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Arbeitsgemeinschaft: QFT and Stochastic PDEs

Organized by
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Abstract. Quantum field theory (QFT) is a fundamental framework for a wide range of phenomena is physics. The link between QFT and SPDE was first observed by the physicists Parisi and Wu (1981), known as Stochastic Quantisation. The study of solution theories and properties of solutions to these SPDEs derived from the Stochastic Quantisation procedure has stimulated substantial progress of the solution theory of singular SPDE, especially the invention of the theories of regularity structures and paracontrolled distributions in the last decade. Moreover, Stochastic Quantisation allows us to bring in more tools including PDE and stochastic analysis to study QFT.

This Arbeitsgemeinschaft starts by covering some background material and then explores some of the advances made in recent years. The focus of this Arbeitsgemeinschaft is QFT models such as the Φ^4, sine-Gordon and Yang–Mills models as examples to discuss stochastic quantisation and SPDE methods and their applications in these models. We introduce the key ideas, results and applications of regularity structure and paracontrolled distributions, construction of solutions of the SPDEs corresponding to these models, and use the PDE method to study some qualitative behaviors of these QFTs, and connections with the corresponding lattice or statistical physical models. We also discuss some other topics of QFT, such as Wilsonian renormalisation group, log-Sobolev inequalities and their implications, and various connections between these topics and SPDEs.

Mathematics Subject Classification (2020): 35R60, 37A50, 37L55, 39A50, 60H15, 60L30, 76M35, 81S20, 81T08, 81T13, 81T25, 81T27.

Introduction by the Organizers

The Arbeitsgemeinschaft *QFT and Stochastic PDEs*, organized by Roland Bauerschmidt (New York), Massimiliano Gubinelli (Oxford), Martin Hairer (London/Lausanne), and Hao Shen (Wisconsin-Madison) was attended by 44 participants (as well as a few remote participants). There was a broad geographic representation from all continents. Most of the participants were in early stages of their careers, with background mostly in the areas of probability theory, analysis, and theoretical physics. All the in-person participants delivered talks, with a total of 22 talks, each coordinated and presented by two speakers.

The talks were organized in a progressive order. The talks on Monday focused on general introductions to Euclidean QFT, and local solutions to SPDEs in the Da Prato–Debussche regime. The example of the stochastic quantisation of the Φ^4_2 model (which is the simplest nontrivial case of a nonlinear SPDE from Euclidean QFT) was discussed. The talks on Tuesday then discussed global solution theory to the stochastic quantisation of Φ^4_2. The talks on Wednesday provided more applications of the Da Prato–Debussche argument, and examples of using PDE methods to study some qualitative behaviors of these QFTs such as integrability of the Φ^4_2 measure, as well as connections with the corresponding statistical physical models. The Wednesday talks introduced the Yang-Mills model and its Langevin dynamics, in continuum and on lattice. On Thursday, the theory of regularity structures was introduced by the participants, and some of the cornerstone theorems of this theory were proved; an application to the stochastic quantisation of the Φ^4_3 model was given. The talks on Friday were focussed on the Wilsonian renormalisation group approach, log-Sobolev inequalities and their implications, and the connections between these topics and SPDEs.

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Arbeitsgemeinschaft: QFT and Stochastic PDEs

Table of Contents

Abstracts

Towards Euclidean quantum field theory

Wei Huang, Weile Weng

The talk consists of two parts: the first part focus on the Feynman-Kac formula that leads to Euclidean quantum mechanics and the second part is about the Ostwalder-Schrader axioms in Euclidean quantum field theory, with a focus on reflection positivity.

We begin our first part with a brief introduction on the basic postulates of quantum mechanics. Then we have a look at one of the simplest quantum mechanics systems, the quantum harmonic oscillator. After rescaling, we get the Hamiltonian $H = 1/2(P^2 + Q^2)$, where $P = i\partial_x, Q = x$. Note that $H = A^*A + 1/2$, where $A = \frac{1}{\sqrt{2}}(Q + iP), A^* = \frac{1}{\sqrt{2}}(Q - iP)$. A is called the annihilation operator, as for any eigenvector Ω of H with eigenvalue λ, $A^*\Omega$(if non-zero) is an eigenvector of H with eigenvalue $\lambda - 1$. A^* is called the creation operator as it increases the eigenvalue by 1 when acting on eigenfunctions. There is a unique ground state which has the lowest eigenvalue 1/2, and the corresponding eigenfunction is $\Omega_0(x) = \pi^{-1/4}e^{-x^2/2}$. The other eigenvectors can be obtained by acting A^* on it and $\Omega_n = (n!)^{-1/2}A^{*n}\Omega_0 = (n!)^{-1/2}P_n(Q)\Omega_0$, where P_n are the Hermite polynomials. The eigenvectors forms an ONS of $\mathcal{H}$, and they can all be obtained from the ground state by multiplying polynomials of Q.

The Feynman path integral expresses the integral kernel of the Schrödinger propagator in terms of a path integral

$$e^{-itH/\hbar}(x, x') = \frac{1}{Z}\int_{\gamma_0 = x, \gamma_t = x'} e^{-\frac{i}{\hbar}\int_0^t \frac{1}{2}\dot{\gamma}_s^2 - V(\gamma_s)ds} d\gamma.$$

It reveals a physical intuition that the particle takes all the possible path with weights given by the classical action, but mathematically the integral is very problematic as it cannot be defined with a measure. We can avoid the problem by running the dynamics in imaginary time(also called Wick rotation). We then get the Feynman-Kac formula(we set $\hbar = 1$):

$$e^{-tH}(x, x') = \int e^{-\int_0^t V(\gamma)ds} dW_{x,x'}(\gamma),$$

where $W_{x,x'}$ is the Wiener measure conditioned on starting at x and ending at x'. The Feynman-Kac formula implies the positivity of kernel and uniqueness and positivity of ground state, which are necessary to construct the measure in the renormalized Feynman-Kac formula.

Assume there exists a ground state Ω and assume the ground state energy to be E_0. Then we subtract the energy to get $\hat{H} = h - E_0$. The ground state transformation($\times\Omega^{-1}$) is a isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \Omega^2 dm)$ and we can transfer $\hat{H}$ to a self-adjoint operator $H^\wedge = \Omega\hat{H}\Omega^{-1}$ on $L^2(\mathbb{R}, \Omega^2 dm)$. Since $e^{-tH^\wedge}$ has positive kernel and $H^\wedge 1 = 0$, it generates a Markov process on $\mathbb{R}$ and we denote its distribution by μ. By the ground state transformation we get the following renormalized Feynman-Kac formula,

$$\langle \Omega, A_1 e^{-(t_2-t_1)\hat{H}} A_2 e^{-(t_3-t_2)\hat{H}} \dots A_n \Omega \rangle = \int A_1(q_{t_1}) \dots A_n(q_{t_n}) d\mu(q), \tag{1}$$

which enables us to express the Wightman function(ground state correlation in Euclidean quantum mechanics) in terms of correlation of a stochastic process. If we construct the measure μ, then we can compute the Wightman function and get Schwinger function(ground state correlation in quantum mechanics) by analytic continuation, and finally retrieve all the information of the quantum dynamics from the Schwinger function.

We now turn our focus to the Euclidean fields. A EQFT is a certain probability measure μ on real distributions $\mathcal{D}' \equiv \mathcal{D}'(\mathbb{R}^d)$, where d is the space-time dimension. Let $\mathcal{D} \equiv C_0^\infty(\mathbb{R}^d)$ be the space of test functions. For $\phi \in \mathcal{D}', f \in \mathcal{D}$, we write $\phi(f) = \langle \phi, f \rangle$ to be the canonical pairing on $\mathbb{R}^d$. The probability measure μ is characterized by the generating functionals $\{S_f, f \in \mathcal{D}\}$, with

$$S_f : \phi \mapsto \int e^{i\phi(f)} d\mu(\phi), \quad \phi \in \mathcal{D}'.$$

Osterwalder-Schrader axioms impose five conditions on μ:

- (OS0) Analyticity: S_f is entire analytic. It ensures the super-exponential decay of $d\mu$.
- (OS1) Regularity: $\log |S_f| \le c(||f||_{L^1} + ||f||_{L^p}^p)$, for $p \in [1, 2]$, and some constant c . If $p = 2$, then the second-order Schwinger function should be locally integrable.
- (OS2) Invariance: S_f is invariant under Euclidean symmetries of $\mathbb{R}^d$, i.e. translation, rotation, and reflection. This is equivalent to the Euclidean invariance of $d\mu$.
- (OS3) Reflection positivity (RP): for every finite sequence $(f_i) \subset \mathcal{D}_{\text{real}}$, the matrix $M_{ij} = S_{f_i - \theta f_j}$ has non-negative eigenvalues, where θ is the time reflection over the point 0.
- (OS4) Ergodicity: the measure space $(D', d\mu)$ is ergodic with respect to the time translation subgroup $T(t)$.

(OS0)-(OS2) are meant for all test function f. For (OS3), there is an equivalent formulation. Let

$$\mathcal{A}_+ = \{A : \phi \mapsto \sum_{j=1}^{N} c_j e^{\phi(f_j)}, \text{for some } c_j \in \mathbb{C}, f_j \in C_0(\mathbb{R}^d_+), N \in \mathbb{N}\},$$

with $\mathbb{R}^d_+$ the half-space of positive time. Let $\mathcal{E} = L^2(\mathcal{D}'(\mathbb{R}^d), d\mu)$, then RP is equivalent to

$$\langle \theta A, A \rangle_{\mathcal{E}} \ge 0, \quad \forall A \in \mathcal{A}_+.$$

The reflection positivity axiom helps us to construct a quantum mechanical Hilbert space $\mathcal{H}$. The construction is based on $\mathcal{A}_+$, and the bilinear form $b(A, B) := \langle \theta A, B \rangle_{\mathcal{E}}$. Specifically, it is constructed in three steps: first, take closure of $\mathcal{A}_+$ in $\mathcal{E}$, and denote it by $\mathcal{E}_+$; second, thanks to the RP, observe that $||\cdot||_b := b(\cdot,\cdot)^{\frac{1}{2}}$ defines a semi-norm on $\mathcal{E}_+$, and thus a norm on $\mathcal{E}_+/\mathcal{N}$, with $\mathcal{N} := \{A \in \mathcal{E}_+, ||A||_b = 0\}$ the null-set; finally, define $\mathcal{H}$ as the closure of the equivalent class $\mathcal{E}_+/\mathcal{N}$ in $(\mathcal{E}, ||\cdot||_b)$, and check the Parallelogram identity, and conclude that $\mathcal{H}$ is a Hilbert space.

For $A \in \mathcal{E}_+$, let $A^\wedge := A + \mathcal{N} \in \mathcal{H}$. To this point, we have $\langle A^\wedge, B^\wedge\rangle_{\mathcal{H}} = \langle \theta A, B\rangle_{\mathcal{E}}$. Next, we wish to transfer an operator S on $\mathcal{E}$ to $S^\wedge$ on $\mathcal{H}$. In order for the following equality to hold, $\langle A^\wedge, S^\wedge B^\wedge\rangle_{\mathcal{H}} = \langle \theta A, SB\rangle_{\mathcal{E}}$, where $S^\wedge B^\wedge := (SB)^\wedge$, S must map $D(S) \cap \mathcal{E}_+$ to $\mathcal{E}_+$, and $D(S) \cap \mathcal{N}$ to $\mathcal{N}$.

Now we are ready to construct the Hamiltonian H via the time translation semi-group $T(t)$.

Theorem (Construction of H). *Suppose (OS3) and (OS2) hold (in particular, the time translation invariance of $d\mu$). Then for $t \geq 0$, $T(t)^\wedge$ is well-defined, and it can be written as $T(t)^\wedge = e^{-tH}$, where H is some positive self-adjoint operator, with ground state $\Omega = 1^\wedge$.*

The idea of the proof is to first show that $R(t) := T(t)^\wedge$ maps $\mathcal{N}$ to $\mathcal{N}$, and satisfies the properties of semi-group, Hermitian, contraction and strong continuity. Hence there exists a positive self-adjoint operator H satisfying the desired relation. H has a ground state $1^\wedge$, i.e. $H1^\wedge = 0$, which follows from $T(t)1 = 1$.

For the lattice models where the space-time is $\mathbb{Z}^d$ instead of $\mathbb{R}^d$, under suitable modified assumptions, we may apply above theorem to construct a self-adjoint matrix K on $\mathcal{H}$, such that $K^n := T(n)^\wedge$, for $n \in \mathbb{N}$. In addition, $1^\wedge$ is an invariant vector for K. Here, K in the lattice model plays the role of e^{-H} in the continuous model.

Reflection positivity for lattice models is a sophisticated topic (see [2], [3, Chap. 10]). To enumerate examples, it is convenient to consider the space-time on a torus $\mathbb{T}_L$, as it has natural reflection symmetry along planes orthogonal to one of the lattice directions. In a broader language, we speak of reflections over one of such hyperplane Π that splits the torus into two halves, and the splitting is either *through sites* or *through edges*. The most simple case is the *product measure*, it is RP with respect to all reflections. *Gibbs measure* of a class of lattice spin systems also possess reflection positivity, such as *Ising*, *Potts* and *Heisenberg* models. We mention a more general result. Given a fixed plane of reflection Π, with the corresponding reflection operator θ, let $A_+(\theta)$ be the algebra of all bounded and measurable functions supported on the positive half of the reflection plane. Now, suppose the torus Hamiltonian takes the form $-H_L = A + \theta A + \sum_\alpha C_\alpha \theta C_\alpha$, with $A, C_\alpha \in \mathcal{A}(\theta)_+$, then the torus Gibbs measure is RP with respect to θ (for any inverse-temperature). Such examples include the torus Hamiltonian for *nearest neighbor (ferromagnetic) interaction*, *Yukawa potentials*, and the *power-law decaying potentials*, which are RP with respect to any plane.

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Gaussian free field and the ϕ_2^4 measure on torus

Aleksandra Korzhenkova

Our goal is to construct a probability measure $\mathbb{P}$ on the space $\mathcal{D}'(\mathbb{T}^2)$ of distributions on the two-dimensional torus heuristically given by

$$\mathbb{P}(\mathrm{d}\phi) \propto \exp\Bigg(-\int_{\mathbb{T}^2}|\phi|^4\mathrm{d}x - \frac{1}{2}\underbrace{\int_{\mathbb{T}^2}(|\nabla\phi|^2+m^2|\phi|^2)\mathrm{d}x}_{=\langle\phi,(-\Delta+m^2)\phi\rangle}\Bigg)\text{“}\mathrm{d}\phi\text{”}$$

for $m^2>0$.

Step 1: Absolute continuity w.r.t. massive GFF measure.

The alternative description of the second integral directly suggests we rewrite $\mathbb{P}$ as $\mathbb{P}(\mathrm{d}\phi)\propto e^{-\int|\phi|^4\mathrm{d}x}\mathbb{Q}(\mathrm{d}\phi)$ for a centered Gaussian measure $\mathbb{Q}$ with the inverse of $-\Delta+m^2$ as covariance, called *massive Gaussian free field (GFF).* One way to define $\mathbb{Q}$ rigorously is by diagonalizing $-\Delta+m^2$. More precisely, let $0=\lambda_1<\lambda_2\le\lambda_3\le\dots$ be the eigenvalues of minus Laplacian and $(e_j)_{j\in\mathbb{N}}$ be a family of the corresponding eigenfunctions that forms an orthonormal basis of $L^2(\mathbb{T}^2)$. In this basis, the desired covariance, called *massive Green's function*, is given by

$$G_{m^2}(x,y)=\sum_{j\ge1}\frac{1}{\lambda_j+m^2}e_j(x)e_j(y)\quad\text{for all } x\neq y\in\mathbb{T}^2,$$

where the right-hand side is a convergent series in $L^2(\mathbb{T}^2\times\mathbb{T}^2)$. Let further $(\alpha_j)_j$ be a sequence of i.i.d. standard normal random variables. We set

$$\Gamma:=\sum_{j=1}^{\infty}\frac{1}{\sqrt{\lambda_j+m^2}}\alpha_j e_j. \tag{1}$$

By Weyl's law or alternatively since when re-indexed by $\mathbb{Z}^2\ni k$, $\lambda_k=|k|^2$ and $e_k(x)=e^{i\langle k,x\rangle}$ are explicit, we can easily check that almost surely $\Gamma\in H^{-\varepsilon}(\mathbb{T}^2)=\{f=\sum_j\langle f,e_j\rangle e_j \mid \sum_{j\ge1}|\langle f,e_j\rangle|^2(\lambda_j+m^2)^{-\varepsilon}<\infty\}$ for any $\varepsilon>0$. That is, almost surely Γ is a random distribution, and its covariance is clearly G_{m^2}, hence, we can set $\mathbb{Q}$ to be the law of Γ.

