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Geproci sets: a new perspective in algebraic geometry

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Geproci sets arise from applying the perspective of inverse scattering problems to algebraic geometry. Analogous to the reconstruction of an object from multiple X-ray images, we aim at a classification of sets with certain algebraic properties under multiple projections.

1 Algebraic geometry and transverse complete intersections

A fundamental goal of algebraic geometry is to understand solution sets of systems of *polynomial equations*, that is, equations obtained by only taking sums, differences, products, and positive-integer powers of variables and numbers. For example, the points in the (x, y)-plane that are solutions to $x^2+y^2-4=0$ form a curve (in fact, it is the circle of radius 2 shown in Figure 1). As another example, there are exactly six points (x, y) that are solutions to both $x^2+y^2-4=0$ and $2x^3-4x-y=0$. We can see this in Figure 1 by graphing the solutions to each equation and seeing where the resulting curves cross.

The highest exponent (or sum of exponents, if multiple variables are involved) in any of the summands of a polynomial is called its *degree*. For example, the polynomial x^2+y^2-4 has degree 2 and $2x^3-4x-y$ has degree 3. The polynomial $x^2y^3+xy^2+y^2+x$ has degree 5 because the exponents in the first summand x^2y^3 add up to 5 and in the other summands, they add up only to 3, 2, and 1,

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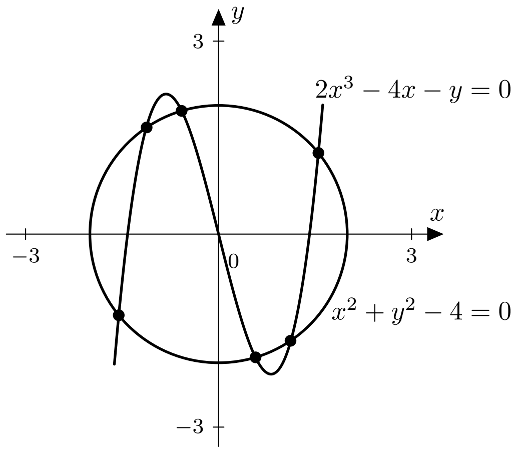

Figure 1: A transverse complete intersection consisting of the six points where the circle (which is a curve of degree 2) crosses a curve of degree 3.

respectively (the exponent of $x = x^1$ counts as 1). The solution set of a polynomial equation of degree n is called a *curve of degree n*, even though it could, for example, also consist of n straight lines (this is the case if the polynomial is the product of n factors of the form $sx + ty - u$).

An important theorem, known as Bézout's Theorem, states that if $f(x, y)$ is a polynomial of degree a and $g(x, y)$ is a polynomial of degree b, then the system of equations $f(x, y) = 0$ and $g(x, y) = 0$ has either infinitely many solutions or at most ab solutions. If the number of solutions is exactly ab, we call the set of solutions a *transverse complete intersection*. The set of six intersection points in Figure 1 is therefore an example of such a transverse complete intersection.

Note that transverse complete intersections are special finite sets: For example, seven points that are not all on a straight line cannot be a transverse complete intersection. Since 7 is prime, the only possibility for $7 = ab$ is for $a = 1$ and $b = 7$. Thus, for seven points to be a transverse complete intersection, g must have degree 7 and f must have degree 1. Thus, $f(x, y)$ must be of the form $sx+ty-u$ for some numbers s, t (not both zero), and u, but $sx+ty-u = 0$ defines a line, so the seven points would all have to be on this line.

2 Inverse scattering problems

The study of inverse scattering problems, while typically rather remote from pure mathematics, has been hugely important in science and technology. For example, in Ernest Rutherford's famous experiment, he measured the angles at

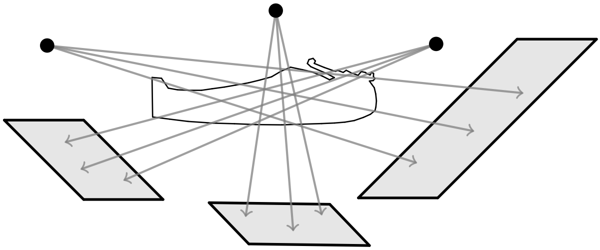

Figure 2: Tomography applied to a sarcophagus as an example for an inverse scattering problem.

which individual alpha particles were scattered after hitting a gold foil. From this scattering pattern, he concluded that the positive charge in an atom must be concentrated in a tiny nucleus rather than spread around the entire atom. Rosalind Franklin's famous experiments, in which she looked at the diffraction pattern of X-rays directed at a DNA sample, led to the double helix model of DNA. Tomography gives another example (see Figure 2); here, X-ray images are taken of a three-dimensional object from various angles.

