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Cluster Algebras and Its Applications

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Abstract. This workshop focused on recent developments in cluster algebras and their applications as well as interactions with other areas of mathematics. In addition to new advances in the theory of cluster algebras themselves, it included applications to knot theory and geometry as well as interactions with representation theory and categorification, Grassmannians, combinatorics, geometric surfaces models and Lie theory.

Mathematics Subject Classification (2020): 13F60, 05Exx, 14Mxx, 16Gxx, 18Gxx, 37Jxx.

Introduction by the Organizers

Cluster algebras are commutative algebras with a special combinatorial structure which are related to various fields in mathematics and physics. Introduced by Fomin and Zelevinsky in 2002 in the context of Lie theory and total positivity, cluster algebras quickly developed deep connections to quite different disciplines including combinatorics, representation theory, algebraic geometry, hyperbolic geometry, group theory, dynamical systems and mathematical physics. The theory of cluster algebras has grown into one of the most active research areas in mathematics; for example the 2002 paper "Cluster Algebras I" by Fomin and Zelevinsky has more than 2300 citations according to Google Scholar.

A particularly active branch is the relation between cluster algebras and the representation theory of finite dimensional algebras. These grew from categorical models for cluster algebras. This connection caused a burst of activity in representation theory which provided a new understanding of classical results and led to many new developments. In the other direction, the categorifications of cluster algebras proved to be powerful tools to obtain deep results in cluster algebras.

This workshop focused on interactions between cluster algebras and representation theory and on interdisciplinary applications of cluster algebras. It brought together researchers from different areas in order to promote interaction between researchers in different fields and to provide a platform for the state of the art on research in cluster algebras and its applications.

The workshop had seven 15 minute talks, ten 30 minute talks and fourteen 60 minute talks for a total of 31. Below they are grouped together according to the specific research area.

Cluster algebras and canonical bases. F. Qin reported on recent advances in finding a unified approach for the construction of dual canonical bases in cluster algebras, and C. Geiss presented a comparison result between two well-known bases for cluster algebras of surface type. A. Garcia Elsener spoke about cluster characters for super cluster algebras.

Categorifications of cluster algebras. B. Keller presented a framework for the categorification of quasi-cluster morphisms. This construction also appeared in the talk by M. Pressland on the relationship between different cluster structures on open positroid varieties. Y. Palu gave an account of relative extriangulated structures on cluster categories and their applications. M. Garcia communicated relations between determinantal semi-invariants and certain thick subcategories in the extriangulated category of 2-term complexes. O. Iyama spoke on semistable torsion classes and canonical decompositions in Grothendieck groups. S. Gratz presented a new approach to infinite rank cluster algebras as ind- and pro-algebras and D. Labardini-Fragoso talked about the mutation of infinite dimensional quiver representations.

Cluster algebras from Grassmannians. A. King explained how Grassmannian cluster categories can shed light on mirror symmetry results for Grassmannians obtained by K. Rietsch and L. Williams, and L. Williams presented a sequel to said work with K. Rietsch. M. Pressland reported on a representation theoretic approach to cluster algebras from positroids, which uses the categorification framework for quasi-cluster morphisms of Keller's talk.

Combinatorial aspects of cluster theory. E. Yıldırım presented a new approach to cluster expansion formulas using posets and their order ideals, and realizing the computations as matrix products. N. Reading reported on a related, but independent result that also uses posets and order ideals to produce cluster expansion formulas in cluster algebras of surface type. H. Thomas gave an account of work on universal F-polynomials and relations to mathematical physics. S. Morier-Genoud presented an application of quantum rational numbers to the Burau representation of the braid group. E. Gunawan spoke about a representation theoretic approach to the study of polytopes, in particular realizing the volume of certain polytopes by counting linear extensions of Auslander-Reiten quivers. M.

Kaipel presented a generalization of tau-cluster morphism categories and picture groups to simplicial fans.

Geometric surface models and representation theory of associated algebras. İ Çanakçı reported on a categorical approach to triangulations of the infinity-gon, realizing them as weak cluster-tilting subcategories in an extriangulated structure. A. Felikson spoke about a three dimensional version of the Farey tessellation and SL_2-tilings over Eisenstein numbers. L. Mou presented a novel approach to skew-symmetrizable cluster algebras via modulated quivers with potential and R. Coelho Simões gave an account of recent advances on maximal almost rigid modules over gentle algebras.

Frieze patterns in cluster algebras and representation theory. T. Holm spoke about a generalization of frieze patterns replacing the ring of integers by an arbitrary quadratic integer ring, and E. Faber presented a reduction technique on friezes. This is an application of a reduction construction on Frobenius extriangulated categories which parallels Iyama-Yoshino reduction.

Applications of cluster algebras. V. Bazier-Matte reported on current advances in the relation between cluster algebras and knot theory, realizing the Alexander polynomial as cluster characters of a specific cluster. M. Reineke presented a new interpretation of Floer potentials in Gromov-Witten theory as cluster characters of quiver representations. N. Williams spoke about recent progress concerning Donaldson–Thomas invariants from a 3-Calabi–Yau category in the context of the Bridgeland–Smith correspondence.

Cluster algebras in Lie theory. T. Nakanishi discussed a fundamental difference between ordinary and quantum cluster algebras that goes back to the pentagon identities of the classical and the quantum dilogarithm. M. Shapiro presented advances on cluster structures on the Teichmüller space of genus 2 surfaces and log-canonical coordinates for the symplectic groupoid. M. Gekhtman spoke about a new approach to constructing cluster structures on simple Lie groups whose main tool is a rational Poisson endomorphism that transforms the Poisson structure. B. Leclerc reported on cluster structures on shifted quantum affine algebras realized as an infinite rank cluster algebra associated with a root system of Dynkin type A,D or E.

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Workshop: Cluster Algebras and Its Applications

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Abstracts

Classical vs Quantum Pentagon Relations

Tomoki Nakanishi

The content of this talk is based on the paper [4].

For the rank 2 cluster algebra $\mathcal{A}$ without coefficients and with the initial exchange matrix

$$B = \begin{pmatrix} 0 & -\delta_1 \\ \delta_2 & 0 \end{pmatrix}, \tag{1}$$

the greedy elements were introduced in [2]. They are natural generalizations of cluster monomials and provide a basis of $\mathcal{A}$. Moreover, each greedy element has a positive expression $x^{\mathbf{g}} F_{\mathbf{g}}(\hat{y})$, where $F_{\mathbf{g}}(\hat{y})$ is a polynomial in $\hat{y}$ with positive integer coefficients. The quantum greedy elements were also introduced in [3]. They are natural quantizations of classical ones; however, it was shown that the positivity property "occasionally" fails. Based on several examples, the following conjecture was given.

Conjecture ([3])**.** *The positive is preserved if $\delta_1|\delta_2$ or $\delta_2|\delta_1$.*

In this talk we explain how the same condition naturally emerges in view of the pentagon relation in the cluster scattering diagram method [1]. Here, we describe the quantum case. (The classical case is obtained in the limit $q \to 1$.) We decompose the initial exchange matrix B as

$$B = \Delta\Omega = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Let $N = \mathbb{Z}^2$, N^+ be the set of the positive elements in N, and $\{\cdot, \cdot\}$ be the skew-symmetric form on N defined by Ω. Let $\mathfrak{g}_q$ be the Lie algebra with the generators X_n $(n \in N^+)$ and the relations

$$[X_n, X_{n'}] = [\{n, n'\}]_q X_{n+n'}, \quad [a]_q = \frac{q^a - a^{-a}}{q - q^{-1}}. \tag{3}$$

Let $\hat{\mathfrak{g}}_q$ be the completion of $\mathfrak{g}_q$ by the degree $\deg(n)$, and let $G_q = \exp(\hat{\mathfrak{g}}_q)$ be the exponential group of $\hat{\mathfrak{g}}_q$ whose product is defined by the BCH formula. For each $n \in N^+$, we introduced a *quantum diagram element* with positive rational parameters a, b,

$$\Psi_{a,b}[n] := \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j[ja]_q} q^{jb} X_{jn} \right). \tag{4}$$

In the limit $q \to 1$ they reduce to the classical one $\Psi[n]^{1/a}$. They satisfy the *(quantum) pentagon relation*: If $\{n_2, n_1\} = c > 0$, we have, for any b_i,

$$\Psi_{c,b_2}[n_2]\Psi_{c,b_1}[n_1] = \Psi_{c,b_1}[n_1]\Psi_{c,b_1+b_2}[n_1 + n_2]\Psi_{c,b_2}[n_2]. \tag{5}$$

They also satisfy the *fission/fusion formulas*:

$$\Psi_{a,b}[n] = \prod_{t=1}^{p} \Psi_{pa,b+(2t-p-1)a}[n], \quad \prod_{t=1}^{p} \Psi_{a,b\pm(2t-p-1)a/p}[n] = \Psi_{a/p,b}[n]. \tag{6}$$

Let us apply them and deduce some consequence for the positivity problem.

Example. (1) $(\delta_1, \delta_2) = (1, 2)$ (type B_2). In the classical case, we have the following consistency relation from the pentagon relation:

$$\begin{aligned} \begin{bmatrix}0\\1\end{bmatrix}^2 \begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \\ &= \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^2\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}^2. \end{aligned} \tag{7}$$

The quantum case is parallel.

$$\begin{aligned} \begin{bmatrix}0\\1\end{bmatrix}_{\frac12,0}\begin{bmatrix}1\\0\end{bmatrix}_{1,0} &= \begin{bmatrix}1\\0\end{bmatrix}_{1,0}\begin{bmatrix}1\\1\end{bmatrix}_{1,-\frac12}\begin{bmatrix}1\\1\end{bmatrix}_{1,\frac12}\begin{bmatrix}1\\2\end{bmatrix}_{1,0}\begin{bmatrix}0\\1\end{bmatrix}_{1,-\frac12}\begin{bmatrix}0\\1\end{bmatrix}_{1,\frac12} \\ &= \begin{bmatrix}1\\0\end{bmatrix}_{1,0}\begin{bmatrix}1\\1\end{bmatrix}_{\frac12,0}\begin{bmatrix}1\\2\end{bmatrix}_{1,0}\begin{bmatrix}0\\1\end{bmatrix}_{\frac12,0}. \end{aligned} \tag{8}$$

In the final expression, there is no negative exponent. This guarantees the positivity of the theta functions in the same way as the classical case [1].

(2) $(\delta_1, \delta_2) = (2, 3)$. This is the simplest example where the nonpositivity was observed in [3]. We concentrate on the relation up to degree 2. In the classical case, we have

$$\begin{bmatrix}0\\1\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix}^2 \equiv \begin{bmatrix}1\\0\end{bmatrix}^2\begin{bmatrix}1\\1\end{bmatrix}^6\begin{bmatrix}0\\1\end{bmatrix}^3. \tag{9}$$

Meanwhile, in the quantum case, we have

$$\begin{aligned} \begin{bmatrix}0\\1\end{bmatrix}_{\frac13,0}\begin{bmatrix}1\\0\end{bmatrix}_{\frac12,0} &\equiv \begin{bmatrix}1\\0\end{bmatrix}_{\frac12,0}\begin{bmatrix}1\\1\end{bmatrix}_{1,-\frac76}\begin{bmatrix}1\\1\end{bmatrix}_{1,-\frac12}\begin{bmatrix}1\\1\end{bmatrix}_{1,-\frac16}\begin{bmatrix}1\\1\end{bmatrix}_{1,\frac16}\begin{bmatrix}1\\1\end{bmatrix}_{1,\frac12}\begin{bmatrix}1\\1\end{bmatrix}_{1,\frac76}\begin{bmatrix}0\\1\end{bmatrix}_{\frac13,0} \\ &= \begin{bmatrix}1\\0\end{bmatrix}_{\frac12,0}\begin{bmatrix}1\\1\end{bmatrix}_{\frac16,-\frac13}\begin{bmatrix}1\\1\end{bmatrix}^{-1}_{\frac16,0}\begin{bmatrix}1\\1\end{bmatrix}_{\frac16,\frac13}\begin{bmatrix}0\\1\end{bmatrix}_{\frac13,0}. \end{aligned} \tag{10}$$

The negative power in the last expression gives rise to the nonpositivity of the theta functions. Observe that the negative power disappears in the limit $q \to 1$.

By a similar consideration, we obtain the following results.

Proposition ([4])**.** *Let $\delta(n)$ be the normalization factor of $n \in N^+$ in* [4].

(1) No negative power of $\Psi_{1/\delta(1,1),b}[1,1]$ appears if and only if $\delta_1|\delta_2$ or $\delta_2|\delta_1$.

(2) If $\delta_1|\delta_2$ or $\delta_2|\delta_1$, no negative power of $\Psi_{1/\delta(n),b}[n]$ appears up to $\deg(n) \leq 4$.

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Poisson maps as a tool for constructing cluster structures

Misha Gekhtman
(joint work with Michael Shapiro and Alek Vainshtein)

In recent years, a lot of progress was made in a study of integrable systems arising in the context of cluster algebras [3, 5, 4]. These systems "live" on Poisson–Lie groups and their Poisson homogeneous spaces, hence it is important to understand interplays between cluster and Poisson structures on such objects. In [6] we introduced the notion of a *Poisson bracket compatible with a cluster structure* and used it to develop, in [6, 7, 8], an approach for constructing cluster structures on algebraic Poisson varieties. In particular, we proved in [7, Ch. 4.3] that the cluster structure constructed in [2] for (double Bruhat cells of) a simply-connected simple Lie group $\mathcal{G}$ is compatible with the standard Poisson–Lie structure on $\mathcal{G}$. This led to a question, posed in [9], of *existence* exotic *cluster structures on $\mathcal{G}$, i.e. cluster structures non-isomorphic to the standard one and compatible with other Poisson-Lie brackets.* Although the answer to this question is negative in general - an example to that effect was constructed in [9] in the case of SL_2 - we conjectured that the answer is affirmative in the case of Poisson-Lie structures corresponding to quasi-triangular solutions of the classical Yang–Baxter equation classified by Belavin and Drinfeld in [1]. These solutions, called r-matrices are parametrized by discrete data consisting of an isometry between two subsets of positive roots in the root system of the Lie algebra of $\mathcal{G}$ and a continuous parameter that satisfies a system of linear equations governed by the discrete data. The discrete data determines a *Belavin–Drinfeld class* of r-matrices and corresponding Poisson–Lie brackets. Given two such brackets on $\mathcal{G}$ associated with representatives of two Belavin–Drinfeld classes, one can define a Poisson–Lie group $\mathcal{G} \times \mathcal{G}$ equipped with the direct product Poisson structure and then construct a Poisson-homogeneous structure on $\mathcal{G}$ with respect to the action of $\mathcal{G} \times \mathcal{G}$ by right and left multiplication.

The conjecture of [9] was restated in [10] to include not just Poisson–Lie brackets but also Poisson-homogeneous brackets of the kind described above. It now claims that for any such bracket associated with an arbitrary pair of Belavin–Drinfeld data there exists a *compatible regular complete*, possibly generalized, cluster structure in the ring of regular functions on $\mathcal{G}$. In [10], we proved this conjecture for a large class of Belavin–Drinfeld data in SL_n called *aperiodic* and *oriented*. Generalized cluster structures are not needed in this case.

The most crucial and difficult step in constructing a cluster structure compatible with a Poisson bracket is finding an initial *log-canonical* coordinate chart consisting of regular functions with particularly simple Poisson brackets between them. In [10], this goal was accomplished in an *ad hoc* way, with the choice of functions in the chart motivated by certain invariance properties, and with Poisson relations between these functions established via lengthy and cumbersome computations.