Remark. One can also define (massive) GFF in higher dimensions as well as on more general domains [1, 4], e.g., on regular subsets of $\mathbb{R}^d$ (for $d=2$, proper subsets) with various boundary conditions. The distinguishing feature of the dimension two, which is of immense importance to our construction of $\mathbb{P}$, is that

G_{m^2} has a logarithmic singularity at the diagonal compared to the polynomial ones in higher dimensions. More precisely, for $d = 2$

$$G_{m^2}(x, y) \sim -c \log(m|x - y|) \quad \text{as } |x - y| \to 0$$

for an explicit constant c (some finite power of 2π).

Step 2: Renormalization of power.

Now that we have $\mathbb{Q}$, for $e^{-\int |\phi|^4 \mathrm{d}x}\mathbb{Q}(\mathrm{d}\phi)$ to be defined we at least need that $\phi \in L^4(\mathbb{T}^2)$ for $\mathbb{Q}$-almost every ϕ. However, $\mathbb{E}_{\mathbb{Q}}[|\phi|^2] = \sum_{k\in\mathbb{Z}^2} \frac{1}{|k|^2+m^2} = \infty$. To cancel this divergence we perform a renormalization of power: instead of $\langle \phi^4, 1\rangle$ we consider $\langle : \phi^4 :, 1\rangle$, where for a centered Gaussian random variable X, $: X^4 :$ is given by

$$: X^4 := X^4 - 6\mathrm{Var}[X]X^2 + 3\mathrm{Var}[X]^2 = (X^2 - 3\mathrm{Var}[X])^2 - 6\mathrm{Var}[X]^2,$$

such that $\mathbb{E}[: X^4 :] = 0$. As $\phi \sim \mathbb{Q}$ is only a random distribution, we still have to make sense of $\langle : \phi^4 :, 1\rangle$ rigorously. For this, consider the truncated Fourier series (cf. (1)) $\phi_N \overset{\text{law}}{=} \sum_{\substack{k\in\mathbb{Z}^2:\\ |k|\le N}} \frac{1}{\sqrt{\lambda_k+m^2}}\alpha_k e_k$. It is almost surely a smooth centered Gaussian field with variance on the diagonal $G_{m^2,N}(0, 0) \sim c \log(N)$ as $N \to \infty$ for a constant $c > 0$. For simplicity set $\chi_N := \langle : \phi_N^4 :, 1\rangle$. Using Wick's formula [2, Lemma 2.4 & Proposition 3.1], [5, Theorem I.3] and the aforementioned fact that G_{m^2} has a logarithmic singularity at the diagonal one can show [2, Section 3] that $(\chi_N)_N$ is uniformly bounded and convergent in $L^2(\mathbb{Q})$. Let us denote the limit by $\chi = \langle : \phi^4 :, 1\rangle$ (it is just notation, $: \phi^4 :$ is not well-defined on its own). An important observation is that each $: \phi_N^4 :$ by definition is an element of the so-called 4th Wiener chaos (on the Gaussian probability space generated by ϕ) [2, Section 2.1], which is a closed subspace of $L^2(\mathbb{Q})$ spanned by "4th Hermite polynomials of the white noise". This in particular implies that also all χ_N and the limit χ are elements of the 4th Wiener chaos, which allows us to use the hypercontractivity result (2) stated below to conclude that the convergence also takes place in $L^p(\mathbb{Q})$ for any $p \ge 2$.

Step 3: $\mathbb{E}_{\mathbb{Q}}[e^{-\langle : \phi^4 :, 1\rangle}] < \infty$.

Now it only remains to verify that $e^{-\langle : \phi^4 :, 1\rangle}$ is integrable w.r.t. $\mathbb{Q}$. We follow Nelson's argument'66 (cf. [3, Section 9]); the idea is to split the field $\phi \sim \mathbb{Q}$ into its truncated Fourier series ϕ_N and the remaining part ψ_N and verify that the latter is negligibly small. The key ingredient of this strategy is the aforementioned *hypercontractivity result* that states that for any element X of the nth ($n \in \mathbb{N}$) Wiener chaos,

$$\mathbb{E}[X^{2p}] \le (2p - 1)^{np}\mathbb{E}[X^2]^p \quad \text{for any } p \ge 1. \tag{2}$$

One can prove (2) either purely combinatorially [5, Lemma I.18], [2, Section 4.1] or even in greater generality using tensorisation property and log-Sobolev inequality [3, Section 7].

By the previous step we know that $\langle : \phi^4 :, 1\rangle := L^p(\mathbb{Q}) - \lim_N \langle : \phi_N^4 :, 1\rangle$ (for any $p \ge 2$) is an element of the 4th Wiener chaos (as well as $\langle : \phi_N^4 :, 1\rangle$). Define

$Y_N = \langle :\phi^4:, 1\rangle - \langle :\phi_N^4:, 1\rangle$. It is possible to show (see [3, Section 9] for a sharper bound or [2, Section 3]) that

$$\mathbb{E}_{\mathbb{Q}}[|Y_N|^2] \le C/N \quad \text{for some } C > 0.$$

Then, by (2), for all $p \ge 1$,

$$\mathbb{E}_{\mathbb{Q}}[|Y_N|^{2p}] \le (2p)^{4p} C^p / N^p.$$

Combining this estimate with the observation that

$$:\phi_N^4: \ge -6\mathrm{Var}[\phi_N]^2 \ge -c(\log N)^2,$$

we get for all $N, t > 1$ sufficiently large such that $\log t - c(\log N)^2 > 0$,

$$\mathbb{Q}[e^{-\langle :\phi^4:, 1\rangle} \ge t] \le \mathbb{Q}[Y_N \le -\log t + c(\log N)^2] \le \frac{(2p)^{4p} C^p / N^p}{(\log t - c(\log N)^2)^{2p}}.$$

We can now adjust p and N to get a faster than polynomial decay for all t sufficiently large, which in turn would conclude our construction. For instance, take N such that $\log t - c(\log N)^2 \in [1, 2]$ and $p = \left\lfloor \frac{1}{2^5 C} e^{\sqrt{(\log t - 2)/c}} \right\rfloor$.

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Hölder-Besov spaces and space-time white noise

Alberto Bonicelli, Fabrizio Zanello

The study of stochastic PDEs encompasses equations with a random forcing to model the behaviour of systems with a large number of interactions, whose evolution displays unpredictable features. Examples of paramount relevance in physics are the KPZ equation, which suitably describes interface dynamics of two competing media, or the so called stochastic φ^4_d equation on $\mathbb{R}^d$, which enters into play when performing the stochastic quantization of a Euclidean self-interacting scalar quantum field theory, as well as to describe phase transitions for systems around the critical threshold. For a pedagogical exposition of these and many more examples of stochastic PDEs we refer the interested reader to the review [3].

The first part of the talk is devoted to defining the random source for the class of stochastic PDEs we are interested in, the so called space-time white noise, as a centred Gaussian random tempered distribution. Starting from its covariance, a direct calculation characterizes its behaviour under scaling. This scaling invariance in law prompts the choice of suitable spaces of functions (and distributions) upon which to construct a suitable solution theory. Focusing on parabolic problems, it is natural to introduce Hölder spaces $\mathcal{C}_s^\alpha$, $\alpha \in (0,1)$ defined in terms of a scaled distance, where time counts twice as space. Yet, as mentioned above, space-time white noise is a distribution, hence we need a natural extension of such spaces of function for negative exponents. The natural candidates are the Hölder-Besov spaces, which with a slight abuse of notation we denote as $\mathcal{C}_s^\alpha$ and whose definition heavily relies on scaling. The second part of the talk focuses on key results of harmonic analysis. An important question is whether a pair of functions with low regularity can be multiplied. The answer goes under the name of Young theorem and states that the product of smooth functions can be extended to a continuous bilinear map over $\mathcal{C}_s^\alpha \times \mathcal{C}_s^\beta$ if and only if $\alpha + \beta > 0$. Another fundamental result is a Schauder estimate for parabolic operators characterizing the gain of regularity in Hölder-Besov spaces.

The final step consists of individuating the space of distributions in which the white noise lies. One has to resort to a Kolmogorov-like criterion that relates the behaviours under scaling of the L^p norm of a distribution to its Hölder regularity. To wit, a direct calculation entails that the space-time white noise lies in $\mathcal{C}_s^\alpha$ for all $\alpha < -\frac{d+2}{2}$.

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The linear stochastic heat equation and some non-linear perturbations

Christopher Janjigian, Xuan Wu

This talk introduces the linear stochastic heat equation (SHE) with additive white noise forcing through its mild (Duhamel) formulation. A proof of existence of solutions in an appropriate Besov space will be sketched based on a version of Kolmogorov's continuity theorem. During this portion of the presentation, graphical notation will be introduced for certain stochastic integrals and associated non-random integrals which appear in the moment estimates. These estimates will be seen to suggest that solutions to the SHE should be functions only in dimensions strictly less than two.

The second portion of the talk will discuss a class of non-linear perturbations of the SHE, introduce the idea of scaling of SPDEs and how this relates to when we should expect to be able to find non-trivial solutions to this class of non-linear SPDEs. In particular, we will discuss what is meant by sub-criticality, criticality, and super-criticality and will state a "meta-theorem" about existence of solutions to sub-critical equations. With this concept in hand, we will see how the mild formulation of a non-linear equation leads to a fixed-point problem that necessitates some renormalization if the solution to the linear SHE is not a function.

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The Da Prato-Debussche argument

Peter Morfe, Florian Schweiger

For some semilinear stochastic PDEs it is possible to construct a solution if one can make sense of the nonlinearity when applied to the solutions of the corresponding linear equation. This method goes back to Da Prato and Debussche [2]. We explain the details using the example of the dynamical Φ^4_d model, where the method can be applied for $d = 2$, but not for $d = 3$. Our presentation follows the review article [1].

In more detail, consider the SPDE formally given by

$$\partial_t \varphi(t,x) = \Delta\varphi(t,x) - \varphi(t,x)^3 + \xi(t,x) \tag{1}$$

on $\mathbb{R}_+ \times \mathbb{T}^d$, where ξ is space-time white noise. Formally, this evolution should have Φ^4_d as its stationary measure, and in fact the idea of parabolic (or Parisi-Wu) stochastic quantization is to construct Φ^4_d as the stationary measure of (1).

The equation (1) is a nonlinear variant of the stochastic heat equation

$$\partial_t \varphi(t,x) = \Delta\varphi(t,x) + \xi(t,x). \tag{2}$$

As soon as $d \geq 2$, solutions of (2) are only distributions, and the same should be true for solutions of (1). However, this means that some renormalization is required to make sense of the term $\varphi(t,x)^3$ in (1). The approach we will take is to renormalize by replacing $\varphi(t,x)^3$ by the Wick power $:\varphi(t,x)^3:$.

To formalize this, consider a regularization ξ_δ of ξ (given by convolution with a suitable mollifier), and consider the SPDE

$$\begin{aligned}\partial_t \varphi_\delta(t,x) &= \Delta\varphi_\delta(t,x) - \,:\varphi_\delta(t,x)^3:\, + \xi_\delta(t,x)\\ &= \Delta\varphi_\delta(t,x) - \varphi_\delta(t,x)^3 + 3C_\delta(t)\varphi_\delta(t,x) + \xi_\delta(t,x),\end{aligned} \tag{3}$$

where C_δ is a suitable renormalization constant. It is clear that for each fixed δ there is a well-defined solution. The key result of the talk is that in dimension 2 and for short times one can pass to the limit $\delta \to 0$.

Theorem 1 ([2])**.** *Let $d = 2$. For any $\kappa > 0$ there is an almost surely positive random variable T such that the solutions of (3) on $[0,T] \times \mathbb{T}^2$ converge, as $\delta \to 0$, in the parabolic Hölder space $\mathcal{C}^{-\kappa}_{\mathfrak{s}}$ to a nontrivial limit φ.*

The key idea of the proof due to Da Prato and Debussche is that the most troublesome part of (3) comes from the solution of the stochastic heat equation. Its powers can be defined via Wick's theorem, and one can hope that the difference of the two solutions has better properties, and can be constructed by a standard fix-point argument.

Sketch of proof. Consider the solution $\uparrow_\delta$ of the regularized stochastic heat equation

$$\partial_t \uparrow_\delta(t,x) = \Delta \uparrow_\delta(t,x) + \xi_\delta(t,x)$$

and its Wick powers $\curlyvee_\delta = \uparrow_\delta^2 - C_\delta$ and $\Psi_\delta = \uparrow_\delta^3 - 3C_\delta \uparrow_\delta$. We make the ansatz $\varphi_\delta = \uparrow_\delta + v_\delta$. Then v_δ should solve

$$\partial_t v_\delta(t,x) = \Delta v_\delta(t,x) - v_\delta^3 - 3\uparrow_\delta v_\delta^2 - 3\curlyvee_\delta v_\delta - \Psi_\delta. \tag{4}$$

It turns out that the inhomogeneities on the right-hand side of (4) all take values in $\mathcal{C}_\mathfrak{s}^{-\kappa'}$ for any $\kappa' > 0$. This allows to construct a solution of (4) via a fix-point argument in $\mathcal{C}_\mathfrak{s}^{2-\kappa}$. The reason this is possible is that we can combine Young's theorem on products of distributions with the Schauder estimate for the heat equation. □

In the case that $d = 3$, one might be tempted to use the same method to construct solutions of (1). The Wick powers up to order 4 still exist, however their Hölder regularity is not good enough to close the fix-point argument to solve (4). One might try to address this with a more elaborate ansatz that includes more terms than just $\uparrow_\delta$, however it turns out that one cannot eliminate all the problematic terms. In fact, more elaborate methods like the theory of regularity structures are necessary to solve (1) in $d = 3$.

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Global solutions and coming down from infinity, I.

Simon Gabriel, Ruoyuan Liu

This session, based on the article [1] by Mourrat–Weber, concerns the well–posedness of the dynamic Φ^4_2 model

$$\partial_t X = \Delta X - X^3 + \xi\,, \qquad X(0,\cdot) = X_0\,, \tag{1}$$

globally in time on $\mathbb{R}^2$. Here ξ denotes a space–time white noise and X_0 lies in a suitable space of distributions.

In previous sessions, we saw that this SPDE is locally well–posed in time on compact tori, using the DaPrato–Debussche trick $X = Z + Y$, i.e. by expanding around the solution of the stochastic heat equation Z. This allowed to reduce the study of (1) to

$$\begin{aligned}\partial_t Y &= \Delta Y - (Y^3 + 3ZY^2 + 3Z^{(2)}Y + Z^{(3)}) \\ &=: \Delta Y - Y^3 + \Psi'(Y, Z, Z^{(2)}, Z^{(3)}) \,,\end{aligned} \tag{2}$$

with vanishing initial datum, where $Z^{(k)}$ are the (renormalised) Wick powers of Z. Likewise, we first present an argument that yields global in time well–posedness on a torus $\mathbb{T}^2_M := [-\frac{M}{2}, \frac{M}{2}]^2$, of arbitrary size M.

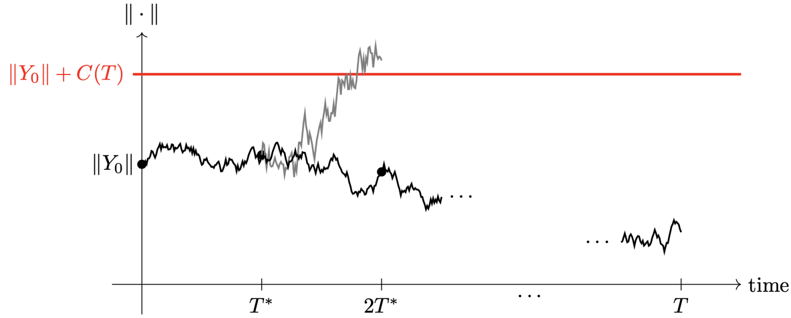

FIGURE 1. Given a time T and an initial condition Y_0, we have the a priori estimate with the constant $C(T)$ (the red line). Moreover, we find a solution up to time $T^*(\|Y_0\| + C(T))$, using the local well–posedness, which will lie below the red threshold. Hence, once more we find a solution on the interval $[T^*, 2T^*]$. Gluing together the two trajectories yields a solution on $[0, 2T^*]$, which by the a priori estimate must again lie below the red line. Thus, a grey trajectory as depicted above is not possible. Iterating this procedure until exhausting the interval $[0, T]$ yields unique solutions on arbitrary large intervals.