In all three examples, we try to work backward from the scattering data (scattering angles of alpha particles, a diffraction image from a DNA sample, or multiple X-ray images) and determine the internal three-dimensional structure of the original object. In this sense, we try to *invert* the scattering process that produced the data from the object, hence the term *inverse scattering problems*.

Reconstructions of a finite set of points in three-dimensional space from a series of projections can also be very useful. This is essentially the basis for depth perception in binocular vision. Radar images of the surface of the earth from satellites can be converted to 3D images by reconstructing the position of specific points in the landscape (such as mountaintops or points along a coast).

3 Inverse scattering problems in algebraic geometry

Given a finite system of polynomial equations (possibly involving more than two variables), we usually cannot find the solution set exactly, so an important aspect of algebraic geometry is to understand qualitative properties of the solution set. For instance, Bézout's Theorem is concerned only with how many solutions there are. We could also ask what is the dimension of the set of solutions (do they comprise a finite set, a curve, or a surface?). If they comprise a surface, for example, we can ask about properties of the surface (is it a sphere, donut-shaped, or something else?). More generally, we can ask which

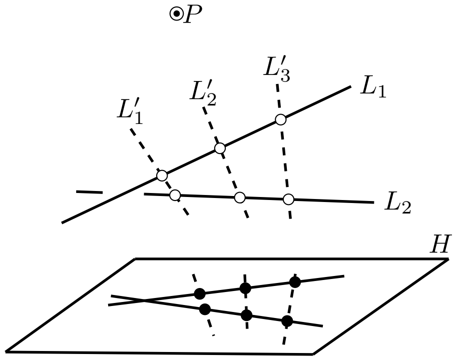

Figure 3: The six white dots give a $(2,3)$-grid, which is also a $(2,3)$-geproci set. The projections (black dots) of the white dots onto the plane H form a transverse complete intersection of the two solid and three dashed lines in H (remember that n lines together form a curve of degree n).

surfaces can arise as solutions to systems of polynomial equations. This is an example of a classification problem in algebraic geometry (for further reading on classification problems, see Snapshot 2/2023 [8]).

The recent work [1] applies the inverse scattering perspective to classification problems in algebraic geometry for the first time. Instead of trying to classify finite point sets with a certain property, the idea of [1] is to try to classify finite point sets in three-dimensional space whose projection from almost any point to almost any plane has the property of interest. This approach is analogous to asking what we know about an object if all of its X-ray-images (as in Figure 2) have a certain property.

Let us first define what we mean by *projection*: For any plane H and any point P not in H, the projection of a point p (other than P) from P to H is where the line $L(P,p)$ through P and p intersects H. The projection of a set S from P to H is obtained by projecting every point of S to H. Intuitively, the projection is the shadow that S casts on H when you place a light source at P.

Depending on P, S, and H, it is possible that the line $L(P,p)$ does not intersect H (it might be parallel to H) or that multiple points of S have the same projection to H. These cases are exceptional because they only occur if P lies in some very specific positions with regard to S and H. Whenever we speak of *general points*, we mean “most points”, but exclude such cases.

The main focus of [1] is to consider finite sets S of points in three-dimensional space whose projections from all general points P are transverse complete intersections in H. Such a set is called an (a,b)-*geproci* set if its *ge*neral *pro*jection is a transverse *c*omplete *i*ntersection of curves of degrees a and b.

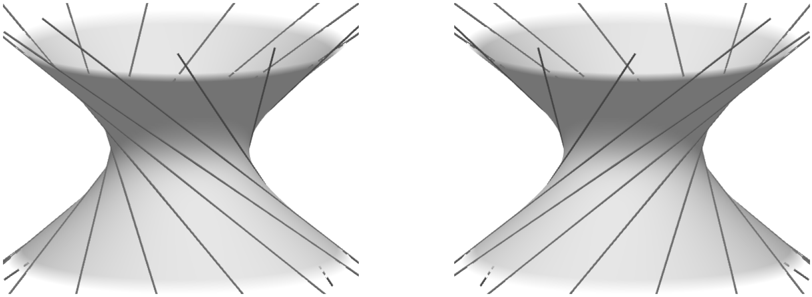

Figure 4: Two views of the same hyperboloid, $x^2 + y^2 - z^2 - 1 = 0$, showing how it consists of two families (called *rulings*) of skew lines. Any line from one ruling intersects almost any line from the other ruling in exactly one point on the surface.

Easy examples of geproci sets S are obtained by taking a transverse complete intersection S contained in a plane H'. Its projection from a general point to any other plane H will also be a transverse complete intersection in H. The interesting question is, what geproci sets can there be that are not contained in any plane? This question was first raised in 2011 [6] by Francesco Polizzi, and again in 2018 [3].

