional nonconvex and nonsmooth functions $\mathcal{E} : \mathbb{R}^d \to \mathbb{R}$, i.e., solving problems of the form

$$x^* \in \arg\min_{x\in\mathbb{R}^d} \mathcal{E}(x).$$

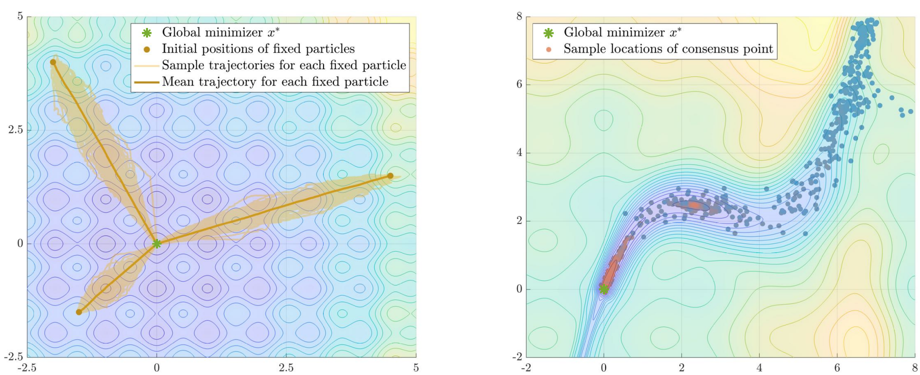
Inspired by consensus dynamics and opinion formation, CBO methods employ a finite number of agents $X^1, \dots, X^N$ to explore the domain and to form a consensus about the global minimizer x^* as time passes. More concretely, for a discrete time step size $\Delta t > 0$ and user-specified parameters $\alpha, \lambda, \sigma > 0$, the time-discrete evolution of the i-th particle X^i is defined according to the iterative update rule

$$X_k^i = X_{k-1}^i - \Delta t\lambda\big(X_{k-1}^i - x_\alpha^{\mathcal{E}}(\widehat{\rho}_{k-1}^N)\big) + \sigma \mathrm{diag}\big(X_{k-1}^i - x_\alpha^{\mathcal{E}}(\widehat{\rho}_{k-1}^N)\big)B_k^i, \tag{1}$$

where $\widehat{\rho}_k^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_k^i}$ is the empirical measure of the particles at time step k and where B_k^i are i.i.d. Gaussian random vectors with zero mean and covariance $\Delta t\mathrm{Id}$. Moreover, $x_\alpha^{\mathcal{E}}$ denotes the so-called consensus point, a weighted average of the particles' positions, which is computed for a measure $\varrho \in \mathcal{P}(\mathbb{R}^d)$ according to

$$x_\alpha^{\mathcal{E}}(\varrho) = \int x \frac{\omega_\alpha^{\mathcal{E}}(x)}{\|\omega_\alpha^{\mathcal{E}}\|_{L_1(\varrho)}}\, d\varrho(x), \quad \text{with} \quad \omega_\alpha^{\mathcal{E}}(x) := \exp(-\alpha\mathcal{E}(x)). \tag{2}$$

In what follows and as illustrated in Figure 1, we provide insights into the internal mechanisms of CBO from two analytical perspectives.



(a) CBO convexifies any nonconvex problem in the mean-field limit, see [2, 3].

(b) CBO can be interpreted as a stochastic relaxation of gradient descent, see [4].

Figure 1. Illustrations of the internal mechanisms of CBO, which are responsible for the success of the method.

First, based on an experimentally supported intuition that, as the number of particles goes to infinity in the continuous-time analogous of (1), i.e., in the mean-field limit, which is captured by the nonlinear nonlocal Fokker-Planck equation

$$\partial_t \rho_t = \lambda \mathrm{div}\big(\big(x - x_\alpha^{\mathcal{E}}(\rho_t)\big)\,\rho_t\big) + \frac{\sigma^2}{2}\sum_{k=1}^{d} \partial_{kk}\left(\big(x - x_\alpha^{\mathcal{E}}(\rho_t)\big)_{kk}^2\,\rho_t\right), \tag{3}$$

CBO always performs a gradient descent of the Wasserstein distance to the global minimizer (see Figure 1a), we present a novel technique for proving global convergence in mean-field law for a rich class of objective functions. More precisely, when analyzing the quantity $W_2^2(\rho_t, \delta_{x^*})$ we observe the following.

Theorem 1 ([2, Theorem 12] and [3, Theorem 2])**.** *Let the objective $\mathcal{E} \in \mathcal{C}(\mathbb{R}^d)$ satisfy $\|x - x^*\|_\infty \leq (\mathcal{E}(x) - \inf \mathcal{E})^\nu/\eta$ for all $x \in \mathbb{R}^d$ with constants $\eta, \nu > 0$. Moreover, let $\rho_0 \in \mathcal{P}_4(\mathbb{R}^d)$ with $x^* \in \mathrm{supp}(\rho_0)$. Then, for any $\varepsilon > 0$, $\gamma \in (0,1)$ and with parameters λ, $\sigma > 0$ obeying $2\lambda > \sigma^2$, there exists $\alpha_0 = \alpha_0(\varepsilon, \gamma, \lambda, \sigma, d, \nu, \eta, \rho_0)$ such that for all $\alpha \geq \alpha_0$ a weak solution $(\rho_t)_{t\in[0,T^*]}$ to (3) satisfies $W_2^2(\rho_T, \delta_{x^*}) = \varepsilon$, where $T \in \left[\frac{1-\gamma}{1+\gamma/2}T^*, T^*\right]$ with $T^* = \frac{1}{(1-\gamma)(2\lambda-\sigma^2)} \log\left(W_2^2(\rho_0, \delta_{x^*})/\varepsilon\right)$. Furthermore, on the time interval $[0,T]$, it holds*

$$W_2^2(\rho_t, \delta_{x^*}) \leq W_2^2(\rho_0, \delta_{x^*}) \exp\left(-(1-\gamma)\left(2\lambda - \sigma^2\right) t\right). \tag{4}$$

From this result it becomes apparent that the hardness of any global optimization problem is necessarily encoded in the mean-field approximation, i.e., in the way how the empirical measure of the finite particle dynamics is used to approximate the mean-field limit. In consideration of the central significance of such approximation with regards to the overall computational complexity of the implemented numerical scheme, we discuss a probabilistic quantitative result about the convergence of the interacting particle system towards the corresponding mean-field dynamics, for which we refer to [2, Proposition 16]. While the observed convergence rate is of order N^{-1} in the number of particles N, the constant in this approximation depends exponentially on the parameter α, which in turn depends in worst-case scenarios linearly on the dimension d. Characterizing more insightfully the dependence of α on properties of specific classes of objectives remains an exciting open problem for future research. A combination of the former results yields a holistic convergence proof of CBO methods on the plane, see [2, Theorem 14]. This analytical framework has allowed to obtain convergence guarantees for several variants of CBO including CBO with memory effects and local gradients [5], CBO with truncated noise [6], constrained CBO [7], CBO for multi-objective optimization problems [8], CBO for saddle point problems [9], and FedCBO for clustered federated learning problems [10]. Moreover, it may permit to prove convergence for other metaheuristics, such as the particle swarm optimization method [11].

Second, by turning our back on the previous mean-field-focused analysis point of view and by leveraging a completely nonsmooth analysis, which combines a novel quantitative version of the Laplace principle (log-sum-exp trick) and the minimizing movement scheme (proximal iteration), we interpret CBO as a stochastic relaxation of gradient descent (see Figure 1b), thereby providing a novel analytical perspective on the theoretical understanding of gradient-based learning algorithms. We observe that through communication of the particles, CBO exhibits a stochastic gradient descent (SGD)-like behavior despite solely relying on evaluations of the objective function. More rigorously, it holds the following.

Theorem 2 ([4, Theorem 1]). *Let the objective $\mathcal{E} \in \mathcal{C}^1(\mathbb{R}^d)$ be L-smooth, Λ-convex and satisfy minimal regularity assumptions. Then, for $\tau > 0$ (satisfying $\tau < 1/(-2\Lambda)$ if $\Lambda < 0$) and with parameters $\alpha, \lambda, \sigma, \Delta t > 0$ such that $\alpha \gtrsim \frac{1}{\tau} d \log d$, the iterates $(x_k^{\mathrm{CBO}})_{k=0,\dots,K}$ with $x_k^{\mathrm{CBO}} := x_\alpha^{\mathcal{E}}(\widehat{\rho}_k^N)$ follow a stochastically perturbed GD, i.e., they obey*

$$x_k^{\mathrm{CBO}} = x_{k-1}^{\mathrm{CBO}} - \tau \nabla \mathcal{E}(x_{k-1}^{\mathrm{CBO}}) + g_k,$$

where g_k is stochastic noise fulfilling for each $k = 1, \dots, K$ with high probability the quantitative estimate $\|g_k\|_2 = \mathcal{O}\big(|\lambda - 1/\Delta t| + \sigma\sqrt{\Delta t} + \sqrt{\tau/\alpha} + N^{-1/2}\big) + \mathcal{O}(\tau)$.

The fundamental value of such link between CBO and SGD lies in the formerly established fact that CBO is provably globally convergent, hence, on the one side, offering a novel explanation for the success of stochastic relaxations of gradient descent, and, on the other side and contrary to the conventional wisdom for which zero-order methods ought to be inefficient or not to possess generalization abilities, unveiling an intrinsic gradient descent nature of such heuristics. With this we furnish insights that explain how stochastic perturbations of gradient descent overcome energy barriers and reach deep levels of nonconvex functions.

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Modeling and management of gas flows across junctions

Massimiliano Daniele Rosini
(joint work with Andrea Corli, Ulrich Razafison)

This presentation focuses on the mathematical theory of flows on networks, which finds diverse applications in areas such as vehicular traffic, supply chains, and data networks. Specifically addressing gas flows, the talk delves into the challenges of modeling and mathematically analyzing these flows at different nodes connecting pipes with possibly different sections. This includes the consideration of various devices like junctions, compressors, valves and control valves.

We underline that control and optimization theory plays a crucial role in gas flows on networks, given its wide-ranging applications in various engineering and industrial systems.

A common challenge encountered in these systems is the phenomenon of chattering of the devices, characterized by a rapid switching on and off at critical states. This behavior corresponds to the mathematical concept of coherence of the corresponding coupling Riemann solver (c-Riemann solver). In control theory, a similar phenomenon is represented by a bang-bang controller.

Our primary objective is to establish a general framework for constructing and studying properties of a c-Riemann solver. The solver's properties, including invariant domains, $\mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}$-continuity, consistency, and coherence, are examined and applied to widely used models.

We consider along the pipes an isothermal plug flow as described by the one-dimensional Euler equations, which express conservation of mass and linear momentum in the absence of viscous effects. The Riemann problem, a critical component in solving these equations, is introduced along with the Rankine-Hugoniot conditions and Lax curves. At the nodes we consider c-Riemann solvers, detailing the notation and definitions.

Special attention is given to self-similar c-Riemann solvers, particularly in scenarios where a gas flows through a device, causing a loss of momentum conservation. On the other the conservation of mass at the node is typically ensured, leading to the definition of coupling functions and the corresponding c-Riemann solvers.

The presentation explores then the coherence of c-Riemann solvers, emphasizing its importance as a stability property. Various properties and sufficient conditions for coherence are discussed, providing insights into the analytical and numerical stability of solutions. The talk concludes with a comprehensive overview of valves, showcasing examples and addressing challenges related to coherence, consistency, and $\mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}$-continuity. Theoretical and practical approaches to mitigate or to avoid chattering in valves are presented, offering a glimpse into the ongoing research endeavors in this fascinating and complex domain.

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The turnpike property for mean-field optimal control problems

Chiara Segala
(joint work with Martin Gugat, Michael Herty)

In this work we study the turnpike phenomenon for optimal control problems with mean field dynamics that are obtained as the limit $N \to \infty$ of systems governed by a large number N of ordinary differential equations. We show that the optimal control problems with with large time horizons give rise to a turnpike structure of the optimal state and the optimal control. For the proof, we use the fact that the turnpike structure for the problems on the level of ordinary differential equations is preserved under the corresponding mean-field limit.

1. The optimal control problem

From a mathematical point of view, a multiagent control problem is described by minimization of an integral objective functional subject to a constraint that is the complex dynamic depicted by a system of ordinary differential equations (ODE). The formulation of an interacting particle system at a microscopic level requires the study of large-scale systems of agents (or particles) and it requires a considerable effort both from a theoretical and numerical point of view. We may consider a different level of description, that is the derivation of mesoscopic or mean-field approximations of the original dynamic. Here, the density of the particles is obtained as the number of particles tends to infinity. Of particular interest is therefore the design of controls in the mean-field control approaches. In this work, we focus on the turnpike phenomenon for optimal control problems. This topic has been studied recently for example in [1, 2, 3], and it concerns relations between the solutions of dynamic optimal control problems with objective functionals of tracking type and the corresponding static optimal control problems. The turnpike property states that the distance between the dynamic and the static optimal solution is small, in particular, for large time intervals. Hence, it allows to use this information about the structure of the dynamic optimal control to reduce the cost to obtain a numerical approximation by using the static optimal control that can be obtained more easily. In this work we consider the turnpike property with interior decay, which describes the situation that in the interior of the time interval, the distance between the dynamic optimal control/state pair and the corresponding static solution is often very small for sufficiently large time horizons. We are interested in particular on the question whether the turnpike property of a system persists in the limit of infinitely many ODEs and under which conditions such a turnpike property holds true on the mean-field level.

We consider the control of high-dimensional nonlinear dynamics accounting for the evolution of N agents at the microscopic level and, the mean-field dynamics given by a non-local transport equation for the density of particles at position $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}^+$. The initial particle density $\mu^0(x)$ is given and the control action is modeled by an additive term in the partial differential equation

(PDE). More specifically, we consider a PDE of the type

$$\partial_t \mu(t,\, x) + \partial_x \Big(((P * \mu)(t,x) + u(t,x))\ \mu(t,\, x) \Big) = 0, \qquad \mu(a,x) = \mu^0(x), \tag{1}$$

where $*$ denotes the convolution operator, the function P is given, and the real positive number a is the initial time. We consider an optimal control problem for a finite large time horizon, subjected to system (1). The objective function that we want to minimize depends both on the control and the state

$$\mathbf{J}_{(a,\,b)}(\mu,\, u) = \int_a^b f(\mu(t,\, x),\, u(t,x))\, dt,$$

for a given real-valued function f

$$f(\mu,\, u) = \int_{\mathbb{R}^d} (L(x) + \Psi(u(t,x)))\ d\mu(t,\, x), \tag{2}$$

and a time interval $[a,b]$ with $a < b$ real positive numbers. We define the parametric mean-field optimization problem

$$\mathbf{Q}(a,\, b,\, \mu^0) : \min_u \mathbf{J}_{(a,\,b)}(\mu,\, u)$$

subject to (1). We define the optimal value of the mean-field limit problem $\mathbf{Q}(a,\, b,\, \mu^0)$ as $\mathbf{V}(a,\, b,\, \mu^0)$. The existence of solutions for $\mathbf{Q}(a,\, b,\, \mu^0)$ is guaranteed by Theorem 5.1 in [4].

2. The strict dissipativity inequality

We assume that the optimal control problem satisfies a strict dissipativity assumption, i.e. for all $\tau \in [a,\, b]$

$$\begin{aligned} &\int_a^\tau f(\mu(t,x),\, u(t,x)) dt \\ &\qquad \geq \int_a^\tau \int_{\mathbb{R}^d} \left(\|x - \psi^{(\sigma)}\|^2 + \|u(t,x) - u^{(\sigma)}\|^2 \right) d\mu(t,\, x)\, dt, \end{aligned} \tag{3}$$

where f is the functional in (2).

3. The cheap control condition

For our analysis, a cheap control condition is essential. It requires that the optimal values are bounded in terms of the distance between the initial state and the desired static state. Given $\mathcal{C}_0 > 0$, for all initial times $a \geq 0$, terminal times $b > a$ and initial states $\mu(a,x) = \mu^0(x) \in P_1(\mathbb{R}^d)$, we have

$$\mathbf{V}(a,\, b,\, \mu^0) \leq \mathcal{C}_0 \int_{\mathbb{R}^d} \|x - \psi^{(\sigma)}\|\, d\mu^0(x) \tag{4}$$

with

$$\mathcal{C}_0 = \frac{1}{\beta} \Big(C_L + \beta C_\Psi + 2 C_P C_\Psi \Big). \tag{5}$$

4. The turnpike property with interior decay

We present a turnpike property for the optimal control problem $\mathcal{Q}(N,\, a,\, b,\, \psi^0)$ that follows from the dissipativity inequality and the cheap control condition. As the name indicates, this property focuses on the situation that the set where the distance between the optimal dynamic and the optimal static solution is small for large b is located in final part of the time interval $[a, b]$.

Theorem 1. *Let $\lambda \in (0,\, 1)$ be given, and the interval $[a, b]$ with $b > 0$. Consider the quantity*

$$\boldsymbol{A}_*(b) = \int_{a+\lambda(b-a)}^{b} \int_{\mathbb{R}^d} \left(\|x - \psi^{(\sigma)}\|^2 + \|\hat{u}_{(a,b,\mu^0)}(t,x) - u^{(\sigma)}\|^2 \right) d\hat{\mu}_{(a,b,\mu^0)}(t,x)\, dt,$$

where we define as $\hat{\mu}_{(a,b,\mu^0)}(t,x)$ and $\hat{u}_{(a,b,\mu^0)}(t,x)$ the density and control respectively at time t with initial condition $\mu(a,x) = \mu^0(x) = \hat{\mu}_{(a,b,\mu^0)}(a,x)$. Then the optimization problem $\boldsymbol{Q}(a,\, b,\, \mu^0)$ has a turnpike property with interior decay in the sense that

$$\boldsymbol{A}_*(b) \leq \frac{\mathcal{C}_0^2}{\lambda(b-a)} \int_{\mathbb{R}^d} \|x - \psi^{(\sigma)}\|\, d\mu(a,x).$$

where $\mathcal{C}_0$ is as in (5).

Providing suitable assumptions to guarantee the existence of solutions in the mean-field limit, we have proven the turnpike property on a mean-field level. Possible future work includes the numerical simulation and the extension e.g. to the case that the microscopic model is governed by a second-order dynamics.

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Regularity and Control for Conservation Laws with Space Discontinuous Flux

Luca Talamini

(joint work with Fabio Ancona)

We consider the Cauchy problem for the scalar conservation law

$$\begin{aligned} \partial_t u(t,x) + \partial_x f(u(t,x),x) &= 0, \qquad (t,x) \in [0,T] \times \mathbb{R}; \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{R} \end{aligned} \tag{1}$$

where f is a discontinuous function

$$f(u,x) = \begin{cases} f_l(u), & x < 0, \\ f_r(u), & x > 0 \end{cases}$$

Here f_l, f_r are strictly convex maps. Conservation laws with discontinuous flux have numerous applications; two well known examples are traffic flow with heterogeneous road conditions and two phase flow in porous media.

The discontinuity of the flux naturally leads to the study of infinitely many $\mathbf{L}^1$ contractive semigroups $\mathcal{S}_t^{AB}$, each one associated to particular pair of values, a *connection* (A,B), such that $f_l(A) = f_r(B)$ and $f_l'(A) \leq 0 \leq f_r'(B)$ (this is a particular example of an $\mathbf{L}^1$ *dissipative germ*, see [5]). Connections (A,B) are introduced in [1], and the corresponding solutions are the ones that dissipate the additional generalized Kružkov entropy

$$\eta^{AB} = \begin{cases} |u-A|, & x < 0, \\ |u-B|, & x > 0. \end{cases}$$

In [3] and [4] we are mainly interested in a theoretical analysis of the solutions of (1) associated to a connection (A,B) and we address both control and regularity problems.

Regularity and Exact Controllability. The *attainable set* at time $T > 0$ is defined by

$$\mathcal{A}^{AB}(T) \doteq \Big\{ \mathcal{S}_T^{AB} u_0 \;\Big|\; u_0 \in \mathbf{L}^\infty(\mathbb{R}) \Big\}. \tag{2}$$

Oleinik Estimates and Regularity. To understand the structure of $\mathcal{A}^{AB}(T)$ we first prove some adapted Oleinik estimates. In order to fix the ideas, assume $x < 0$ and introduce the *auxiliary characteristics lines*:

$$\vartheta_x(t) \doteq \begin{cases} x - (T-t)\cdot f_l'(\omega(x)), & \text{if } \tau(x) \leq t \leq T, \\ (t-\tau(x))\cdot f_r'\big(\pi^l_{r,-}(\omega(x))\big), & \text{if } 0 \leq t < \tau(x) \end{cases}, \qquad t \in [0,T].$$

These are lines which proceed with the characteristic speed until they are refracted by the interaction with the interface $\{x=0\}$ at time $\tau(x)$, according to the unique transition map $\pi^l_{r,-}$ such that the Rankine-Hugoniot conditions are satisfied. It should be noted that these are not real characteristic of the solution due to the presence of undercompressive zones at $\{x=0\}$.

A first point that we make is that if $\omega \in \mathcal{A}^{AB}(T)$, then the lines $\vartheta_x(t)$ are monotone in x, although they are not characteristics. In turn, this yields a bound of the form

$$\partial_x \omega \leq g_T \mathcal{L}^1 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^-) \tag{3}$$

where $g_T : \mathbb{R}^- \to \mathbb{R}^+$ is a continuous function with possibly a non integrable singularity in $x=0$, i.e. it can happen that $\lim_{x\to 0-} g(x) = +\infty$ and moreover

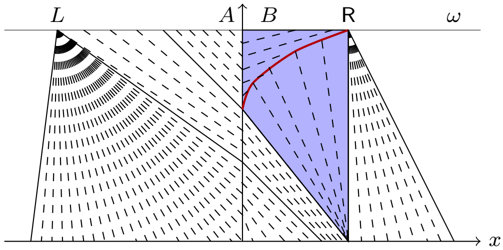

FIGURE 1. A shock in the solution is necessary to create the discontinuity at (T, R).

$g \notin \mathbf{L}^1(\mathbb{R}^-)$ (only when $f_l'(A) \cdot f_r'(B) = 0$). In fact in general $\omega \notin BV(\mathbb{R})$. Nevertheless, a calculation shows that

$$f_l'(x) \cdot g_T(x) \leq \mathcal{O}(1) \cdot \frac{1}{T|x|^{1/3}}$$

which is integrable. Repeating the argument for $x > 0$ yields $f(\omega) \in BV(\mathbb{R})$. From this one also deduces Lipschitz in time regularity for the solution map $t \mapsto u(t, \cdot)$ for uniformly positive times.

Structure of $\mathcal{A}^{AB}(T)$. As a second result, we fully characterize the attainable set $\mathcal{A}^{AB}(T)$ by the above Oleinik-like estimates plus additional geometric constraints. To give a flavour of the geometric constraints, consider a profile $\omega \in \mathcal{A}^{AB}(T)$ as in Figure 1. To create the discontinuity at the point (T, R), a shock must be present in the solution before time T. To construct this shock one needs the shaded area to be big enough, and this translates into a condition like $\omega(\mathsf{R}+) \leq u_{\mathsf{R}}$, for some state u_{R}. The tricky part is to characterize u_{R} and to show that $\omega(\mathsf{R}+) \leq u_{\mathsf{R}}$ is also a necessary condition. In [3] we characterize u_{R} by a duality procedure introducing a natural backward semigroup (see (4)) and we show that $\omega(\mathsf{R}+) \leq u_{\mathsf{R}}$ is also a necessary condition by using an elementary comparison argument.

Initial Data Identification and Backward Semigroup. For $\omega \in \mathbf{L}^\infty(\mathbb{R})$, our goal will be to characterize the set

$$\mathcal{I}_T^{AB}\omega = \Big\{u_0 \in \mathbf{L}^\infty(\mathbb{R}) \;\big|\; \mathcal{S}_T^{AB} u_0 = \omega\Big\}.$$

In particular, in [4] we prove that $\mathcal{I}_T^{AB}\omega$ is either a singleton or an infinite dimensional cone. A distinctive feature is that this cone can be non-convex: this is in contrast with the classical case of a single conservation law with convex flux, or even a flux smoothly depending on space, in which the corresponding set of initial data is always a convex cone (see [6], [7]).

An important point is the construction of a *backward semigroup operator* $\mathcal{S}_T^{[AB]-}$, through which the vertex of the cone $\mathcal{I}_T^{AB}\omega$ can be characterized as the backward evolution of ω: we prove that the vertex is exactly $\mathcal{S}_T^{[AB]-}\omega$. In addition to helping in the proof, we believe that the backward operator has an independent theoretical interest by itself, therefore we sketch its construction. For a connection (A, B) we define the dual objects:

(i) a *dual flux*, defined by $\overline{f}(u,x) \doteq f(-x,u)$;

(ii) a *dual connection* $(\overline{A}, \overline{B})$, which is uniquely determined by being a connection for the flux $\overline{f}(u,x)$ with $f_r(B) = f_r(\overline{A})$;

(iii) and a *dual semigroup* $\widetilde{\mathcal{S}}_T^{\bar{B}\bar{A}}$, of $\overline{AB}$ entropy solutions for $\overline{f}$.

Then the *backward AB-semigroup* $\mathcal{S}_T^{[AB]-} : \mathbf{L}^\infty \to \mathbf{L}^\infty$ is defined by

$$\mathcal{S}_T^{[AB]-}\omega(x) := \widetilde{\mathcal{S}}_T^{\bar{A}\bar{B}}(\omega(-\cdot))(-x). \tag{4}$$

Finally, using the backward AB semigroup, we prove the equivalence

$$\omega \in \mathcal{A}^{AB}(T) \quad \Longleftrightarrow \quad \mathcal{S}_T^{AB} \circ \mathcal{S}_T^{[AB]-}\omega = \omega$$

providing a more intrinsic, alternative characterization of the attainable set.

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Optimal control of scalar conservation law with particle approximations

Oliver Tse

This talk reports on recent (unpublished) work that introduces a convenient approach to analyzing and numerically solving optimal control problems of the form

$$\inf \mathcal{J}(u) := \Phi(\rho(T)\mathsf{Leb}), \qquad u \in \mathscr{U}(\Omega, M), \tag{\textbf{OCP}}$$

where $[0,\infty) \ni t \mapsto \rho(t) \in L^1(\mathbb{R})$ is a solution to a scalar conservation law

$$\partial_t\rho + \partial_x f(\rho) = 0, \qquad \rho(0) = u, \tag{\textbf{SCL}}$$

and

$$\mathscr{U}(\Omega, M) := \left\{ u \in BV(\Omega) \;:\; 0 \le u \le M,\ \int_\Omega u\,dx = 1 \right\}, \quad M \in (0, +\infty),$$

is the family of admissible controls. Here, the *terminal cost* $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ is a functional on the space of probability measures $\mathcal{P}(\mathbb{R})$ over $\mathbb{R}$, which we assume to be $\mathbb{W}_p$-continuous, i.e. continuous w.r.t. the p-Wasserstein metric, and $f(r) = r\beta(r)$ is the *flux* function, with $\beta : [0, \infty) \to [0, \infty)$ being a Lipschitz non-increasing *mobility* function with $\beta(0) = \beta_{\max} > 0$, $\beta \equiv 0$ on $[M, +\infty)$ and smooth on $[0, M)$.

An example of a terminal cost Φ and mobility β to keep in mind is

$$\mathcal{P}_p(\mathbb{R}) \ni \mu \mapsto \Phi(\mu) = \frac{1}{p}\mathbb{W}_p^p(\mu, \nu), \qquad \nu \in \mathcal{P}_p(\mathbb{R}),$$
$$[0, \infty) \ni r \mapsto \beta(r) = (1 - r)^+,$$

where $\mathcal{P}_p(\mathbb{R})$, $p \in [1, \infty]$ is the space of probability measures with finite p-moments.

While the solution theory for (**SCL**) is by now rather mature, the development of optimal control theories for such equations has been slow due to issues regarding the non-differentiability, in L^1, of flows generated by solutions to (**SCL**), thus rendering standard PDE constrained optimal control theory incompatible. Indeed, although the map $L^1 \cap L^\infty(\mathbb{R}) \ni u \mapsto \rho \in \mathcal{C}([0, T]; L^1(\mathbb{R}))$ can be shown to be locally Lipschitz, it is in general not directionally differentiable if $\rho = \rho[u]$ contains shocks [8]. For this reason, new notions of differentiability (eg. shift-differentiability) had to be developed to deal with the issues [1, 2, 3, 6, 7], which subsequently led to rigorous studies of optimal control problems for conservation laws, and for conservation laws on networks used in modeling, i.a., traffic flow, gas networks, and product flow in supply chains. However, these new notions of differentiability are often difficult to work with and this study originates from the desire to address these issues.

Using a *(follow-the-leader) discrete particle approximation* of (**SCL**), one obtains a discrete optimal control problem

$$(\mathbf{OCP}_n) \qquad \inf_{\bar{\mathbf{x}} \in \mathscr{K}^n(\Omega, M)} \mathcal{J}^n(\bar{\mathbf{x}}) := \Phi(\mathfrak{p}^{\mathbf{x}}(T)\mathsf{Leb}), \qquad M \in (0, \infty),$$

where $\mathbf{x} = \mathbf{x}(t)$ is the unique solution of deterministic particle approximation

$$(\mathbf{DPA}_n) \qquad \begin{aligned} \dot{x}_i &= \beta(\rho_i^{\mathbf{x}}), \qquad i = 0, \ldots, n-1\,, \\ \dot{x}_n &= \beta_{\max}\,, \end{aligned} \qquad \mathbf{x}(0) = \bar{\mathbf{x}},$$

and the *discrete density* is defined by

$$\mathfrak{p}^{\mathbf{x}} := \sum_{i=0}^{n-1} \rho_i^{\mathbf{x}} \mathbf{1}_{K_i^{\mathbf{x}}}, \qquad K_i^{\mathbf{x}} = [x_i, x_{i+1}),$$

where

$$\rho_i^{\mathbf{x}} := \frac{h}{x_{i+1} - x_i} \qquad i = 0, \ldots, n-1, \qquad \rho_n = 0, \qquad h = 1/n.$$

Here,

$$\mathscr{K}^n(\Omega, M) := \Big\{ \mathbf{x} \in \mathbb{R}^{n+1} : M\,(x_{i+1} - x_i) \geq h,\ i = 0, \ldots, n-1 \Big\}.$$

denotes the family of admissible controls $\bar{\mathbf{x}}$.

The discrete particle approximation ($\mathbf{DPA}_n$) is known to converge to the Kružkov entropy solution of ($\mathbf{SCL}$) [4, 5] and allows one to establish stability estimates w.r.t. the p-Wasserstein metric. In particular, one obtains the following result:

Result 1: Let $\mathbf{x}, \mathbf{y}$ be solutions of ($\mathbf{DPA}_n$) with initial data $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \mathscr{K}^n(\Omega, M)$, respectively. Then,

$$\mathbb{W}_p(\mathfrak{p}^{\mathbf{y}}(t)\mathsf{Leb}, \mathfrak{p}^{\mathbf{x}}(t)\mathsf{Leb}) \leq \mathbb{W}_\infty(\mathfrak{p}^{\bar{\mathbf{y}}}(t)\mathsf{Leb}, \mathfrak{p}^{\bar{\mathbf{x}}}(t)\mathsf{Leb}) \qquad \forall\, t \geq 0.$$

The convergence of ($\mathbf{DPA}_n$) consequently provides the continuous counterpart of the stability estimate:

$$\mathbb{W}_p(S_t(w)\mathsf{Leb}, S_t(u)\mathsf{Leb}) \leq \mathbb{W}_\infty(w\,\mathsf{Leb}, u\,\mathsf{Leb}) \qquad \forall\, t \geq 0,$$

where $u \mapsto S_t(u)$ is the solution operator associated to ($\mathbf{SCL}$).

Moreover, the discrete optimal control problem ($\mathbf{OCP}_n$) is shown to admit a minimizer $\bar{\mathbf{x}}^n \in \mathscr{K}^n(\Omega, M)$ for each $n \geq 1$ such that $\mathfrak{p}^{\bar{\mathbf{x}}^n} \in \mathscr{U}(\Omega, M)$. In addition, the sequence $(\mathfrak{p}^{\bar{\mathbf{x}}^n})_{n\geq 1}$ admits an accumulation point $u^* \in \mathscr{U}(\Omega, M)$ that turns out to be a minimizer of the continuous optimal control problem ($\mathbf{OCP}$), which is a consequence of the following Γ-convergence result, justifying the role of the deterministic particle approximation ($\mathbf{DPA}_n$) as a surrogate model for the optimal control problem:

Result 2: The family of functionals $\{\widehat{\mathcal{J}}^n\}_{n\geq 1}$ defined by

$$\widehat{\mathcal{J}}^n(u) := \begin{cases} \mathcal{J}^n(\bar{\mathbf{x}}) & \text{if } u = \mathfrak{p}^{\bar{\mathbf{x}}},\ \ \bar{\mathbf{x}} \in \mathscr{K}^n(\Omega, M), \\ +\infty & \text{otherwise,} \end{cases} \qquad n \geq 1,$$

is (weakly) equi-coercive and $\Gamma_{\text{weak-}L^1}$-converges to $\mathcal{J}$.

In practice, one would like to numerically compute a minimizer of ($\mathbf{OCP}$). The previous result suggests that the numerical solution of ($\mathbf{OCP}_n$) can be used as a proxy for obtaining a minimizer of ($\mathbf{OCP}$). One way of obtaining a minimizer of ($\mathbf{OCP}_n$) is utilizing an adjoint-based approach, which can be easily executed due to the finite-dimensional nature of the problem.

The last part of the talk was devoted to an initial attempt to link the discrete adjoint with its continuous counterpart and highlight the issues encountered along the way. Under very restrictive assumptions on the behavior of the state ρ, one obtains an equation governing the continuous adjoint equation. This leads to the following conjecture for general states:

Conjecture: Let $u \in \mathscr{U}(\Omega, M)$ and ρ be its corresponding state satisfying ($\mathbf{SCL}$). Then, the associated adjoint equation for the adjoint state η reads

$$\partial_t(\eta\rho) + \partial_x(\eta P(\rho)) = 0,$$

subjected to an appropriate terminal condition, where $P(s) = sf'(s) - f(s)$.

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Neural Network approaches for High-dimensional Optimal Control Problems

Deepanshu Verma

Optimal Control (OC) problems are pervasive in various fields, from finance to robotics, aiming to find a control policy minimizing a defined control objective functional. Traditionally, Dynamic Programming is employed to solve these problems, seeking the value function that assigns each system state the optimal cost-to-go and satisfies the Hamilton-Jacobi-Bellman (HJB) equation.

The challenge lies in solving the HJB PDE for the value function. Traditional numerical schemes suffer from the Curse-of- Dimensionality (CoD), where computational complexity increases exponentially with problem dimension. In order to mitigate the CoD, we not only need to alleviate the need for spatial discretization but also to effectively parameterize the value function in high-dimensions. This is achieved by parameterizing the value function using NNs.

In [1, 2], we present *neural-HJB* approach for solving high-dimensional OC (stochastic and deterministic) problems informed by control theory. Our method leverages Pontryagin's maximum principle to guide system sampling and obtain the optimal control in real-time from the value function via feedback form. Notably, our learning is *unsupervised*, requiring no prior data to learn value function. In comparison with Reinforcement Learning, a popular unsupervised learning approach for approximate policies. Our neural-HJB approach demonstrates improved accuracy, reduced time-to-solution, and fewer PDE solves compared to RL while approximating superior policies.

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Kinetic Modelling and Control of Multiagent Systems with Missing Information

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Kinetic equations play a leading role in the modelling of large systems of interacting particles/agents with a recognized effectiveness in describing real world phenomena ranging from plasma physics to multi-agent dynamics. The derivation of these models has often to deal with physical, or even social, forces that are deduced empirically and of which we have limited information [1]. To produce realistic descriptions of the underlying systems, it is of paramount importance to quantify the propagation of uncertain quantities across the scales.

We concentrate on the interplay of this class of models with collective phenomena in life and social sciences, where the assessment of uncertainties in data assimilation is crucial to design efficient interventions. Furthermore, to discuss the mathematical interface of this class of models with available data, we derive the evolution of observable quantities based on suitable macroscopic limits of classical kinetic theory [2, 3]. Finally, we analyze how the introduction of robust control strategies leads to the damping of the uncertainties characterizing the system at the macroscopic level [4].

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Sidewise control

Enrique Zuazua

Inspired on [1] and [4], we explored the lateral or sidewise control properties of 1-dimensional waves, a concept applicable to any spatial dimension. Drawing parallels with classical control and inverse problems in wave propagation, our focus lies in influencing the behavior of waves on a portion of the domain boundary through localized control actions on a distinct subset of the boundary. Unlike classical problems, our objective is not the control of wave dynamics within the domain but rather their boundary traces, constituting a goal-oriented controllability problem.

In the one-dimensional scenario, the typical aim is to govern the trace on one end of the string by means of a control action at the other end.

Utilizing duality, we reformulated the lateral control problem, into a pertinent observability inequality. This is applicable in any space dimension. This inequality involves estimating non-homogeneous boundary traces of waves on a specific subset of the boundary using measurements acquired on a different one. These inequalities pose novel challenges that diverge from traditional techniques in the field, such as Carleman inequalities, non-harmonic Fourier series, microlocal analysis, and multipliers.

We introduced a distinctive one-dimensional solution method grounded in sidewise energy propagation estimates, leading to a complete and precise solution. This methodology extends to address 1-dimensional wave equations featuring BV-variable coefficients. By combining it with fixed-point techniques as in [5], this allows handling 1-dimensional semilinear wave equations.

In the multi-dimensional scenario, building upon [3], we demonstrated how Fourier series decomposition facilitates addressing the problem, resulting in lateral controllability properties in rectangular domains. Controls are applied on one side, influencing dynamics on the opposite side. However, the attained results exhibit an infinite loss (in Sobolev terms) on observed norms and controlled sources. To deal with more general geometries, we introduced a geometric control condition of microlocal nature, ensuring control towards targets defined on concave subsets of the boundary of suitable domains.

Finally, we delved into the parabolic counterpart, introducing a novel transmutation formula and establishing a connection between wave and heat equations, as presented in [2].

Acknowledgments

E. Zuazua has been funded by the Alexander von Humboldt-Professorship program, the ModConFlex Marie Curie Action, HORIZON-MSCA-2021-DN-01, the COST Action MAT-DYN-NET, the Transregio 154 Project "Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks" of the DFG, grants PID2020-112617GB-C22 and TED2021-131390B-I00 of MINECO (Spain) and the Madrid Government - UAM Agreement for the Excellence of the

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