In order to convey the general idea of the argument, the following two ingredients are necessary: Considering a suitable norm $\|\cdot\|$ on the state space of solutions (which we shall fix below), we require **local in time well–posedness** of the type

$$\forall K > 0 \; \exists T^* > 0 \; \forall Y_0 \text{ with } \|Y_0\| \leq K \Rightarrow \exists! \text{ solution } (Y_t)_{t \in [0,T^*]} \,.$$

and an **a priori estimate:**

$$\forall T > 0 \; \exists C > 0 \; \forall T^* \leq T \Rightarrow \forall \text{ solutions } (Y_t)_{t\in[0,T^*]} : \sup_{t\in[0,T^*]} \|Y_t\| \leq \|Y_0\| + C \,.$$

Note that the local in time well–posedness result is slightly stronger than the one presented previously, because the random time T^* only depends on the upper bound K of $\|Y_0\|$. Now, having these two results at hand, the global in time well–posedness can be summarised pictorially, see Figure 1.

The underlying function spaces used in the well-posedness argument are Besov spaces defined by the norm

$$\|f\|_{B^{\alpha}_{p,q}} = \big\|\big(2^{\alpha k}\|\delta_k f\|_{L^p}\big)_{k\geq -1}\big\|_{\ell^q},$$

where δ_{-1} is a smooth frequency cutoff onto $\{|\zeta| \leq 1\}$ and δ_k for $k \geq 0$ is a smooth frequency cutoff onto $\{|\zeta| \sim 2^k\}$. The Besov spaces enjoy the following useful properties:

(1) **Embeddings:** For $1 \leq p, q_1, q_2 \leq \infty$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $q_1 \geq q_2$ and $\alpha_1 \leq \alpha_2$, we have

$$\|f\|_{B^{\alpha_1}_{p,q_1}} \leq \|f\|_{B^{\alpha_2}_{p,q_2}}.$$

For $1 \leq p, q, r \leq \infty$ and $\alpha, \beta \in \mathbb{R}$ with $p \geq r$ and $\beta = \alpha + d(\frac{1}{r} - \frac{1}{p})$, we have

$$\|f\|_{B^{\alpha}_{p,q}} \leq C\|f\|_{B^{\beta}_{r,q}}.$$

(2) **Interpolation:** For $1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty$, $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$, and $\theta \in [0,1]$ satisfying $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$, and $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$, we have

$$\|f\|_{B^{\alpha}_{p,q}} \leq C\|f\|^{\theta}_{B^{\alpha_1}_{p_1,q_1}}\|f\|^{1-\theta}_{B^{\alpha_2}_{p_2,q_2}}.$$

(3) **Multiplicative inequalities:** For $1 \leq p, p_1, p_2, q \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\alpha > 0$, we have

$$\|fg\|_{B^{\alpha}_{p,q}} \leq C\|f\|_{B^{\alpha}_{p_1,q}}\|g\|_{B^{\alpha}_{p_2,q}}.$$

For $1 \leq p, p_1, p_2 \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 \leq q \leq \infty$, and $\beta < 0 < \alpha$ with $\alpha + \beta > 0$, we have

$$\|fg\|_{B^{\beta}_{p,q}} \leq C\|f\|_{B^{\beta}_{p_1,q}}\|g\|_{B^{\alpha}_{p_2,q}}.$$

(4) **Duality:** For $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1$ and $\alpha \in \mathbb{R}$, we have

$$|(f,g)| \leq C\|f\|_{B^{\alpha}_{p_1,q_1}}\|g\|_{B^{-\alpha}_{p_2,q_2}}.$$

(5) **Smoothing of the heat flow:** For $1 \leq p, q \leq \infty$, $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq \beta$, $t > 0$, and f supported on an annulus, we have

$$\|e^{t\Delta}f\|_{B^{\alpha}_{p,q}} \leq Ct^{\frac{\beta-\alpha}{2}}\|f\|_{B^{\beta}_{p,q}}.$$

We shall now focus on the a priori estimate only. Using a classical Schauder estimate, one can guess that mild solutions $(Y_t)_{t\in[0,T^*]}$ of (2) take values in $B^{2-}_{p,q}$. Indeed, Y_t will be function–valued and it suffices to consider $\|\cdot\| := \|\cdot\|_{L^p}$, the L^p norm on the torus with periodic boundary condition. The a priori estimate then requires control of the difference

$$\frac{1}{p}\left(\|Y_t\|^p_{L^p} - \|Y_0\|^p_{L^p}\right) = \int_0^t (\Psi'_s, Y_s^{p-1}) - \big((p-1)(|\nabla Y_s|^2, Y_s^{p-2}) + \|Y_s^{p+2}\|_{L^1}\big)\,\mathrm{d}s\,,$$

where p is an even, large natural number. Here, we conveniently expressed the difference in terms of an L^p–energy identity, derived by testing a mild solution Y_t against Y_t^{p-1}, cf. [1, Proposition 6.8].

The a priori estimate is an immediate consequence, once we show integrability of the right–hand side of the above energy identity. To this end, we analyse each summand in Ψ'_s separately. As an example, we use duality, interpolation and Young's inequalities to obtain

$$\begin{aligned} |(Z_s Y_s^2, Y_s^{p-1})| &\leq C\|Y_s^{p+1}\|_{B^{\varepsilon}_{1,1}}\|Z_s\|_{B^{-\varepsilon}_{\infty,\infty}} \\ &\leq C\left(\|Y_s^{p+1}\|_{L^1}^{1-\varepsilon}\|Y_s^p\nabla Y_s\|_{L^1}^{\varepsilon} + \|Y_s^{p+1}\|_{L^1}\right)\|Z_s\|_{B^{-\varepsilon}_{\infty,\infty}} \\ &\leq c\left((p-1)(|\nabla Y_s|^2, Y_s^{p-2}) + \|Y_s^{p+2}\|_{L^1}\right) + f(s)\,, \end{aligned} \tag{3}$$

for some integrable function f and $c < 1$ small enough. Equivalently, we find bounds for the other terms in (Ψ'_s, Y_s^{p-2}) such that the sum of all such c's lies in $(0,1)$. Lastly, exploiting the two negative terms in the integrand of the energy identity together with estimates of the form (3), we conclude integrability of the bound for the a priori estimate. Here we shall stress the importance of the term $-\|Y_s^{p+2}\|_{L^1}$ in the energy identity, which is due to the negative sign of the non–linearity $-X^3$ in (1).

On the other hand, global well-posedness of (1) on the whole plane $\mathbb{R}^2$ can be proved in three main steps. Firstly, one shows global well-posedness of (1) on the large torus $\mathbb{T}^2_M$, where we d enote the global solution by Y_M. Secondly, one establishes a priori estimates for Y_M in (weighted) Besov spaces, uniformly in M. By using compact embeddings of weighted Besov spaces, one can extract a converging subsequence of $\{Y_M\}_{M\in\mathbb{N}}$ as $M \to \infty$. Lastly, one proves uniqueness of the solution Y to the equation on the whole plane, which then shows the convergence of the whole sequence $\{Y_M\}_{M\in\mathbb{N}}$.

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Global solutions and coming down from infinity, II.

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We discuss the "coming down from infinity" property for the Φ^4_2 model on $\mathbb{R}^2$ based on [1]. In other words, we establish a priori estmates for the global-in-time solution of Φ^4_2 in suitable weighted Hölder spaces, uniformly over the initial data. This problem is strongly related to the construction of Φ^4 measure since global-in-time bounds for the solution can be applied to the construction.

In order to prove the global bounds, we introduce localization operators to decompose the solution into singular and regular parts. In particular, the regular part can be controlled by the help of a minus cubic term of Φ^4 model. We then show the uniform bounds on solutions to regularised equation driven by a regularised white noise, with renomalization constants which diverges as regularising parameter $\epsilon \to 0$. Using compactness argument, we shall prove the existence of a solution and its uniform bounds.

In the end, we use a further time weight to get the coming down from infinity property. One of key ideas is that the time weight is zero at the initial time so that it removes in some sense the dependency of the time from estimates. However, we should modify the tools such as the Schauder estimate to be fit with the time weights.

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Tightness via the energy method for the φ_2^4 measure

Azam Jahandideh

Formally the measure of the dynamical φ_2^4 model on $\mathcal{S}'(\mathbb{R}^2)$ is given by

$$\nu(d\varphi) = \frac{1}{\mathcal{Z}} \exp\Big[- \int_{\mathbb{R}^2} (\frac{1}{4}\lambda\varphi^4 - \frac{3}{2}\lambda\infty\varphi^2)\Big]\mu(d\varphi)\,, \tag{1}$$

where $\lambda > 0$ is the coupling constant, $\mathcal{Z} \geq 1$ is normalization factor, $\mu(d\varphi)$ is the Gaussian measure with covariance $(m^2 - \Delta)^{-1}$ and Δ is the Laplacian on $\mathbb{R}^2$. The corresponding semilinear parabolic partial differential equation to the measure $\nu(d\varphi)$ according to the Langevin dynamics is given by

$$\begin{cases} (\partial_t - \Delta + m^2)\,\Phi(t,x) = -\lambda\Phi^3(t,x) + 3\lambda\infty\Phi + \xi(t,x) \\ \Phi(0,x) = \varphi(x)\,, \end{cases} \tag{2}$$

where $\xi(t,x)$ is the unique space-time Gaussian white noise on $\mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^2)$. The above SPDE has the φ_2^4 measure, i.e., $\nu(d\varphi)$ as its invariant measure. This implies that if Φ is a solution to Eq. (2) with the initial condition $\Phi(0,\cdot) = \varphi(\cdot)$ distributed according to the measure $\nu(d\varphi)$, then for all $t \in \mathbb{R}$ the random field $\Phi(t,\cdot)$ is also distributed according to this measure. Consequently, one has $\mathrm{Law}(\Phi(t,\cdot)) = \mathrm{Law}(\Phi(0,\cdot)) = \nu(d\varphi)$.

By the parabolic scaling, the sample paths of ξ belong almost surely to the space of regularity $-2-\kappa$ for all $\kappa > 0$. The heat kernel is 2 regularizing, which implies that the solution to SPDE (2) has regularity $-\kappa$. Hence, we expect the regularity of the renormalized non-linear term to be $-\kappa$ for all $\kappa > 0$.

Observe that the measure of the dynamical φ_2^4 model as given in (1) is ill-defined. Firstly, a typical field φ in the support of the Gaussian measure $\mu(d\varphi)$ lacks integrability, i.e., it does not decay at infinity. Secondly, such a field does not have enough regularity as oftentimes it is a distribution. Consequently, the non-linear term, in Eq. (2) is ill-defined. These two problems are known as IR and UV problems, respectively. To get around these problems, we first introduce the discrete family of measures $(\nu_{M,\epsilon})_{M,\epsilon}$ corresponding to the φ_2^4 model on lattice $\Lambda_{M,\epsilon} = (\epsilon\mathbb{Z})^2 \cap [-M/2, M/2)^2$, where ϵ and M play the role of UV and IR cut-offs respectively.

The lattice approximation of the φ_2^4 measure

Let $\Lambda_{M,\epsilon}$ be a periodic lattice with mesh size ϵ and size length M. Consider the scalar filed $\phi : \Lambda_{M,\epsilon} \to \mathbb{R}$. The corresponding regularized family of measures to the φ_2^4 model on $\mathbb{R}^{\Lambda_{M,\epsilon}}$ is given by

$$\nu_{M,\epsilon}(d\varphi) := \frac{1}{\mathcal{Z}_{M,\epsilon}} \exp\Big[-2\epsilon^2 \sum_{x\in\Lambda_{M,\epsilon}} \mathcal{U}_{M,\epsilon}(\varphi)\Big] \mu_{M,\epsilon}(d\varphi)\,, \tag{3}$$

where $\mathcal{Z}_{M,\epsilon}$ is normalization factor, $\mathcal{U}_{M,\epsilon}(\varphi) = \frac{\lambda}{4}|\varphi|^4 - \frac{3}{2}\,\lambda\, a_{M,\epsilon}\,|\varphi|^2 + \frac{3}{4}a_{M,\epsilon}^2$, $\mu_{M,\epsilon}(d\varphi)$ is the discrete Gaussian measure with mean zero and covariance $K^{M,\epsilon} := (m^2 - \Delta_\epsilon)^{-1}$, Δ_ϵ is the discrete Laplacian, and $a_{M,\epsilon} = Tr(K^{M,\epsilon}) = (m^2 - \Delta_\epsilon)^{-1}(x,x)$, which diverges like $\log(\epsilon^{-1})$. Observe that $a_{M,\epsilon}$ is independent of $t \in \mathbb{R}$ as we deal with the stationary solutions.

Remark 1. *For fixed ϵ, M the measure $\nu_{M,\epsilon}(d\varphi)$ is well-defined.*

Definition 2. *A φ_2^4 measure is any non-Gaussian, Euclidean invariant and reflection positive accumulation point of the family of regularized measures $\nu_{M,\epsilon}(d\varphi)$ as $\epsilon \to 0$ and $M \to \infty$, where $\mathcal{U}_{M,\epsilon}(\varphi)$ is any 4-th order polynomial, which is bounded from below with ϵ, M dependent coefficient* [Gub21].

Discrete stochastic quantization equation

Utilizing the parabolic stochastic quantization, we obtain the discrete stochastic quantization equation corresponding to the measure $\nu_{M,\epsilon}(d\varphi)$ on $\mathcal{S}'(\mathbb{R}\times\Lambda_{M,\epsilon})$ as follows

$$(\partial_t + m^2 - \Delta_\epsilon)\Phi_{M,\epsilon}(t,x) = -\lambda\Phi_{M,\epsilon}^3(t,x) + 3\,\lambda\, a_{M,\epsilon}\,\Phi_{M,\epsilon} + \xi_{M,\epsilon}(t,x) \tag{4}$$

such that $\mathrm{Law}(\Phi_{M,\epsilon})(t,\cdot) = \mathrm{Law}(\Phi_{M,\epsilon})(0,\cdot) = \nu_{M,\epsilon}(d\varphi)$ for all $t\in[0,\infty)$ and $\xi_{M,\epsilon}$ is the discrete space-time Gaussian white noise defined on $\mathbb{R}\times\Lambda_{M,\epsilon}$, which is of regularity $-2-\kappa$ for all $\kappa > 0$.

Our aim is to show the existence of the infinite volume measure associated to the φ_2^4 model using tightness of the family of the regularized Gibbs measures $\nu_{M,\epsilon}(d\varphi)$ defined on $\mathcal{S}'(\Lambda_{M,\epsilon})$. To this end, we shall utilize the energy method in $L^2(\Lambda_{M,\epsilon})$. Note that we cannot apply the energy method directly to the Eq. (4), since as $\epsilon \to 0$ it becomes singular. That is why we need to come up with an appropriate decomposition of the random field $\Phi_{M,\epsilon}$.

Decompose the solution into singular and regular parts

Using the Da Prato and Debussche decomposition [DD03], one writes $\Phi_{M,\epsilon} = X_{M,\epsilon} + \eta_{M,\epsilon}$ with $X_{M,\epsilon}$ solving

$$(\partial_t + m^2 - \Delta_\epsilon)X_{M,\epsilon}(t,x) = \xi_{M,\epsilon}(t,x)\,. \tag{5}$$

Set $X_{M,\epsilon}^{:2:} := X_{M,\epsilon}^2 - a_{M,\epsilon}$ and $X_{M,\epsilon}^{:3:} := X_{M,\epsilon}^3 - 3\,a_{M,\epsilon}\,X_{M,\epsilon}$, where $a_{M,\epsilon}$ is chosen in a such way that for all $\kappa > 0$ the stochastic objects $X_{M,\epsilon}$, $X_{M,\epsilon}^{:2}$ and $X_{M,\epsilon}^{:3}$ can be almost surely bounded in some Besov space of regularity $-\kappa$ for all $\kappa > 0$. Let ρ denote a polynomial weight of the form $\rho(x) = \langle hx\rangle^{-\nu} = (1+|hx|^2)^{-\nu/2}$, where

$\nu \geq 0$ and $h > 0$, $B^{\alpha,\epsilon}_{p,q}(\rho)$ denote the discrete weighted Besov spaces on $\Lambda_{M,\epsilon}$ and $\mathbb{X} = \{X_{M,\epsilon}, X^{:2:}_{M,\epsilon}, X^{:3:}_{M,\epsilon}\}$.

Proposition 3. *Let $p \in [2,\infty)$, $\kappa > 0$ and $a_{M,\epsilon} = \mathbb{E}\Big[X_{M,\epsilon}(t,x)^2\Big]$. There exists $C > 0$ such that for all ϵ, M and $t \in \mathbb{R}$ it holds $\mathbb{E}\Big[\|\rho\, \mathbb{X}(t,\cdot)\|^p_{B^{-\kappa,\epsilon}_{p,p}}\Big] \leq C$.*

Proof. The proof follows from the Kolmogorov type estimate and hypercontractivity estimate with the use of the discrete semigroup property. Similar bounds can be found in [GH18, MWX16]. □

For later use let

$$\mathcal{Q}^{M,\epsilon}(\mathbb{X})(t) = \|\rho X_{M,\epsilon}(t,\cdot)\|^8_{B^{-\kappa,\epsilon}_{8,8}} + \|\rho\, X^{:2:}_{M,\epsilon}(t,\cdot)\|^4_{B^{-\kappa,\epsilon}_{4,4}} + \|\rho\, X^{:3:}_{M,\epsilon}(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}}\,.$$

Observe that by Prop. 3 one has $\mathbb{E}[\mathcal{Q}^{M,\epsilon}(\mathbb{X})(t)] \leq C$, where $C \in (0,\infty)$ is some constant independent of ϵ and M.

Application of the energy method

In this section we aim to show that for all $\kappa > 0$ there exists $C > 0$ such that for all ϵ, M and $t \in \mathbb{R}$ it holds

$$\mathbb{E}\Big[\|\eta_{M,\epsilon}(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}(\rho)}\Big] \leq C\,. \tag{6}$$

The remainder $\eta_{M,\epsilon}(t,x)$ solves

$$\begin{aligned}(\partial_t + m^2 - \Delta_\epsilon)\eta_{M,\epsilon}(t,x) = -\lambda\Big[&\eta^3_{M,\epsilon}(t,x) + 3\,\eta_{M,\epsilon}(t,x)\, X^{:2:}_{M,\epsilon}(t,x) \\ &+ 3\,\eta^2_{M,\epsilon}(t,x)\, X_{M,\epsilon}(t,x) + X^{:3:}_{M,\epsilon}(t,x)\Big]\,.\end{aligned} \tag{7}$$

Observe that in the limit as $\epsilon \to 0$ all the product terms in Eq. (7) are well-defined as the sums of their regularities are positive. To obtain the uniform bound (6), we shall apply the energy method to Eq. (7) in $L^2(\Lambda_{M,\epsilon})$. To this end, we multiply both sides of Eq. (7) by $\rho(x)^4\, \eta_{M,\epsilon}(t,x)$ and perform the sum over $\Lambda_{M,\epsilon}$.

Proposition 4. *There exist $\kappa \in (0,\infty)$, $\delta \in (0,1)$ sufficiently small, $C \in (0,\infty)$, $p \in [2,\infty)$, an appropriate polynomial weight ρ such that for all ϵ and M it holds*

$$\begin{aligned}&\frac{1}{2}\partial_t \|\rho^2\, \eta_{M,\epsilon}(t,.)\|^2_{L^{2,\epsilon}} \\ &\quad + \lambda(1-\delta)\|\rho\, \eta_{M,\epsilon}(t,.)\|^4_{L^{4,\epsilon}} + (m^2 - C_\delta\, C^2_\rho)\|\rho^2\, \eta_{M,\epsilon}(t,\cdot)\|^2_{L^{2,\epsilon}} \\ &\quad\quad + (1-\delta-\lambda\,\delta)\|\rho^2\, \nabla_\epsilon \eta_{M,\epsilon}(t,\cdot)\|^2_{L^{2,\epsilon}} \leq \lambda\, C\, \mathcal{Q}^{M,\epsilon}(\mathbb{X})(t)\,.\end{aligned} \tag{8}$$

Proof. The proof is an application of the energy method in $L^2(\Lambda_{M,\epsilon})$ as outlined above, integration by part, the discrete Leibniz rule, Hölder's and Young's inequalities. To conclude one uses Lemma 5 with $n = 3$ for $X = X_{M,\epsilon}$, $n = 2$ for $X = X^{:2:}_{M,\epsilon}$, $n = 1$ for $X = X^{:3:}_{M,\epsilon}$. This finishes the proof. □

Lemma 5. *Let $n \in \{1,2,3\}$, $\delta \in (0,1)$ and $\kappa > 0$. There exists $C_\delta \in (0,\infty)$ such that for $p \in [2,\infty)$ it holds*

$$\left|\langle \rho^4 \, \eta^n_{M,\epsilon}, X\rangle_{L^{2,\epsilon}}\right| \leq \delta \|\rho^2 \nabla_\epsilon \, \eta_{M,\epsilon}\|^2_{L^{2,\epsilon}} + \delta \|\rho \, \eta_{M,\epsilon}\|^4_{L^{4,\epsilon}} + C_\delta \|\rho \, X\|^p_{B^{-\kappa,\epsilon}_{p,p}} \, .$$

Proposition 6. *Let $\kappa \in (0,\infty)$. There exists a constant $C \in (0,\infty)$ such that for all ϵ, M, $\lambda > 0$ and all $t \in \mathbb{R}$ it holds*

$$\mathbb{E}\Big[\|\eta_{M,\epsilon}(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}(\rho^2)}\Big] \leq C \, .$$

Proof. By Prop. 4 combined with the fact that $\Phi_{M,\epsilon}$, $X_{M,\epsilon}$ and $\eta_{M,\epsilon}$ are jointly stationary one has

$$\mathbb{E}\Big[\|\eta_{M,\epsilon}(t,.)\|^2_{L^{2,\epsilon}(\rho^2)}\Big] \leq C \, \mathbb{E} \mathcal{Q}^{M,\epsilon}(\mathbb{X})(t) \, , \tag{9}$$

where we used the fact that for suitably chosen ρ and δ, all the coefficients in the LHS of Eq. (8) are positive. To conclude, recall that $\mathbb{E}[\mathcal{Q}^{M,\epsilon}(\mathbb{X})(t)] \leq C$, and $\|\cdot\|_{B^{-\kappa,\epsilon}_{2,2}(\rho^2)} \lesssim \|\cdot\|_{B^{0,\epsilon}_{2,\infty}(\rho^2)} \lesssim \|\cdot\|_{L^{2,\epsilon}(\rho^2)}$. This finishes the proof. □

Tightness

In this section we aim to verify that the family of measures $(\nu_{M,\epsilon})_{M,\epsilon}$ is tight, i.e., it is sequentially compact in the topology of weak convergence of measures. Specifically, we want to prove the following.

Theorem 7 (Main Result)**.** *Let $k \in (0,\infty)$. There exists a choice of the sequence $(a_{M,\epsilon})_{M,\epsilon}$ such that the family of measures $(\nu_{M,\epsilon})_{M,\epsilon}$ appropriately extended to $\mathcal{S}'(\mathbb{R}^2)$ is tight. In particular, for every accumulation point ν it holds*

$$\int \|\varphi\|^2_{B^{-3\kappa}_{2,2}(\rho^{2+\kappa})} \, \nu(d\phi) < \infty. \tag{10}$$

Proof. Using Prop. 3 with $\mathbb{X} = X_{M,\epsilon}$ for $p = 2$, Prop. 6 and the triangle inequality one has

$$\mathbb{E}\Big[\|(\Phi_{M,\epsilon})(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}(\rho^2)}\Big] \leq C \tag{11}$$

for some constant $C \in (0,\infty)$ uniformly in ϵ and M and for any $\kappa > 0$. To go from $\mathcal{S}'(\Lambda_{M,\epsilon})$ to $\mathcal{S}'(\mathbb{R}^2)$, one utilizes the extension operator $\mathcal{E}^\epsilon$, which is bounded uniformly in ϵ [GH21, Lemma A.15]. Hence,

$$\mathbb{E}\Big[\|\mathcal{E}^\epsilon \Phi_{M,\epsilon}(t,\cdot)\|^2_{B^{-\kappa}_{2,2}(\rho^2)}\Big] \leq \mathbb{E}\Big[\|(\Phi_{M,\epsilon})(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}(\rho^2)}\Big] \leq C \, . \tag{12}$$

Note that up to a subsequence one can pass to the limits as $\epsilon \to 0$ and $M \to \infty$. Evoking the fact that $\mathrm{Law}(\Phi_{M,\epsilon})(t,\cdot) = \nu_{M,\epsilon}$ for all $t \in [0,\infty)$ yields

$$\int \|\varphi\|^2_{B^{-\kappa}_{2,2}(\rho^2)} \, \nu(d\phi) = \lim_{\substack{\epsilon\to 0\\ M\to\infty}} \mathbb{E}\Big[\|\mathcal{E}^\epsilon \Phi_{M,\epsilon}(t,.)\|^2_{B^{-\kappa}_{2,2}(\rho^2)}\Big] \leq C \tag{13}$$

uniformly in ϵ and M. Note that $B^{-\kappa}_{2,2}(\rho^2) \hookrightarrow B^{-2\kappa}_{2+2\kappa,2}(\rho^{2+\kappa})$ continuously and $B^{-2\kappa}_{2+2\kappa,2}(\rho^{2+\kappa}) \hookrightarrow B^{-3\kappa}_{2,2}(\rho^{2+\kappa})$ compactly. It holds

$$\int \|\varphi\|^2_{B^{-3\kappa}_{2,2}(\rho^{2+\kappa})}\, \nu(d\phi) < \infty\,. \tag{14}$$

Use Prokhorov's theorem to infer the tightness. This concludes the proof. □

Let $\mathcal{E}^\epsilon : B^{\alpha,\epsilon}_{p,q}(\rho) \to B^{\alpha}_{p,q}(\rho)$. By $\mathcal{E}^\epsilon \sharp \nu_{M,\epsilon}$ we indicate the measure on $\mathcal{S}'(\mathbb{R}^2)$ obtained from the measure $\nu_{M,\epsilon}$ on $\mathcal{S}'(\Lambda_{M,\epsilon})$.

Reflection Positivity

Let F be some cylindrical function on $\mathcal{S}'(\mathbb{R}^2)$ depending on φ's which are supported in $\{(x_1,x_2) \in \mathbb{R}^2\ ;\ x_1 > 0\}$. We denote the algebra of all such functionals by $\mathcal{F}_+$. Let $\varphi \in \mathcal{S}'(\mathbb{R}^2)$ and $f \in \mathcal{C}^\infty(\mathbb{R}^2)$. We set $\langle \Theta\varphi, f\rangle := \langle \varphi, \Theta f\rangle$ and $(\Theta f)(x_1,x_2) = f(-x_1,x_2)$ for all $f \in \mathcal{C}^\infty(\mathbb{R}^2)$.

Proposition 8. *Let ν be a weak limit of a subsequence of the sequence of measures $(\mathcal{E}^\epsilon \sharp \nu_{M,\epsilon})_{M,\epsilon}$ on $\mathcal{S}'(\mathbb{R}^2)$. For all $F \in \mathcal{F}_+$ it holds $\int \overline{F(\Theta\varphi)} F(\varphi)\, \nu(d\varphi) \geq 0$.*

The preceding proposition implies the reflection positivity axiom in [OS75]. To prove it, one can start off by verifying an analogous property on $\Lambda_{M,\epsilon}$. Then, use the extension operator $\mathcal{E}^\epsilon$ and take the limits $M \to \infty$, $\epsilon \to 0$. It is believed that one needs to start from finite volume lattice measures, as the only concrete way, to prove the reflection positivity axiom for the infinite volume measure [GH21].

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Integrability of Φ^4_2

Lucas Broux, Wenhao Zhao

This talk is a survey of the article [5] by Martin Hairer and Rhys Steele. In the talk we show that the Φ^4_2 measure[1] on the 2-dimensional torus $\mathbb{T}^2_M := (\mathbb{R}/M\mathbb{Z})^2$ of length M admits quartic exponential tails, as expected from its formal expression

$$\mu \sim \exp\Big(- \int_{\mathbb{T}^2} \Big(\frac{1}{2}|\nabla\Phi|^2 + \frac{1}{4}\Phi^4\Big)\,\mathrm{d}x\Big) \prod_{x\in\mathbb{T}^2} \mathrm{d}\Phi(x), \qquad \text{over } \Phi \in \mathcal{S}'(\mathbb{T}^2_M).$$

Even though this expression is purely formal, it is known since the 1970's that μ can actually be rigorously constructed via a suitable procedure of regularization and renormalization. Now, the main theorem of [5] reads as follows.

Theorem 1 ([5, Theorem 1.1])**.** *For any $\psi \in C^\infty_c(\mathbb{R}^2)$, $M > 0$ large enough to accomodate the support of ψ, and $\beta > 0$ small enough depending only on ψ,*

$$\mathbb{E}_{\Phi\sim\mu}\Big[\exp\big(\beta\langle\Phi,\psi\rangle^4\big)\Big] < \infty.$$

Let us quickly comment on this result:

(1) Such a bound was already known in the QFT literature, although with different methods [2]. The novelty of [5] is to establish this result in the three-dimensional case of the Φ^4_3 measure.
(2) This implies that the Φ^4_2 measure is not gaussian, since no gaussian measure satisfies such a quartic exponential integrability estimate.
(3) The same method would in principle apply to other quartic functionals of Φ, for instance also $\mathbb{E}_{\Phi\sim\mu}\big[\exp\big(\beta|\Phi|^4_{-\kappa}\big)\big] < \infty$ would hold for any $\kappa > 0$, where $|\Phi|_{-\kappa}$ denotes the (homogeneous) Sobolev norm.
(4) In the context of QFT, such estimates are used for estabishing the regularity axiom of Osterwalder–Schrader.
(5) The bound is independent of the size of the torus, which may give some result for the Φ^4 measure on the full space.

In the remainder of this extended abstract, we wish to sketch some ideas of the proof.

A first attempt by stochastic quantization of Φ^4_2. A first idea is to argue by *stochastic quantization*, namely to realize μ as an invariant measure to the SPDE

$$(\Phi^4_2) \qquad \partial_t\Phi = \Delta\Phi - \Phi^3 + \infty\,\Phi + \xi, \qquad (t \in \mathbb{R}_+, x \in \mathbb{T}^2),$$

where ξ denotes space-time white noise. Rigorously, one takes mollifiers $(\rho_\epsilon)_{\epsilon>0} \subset C^\infty_c(\mathbb{R}^2)$ and considers rather the equation with smooth noise

$$\partial_t\Phi_\epsilon = \Delta\Phi_\epsilon - \Phi^3_\epsilon + c_\epsilon\,\Phi_\epsilon + \xi * \rho_\epsilon.$$

As is by now well-known, by suitably choosing the diverging sequence $(c_\epsilon)_{\epsilon>0}$ of deterministic constants, one can make $(\Phi_\epsilon)_{\epsilon>0}$ converge (in probability in a suitable Hölder space of distributions) as $\epsilon \to 0$ to a random distribution Φ which

[1]in fact the article presents the (more difficult) case of the Φ^4_3 measure

does not depend on the choice of $(\rho_\epsilon)_{\epsilon>0}$, and which admits μ as a unique invariant measure. See e.g. [1, 7] for the two-dimensional case of Φ^4_2. One crucial point is that Φ_ϵ is controlled by a *finite* number of stochastic objects, given for Φ^4_2 by the random distributions

$$Z_\epsilon \stackrel{\text{def}}{=} (\partial_t - \Delta)^{-1}\xi * \rho_\epsilon, \qquad {:}Z_\epsilon^2{:} \stackrel{\text{def}}{=} Z_\epsilon^2 - c_\epsilon, \qquad {:}Z_\epsilon^3{:} \stackrel{\text{def}}{=} Z_\epsilon^3 - 3c_\epsilon Z_\epsilon.$$

One may now use PDE techniques to deduce properties of the measure μ. For instance, by combining (amongst other) techniques of Schauder theory and a maximum principle for the damped heat operator $u \mapsto \partial_t u - \Delta u + u^3$, the authors of [6] were able to obtain *a priori estimates* of the form: for all $N > 0$, $R \in (0,1)$, $\kappa > 0$ small,

$$\begin{aligned}&\sup_{\substack{t\in(R^2,1)\\ x\in[-N+R,R-N]^2}} |(\Phi - Z)(t,x)| \\ &\le C \max\Big(R^{-1}, \limsup_{\epsilon\to 0} \|Z_\epsilon\|_{C^{-\kappa}}^{\frac{2}{1-\kappa}}, \limsup_{\epsilon\to 0} \|{:}Z_\epsilon^2{:}\|_{C^{-2\kappa}}^{\frac{2}{2-2\kappa}}, \limsup_{\epsilon\to 0} \|{:}Z_\epsilon^3{:}\|_{C^{-3\kappa}}^{\frac{2}{3-3\kappa}}\Big),\end{aligned}$$

where the Hölder norms are over $t \in (0,1), x \in [-N,N]^2$. Let us denote by Y the right-hand side of this display. It is possible to bound the stochastic objects appearing in Y and obtain $\mathbb{E}\big[\exp(\beta Y^{1-\kappa})\big] < \infty$ for any $\kappa > 0$ and $\beta > 0$ small enough. Then, starting Φ at time $t = 0$ according to its invariant measure μ, at later times $\Phi(t,\cdot)$ is still distributed according to μ and it is straigtforward to deduce from the above the stretched-exponential estimate

$$\mathbb{E}_{\Phi\sim\mu}\Big[\exp\big(\beta\langle\Phi,\psi\rangle^{1-\kappa}\big)\Big] < \infty.$$

Unfortunately, this approach only yields the exponent $1-\kappa$ rather than the desired 4.

The Hairer–Steele argument. The idea at this point is to focus instead on the *tilted measure*

$$\mathrm{d}\nu := \exp\big(\beta\langle\Phi,\psi\rangle^4\big)\,\mathrm{d}\mu,$$

so that the theorem reduces to proving that ν is a finite measure. One naturally argues by stochastic quantization on ν: Formally, it should be invariant for

$$(\Psi^4_2) \qquad \partial_t\Psi = \Delta\Psi - \Psi^3 + \infty\,\Psi + \beta\langle\Psi,\psi\rangle^3\psi + \xi,$$

which can be seen as a perturbation of the Φ^4_2 equation. In particular, when $\beta > 0$ is small enough, the contribution of $\beta\langle\Psi,\psi\rangle^3\psi$ should be absorbed in that of the damping term $-\Psi^3$, which motivates that the same a priori estimates as for Φ should hold.

In fact, it is convenient to rather work with a sequence $(\nu_n)_n$ of probability measures where the fourth power is replaced by a bounded approximation:

$$\mathrm{d}\nu_n := \mathcal{Z}_n^{-1}\exp\big(\beta F_n(\langle\Phi,\psi\rangle)\big)\,\mathrm{d}\mu,$$

for some smooth $F_n\colon \mathbb{R} \to \mathbb{R}$ with

$$F_n(x) = \begin{cases} \frac{x^4}{4}, & |x| \le n \\ \frac{n^4}{4} + 1, & |x| \ge n+1 \end{cases}, \qquad 0 \le F_n' \le n^3,$$

and where $\mathcal{Z}_n = \mathbb{E}_{\Phi\sim\mu}\big[\exp\big(\beta F_n(\langle\Phi,\psi\rangle)\big)\big]$ denotes the corresponding normalization. By Fatou's lemma,

$$\mathbb{E}_{\Phi\sim\mu}\big[\exp\big(\beta\langle\Phi,\psi\rangle^4\big)\big] \le \liminf_{n\to\infty} \mathcal{Z}_n,$$

so that the theorem follows once one establishes the boundedness of $(\mathcal{Z}_n)_n$. The article [5] proceeds to prove the following properties:

(1) The measure ν_n is invariant for the SPDE

$$\partial_t \Psi^{(n)} = \Delta\Psi^{(n)} - (\Psi^{(n)})^3 + \infty\,\Psi^{(n)} + \beta\langle\Psi^{(n)},\psi\rangle^3\psi + \xi. \tag{$\Psi^{4,n}_2$}$$

(2) The following a priori estimate holds for all $N > 0$, $R \in (0,1)$, and $\beta > 0$ small enough:

$$\sup_{\substack{t\in(R^2,1)\\ x\in[-N+R,R-N]^2}} |(\Psi^{(n)} - Z)(t,x)| \le Y,$$

where Y is the same right-hand side as in the a priori estimate of Φ above.

The proof of property (1) follows from a discretisation argument, which is also used to prove that the Φ^4_3 measure is an invariant measure of the corresponding SPDE in [3]. At this point it is convenient to work with a bounded density, which is a reason to introduce ν_n rather than to work with ν. The exponential mixing property proved in [4] is used in the argument to prove the convergence of the discretised measure.

As for property (2), it follows along the same argument as in [6]. Note that the bound is independent of the size of the torus, which is the key for the independence in the torus size of the tail bound for the measure.

We may conclude from there. Starting $\Psi^{(n)}$ from its invariant measure ν_n, and appealing to the a priori estimate (2) and the fact that the stochastic objects in Y are almost surely finite, we deduce that for some $K > 0$, denoting B_K the centered ball of radius K in the Hölder space $C^{-\kappa}$,

$$\frac{1}{2} \le \mathbb{P}\big[\|\Psi^{(n)}\|_{C^{-\kappa}} \le K\big] = \nu_n(B_K) = \mathcal{Z}_n^{-1}\int_{B_K} \exp\big(\beta F_n(\langle\Phi,\psi\rangle)\big)\,\mathrm{d}\mu.$$

But for $\Phi \in B_K$ one bounds $F_n(\langle\Phi,\psi\rangle) \le CK^4$ for some constant C uniform in n, yielding the desired uniform bound $\mathcal{Z}_n \le 2\exp(\beta CK^4)$, and concluding the proof of the theorem.

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More Applications of the Da Prato-Debussche Argument

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Besides the effectiveness of the Da Prato-Debussche trick for the stochastic quantization equations, it can also be adapted to several other singular SPDEs to derive local existence and uniqueness. The first example we discuss is the parabolic sine-Gordon model in the range $0 < \beta^2 < 4\pi$ following [1], where the regularity of imaginary Gaussian multiplicative chaos is assumed. Another variation of the Da Prato-Debussche trick is the exponential Ansatz initiated in [2], where it is applied to prove local existence and uniqueness of the parabolic Anderson model (PAM) on $\mathbb{R}^2$ in a relatively simple way. The exponential Ansatz is then further modified in [3] for the simple construction of the Φ^4_3 model on 3d torus.

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Convergence of the Two-Dimensional Dynamic Ising-Kac Model to Φ^4_2

Benoit Dagallier and Markus Tempelmayr

The Ising model with Kac interactions is a model of magnetism on a lattice, where elementary components of magnetism called spins interact in a way made precise below. Let $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^2$ denote the two-dimensional discrete torus with linear size $N \geq 1$. The Ising model is a measure on spin configurations, i.e. elements $\sigma \in \{-1, 1\}^{\Lambda_N}$, defined for $\gamma \in (0,1)$ and $\beta \geq 0$ by:

$$\lambda_\gamma(\sigma) \propto \exp\big[-\beta H_\gamma(\sigma)\big],$$

where β plays the role of an inverse temperature and H_γ is the Hamiltonian:

$$H_\gamma(\sigma) = -\frac{1}{2}\sum_{i,j} \sigma_i\sigma_j K_\gamma(i-j) = -\frac{1}{2}\sum_i \sigma_i h_\gamma(\sigma, i).$$

Above, $K_\gamma : \mathbb{R}^2 \to [0,1]$ is the kernel encoding the Kac interaction, defined by $K_\gamma(x) = K(x/\gamma)$ for some smooth, compactly supported $K : \mathbb{R}^2 \to [0,1]$ with unit integral. The quantity $h_\gamma(\sigma, i) = (K_\gamma * \sigma)(i)$ $(i \in \Lambda_N)$ is called the magnetisation field, with $*$ the discrete convolution on Λ_N.

The parameter γ tunes the range of the interaction. For fixed γ, it is well known that the above Ising model admits a phase transition at a certain value $\beta_c(\gamma)$ of the inverse temperature. It is expected, with known partial results [2], that the magnetisation field has non-Gaussian fluctuations close to $\beta_c(\gamma)$, and that this critical point satisfies $\beta_c(\gamma) = 1 + c\gamma^2 \log \gamma^{-1} + O(\gamma^2)$ for an explicit constant $c = c(K)$ ($\beta_c = 1$ is the mean-field value, corresponding to the model with $\gamma = 1/N$). Following [1], we explain how the $\gamma^2 \log \gamma^{-1}$ shift in the critical inverse temperature naturally arises as the appropriate counterterm for a suitably rescaled version of the magnetisation field undergoing Glauber dynamics to converge, when γ is small and N is large, to the solution of the dynamical Φ^4_2 model on the torus.

The Glauber dynamics is defined as follows. Put independent Poisson clocks on all sites of Λ_N, and if the clock rings at position $j \in \Lambda_N$ flip the corresponding spin with the jump rate

$$c_\gamma(\sigma, j) = \frac{\lambda_\gamma(\sigma^j)}{\lambda_\gamma(\sigma) + \lambda_\gamma(\sigma^j)}.$$

Here, σ^j denotes the spin configuration that coincides with σ except for a flipped spin at position j. This defines a (jump) Markov process $(\sigma(t))_{t \geq 0}$ with λ_γ as its reversible measure.

With the slight abuse of notation $h_\gamma(t,k) = h_\gamma(\sigma(t), k)$, we write for $t \geq 0$ and $k \in \Lambda_N$ the Martingale decomposition

$$h_\gamma(t,k) = h_\gamma(t=0,k) + \int_0^t \mathcal{L}_\gamma h_\gamma(s,k)\, ds + m_\gamma(t,k),$$

where $\mathcal{L}_\gamma$ denotes the generator of the Markov process $\sigma(\cdot)$, and $m_\gamma(\cdot, k)$ is a martingale. We remark that a short calculation using the definitions of λ_γ, H_γ and h_γ yields

$$\mathcal{L}_\gamma h_\gamma(\sigma,k) = -h_\gamma(\sigma,k) + K_\gamma * \tanh(\beta h_\gamma(\sigma,k)).$$

By Taylor's approximation $\tanh(\beta h) = \beta h - (\beta h)^3/3 + \dots$, we obtain

$$\mathcal{L}_\gamma h_\gamma(\sigma,k) = -h_\gamma(\sigma,k) + \beta K_\gamma * h_\gamma(\sigma,k) - \tfrac{\beta^3}{3} K_\gamma * h_\gamma^3(\sigma,k) + K_\gamma * \dots,$$

and plugging this into the Martingale decomposition yields

$$\begin{aligned} h_\gamma(t,k) = h_\gamma(t=0,k) + \int_0^t \Big(& K_\gamma * h_\gamma(s,k) - h_\gamma(s,k) + (\beta-1) K_\gamma * h_\gamma(s,k) \\ & - \tfrac{\beta^3}{3} K_\gamma * h_\gamma^3(s,k) + K_\gamma * \dots \Big)\, ds + m_\gamma(t,k). \end{aligned}$$

We now aim to rescale the lattice Λ_N to a box of size 1. Hence for $\epsilon = 1/N$ we denote $\Lambda_\epsilon = \epsilon \Lambda_N \approx \mathbb{T}^2$. Furthermore, let $\alpha, \delta > 0$. Then for $t \geq 0$ and $x \in \Lambda_\epsilon$ the rescaled locally averaged field $X_\gamma(t,x) := \delta^{-1} h_\gamma(\alpha^{-1}t, \epsilon^{-1}x)$ satisfies

$$\begin{aligned} X_\gamma(t,x) = X_\gamma(0,x) + \int_0^t \Big(&\tfrac{\epsilon^2}{\gamma^2\alpha}\Delta_\gamma X_\gamma(s,x) + \tfrac{\beta-1}{\alpha} K_\gamma^{(\epsilon)} *_\epsilon X_\gamma(s,x) \\ &- \tfrac{\beta^3}{3}\tfrac{\delta^2}{\alpha} K_\gamma^{(\epsilon)} *_\epsilon X_\gamma^3(s,x) + K_\gamma^{(\epsilon)} *_\epsilon E_\gamma(s,x)\Big)\, ds \\ &+ \tfrac{1}{\delta} m_\gamma(\tfrac{t}{\alpha}, \tfrac{x}{\epsilon}), \end{aligned} \tag{1}$$

where $*_\epsilon$ denotes convolution on Λ_ϵ, $K_\gamma^{(\epsilon)}$ is a kernel at scale ϵ/γ approximating a Dirac in the regime $\epsilon \ll \gamma$, $\Delta_\gamma X = \gamma^2/\epsilon^2(K_\gamma^{(\epsilon)} *_\epsilon X - X)$ is an approximation of the Laplacian, and $E_\gamma(t,x) = (\alpha\delta)^{-1}(\tanh(\beta\delta X_\gamma(t,x) - \beta\delta X_\gamma(t,x) + (\beta\delta X_\gamma(t,x))^3/3)$.

In order for the scaling factors in front of the discrete Laplacian and the cubic term to stay of order one, imposes $\epsilon^2/(\gamma^2\alpha) \approx 1 \approx \delta^2/\alpha$. Similarly, one can check that the predictable quadratic co-variation of the martingale term approximates a cylindrical Brownian motion which is delta-correlated in space, provided $\epsilon^2/(\delta^2\alpha) \approx 1$. We thus choose the scaling

$$\epsilon = \gamma^2, \quad \alpha = \gamma^2, \quad \delta = \gamma, \quad N = 1/\gamma^2.$$

Note that $E_\gamma \approx (\alpha\delta)^{-1}(\beta\delta X_\gamma)^5$, which is of the order $\delta^4/\alpha = \gamma^2$ provided βX_γ is of order one, and is thus expected to disappear in the limit $\gamma \to 0$.

It remains to control the linear term in (1) with the pre-factor $(\beta-1)/\alpha$. This is where the inverse temperature needs to be chosen in a suitable window around the critical temperature. A naive guess would be to take $\beta = 1 + \alpha A = 1 + \gamma^2 A$ ($A \in \mathbb{R}$) for this term to be of order 1. However if one believes that the limit X of X_γ as $\gamma \to 0$ should solve the dynamical Φ^4_2 model, then we know a diverging counterterm must be added to (1) for X to be non-trivial. This corresponds to taking $\beta = 1 + c(K)\gamma^2 \ln \gamma^{-1} + A\gamma^2$ as guessed earlier.

The main result of [1] can be paraphrased as follows.

Theorem. Under the above scaling, the rescaled locally averaged field X_γ converges in law[1] to the dynamical Φ^4_2 model X on $\mathbb{T}^2$, i.e. the solution of

$$\partial_t X = \Delta X - \tfrac{1}{3}\big(X^3 - 3\infty X\big) + AX + \sqrt{2}\xi \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^2, \tag{2}$$

provided the respective initial conditions converge[2]. Here, ξ denotes a space-time white noise.

We refer to [1, Section 3] for details on how to interpret (2) and conclude with some ideas of the proof in [1]. The main idea is to use a suitable version of the Da Prato-Debussche decomposition, writing X_γ as a deterministic function of the solution Z_γ of a discrete heat equation:

$$dZ_\gamma = \Delta_\gamma Z_\gamma\, dt + dM_\gamma,$$

with $M_\gamma(t,x) = \gamma^{-1} m_\gamma(\gamma^{-2}t, \gamma^{-2}x)$ the rescaled martingale appearing in (1). A careful study allows one to obtain tightness for Z_γ and its suitably interpreted Wick

[1] w.r.t. the Skorokhod topology of $C^{-\nu}$-valued cadlag functions for $\nu > 0$ small enough

[2] in $C^{-\nu}$, and are uniformly bounded in $C^{-\nu+\kappa}$ for an arbitrarily small $\kappa > 0$

powers (special care has to be taken as Z_γ is not Gaussian). Convergence to a solution of the stochastic heat equation then relies on a martingale characterisation of such solutions. This convergence is the main input. The convergence for X_γ then follows, after a number of technical steps, by checking how close the discrete convolution and Laplacian appearing in the right-hand side of (1) are to their continuous counterparts.

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Gaussian Multiplicative Chaos and Liouville Quantum Gravity

Xingjian Di and Michael Hofstetter

Let (Σ, g) be a Riemannian manifold. It is possible to derive the area measure $\mathrm{v}_g(dx)$, the scalar curvature R_g and other quantities from the metric. We take $\hat{g}$ to be the round metric on the Riemann sphere $\hat{\mathbb{C}}$. Following [1, Section 2], we define the Gaussian Free Field (GFF) on the Riemann sphere to be the zero-mean Gaussian process with covariance function

$$G_{\hat{g}}(z,z') := \mathbb{E}[X_{\hat{g}}(z)X_{\hat{g}}(z')] = \ln\frac{1}{|z-z'|} - \frac{1}{4}(\ln\hat{g}(z) + \ln\hat{g}(z')) + \ln 2 - \frac{1}{2}. \tag{1}$$

The celebrated Gaussian Multiplicative Chaos (GMC) theory defines in great generality the following measure as a weak limit (for $0 < \gamma < 2$)

$$\mu_h(dx) = \lim_{\epsilon\to 0} \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(x)}\sigma(dx), \tag{2}$$

where h is a log-correlated Gaussian field (in particular the GFF), h_ϵ the ϵ-circle average of h and σ some reference Radon measure. We take h to be the GFF on the Riemann sphere and σ the area measure associated to the spherical metric, and refer to μ_h the quantum area measure.

Tentatively, the Liouville action functional is defined as

$$S(X,g,\mu) = \frac{1}{4\pi}\int_\Sigma \left(|\partial_g X|^2 + QR_gX + 4\pi\mu e^{\gamma X}\right) d\mathrm{v}_g, \tag{3}$$

and the path integral measure is defined as

$$\langle \mathcal{O}(X)\rangle^{\text{tent.}}_{g,\mu} = \int \mathcal{O}(X)e^{-S(X,g,\mu)}\mathcal{D}X, \tag{4}$$

where $\mathcal{O}$ is some generic observable associated to the field.

It has been known to physicists via renormalization arguments that if we take $Q = \gamma/2 + 2/\gamma$, the resulting quantum field theory is conformally invariant. We keep this choice. Note that GFF on the Riemann sphere is defined up to a global additive constant. We now let $\phi = h + \mathbf{c}$ where ϕ is required to have zero mean. The key construction in [1, Section 3.1] is to integrate out $\mathbf{c}$ with respect to the

Lebesgue measure. Also observe that in the spherical metric, $R_g \equiv 2$. Let $\mathbb{P}$ be the law of zero-mean GFF on the Riemann sphere. We have the rigorous definition

$$\langle \mathcal{O}(X) \rangle_{\hat{g},\mu} = \int_{-\infty}^{\infty} \int \mathcal{O}(\phi) e^{-\mu_\phi(\mathbb{C})+2Qc} \mathbb{P}(dh) d\mathbf{c}. \tag{5}$$

The observables of physical interest are the vertex operators $e^{\alpha\phi(x)}$ for some $\alpha \in \mathbb{R}$ and $x \in \mathbb{C}$. Since ϕ not defined pointwise, we follow a similar regularization and renormalization process. It follows that the convergence and nontriviality criterion of the partition function is the Seiberg bounds,

$$\alpha_i < Q, \quad \text{and} \quad \sum_i \alpha_i > 2Q. \tag{6}$$

The latter inequality follows easily as we analyze the integral near $c = \pm\infty$, while the former follows essentially states the condition that the quantum area of an infinitesimal neighborhood of an insertion point does not blow up. The proof uses the multifractal spectrum estimate [2, Section 3.8]

$$\mathbb{E}[\mu_h(B_r)^q] \asymp r^{(2+\gamma^2/2)q-\gamma^2 q^2/2} \tag{7}$$

and the Chebyshev inequality to bound $\mu_\phi(B_r)$ as $r \to 0$.

We also briefly presented some properties of the resulting measure, including the KPZ formula and Weyl anomaly, which allows us to generalize to other background metrics. Lastly, we discussed some recent development and application of the theory, including the compactified imaginary Liouville theory [3] and the backbone exponent in critical percolation [4].

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Introduction to continuum and lattice Yang Mills theory

Léonard Ferdinand, Sarah-Jean Meyer

Yang-Mills (YM) theory is central to the description of elementary particles in the standard model but unfortunately a rigorous mathematical foundation is lacking. As such, the rigorous construction of YM in the physically relevant $3+1$ dimensional space-time is an important unsolved problem in mathematics [8]. The goal of this talk is to introduce the core ideas to understand the problem of constructing a (Euclidean) YM theory and present some interesting open questions concerning the mass gap, quark confinement, the area law as well as the large N-factorization. As even the correct spaces to consider are up to debate, the discussions are almost exclusively at an informal level. We mainly follow [4], but also refer to the surveys [9, 6] and the recent works [3, 2, 1, 5, 10, 7] for more details and further reading.

The Yang-Mills measure. Fix a semi-simple Lie group $G \subset SU(N)$ and denote its Lie algebra by $\mathfrak{g}$. We equip $\mathfrak{g}$ Ad-invariant inner product $\langle \cdot, \cdot \rangle$ and the induced norm. Here, the adjoint action is given by $g \cdot X = \mathrm{Ad}_g X := gXg^{-1}$. For example, in the case $\mathfrak{g} = \mathfrak{su}(N)$, we may use $\langle X, Y \rangle = -\mathrm{Tr}(XY)$. Consider a trivial G-principal bundle P over $\mathbb{R}^d$, where $G \subset SU(N)$. The space of $\mathfrak{g}$-connections $\mathcal{A}$ is the affine space of all elements of the form $\mathrm{d}_A := \mathrm{d} + A$ for A a $\mathfrak{g}$ valued 1-forms $A = (A_1, \ldots, A_d) : \mathbb{R}^d \to \mathfrak{g}^d$, and d an arbitrarily fixed trivial connection. On the Euclidean space $\mathbb{R}^d$, the Yang-Mills measure is formally defined on $\mathcal{A}$ for $\beta > 0$ as

$$\mu_\beta(\mathrm{d}A) := Z_\beta^{-1} \exp\big(- \beta S_{\mathrm{YM}}(A)\big) \mathrm{d}A \,,$$

where d formally corresponds to the Lebesgue measure on $\mathcal{A}$, and Z is a normalization constant. Here, S_{YM} is the YM action formally defined as

$$S_{\mathrm{YM}}(A) := \int_{\mathbb{R}^d} |F(A)|_{\mathfrak{g}}^2 \,,$$

where $F(A) := \mathrm{d}_A A$ is the curvature 2-form of the connection A, in coordinates given by $F_{ij}(A) = \partial_i A_j - \partial_j A_i + [A_i, A_j]$, and $|\cdot|_{\mathfrak{g}}$ is the Euclidean norm associated with the Ad-invariant inner product.

In addition to the usual UV and IR problem arising in the definition of any singular EQFT, the Yang-Mills action turns out to be invariant under an infinite dimensional group of "gauge transformations", the group of G valued 0-forms corresponding to the changes of coordinates on P. This last fact makes its definition even more subtle, since it involves working on the non-linear(!) quotient space of connections modulo these gauge transformations.

Lattice Yang-Mills theories. As for the scalar theories, one attempt to rigorously define the Yang-Mills measure is to start from a finite dimensional approximation defined on the discrete torus $\Lambda = \Lambda_{\epsilon, L}$. In this setting, the connection is approximated by its holonomies U_{xy} along the edges of Λ. The discrete Yang-Mills measure is defined by

$$\mu_{\beta,\Lambda}(\mathrm{d}U) := Z_{\beta,\Lambda}^{-1} \exp\big(\beta \sum_p \chi_\epsilon(U_p)\big) \mathrm{d}U \,,$$

where the sum runs over all *plaquettes* p, that over all squares with edges in Λ and χ_ϵ is suitably chosen to recover the continuum Yang-Mills measure in the limit. Finally, $\mathrm{d}U$ denotes the Haar measure on the field configuration. Some examples of discrete actions are given by

$$\chi_\epsilon(g) = \begin{cases} \epsilon^{d-4} \Re \mathrm{Tr}(\mathrm{id} - g) & \text{(Wilson action)}\,, \\ - \log e^{\frac{1}{4}\epsilon^{4-d} \Delta_G}(\mathrm{id}, g) & \text{(Villain action)}\,. \end{cases}$$

The discrete measure is invariant under the action of the discrete gauge group G^Λ that acts on U_{xy} via conjugation $g \cdot U_{xy} = g_x U_{xy} g_y^{-1}$. To make the connection to the continuous setting, one can heuristically always identify U with a connection A in the continuum via $U_{x,x+\epsilon e_i} \approx \mathrm{e}^{\epsilon A_i(x)}$.

Wilson loops. The invariance of the Yang-Mills measure under the group of gauge transformations makes it necessary to work with gauge invariant observables. A natural choice is to consider the traces of the holonomies of the connection along closed loops $\gamma : [0,1] \to \mathbb{R}^d$, known as Wilson loops, and often denoted by $W_\gamma(A)$. An important conjecture (see also [8]) about non-Abelian Yang-Mills theories, known as "mass gap", is that, in the infinite volume limit $L \nearrow \infty$, the correlation length of the Wilson loops

$$\xi^{-1}(\epsilon,\beta) := -\lim_{d(\gamma_1,\gamma_2)\to\infty} \frac{\log\big(\mathrm{Cov}(W_{\gamma_1}, W_{\gamma_2})\big)}{d(\gamma_1,\gamma_2)}$$

takes a finite non-zero value for all finite β, and diverges as $\beta \nearrow \infty$. This indicates that it should be possible to obtain non-trivial correlations in the continuum limit.

Parabolic stochastic quantisation for YM. A nice reference for this part is [6]. One way to try to rigorously define the continuous Yang-Mills measure is by studying its Langevin dynamic, or noisy gradient descent, that formally reads

$$\partial_t A = -\nabla_A S_{\mathrm{YM}}(A) + \xi\,,$$

where ξ is a space-time white-noise. A consequence of gauge invariance is that the linear part of $\nabla_A S_{\mathrm{YM}}(A) = \mathrm{d}_A^* \mathrm{d}_A A$ is not elliptic. One way to circumvent this issue is to introduce by hand a DeTurck-Zwanziger-term $-\mathrm{d}_A \mathrm{d}_A^* A$ on the r.h.s. In coordinates, the new equation (noisy YM heat flow) reads

$$(\partial_t - \Delta) A_i = [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + \xi_i\,.$$

While this equation is no longer gauge invariant, since the gauge breaking term is tangent to the gauge orbits at A, the equation still exhibits a gauge covariance property. Indeed, denoting by $\Phi_t A$ the flow of some initial condition A under the noisy YM-heat flow, we verify that $\Phi_t(A^{g_0}) \overset{\text{Law}}{=} (\Phi_t A)^{g(t)}$ provided we choose the gauge $g(t)$ *dynamically*, so that

$$(\partial_t g) g^{-1} = -\mathrm{d}^*_{A^g}((\mathrm{d}g) g^{-1}).$$

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Langevin Dynamics of Lattice Yang-Mills Model

Jiasheng Lin, Kihoon Seong

The goal of the talk is to introduce Yang-Mills measures and Langevin dynamics for the lattice Yang-Mills model in [9, 10]. We first explain the set-up and preliminaries required to understand this lattice Yang-Mills model. Let $\Lambda_L \subset \mathbb{Z}^d$ be (vertices of) a finite lattice with side length L and unit lattice spacing. Orient the edges in *lexographic direction* (for a careful description see Chatterjee [1] section 2). Denote by $E^+_{\Lambda_L}$ the set of positively oriented edges whose end points belong to Λ_L. Denote by $\mathcal{P}^+_{\Lambda_L}$ the set of positively oriented *plaquettes*, concatenation of four edges tracing out the boundary of a unit square (in $d = 2$ positively oriented means going anticlockwise viewed from the reader). Let $N \in \mathbb{N}$ and G be the Lie group $SO(N)$, $U(N)$ or $SU(N)$. We work in the so-called *configuration space*

$$\mathcal{Q}_L := G^{E^+_{\Lambda_L}} = \{(Q_e)_{e\in E^+_{\Lambda_L}} \mid Q_e \in G\} \tag{1}$$

of "configurations" of matrices from the Lie group, one for each edge. Given a configuration $Q = (Q_e)_{e\in E^+_{\Lambda_L}}$ and for $\ell = e_1e_2\cdots e_n$ a path or loop consisting of concatenation of successive edges, we impose the matrix $Q_\ell := Q_{e_n}\cdots Q_{e_1}$, where we also set $Q_e := Q^{-1}_{e^{-1}}$ if e is negatively oriented. Let $\mathfrak{g}$ be the Lie algebra of G and we note that a *tangent vector* to $\mathcal{Q}_L$ at a configuration $(Q_e)_{e\in E^+_{\Lambda_L}}$ would be a configuration of the form $(X_eQ_e)_{e\in E^+_{\Lambda_L}} =: XQ$, where $X_e \in \mathfrak{g}$, lying in the full space $M_N(\mathbb{C})^{E^+_{\Lambda_L}}$ which is finite dimensional Euclidean and where $\mathcal{Q}_L$ embeds. For two such tangent vectors XQ, YQ we define the inner product $\langle XQ, YQ\rangle := \sum_e \mathrm{Tr}(X_eY_e^*)$ where A^* denotes the adjoint of A.

The main object of study is the probability measure $\mu_{\Lambda_L,N,\beta}$ on $\mathcal{Q}_L$ given by the density expression

$$\mathrm{d}\mu_{\Lambda_L,N,\beta}(Q) := Z^{-1}_{\Lambda_L,N,\beta}\mathrm{e}^{\mathcal{S}_{\Lambda_L,N,\beta}(Q)} \prod_{e\in E^+_{\Lambda_L}} \mathrm{d}\sigma_G(Q_e), \tag{2}$$

where $Z_{\Lambda_L,N,\beta}$ is the normalization constant, $\beta > 0$ is the inverse coupling constant, σ_G is the Haar measure on the Lie group G, and $\mathcal{S}_{\Lambda_L,N,\beta}$ is the Yang-Mills action

$$\mathcal{S}_{\Lambda_L,N,\beta}(Q) := N\beta \sum_{p\in\mathcal{P}^+_{\Lambda_L}} \mathfrak{Re}\mathrm{Tr}(Q_p). \tag{3}$$

The main method of study is to exhibit (2) as the invariant measure of a stochastic differential equation (SDE) on $\mathcal{Q}_L$,

$$\mathrm{d}Q_t = \frac{1}{2}\nabla\mathcal{S}_{\Lambda_L,N,\beta}(Q_t)\mathrm{d}t + \mathrm{d}\vec{\mathcal{B}}_t, \tag{4}$$

called the *lattice Yang-Mills SDE*, where $\nabla\mathcal{S}_{\Lambda_L,N,\beta}(Q_t)$ is the gradient of $\mathcal{S}_{\Lambda_L,N,\beta}$ valued at Q_t, taken under the inner product described above, and $\vec{\mathcal{B}}_t$ an edgewise independent tuple of Brownian motions on G, discussed below. This SDE describes the *stochastic gradient flow* of $\mathcal{S}_{\Lambda_L,N,\beta}$ with noise produced by $\vec{\mathcal{B}}_t$.

To show long time stochastic well-posedness of (4) one first show it in the Euclidean space $M_N(\mathbb{C})^{E^+_{\Lambda_L}}$ following the ordinary procedure and then use Itô formula to show the solution lies a.s. in $\mathcal{Q}_L$. To show $\mu_{\Lambda_L,N,\beta}$ is invariant is also a standard argument, by computing explicitly the Feller generator, see Shen, Smith and Zhu [9]. These are in parallel with the treatment of Brownian motion on G.

To define Brownian motion (BM) on G (starting at the identity I_N) first define the BM, B_t, on $\mathfrak{g}$ (with the above inner product) which is the ordinary BM, starting at zero. Then the BM on G is defined by solving in $M_N(\mathbb{C})$ the SDE

$$\mathrm{d}\mathcal{B}_t = \frac{1}{2}c_{\mathfrak{g}}\mathcal{B}_t\mathrm{d}t + \mathrm{d}B_t \cdot \mathcal{B}_t, \tag{5}$$

where $c_{\mathfrak{g}}$ is the constant making $\sum_i e_i^2 = c_{\mathfrak{g}} I_N$ for an o.n. basis $\{e_i\}_i$ of $\mathfrak{g}$, and the solution lies a.s. in G. This corresponds to the following intuitive picture: compare G to a sphere and $\mathfrak{g}$ to a tangent plane to the sphere at a point denoted 0; pick a trajectory of B_t starting at 0, roll the sphere *without slipping* on the plane so that the contact point traces out the trajectory, then the corresponding trajectory on the sphere would be one of the BM on G. This picture is made rigorous by the McKean "injection method", see McKean [8] sections 4.7-4.8. More comprehensively see the monograph by Liao [7]. See also the first section of Dahlqvist [3] (in French) for a nice, shorter summary and an Itô formula.

By applying Itô's formula to the dynamics of (4) Shen, Smith and Zhu [9] managed to obtain a version of the so-called Makeenko-Migdal (MM) equations on the lattice. We explain (MM) in the continuum which is simpler. There, instead of on $\mathcal{Q}_L$ one considers a measure on $\mathcal{A}$, the space of connections on the (trivial) principal G-bundle over $\mathbb{R}^2$, which is formally $\mu_{\mathrm{YM}} \propto \exp(-\frac{1}{2}\mathcal{S}_{\mathrm{YM}}(A))\mathrm{d}\mathcal{L}(A)$, $\mathcal{S}_{\mathrm{YM}}$ being the *Yang-Mills action* defined in the previous talk and $\mathcal{L}$ the nonexistent "Lebesgue" measure on $\mathcal{A}$. For a piecewise smooth loop ℓ in $\mathbb{R}^2$, the matrix Q_ℓ is defined accordingly to be the holonomy matrix along ℓ.[1] Then $\mathbb{E}_\mu[\mathrm{Tr}(Q_\ell)]$ defines a function of ℓ. If on ℓ we perform a "surgery" turning it into finitely many loops $\ell_1, \ldots, \ell_n$, then (MM) gives a set of PDEs relating the function $\mathbb{E}_\mu[\mathrm{Tr}(Q_\ell)]$ to $\mathbb{E}_\mu[\mathrm{Tr}(Q_{\ell_1})\cdots\mathrm{Tr}(Q_{\ell_n})]$. The key to the formal derivation lies in writing down a formal integration-by-parts formula for μ_{YM} and differentiating a clever functional in a clever direction. But in finite dimensions integration-by-parts formula of the ordinary Gaussian measure is also a consequence of it being the invariant measure

[1] In fact, our lattice matrix configuration Q should be seen as the parallel transport matrices induced by a connection over the background continuum.

of the Ornstein-Uhlenbeck process and Itô's formula. Inspired by this fact, [9] obtain a new proof of lattice (MM) which was previously obtained by Chatterjee [1]. Another interesting aspect is that when the matrix size N tends to infinity, the random variable $\mathrm{Tr}(Q_\ell)$ converge in law to a deterministic number $\Phi(\ell)$, thus defining a function Φ on the space of loops, called the *master field*. The (MM) equations turn then into PDEs describing Φ. For fuller treatment of (MM) in the continuum see Lévy [5] and [6], on the lattice [1], and also Singer [11] for a broader perspective.

Let us explore additional outcomes related to the lattice Yang–Mills measure. As long as the smallness assumption for β holds (i.e. strong coupling regimes), the infinite volume (tight) limit $\mu^{\mathrm{YM}}_{\beta,N}$ of the finite volume Yang–Mills measures $\mu_{\Lambda_L,\beta,N}$ as $L \to \infty$ is unique, which is also the unique invariant measure under the solution to the Yang-Mills SDE (on entire $\mathbb{Z}^d$). The proof of uniqueness is obtained by a variation of the Kendall–Cranston coupling. In addition to uniqueness, we can obtain various properties of the infinite volume measure $\mu^{\mathrm{YM}}_{\beta,N}$ by establishing functional inequalities associated with the measure. We first consider the finite volume Yang–Mills measures $\mu_{\Lambda_L,\beta,N}$. Then, under the smallness assumption for β, the Bakry–Émery condition is satisfied: for any tangent vector v (of the product of Lie group i.e. $\mathcal{Q}_L = G^{E^+_{\Lambda_L}}$),

$$\mathrm{Ricc}(v,v) - \mathrm{Hess}_{\mathcal{S}}(v,v) \geq K_{\mathcal{S}}|v|^2$$

where $K_{\mathcal{S}} > 0$ does not depend on the volume parameter size L. In these approaches, the Ricci curvature properties of the Lie groups are importantly used through the verification of the Bakry–Émery condition. In other words, in strong coupling regimes, the Hessian of the Yang Mills action $\mathcal{S}$ can be controlled by the Ricci curvatures of the configuration space $\mathcal{Q}_L = G^{E^+_{\Lambda_L}}$ to guarantee $K_{\mathcal{S}} > 0$. Note that the Barkly Émery criterion implies the log-Sobolev and Poincaré inequalities for the measure $\mu_{\Lambda_L,\beta,N}$. This gives that the dynamics (lattice Yang-Mills SDE) on $\mathcal{Q}_L$ is exponentially ergodic. Moreover, the log-Sobolev and Poincaré inequalities for the measure $\mu_{\Lambda_L,\beta,N}$ extend to the infinite volume measure $\mu^{\mathrm{YM}}_{\beta,N}$ as they are independent of dimension. We point out that in the strong coupling regime one of the important parts is to be able to take β small uniformly in the large N parameter, which allows us to take the large N limit with the infinite volume measure $\mu^{\mathrm{YM}}_{\beta,N}$ in the below applications.

We present various applications of the Poincaré inequality. For cylinder functions $F \in C^\infty_{\mathrm{cyl}}(\mathcal{Q})$, we have

$$\mathrm{Var}_{\mu^{\mathrm{YM}}_{\beta,N}}(F) = \int |F - \int F d\mu_{\beta,N}|^2 d\mu^{\mathrm{YM}}_{\beta,N} \leq \frac{1}{K_S} \int |\nabla F|^2 d\mu^{\mathrm{YM}}_{\beta,N},$$

which implies that (i) the rescaled Wilson loop converges to a deterministic limit and (ii) the factorization property of Wilson loops holds as follows:

$$\left| \frac{W_\ell}{N} \to \mathbb{E}_{\mu^{\mathrm{YM}}_{\beta,N}} \frac{W_\ell}{N} \right| \to 0 \qquad \text{and} \qquad \left| \mathbb{E}_{\mu^{\mathrm{YM}}_{\beta,N}} \frac{W_{\ell_1} \cdots W_{\ell_m}}{N^m} - \prod_{i=1}^m \mathbb{E}_{\mu^{\mathrm{YM}}_{\beta,N}} \frac{W_{\ell_i}}{N} \right| \to 0$$

in probability as $N \to \infty$, where $W_\gamma = \mathrm{Tr}(Q_{e_1} \cdots Q_{e_n})$ with a loop $\gamma = e_1 e_2 \cdots e_n$ is called a Wilson loop.

The other application is to exhibit the existence of mass gap for lattice Yang–Mills. By exploiting the Poincaré inequality, one obtains the mass gap as follows: for any $f, g \in C^\infty_{\mathrm{cyl}}(\mathcal{Q})$ with supports $\Lambda_f \cap \Lambda_g = \emptyset$, we have the exponential decay of correlations

$$\mathrm{Cov}_{\mu^{\mathrm{YM}}_{\beta,N}}(f,g) \leq c_1 e^{-c_2 d(\Lambda_f, \Lambda_g)}$$

where $d(A,B)$ means the distance between A and $B \in E^+$. In particular, selecting the functions f and g as Wilson loops is of particular interest in physics.

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Basic Concepts and Reconstruction Theorem

Sefika Kuzgun, Ilya Losev

Theory of regularity structures is a very important tool in the modern theory of Stochastic Partial Differential Equations, which allows one to make sense and study basic properties of certain SPDEs. In particular, they play crucial role in the solution theory of KPZ and Φ^4_3 equations.

The notion of regularity structures is a generalization of such well-known things as Taylor polynomials and rough paths. In our talk we discuss the basic concepts in the theory of regularity structures and illustrate them using an example of polynomial regularity structure, which is a regularity structure designed to describe the theory of Taylor polynomials.

A *regularity structure* consists of a *structure space* (a graded linear Banach space) together with a *structure group*, which encodes how elements of the structure space change when one shifts an argument. In the case of polynomial regularity structure the structure space consists of all polynomials with a natural grading given by degree.

A regularity structure is an abstract notion, and it needs to be endowed with a model, which allows one to represent elements of its structure space as concrete distributions on $\mathbb{R}^d$. A *model* consists of *realisation map* and *reexpansion map*. Realisation map turns elements of structure space into distributions which form an expansion around a given point. For polynomial regularity structure the realisation map returns a Taylor polynomial with given coefficients around a given point. The reexpansion map, in turn, tells you how to turn a given expansion around a point into a similar expansion around a different point.

Finally, we introduce a notion of *modelled distribution*. Essentially, a modelled distribution is an analogue of a function in the theory of regularity structures. In the case of polynomial regularity structure we have that the space of modelled distributions exactly coincides with the classical Hölder class.

In our talk we also discuss the Reconstruction Theorem. This theorem allows one to represent any modelled distribution as a concrete distribution on $\mathbb{R}^d$.

Theorem 1 ([2]). *Let $\mathcal{T}$ be a regularity structure and let (Π, Γ) a model for $\mathcal{T}$ on $\mathbb{R}^d$. Then for $\gamma > 0$, there exists a unique linear map $R : D^\gamma \to D'(\mathbb{R}^d)$ such that*

$$|(Rf - \Pi_x f(x))(\psi_x^\Lambda)| \lesssim \lambda^\gamma \tag{1}$$

uniformly over $\psi \in B_r$ and $\lambda \in (0, 1]$, and locally uniformly in x.

The second part of our presentation is devoted to proving this fundamental theorem. We closely follow the proof as presented in second edition of the book [1], which is based on the proofs given in [5] and [4]. Hairer's original proof in [2] is based on the wavelength analysis, the former presentation is self-contained.

Let $\alpha > 0$. The proof relies on the existence of an even smooth function $\rho : \mathbb{R}^d \to \mathbb{R}$ that is compactly supported in the unit ball and satisfies

$$\int_{\mathbb{R}^d} x^k \rho(x) dx = \delta_{k,0}, \qquad 0 < |k| \le \alpha,$$

where k denotes a d-dimensional multi-index and δ Kronecker's delta. Detailed construction of such a function can be found in [6].

To construct an approximation scheme, define $\rho^n(x) := 2^{nd} \rho(2^n x)$ for $n \in \mathbb{N}$, and $\rho^{n,m} := \rho^n * \cdots * \rho^m$ for $n, m \in \mathbb{N}$, $m \ge n$. It can be shown that $\varphi^n := \lim_{m\to\infty} \rho^{n,m}$ exists, is compactly supported and satisfies a similar scaling as ρ.

Using these smooth functions, it is possible to construct a two layer approximation to obtain Rf as limit of

$$R^{n,m} := \rho^{n,m-1} * \left((\Pi_y f(y)) \left(\varphi_y^m \right) \right)$$

first sending $m \to \infty$ and then $n \to \infty$. The final step in the proof is to show that the distribution constructed this way satisfies (1). Some details of these steps are provided in our presentation.

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Fixed point problem in the space of modelled distributions

Sky Cao, Fabian Höfer

A standard way to solve a semilinear parabolic PDE

$$\begin{cases} \partial_t u = Au + F(u) \\ u(0) = u_0 \end{cases} \tag{1}$$

locally in time, where A generate a semigroup $S(t) = e^{At}$, is to set up a fixed-point problem. By Duhamel's formula the mild form of (1) is

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds =: (\mathcal{M}u)(t).$$

The strategy then is to find a complete metric space X_T consisting of space-time functions up to time T, such that $\mathcal{M}\colon X_T \to X_T$ is a contraction for sufficiently small T.

In many cases the same methodology can be applied when we are looking for solutions in the space of modelled distributions. To motivate the necessary ingredients needed for this, we consider the Φ^4_3 model

$$\partial_t \Phi = \Delta\Phi - \Phi^3 + \xi \tag{2}$$

where ξ denotes space-time white noise and the spatial variable takes values in the 3-dimensional torus. The mild formulation of (2) is then given by

$$\Phi = K * (\xi - \Phi^3) + K\Phi_0 \tag{3}$$

where K denotes the heat kernel, $*$ the space-time convolution and $K\Phi_0$ the harmonic extension of the initial data Φ_0, i.e. the solution to the linear heat equation with initial data Φ_0.

The "abstract" formulation of (3), where Φ is now a modelled distribution, should then be given by

$$\Phi = \mathcal{K}(\Xi - \Phi^3) + K\Phi_0. \tag{4}$$

Here Ξ is a symbol representing the noise ξ and $\mathcal{K}$ is a linear operator acting on the space of modelled distributions correspoding to the convolution with the heat kernel.

In order to make sense of (4) we need

(1) Make sense of products of modelled distributions, e.g. of $\Phi \mapsto \Phi^3$.
(2) Schauder theorem: Given a β-regularising kernel K, build $\mathcal{K}\colon \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+\beta}$ such that $\mathcal{R}\mathcal{K}f = K * \mathcal{R}f$.

Here $\mathcal{R}$ denotes the reconstruction operator. The Schauder theorem will be the key ingredient to get a gain T^κ in estimates for $\mathcal{M}$ and thus making it a contracting self-map for small T.

In order to state the multiplication theorem, we need to assume that our regularity structure is equipped with a product itself.

Definition 1. *Given a regularity structures (T, G) and two sectors $V, \bar{V} \subset T$, a continuous bilinear map $\star\colon V \times \bar{V} \to T$ is called a* product *on $(V, \overline{V})$ if for any $\tau \in V_\alpha$ and $\bar{\tau} \in \bar{V}_\beta$, one has $\tau \star \bar{\tau} \in T_{\alpha+\beta}$ and if for any $\Gamma \in G$ one has $\Gamma(\tau \star \bar{\tau}) = \Gamma\tau \star \Gamma\bar{\tau}$.*

Using the notation $f \in \mathcal{D}^\gamma_\alpha(V)$ iff $f \in \mathcal{D}^\gamma$ and $f(x) \in V_{\geq\alpha}$ for all $x \in \mathbb{R}^d$ and letting $\mathcal{Q}_{<\gamma}$ denote the projection onto $T_{<\gamma}$, we have

Theorem 1. *Let $f_1 \in \mathcal{D}^{\gamma_1}_{\alpha_1}(V)$ and $f_2 \in \mathcal{D}^{\gamma_2}_{\alpha_2}(\bar{V})$. Then the function*

$$f(x) := \mathcal{Q}_{<\gamma}(f_1(x) \star f_2(x))$$

belongs to $\mathcal{D}^\gamma_\alpha$ with

$$\alpha = \alpha_1 + \alpha_2, \qquad \gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1).$$

In the second half of the talk, we discussed the multilevel Schauder theorem for modelled distributions. In particular, we defined β-regularizing kernels and admissible models. Then given such a model, we described how to realize convolution with a regularizing kernel on the space of modelled distributions. Finally, we stated the multi-level Schauder estimate.

Theorem 2 (Multi-level Schauder estimate)**.** *Let K be a β-regularizing kernel. Let $\mathcal{T}$ be a regularity structure satisfying certain assumptions. Let (Π, Γ) be an admissible model for $\mathcal{T}$. For $\gamma > 0$, there exists a bounded operator $\mathcal{K} : \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+\beta}$ such that $\mathcal{R}\mathcal{K}f = K * \mathcal{R}f$ for all $f \in \mathcal{D}^\gamma$.*

We emphasize the two key features of this operator $\mathcal{K}$: (1) it increases "homogeneity" by β, and (2) it plays well with the reconstruction operator, so that we may indeed think of $\mathcal{K}$ as an abstract version of convolution with K.

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Stochastic quantisation of Φ_3^4

Salvador Cesar Esquivel Calzada, Huaxiang Lu

In this tall, we will provide a concise overview of the Φ_3^4 model, highlighting the main result [1, Proposition 4.9] and the renormalization constants. Then we will explain how to associate a regularity structure to this SPDE, referring to [1, Section 4.1-4.5]. We will introduce the model for mollified noise and discuss the non-convergence of the mollified model, which leads to the brief introduction to the renormalisation group. Then we will derive the renormalization equations for the Φ_3^4 model, with a focus on proving [1, Proposition 4.9].

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Convergence of renormalized models

David Lee, Harprit Singh

We recall the notion of Wiener chaos and Nelson's hypercontractive estimate. Then, after introducing some diagrammatic notation, we explain how this can be used to obtain convergence of renormalised models for the ϕ_3^4 equation.

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Hyperbolic stochastic quantization

Petri Laarne, Rui Liang

1. Background

In this talk, we will initially transition from the concept of path integrals to Gibbs measures by incorporating fictitious time and formulating a new Lagrangian [1]. Following this, we will delve into the concept of canonical stochastic quantization [6], presenting a heuristic argument that anticipates the invariance of the Gibbs measure under the flow of the canonical stochastic quantization equation. Subsequently, we will examine the hyperbolic φ_2^4 model as a specific case study.

By these considerations (see also the introduction of [7]), it is possible to show that the corresponding Gibbs measure is formally invariant under the stochastic damped nonlinear wave equation

$$(\partial_t^2 + \partial_t + 1 - \Delta)u + u^3 = \xi,$$

which is posed on $\mathbb{R} \times \mathbb{T}^2$, where ξ is the spacetime white noise. A related question is to solve the undamped equation

$$(\partial_t^2 + 1 - \Delta)u + :u^3: = \xi,$$

where such an invariance does not hold.

2. Solving the damped equation

2.1. **Local-in-time solution.** This is based on Sections 1.3 and 4 in [5]; see also Section 4 in [2], which presents the similar deterministic equation in detail.

The idea is to apply the Da Prato–Debussche trick. We decompose the solution as $u = v + w$, where w solves the linear equation $\partial_{tt} w + \partial_t w + (1 - \Delta)w = \sqrt{2}\xi$ with the given initial data (u_0, u_1). This equation is solved in $H^{-\varepsilon}$ by

$$w(t) = \mathcal{D}_t u_0 + \mathcal{D}_t'(u_0 + u_1) + \sqrt{2} \int_0^t \mathcal{D}_{t-s}\xi(s)\,\mathrm{d}s, \quad \mathcal{D}_t = \frac{e^{-t/2}\sin(t\sqrt{3/4 - \Delta})}{\sqrt{3/4 - \Delta}},$$

for arbitrarily large times. The remainder v solves the coupled nonlinear equation $\partial_{tt} w + \partial_t w + (1 - \Delta)w = :(v + w)^3:$ with zero initial data. We solve this part with a fixed-point argument in the more regular space $C([0, \tau], H^{1-\varepsilon}(\mathbb{T}^2))$.