Until 2018, the only known examples of geproci sets not contained in a plane were those pointed out by Dimitri Panov [6], namely *grids.* Figure 3 shows an example of a grid. In particular, it shows a $(2, 3)$-grid, consisting of the $2 \cdot 3 = 6$ points represented as white dots in the figure, with three points on each of the two skew lines, L_1 and L_2. (*Skew* means L_1 and L_2 are not parallel but do not meet.) To construct a $(2, 3)$-grid, we can choose any three different points on L_1 and three different points on L_2, and connect pairs of them by lines; these are the lines L'_1, L'_2, and L'_3 in the figure.

The white dots in Figure 3 project from the general point P to the black dots in the plane H, giving a transverse complete intersection in H. The lines L_1 and L_2 could in a similar way each have an arbitrary number $b \geq 2$ of points, which would give a $(2, b)$-grid.

It is also possible to have an (a, b)-grid with $a > 2$ skew lines $L_1, \ldots, L_a$, but then you have to be more careful how you pick these lines and how you pick the lines $L'_1, \ldots, L'_b$. You want the lines $L'_1, \ldots, L'_b$ also to be skew, and you want that each line L_i meets every line L'_j at exactly one point. The (a, b)-grid then consists of the ab points where each L_i crosses each L'_j. Two sets of lines, $L_1, \ldots, L_a$ and $L'_1, \ldots, L'_b$, that meet these conditions can be obtained by picking a lines from one side and b lines from the other side of Figure 4.

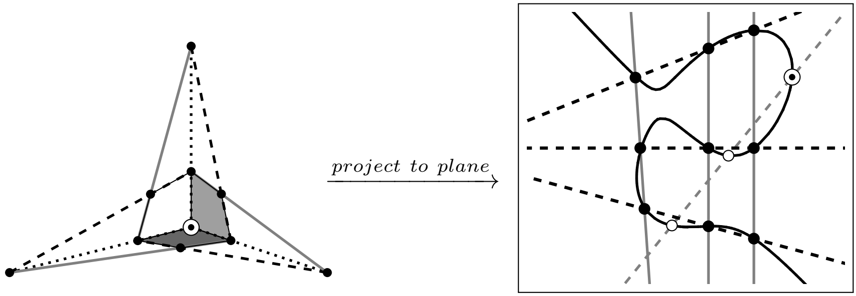

Figure 5: The D_4 configuration of 12 points and its projection.
The center of the cube and the back top corner of the cube (white points) are not shown in the left figure.

Until 2018, no nonplanar examples of geproci sets were known other than grids. But at a math conference in Levico Terme in 2018, it was noticed, based on the work of [4], that nonplanar non-grid examples exist, coming from symmetrical sets of points known as *root systems* (see the appendix to [2]). The simplest example is given by the D_4 configuration of 12 points. The 12 points consist of the eight vertices of a cube, the center of the cube, and the three points of perspective of the cube. (These three points will be at infinity unless one deforms the cube a bit.) This set of 12 points is $(3, 4)$-geproci since its projection, as shown in Figure 5, is a transverse complete intersection of a curve of degree 3 (shown in black) and a curve of degree 4 consisting of four lines (the three solid gray lines in the right figure and the dashed gray line). These four lines are the projections of the solid gray lines in the left figure together with the diagonal of the cube (not shown in the figure) going through the front bottom corner of the cube, the center of the cube, and the back top corner of the cube.

It is less obvious where the black curve of degree 3 through the projections of the 12 points comes from. The three black dashed lines in the left figure project to the three black dashed lines in the right figure, and we can see that the projections of the nine black points are a transverse complete intersection of the three black dashed lines with the three gray solid lines in the right figure. If the curve given by the gray lines has the equation $f(x, y) = 0$ and the curve given by the black dashed lines has the equation $g(x, y) = 0$, then any curve of the form $s \cdot f(x, y) + t \cdot g(x, y) = 0$, with some numbers s and t (not both zero), has degree 3 and goes through the black points. For some values of s and t, it also goes through the marked point, and it turns out, due to the symmetry of the D_4 configuration, that this curve goes through all 12 points. In fact, it is the curve shown in black in the right figure.

Other root systems also give examples, such as F_4 [2] and H_4 [7]. A non-root system example was also found [5]. Understanding these examples led to one of the main results of [1]: for each a, b with $4 \leq a$ and $a \leq b$, there exists a nonplanar non-grid (a, b)-geproci set. One of the other main results of [1] is to show, up to the choice of coordinates, that there is only one nonplanar, non-grid (a, b)-geproci set with $a \leq 3$ and $a \leq b$, namely the D_4 configuration.

Many questions remain. One of the biggest is whether every nonplanar non-grid geproci set must have at least three points that are on a straight line. All currently known examples do, but it is not known if being geproci forces this to be the case.

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