In [11], we propose a new approach to building log-canonical coordinate charts for any simply-connected simple Lie group $\mathcal{G}$ and arbitrary Belavin–Drinfeld data. The main ingredient is a rational Poisson map $h^{r,r'}$ between two copies of $\mathcal{G}$ endowed with two different Poisson-homogeneous structures. One is $\{\cdot,\cdot\}_{r,r'}$ determined by a pair of r-matrices from two arbitry Belavin–Drinfeld classes. The other, which we denote here by $\{\cdot,\cdot\}^{\mathrm{st}}_{r,r'}$, corresponds to two r-matrices from the standard Belavin–Drinfeld class whose Cartan parts match those of r, r'. The rational map $h^{r,r'}$ maps $(\mathcal{G}, \{\cdot,\cdot\}^{\mathrm{st}}_{r,r'})$ to $(\mathcal{G}, \{\cdot,\cdot\}_{r,r'})$. In the context of construction of an initial seed for a cluster structure compatible with $\{\cdot,\cdot\}_{r,r'}$, the map's utility is that by inverting $h^{r,r'}$ and using the inverse to pull back any of the clusters in the standard cluster structure on $\mathcal{G}$, one obtains a log-canonical parametrization for $(\mathcal{G}, \{\cdot,\cdot\}_{r,r'})$. In particular, when $h^{r,r'}$ has a rational inverse, one can build a regular log-canonical coordinate chart this way and then use it as an initial seed for a cluster structure. We illustrate this point in Section 4, where we use the current approach not only to recover all the results of [10] in a much more conceptual way, but also to drop the orientability condition which was imposed in [10] and which does not appear to be natural in a general Lie-theoretic framework. The aperiodicity condition is retained, however, since it is precisely the one that guarantees that the map $h^{r,r'}$ has a rational inverse. If this condition is not satisfied, finding the inverse involves considering certain polynomials in one variable whose roots allow to restore frozen variables for the standard cluster structure in terms of elements of $(\mathcal{G}, \{\cdot,\cdot\}_{r,r'})$ and whose coefficients serve as coefficients for *generalized exchange relations* in a compatible generalized cluster structure on $(\mathcal{G}, \{\cdot,\cdot\}_{r,r'})$.

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Symplectic groupoid and cluster structure on the Teichmüller space of the closed genus two surfaces

Michael Shapiro

(joint work with Leonid Chekhov)

This is a report on the arXiv preprint [1]. Denote by $\mathcal{A}_n$ the space of unipotent real upper-triangular $n \times n$ matrices. The symplectic groupoid of the unipotent upper-triangular matrices is defined as the variety $\mathcal{M}_n := \{(B,A)|B \in GL_n(\mathbb{R}), A \in \mathcal{A}_n, BAB^T \in \mathcal{A}_n\}$. $\mathcal{M}_n$ is equipped with two projections $s,t : \mathcal{M}_n \to \mathcal{A}_n$, $s : (B,A) \mapsto A, t : (B,A) \mapsto BAB^T$.

The fiber product $\mathcal{M}_n^{(2)} = \{(C, BAB^T), (B,A)\}$. The multiplication map $m : \mathcal{M}_n^{(2)} \to \mathcal{M}_n$ is defined as $m : \big((C,BAB^T),(B,A)\big) \mapsto (CB,A)$, and two projections $p_1, p_2 : \mathcal{M}_n^{(2)} \to \mathcal{M}_n$ are defined as $p_1 : \big((C,BAB^T),(B,A)\big) \mapsto (C,BAB^T)$ and $p_1 : \big((C,BAB^T),(B,A)\big) \mapsto (B,A)$. Weinstein's theorem [3] claims that there exists a unique up to a nonzero scalar symplectic form ω on $\mathcal{M}_n$ that satisfies the relation $m^*\omega = p_1^*\omega + p_2^*\omega$. The push forward by s or t of the Poisson structure dual to ω determines Dubrovin Poisson brackets on $\mathcal{A}_n$. The map $s \oplus t : \mathcal{M}_n \to \mathcal{A}_n \oplus \mathcal{A}_n$ is Poisson.

$GL_n(\mathbb{R})$ is equipped with the standard trigonometric Poisson-Lie structure. We denote by GL_n° a particular symplectic leaf of $GL_n(\mathbb{R})$ and show that for $B \in GL_n^\circ$ there exists a unique $A \in \mathcal{A}_n$ satisfying the groupoid condition $BAB^T \in \mathcal{A}_n$. Clearly, trigonometric Poisson structure can be restricted to GL_n°. Therefore we obtain the map $\phi : GL_n^\circ \to \mathcal{A}_n \times \mathcal{A}_n$, where ϕ is a Poisson map from GL_n° equipped with the trigonometric Poisson structure to $\mathcal{A}_n \times \mathcal{A}_n$ equipped with the product of Dubrovin Poisson brackets.

Theorem 1. *The standard cluster structure on GL_n compatible with the trigonometric Poisson structure induces cluster structure on $\mathcal{A}_n \times \mathcal{A}_n$ compatible with Dubrovin Poisson structure.*

We assume now that $n = 3$. The dimension $\dim(GL_3^\circ) = 6, \dim(\mathcal{A}_3 \times \mathcal{A}_3) = 6$, however $\dim(im(\phi)) = 5$ because A and BAB^T satisfy relation $M(A) = M(BAB^T)$ where Markov function $M(A) = A_{12}A_{13}A_{23} - A_{12}^2 - A_{13}^2 - A_{23}^2 + 4$.

For $n = 3$ Fock-Chekhov construction [2] associates with each unipotent upper-triangular A a hyperbolic torus with one hole whose length is described by Markov function. Therefore, each of A and BAB^T corresponds to a torus with a hole whose boundaries have equal hyperbolic length. Gluing these two tori along the boundaries creates a genus two hyperbolic surface described by GL_3°. Therefore it inherits the cluster structure from GL_3° associated with finite mutation type quiver X_6. Moreover, some elements of the mapping class group are representable as a sequence of cluster mutations, or are *cluster*. But not all elements are cluster.

To make all elements cluster we extend the quiver X_6 to X_7 with additional condition that the unique Casimir function of X_7 equals 1. This observation leads to the next statement.

Theorem 2.

- *The Teichmüller space of genus 2 closed surfaces possesses the cluster structure described by quiver X_7 modulo relation that the Casimir function equals one.*
- *The mapping class group has cluster representation, i.e. all its elements can be presented as rational transformations obtained by sequences of cluster mutations.*

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Analogs of dual canonical bases for cluster algebras from Lie theory

Fan Qin

Let $\Bbbk$ denote the base ring, which could be $\mathbb{Z}$, $\mathbb{C}$ (the classical case), $\mathbb{Z}[v^{\pm}]$, or $\mathbb{C}(v)$ (the quantum case). Fomin and Zelevinsky [FZ02] invented cluster algebras to study the total positivity and the dual Canonical basis $\mathbf{B}^*$ of the quantum group $\Bbbk[N]$. They expect that, for many varieties $\mathcal{A}$ from Lie theory, $\Bbbk[\mathcal{A}]$ is a cluster algebra and, moreover, $\Bbbk[\mathcal{A}]$ has a basis: it is an analog of $\mathbf{B}^*$ and contains all cluster monomials. In particular, we have the following conjecture following [FZ02] and later developments [Kim12][GLS13][GY16].

Conjecture (FZ-conjecture)**.** *The dual canonical basis $\boldsymbol{B}^*$ of $\Bbbk[N]$ (or $\Bbbk[N(w)]$ for for Kac-Moody types) contains all quantum cluster monomials.*

Remark. *Partial results were due to* [Lam11] *(Kronecker),* [KQ14] *(acyclic),* [Qin17] *(ADE type; symmetric Kac-Moody type with adaptable w), and* [KKKO18] *(symmetric Kac-Moody with any w).* [Qin20b] *gave the first proof for all cases. Most recently,* [McN21] *generalized it to p-canonical bases.*

Question. What are analogs of $\mathbf{B}^*$ for other cluster algebras $\Bbbk[\mathcal{A}]$? Can we generalize FZ-conjecture to general $\mathcal{A}$, which are not contained in N?

The analogs are provided by the common triangular bases $\mathbf{L}$ introduced in [Qin17]. These are Kazhdan-Lusztig type bases and generalize $\mathbf{B}^*$ for cluster algebras [Qin17, Qin20a]. They include the bases $\mathbf{L}^{BZ}$ for acyclic seeds in [BZ14].

Theorem. *For almost all known (quantum) upper cluster algebras arising from Lie theory, they possess the common triangular bases $\boldsymbol{L}$. Moreover, when the generalized Cartan matrix C is symmetric, $\boldsymbol{L}$ are strongly positive and are quasi-categorified by non-semisimple monoidal categories $(\mathcal{M}, \otimes)$ (i.e, categorified after minor changes).*

To be more precise, the theorem applies to $\Bbbk[\mathcal{A}]$ where $\mathcal{A}$ include unipotent subgroups, double Bruhat cells, double Bott–Samelson cells, Grassmannians, Positroids, open Richardson varieties, braid varieties, semisimple algebraic groups. The exception are the exotic cluster algebras in [GSV23], which have yet to be examined.

Our approach is as follows: While $\mathcal{A}$ from Lie theory can vary significantly, the coordinate rings $\Bbbk[\mathcal{A}]$ are closely related in cluster theory. This relationship enables us to construct the common triangular bases $\mathbf{L}$ for $\Bbbk[\mathcal{A}]$ from known dual canonical bases $\mathbf{B}^*$ for the quantum unipotent cells $\Bbbk[N^w]$ by employing cluster operations (for which we introduce the concept of freezing operators).

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3D Farey graph, lambda lengths and SL_2-tilings

Anna Felikson

(joint work with Oleg Karpenkov, Khrystyna Serhiyenko and Pavel Tumarkin)

We study geometric aspects of the Farey graph over Eisenstein integers and its realisation in the hyperbolic three-dimensional space obtained from the reflection group of the regular ideal hyperbolic tetrahedron. We generalise relations between Penner's λ-lengths and $SL_2(\mathbb{Z})$-tilings and prove a three-dimensional version of Ptolemy relation. Furthermore, based on the ideas of Short [Sh] we classify tame SL_2-tilings over Eisenstein integers in terms of pairs of paths in the 3D Farey graph.

The classical notion of the Farey graph, together with its close relatives such as circle packings, continued fractions, Conway-Coxeter friezes and SL_2-tilings, is a subject of large literature, see [MG] for an overview. Moreover, the Farey graph also appears in the study of discrete group of symmetries of the hyperbolic plane $\mathbb{H}^2$. It is then natural to ask which features of this theory can be generalised or extended to higher dimensions. There are many natural generalisations of the above mentioned classical notions based on the substitution of the ring of integer $\mathbb{Z}$ with other rings (see works of A. Hurwitz, Schmidt, Ford, Stange, Coxeter [C], Holm and Jorgensen [HJ] and many others).

We consider a 3-dimensional analogue of the Farey graph arising from a tessellation of hyperbolic space $\mathbb{H}^3$ by regular hyperbolic ideal simplices (used in place of a tessellation of $\mathbb{H}^2$ by ideal triangles). We call it the *tetrahedral graph* $\mathcal{T}$. The graph $\mathcal{T}$ inherits many good properties of the classical Farey graph $\mathcal{F}$. In particular, the vertices of $\mathcal{T}$ are precisely points of $\widehat{\mathbb{Q}}(\sigma) = \mathbb{Q}(\sigma) \cup \{\infty\}$, where $\sigma = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, the group of symmetries of $\mathcal{T}$ is the Bianchi group $Bi(3)$, and the edges of $\mathcal{T}$ can be described, similarly to the ones of the Farey graph, via determinants: two irreducible fractions p/q and $r/s \in \widehat{\mathbb{Q}}(\sigma)$ are joined by an edge if and only if $|ps - rq| = 1$. Furthermore, as for the Farey graph, faces of $\mathcal{T}$ can be described via Farey addition.

Another property inherited by the tetrahedral graph is the relation with λ-*lengths*. Given two points $x, y \in \partial\mathbb{H}^d$ and a choice of horospheres h_x, h_y centred at x and y, Penner [P] introduced the notion of λ-length λ_{xy} between x and y as $\lambda_{xy} = e^{d/2}$, where d is the signed distance between h_x and h_y. Penner also showed that for an ideal quadrilateral $xyzt$, the corresponding λ-lengths satisfy Ptolemy relation $\lambda_{xz}\lambda_{yt} = \lambda_{xy}\lambda_{zt} + \lambda_{yz}\lambda_{xt}$.

Given two irreducible fractions $p/q, r/s \in \widehat{\mathbb{Q}}(\sigma)$, we can also define the *det-length* $l(p/q, r/s)$ as the absolute value of the determinant $l(p/q, r/s) = |ps - rq|$. We then choose a distinguished set of horospheres at points of $\widehat{\mathbb{Q}}(\sigma)$ and show that λ-lengths computed with respect to these horospheres coincide with det-lengths.

Theorem 1. *Let $X, Y \in \widehat{\mathbb{Q}}(\sigma)$ be two irreducible fractions. Let the standard horosphere be chosen at every point of $\widehat{\mathbb{Q}}(\sigma)$. Then $\lambda_{XY} = l_{XY}$.*

To prove Theorem 1, we first show an analogue of the Ptolemy relation:

Theorem 2. *Let $A_1A_2A_3A_4$ be a fundamental tetrahedron of $\mathcal{T}$ with vertices in $\widehat{\mathbb{Q}}(\sigma)$, choose any $X \in \widehat{\mathbb{Q}}(\sigma)$ distinct from A_i. Let $\lambda_i = \lambda_{XA_i}$ be the λ-length of XA_i, $i = 1, \ldots, 4$. Then*

$$\sum_{i=1}^{4} \lambda_i^4 = \sum_{1 \le i < j \le 4} \lambda_i^2 \lambda_j^2.$$

We then prove a 3-dimensional counterpart of Ptolemy relation which can be applied to any 5 points in $\widehat{\mathbb{C}}$:

Theorem 3. *Let $X_1, \ldots, X_5 \in \widehat{\mathbb{C}} = \partial \mathbb{H}^3$ be 5 distinct points. Suppose that there are horospheres chosen in these points. Let $\lambda_{ij} = \lambda_{X_i X_j}$. Then*

$$\sum_{i<j} \lambda_{ij}^4 \lambda_{kl}^2 \lambda_{lm}^2 \lambda_{mk}^2 = \sum_{\text{cycles } (ijklm)} \lambda_{ij}^2 \lambda_{jk}^2 \lambda_{kl}^2 \lambda_{lm}^2 \lambda_{mi}^2,$$

where all indices i, j, k, l, m are distinct.

Next, we apply $\mathcal{T}$ to generalise results of Short [Sh] to classify $SL_2(\mathbb{Z}[\sigma])$-tilings. A *path* (v_i) in $\mathcal{T}$ is a (bi-infinite) sequence of vertices of $\mathcal{T}$ such that v_i and v_{i+1} are connected by an edge of $\mathcal{T}$. We normalise the paths by requiring that the expressions $v_i = p_i/q_i$ satisfy the condition $p_i q_{i+1} - p_{i+1} q_i = 1$. We then prove the following result.

Theorem 4. *Given two normalised paths $v_i = p_i/q_i$ and $u_j = r_j/s_j$, the map $(u_i, v_j) \mapsto m_{ij} = p_i s_j - q_i r_j$ provides a bijection between equivalence classes of the tame $SL_2(Z[\sigma])$-tilings and pairs of paths in $\mathcal{T}$ considered up to simultaneous action of $SL_2(Z[\sigma])$ on both paths.*

Here, we say that two $SL_2(Z[\sigma])$-tilings are equivalent if one is obtained from the other by multiplication of even rows by σ^k and of odd rows by σ^{-k}, together with multiplication by σ^l (resp. σ^{-l}) of even (resp. odd) columns.

Given a path in $\mathcal{T}$, we construct a sequence of numbers from $\mathbb{Z}[\sigma]$ called $\mathcal{T}$-angle sequence of the path (also known as quiddity sequences for friezes or as itinerary in [Sh]). We then use Theorem 4 to provide a geometric interpretation of the classification of SL_2-tilings obtained in [BR].

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Representations of shifted quantum affine algebras and cluster algebras

Bernard Leclerc
(joint work with Christof Geiss and David Hernandez)

Let $\mathfrak{g}$ be a simple Lie algebra of type A, D, E over $\mathbb{C}$, and let $U_q(\widehat{\mathfrak{g}})$ be the corresponding untwisted quantum affine algebra for a generic quantum parameter q. In recent years, cluster algebras have become a powerful new tool for studying the tensor structure of the category of finite-dimensional modules over $U_q(\widehat{\mathfrak{g}})$, see for instance [7, 8].

Shifted quantum affine algebras $U_q^\mu(\widehat{\mathfrak{g}})$ are a larger class of algebras introduced by Finkelberg and Tsymbaliuk [3] in their study of quantized K-theoretic Coulomb branches of $3d$ $N = 4$ SUSY quiver gauge theories. They depend on an integral weight μ of $\mathfrak{g}$. When $\mu = 0$, the algebra $U_q^0(\widehat{\mathfrak{g}})$ is a central extension of $U_q(\widehat{\mathfrak{g}})$ and it has essentially the same representation theory. When $\mu \neq 0$ is anti-dominant, $U_q^\mu(\widehat{\mathfrak{g}})$ does not have any non-trivial finite-dimensional representation. In [5], Hernandez has started a systematic study of the representation theory of $U_q^\mu(\widehat{\mathfrak{g}})$. He has introduced a category $\mathcal{O}^\mu$ containing infinite-dimensional representations, and shown that the Grothendieck group of $\mathcal{O}^{\mathrm{sh}} := \bigoplus_{\mu \in P} \mathcal{O}^\mu$ has a natural ring structure coming from an operation on representations called fusion product.

The aim of this talk was to present the main result of [4], which states that the Grothendieck ring $K_{\mathbb{Z}}$ of a full subcategory of $\mathcal{O}^{\mathrm{sh}}$ (defined by certain integrality conditions on the loop-weights of the representations) is isomorphic to a suitable completion of an infinite rank cluster algebra $\mathcal{A}_{w_0}$.

The first part of the talk was devoted to the description and main properties of $\mathcal{A}_{w_0}$. This algebra is a modification of the cluster algebra $\mathcal{A}_e$ of [6], whose quiver Γ_e coincides with the Auslander-Reiten quiver of the derived category $D^b(KQ)$ of a Dynkin quiver Q of the same type as $\mathfrak{g}$, with added vertical arrows corresponding to the Auslander-Reiten translation. The quiver Γ_{w_0} of $\mathcal{A}_{w_0}$ is defined as follows. Let G_Q denote the finite subquiver of Γ_e corresponding to the Auslander-Reiten quiver of the abelian module category $\mathrm{mod}(KQ)$. Replace each vertex of G_Q by a pair of a red and a green vertex connected by a down arrow, and rearrange the incident arrows so that 3-cycles of Γ_e involving at least two vertices of G_Q become 4-cycles in the new quiver Γ_{w_0}. This is illustrated in Figure 1 for an equi-oriented quiver Q of type A_3.

A remarkable feature of Γ_{w_0} is that if we perform a sequence of quiver mutations at all green vertices, we get an isomorphic quiver in which the middle finite part consisting of red and green vertices has been shifted one step down. Iterating infinitely many times this sequence of mutations, we obtain in the limit the quiver Γ_e. This allows us to regard Γ_e as a "reference seed at infinity", and to attach certain "stabilized g-vectors" to the cluster variables of $\mathcal{A}_{w_0}$.

In type A_1, the cluster algebra $\mathcal{A}_{w_0}$ has a nice combinatorial model in terms of triangulations of an infinity-gon. Similar models have been studied recently [1] in relation with completions of the cluster categories of type A_N in the limit $N \to \infty$.

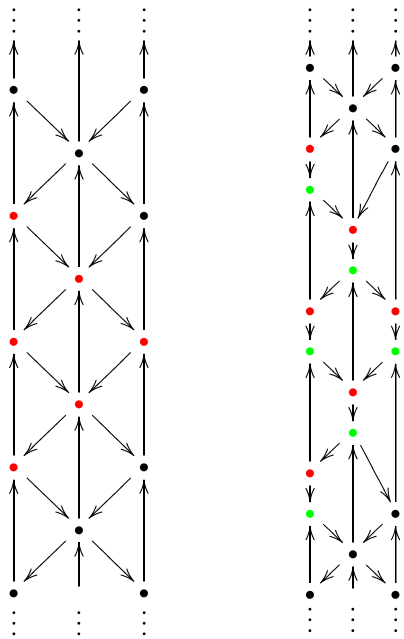

FIGURE 1. *The quiver Γ_e (with its red subquiver G_Q), and the quiver Γ_{w_0} in type A_3.*

In the second part of the talk, I gave a brief review of the representation theory of shifted quantum affine algebras [5], and of the QQ-system relations [2] which correspond to the mutations at green and red vertices of the initial seeds of $\mathcal{A}_{w_0}$.

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On the categorification of quasi-cluster morphisms

Bernhard Keller
(joint work with Chris Fraser)

Quasi-cluster morphisms were introduced by Chris Fraser [2]. They are certain algebra morphisms between cluster algebras all of whose coefficients are invertible. For example, as discovered by Fraser and Sherman-Bennett [3], mutations at frozen sinks or sources give rise to (generalized) quasi-cluster isomorphisms. Donaldson–Thomas transformations (=twist automorphisms) also naturally lift to quasi-cluster automorphisms. In this talk, we would like to propose a framework for categorifying quasi-cluster morphisms using derived categories of stably 2-Calabi–Yau Frobenius categories. This framework has grown out of joint work with Yilin Wu [4] and was recently applied by Matthew Pressland [6] in his proof of Muller–Speyer's quasi-coincidence conjecture [5].

Let Q be a finite quiver without loops nor 2-cycles (and without frozen part). Suppose its set of vertices is the set of natural numbers $1 \leq i \leq n$. Let N be the lattice $\mathbb{Z}^n$ and $\beta : N \times N \to \mathbb{Z}$ the skew-symmetric bilinear form whose matrix in the standard basis is the exchange matrix B associated with Q (the coefficient b_{ij} is the difference between the number of arrows from i to j and the number of arrows from j to i). Let M be the $\mathbb{Z}$-dual lattice of N. We also denote by β the map $N \to M$ taking a vector x to $\beta(x, ?)$. We consider N as the 'c-vector lattice' and M as the 'g-vector lattice'. Now let $\widetilde{Q}$ be an ice quiver whose frozen part is Q and whose set of vertices is the set of integers $1 \leq i \leq m$ for some $m \geq n$. Here, by saying that $\widetilde{Q}$ is an ice quiver, we mean that it is endowed with a (not necessarily full) subquiver $\widetilde{Q}_{fr}$, called its frozen part, such that $\widetilde{Q}$ does not contain any loops nor half-frozen 2-cycles nor non frozen 2-cycles. However, $\widetilde{Q}_{fr}$ may contain frozen 2-cycles. We denote by $\widetilde{M}$ the direct sum of M and the dual of the free abelian group on the frozen vertices of $\widetilde{Q}$. We have the canonical projection $\widetilde{M} \to M$ and the map $\widetilde{\beta} : N \to \widetilde{M}$ lifting β and given by the arrows between the unfrozen and the frozen vertices of $\widetilde{Q}$ (notice that arrows between frozen vertices are not taken into account here).

Let $\mathcal{U}$ denote the upper cluster algebra with invertible coefficients associated with $\widetilde{Q}$ and $\mathcal{U}'$ the upper cluster algebra associated with another ice quiver $\widetilde{Q}'$. We write $\mathbb{P}$ for the coeffient group of $\mathcal{U}$, i.e. the group of Laurent monomials in the frozen cluster variables. Denote by $\underline{\mathcal{U}}$ and $\underline{\mathcal{U}}'$ the corresponding upper cluster algebras without coefficients associated with Q and Q'. We have a canonical isomorphism between $\underline{\mathcal{U}}$ and the quotient of $\mathcal{U}$ where all coefficients are specialized to 1. A *quasi-cluster morphism* from $\mathcal{U}$ to $\mathcal{U}'$ is an algebra morphism $f : \mathcal{U} \to \mathcal{U}'$ such that

a) f takes $\mathbb{P}$ to $\mathbb{P}'$ and induces a cluster algebra isomorphism $\underline{f} : \underline{\mathcal{U}} \xrightarrow{\sim} \underline{\mathcal{U}}'$;
b) f takes each cluster variable of $\mathcal{U}$ to the product of a cluster variable of $\mathcal{U}'$ with an element in $\mathbb{P}'$;

c) we have $\phi \circ \widetilde{\beta} = \widetilde{\beta}' \circ \psi$, where ψ is the isomorphism induced by f between the c-vector lattices and ϕ the morphism induced by f between the (extended) g-vector lattices.

To categorify this notion, we assume that $\mathcal{U}$ admits a Frobenius categorification, i.e. there is a pair $(\mathcal{E}, T)$ such that

a) $\mathcal{E}$ is a Krull–Schmidt Frobenius exact category (which may be Hom-infinite but in that case should be enriched over the category of pseudo-compact vector spaces);
b) the associated stable category $\underline{\mathcal{E}}$ is Hom-finite and 2-Calabi–Yau;
c) $T = T_1 \oplus \cdots \oplus T_n \oplus \cdots \oplus T_m$ is a basic cluster-tilting object whose projective indecomposable summands are in bijection with the frozen vertices of $\widetilde{Q}$ and whose non projective summands with the non frozen vertices;
d) the basic cluster-tilting objects of $\mathcal{E}$ determine a cluster structure on $\mathcal{E}$ in the sense of [1];
e) the endomorphism algebra A of T has $\widetilde{Q}$ as its ice quiver, i.e. the frozen vertices are $n+1, \ldots, m$, the number of non frozen arrows from i to j equals $\dim \operatorname{Ext}^2_A(S_i, S_j)$ and the number of non frozen arrows equals $\dim \operatorname{Ext}^1_A(S_j, S_i) - \dim \operatorname{Ext}^2_A(S_i, S_j)$.

Let $\mathcal{P} \subseteq \mathcal{E}$ be the full subcategory of projective-injective objects. With the obvious notations, the notion of quasi-cluster morphism is then categorified by that of a *quasi-cluster functor*, i.e. a triangle functor $\mathcal{D}^b(\mathcal{E}) \to \mathcal{D}^b(\mathcal{E}')$ taking $\mathcal{D}^b(\mathcal{P})$ to $\mathcal{D}^b(\mathcal{P}')$, inducing a triangle equivalence $\underline{\mathcal{E}} \xrightarrow{\sim} \underline{\mathcal{E}}'$ taking T to a cluster tilting object FT reachable from T' and such that there is a triangle functor $\widetilde{F} : \mathcal{D}^b(\operatorname{add} T) \to \mathcal{D}^b(\operatorname{add} T')$ compatible with F via the canonical functors induced the the inclusion of $\operatorname{add}(T)$ in $\mathcal{E}$ and $\operatorname{add}(T')$ in $\mathcal{E}'$.

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Positroid varieties and quasi-coincidence via representation theory

Matthew Pressland

The problem. The totally non-negative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}(\mathbb{C}^n)$ is an important object in the classical study of total positivity. It is a moduli space whose points are those k-dimensional subspaces of $\mathbb{C}^n$ realisable as the row span of a $k \times n$ matrix all of whose maximal minors—the Plücker coordinates of the subspace—are non-negative real numbers. Postnikov [6] describes a cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}(\mathbb{C}^n)$ in which each open cell has the form $\Pi^\circ \cap \mathrm{Gr}_{k,n}^{\geq 0}(\mathbb{C}^n)$ for a subvariety $\Pi^\circ \subseteq \mathrm{Gr}_{k,n}(\mathbb{C}^n)$ of the full Grassmannian, called an *open positroid variety*.

Cluster algebras are a useful tool for studying totally positive spaces, and indeed this motivated their original definition. To apply them to an open positroid variety Π°, one needs a cluster algebra structure on the homogeneous coordinate ring $\mathbb{C}[\widehat{\Pi}^\circ]$ with the property that $\Pi^\circ \cap \mathrm{Gr}_{k,n}^{\geq 0}(\mathbb{C}^n)$ is the locus on which all cluster variables take non-negative real values. For the top-dimensional cell, this was achieved by Scott [9], who describes a cluster algebra structure in which all Plücker coordinates appear among the (usually infinitely many) cluster variables.

For a general open positroid variety, the cluster algebra structure was obtained only recently by Galashin and Lam [4]. In general, the situation is not as nice as for the top-dimensional cell: one obtains two different isomorphisms $\eta^{\pm} \colon \mathcal{A} \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^\circ]$ for a cluster algebra $\mathcal{A}$, such that two sets of cluster variables contain different subsets of the non-zero Plücker coordinates on Π°. However, a conjecture attributed to Muller and Speyer [5] asserts that these two cluster structures quasi-coincide: among other properties, this means that for each cluster variable x in $\mathcal{A}$, there exists a cluster variable x' and frozen variables p and q such that

$$\eta^{+}(x) = \eta^{-}(x')\frac{\eta^{-}(p)}{\eta^{-}(q)}. \tag{1}$$

In particular, the two cluster structures have the same cluster *monomials*, and hence define the same totally non-negative part.

The solution. In [8], we give a proof of this conjecture via additive categorification (see also [2] for an independent proof). The key steps are as follows.

Step 1: Categorify the abstract cluster algebra $\mathcal{A}$; this was done in [7]. The categorification has the form $\operatorname{gproj} \mathrm{CM}(B)$, the category of Gorenstein projective objects in the category of maximal Cohen–Macaulay modules over a certain Gorenstein order B. By duality, the category $\operatorname{ginj} \mathrm{CM}(B)$ of Gorenstein injective objects in $\mathrm{CM}(B)$ can be used instead: this is equivalent to $\operatorname{gproj} \mathrm{CM}(B)$ as an exact category, but sits inside $\mathrm{CM}(B)$ differently.

Step 2: Relate these categorifications of $\mathcal{A}$ to the Grassmannian cluster category $\mathrm{CM}(C)$ defined by Jensen, King and Su; here C is another, explicitly defined, Gorenstein order. This is done in joint work with Çanakçı and King [1], in which we show that $\mathrm{CM}(B)$ is a full subcategory of $\mathrm{CM}(C)$, and hence the same goes for $\operatorname{gproj} \mathrm{CM}(B)$ and $\operatorname{ginj} \mathrm{CM}(B)$. Jensen, King and Su describe a particular set of 'rank one' objects in $\mathrm{CM}(C)$, in bijection with the Plücker coordinates. A further result of [1] is that the rank one objects in the full subcategory $\mathrm{CM}(B)$ are those corresponding to non-zero Plücker coordinates on the open positroid variety Π°.

Step 3: To complete the proof of Muller–Speyer's conjecture, we first observe that the categorifications $\operatorname{gproj}\mathrm{CM}(B)$ and $\operatorname{ginj}\mathrm{CM}(B)$ are adapted to the isomorphisms η^+ and η^- respectively: for example, the rank one objects in $\operatorname{gproj}\mathrm{CM}(B)$ correspond to the Plücker coordinates which are cluster variables under η^+, while those in $\operatorname{ginj}\mathrm{CM}(B)$ correspond to the Plücker cluster variables for η^-.

The key homological statements are then that there are canonical equivalences

$$\mathcal{D}^{\mathrm{b}}(\operatorname{gproj}\mathrm{CM}(B)) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathrm{CM}(B)) \xleftarrow{\sim} \mathcal{D}^{\mathrm{b}}(\operatorname{ginj}\mathrm{CM}(B)),$$

of derived categories, and that the composed equivalence $\mathcal{D}^{\mathrm{b}}(\operatorname{gproj}\mathrm{CM}(B)) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\operatorname{ginj}\mathrm{CM}(B))$ is a quasi-cluster functor (see [3] and the extended abstract by B. Keller in this volume); the quasi-coincidence conjecture follows from this.

In practice, given $X \in \operatorname{ginj}\mathrm{CM}(B)$, one can construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega X \arrow[r] & P \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & \Omega X \arrow[r] \arrow[u, equal] & Q \arrow[r] \arrow[u] & \Sigma\Omega X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

by first taking a syzygy of X, yielding $\Omega X \in \operatorname{gproj}\mathrm{CM}(B)$, and then a cosyzygy of ΩX in the Frobenius category $\operatorname{gproj}\mathrm{CM}(B)$, yielding $X' = \Sigma\Omega X \in \operatorname{gproj}\mathrm{CM}(B)$. Then

$$\eta^+(X) = \eta^-(X')\frac{\eta^-(P)}{\eta^-(Q)},$$

for lifts [3] of $\eta^\pm$ to cluster characters on $\mathcal{D}^{\mathrm{b}}(\mathrm{CM}(B))$. This recovers Equation (1).

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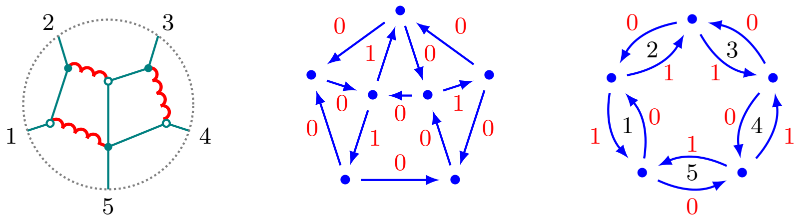
Categorification and Mirror Symmetry for Grassmannians

Alastair King
(joint work with Bernt Tore Jensen and Xiuping Su)

This talk aims to show how mirror symmetry, in the sense of [4], can be used to characterise g-vectors for the Grassmannian cluster category $\mathrm{CM}\, C$ from [2].

(1)



plabic graph Γ $\quad$ $\mu \in \mathbb{M}$ on quiver Q_A $\quad$ $\partial\mu = 14$ on Q_C

The coordinate ring $\mathbb{C}\big[\widehat{\mathrm{Gr}}_{k,n}\big]$ of the Grassmannian of k-planes in $\mathbb{C}^n$ is generated by Plücker coordinates Δ_J, for each k-subset $J \subseteq \{1..n\}$. Clusters of Plücker coordinates are famously controlled by plabic graphs.

Let Γ be a plabic graph of 'rank' n and 'helicity' k, and let $Q = Q_A$ be its dual quiver with faces, as in [1], and illustrated in (1) in the case $(k,n) = (2,5)$. The *matching lattice* $\mathbb{M}$ consists of 1-cochains $m \in \mathbb{Z}^{Q_1}$ such that dm is a constant, which we denote by $\deg m$. A *matching* μ is such a cochain with $\mu_a \geq 0$, for all $a \in Q_1$, and $\deg \mu = 1$. We write $\mathbb{C}[\mathbb{M}]$ for the coordinate ring of the torus whose character lattice is $\mathbb{M}$ and write z^m for the function on the torus given by $m \in \mathbb{M}$.

The homogeneous network chart for Γ is given algebraically by the map (cf. [3])

$$\widehat{\mathrm{net}}_\Gamma \colon \mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}] \to \mathbb{C}[\mathbb{M}] \colon \Delta_J \mapsto P_J := \textstyle\sum_{\mu\,:\,\partial\mu = J} z^\mu \tag{2}$$

In [2], an additive categorification of $\mathbb{C}\big[\widehat{\mathrm{Gr}}_{k,n}\big]$ is given by the category $\mathrm{CM}\, C$ for an algebra C whose quiver is Q_C as in (1). Every $M \in \mathrm{CM}\, C$ has a cluster character $\Psi_M \in \mathbb{C}\big[\widehat{\mathrm{Gr}}_{k,n}\big]$ and, for rank 1 modules M_J, we have $\Psi_{M_J} = \Delta_J$.

For more general clusters, the plabic graph Γ can be replaced by a cluster tilting object $T \in \mathrm{CM}\, C$ and the algebra $A = \mathrm{End}(T)^{\mathrm{op}}$. The matching lattice $\mathbb{M}$ is replaced by $K(\mathrm{CM}\, A)$ and the boundary value map $\mu \mapsto \partial\mu$ is replaced by the restriction functor $e\colon\ \mathrm{CM}\, A \to \mathrm{CM}\, C$, which has a right adjoint $G = \mathrm{Hom}(T, -)$. The class $[GM] \in K(\mathrm{CM}\, A)$ is the g-vector of $M \in \mathrm{CM}\, C$, often expressed in components in basis of indecomposable projective A-modules.

Following [1], we categorify the partition function P_J in (2) by

$$\mathcal{P}_M = \sum_{X\,:\, eX = M} z^{[X]} = \sum_{Y \leqslant \underline{GM}} z^{[\widehat{Y}]} \tag{3}$$

for $M \in \mathrm{CM}\, C$, where $\underline{GM} = \underline{\mathrm{Hom}}(T, M)$, while $\widehat{Y} \leqslant GM$ is the lift of $Y \leqslant \underline{GM}$. The sum is to be interpreted motivically, taking Euler characteristics of families of submodules.

In a forthcoming paper, we prove the following results.

Theorem 1. *For any cluster tiling object T in $\mathrm{CM}\,C$, there is a map*

$$\widehat{\mathrm{net}}_T\colon \mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}] \to \mathbb{C}[K(\mathrm{CM}\,A)]\colon \Psi_M \mapsto \mathcal{P}_M$$

Theorem 2. *The set $\{\Psi_M : M \text{ generic}\}$ is a basis of $\mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}]$. In the network chart $\widehat{\mathrm{net}}_T$, these have distinct leading exponents $[GM] \in K(\mathrm{CM}\,A)$.*

Theorem 3. *Every g-vector is a generic g-vector, that is,*

$$\mathsf{Mon}_{\mathrm{GV}}(T) := \{[GM] : M \in \mathrm{CM}\,C\} = \{[GM] : \textit{generic } M \in \mathrm{CM}\,C\}$$

Theorem 4. $\mathsf{Mon}_{\mathrm{GV}}(T)$ *is (the integral points of) a rational polyhedral cone.*

Theorem 5. $\mathsf{Mon}_{\mathrm{GV}}(T)$ *is determined by the conditions*

$$[U]\cdot x \geq 0, \quad \textit{for all } U \leq \mathrm{Ext}^1(S_i, T), \textit{ for each simple } C \textit{ module } S_i.$$

Note that $[U] \in K(\mathrm{fd}\,A^{\mathrm{op}})$, which is dual to $K(\mathrm{CM}\,A)$. This result is proved as in [4] by showing that $\mathsf{Mon}_{\mathrm{GV}}(T)$ is the superpotential cone for a superpotential W_T, which is a sum of the F-polynomials of $\mathrm{Ext}^1(S_i, T)$. As such it can be considered an instance of mirror symmetry for the Grassmannian.

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Thick subcategories arising from semi-invariants

Monica Garcia

In this talk we present the results in [5] as well as work in progress. Let Λ be a finite dimensional algebra over a field $\Bbbk$. Inspired in part by the additive categorification of cluster algebras [1, 4], T. Adachi, I. Reiten and O. Iyama introduced τ-tilting theory [2], a generalization of tilting theory where mutation is always possible. A central class of objects in τ-tilting theory is the set of *2-term (pre)silting complexes*, which are ext-rigid objects in the extriangulated category $\mathcal{K}_\Lambda = \mathcal{K}^{[-1,0]}(\mathrm{proj}\Lambda)$ of complexes of projective modules concentrated in degrees -1 and 0. Basic 2-term silting objects (up to isomorphism) are in bijection with *support τ-tilting modules*, which parametrize *functorially finite torsion classes* and *left-finite wide subcategories* (see Figure 1). Left-finite wide subcategories can be realized as categories of θ-semistable modules [7], where θ is the g-vector of certain 2-term presilting object [11]. This fact can be stated in the following way :

Theorem 1 ([11])**.** *For any left finite wide subcategory $\mathcal{W}$, there exists a 2-term presilting object U of g-vector θ such that*

$$\mathcal{W} = \mathscr{W}(U) = \{M \in \mathit{mod}\ \Lambda \mid \langle \theta, \mathit{dim}\ M\rangle = 0 \text{ and } s(U, M) \neq 0\}$$

where $s(U,-)$ is the ***determinantal semi-invariant*** [10, 6] *associated to U and defined by :*

$$s(U, M) = \mathit{det}\left(\mathit{Hom}(U^0, M) \xrightarrow{\textbf{\textit{Hom(f,-)}}} \mathit{Hom}(U^{-1}, M)\right).$$

Determinantal semi-invariants also give rise to subcategories of $\mathcal{K}_\Lambda$ with interesting properties :

Proposition 1. *Let $M \in \mathit{mod}\ \Lambda$, then the full subcategory*

$$\mathscr{T}(M) = \{X \in \mathcal{K}_\Lambda \mid s(X, M) \neq 0\}$$

is ***thick****, that is, it is additive and satisfies that for all triangles $X \to Y \to Z \dashrightarrow X[1]$ with $X, Y, Z \in \mathcal{K}_\Lambda$, if two of the objects appearing in the triangle lays in $\mathscr{T}(M)$, then the third does as well. In other words, $\mathscr{T}(M)$ is closed under extensions, cones and cocones in $\mathcal{K}_\Lambda$.*

Inspired by the previous proposition, we introduce new bijections between thick subcategories and known classes of objects in $\mathcal{K}^{[-1,0]}(\mathrm{proj}\Lambda)$ which mirror those between support τ-tilting modules, f.f. torsion classes and l.f. wide subcategories. Moreover, we obtain new equivalent definitions of being g-finite for any finite dimensional $\Bbbk$-algebra Λ.

Theorem 2 ([5])**.** *There are explicit bijections between:*

(i) Isomorphism classes of basic silting objects in $\mathcal{K}^{[-1,0]}(\mathit{proj}\Lambda)$.
(ii) Complete cotorsion pairs in $\mathcal{K}^{[-1,0]}(\mathit{proj}\Lambda)$.
(iii) Thick subcategories in $\mathcal{K}^{[-1,0]}(\mathit{proj}\Lambda)$ with enough ext-injectives.

These bijections fit in the following commutative diagram:

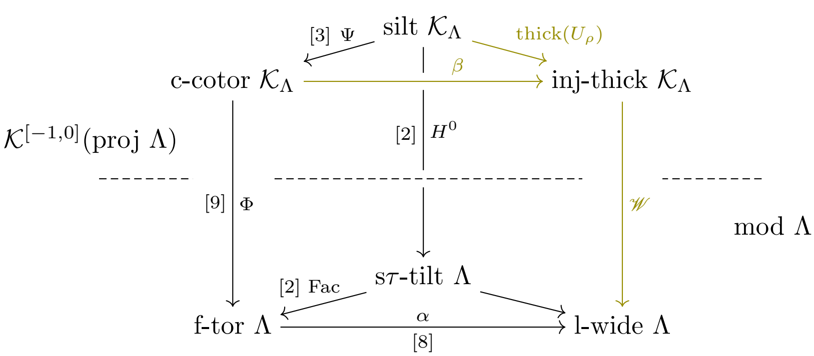

FIGURE 1. Summary of our contributions and references for known results.

Theorem 3 (In preparation). *Let Λ be a finite dimensional algebra. The following are equivalent :*

(1) Λ is g-finite.
(2) All cotorsion pairs are complete.
(3) There exist a finite number of cotorsion pairs.

Theorem 4 (In preparation). *If Λ is g-finite, then all thick subcategories have enough injective objects.*

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Pro-cluster algebras and the Sato Grassmannian

Sira Gratz
(joint work with Christian Korff)

Classically, cluster algebras have finite clusters, each of which can be reached from any other by a finite sequence of mutations. When we pass to an infinite rank setting, allowing clusters to be infinite, we obtain *cluster algebras of infinite rank* as introduced in [5]. These locally behave like cluster algebras of finite rank: In fact, in [6] we show that they are precisely the ind-objects in an appropriate category of cluster algebras—the category of *melting cluster morphisms* as introduced in [1] under the name *category of rooted cluster algebras.*

From a combinatorial ([3], [4]) as well as from a representation theoretic perspective ([2], [7]) we are confronted with situations where we have infinite rank "clusters" which are not connected by finite mutations. On the side of cluster algebras, this phenomenon can be captured by *pro-cluster algebras*. We work in the category fCl of *freezing cluster morphisms*, introduced by [10].

Definition 1. A *pro-cluster algebra* is a pro-object in the category fCl of rooted cluster algebras and freezing cluster morphisms.

This is a modification of the definition given in [10]. Passing to an appropriate subcategory of fCl, the wide subcategory fCl_{id} of so-called *ideal cluster morphisms*, we obtain *ideal pro-cluster algebras*, which are topologically controlled by a "cluster structure" we will formalize below.

Definition 2. Let $F\colon D \to \mathrm{fCl}_{id}$ be a cofiltered system in the category of finite rank rooted cluster algebras with *ideal* freezing cluster morphisms. Then we call the pro-cluster algebra $\lim F$ an *ideal pro-cluster algebra*.

Passing to ideal freezing cluster morphisms is a reasonable restriction; in fact, we can show that all freezing cluster morphisms which are *controlled*, that is, those which in a formal sense meaningfully preserve the cluster structure, are also ideal.

A cofiltered system $F\colon D \to \mathrm{fCl}$ induces several cofiltered systems $\mathbb{X}\colon D \to \mathrm{Set}$ in the category of sets, where eventually, each level $\mathbb{X}(i)$ is described by a cluster, and the maps $\mathbb{X}(i \to j)$ are restrictions of $F(i \to j)$. Here $i, j \in D$ and $i \to j$ is a map in D. The set $\lim \mathbb{X} \setminus \mathbb{Z}$ is a *pro-cluster* in the pro-cluster algebra $\lim F$.

Example. [10] features a construction of a pro-cluster algebra, where the pro-clusters are precisely triangulations of the "completed ∞-gon" (cf. [3]).

We can now make precise the cluster structure on ideal pro-cluster algebras.

Theorem 1. *Let $\mathcal{A}$ be an ideal pro-cluster algebra, considered as a topological space under the limit topology. The subring of $\mathcal{A}$ generated by its pro-cluster variables is dense in $\mathcal{A}$.*

While each pro-cluster is determined by its finite levels, which live in finite rank cluster algebras where we have only finite mutations available, the limit construction allows us to see many more clusters which, in general, are no longer connected by finite mutations (as illustrated in Example).

We end with an important example of a pro-cluster algebra. The points of the *Sato Grassmannian* $\widetilde{\mathrm{UGM}}$ parametrise (up to a factor) the solutions of the Kadomtsev-Petviashvili-hierarchy [8]. The Sato Grassmannian can be viewed as an ind-variety, with a coordinate ring defined as the appropriate limit of the coordinate rings of its finite pieces. Said finite pieces, i.e. the homogeneous coordinate rings $\mathbb{C}[\widehat{\mathrm{Gr}_{\mathrm{k,n}}}]$ of the Grassmannians $\mathrm{Gr}(k, n)$ for $k \leq n$ under the Plücker embedding, have natural cluster algebra structures as shown in [9].

Theorem 2 ([6])**.** *The coordinate ring of the Sato Grassmannian $\widetilde{\mathrm{UGM}}$ viewed as an ind-variety has a pro-cluster algebra structure induced by the cluster algebra structures on $\mathbb{C}[\widehat{\mathrm{Gr}_{\mathrm{k,n}}}]$.*

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Floer potentials, cluster algebras and quiver representations

Markus Reineke

(joint work with Peter Albers and Maria Bertozzi)

To any Markov triple $\mathbf{m}$, that is, a triple (a, b, c) of positive integers such that $a^2 + b^2 + c^2 = 3abc$, R. Vianna [4] associates a monotone Lagrangian torus $T_\mathbf{m} \subset \mathbb{C}P^2$. To distinguish these tori up to Hamiltonian isotopy, one considers their Floer potentials $\mathrm{FP}_{T_\mathbf{m}} \in \mathbb{Z}[z_1^\pm, z_2^\pm]$ encoding their Gromov-Witten invariants, more precisely counts of Maslov index 2 holomorphic disks bounded by $T_\mathbf{m}$. All these objects are compatible with mutation: Markov triples are mutated by $(a, b, c) \rightsquigarrow (3bc - a, b, c)$, leading to the Markov tree, Vianna tori are constructed by induction along the Markov tree using geometric operations called nodal trade/nodal slide of almost toric fibrations, and the Floer potentials mutate by applying algebraic mutation operators μ_i on Laurent polynomials as proved by Pascaleff and Tonkonog [3].

On the other hand, to a Markov triple $\mathbf{m}$ we can associate a quiver with three vertices $1 \overset{(3a)}{\to} 2 \overset{(3b)}{\to} 3 \overset{(3c)}{\to} 1$ together with a non-degenerate potential, for which mutation of quivers with potential is defined [2]. Starting from a particular choice $V_{(1,1,1)}$, (virtual, decorated) QP-representations $V_\mathbf{m}$ of $Q_\mathbf{m}$ can be defined inductively by mutation of QP-representations. To such representations their cluster characters $\mathrm{CC}_{V_\mathbf{m}} \in \mathbb{Z}[x_1^\pm, x_2^\pm, x_3^\pm]$, encoding Euler characteristic of Grassmannians of quotient representations, can be associated.

The main result of [1] is: There exist comparison maps $\Phi_\mathbf{m} : \mathbb{Z}[z_1^\pm, z_2^\pm] \to \mathbb{Z}[x_1^\pm, x_2^\pm, x_3^\pm]$ such that

$$\Phi_\mathbf{m}(\mathrm{FP}_{T_\mathbf{m}}) = \mathrm{CC}_{V_\mathbf{m}},$$

thus providing a link between the symplectic geometry of Vianna tori and the algebraic geometry of quiver representations.

This result easily follows from the choice of $V_{(1,1,1)}$ and the compatibility of the maps $\Phi_{\mathbf{m}}$ with all mutations.

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Pattern-avoiding polytopes and Cambrian lattices

Emily Gunawan

(joint work with Esther Banaian, Sunita Chepuri, and Jianping Pan)

Our project was inspired by the OEIS sequence A003121 which counts shifted standard tableaux of staircase shape, longest chains in the Tamari lattice, linear extensions of a certain poset, and reduced words in a certain commutation class of the longest permutation w_0. Recently, it was shown by Davis and Sagan [1] that this sequence gives the normalized volume of a certain "pattern-avoiding polytope," a subpolytope of the Birkhoff polytope whose vertices are 132 and 312 avoiding permutations. Since these permutations form a distributive sublattice of the right weak order, Davis and Sagan asked whether their polytope might be unimodularly equivalent to the (Stanley's) order polytope ([4]) of a poset.

In our work, we associate a pattern-avoiding polytope $\mathrm{Birk}(c)$ to each Coxeter element c in the symmetric group and prove that $\mathrm{Birk}(c)$ is unimodularly equivalent to the order polytope of a poset. For the Coxeter element corresponding to the Tamari lattice, our result answers Davis and Sagan's question in the affirmative.

Example. Consider the type A_4 symmetric group W, and a Coxeter element $c = s_1 s_4 s_2 s_3$. This gives a quiver Q which is an orientation of the type A_4 Dynkin diagram. Consider the Auslander–Reiten quiver of $\mathrm{rep}\, Q$ drawn vertically, and let H be the "heap" obtained from it by replacing all representations in the τ^{-1}-orbit of $P(j)$ with the label s_j. See Figure 1.

In general, H is the heap of the c-sorting word of w_0, denoted by $\mathrm{sort}_c(w_0)$. The c-singletons [2] are prefixes of words in the commutation class of $\mathrm{sort}_c(w_0)$; equivalently, order ideals of H; and permutations which avoid certain four patterns. For the "Tamari" c, these patterns collapse to just two patterns 132 and 312.

Definition. We define our "pattern-avoiding" polytope $\mathrm{Birk}(c)$ to be the convex hull of the permutation matrices of c-singletons.

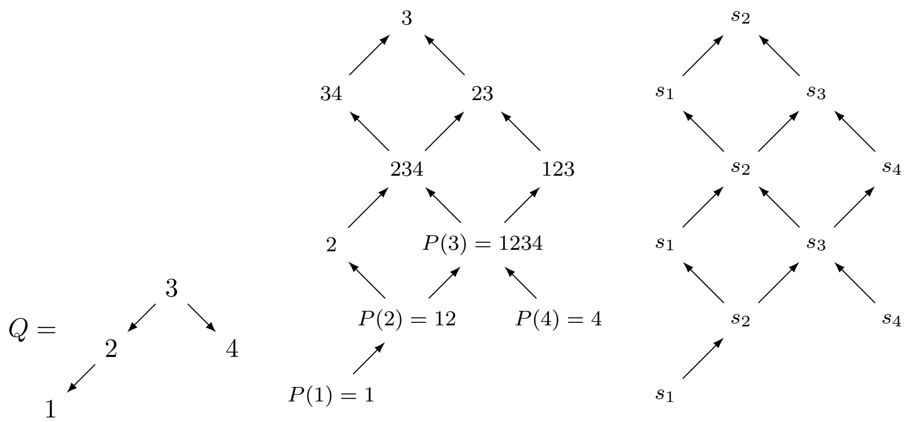

FIGURE 1. From left to right: Quiver Q; The Auslander–quiver of $\operatorname{rep} Q$; The heap H of $\operatorname{sort}_c(w_0)$

Theorem. $\operatorname{Birk}(c)$ *is unimodularly equivalent to the order polytope of* H.

Corollary. The normalized volume of $\operatorname{Birk}(c)$ is equal to the number of linear extensions of H which is the number of longest chains in (Reading's [3]) c-Cambrian lattice. For our example c, this number is 41.

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The category of a partitioned fan

MAXIMILIAN KAIPEL

Let A be a finite-dimensional algebra whose category of finitely generated right modules is denoted $\operatorname{mod} A$. A *brick* is a module $B \in \operatorname{mod} A$ such that $\operatorname{End}_A(B)$ is a division ring. Denote by $\operatorname{brick} A$ the full subcategory of $\operatorname{mod} A$ consisting of bricks. Motivated by the combinatorics of cluster algebras studied in [7] the authors of [1] introduce τ-tilting theory in the early 2010s. This new theory may be viewed as a completion of classical tilting theory with respect to mutation. The central objects of τ-tilting theory are τ-tilting modules $M \in \operatorname{mod} A$ which are those satisfying $\operatorname{Hom}_A(M, \tau M) = 0$, where τM denotes the Auslander-Reiten translation of M.

Bricks and τ-rigid modules are closely connected by [8], in particular, if the number of bricks of A is finite, then the number of τ-rigid modules of A is finite and vice-versa. Such an algebra is called *τ-tilting finite.* A torsion class $\mathcal{T}$ is a full subcategory of $\operatorname{mod} A$ which is closed under extensions and quotients and these are also closely connected to τ-rigid modules by [1]. A chain $\emptyset \subset \mathcal{T}_1 \subset \cdots \subset \operatorname{mod} A$ of proper minimal inclusions of torsion classes is called a *maximal green sequence* and each minimal inclusion $\mathcal{T}_i \subset \mathcal{T}_{i+1}$ may be labelled by a brick B by [9].

To a τ-tilting finite algebra A, we associate a group $G(A)$, called the *picture group*, whose generators are given by $\{X_S : S \in \operatorname{brick} A\}$ and which has a relation given by

$$X_{S_1} \dots X_{S_k} = X_{S_1'} \dots X_{S_\ell'}$$

whenever $(S_1, \dots, S_k)$ and $(S_1', \dots, S_\ell')$ label maximal green sequences [10][14]. This group has close connections to maximal green sequences [13]. In order to study the picture group, the authors of [12], [6] and [5] sequentially introduced the *τ-cluster morphism category* $\mathfrak{C}(A)$. This category has the property that the fundamental group of its classifying space $\mathcal{B}\mathfrak{C}(A)$ is isomorphic to the picture group. In particular, our guiding question is whether the classifying space is a $K(\pi, 1)$ space. Such a space has no non-trivial higher homotopy groups and thus there are isomorphisms between the cohomology groups of the space and the group.

The τ-cluster morphism category may also be defined from the g-vector fan $\Sigma(A)$ of the algebra [16]. Roughly speaking, the g-vector fan encodes the τ-tilting theory of the algebra [2][4]. More generally a *fan* Σ is a collection of non-negative linear combinations of vectors, called cones, such that two cones intersect in a shared face. If for every cone, the generating vectors are linearly independent, we say Σ is *simplicial.* In my work I generalise the geometric construction of the τ-cluster morphism category to an arbitrary simplicial fan with an admissible partition of its cones.

Loosely speaking, an admissible partition $\mathfrak{P}$ is an identification of cones $\kappa_1, \kappa_2 \in \Sigma$ such that two cones which are identified span the same linear subspace, have the same relative fan structure around them and such that cones in the same relative position are also identified. Now the *category of a partition fan* $\mathfrak{C}(\Sigma, \mathfrak{P})$ has as objects the equivalence classes of $\mathfrak{P}$ and as morphisms the union of morphisms between the representatives in the poset category of the fan modulo a relation setting morphisms going to cones in the same relative position equal.

The collection of admissible partitions defines a lattice of categories associated to any simplicial fan. Moreover, the classifying space of each category is a cube complex and thus we obtain two sufficient conditions for $\mathcal{B}\mathfrak{C}(A)$ to be a $K(\pi, 1)$ space by [11]. I generalise the definition of the picture group to this setting by assuming the existence of a weaker version of a fan poset [15]. The lattice structure of the categories induces maps between the picture groups and classifying spaces and functors between the categories.

Let A be the Brauer graph algebra whose underlying Brauer graph is the 3-cycle. As an application of the lattice structure I prove that the classifying space $\mathcal{B}\mathfrak{C}(A)$ is a $K(\pi, 1)$ space by showing that one of the sufficient conditions is satisfied for the maximal element in the lattice, which implies it holds for all categories in the lattice. The second sufficient condition holds for the τ-cluster morphism category by [3] and thus we obtain the desired result.

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Maximal almost rigid modules over gentle algebras

Raquel Coelho Simões

(joint work with Emily Barnard, Emily Gunawan and Ralf Schiffler)

Let S be an oriented Riemann surface with non-empty boundary, M be a finite set of marked points in the boundary of S, P be a *dissection* of S, i.e. a collection of curves in S which do not intersect themselves or each other in their interiors and whose only intersection with M is at their endpoints, and M^* be a set of marked points in one-to-one correspondence with the tiles of the dissection P, such that the marked point lies in the boundary segment if the tile has one, otherwise it is a puncture. The data $(\mathsf{S}, \mathsf{M}, \mathsf{P}, \mathsf{M}^*)$ is called a *tiling* and we can associate to it an algebra $A = kQ_A/I_A$, called *tiling algebra*, as follows.

The set of vertices of its quiver Q_A is in bijection with the set of curves in P. There is an arrow $i \to j$ if the curves associated to i and j share an endpoint x and j follows i in the clockwise direction around x. Note that each arrow is defined by a pair of half-edges sharing a common endpoint. The ideal I_A is generated by paths ab of length 2 for which the half-edge of $t(a)$ differs from the half-edge of $s(b)$. It is known that the class of tiling algebras coincide with the class of gentle algebras [2, 3]. The definition of tiling algebra can be extended to the case where M has punctures, in which case the tiling algebra may be infinite-dimensional but it is still locally gentle [4].

One can describe the module category [2] and the bounded derived category [3] of a gentle algebra using the associated surface. In particular, the string modules over a tiling algebra A are in one-to-one correspondence with *permissible arcs*, i.e. curves in S whose endpoints lie in M^*, the only intersection with the boundary and with M^* is at its endpoints, and such that each segment of γ between two consecutive crossings with P reads off an arrow in Q_A.

It is natural to ask what is the representation theoretic interpretation of *permissible triangulations*, i.e. maximal collection of noncrossing permissible arcs.

Maximal almost rigid modules were introduced in [1] in connection with Cambrian lattices of type A [5]. We can extend the definition in [1] to any gentle algebra.

Definition 1. Let A be a gentle algebra and T be a basic A-module.

(1) T is *almost rigid* if it is a direct sum of string modules and for each pair M, N of indecomposable summands of T and each non-split short exact sequence $0 \to N \to E \to M \to 0$, the middle term E is indecomposable.

(2) T is *maximal almost rigid*, MAR for short, if it is almost rigid and if $T \oplus L$ is almost rigid, then $L = 0$.

Theorem 1. *The set* $\operatorname{mar}(A)$ *of MAR A-modules is in one-to-one correspondence with the set of permissible triangulations of* $(\mathsf{S}, \mathsf{M} \cup \mathsf{M}^*)$.

In order to describe the endomorphism algebra of MAR modules, we need the following data. Given a gentle algebra A, we define a *bigger* gentle algebra $\bar{A}$ as follows. We add a vertex v_a, for each arrow $a \in Q_A$; the arrows are given by $a_1 \colon s(a) \to v_a$ and $a_2 \colon v_a \to t(a)$, for each arrow $a \in Q_A$; $a_2 b_1$ is a relation in $I_{\bar{A}}$ if and only if ab is a relation in I_A. The corresponding tiling $(G(\mathsf{S}), G(\mathsf{M}), G(\mathsf{P}, G(\mathsf{M}^*))$ can be obtained from the tiling of A by replacing each puncture in M^* with a boundary component, adding new arcs associated to the new vertices, and new marked points to $\mathsf{M} \cup \mathsf{M}^*$ accordingly.

We can also naturally define a functor $G \colon \operatorname{mod} A \to \operatorname{mod} \bar{A}$ which is fully faithful but not dense.

Theorem 2. *Let A be a gentle algebra, T be a MAR A-module, $\mathcal{T}$ the corresponding triangulation in* $(\mathsf{S}, \mathsf{M} \cup \mathsf{M}^*)$ *and* $\mathsf{C} = \operatorname{End}_A(T)$.

(1) $\mathsf{C} \cong \operatorname{End}_{\bar{A}}(G(T))$, *and $G(T)$ is a tilting module over $\bar{A}$.*

(2) C *is a gentle algebra of global dimension at most 2, with corresponding tiling* $(G(\mathsf{S}), G(\mathsf{M}), G(\mathcal{T}), G(\mathsf{M}^*))$.

(3) The tiling algebra associated to $\mathcal{T}$ is isomorphic to the tensor algebra $T_{\mathsf{C}}(\mathrm{Ext}^2_{\mathsf{C}}(DC, C))$.

In [1], the authors define a poset structure in the set $\mathrm{mar}(A)$, where A is of type A_n, linked to mutation, which is isomorphic to the Cambrian lattice of the same type. The next step in our project is to generalise this poset structure to any gentle algebra and study its properties.

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MSW-bangle functions as generic basis for surface cluster algebras

Christof Geiss

(joint work with Daniel Labardini-Fragoso and Jon Wilson)

Let $\mathbf{\Sigma} := (\Sigma, \mathbb{M}, \mathbb{P})$ be a marked surface where Σ is a connected Riemann surface of genus $g(\Sigma)$ and with $b(\Sigma)$ boundary components. Moreover, $\mathbb{M} \subset \partial\Sigma$ denotes the marked boundary points and $\mathbb{P} \subset (\Sigma \setminus \partial\Sigma)$ denotes the punctures. Fomin, Shapiro and Thurston introduced in their landmark paper [1] the cluster algebra with trivial coefficients $\mathcal{A}(\mathbf{\Sigma})$. It is of rank $n(\mathbf{\Sigma}) = 6(g(\Sigma)-1)+3(|\mathbb{P}|+b(\Sigma))+|\mathbb{M}|$ and its clusters are in natural bijection with the tagged triangulations of $\mathbf{\Sigma}$. The totally positive part $\mathrm{Spec}_+(\mathcal{A}(\mathbf{\Sigma}))$ of its spectrum can be identified with the decorated Teichmüller space of $\mathbf{\Sigma}$.

We suppose here $|\mathbb{M}| \geq 2$, so in particular $\partial\Sigma \neq \emptyset$. In this situation the cluster algebra $\mathcal{A}(\mathbf{\Sigma})$ is locally acyclic and admits a reddening sequence. It follows that the generic Caldero–Chapoton functions form the *generic basis* of $\mathcal{A}(\mathbf{\Sigma})$, see [6]. Recall that the set of generic Caldero–Chapton functions contains all cluster monomials.

In [9], Musiker, Schiffler and Williams introduced the bangle functions for $\mathcal{A}(\mathbf{\Sigma})$ in terms of perfect matchings on snake- and band graphs, see also [10]. In [9] it was shown that for $\mathbb{P} = \emptyset$ and $|\mathbb{M}| \geq 2$ the bangle functions form the so called bangle *basis* of $\mathcal{A}(\mathbf{\Sigma})$, which also contains all cluster monomials. We can now remove the restriction $\mathbb{P} = \emptyset$:

Theorem. [8]. *Suppose that $|\mathbb{M}| \geq 2$, then the generic basis coincides with the set of MSW-bangle functions. In particular, the bangle functions form in this situation a basis of $\mathcal{A}(\mathbf{\Sigma})$.*

The equality was previously only known for $\mathbb{P} = \emptyset$, see [5]. Important ingredients of the proof are the following considerations:

(1) $\mathbf{\Sigma}$ admits a tagged triangulation T of signature 0. The corresponding non degenerate Jacobian algebra $A(T) := \mathcal{P}_{\mathbb{C}}(Q(T), W(T))$ is skewed-gentle.
(2) The description of homomorphisms between finite dimensional indecomposable representations of skewed-gentle algebras from [2] was recently completed [3]. This allows us, in view of (1), to conclude that for *each* tagged triangulation T' there is an isomorphism of partial KRS-monoids

$$\pi_{T'}\colon\ \mathrm{Lam}(\mathbf{\Sigma}) \to \mathrm{DecIrr}^{\tau}(A(T')),$$

which intertwines shear coordinates with respect to T' and generic g-vectors, see [7]. This requires also the tools from [6] and from [4]. Note that the Jacobian algebra $A(T')$ is tame, but in general it is not skewed-gentle.
(3) For each primitive loop λ on $\mathbf{\Sigma}$ there exists a (tagged) triangulation T'', where we can verify directly that the generic CC-function for $\pi_{T''}(\lambda)$ is the same as the MSW-function for λ with respect to T''.
(4) Since generic CC-functions and MSW-functions transform in the same way under flips of triangulations, our claim follows by combining (2) and (3).

Note that (2) implies by Plamondon's theorem, that for each triangulation T the shear coordinates with respect to T yield a bijection between $\mathrm{Lam}(\mathbf{\Sigma})$ and $\mathbb{Z}^{n(\mathbf{\Sigma})}$.

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Caldero-Chapoton map for super algebras of type A

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(joint work with İlke Çanakçı, Francesca Fedele and Khrystyna Serhiyenko)

Musiker, Ovenhouse and Zhang [2] defined a super algebra of type A_n arising from decorated super Teichmüller theory [1, 3]. This generalizes the case of cluster algebras of surface type. The super algebras are generated by even variables x_i, associated to super lambda lengths, and odd variables θ_j which anticommute with each other and commute with the even ones. The authors show how these super lambda lengths over a polygon, that occur as rational functions on the even variables and their square roots and odd variables, can be computed combinatorially using double dimer covers of snake graphs.

Theorem 1. [2, Theorem 6.2] *Consider a triangulated polygon with no internal triangles. Let $\mathcal{G}$ be the snake graph corresponding to an arc $\gamma \notin T$. Then the super lambda length x_γ is given by*

$$x_\gamma = \frac{1}{\operatorname{cross}(\gamma)} \sum_{D \in \mathcal{DD}(\mathcal{G})} \operatorname{wt}_2(D).$$

We give a representation theoretic interpretation for the super algebras of type A_n. Each triangulation of the polygon defines a gentle algebra Λ. We define an algebra $\widetilde{\Lambda} = \Lambda \otimes_K K[\epsilon]/(\epsilon^2)$, tensoring Λ with the dual numbers. In particular we are interested in *induced modules*, that is modules in $\operatorname{mod}\widetilde{\Lambda}$ of the form $\widetilde{M} = M \otimes_K K[\epsilon]/(\epsilon^2)$. We consider a string module $M_{\mathcal{G}}$ in $\operatorname{mod}\Lambda$ corresponding to a snake graph $\mathcal{G}$, and a double dimer cover of $\mathcal{G}$.

Theorem 2. *The lattice of the double dimer covers of $\mathcal{G}$ is in bijection with the submodule lattice of $\widetilde{M_{\mathcal{G}}}$.*

We construct a *super Caldero-Chapoton map* from the induced modules to the set of super lambda lengths.

Theorem 3. *Let $\widetilde{\Lambda} = \Lambda \otimes_K K[\epsilon]/(\epsilon^2)$ where Λ is a Jacobian algebra coming from a triangulation with no internal triangles of an $(n+3)$-gon. For an arc γ in the polygon, let M_γ be the corresponding indecomposable in* $\operatorname{mod}\Lambda$. *Then, the corresponding super lambda length is*

$$x_\gamma = X^{\operatorname{ind}_{\widetilde{\Lambda}}(\widetilde{M_\gamma})} \sum_{\mathbf{e} \in \mathbb{Z}^n} \chi(\operatorname{Gr}_{\mathbf{e}}(\widetilde{M_\gamma})) \prod_{i=1}^{n} \sqrt{x_i}^{\langle S_i, \oplus_j S_j^{m_j} \rangle_{\widetilde{\Lambda}}} \mu_{\mathbf{e}},$$

where $\mathbf{e} = \underline{\dim}(\bigoplus_j S_j^{m_j})$.

Almost all of the terms resemble the ones appearing in the classic Caldero–Chapoton map. Apart from the appearance of square roots, the only surprising term is $\mu_{\mathbf{e}}$, which is the term associating the correct product of odd variables to a given vector $\mathbf{e}$. The above formula can be rewritten to reduce most of the calculations to calculations over the algebra Λ as follows

$$x_\gamma = X^{\mathrm{ind}_\Lambda(M_\gamma)} \sum_{\substack{N \subseteq \widetilde{M}_\gamma \\ \dim(N)=\mathbf{e}}} \prod_{i=1}^{n} \sqrt{x_i}^{\langle S_i, \oplus_j S_j^{m_j} \rangle_\Lambda} \mu(N),$$

where $\mathbf{e} = \underline{\dim}(\bigoplus_j S_j^{m_j})$. In this version both the index and the antisymmetrized bilinear form are computed over the algebra Λ, while the product of odd variables is still over $\widetilde{\Lambda}$-modules.

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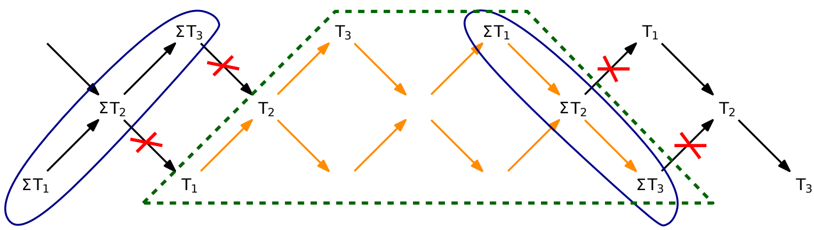
Using relative extriangulated structures on cluster categories

Yann Palu

(joint work with Xin Fang, Misha Gorsky, Hiroyuki Nakaoka, Arnau Padrol, Vincent Pilaud, Pierre-Guy Plamondon and Matt Pressland)

Quick overview: In [7], some new structure was introduced on cluster categories in order to study polytopal realizations of g-vector fans for cluster algebras of finite type. Some specific properties of this structure where made explicit. It was then realized in [4] that those properties arise very often in representation theory and account for the nice behavior of mutations. Finally, the same structure is again used in [3] in order to revisit, and slightly improve on, a result by Thomas Brüstle and Dong Yang relating cluster categories to two-term homotopy categories.

Toy example: The following picture shows the Auslander–Reiten quiver of the cluster category of type A_3:



The object $T_1 \oplus T_2 \oplus T_3$ is a cluster tilting object, hence categorifies some cluster of the cluster algebra of type A_3. Circled in dashed green is the Auslander–Reiten quiver of the homotopy category $K^{[-1,0]}(\mathrm{proj}\, \mathbb{K}\overrightarrow{A_3})$ of two-term complexes of projectives. This illustrates the following result by Brüstle–Yang:

Theorem 1. *([1]): Let $\mathcal{C}$ be the cluster category of a Jacobi-finite quiver with potential and $T \in \mathcal{C}$ a cluster tilting object. There is an additive equivalence of categories*

$$\mathcal{C}/(\Sigma T \to T) \simeq_{add} K^{[-1,0]}(\operatorname{proj}\operatorname{End}(T)),$$

where the quotient is with respect to the ideal of morphisms factoring first through an object in $\operatorname{add}\Sigma T$*, then through an object in* $\operatorname{add} T$*.*

Aim: (1) Understand why this unusual ideal quotient arises.

(2) Prove that the equivalence preserves more structure than mere additivity.

Fix a cluster tilting object T in a Hom-finite cluster category $\mathcal{C}$ and consider the class Δ_T of triangles of the form $X \to Y \to Z \xrightarrow{\varepsilon} \Sigma X$ where ε belongs to the ideal (ΣT) of morphisms factoring through an object in $\operatorname{add}\Sigma T$. By [5], $(\mathcal{C}, \Delta_T)$ is an extriangulated category in the sense of [6], and Δ_T is called a relative extriangulated structure on $\mathcal{C}$.

Since T is cluster tilting, there are triangles $T_1 \to T_0 \to X \to \Sigma T_1$, for each $X \in \mathcal{C}$, and those triangles belong to Δ_T. The index of X with respect to T is the class $\operatorname{ind}_T X = [T_0] - [T_1]$ in the split Grothendieck group $K_0^{\mathrm{sp}}(\operatorname{add} T)$. As shown in [2], ind_T categorifies the g-vectors with respect to the initial seed associated with T. The main reason for introducing the class Δ_T in [7] is that ind_T becomes additive on the triangles in Δ_T. More precisely, the index induces an isomorphism of groups $K_0(\mathcal{C}, \Delta_T) \xrightarrow{\cong} K_0^{\mathrm{sp}}(\operatorname{add} T)$.

Remark. *Some of the properties enjoyed by $(\mathcal{C}, \Delta_T)$ where axiomatized in [4] under the name of 0-Auslander extriangulated categories. It was then proved that silting objects in 0-Auslander extriangulated categories have a nice theory of mutation. This result recovers many of the mutations arising in representation theory: cluster tilting, relative tilting, 2-term silting, maximal almost rigid modules (in type A), intermediate co-t-structures, flips of accordions for gentle algebras...*

It turns out that the relative structure Δ_T is also key to achieving our two aims. We first note that $K^{[-1,0]}(\operatorname{proj}\operatorname{End} T)$ is extension-closed in the triangulated category $K^{\mathrm{b}}(\operatorname{proj}\operatorname{End} T)$. This implies that the triangles of $K^{\mathrm{b}}(\operatorname{proj}\operatorname{End} T)$ define an extriangulated structure on $K^{[-1,0]}(\operatorname{proj}\operatorname{End} T)$. Moreover, in $(\mathcal{C}, \Delta_T)$, the projective objects are precisely those in $\operatorname{add} T$ and the injectives those in $\operatorname{add}\Sigma T$. The answer to our main aim is thus given by the following:

Lemma 1. [3]: *For any extriangulated category $\mathcal{C}$, the ideal quotient $\mathcal{C}/(\mathrm{inj} \to \mathrm{proj})$ canonically inherits an extriangulated structure from $\mathcal{C}$.*

The ideal quotient appearing in the theorem by Brüstle–Yang is now meaningful: informally, it is precisely the quotient that arises when one wants to kill as many morphisms as possible while preserving the extriangulated structure. Our main result in [3] implies that the equivalence of additive categories $\mathcal{C}/(\Sigma T \to T) \simeq_{\mathrm{add}} K^{[-1,0]}(\operatorname{proj}\operatorname{End}(T))$ is in fact an equivalence of extriangulated categories: It sends those triangles that belong to Δ_T to triangles in the homotopy category. Our result also applies to the setting of Yilin Wu's Higgs categories [8].

Theorem 2. [3]*: Let (Q, W, F) be a Jacobi-finite frozen quiver with potential. Endow the Higgs category $\mathcal{H}$ with the relative extriangulated structure Δ_T, where T is the image of the frozen Ginzburg dg-algebra Γ in $\mathcal{H}$. Then there is an equivalence of extriangulated categories*

$$(\mathcal{H}, \Delta_T)/(Inj \to Proj) \simeq_{extri} K^{[-1,0]}(\operatorname{proj} \underline{\operatorname{End}}T),$$

where the stable endomorphism algebra $\underline{\operatorname{End}}T$ is the Jacobian algebra of the quiver with potential $(\overline{Q}, \overline{W})$ obtained by deleting the frozen part.

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From reduction of Frobenius extriangulated categories to reduction of friezes

Eleonore Faber

(joint work with Bethany Marsh and Matthew Pressland)

This talk is about a reduction result for Frobenius extriangulated categories, which yields a reduction for so-called *Conway–Coxeter friezes*: Iyama and Yoshino introduced reduction for 2-Calabi–Yau triangulated categories in their 2008 paper. This produces a new 2-Calabi–Yau triangulated category as a subquotient of the original category in a way that works particularly nicely for cluster categories. We give a reduction technique that applies to stably 2-Calabi–Yau Frobenius extriangulated categories. As an application, we show that this provides a categorical model for the reduction of Conway-Coxeter friezes, see e.g. [CC], to *frieze patterns with coefficients* in the sense of Cuntz, Holm, and Jørgensen [CHJ].

More precisely, in [FMP] we show the following

Theorem 1. *Let $\mathcal{F}$ be a stably 2-Calabi–Yau Frobenius extriangulated category, and $\mathcal{X} \subseteq \mathcal{F}$ a functorially finite rigid subcategory.*

(1) The full extension-closed subcategory

$$\mathcal{X}^{\perp_1} = \{M \in \mathcal{F} : \mathbb{E}_{\mathcal{F}}(X, M) = 0 \text{ for all } X \in \mathcal{X}\}$$

is itself a stably 2-Calabi–Yau Frobenius extriangulated category, and its cluster-tilting subcategories are those of $\mathcal{F}$ which contain $\mathcal{X}$.

(2) *There is a triangle equivalence*

$$\underline{\mathcal{X}^{\perp_1}} = \mathcal{X}_{\underline{\mathcal{F}}}^{\perp_1} / \operatorname{add}(\mathcal{X})$$

between the stable category of the reduction of $\mathcal{F}$ at $\mathcal{X}$, and the Iyama–Yoshino reduction of the stable category $\underline{\mathcal{F}}$ at $\mathcal{X}$.

For the application to friezes, we note that for a Frobenius category $\mathcal{F}$ as in the theorem above, which also has a cluster tilting object T, there exists a cluster character $\Phi_{\mathcal{F}}^T : \mathcal{F} \to \mathbb{Q}[K_0(\operatorname{add} T)]$, see [WWZ], building on work by Palu, Fu–Keller and many others. This cluster character can be restricted to the subcategory $\mathcal{X} = M^{\perp_1}$ for M a rigid indecomposable non-projective object in $\mathcal{F}$:

Theorem 2. *For $\mathcal{F}$ and $\mathcal{X} = M^{\perp_1}$ as above, we have*

$$\Phi_{\mathcal{F}}^T|_{M^{\perp_1}} = \Phi_{M^{\perp_1}}^T \ .$$

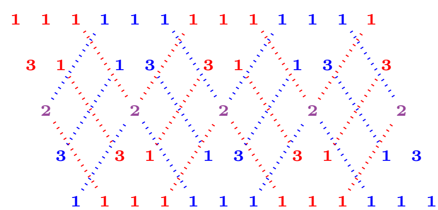
Now taking $\mathcal{F} = \mathcal{C}_{2,n}$, the Grassmannian cluster category of [JKS], the evaluation of the cluster character $\Phi_{\mathcal{F}}^T(T_i) = 1$ for a cluster tilting object $T = \bigoplus_{i=1}^{2n+3} T_i$ yields a Conway–Coxeter frieze (this is well-known by work of Caldero–Chapoton, and for this setting see [BFG+]). Then the reduction with respect to $\mathcal{X} = T_i^{\perp_1}$ for a non-projective T_i yields two smaller Conway–Coxeter friezes. In [CHJ, §4], it is pointed out that so-called frieze patterns with coefficients can be obtained from Conway–Coxeter friezes by cutting out subpolygons from the corresponding triangulation of a regular polygon. We obtain such a frieze with coefficients if we reduce with respect to $\mathcal{X} = M^{\perp_1}$, where M is a rigid non-projective indecomposable not contained in $\operatorname{add}(T)$.

Moreover, we apply our reduction method to *mesh friezes* coming from Grassmannian cluster categories of finite type and use it to give an alternative proof of the following result [BFG+, Prop. 5.3], which was proved in an ad hoc way in loc. cit.

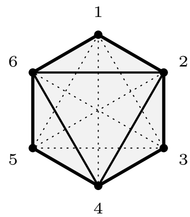
Theorem 3. *Let F be a mesh frieze coming from a Grassmannian cluster category of finite type $\mathcal{F}$ and let M be a rigid indecomposable non-projective object in $\mathcal{F}$. If $F(M) = 1$, then $F|_{M^{\perp_1}}$ is a mesh frieze for the category $M^{\perp_1}$.*

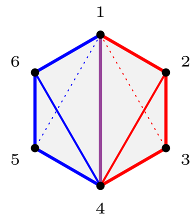
Below there is an example of the reduction of a Conway–Coxeter frieze of width 3 into two friezes with coefficients of width 1. Here the reduction is with respect to the module corresponding to the arc $(1, 4)$ (an entry 2 in the frieze):

1 1 1 1 1 1 1 1 1 1 1 1 1
1 3 1 3 1 3 1 3 1 3 1 3 1
2 2 2 2 2 2 2 2 2 2 2 2 2
1 3 1 3 1 3 1 3 1 3 1 3 1
1 1 1 1 1 1 1 1 1 1 1 1 1



The first Conway–Coxeter frieze corresponds to the triangulation of a hexagon, where the boundary edges are evaluated to 1, whereas the second frieze with coefficients corresponds to two triangulated rectangles, where the common boundary (the arc $(1,4)$) is evaluated to 2:





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Modulated graphs with potentials and skew-symmetrizable cluster algebras

Lang Mou

(joint work with Daniel Labardini-Fragoso and Pierre-Guy Plamondon)

Modulated graphs are natural generalizations of (linearizations of) quivers and modulations of weighted quivers serve as prototypical examples. Despite the remarkable success in categorifying cluster algebras associated to quivers by their representations (e.g. [3]), much is to be clarified about the relation between cluster algebras from weighted quivers (i.e. *skew-symmetrizable* cluster algebras) and representations of carefully selected modulations. Geiss, Leclerc and Schröer have laid foundational work by constructing modulated graphs for acyclic weighted quivers [4] and related their representation theory to cluster algebras [5]. However the theoretic framework remains incomplete, especially regarding the search of suitable relations for non-acyclic GLS-type modulations. Moreover, even in the acyclic case whether the locally free version of the Caldero–Chapoton formula [1] for cluster variables proposed in [5] holds is not known in general beyond finite types.

We generalize the notion of potentials on quivers of Derksen, Weyman and Zelevinsky [2] to modulated graphs so that the derivatives ought to give the correct relations. Let $(R_i, A_{i,j})$ be a modulated graph where R_i are symmetric K-algebras and $A_{i,j}$ are (R_i, R_j)-bimodules. The *tensor algebra* of the bimodule $A = \bigoplus A_{i,j}$ over $R = \prod R_i$ is $R\langle A\rangle = \bigoplus_{d\geq 0} A^{\otimes_R^d}$. A *potential* S is defined to be an (R,R)-bimodule map

$$S\colon R \to \bigoplus_{d\geq 1} A^{\otimes_R^d}$$

such that its left and right derivatives (defined through the symmetric structures of R_i) equal. Hence for a potential S there is a well-defined *derivative*, which is an (R,R)-bimodule map $\partial_\bullet S\colon \operatorname{Hom}_K(A,K) \to R\langle A\rangle$. The derivatives of all K-linear forms on A generate the *Jacobian ideal* $J(S)$ in $R\langle A\rangle$. The *Jacobian algebra* $\mathcal{P}(A,S)$ is defined to be the quotient of $R\langle A\rangle$ be $J(S)$.

We move on to explicitly write down potentials on the modulated graphs associated to non-acyclic weighted quivers of type B_3 and C_3 cluster algebras respectively as follows. The potential S is represented by the value $S(1)$.

$$\begin{tikzcd}
& K[\varepsilon_2]/\varepsilon_2^{2d} \arrow[dl, "\varepsilon_1 a = a\varepsilon_2"'] & \\
K[\varepsilon_1]/\varepsilon_1^{2d} \arrow[rr, "\varepsilon_3 c = c\varepsilon_1^2"] & & K[\varepsilon_3]/\varepsilon_3^{d} \arrow[ul, "\varepsilon_2^2 b = b\varepsilon_3"']
\end{tikzcd}
\qquad
\begin{gathered}
S = (\varepsilon_1 abc + abc\varepsilon_1) + (\varepsilon_2 bca + bca\varepsilon_2) + cab,\\
J(S) = (\varepsilon_2 bc + bc\varepsilon_1, ab, ca).
\end{gathered}$$

$$\begin{tikzcd}
& K[\varepsilon_2]/\varepsilon_2^{2d} \arrow[dl, "\varepsilon_1 a = a\varepsilon_2^2"'] & \\
K[\varepsilon_1]/\varepsilon_1^{d} \arrow[rr, "\varepsilon_3 c = c\varepsilon_1"] & & K[\varepsilon_3]/\varepsilon_3^{d} \arrow[ul, "\varepsilon_2^2 b = b\varepsilon_3"']
\end{tikzcd}
\qquad
\begin{gathered}
S = abc + (\varepsilon_2 bca + bca\varepsilon_2) + cab,\\
J(S) = (bc, ab, ca).
\end{gathered}$$

More generally for type B_n and C_n, there is at most one such 3-cycle as above in a non-acyclic modulation in the mutation class. Our main result reads as follows.

Theorem ([8])**.** *For any weighted quiver Q of finite cluster type, there is a potential S on the GLS modulation A of Q such that there is a bijection between indecomposable τ-rigid modules of the Jacobian algebra $\mathcal{P}(A,S)$ and non-initial cluster variables of the cluster algebra $\mathcal{A}(Q)$. This bijection is given explicitly by the locally free Caldero–Chapoton formula.*

This theorem can be seen as a generalization of the categorification result obtained by Geiss–Leclerc–Schröer for acylic weighted Dynkin quivers [5] to non-acyclic ones. For a class of cluster algebras associated to orbifold triangulations (which includes the finite type C_n and the affine type $\widetilde{C}_n$ with minimal symmetrizers) the Jacobian algebras are constructed in [6] and the Caldero–Chapoton formula as in the above theorem is proven in [7].

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Knots and cluster algebras

Véronique Bazier-Matte
(joint work with Ralf Schiffler)

The aim of this talk is to explain new connections between knot theory and cluster algebras.

Introduction. A *link* can be studied via its *link diagram*. Then, from the link diagram, one can compute the *Alexander polynomial* of the knot. One method for this computation involves *Kauffman states*, [2]. In a previous work, we constructed a quiver Q from the link diagram with n segments. From this quiver Q, we obtained a Jacobian algebra and we considered some indecomposable modules, called the *link modules* and denoted by T_i for $i = 1, \dots, 2n$, over this algebra. Then, we proved that the F-polynomials of the link modules specialize to the Alexander polynomial, [1]. In our recent work, we take the cluster algebra given by the quiver Q. By applying a sequence of mutations given by bigons in the link diagram, we showed that we obtain a cluster whose F-polynomial for each cluster variable is equal to the F-polynomial of one of the link modules.

Knot theory. We introduce some terminology in knot theory: *link*, *link diagram*, *bigon* in a link diagram and *Alexander polynomial*. An illustrative example is also provided.

Cluster algebra. We then outline our construction for obtaining the link modules. Starting with a link diagram, we associate a quiver where vertices represent segments of the link diagram, and arrows correspond to crossing points. A Jacobian algebra is defined from this quiver, by defining a potential with the crossing points and the regions of the link diagram. The dimension in each vertex of a link module is given by the topology of the link.

Cluster corresponding to the link modules. We introduce a new operation on a link diagram where a bigon is substituted with a single crossing point. This operation, along with an algorithm for creating a bigon in a link diagram without increasing the number of crossing points, enables the reduction of any link to the Hopf link. We define a sequence of mutations μ from this reduction procedure.

Applying μ to the cluster algebra $\mathcal{A}(Q)$ with principal coefficients yields a cluster $\mathbf{x} = x_1, \ldots, x_{2n}$ such that the F-polynomial of x_i is the same as the F-polynomial of $T_{\sigma(i)}$, where σ is a permutation in $\mathcal{S}_{2n}$ such that $\sigma^2 = \mathrm{id}$.

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Mirror symmetry and cluster duality for Schubert varieties in the Grassmannian

Lauren Williams

(joint work with Konstanze Rietsch)

In this work we use network charts and the cluster structure on open Schubert varieties to propose a superpotential for Schubert varieties. Given a partition λ contained in a $(n-k) \times k$ rectangle, let $\mathbb{X}_\lambda \subset Gr_{n-k,n}$ be the corresponding Schubert variety in the Grassmannian of $(n-k)$-planes in an n-dimensional vector space, and let $\check{\mathbb{X}}_\lambda \subset Gr_{k,n}$ be the corresponding Schubert variety in the dual Grassmannian of k-planes in an n-dimensional vector space. Let G index a cluster seed for the open Schubert variety (G could be a plabic graph but we also let G denote a more general seed). Then using the corresponding network chart on the open Schubert variety $\mathbb{X}_\lambda^\circ$, we can compute the associated *Newton-Okounkov body* $\Delta_G(\mathbb{X}_\lambda)$, which we show is a rational polytope. (When G is the rectangles cluster, this polytope is unimodularly equivalent to an order polytope.) We also define for each λ a regular function W^λ on the open Schubert variety $\check{\mathbb{X}}_\lambda^\circ$, which is an element of $\mathcal{A}[q_1, \ldots, q_d]$, where $\mathcal{A}$ is the cluster algebra associated to $\check{\mathbb{X}}_\lambda^\circ$ and d is the number of outer corners of λ. Since W^λ lies in $\mathcal{A}[q_1, \ldots, q_d]$, for any cluster seed G we can express W^λ as a Laurent polynomial $W^\lambda|_G$ in the cluster variables of the seed of G. We can then "tropicalize" $W^\lambda|_G$, obtaining a polytope that we call the *superpotential polytope* $\Gamma_G(\check{\mathbb{X}}_\lambda)$.

Our first "mirror" theorem is that for all cluster seeds G, the Newton Okounkov body $\Delta_G(\mathbb{X}_\lambda)$ coincides with the superpotential polytope $\Gamma_G(\check{\mathbb{X}}_\lambda)$.

We also show that associated to each chart G we obtain a degeneration of $\mathbb{X}_\lambda \subset Gr_{n-k,n}$ to a toric variety C_λ. Now let G be the "rectangles cluster." Our second theorem is that the toric variety Y_λ associated to the face fan of the *Newton polytope* of W_G^λ is a partial desingularization of C_λ. Moreover the polytope of Y_λ is reflexive and terminal and hence Y_λ is Gorenstein Fano.

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Mutations of infinite-dimensional quiver representations

Daniel Labardini-Fragoso
(joint work with Rosie Laking, Bea de Laporte and Lang Mou)

Let $Q = (Q_0, Q_1, t, h)$ be a 2-acyclic finite quiver and $k \in Q_0$. The *mutation* $\mu_k(Q)$ is the quiver obtained after applying the following three combinatorial steps:
1^{st} : For each pair $a : j \longrightarrow k, b : k \longrightarrow i$, introduce a new arrow $[ba] : j \longrightarrow i$;
2^{nd} : replace each arrow a incident to k with an arrow a^* in the opposite direction;
3^{rd} : remove a maximal collection of disjoint 2-cycles.
We denote by $\widetilde{\mu}_k(Q)$ the quiver after applying only the first two steps.

Let $a_1, \ldots, a_s$ be the arrows having k as head, and $b_1, \ldots, b_r$ the arrows having k as tail. Following [1], for any representation M of Q we can consider the maps

$$\begin{tikzcd}[column sep=small] & M_k \arrow[dr, "\beta:=\begin{bmatrix} M_{b_1} \\ \vdots \\ M_{b_r} \end{bmatrix}"] & \\ M_{\text{in}} := \bigoplus_{i=1}^{s} M_{t(a_i)} \arrow[ur, "\alpha:=\begin{bmatrix} M_{a_1} & \cdots & M_{a_s} \end{bmatrix}"] & & \bigoplus_{j=1}^{r} M_{h(b_j)} =: M_{\text{out}} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \operatorname{coker}\beta \oplus \ker\alpha \arrow[dl, "\begin{bmatrix} 0 & \text{incl} \end{bmatrix}"'] & \\ M_{\text{in}} := \bigoplus_{i=1}^{s} M_{t(a_i)} \arrow[rr, "\beta\alpha"] & & \bigoplus_{j=1}^{r} M_{h(b_j)} =: M_{\text{out}} \arrow[ul, "\begin{bmatrix} \text{proj} \\ 0 \end{bmatrix}"'] \end{tikzcd}$$

Now, from a cluster-algebraic perspective, the quiver representation of $\widetilde{\mu}_k(Q)$ thus obtained is not the correct one: its representation-theoretic F-polynomials and g-vectors may fail to match the desired cluster-theoretic F-polynomials and g-vectors. The reason is that $\operatorname{coker}\beta \oplus \ker\alpha$ is too big. We would like to shrink it. For that, we will amalgamate $\operatorname{coker}\beta$ and $\ker\alpha$ along a common vector subspace.

A *potential* on Q is an element S of the complete path algebra $\Bbbk\langle\langle Q\rangle\rangle$ that can be written as a possibly infinite linear combination of cycles. The need to work over $\Bbbk\langle\langle Q\rangle\rangle$ rather than over the usual path algebra $\Bbbk\langle Q\rangle$ stems from the desire to perform the 3^{rd} step of quiver mutation algebraically, which requires certain algebra homomorphisms to be actually isomorphisms (cf. [2, Proposition 2.4, Theorem 4.6]), a requirement satisfied by $\Bbbk\langle\langle Q\rangle\rangle$ but not always by $\Bbbk\langle Q\rangle$.

Given a potential S and an arrow $a \in Q_1$, the *cyclic derivative* $\partial_a(S)$ is certain possibly infinite linear combination of paths from $h(a)$ to $t(a)$. For each 2-path $a : t(a) \to h(a), b : h(a) \to h(b)$, the *second order cyclic derivative* $\partial_{ba}(S)$ is certain possibly infinite linear combination of paths from $h(b)$ to $t(a)$. The *Jacobian ideal* $J(S)$ is the $\mathfrak{m}$-adic topological closure of the two-sided ideal of $\Bbbk\langle\langle Q\rangle\rangle$ generated by $\{\partial_a(S) \mid a \in Q_1\}$. The *Jacobian algebra* is the quotient $\mathcal{P}(Q, S) := \Bbbk\langle\langle Q\rangle\rangle/J(S)$. For a left $\mathcal{P}(Q, S)$-module M, we can form the linear maps

$$\begin{tikzcd}[column sep=small]
& M_k \arrow[dr, "\beta:=\begin{bmatrix} M_{b_1} \\ \vdots \\ M_{b_r} \end{bmatrix}"] & \\
M_{\text{in}} := \bigoplus_{i=1}^{s} M_{t(a_i)} \arrow[ur, "\alpha:=\begin{bmatrix} M_{a_1} & \cdots & M_{a_s} \end{bmatrix}"] & & \bigoplus_{j=1}^{r} M_{h(b_j)} =: M_{\text{out}} \arrow[ll, "\gamma:=\left[M_{\partial_{b_j a_i}(S)}\right]_{(i,j)}"']
\end{tikzcd}$$

Since $J(S) \cdot M = 0$, we have $\gamma\beta = 0 = \alpha\gamma$, hence there is a commutative diagram

$$\begin{tikzcd}
M_{\text{out}} \arrow[r, two heads, "\text{proj}"'] \arrow[rr, bend left, "\gamma"] & \operatorname{coker}\beta \arrow[r, two heads, "\overline{\gamma}"'] & \operatorname{im}\gamma \arrow[r, hook, "\text{incl}"'] & \ker\alpha.
\end{tikzcd}$$

So, if we choose a section of $\overline{\gamma}$, the common subspace we are looking for is $\operatorname{im}\gamma$. Thus, the *pre-mutation* $\widetilde{\mu}_k(M)$ is the $\Bbbk\langle\widetilde{\mu}_k(Q)\rangle$-module dictated by:

$$\begin{tikzcd}[column sep=small]
& \frac{\ker\gamma}{\operatorname{im}\beta} \oplus \operatorname{im}\gamma \oplus \frac{\ker\alpha}{\operatorname{im}\gamma} \arrow[dl, "\overline{\beta}:=\begin{bmatrix} 0 & \text{incl}_{\operatorname{im}\gamma} & \text{incl}_{\ker\alpha} \circ\sigma \end{bmatrix}"'] & \\
M_{\text{in}} \arrow[rr, "\beta\alpha"] & & M_{\text{out}}, \arrow[ul, "\overline{\alpha}:=\begin{bmatrix} -\operatorname{proj}\circ\rho \\ -\gamma \\ 0 \end{bmatrix}"']
\end{tikzcd}$$

being σ any section of $\ker\alpha \to \ker\alpha/\operatorname{im}\gamma$ and ρ any retraction for $\ker\gamma \hookrightarrow M_{\text{out}}$. We have $(\ker\gamma/\operatorname{im}\beta) \oplus \operatorname{im}\gamma \cong \operatorname{coker}\beta$ and $\operatorname{im}\gamma \oplus (\ker\alpha/\operatorname{im}\gamma) \cong \ker\alpha$, $-\overline{\alpha}$ represents the projection $M_{\text{out}} \to \operatorname{coker}\beta$, $\overline{\beta}$ represents the inclusion $\ker\alpha \hookrightarrow M_{\text{in}}$. Now, as we said, $\widetilde{\mu}_k(M)$ must be a module over $\Bbbk\langle\langle\widetilde{\mu}_k(Q)\rangle\rangle$, not only over $\Bbbk\langle\widetilde{\mu}_k(Q)\rangle$.

Theorem 1. [2] *If* $\dim_{\Bbbk}(M) < \infty$*, then this action of* $\Bbbk\langle\widetilde{\mu}_k(Q)\rangle$ *on* $\widetilde{\mu}_k(M)$ *extends to an action of* $\Bbbk\langle\langle\widetilde{\mu}_k(Q)\rangle\rangle$ *on* $\widetilde{\mu}_k(M)$ *that in turn induces on* $\widetilde{\mu}_k(M)$ *a module structure over* $\mathcal{P}(\widetilde{\mu}_k(Q,S)) \cong \mathcal{P}(\mu_k(Q,S))$*, where* $\mu_k(Q,S)$ *is as in* [2, Definition 5.5].

The *mutation* $\mu_k(M)$ is defined as $\widetilde{\mu}_k(M)$ with the induced action of $\mathcal{P}(\mu_k(Q,S))$.

Theorem 2. [3] *Theorem 1 holds without the need to assume that* $\dim_{\Bbbk}(M) < \infty$. *Furthermore, if* ℓ *is a vertex different from* k*, then* $\mu_k(P_{(Q,S)}(\ell)) = P_{\mu_k(Q,S)}(\ell)$ *and* $\mu_k(I^{\text{loc.nil.}}_{(Q,S)}(\ell)) = I^{\text{loc.nil.}}_{\mu_k(Q,S)}(\ell)$*, where* $P_{(Q,S)}(\ell)$ *is the* ℓ^{th} *indecomposable projective* $\mathcal{P}(Q,S)$*-module, and* $I^{\text{loc.nil.}}_{(Q,S)}(\ell)$ *is the injective envelope of the* 1*-dimensional nilpotent simple* $S(\ell)$ *in the category of locally nilpotent* $\mathcal{P}(Q,S)$*-modules.*

In the finite dimensional case, the second statement in Theorem 2 was proved in [5], and re-proved in [4]. In [3] we investigate the mutation behavior of F*-series*, as well as of other relevant classes of infinite-dimensional modules.

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Cluster Expansion via matrices

Emine Yildirim
(joint work with Ezgi Kantarcı Oğuz)

Cluster algebras are generated by so-called cluster variables which can be obtained recursively from a set of initial data. Writing cluster algebra elements in terms of the initial ones has been an important question in the literature. There have been different ways to answer this question, i.e. one can compute the cluster expansion formulas using snake graphs, T-paths or CC-map in the representation theory of algebras. In a joint work with E. Kantarcı Oğuz, we compute the cluster expansion formulas using labelled posets and their order ideals. After writing a labelled poset for an arc coming from a punctured surface, we efficiently compute the corresponding cluster expansion using 2 by 2 matrices. Furthermore, we give a uniform way of writing a generalised T-path formula, "T-walks", for every arc on a possibly punctured surface. We can state our theorems (in a compact way without going into technicalities) as follows. Assume we are in the setting of cluster algebras from surface as in [1, 3].

Theorem 1. *Let γ an be arc on a triangulation T_0. Let P_γ be the labelled poset associated to γ. Then the expansion of x_γ with respect to the triangulation T_0 is given by:*

$$x_\gamma = x(T_0) \sum_{I \in J(P_\gamma)} x(I)y(I)$$

where $J(P_\gamma)$ is the set of all order ideals in P_γ.

Theorem 2. *The expansion formula for an arc γ can be calculated via T-walks as follows:*

$$x_\gamma = \sum_{T_{\vec{v}} \in \mathrm{TW}(\gamma)} x(T_{\vec{v}})y(T_{\vec{v}}).$$

where $\mathrm{TW}(\gamma)$ *is the set of all T-walks for γ.*

The idea of calculating the expansion formulae via matrices is not a new, in fact it might be thought of as predating the combinatorial methods such as looking at matchings in snake graphs or ideals of posets. It has not been the go-to method for doing the calculations possibly because the framework of snake graphs proved more accessible. In the meantime, with the definition of the new q-deformations of rational numbers by Morier-Geoaud and Ovsienko [2], the combinatorics of fence posets and corresponding polynomials came into a new attention of mathematical community, prompting new works on the combinatorical aspects and calculation methods ([4],[5]). This work aims to bring these developments back into the cluster setting and give a matrix characterization that can be fully visualized as building posets step by step. The biggest advantage of this method is that the ideas are easy to extend to new settings in cluster algebras and beyond. We note that, in addition to the classical cases of snake and band graph, we give a characterization of calculating expansion formulae for Wilson's loop graphs [6] using matrices.

Schiffler and Thomas gave expansion formulae of cluster variables using certain paths, (complete) T-paths, on a triangulation of an unpunctured surface. Schiffler generalized this to the cluster algebras with coefficients again in the unpunctured setting. Later on, Gunawan and Musiker studied these paths for a surface only with one puncture. In this work, we consider a generalization of T-paths associated to arcs on all surfaces possibly with punctures. As already in literature, there is a bijection between perfect matchings and T-paths for the unpunctured surface is shown by Musiker-Schiffler, we furthermore show that this bijection holds in general.

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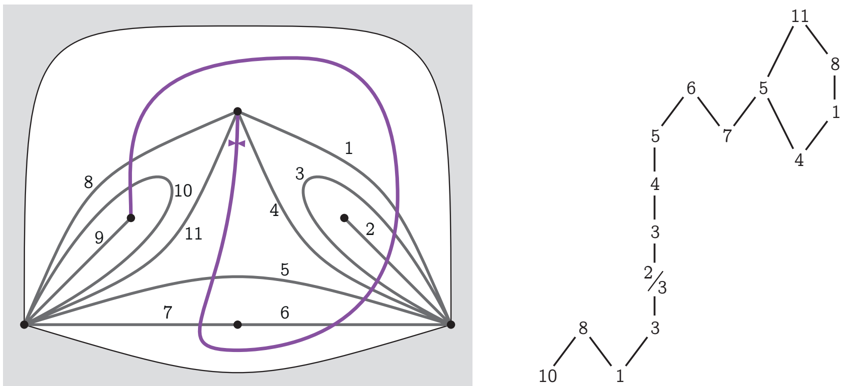
Posets for F-polynomials in marked surfaces

Nathan Reading

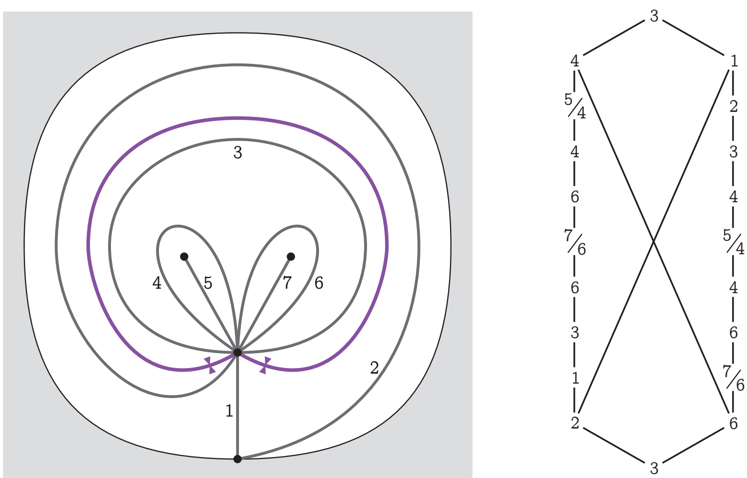
(joint work with Vincent Pilaud and Sibylle Schroll)

We prove a simple formula for arbitrary cluster variables in the marked surfaces model. As part of the formula, we associate a labeled poset to each tagged arc, such that the associated F-polynomial is a weighted sum of order ideals. Each element of the poset has a weight, and the weight of an ideal is the product of the weights of the elements of the ideal. In the unpunctured case, the weight on each element is a single $\hat{y}_i$, in the usual sense of principal coefficients. In the presence of punctures, some elements may have weights of the form $\hat{y}_i/\hat{y}_j$. Our search for such a formula was inspired by the Fundamental Theorem of Finite Distributive Lattices combined with work of Gregg Musiker, Ralf Schiffler, and Lauren Williams that, in some cases, organized the terms of the F-polynomial into a distributive lattice. The proof consists of a simple and poset-theoretically natural argument in a special case, followed by a hyperbolic geometry argument using a cover of the surface to prove the general case. References to related work are found in our preprint [1].

Example. The pictures below show a marked surface triangulated by arcs numbered 1 through 11 and a tagged arc α (shown thicker, in purple). The corresponding cluster variable is $\frac{x_5x_6x_8}{x_1x_4x_7x_9}$ times the weighted sum of all order ideals in the poset P_α shown. The weight of an element is $\hat{y}_i$ if the element is labeled i. The weight of the element labeled $\frac{i}{j}$ is $\frac{\hat{y}_i}{\hat{y}_j}$.



Example. In the pictures below, the arc α coincides with arc 3 except that α is tagged notched at both endpoints. The corresponding cluster variable is $\frac{1}{x_3}$ times the weighted sum of all order ideals in the poset P_α shown in the right picture. The weights of elements are as described in the previous example.



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Triangulations versus cluster-tilting in completed infinity-gons

İlke Çanakçı
(joint work with Martin Kalck and Matthew Pressland)

Paquette and Yıldırım [4] recently introduced cluster-type categories for completed infinity-gons, which are discs with an infinite closed set of marked points on their boundary. They classify (weak) cluster-tilting subcategories, which turn out to be in bijection with very special triangulations of the disc. This is in contrast to Igusa–Todorov's [2] earlier work in the uncompleted case, in which every triangulation corresponds to a weak cluster-tilting subcategory.

In this work, we replace the triangulated structure of Paquette–Yıldırım's category by an extriangulated substructure and prove that, with this structure, the weak cluster-tilting subcategories are once again in bijection with triangulations. We further show that functorial finiteness of such a subcategory is equivalent to a very mild condition on the triangulation. This condition also appears in Çanakçı and Felikson's [1] study of infinite rank cluster algebras from Teichmüller theory. By comparison with the combinatorics of triangulations, we are also able to characterise when weak cluster-tilting subcategories can be mutated in this new extriangulated category.

Paquette and Yıldırım [4]'s triangulated category $\overline{\mathcal{C}}_n$ is associated to a disc $\mathbb{D}$ with a closed set $\overline{\mathbb{M}}_n$ of infinitely many marked points in $\partial\mathbb{D}$, such that $\overline{\mathbb{M}}_n$ has n accumulation points, all of which are two-sided. The indecomposable objects of $\overline{\mathcal{C}}_n$ are in one-to-one correspondence with arcs in $\mathbb{D}$ connecting points of $\overline{\mathbb{M}}_n$, allowing Paquette–Yıldırım to classify the cluster-tilting subcategories of $\overline{\mathcal{C}}_n$ in terms of triangulations. However, only very few triangulations turn out to correspond to cluster-tilting subcategories (or even to weak cluster-tilting subcategories, which are not required to be functorially finite). Indeed, the failure of $\overline{\mathcal{C}}_n$ to be 2-Calabi–Yau means there are significant restrictions on the 'limit arcs', defined in [4] to be those arcs incident with an accumulation point of $\overline{\mathbb{M}}_n$, that may appear in a triangulation corresponding to a cluster-tilting subcategory. Thus the cluster-tilting theory of $\overline{\mathcal{C}}_n$ appears to describe only a very small part of the combinatorics of triangulations of $\mathbb{D}$, in contrast to categories associated to surfaces with discrete sets of marked points [2].

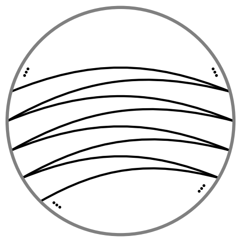
In this work, we demonstrate that in fact $\overline{\mathcal{C}}_n$ does encode the combinatorics of all triangulations of the marked surface $(\mathbb{D}, \overline{\mathbb{M}}_n)$, and surprisingly even does so through cluster-tilting theory. The key observation is that this is achieved not with the triangulated structure on $\overline{\mathcal{C}}_n$, but by passing to a certain natural extriangulated substructure in the sense of Nakaoka and Palu [3]. Roughly speaking, this corresponds to removing certain extensions from $\overline{\mathcal{C}}_n$, while keeping the underlying additive category the same, thus allowing more objects and subcategories to become rigid.

Theorem 1. *The map $\mathcal{T} \mapsto \operatorname{indec} \mathcal{T}$ is a bijection between weak cluster-tilting subcategories of $\overline{\mathcal{C}}_n$, with the appropriate extriangulated structure, and triangulations of the disc $(\mathbb{D}, \overline{\mathbb{M}}_n)$.*

Intriguingly, the definition of this extriangulated structure on $\overline{\mathcal{C}}_n$ uses its description as the Verdier quotient of a second triangulated category, first introduced by Igusa and Todorov [2]. The construction in fact makes sense for any Verdier quotient, and yet gives an extriangulated structure that is typically different from Verdier's triangulated structure [5].

Having recovered the dictionary between triangulations and weak cluster-tilting subcategories, we continue by translating further phenomena between the two languages. We characterise the triangulations for which the corresponding subcategory is cluster-tilting (i.e. functorially finite).

Theorem 2. *The map from the Theorem 1 restricts to a bijection between cluster-tilting subcategories and fan triangulations, i.e. those with no configurations as in the figure below.*



Finally, we show that our bijection between weak cluster-tilting subcategories and triangulations is compatible with mutation. Note that, in contrast to surfaces with finitely many marked points, an arc may be mutable (i.e. flippable) in one triangulation but not in another, and we explain how this feature is reflected categorically in terms of the existence of certain approximations.

Theorem 3. *Let $\mathcal{T}$ be a cluster-tilting object of $\overline{\mathcal{C}}_n$ (with the appropriate extriangulated structure), and let $\alpha \in \operatorname{indec}\mathcal{T}$ be an arc. Then the following are equivalent:*

(1) $\mathcal{T}$ is mutable at α;
(2) α is the diagonal of a quadrilateral in the corresponding triangulation;
(3) α admits left and right approximations by $\mathcal{T} \setminus \{\alpha\} = \operatorname{add}(\operatorname{indec}(\mathcal{T}) \setminus \{\alpha\})$.

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Variations on frieze patterns

Thorsten Holm

(joint work with Michael Cuntz, Peter Jørgensen and Carlo Pagano)

1. Classic frieze patterns [3] A *frieze pattern* of height n over a subset R of a ring is an array of the form

$$\begin{array}{ccccccccccc}
 & & \ddots & & & & \ddots & & & & \\
0 & 1 & c_{i-1,i+1} & c_{i-1,i+2} & \cdots & \cdots & c_{i-1,n+i} & 1 & 0 & & \\
 & 0 & 1 & c_{i,i+2} & c_{i,i+3} & \cdots & \cdots & c_{i,n+i+1} & 1 & 0 & \\
 & & 0 & 1 & c_{i+1,i+3} & c_{i+1,i+4} & \cdots & \cdots & c_{i+1,n+i+2} & 1 & 0 \\
 & & & & \ddots & & & & \ddots & &
\end{array}$$

where $c_{i,j} \in R$, and every adjacent 2×2 submatrix has determinant 1. