Mathematisches Forschungsinstitut Oberwolfach

Report No. 6/2024

DOI: 10.4171/OWR/2024/6

# Mini-Workshop: Permutation Patterns

Organized by
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28 January – 02 February, 2024

Abstract. The study of permutation patterns has recently seen several surprising results, and the purpose of this mini-workshop was to bring together researchers from across the field to focus on four hot topics related to these recent developments. The topics covered the nature of generating functions that enumerate permutation classes, the structure of permutation classes and the impact this has on their growth rates, and the study of permutons, which lies at the interface of permutation patterns and discrete probability. The workshop offered an opportunity for knowledge exchange, but also time and space to initiate group collaborations on open problems related to these topics.

*Mathematics Subject Classification (2020):* 05A05, 05A15, 05A16.



## Introduction by the Organizers

The mini-workshop *Permutation Patterns*, organized by Miklós Bóna (Gainesville), Mathilde Bouvel (Vandoeuvre-lès-Nancy), Robert Brignall (Milton Keynes) and Jay Pantone (Milwaukee) was attended by 17 participants, one of whom attended remotely. Amongst the participants were researchers of varying levels of academic seniority from a broad variety of geographic locations, and included several early career researchers, and several females.

The aims of the mini-workshop were to bring together researchers with varied interests in permutation patterns to tackle four hot topics in the study of permutation classes, building on several recent and surprising results. A *permutation class* is a downset of permutations under the permutation pattern order, and the questions of interest here include those concerning asymptotic and precise enumeration, structural considerations and permutation limits. The four topics for the workshop were as follows.

**Topic 1: Negative results concerning generating functions** Until recently, relatively few techniques have been available to prove, for example, that a particular generating function *cannot* have a particular form, such as rational, algebraic, or D-finite. Some methods for classifying generating functions are now coming on-stream, and – furthermore – there are several candidate permutation classes whose enumeration has proved elusive.

**Topic 2: Structure and positive classification of generating functions** This, in a sense, is the converse approach to Topic 1, whereby one can exploit understanding of the structure present in a given permutation class or family of permutation classes to make conclusions about the nature of their generating function. For example, it is known that every class with growth rate less than $\kappa \approx 2.206$ has a rational generating function.

**Topic 3: Classification of growth rates** Some permutation classes, most notably Av(1324), have proved so difficult to enumerate that only rough bounds on the growth rate are known. Meanwhile, there also exists the question of whether a given positive real number is the growth rate of *some* permutation class. It is known which real numbers below $\xi \approx 2.305$ are growth rates of permutation classes, and also that every real number $\geq \lambda_B \approx 2.357$ is the growth rate of some permutation class, which leaves a gap of just over 0.05 to complete the classification.

**Topic 4: Permutons** A more recent, but now well-established topic lies at the interface of permutation patterns and probability theory, and addresses questions of the form: what does a permutation in a given class typically look like?

The workshop schedule was designed to balance opportunities for knowledge exchange (in the form of formal or informal talks) with time for collaborative work in groups. Each day began at 9am with a (sometimes very short) plenary session, allowing an opportunity for knowledge exchange as well as to make administrative announcements. On Monday, three one-hour talks were given (two in the morning, one in the evening) to introduce the four topics (one talk covered both Topics 2 and 3) and pose some open problems. After the final talk, an open problem session was held in which participants were invited to present problems that could be worked on during the week. The format of the open problem session was as follows: one organizer (Jay Pantone) acted as 'scribe' by writing the open problem and any relevant background material on the board, as directed by the setter of the open problem from their chair. This ensured that each problem was presented in a clear yet concise way, and at a pace that ensured all participants could follow what was being presented.

Tuesday began by concluding the open problem session. All problems that were presented during the sessions on Monday and Tuesday were written up by Justin Troyka, and have been included in this report. Until 4pm, the day was given over to working on open problems. In order to ensure openness in forming groups, participants were encouraged to write on a blackboard in the main lecture theatre what they planned to work on, so that others could join if they wished. At the end of the day, an expository talk on 'twin width' was given by Opler at 4pm, upon the request of several participants who were interested to hear how this hot topic might apply to the study of permutation classes.

By Wednesday morning, a counterexample to one question presented during Bevan's introductory talk on topic 4 had been found by Troyka, so our day began with a brief presentation of this. The rest of the morning was devoted to time for group collaboration, and the afternoon, of course, was given over to the walk to Sankt Roman. Thursday was entirely devoted to group collaborative work.

On Friday morning, each collaborative group was invited to give a short presentation regarding progress that had been made during the week. The speakers have recorded their talks in this report, but there are several highlights worth mentioning here: Blitvić, Elvey Price and Troyka established a result relating to Hamburger moment sequences of principal matching classes. Bousquet-Mélou, Bouvel and Pantone found an explicit expression for the generating function of $\mathrm{Av}(4123, 4231, 4312)$, a class for which 5 000 terms of the enumeration sequence are known, and whose generating function is conjectured to be non-D-finite. Bóna, Brignall, Defant, Opler and Vatter identified several promising avenues of research related to classifying generating functions in subclasses of the separable permutations. Finally, at the start of the week, Bouvel introduced an open problem concerning *$k$-shuffles* which attracted a lot of interest from the workshop participants, and Bevan, Elvey Price and Felsner established a characterisation of permutations that are *not* $k$-shuffles of 21 (or, symmetrically, 12).

Overall, the mini-workshop offered a rare opportunity for a group of researchers to come together and to spend time discussing mathematical problems. Many participants expressed their gratitude for having long periods of unbroken time, especially on Tuesday and Thursday, to devote to group collaboration. We anticipate several outputs will become available in due course, as a direct result of work that was inspired by, or started during the mini-workshop.

We are grateful to Colin Defant for acting as Video Conference Assistant to enable to remote participant to attend talks. We would, of course, like to thank all the staff at Oberwolfach for enabling this mini-workshop to take place in such a smooth and satisfactory way.

*Acknowledgement:* The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

## Mini-Workshop: Permutation Patterns

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# Abstracts

## Negative Results

Miklós Bóna

When we want to solve a combinatorial enumeration problem and we fail, we may want to have some form of measurement of the difficulty level of the problem. One way to achieve that is by proving *negative results* on the generating function of the counting sequence at hand, like proving that the generating function is not rational, not algebraic, not differentiably finite or maybe not even differentially algebraic.

Let $A_q(z)$ be the ordinary generating function for the sequence counting permutations of length $n$ that avoid the pattern $q$. In this talk, we show that for most patterns $q$, the power series $A_q(z)$ is not rational. The only patterns for which our proof does not work are, up to trivial symmetries, the patterns of length $k$ that start with the entry 1, end with the entry $k$, and are not Wilf-equivalent with patterns that do not share those properties. It is worth pointing out that the shortest pattern $q$ for which we cannot prove nonrationality of $A_q(z)$ is 1324. (For the trivial pattern $q = 12$, we have $A_q(z) = 1/(1-z)$, which is rational.)

One tool we use here is the theory of *supercritical relations* [5]. Let $F$ and $G$ be two generating functions with nonnegative real coefficients that are analytic at 0, and let us assume that $G(0) = 0$. Then the relation

$$F(z) = \frac{1}{1 - G(z)}$$

is called *supercritical* if $G(R_G) > 1$, where $R_G$ is the radius of convergence of $G$. It is easy to prove that if the $F$ and $G$ are rational, then their relation displayed above is supercritical. Therefore, in order to prove that a generating function is not rational, it suffices to prove that the corresponding relation is not supercritical. And a way to prove that is by showing that the exponential order of the number of $q$-avoiding permutations of length $n$ is the same as that of *indecomposable* $q$-avoiding permutations of length $n$.

Another approach is the following. Let $A_{q,i}(z)$ be the ordinary generating function for the number of permutations of length $n$ that avoid $q$ and have exactly $i$ skew components. It can be proved that if $q$ is as described two paragraphs earlier, then for all $n$, the number of such permutations of length $n$ with $i = 1$ is at least as large as the number of those with $i = 2$. So the chain of inequalities

$$A_{q,2}(z_0) = (A_{q,1}(z_0))^2 \leq A_{q,1}(z_0)$$

must hold for all $z_0$ inside the circle of convergence of $A_{q,1}$. However, that means that at such points $z_0$, the inequality

$$A_{q,1}(z_0) < 1$$

must hold, implying that the dominant singularity of $A_{q,1}(z)$ cannot be a pole, so in particular, $A_{q,1}(z_0)$, and therefore, $A_q(z)$, cannot be rational.

For the remaining cases, (those in which we could not prove that $A_q(z)$ is not rational), we show that the dominant singularity of $A_q(z)$ cannot be a multiple pole. Indeed, if $a_n$ denotes the number of permutations avoiding a given single pattern $q$, then it is known that $a_n a_m \leq a_{n+m}$. On the other hand, if the sequence $a_n$ has a rational generating function then $a_n \simeq \alpha^n \cdot n^d$ must hold, for a nonnegative integer $d$. These two conditions can simultaneously hold only if $d = 0$, which implies that if the dominant singularity of $A_q(z)$ is a pole, it is a simple pole.

A power series $A(z)$ is called *algebraic* if there are polynomials $P_0(z), \cdots, P_d(z)$ that are not all identically zero so that the equality

$$P_0(z) + P_1(z)A(z) + P_2(z)A^2(z) + \cdots + P_d(z)A^d(z) = 0$$

holds. See Section 6 of [7] for a high-level introduction to the theory of algebraic power series. Until recently, the only general, direct method to prove non-algebraicity of a generating function $A_q(S)$ was the following theorem of Jungen [6]. Let $m$ be a positive integer, let $c$ and $\gamma$ be positive constants, and let $A(z) = \sum_{n\geq 0} a_n z^n$ be a power series with complex coefficients. If

$$a_n \simeq c\frac{\gamma^n}{n^m},$$

then $A(z)$ is not an algebraic power series.

However, the following tool that was recently developed by Alin Bostan is stronger. Let $A(z) = \sum_{n\geq 0} a_n z^n$ be a power series with nonnegative real coefficients that is analytic at the origin. Let us assume that constants $c$, $C$, $K$ and $m$ exist so that $m > 1$ is an integer, and for all positive integers $n$, the chain of inequalities

$$c\frac{K^n}{n^m} \leq a_n \leq C\frac{K^n}{n^m} \tag{1}$$

holds. Then $A(z)$ is not an algebraic power series.

We show an application of this method to prove that the class of permutations avoiding the patterns 12354, 12453, 13452, and 23451 is not algebraic. This result can be generalized to classes defined by longer patterns in an analogous way. We also show that the same result holds for the class of permutations avoiding the patterns 21354, 21453, 31452, and 32451.

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## Permutons and permutation class limit shapes

David Bevan

Permutons are the analytic limits of convergent sequences of permutations, and provide an effective way of describing the shape of a typical large element of a permutation class. We present the relevant theory concerning permutons before surveying what is currently known about permutation class limit shapes and putting forward conjectures concerning some of what remains to be established.

Formally, a permuton is a probability measure $\mu$ on the unit square that has uniform marginals:

$$\mu([a,b]\times[0,1]) \;=\; \mu([0,1]\times[a,b]) \;=\; b-a \text{ for every } 0\leqslant a\leqslant b\leqslant 1.$$

A permuton can be used to randomly sample $k$ points. Given that the probability of sharing a coordinate is zero, the order of the $y$-coordinates gives a *$\mu$-random* permutation of length $k$. This enables us to define the notion of the *density* $\rho(\tau,\mu)$ of a pattern $\tau$ of length $k$ in a permuton $\mu$:

$$\rho(\tau,\mu) \;=\; \mathbb{P}\big[\text{a } \mu\text{-random permutation of length } k \text{ equals } \tau\big].$$

We also have the analogous notion of the density $\rho(\tau,\pi)$ of a pattern $\tau$ of length $k$ in a permutation $\pi$ of length $n$ being $\nu(\tau,\pi)/\binom{n}{k}$, where $\nu(\tau,\pi)$ is the number of occurrences of $\tau$ in $\pi$.

With these notions, we can define convergence for a sequence of permutations to a limit permuton. If $|\pi_j|\to\infty$, then $(\pi_j)_{j\in\mathbb{N}}$ is convergent if $\rho(\tau,\pi_j)$ converges for every pattern $\tau$. The permuton $\mu$ is the limit if $\lim_{j\to\infty}\rho(\tau,\pi_j)=\rho(\tau,\mu)$ for every pattern $\tau$.

Our main interest is permutation class *limit shapes*. Given a permutation class $\mathcal{C}$, let $\boldsymbol{\sigma}_n^{\mathcal{C}}$ be a random permutation of size $n$ drawn uniformly from $\mathcal{C}_n$. If $\mu_{\boldsymbol{\sigma}_n^{\mathcal{C}}}$ converges to some (possibly random) permuton $\boldsymbol{\mu}_{\mathcal{C}}$, what can we say about its scaling limit $\boldsymbol{\mu}_{\mathcal{C}}$? Obviously, every permutation class scaling limit is singular, since the density of any pattern in a nonsingular permuton is positive.

Existing results include the following:

- The limit shape of Av(321) and Av(312) is the increasing permuton. See [9].
- The limit shape of Av($12\ldots k$) and Av($12\ldots\ell k(k-1)\ldots(\ell+1)$) is the decreasing permuton. See [8].
- The deterministic limit shape of any connected monotone grid class is well understood. See [5] and [1].
- The random limit shape of the class of square permutations (in which every point is a left-to-right or right-to-left minimum or maximum) is a randomly chosen rotated rectangle, with fluctuations are described by certain coupled Brownian motions. See [7].

- Given a class of permutations with a *finite specification* whose dependency graph is strongly connected. Then its limit shape is a deterministic *X-permuton*, if the specification is *essentially linear*, or a random *Brownian separable permuton*, if it is *essentially branching*. See [4, 2, 3, 6].

We make the following three, increasingly general, conjectures:

**Conjecture 1** (Bevan, Dijon 2023)**.** *If every pattern in $B$ ends in 1, then the scaling limit of* $\mathrm{Av}(B)$ *is the increasing permuton.*

**Conjecture 2** (Bevan, Oberwolfach 2024)**.** *If every pattern in $B$ ends in 1 or begins with its largest value, then the scaling limit of* $\mathrm{Av}(B)$ *is the increasing permuton.*

**Conjecture 3** (Troyka)**.** *If every pattern in $B$ is skew decomposable, then the scaling limit of* $\mathrm{Av}(B)$ *is the increasing permuton.*

To prove any of these, it would be sufficient to establish that $\mathbb{E}\big[\rho(21, \boldsymbol{\sigma}_n)\big] \to 0$, or equivalently, that $\mathbb{E}\big[\mathrm{inv}(\boldsymbol{\sigma}_n)\big] \ll n^2$. However, in all the cases in which we know the scaling limit, we have a nice structural specification of the class.

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## Exact and asymptotic enumeration of permutation classes


Vincent Vatter

(joint work with Albert, Atkinson, Bouvel, Brignall, Ruškuc, and Pantone)


This talk attempts to summarize some results and questions concerning the classification of generating functions of permutation classes, one of the four topics of the mini-workshop. Here a *permutation class* is a downset of permutations under the permutation pattern order (for example, $\pi = 372694185$ contains $\sigma = 32514$, as witnessed by its subsequence 32918, but $\pi$ avoids 54321 because it has no decreasing subsequence of length five). I refer to my survey [9] for any terms or concepts not defined herein.

We begin by recalling a result that shows that the rationality of the generating function of a permutation class cannot in general imply that the class has nice structure. Note that the proof of this result uses the Marcus–Tardos theorem [8].

**Theorem 1** (Albert, Brignall, and Vatter [3, Theorem 3.2]). *Every permutation class except for the class of all permutations is contained in a class with a rational generating function.*

Instead, one may want to consider the *strongly rational classes*—those permutation classes for which themselves and all their subclasses have rational generating functions. The following result characterizes the strongly rational subclasses of the separable permutations (the smallest class of permutations that can be constructed via direct and skew sums of smaller permutations from the class, starting with the permutation 1).

**Theorem 2** (Albert, Atkinson, and Vatter [2, Theorem 4.1]). *If a subclass of the separable permutations does not contain* Av(231) *or any symmetry of this class, then it is strongly rational.*

A component in the proof of Theorem 2 was a result that the inflation of the class $\mathcal{X}$—defined as the smallest class of permutations that can be constructed via direct and skew sums *with the permutation* 1—by an arbitrary strongly rational class is also strongly rational. As the class $\mathcal{X}$ is geometrically griddable (in the sense of Albert, Atkinson, Bouvel, Ruškuc, and Vatter [1]), this result was later generalized to the following theorem.

**Theorem 3** (Albert, Ruškuc, and Vatter [4, Theorem 7.6]). *The inflation of a geometrically griddable class by a strongly rational class is also strongly rational.*

The nonrationality results of Bóna [5], suggest that it may be possible to ask about a converse to Theorem 2.

**Question 4.** *If a subclass of the separable permutations does contain all of* Av(231) *or one of its symmetries, must it have a nonrational generating function?*

By an elementary counting argument, such classes cannot contain infinite antichains, and thus they are *well-quasi-ordered* (*wqo*). Indeed, if a permutation class is *not* wqo, then it cannot even be *strongly algebraic*, that is, it contains subclasses with nonalgebraic generating functions (or strongly D-finite, or strongly D-algebraic, and so on).

**False Conjecture 1** (Vatter [9, Conjecture 12.3.4]). *A permutation class is strongly algebraic if and only if it is well-quasi-ordered.*

This conjecture has been disproved in a preprint of Brignall and Vatter [6]. They construct a family of well-quasi-ordered permutation classes that has uncountably many different enumeration sequences, thereby ensuring that there are well-quasi-ordered permutation classes without algebraic generating functions (as there are only countably many algebraic generating functions of combinatorial objects).

It is possible that one direction of False Conjecture 1 could be rescued by strengthening the hypotheses to require *labelled* well-quasi-ordering; see Brignall and Vatter [7], where the following question is stated.

**Question 5.** *Does every labelled well-quasi-ordered permutation class have an algebraic generating function?*

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## Open Problems

Justin Troyka (reporter)

### 1. Anders Claesson

Growth rate of Av(1324). The current bounds are

$$10.27 \le \mathrm{gr}(\mathrm{Av}(1324)) \le 13.5.$$

Consider $\mathrm{Av}_n^k(1324)$, the set of permutations of size $n$ with $k$ inversions.

**Conjecture:**

$$|\mathrm{Av}_n^k(1324)| \le |\mathrm{Av}_{n+1}^k(1324)|.$$

If true, we would have $\mathrm{gr}(\mathrm{Av}(1324)) \le e^{\pi\sqrt{2/3}} \approx 13.002$ (2012).
Another conjecture:

$$|\mathrm{Av}_n^k(1324, 231)| \le |\mathrm{Av}_{n+1}^k(1324, 231)|.$$

We may be able to use a bivariate generating function.

**Theorem** (Claesson–Ulfarsson):

$$|\mathrm{Av}_n^k(1324, 231, 4321)| \leq |\mathrm{Av}_{n+1}^k(1324, 231, 4321)|.$$

The domino subclass [Av(132) | Av(231)] (where we also require the whole thing to avoid 1324) is one of the main things used in the proof of the known bounds. Can we prove the inversion conjecture on the domino subclass?

## 2. Miklós Bóna

Consider patterns of size 6. Look at these three principal classes:

$$\mathcal{A} = \mathrm{Av}(132456), \qquad \mathcal{B} = \mathrm{Av}(124356), \qquad \mathcal{C} = \mathrm{Av}(123546).$$

All three have the same growth rate, which can be computed using Av(13245). But $\mathcal{A}$ and $\mathcal{B}$ have different counting sequences, whereas $\mathcal{C}$ has the same counting sequence as $\mathcal{A}$.

The question is: how different are the counting sequences of $\mathcal{A}$ and $\mathcal{B}$? Is their ratio a constant? Is their ratio a polynomial? What is that polynomial?

In general, I am interested in finding principal permutation classes that are very close to each other but not identical. How close can they be? How different can they be?

## 3. Natasha Blitvic and Andrew Elvey-Price

**Definition:** A *moment sequence* is a sequence $(a_0, a_1, \ldots)$ such that there exists a measure $\mu$ on $\mathbb{R}$ such that

$$a_n = \int_{\mathbb{R}} x^n \, d\mu(x).$$

And the sequence is a *Stieltjes moment sequence* if the support of $\mu$ is contained in $(0, \infty)$.

Example: $n! = \int_0^\infty x^n e^{-x} \, dx$; so $n!$ is a Stieltjes moment sequence with $d\mu(x) = e^{-x} \, dx$.

Question: When are the counting sequences of a permutation class a moment sequence (or Stiltjes moment sequence)?

Conjecture: It is a Stiltjes moment sequence for all principal classes.

This conjecture is supported by computational intuition from Elvey-Price and Tony Guttmann, and it is also supported by structural intuition from Blitvic and Steingrímsson.

If you have a Stiltjes moment sequence, then you get surprisingly good lower bounds on the growth rate. (This includes a better lower bound for 1324-avoiders!)

Next, define a *k-arrangement* to be a permutation with a $k$-coloring of its fixed points. For $k = 0$, this is interpreted to be a derangement. Let $A_n^k$ denote the number of $k$-arrangements of size $n$. It is a fact that

$$A_n^k = \int_{k-1}^\infty x^n e^{-x+(k-1)} \, dx.$$

This has a combinatorial interpretation when $k$ is a non-negative integer. We ask, what is a combinatorial interpretation when $k$ is not a positive integer?

Note that there is an alternative characterization of moment sequences, using continued fractions:

**Theorem:** $(a_n)$ is a Stiltjes moment sequence if and only if the generating function $\sum_{n\geq 0} a_n t^n$ is equal to

$$\frac{\alpha_0}{1-\frac{t\alpha_1}{1-\frac{t\alpha_2}{1-\cdots}}}$$

for $\alpha_i \geq 0$. And $(a_n)$ is a moment sequence if and only if the generating function is equal to

$$\frac{\delta_0}{1-\gamma_0 t-\frac{\delta_1 t^2}{1-\gamma_1 t-\frac{\delta_2 t^2}{\cdots}}}$$

for $\gamma_i \in \mathbb{R}$ and $\delta_i \geq 0$.

Usually, there is a unique way to write a generating function as a continued fraction in this way.

On the OEIS, about 5% of sequences are moment sequences, and about 2% are Stiltjes moment sequences.

## 4. Justin Troyka et al.

We have $\mathrm{Av}(\pi) \subseteq \mathrm{Av}(\pi \oplus 1, 1 \oplus \pi) \subseteq \mathrm{Av}(1 \oplus \pi)$.

**Conjecture:** $\mathrm{gr}(\mathrm{Av}(\pi)) < \mathrm{gr}(\mathrm{Av}(\pi \oplus 1, 1 \oplus \pi)) < \mathrm{gr}(\mathrm{Av}(1 \oplus \pi))$.

We found an example where $\mathrm{gr}(\mathrm{Av}(\pi)) = 4$ and $\mathrm{gr}(\mathrm{Av}(\pi \oplus 1, 1 \oplus \pi)) \approx 4.002$.

Here is what we proved:

**Theorem:** $\mathrm{gr}(\mathrm{Av}(\pi) + 1 \leq \mathrm{gr}(\mathrm{Av}(1 \oplus \pi))$.

The proof is by looking at the vertical juxtaposition of $\mathrm{Av}(\pi)$ and the class of decreasing permutations. It can be generalized to prove that $\mathrm{gr}(\mathrm{Av}(\pi) + 1 \leq \mathrm{gr}(\mathrm{Av}(\rho))$ if $\pi$ is obtained from $\rho$ by deleting the first entry of $\rho$ and standardizing.

Our more general conjecture is that, if $\pi$ contains $\rho$, and $\pi \neq \rho$, then $\mathrm{gr}(\mathrm{Av}(\pi)) < \mathrm{gr}(\mathrm{Av}(\rho))$.

## 5. Natasha Blitvić

What do the permutons of consecutive-pattern avoiders look like?

## 6. Robert Brignall

**Conjecture:** Every finitely based class with growth rate $< 4$ is rational.

Evidence:

- If $\mathcal{C}$ is contained in the separable permutations and $\mathcal{C}$ does not contain $\mathrm{Av}(132)$ or its symmetries, then $\mathcal{C}$ is strongly rational (every subclass is rational).

- Every finitely based subclass of Av(123) is rational (Albert, Brignall, Rusku'vc, Vatter).

## 7. Mireille Bousquet-Mélou

Does $\sigma_{n,\mathcal{C}}$ (uniformly random size-$n$ permutation in $\mathcal{C}$) converge to something for every class $\mathcal{C}$? What happens in other contexts, like graphons?

I am looking for functional equations with the property that $F(t,u)$ satisfies a polynomial equation in $F(t,u)$, $F(t,0)$, $F_u(t,0)$, t, u, etc. Under natural assumptions, $F(t,u)$ is algebraic.

I am also looking for series $F(t,u,v)$ given by a polynomial in $F(t,u,v)$, $G_i(t,u)$ depending only on $t$ and $u$, and $H_j(t)$ depending only on $t$. These are algebraic under natural assumptions.

## 8. Colin Defant

A *uniquely-sorted permutation* is a permutation that has one pre-image under the West stack-sorting map. We can also look at uniquely-sorted permutations avoiding certain patterns.

What does the permuton of the set of uniquely-sorted permutations look like? What about the permuton of the set of uniquely-sorted permutations that avoid a given pattern?

We can do the same thing with *sorted permutations*, which are permutations in the image of the West stack-sorting map.

## 9. Mathilde Bouvel

"Unavoidability of shuffles in permutations"

Let $\pi$ be a permutation of size $pk$. We say $\pi$ is a *$k$-shuffle* if it can be partitioned into $k$ subsequences such that every subsequence is an occurrence of the *same* pattern (of length $p$).

Example: $\underline{6}\overline{52}\underline{13}\overline{4}$ is a 2-shuffle of 312.

**Conjecture:** For each $k$, there exists a constant $n_0(k)$ such that every permutation of size $n > n_0(k)$ contains a factor (consecutive subsequence) of length $\geq 2k$ that is a $k$-shuffle. (In all known examples, it can be done with length exactly $2k$.)

Case analysis: $n_0(2) = 6$; $n_0(3) = 12$; $n_0(4) \leq 26$

We note that a permutation of size $2k$ is a 2-shuffle if and only if its inversion graph has a perfect matching or its complement has a perfect matching.

## 10. Mireille Bousquet-Mélou

Certain permutation classes are conjectured to be not D-finite: three classes each avoiding 2 patterns of length 4, and one class avoiding 3 patterns of length 4. Each one has functional equations with 2 catalytic variables. These classes are $\mathrm{Av}(1234, 1324)$, $\mathrm{Av}(1243, 1432)$, $\mathrm{Av}(1324, 1432)$, and $\mathrm{Av}(1243, 1324, 1432)$.

## Twin Width

Michal Opler

In this talk, we introduce the notion of twin-width and review recent developments with emphasis on permutations.

The most natural way to define twin-width for permutations comes from defining twin-width of an arbitrary point set in the plane. The twin-width of permutations then follows by considering for each permutation $\pi = \pi_1, \pi_2, \ldots, \pi_n$ the point set $P_\pi = \{(i, \pi_i) \mid i \in [n]\}$ called *permutation diagram.*

A *rectangle family* $\mathcal{R}$ is a set of axis-parallel rectangles. A *merge sequence* of a point set $P \subset \mathbb{R}^2$ of size $n$ is a sequence $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$ of rectangle families where $\mathcal{R}_1 = P$ is the original point set, $\mathcal{R}_n$ contains a single rectangle, and each $\mathcal{R}_{i+1}$ is obtained by replacing two rectangles of $\mathcal{R}_i$ by their bounding box. Figure 1 shows an example merge sequence.

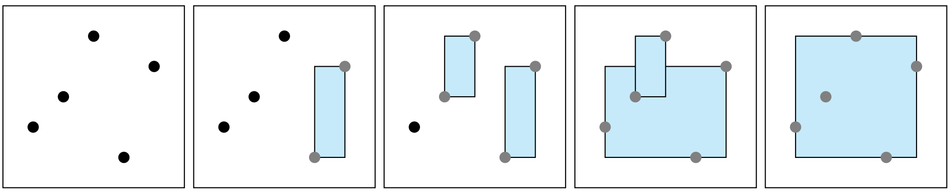

Figure 1. A 3-wide merge sequence of 23514. Black points are also degenerate rectangles.

Two rectangles $S$ and $T$ are called *homogeneous* if their projections onto both $x$- and $y$-axis are disjoint. Given a rectangle family $\mathcal{R}_i$, we consider an auxiliary graph $R_i$, called *red graph*, where the rectangles of $\mathcal{R}_i$ are vertices, and there is a (red) edge between every pair of distinct *non-homogeneous* rectangles $S, T$. Let $d$ be a positive integer. We say that a merge sequence $\mathcal{R}_1, \ldots, \mathcal{R}_n$ is *d-wide* if the maximum degree over all red graphs associated to this sequence is strictly less than $d$. The *twin-width* $\mathrm{tww}(P)$ of a point set $P$ is then the minimum integer $d$ such that there exists a $d$-wide merge sequence of $P$. For a permutation $\pi$, we let $\mathrm{tww}(\pi) = \mathrm{tww}(P_\pi)$.

**Observation 1.**

- *The singleton permutation has twin-width* 1*.*
- *If* $\pi$ *contains* $\rho$*, then* $\mathrm{tww}(\rho) \le \mathrm{tww}(\pi)$*.*
- $\mathrm{tww}(\pi \oplus \rho) = \mathrm{tww}(\pi \ominus \rho) = \max(\mathrm{tww}(\pi), \mathrm{tww}(\rho))$*. In particular, the separable permutations are precisely the permutations of twin-width 1.*
- *More generally, if* $\sigma$ *is the inflation of* $\pi$ *by* $\rho_1, \rho_2, \ldots, \rho_k$*, then*

$$\mathrm{tww}(\sigma) = \max(\mathrm{tww}(\pi), \mathrm{tww}(\rho_1), \mathrm{tww}(\rho_2), \ldots, \mathrm{tww}(\rho_k)).$$

  *In particular, any permutation class with finitely many simple permutations has bounded twin-width.*

It turns out that there is a tight connection between pattern-avoidance and bounded twin-width. This correspondence arises as a consequence of the celebrated Marcus-Tardos theorem. Given a permutation $\pi$, let $\mathrm{ex}_\pi(n)$ be the maximum number of ones in an $n \times n$ 0-1 matrix that avoids $\pi$. Marcus and Tardos [5] proved that $\mathrm{ex}_\pi(n) \in O(n)$ for each fixed $\pi$ (the *Füredi-Hajnal conjecture* [3]). The *Füredi-Hajnal limit* $c_\pi$ of $\pi$ is the constant hidden in the $O$-notation, i.e. $c_\pi = \lim_{n\to\infty} \frac{1}{n}\,\mathrm{ex}_\pi(n)$.

**Theorem 1** (Guillemot and Marx [4])**.** *Every $\pi$-avoiding permutation has twin-width at most $8c_\pi$. A corresponding merge sequence can be found in time $O(n)$.*

The original application of this result was a win-win type algorithm for the *Permutation Pattern Matching (PPM)* problem, i.e., deciding for two given permutations $\pi$ and $\tau$, whether $\tau$ contains $\pi$.

**Theorem 2** (Guillemot and Marx [4])**.** *There is an algorithm solving PPM in time $O(2^{O(k^2)} \cdot n)$ where $n$ is the size of $\tau$ and $k$ is the size of $\pi$.*

Later, twin-width was generalized to graphs and even arbitrary binary relational structures by Bonnet, Kim, Thomassé and Watrigant [2] and ever since, it became the focus of very active development. On the other hand, there have been far less results exploring the structure of pattern-avoiding permutations through the lens of twin-width. As one of the few exceptions, it has been recently showed that pattern-avoiding permutations can be expressed as a composition of constantly many separable permutations (permutations of twin-width 1).

**Theorem 3** (Bonnet, Bourneuf, Geniet and Thomassé [1])**.** *Every $\pi$-avoiding permutation can be expressed as a composition of $2^{O(c_\pi)}$ separable permutations.*

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## An example of the permuton of the intersection of two classes

Justin M. Troyka

In a conversation ensuing from David Bevan's presentation, the question arose of whether it is possible for two classes $\mathcal{C}$ and $\mathcal{D}$ to have the same limiting permuton as each other but for the intersection $\mathcal{C} \cap \mathcal{D}$ to have a different limiting permuton. In this talk (and in this abstract), I present an example showing that it is possible.

We define $\mathcal{C} = \text{Av}(321) \cup \text{Av}(1243)$ and $\mathcal{C} = \text{Av}(321) \cup \text{Av}(3214)$. Each of $\mathcal{C}$ and $\mathcal{D}$ is a finitely based class whose basis permutations have size at most 7. The growth rate of Av(321) is 4, and the growth rate of Av(1243) is 9 [3, 1], so almost all elements of $\mathcal{C}$ are in Av(1243). Therefore, the limiting permuton of $\mathcal{C}$ is equal to the limiting permuton of Av(1243), which is the anti-diagonal permuton (the permuton with support $y = 1 - x$ in the unit square). Similarly, since the growth rate of Av(3214) is 9 [3, 1], the same argument shows that the limiting permuton of $\mathcal{C}$ is the anti-diagonal permuton.

However, $\mathcal{C} \cap \mathcal{D} = \text{Av}(321) \cup \text{Av}(1243, 3214)$. The growth rate of $\text{Av}(1243, 3214)$ is only $\approx 3.87$ [2], so almost all elements of $\mathcal{C} \cap \mathcal{D}$ are in Av(321). Therefore, the limiting permuton of $\mathcal{C} \cap \mathcal{D}$ is equal to the limiting permuton of Av(321), which is the diagonal permuton (the permuton with support $y = x$ in the unit square).

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# Enumeration in subclasses of the separable permutations, and related questions


Robert Brignall

(joint work with Miklós Bóna, Colin Defant, Michal Opler and Vincent Vatter)


This talk reports on discussions and findings made by various combinations of M. Bóna (University of Florida), R. Brignall (Open University), C. Defant (Harvard University), M. Opler (Czech Technical University) and V. Vatter (University of Florida) during the “Permutation Patterns” mini-workshop, held at Mathematisches Forschungsinstitut Oberwolfach from 28 January 2024 – 2 February 2024.

The *separable permutations* are the well-known class of permutations that avoid 2413 and 3142, or that can alternatively be characterised as being the class of permutations that can be constructed via direct and skew sums of smaller permutations from the class, starting with the permutation 1. The class is enumerated by the Schröder numbers (sequence A006318 in the OEIS), and thus they have the following generating function.

$$f(z) = \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z}.$$

During his introductory talk earlier in the week, Vince Vatter asked the following question:

**Question 1.** *If a subclass of the separable permutations has a rational generating function, must it be the case that it cannot contain all of* Av(231) *and its symmetries?*

The converse to this question is known to be true, namely:

**Theorem 2** (Albert, Atkinson and Vatter [2]). *Every subclass of the separable permutations that does not contain all of* $Av(231)$ *or a symmetry of this class has a rational generating function.*

Our explorations into Question 1 resulted in identifying four distinct avenues for future research, alongside some related results. We briefly summarise each of these avenues in the following four sections.

## 1. Tackling Question 1 directly

Our work in exploring this question primarily circulated around studying the methodology used to prove the converse Theorem 2, and investigating what elements of it could be applied to *any* subclass of the separable permutations. That is, what happens to the argument if we drop the requirement that the class can't contain all of the 231-avoiders or a symmetry of this class? Our working hypothesis became the following. The notation $\mathcal{X}[\mathcal{U}]$ denotes the permutation class in which points of permutations from the class $\mathcal{X}$ are replaced by sequences of points that are order isomorphic to an element from $\mathcal{U}$.

**Conjecture 3.** *Let* $\mathcal{C}$ *be a subclass of the separable permutations. Then either* $\mathcal{C}$ *is sum or skew closed (or the finite union of such classes), or* $\mathcal{C} \subseteq \mathcal{X}[\mathcal{U}]$ *where* $\mathcal{X} = Av(2413, 3142, 2143, 3412)$ *and* $\mathcal{U}$ *is a proper subclass of* $\mathcal{C}$*.*

The reason this would help us is due to the following result, which is a slight reinterpretation of Theorem 5.1 of Bóna [3].

**Theorem 4.** *If* $\mathcal{C}$ *is a permutation class such that* $\mathcal{C} \oplus \mathcal{C}$ *and* $1 \ominus \mathcal{C}$ *are both contained in* $\mathcal{C}$ *itself, then* $\mathcal{C}$ *does not have a rational generating function.*

In particular, note that Av(231) is the smallest class that satisfies the hypotheses of Theorem 4, which strongly suggests that the above machinery could be sufficient to characterise nonrationality amongst subclasses of the separable permutations.

The proof of Conjecture 3 remains incomplete, but we have established that many of the steps needed can use similar methods to those from [2].

## 2. The form of generating functions for subclasses of the separables

During our investigations, it was noted that when a proper subclass of the separables contains all of Av(231) (or a symmetry of this), then its generating function belongs to $\mathbb{Q}(z, \sqrt{1-4z})$. That is, the generating function is 'rational' in $z$ and the radical $\sqrt{1-4z}$, which features in the generating function for the Catalan numbers.

It was asked whether this was always the case. That is, is it the case that every proper subclass of the separables has a generating function that belongs to $\mathbb{Q}(z, \sqrt{1-4z})$?

Shortly before presenting the talk, it was observed that this turns out to be false, slightly embarrasingly as a result of an enumeration first carried out in a paper authored by three participants of this mini-workshop [5]. Specifically, the class Av(2143, 2413, 3142) has an enumeration sequence given by OEIS sequence A033321, and generating function

$$f(z) = \frac{2}{1 + z + \sqrt{1 - 6z + 5z^2}}.$$

A possible explanation for the form of this generating function is that this class $\mathcal{C}$ is the smallest for which $\mathcal{C} \oplus \mathcal{C}$ and $1 \ominus \mathcal{C} \ominus 1$ are contained in $\mathcal{C}$ itself, thereby indicating a new type of recursive structure that introduces a slightly different combinatorial specification. With this counterexample in mind, we pose the following question.

**Question 5.** *Is it the case that every proper subclass of the separable permutations has a generating function that belongs to* $\mathbb{Q}(z, \sqrt{1-4z}, \sqrt{1-6z+5z^2})$*?*

## 3. Extending the theory to bigger classes

We investigated whether the methodology used to prove Theorem 2 could be extended to larger classes, providing the simple permutations in the class are sufficiently well-behaved. A good candidate for exploration here are classes whose simple permutations are "geometrically griddable" (see [1]), since this still gives us some control of the structure, enumeration and well-quasi-ordering.

It was observed that in order to generalise the case where one can express a subclass of the separable permutations as an inflation of the $\mathcal{X}$ class by a proper subclass, $\mathcal{C} \subseteq \mathcal{X}[\mathcal{U}]$, the class $\mathcal{X} = \text{Av}(2413, 3142, 2143, 3412)$ would likely need to be replaced by a suitable class that can capture the structure of the simple permutations in $\mathcal{C}$. This may have consequences in other parts of a possible argument, but it was felt that these complexities could be overcome with further in-depth work.

## 4. Linear clique-width in hereditary graph classes

The final topic goes beyond permutation patterns entirely. An earlier paper by Brignall, Korpelainen and Vatter [4] applied similar arguments to those in [2] to show that amongst subclasses of cographs (which can be thought of as the graph-theoretic equivalent of the separable permutations), the ones that have bounded linear clique-width are precisely those that do not contain every quasi-threshold graph or the complement of this class. In other words, boundedness of "linear clique-width" in a hereditary graph class often requires similar structural properties to those that guarantee a rational generating function for a permutation class.

While we did not explore this topic in any detail, it seemed worth recording that this topic is closely connected to that of subclasses of the separable permutations, and thus it might be possible to extend the results of [4] to hereditary classes that are more general than cographs.

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## The inversion conjecture for Av(1324)

Christian Bean
(joint work with Anders Claesson)

An inversion is a pair of indices $(i, j)$ such that $i < j$ and $\pi(i) > \pi(j)$ for a permutation $\pi$. Claesson, Jelínek, and Steingrímsson [1] conjectured that for Av(1324) if we fix $k$, then the number of permutations in Av(1324) with exactly $k$ inversions grows monotonically with the size of the permutation.

If true, Claesson, Jelínek, and Steingrímsson [1] showed that it will improve the upper bound on the growth rate of the number of permutations in Av(1324). In particular, it would imply

$$gr(\mathrm{Av}(1324)) \leq e^{\pi\sqrt{\frac{2}{3}}} \approx 13.002.$$

For permutation classes that avoid only patterns of the form $\pi \ominus 1$ or only $1 \ominus \pi$, the inversion conjecture trivially holds, as can be seen by either prepending a 1 or appending a new maximum to the permutation.

In this talk, we discuss progress on the subclass Av(231, 1324), where the same monotonicity property appears to hold. Our approach was to find a two-variable generating function $F(x, t)$ that tracks both the size of the permutation and the number of inversions of the permutation. We found functional equations for this bivariate generating function, which verified the conjecture to be true for $n \leq 70$.

We then turned our attention to finding a $q$-exponential generating function

$$A(x, q) = \sum_{\pi \in \mathrm{Av}(231,1324)} \frac{x^{|\pi|} q^{\mathrm{inv}(\pi)}}{[n]_q!}$$

where $[n]_q!$ is the $q$-analog of the factorial. This generating function has the nice benefit that when taking products, it can neatly keep track of the number of inversions. We first found a functional equation for the subclass Av(132, 231)

$$B(x, q) = 1 + \int_0^x B(tq, q) B(t, q) d_q t$$

where here we have the Jackson integral. This relies on the fact that these permutations are of the form $\sigma 1 \tau$ where $\sigma$ is a decreasing sequence and $\tau$ is an increasing sequence.

From the functional equation we can find a formula for the polynomial that records the distribution of inversions for the permutations of size $n$

$$b_n = \sum_{n=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{\binom{k}{2}+k}.$$

For $\mathrm{Av}(231, 1324)$, we get the functional equation

$$A(x,q) = 1 + \int_0^x A(tq,q) d_q t + \int_0^x A(t,q) B(q(x-t),q) d_q t.$$

This functional equation relies on the structure of the permutations in the class. We need to consider if a permutation begins with its maximum element $n$ or not. If it is the form $n\sigma$ then $\sigma$ can be any permutation in $\mathrm{Av}(231, 1324)$, and if it is of the form $\sigma n \tau$ then $\sigma$ can be any permutation in $\mathrm{Av}(132, 231)$, $\tau$ can be any permutation in $\mathrm{Av}(213, 231)$ and all of the values to the left of $n$ are lower than the values to the right of $n$. Both of these subclasses are counted by $B(x, q)$.

Although we were not able to solve the equations explicitly, we were able to use them to verify that the conjecture holds for $\mathrm{Av}(231, 1324)$ for $n \leq 240$. This could easily be pushed further given more computation time.

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## Moment sequences for pattern avoiding matchings

Natasha Blitvić, Andrew Elvey Price
(joint work with Justin Troyka)

A sequence $\mathbf{a} = (a_n)_{n\geq 0}$ of real numbers is called a *(Hamburger) moment sequence* if there is some positive Borel measure $\rho$ on $\mathbb{R}$ such that each term $a_n$ can be expressed as the integral

$$a_n = \int_{\mathbb{R}} x^n d\rho(x) \quad \text{for all } n \in \mathbb{N}_0, \tag{1}$$

and more specifically a *Stieltjes moment sequence* if the support of $\rho$ is included in the positive real line. The existence of a positive $\rho$ satisfying (1) is equivalent to the sequence $(a_n)_{n\geq 0}$ being positive-definite [11].

In recent years there has been significant interest in combinatorial moment sequences, and in particular it is now believed that the counting sequence $Av_n(\pi)$ of permutations avoiding any single pattern $\pi$ is a Stieltjes moment sequence [1, 2, 6, 8, 9]. During the week of the Oberwolfach conference, we worked on the question of whether the same property holds for matchings.

A *matching* (aka perfect matching, pairing, or pair-partition) of a set $[n]$ is a partition of $[n]$ into pairs. Clearly, there are no matchings on $[n]$ when $n$ is odd, whereas there are exactly $(n-1)!! := (n-1)(n-3)\cdots 3\cdot 1$ of these when $n$ is even. In other words, matchings enumerate moments of the standard Gaussian random variable, i.e.

$$\int_{\mathbb{R}} x^{2n-1}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = 0 \quad \text{and} \quad \int_{\mathbb{R}} x^{2n}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = (2n-1)!! \quad \text{for all } n \in \mathbb{N}.$$

Surprisingly, some of the basic combinatorial statistics on matchings can be seen to correspond to moments of Gaussian-type random variables by moving to the setting of noncommutative probability. In particular, the passage from classical probability to free probability of Voiculescu [12] (roughly speaking, the theory of 'generic' infinite-dimensional random matrices) is reflected in the passage from matchings to *non-crossing matchings*, enumerated by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. Concretely, in free probability, the role of the Gaussian distribution is played by the *semicircle law* $\frac{\sqrt{4-x^2}}{2\pi}dx$ with moments

$$\int_{[-2,2]} x^{2n-1}\frac{\sqrt{4-x^2}}{2\pi}dx = 0 \quad \text{and} \quad \int_{[-2,2]} x^{2n}\frac{\sqrt{4-x^2}}{2\pi}dx = C_{2n}.$$

We can further refine this result by moving to more general noncommutative probabilistic frameworks. For instance, keeping track of the number of *crossings* in a matching (i.e. pairs $(i_1, j_1)$ and $(i_2, j_2)$ s.t. $i_1 < i_2 < j_1 < j_2$), we obtain

$$\int_{\mathbb{R}} x^{2n} d\rho_q = \sum_{\pi \text{ matching on } [n]} q^{\#\text{crossings}(\pi)},$$

where $\rho_q$ is the *q-Gaussian* law [4], parametrized by $-1 \leq q \leq 1$. Furthermore, considering also *nestings* in matchings (i.e. pairs $(i_1, j_1)$ and $(i_2, j_2)$ s.t. $i_1 < i_2 < j_2 < j_1$), we instead obtain

$$\int_{\mathbb{R}} x^{2n} d\rho_{q,t} = \sum_{\pi \text{ matching on } [n]} q^{\#\text{crossings}(\pi)} t^{\#\text{nestings}(\pi)},$$

where $\rho_{q,t}$ is a two-parameter family of noncommutative Gaussian laws defined for $|q| \leq t$ [3].

Note that while the semicircle law, $\rho_q$, and $\rho_{q,t}$ are probability measures on the real line, they can be interpreted as "noncommutative analogues" of the classical Gaussian distribution, and naturally arise from certain algebraically-flavored constructions [12, 4, 3]. We are therefore interested in studying pattern avoidance in matchings, as a potentially more natural setting in which to consider the question of when the *avoidance* of a specific sub-diagram (aka *pattern*) preserves positivity. For example, as per the discussion of the preceding paragraph, the avoidance of crossings in matchings is well-understood, preserving not only positivity but also the "Gaussianity" of the law. What can can be deduced for more general patterns in matchings?

## 1. Statement of the problem

**Definition 1.** *We say that a matching* $\pi$ contains *a matching* $\tau$ *(as a pattern) if some subset of the pairs of* $\pi$ *has elements in the same relative order as the elements of* $\tau$.

For example the matching $\{(1,3),(2,6),(4,5)\}$ contains $\{(1,3),(2,4)\}$ as a pattern as the elements from the pairs $\{(1,3),(2,6)\}$ lie in the same relative order as those in $\{(1,3),(2,4)\}$. If $\pi$ does not contain $\tau$ we say that $\pi$ avoids $\tau$, and we denote by $M_n(\tau)$ the number of matchings of $[n]$ avoiding $\tau$. The determination of the values $M_n(\tau)$ has been considered by several different authors, see for example [5, 7, 10]. We consider the following question:

**Question 1.** *For which matchings* $\tau$ *is the sequence* $(M_n(\tau))_{n\geq 0}$ *a Hamburger moment sequence?*

Since $M_n(\tau) = 0$ for $n$ odd, we could equivalently ask when the sequence $(M_{2n}(\tau))_{n\geq 0}$ is a Stieltjes moment sequence. The evidence that we have gathered suggests that the answer may be that $(M_n(\tau))_{n\geq 0}$ is a Hamburger moment sequence for any matching $\tau$.

## 2. Exact enumeration for principal matching classes

Below we list several cases where the sequence $(M_n(\tau))_{n\geq 0}$ or its generating function is known exactly (see [5]). In each case we give an integral representation showing that $(M_n(\tau))_{n\geq 0}$ is a Hamburger moment sequence, or equivalently that $(M_{2n}(\tau))_{n\geq 0}$ is a Stieltjes moment sequence.

$$M_{2n}(\{(1,3),(2,4)\}) = C_n = \int_{-2}^{2} x^{2n}\frac{\sqrt{4-x^2}}{2\pi}dx,$$

$$M_{2n}(\{(1,4),(2,3)\}) = C_n = \int_{-2}^{2} x^{2n}\frac{\sqrt{4-x^2}}{2\pi}dx,$$

$$M_{2n}(\{(1,2),(3,4)\}) = n! = \int_{-\infty}^{\infty} x^{2n}|x|e^{-x^2}dx,$$

$$M_{2n}(\{(1,4),(2,6),(3,5)\}) = [t^n]\frac{54t}{1+36t-(1-12t)^{3/2}} = \int_{-\sqrt{12}}^{\sqrt{12}} x^{2n}\frac{(12-x^2)^{3/2}}{2\pi(x^2+4)^2}dx,$$

$$M_{2n}(\{(1,5),(2,4),(3,6)\}) = [t^n]\frac{54t}{1+36t-(1-12t)^{3/2}} = \int_{-\sqrt{12}}^{\sqrt{12}} x^{2n}\frac{(12-x^2)^{3/2}}{2\pi(x^2+4)^2}dx,$$

$$M_{2n}(\{(1,4),(2,5),(3,6)\}) = C_{n+2}C_n - C_{n+1}^2 = \int_{-\infty}^{\infty} x^{2n}J(x)dx,$$

$$M_{2n}(\{(1,5),(2,6),(3,4)\}) = C_{n+2}C_n - C_{n+1}^2 = \int_{-\infty}^{\infty} x^{2n}J(x)dx,$$

$$M_{2n}(\{(1,6),(2,5),(3,4)\}) = C_{n+2}C_n - C_{n+1}^2 = \int_{-\infty}^{\infty} x^{2n}J(x)dx,$$

where

$$J(x) = \int_{|x|/2}^{2} \frac{(z^4-x^2)^2\sqrt{(4-z^2)(4z^2-x^2)}}{4\pi^2 z^4}dz.$$

## 3. Avoiding direct sum patterns

If $\pi$ is a matching of $[n]$ and $\tau$ is a matching of $[m]$, denote by $\pi \oplus \tau$ the matching of $[n+m]$ given by $\pi \cup (\tau + n)$, where $(\tau + n)$ denotes the matching of $\{n+1, n+2, \ldots, n+m\}$ given by increasing each value in $\tau$ by $n$. for example, $\{(1,4),(2,3)\} \oplus \{(1,3),(2,4)\} = \{(1,4),(2,3),(5,7),(6,8)\}$. We derive the following formula for enumerating $M_n((1,2)\oplus\pi)$ in terms of $M_n(\pi)$:

$$M_{2n}((1,2)\oplus\pi) = \sum_{k=0}^{n-1} M_{2k}(\pi)\frac{(n+k-1)!}{(2k)!}(n-k),$$

moreover, using this equation we have deduced the following theorem:

**Theorem 2.** *If $(M_n(\pi))_{n\geq 0}$ is a Hamburger moment sequence then $(M_n((1,2)\oplus \pi))_{n\geq 0}$ is also Hamburger moment sequence.*

It follows that $(M_n(\tau))_{n\geq 0}$ is a Hamburger moment sequence for both $\tau = \{(1,2),(3,4),(5,6)\}$ and $\tau = \{(1,2),(3,5),(4,6)\}$. Moreover,

$$\begin{aligned} &M_n(\{(1,2),(3,5),(4,6)\}) = M_n(\{(1,2),(3,6),(4,5)\}) \\ &= M_n(\{(1,3),(2,4),(5,6)\}) = M_n(\{(1,4),(2,3),(5,6)\}). \end{aligned}$$

We also deduced the following, more general formula, however we have not yet been able to use it to show that other sequences are moment sequences:

$$M_{2n}(\pi\oplus\tau) =$$

$$\sum_{\ell=0}^{n-1}\sum_{k=0}^{\max(n-\ell,n-1)} M_{2k}(\pi)M_{2\ell}(\sigma)\frac{(n+\ell-k-1)!(n+k-\ell-1)!}{(2k)!(2\ell)!(n-k-\ell)!}((n-k-\ell)^2-4k\ell).$$

We note that an equivalent formula was found in [7], albeit in a slightly more complicated form.

## 4. Remaining patterns of size 3

There are five matchings $\tau$ of [6] for which we have no exact formula, so we have not yet been able to prove that $(M_n(\tau))_{n\geq 0}$ is a Hamburger moment sequence. Namely, the remaining cases are $\{(1,3),(2,5),(4,6)\}$, $\{(1,6),(2,3),(4,5)\}$, $\{(1,6),(2,4),(3,5)\}$, $\{(1,3),(2,6),(4,5)\}$ and $\{(1,5),(2,3),(4,6)\}$, the last two of which are equivalent. In all but the first case our numerical analysis of the known exact terms strongly suggest that the sequence is a moment sequence, while for the case $\{(1,3),(2,5),(4,6)\}$, no significant computation seems to have been done.

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## Permuton Limits of Permutation Classes and Rare Regions

Justin M. Troyka
(joint work with David Bevan)

Bevan has conjectured the following:

**Conjecture.** The permuton limit of $\mathrm{Av}(\alpha \ominus 1)$ is the diagonal permuton, for any non-empty permutation $\alpha$.

In trying to prove this conjecture, we have revisited work by Atapour and Madras [1] and Madras and Yıldırım [2] on exponentially rare regions of permutation classes. For a pattern $\tau \in S_k$, they define a point $(x, y) \in [0,1]^2$ to be $\tau$-*rare* if, for every sequence $(i_n, j_n) \in [n]^2$ such that $(i_n/n, j_n/n) \to (x, y)$, there exists $r \in (0, 1)$ such that

$$\mathbb{P}_n^\tau[\pi(i_n) = j_n] = o(r^n)$$

as $n \to \infty$, where $\pi$ is chosen from the uniform distribution on $\mathrm{Av}_n(\tau)$ and where $\mathbb{P}_n^\tau$ denotes the probability measure of this distribution. In words, $(x, y)$ is $\tau$-rare if every sequence of points converging to $(x, y)$ has exponentially decaying probabilities of being a point in a $\tau$-avoiding permutation. Then the $\tau$-rare region $\mathcal{R}$ is defined to be the set of $\tau$-rare points.

There are some interesting results about $\tau$-rare points, some of which we would potentially like to adapt for proving Bevan's conjecture:

**Theorem** [1, Thm. 8.1], [2, Thm. 1]. If $\tau = \alpha \ominus 1$ for some non-empty $\alpha$, then the corner point $(1, 0)$ has an open neighborhood contained in $\mathcal{R}$.

**Theorem** [2, Thm. 17]. Suppose $\tau = \alpha \ominus 1$ for some non-empty $\alpha$. If $(x, y) \in [0, 1] \setminus \mathcal{R}$, then the convex hull of $\{(x, y), (0, 0), (0, 1)\}$ is contained in $[0, 1] \setminus \mathcal{R}$.

If we could adapt the proof of this last theorem to show that the convex hull of $\{(x, y), (0, 0), (0, 1)\}$ is in the support of the permuton limit, then this would prove Bevan's conjecture, since the support cannot have positive measure if the permuton is the limit of a proper permutation class.

Both of those theorems involve exponentially rare regions, but we have found that proving results about permutons can be done with a significantly weaker condition than being exponentially rare. We have proved the following theorem, which replaces $o(r^n)$ with $o(1/n)$:

**Theorem.** Let $U$ be an open set in $[0, 1]^2$. If there exists a function $f(n) = o(n)$ such that $\mathbb{P}_n^\tau[\pi(i) = j] \leq f(n)$ whenever $(i/n, j/n) \in U$, then $U$ is contained in the complement of the support of the permuton limit of $\mathrm{Av}(\tau)$.

This theorem is proved using weak convergence of probability measures. That is, if $\mu^\tau$ is the permuton limit of $\mathrm{Av}(\tau)$, and if $\mu_n^\tau$ is the measure on $[0, 1]$ associated to a uniformly random permutation in $\mathrm{Av}_n(\tau)$, then $\mu_n^\tau$ weakly converges to $\mu^\tau$, meaning that $\lim_{n\to\infty} \mu_n^\tau(A) = \mu^\tau(A)$ for every measurable set $A$. Thus, to prove the conclusion of the theorem, we prove that $\lim_{n\to\infty} \mu_n^\tau(U) = 0$.

We hope that this condition with $o(1/n)$ is the "right" condition for studying permutons and proving Bevan's conjecture. The condition is consistent with what we know about the uniformly random permutation (chosen from the set of all size-$n$ permutations), since in that case the probability of $\pi(i) = j$ is exactly $1/n$ (rather than $o(1/n)$) for all $i$ and $j$.

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## Solution to a functional equation for Av(1243, 1324, 1432)


Jay Pantone

(joint work with Mireille Bousquet-Mélou, Mathilde Bouvel)


### 1. Background on $\mathcal{C}$-Machines

The paper *Generating Permutations with Restricted Containers* by Albert, Homberger, Pantone, Shar, and Vatter [1] defines a sorting (really, generating) machine that generalizes a stack. For any permutation class $\mathcal{C}$, the $\mathcal{C}$*-machine* is a box into which we place the entries $1, 2, 3, \ldots$, in order, in any arrangement such that the entries **currently in the machine** form a permutation in the class $\mathcal{C}$. This is called a *push* operation. The *pop* operation removes the leftmost entry from the machine and records it as the next entry of the output permutation. Finally, the *bypass* operation takes the smallest entry that has not been pushed into the machine, and records it directly in the output, bypassing the machine entirely. Note that there may be places where a new entry cannot be inserted because it would create a forbidden pattern, but after popping some of the entries on the left side of the machine, those insertion locations are now allowed.

**Theorem 1.** *Suppose* $\mathcal{C} = \mathrm{Av}(\beta_1, \ldots, \beta_k)$. *The set of permutations that the* $\mathcal{C}$-*machine can generate is precisely the class* $\mathcal{D} = \mathrm{Av}(1 \ominus \beta_1, \ldots, 1 \ominus \beta_k)$.

For example, the Av(231, 1234)-machine generates the class Av(4231, 51234). This theorem is most useful in reverse: If $\mathcal{D}$ is a permutation class of the form $\mathrm{Av}(1\ominus\beta_1, \ldots, 1\ominus\beta_k)$, then we can use the structure of the class $\mathcal{C} = \mathrm{Av}(\beta_1, \ldots, \beta_k)$ to describe the structure of $\mathcal{D}$. This is done by, in essence, creating a set of succession rules where the labels are permutations of $\mathcal{C}$.

For most permutations $\pi \in \mathcal{D}$ there are many different push/pop/bypass operation sequences on the $\mathcal{C}$-machine that generate $\pi$, but a simple restriction ensures that each permutation in $\mathcal{D}$ is generated exactly once: after each push, popping is forbidden until a bypass has occurred. Phrased differently, pushing "locks" the pop operation, while bypassing "unlocks" it.

Sometimes $\mathcal{C}$ is simple enough that it gives a structural description of $\mathcal{D}$ that is actually useful. The following are proved in [1].

- If $\mathcal{C}$ is finite, then $\mathcal{D}$ has a rational generating function.
- If the enumeration of $\mathcal{C}$ is bounded by a constant, then $\mathcal{D}$ has an algebraic generating function.
- If the enumeration of $\mathcal{C}$ is bounded by a polynomial, then there is a polynomial-time counting algorithm for the enumeration of $\mathcal{D}$.

When none of these hypotheses hold for $\mathcal{C}$, it may still be possible to use the $\mathcal{C}$-machine to learn something about $\mathcal{D}$. For instance, the class $\mathcal{C} = \mathrm{Av}(231, 321)$ has $2^{n-1}$ permutations of length $n$, but [1] observes that a polynomial-time algorithm (and a set of functional equations) can still be derived for $\mathcal{D} = \mathrm{Av}(4231, 4321)$.

## 2. The class Av(1243, 1324, 1432)

The class Av(1243, 1324, 1432) is symmetrically equivalent to the class Av(4123, 4231, 4312). By Theorem 1, this class is generated by the Av(123, 231, 312)-machine. The class Av(123, 231, 312) consists of permutations of the form $\alpha \oplus \beta$ where $\alpha$ and $\beta$ are strictly decreasing. The highly-structured nature of this class makes it possible to write down the following system of functional equations for Av(4123, 4231, 4312):

$$A(u,t) = 1 + uA(u,t) + t\frac{A(u,t) - A(0,t)}{u} + tB(0,u,t),$$

$$B(u,v,t) = \frac{vt}{(1-v)(1-t)}B(u,v,t) + \frac{t}{1-t}\frac{B(u,v,t) - B(0,v,t)}{u} + \frac{vt}{(1-v)(1-t)}\frac{A(u,t) - A(0,t)}{u}.$$

The generating function for Av(4123, 4231, 4312) is then $A(0,t)$. In [1], the authors use the same structure to compute the first 5000 terms in the counting sequence of this class, and conjecture that the generating function $A(0,t)$ is non-D-finite, and perhaps even non-D-algebraic. It is this conjecture that we worked on during the week of our mini-workshop.

## 3. An Explicit Solution

The starting point in our search for an explicit expression for $A(0,t)$ is an application of the kernel method to the equation for $B(u,v,t)$, which produces the following single equation:

$$(1-u)A(u,t) = 1 + \frac{t}{u}(A(u,t) - A(0,t)) + \frac{tu}{1-u}(A(R(u,t),t) - A(0,t)) \tag{1}$$

where

$$R(u,t) = \frac{t}{1 - \frac{t}{1-u}}.$$

Standard iteration techniques suggest that one should replace $u$ by $R(u,t)$ in Equation (1), obtaining an equation for $A(R(u,t),t)$ in terms of $A(0,t)$ and $A(R^{(2)}(u,t),t)$, and then substitute this back into Equation (1) to obtain an equation for $A(u,t)$ in terms of $A(0,t)$ and $A(R^{(2)}(u,t),t)$. It is sometimes possible to repeat this process infinitely and, in the limit, obtain an equation to which the kernel method can then be applied (see, e.g., [2]). Unfortunately, we discovered that in this case, the iterated functions $R^{(k)}(u,t)$ converge to the (shifted) Catalan generating function, which is also a root of the kernel of (1), namely $1-u-t/u$, that would need to be canceled in order to apply the kernel method.

At this juncture, we spent an afternoon creating a simpler functional equation that encounters the same difficulties and proceeded to review some techniques used successfully in earlier works to circumvent the issue. After some trial-and-error, we discovered that a clever change in variables would allow us to "decouple" the main variable $t$ from its simultaneous usage in the iterated $R(u,t)$. We introduced a new variable $s$, and replaced $R(u,t)$ in (1) by $R(u,s)$. We further made the changes of variables

$$t \to \frac{q}{(1+q)^2} \qquad \text{and} \qquad s \to \frac{x}{(1+x)^2}.$$

In particular, the fixed point of $R(u,s)$ is now $x/(1+x)$. These alterations then permitted us to successfully carry out the "iterate and apply the kernel method" procedure to produce an explicit expression for $A(0,t)$. Recall the $q$-Pochhammer notation

$$(r;q)_k = \prod_{i=0}^{k-1}(1-rq^i).$$

Define $a = \dfrac{q-x}{1-qx}$ and let

$$N(x,q) = \frac{x}{q} \sum_{k\geq 0} \left( \frac{xq^2(1+q)}{(q-x)(1-qx)} \right)^k \left( \frac{1-aq}{1-aq^{2k-1}} \right) \left( \frac{(aq;q)_{2k}}{(q^2;q^2)_k (a^2q^2;q^2)_k} \right).$$

Further define

$$D(x,q) = 1 - \frac{x}{(1+x)^2} N(x,q) + \frac{x^2}{(1+x)^2} \cdot \frac{(1+q)^2}{q^2} \cdot \sum_{k\geq 1} \left( \frac{xq^2(1+q)}{(q-x)(1-qx)} \right)^k \left( \frac{(1-aq)(1-aq^{2k})^2}{(1-aq^{2k-1})^2(1-aq^{2k+1})} \right) \left( \frac{(aq;q)_{2k}}{(q^2;q^2)_k (a^2q^2;q^2)_k} \right).$$

Then, an explicit univariate expression for $A(0,t)$ is, after accounting for the several changes in variable,

$$\left[ \frac{N(x,q)}{D(x,q)} \right]_{q=x}.$$

Carrying out this computation and then verifying its correctness took the remainder of the mini-workshop. The next step of the project is to determine whether one can use this explicit expression to prove non-D-finiteness, either by using it to demonstrate that the generating function possesses infinitely many singularities (as empirical evidence strongly suggests), or by other means.

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## A probabilistic interpretation points to new combinatorial results for $k$-arrangements

Natasha Blitvić and Miklós Bóna

Let $A(n,k)$ be the number of permutations of length $n$ whose fixed points are colored using $k$ colors. Such a structure is called a *$k$-arrangement*, introduced by Blitvić and Steingrímsson [1] as an interpolation between permutations ($k=1$), derangements ($k=0$), and Comtet's arrangements ($k=2$) [2]. (For $k=2$, these are also Postnikov's decorated permutations [3].) Several combinatorial properties shared by the special cases $k=0,1,2$ generalize naturally to the $k$-arrangements. For example, it is known [1] that

$$A(n,k) = nA(n-1,k) + (k-1)^n,$$

and that for fixed $k$,

$$\sum_{n\geq 0} A(n,k) \frac{x^n}{n!} = \frac{\exp((k-1)x)}{1-x}. \tag{1}$$

Furthermore, the notion of pattern avoidance in permutations has a natural generalization to $k$-arrangements. (See [1] for the definition of classical pattern avoidance in this setting.) Let $\mathrm{Av}_\pi(n,k)$ denote the number of $k$-arrangements avoiding the classical permutation pattern $\pi$. For any classical pattern $\pi$ of size 3, we have

$$\mathrm{Av}_\pi(n,1) = \frac{1}{n+1}\binom{2n}{n} = C_n$$

which follows by the classical result of [4] as $k = 1$ recovers all permutations. For $k = 2$ (decorated permutations), it can be shown [1] that

$$\mathrm{Av}_\pi(n,2) = C(n+1),$$

while for $k = 3$ it was conjectured by [1] and subsequently proven in [5] that

$$\mathrm{Av}_\pi(n,3) = C(n+2) - 2^n.$$

Overall, the $k$-arrangements appear to provide a combinatorially rich general setting for several types of enumerative questions.

Intriguingly, the $k$-arrangements are also closely connected to a fundamental object in probability theory. Specifically, (1) can be seen to be equivalent to

$$A(n,k) = \int_{k-1}^{\infty} x^n e^{-x+(k-1)}\,dx. \tag{2}$$

In other words, the number of $k$-arrangements on $[n]$ equals the $n$th moment of a rate-one exponential random variable shifted to the right by $k-1$ units. Letting $k = 1$ recovers the exponential law. The $k = 0$ case appears in [6], while the $k = 2$ case was studied in detail by [7].

The starting point for this collaboration is the following observation. The integral on the right-hand side of (2) is defined for any $k \in \mathbb{R}$, whereas its combinatorial interpretation (i.e. the right-hand side of (2), given in terms of $k$-arrangements) is currently only available for $k \in \mathbb{Z}_{\geq 0}$.

**Question.** (Remark 3.8 in [1]) *Extend the definition of $A(n,k)$ via (2) to any $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Does $A(n,k)$ have a combinatorial interpretation when $k$ is an arbitrary negative integer?*

Furthermore, it follows by (2) that for $n$ even, $A(n,k) \geq 0$ for any $k \in \mathbb{R}$. (We are integrating $x^2$ against a non-negative function over $\mathbb{R}$.) A potentially easier question is therefore:

**Question′.** *Does $A(2n,k)$ have a combinatorial interpretation for all $k \in \mathbb{Z}$?*

For fixed $n$, let the polynomial $B_n(x)$ denote the generating polynomial of permutations of length $n$ with respect to the number of their fixed points. For example,

$B_3(x) = x^3 + 3x + 2.$

Then the equality $B_n(k) = A(n,k)$ holds. We investigated what happens when $k$ is a *negative* integer.

First, we noticed that $B_n(-k)$ is the number of $k$-arrangements of length $n$ with an even number of fixed points minus the number of $k$ arrangements with an odd number of fixed points. In the above example, $B_3(-1) = -2$, $B_3(-2) = -12$, and $B_3(-3) = -34$.

Our main approach was the following. We tried to find an "almost bijection" that matches $k$-arrangements on $[n]$ with an even number of fixed points to $k$-arrangements on $[n]$ with

an odd number of fixed points. The $k$-arrangements missed by this almost-bijection would then be counted by $B_n(-k)$. A potential start for such a map could be some operation that changes the number of fixed points of a permutation by exactly one.

Note that even for $k = -1$, when we get the exponential generating function

$$\sum_{n\geq 0} A(n,-1)\frac{x^n}{n!} = \frac{\exp(-2x)}{1-x},$$

we could not find a purely combinatorial interpretation for the sequence of coefficients, even though that sequence is in the Encyclopedia of Integer sequences [8].

One possible starting point is a paper of Herbert Wilf [9] that sets up an almost-bijection between derangements of length $n$ and permutations of length $n$ that have exactly one fixed point.

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## Progress on $k$-shuffle conjecture

Andrew Elvey Price
(joint work with David Bevan and Stefan Felsner)

A *$k$-shuffle* of a permutation $\tau$ is a permutation $\pi$ of length $k|\tau|$ whose elements can be partitioned into $k$ sub-permutations, each with shape $\tau$. For example, 219634785 is a 3-shuffle of 132 as the subwords 297, 164 and 385 all have the same shape as 132. During the week of the Oberwolfach conference, we studied the following conjecture, posed by Mathilde Bouvel:

**Conjecture 1.** *For a fixed $k \in \mathbb{N}$, any sufficiently long permutation contains a factor which is a $k$-shuffle of* 12 *or* 21*.*

Although we did not prove this conjecture, we made progress in a number of directions. First, we reinterpreted the conjecture as a question about the existence of a cycle in the following graph:

**Definition 1.** *Define the factor graph $\Gamma_k = (V, E)$ as follows: The vertex set $V$ is the set of all permutations of length $2k-1$. For each permutation $\pi$ of length $2k$ which is neither a $k$-shuffle of* 12 *nor of* 21*, there is a directed edge from $\pi_1$ to $\pi_2$, where $\pi_1$ and $\pi_2$ are the two factors of $\pi$ of length $2k-1$, with $\pi_1$ containing the first element of $\pi$ and $\pi_2$ containing the last element of $\pi$.*

Conjecture 1 holds for some $k$ if and only if the factor graph $\Gamma_k$ is acyclic, which we confirmed up to $k = 5$.

Furthermore, we characterised permutations of length $2k$ which are not $k$-shuffles of 12 or 21 using the observation that a permutation $\pi$ is a $k$ shuffle of 21 if and only if the inversion graph of $\pi$ contains a perfect matching, along with the Tutte theorem [1] which characterises graphs containing a perfect matching. In our context the Tutte theorem is equivalent to the following Proposition:

**Proposition 1.** *A permutation $\pi$ of length $2k$ is not a $k$-shuffle of* 21 *if and only if there is some $\ell$ and some sub-permutation $\hat{\pi}$ of $\pi$ of length $2k-\ell$ such that $\hat{\pi}$ can be written as the direct-sum of $\ell+2$ permutations of odd size.*

An analogous result holds for permutations which are not $k$-shuffles of 12 involving a skew-sum rather than direct-sum. Intuitively, this means that the permutations $\pi$ which are neither a $k$-shuffle of 12 nor of 21 somewhat resemble an $X$ in that they can be mostly decomposed into both a long direct-sum and a long skew-sum. Hence if the conjecture were false, one may expect that each factor starts or ends with either its largest or smallest element, however we rule out this extreme case in the following proposition:

**Proposition 2.** *Let $\pi$ be a permutation of length $2k^2+2k$ such that each factor of $\pi$ of length $2k$ begins or ends with its largest or smallest element. Then $\pi$ contains a factor $\tau$ of length $2k$ which is a $k$-shuffle of* 12 *or* 21*.*

For comparison, the longest permutation $\pi$ which does not contain a 5-shuffle of 12 or 21 has length 43 and satisfies the property that each factor of $\pi$ of length 10 begins or ends with its largest or smallest element.

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