There is only the Duhamel term in v, and we can estimate its norm by

$$\sup_{0 \le t \le \tau} \left\| \int_0^t \mathcal{D}_{t-s} :(v + w)^3:(s)\,\mathrm{d}s \right\|_{H^{1-\varepsilon}} \le C\tau^{1/2} \left\| :(v + w)^3: \right\|_{L^2([0,\tau];\, H^{-\varepsilon})}.$$

Here we used Cauchy–Schwarz in time and uniform boundedness of $\mathcal{D}_t$ from $H^{-\varepsilon}$ to $H^{1-\varepsilon}$. We then apply the binomial formula and estimate each term with Besov space properties (as presented by Gabriel and Liu); for example

$$\begin{aligned}\left\| v^2 w \right\|_{L^2([0,\tau];\, H^{-\varepsilon})} &\le C \left\| v \right\|^2_{L^\infty([0,\tau];\, B^{2\varepsilon}_{6,6})} \left\| w \right\|_{L^2([0,\tau];\, B^{-\varepsilon}_{6,6})} \\ &\le C \left\| v \right\|^2_{L^\infty([0,\tau];\, H^{1-\varepsilon})} \left\| w \right\|_{L^2([0,1];\, B^{-\varepsilon}_{6,6})}.\end{aligned}$$

In the end, we see that a radius-R ball is mapped into a radius $C\tau^{1/2}M(1 + R^3)$ ball, where M is sum of $L^2([0, 1];\ B^{-\varepsilon}_{p,p})$ norms of Wick powers of w. We can then choose $R = M$ and local solution time $\tau = cM^{-10}$. Contractivity follows similarly.

2.2. **Global-in-time solution.** Bourgain's argument [3] gives almost sure solution up to time $T > 0$. If the stochastic linear part w has norms bounded by $M > 0$, then u exists on $[0, \tau_M]$. If we restart the linear part from $u(\tau_M)$, and the norm bound also holds on $[\tau_M, 1+\tau_M]$, then we can continue u to $[\tau_M, 2\tau_M]$. Repeating this, we can estimate the probability of finding a solution by

$$\begin{aligned}\mathbb{P}(u \text{ exists on } [0,T]) &\geq 1 - \mathbb{P}\left(\bigcup_{j=0}^{T/\tau_M}\bigcup_{k=1}^{3} \|{:}w^k{:}\|_{L^2([k\tau_M, 1+k\tau_M];\, B^{-\varepsilon}_{p,p})} > M\right)\\ &\geq 1 - \sum_{j=0}^{T/\tau_M}\sum_{k=1}^{3}\mathbb{P}\left(\|{:}w^k{:}\|_{L^2([0,1];\, B^{-\varepsilon}_{p,p})} > M\right)\\ &\geq 1 - CTM^{10}\sum_{k=1}^{3}\frac{\mathbb{E}\,\|{:}w^k{:}\|^p_{L^2([0,1];\, B^{-\varepsilon}_{p,p})}}{M^p}.\end{aligned}$$

Here we used invariance of measure, the choice of τ, and Markov's inequality. The Wick powers of ϕ^4 have bounded moments for any $p < \infty$, and this also translates to the linear part w. Thus we can choose p and M large to get an arbitrarily high probability of solution.

To be precise, the invariance only holds in a finite-dimensional system. All of the previous estimates are uniform in Fourier truncation. It then remains to perform a (technical) limit argument; see Section 4.4 in [2].

3. Solving the undamped equation

Apart from Bourgain's globalisation argument, we will also see how combining the I-method in a stochastic setting with a Gronwall-type argument can establish the norm's double exponential growth. We will go over the difficulties encountered in this process and then present the ideas used to overcome these challenges.

By using the Da Prato–Debussche trick,

$$u = v + \Psi,$$

where Ψ is the stochastic convolution, we then consider the following equation:

$$(\partial_t^2 + 1 - \Delta)v + v^3 + \underbrace{3v^2\Psi + 3v{:}\Psi^2{:} + {:}\Psi^3{:}}_{\text{perturbation}} = 0.$$

There are two difficulties coming from the perturbation and roughness of v. If there is no perturbation, then we can use conservation of the Hamiltonian

$$H(\partial_t v, v) = \frac{1}{2}\int (|v|^2 + |\nabla v|^2)dx + \frac{1}{2}\int (\partial_t v)^2 dx + \frac{1}{4}\int v^4 dx$$

to get the globalisation. However, we have perturbation here. Nevertheless, we can see how the the Hamiltonian grows by taking derivative

$$\begin{aligned}\partial_t H(v) &= \int_{\mathbb{T}^2} \partial_t v \, \underbrace{(\underbrace{(\partial_t^2 + 1 - \Delta)v}_{=-(v+\Psi)^3} + v^3)}_{\sim v^2\Psi + v\Psi^2 + \Psi^3} \, dx \\ &\overset{\text{C-S}}{\lesssim} (H(v))^{\frac{1}{2}} \left(\|\Psi\|^2_{C_T L^\infty_x} \int v^4 dx + \|\Psi\|^6_{C_T L^6_x} \right)^{\frac{1}{2}} \\ &\leq C(T, \Psi)(1 + H(v)),\end{aligned}$$

provided that the noise is smoother. Then by Gronwall's inequality, we have

$$\|v(t)\|^2_{H^1} \leq H(t) \leq H(0)\, e^{2C(T,\Psi)T}, \text{for } 0 < t \leq T,$$

which gives global solution. However, as v is not in H^1, we need to remedy by using the I-method [4] given by an operator $I = I_N$ such that

$$\|v\|_{H^s} \lesssim \|Iv\|_{H^1} \lesssim N^{1-s}\|v\|_{H^s},$$

from which we are led to try using Iv to replace the role of v in the Gronwall-type argument stated above. After some estimates for some commutators and some analytical techniques, we get the norm's double exponential growth which contradicts the blowup criteria.

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On the Polchinski Equation

Zhituo Wang

The Polchinski equation [1] is a partial differential equation for the renormalized effective action in quantum field theory. It is a powerful tool for proving renormalizability of quantum field theory models and has been applied successfully in the study of the scalar ϕ^4_4 model [1, 2], the QED [3], the Gross-Neveu model [4], the noncommutative Grosse-Wulkenhaar model [5], ect. In this short presentation I will derive the Polchinski equation for the scalar Φ^4 model. An explicit smooth cutoff function for the momentum has been introduced and the integration-by-parts formula has been used.

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Stochastic control approach to Φ^4_2 measure

Abdulwahab Mohamed

We discuss the stochastic control approach for Φ^4_2 measure in [1]. We introduce a convenient way of regularising the measure which naturally leads to a stochastic process. The regularised measure is given by ν_t and has the form

$$\nu_t(\mathrm{d}\varphi) = \mathcal{Z}_t^{-1} e^{-V_t(\varphi_t)} \mu(\mathrm{d}\varphi),$$

where $\varphi_t = \rho_t * \varphi$ is a mollified distribution with $\rho_t \to \delta_0$, μ is the Gaussian free field on $\mathbb{T}^2$, $\mathcal{Z}_t$ is the partition function and V_t is a suitable function. We construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a process $(Y_t)_{t\geq 0}$ such that $\mathrm{Law}_{\mathbb{P}}(Y_t) = \mathrm{Law}_{\mu}(\varphi_t)$. With this formulation, we have for any $A \in \mathcal{F}$

$$\nu_t(A) = \mathcal{Z}_t^{-1} \mathbb{E}[\mathbf{1}_A(Y_t) e^{-V_t(Y_t)}].$$

From there we see that it is enough to study the process $(Y_t)_{t\geq 0}$ and the density $\mathcal{Z}_t^{-1} e^{-V_t(Y_t)}$.

The process $(Y_t)_{t\geq 0}$ is constructed by an Itô integral with respect to a Brownian motion which enables us to use techniques from stochastic calculus. For instance, using Girsanov's transform, we establish a direct link between the measure ν_t and a stochastic control problem. The control problem is based on Boué–Dupuis formula which allows us to express $-\log \mathbb{E}[e^{-pV_t(Y_t)}]$ in terms of a minimisation problem. The functional that we minimise can be easily bounded from above and below after suitable renormalisation. These bounds are based on fairly standard inequalities in Sobolev spaces, for example duality, product rule and interpolation. In the bounds, we exploit the fact that the Gaussian free field on $\mathbb{T}^2$ has all its Wick powers in $C^{-\kappa}$ for any $\kappa > 0$. The bounds obtained for the minimisation problem then leads to a lower and upper bound for the quantity $\mathbb{E}[e^{-pV(Y_t)}]$. From there we can show that the Radon-Nikodym derivative $\mathcal{Z}_t^{-1}e^{-V(Y_t)}$ is bounded in $L^p(\mathbb{P})$ uniformly in $t \geq 0$. This yields tightness of the measure ν_t for which the limit as $t \to \infty$ is going to be a candidate for the Φ_2^4-measure.

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Log-Sobolev Inequality

Henri Elad Altman, Chunqiu Song

Consider our models on finite lattice $\Lambda = \Lambda_{\varepsilon,L} \subset L\mathbb{T}^d \cap \varepsilon\mathbb{Z}^d$ and field configurations are denoted by $\varphi : \Lambda \to \mathbb{R}$. The Hamiltonian $H(\varphi)$ is of the form $H(\varphi) = \frac{1}{2}(\varphi, A\varphi) + V(\varphi)$ where $(f,\, g) := \varepsilon^d \sum_{x\in\Lambda} f_x \cdot g_x, V(\varphi) = \varepsilon^d \sum_{x\in\Lambda} \mathcal{V}(\varphi_x)$ with $\mathcal{V}$ bounded below, and A is positive. The statistical property of the system is given by the Gibbs measure $\frac{1}{Z}e^{-H(\varphi)}d\varphi$, where $d\varphi$ is the Lebesgue measure on $\mathbb{R}^\Lambda$, and $Z = \int_{\mathbb{R}^\Lambda} e^{-H(\varphi)}d\varphi$. The Gibbs measure is attained by the equilibrium measure of the Glauber-Langevin dynamics

$$d\varphi_t = -\nabla H(\varphi_t)\, dt + \sqrt{2}dW_t$$

whose solution is a Markov process with the Markovian semigroup P_t. How fast does it equilibrate? Denote $\mathrm{Law}(\varphi_t) = \nu_t$, hence $\nu_\infty = \frac{1}{Z}e^{-H(\varphi)}d\varphi$. For a positive function G, a probability measure ν and $\Phi(x) = x\ln(x)$, define the entropy and Fisher information to be

$$\mathrm{Ent}_\nu(G) := \mathbb{E}_\nu[\Phi(G)] - \Phi(\mathbb{E}_\nu[G]), \quad \mathbb{I}_\nu(G) := 4\mathbb{E}_\nu\left[(\nabla\sqrt{G})^2\right].$$

The log-Sobolev inequality

$$\mathrm{Ent}_{\nu_\infty}\left(\frac{d\nu_t}{d\nu_\infty}\right) = \mathbb{H}(\nu_t \mid \nu_\infty) \leq \frac{1}{2\gamma}\mathbb{I}(\nu_t \mid \nu_\infty) = \frac{1}{2\gamma}\mathbb{I}_{\nu_\infty}\left(\frac{d\nu_t}{d\nu_\infty}\right)$$

implies that te dynamics equilibrates exponentially fast with rate γ:

$$\|\nu_t - \nu_\infty\|_{\mathrm{TV}}^2 \leq 2\mathbb{H}(\nu_t \mid \nu_\infty) \leq 2e^{-2\gamma t}\mathbb{H}(\nu_0 \mid \nu_\infty)$$

where the first inequality is given by Pinsker.

Definition 1. (Log-Sobolev Inequality) A probability measure ν on $\mathbb{R}^\Lambda$ is said to satisfy the log-Sobolev inequality with constant γ, if for all bounded smooth positive function $G : \mathbb{R}^\Lambda \to \mathbb{R}^+$, the inequality $\mathrm{Ent}_\nu(G) \leq \frac{1}{2\gamma}\mathbb{I}_\nu(G)$ is true. The largest choice of γ is called the log-Sobolev constant of ν.

One natural question is: when does the Gibbs measure satisfy the log-Sobolev inequality? By decomposing the entropy along the Langevin dynamics, one can show the following:

Theorem 2. (Bakry-Emery) If there is a constant $\lambda > 0$ such that for all $\varphi \in \mathbb{R}^\Lambda$, the inequality Hess $H(\varphi) \geq \lambda \,\mathrm{id}$ (where id denotes the identity matrix) is true, then the Gibbs measure satisfies the LSI with log-Sobolev constant $\gamma \geq \lambda$.

In the UV limits of continuum models, the divergent counterterms break the convexity of the Hamiltonian, hence the theorem does not apply. One way to generalise the theorem is given by decomposing the entropy along another process inspired by Wilson's RG (see [1], [2]). The kinetic part $\frac{1}{2}(\varphi, A\varphi)$ in the Hamiltonian provides a Gaussian in the Gibbs measure with covariance A^{-1}. We assume there is a scale decomposition of covariance A^{-1} with the form $A^{-1} = C_\infty = \int_0^\infty \dot{C}_s ds$ where $\dot{C}_t$ are assumed to be positive definite with $C_0 = 0$. The idea is to build up a dynamical system by keep averaging out the part of the field with smaller scale interactions (corresponding to C_s) which result in an updating of the effective interaction in large scale (corresponding to $C_\infty - C_s$), and hence an updating of the measure. That is to consider the renormalised measure ν_s defined as

$$\frac{1}{Z} e^{-\frac{1}{2}\left(\varphi,(C_\infty - C_s)^{-1}\varphi\right)} \int e^{-V(\varphi+\psi)} e^{-\frac{1}{2}\left(\psi, C_s^{-1}\psi\right)} d\psi d\varphi$$
$$= \frac{1}{Z} e^{-\frac{1}{2}\left(\varphi,(C_\infty - C_s)^{-1}\varphi\right) - V_s(\varphi)} d\varphi$$

where we define the renormalised potential

$$V_s(\varphi) = -\ln \int e^{-V(\varphi+\psi)} e^{-\frac{1}{2}\left(\psi, C_s^{-1}\psi\right)} d\psi.$$

Theorem 3. (Bauerschmidt&Bodineau 21) Suppose $\dot{C}_t$ is differentiable, and there is some real-valued functions $\dot{\lambda}_t$ such that

$$\dot{C}_t \,\mathrm{Hess}\, V_t(\varphi) \dot{C}_t - \frac{1}{2}\ddot{C}_t \geq \dot{\lambda}_t \dot{C}_t \quad \forall \varphi \in \mathbb{R}^\Lambda \text{ and } t > 0$$

and define $\lambda_t = \int_0^t \dot{\lambda}_s ds$ and $\frac{1}{\gamma} = \int_0^\infty e^{-2\lambda_t} dt$. Then ν_0 satisfies the LSI $\mathrm{Ent}_{\nu_0}[G] \leq \frac{1}{2\gamma} \mathbb{I}_{\nu_0}(G)_{\dot{C}_0}$.

The above result, known as a multi-scale Bakry-Emery criterion, can be used to derive log-Sobolev inequalities for the continuum Sine-Gordon model, which is a 2-dimensional model ($d = 2$) described by the probability measure on $\mathbb{R}^\Lambda$ given by

$$\nu_{\varepsilon,L} \propto \exp\left(-\frac{1}{2}(\varphi, A\varphi) - V_0(\varphi)\right) d\varphi,$$

where $A\varphi = (-\Delta^\varepsilon \varphi + m^2 \varphi)$, with Δ^ε the discrete Laplacian on $\mathbb{R}^\Lambda$ and

$$V_0(\varphi) = 2\varepsilon^2 \sum_{x \in \Lambda} \varepsilon^{-\beta/4\pi} z \cos(\sqrt{\beta}\varphi_x).$$

Here, $m > 0$ is a mass term, $z \in \mathbb{R}$ is the coupling constant, and $\beta \in (0, 8\pi)$. The above potential is highly non-convex, all the more so as the non-convexity is amplified by the diverging renormalisation parameter $\varepsilon^{-\beta/4\pi}$ entering the picture as we take the continuum UV limit $\varepsilon \to 0$. However, for $\beta < 6\pi$, the multi-scale Bakry-Emery criterion applies to the effective potential V_t associated with the renormalisation semi-group, and provides a LSI with a constant that is uniform in ε. To show the required bounds on $\operatorname{Hess} V_t$ one exploits the fact that this effective potential solves the so-called Polchinski PDE, in order to represent it using an Ansatz due to Brydges and Kennedy [3], with coefficients that can be bounded uniformly in ε.

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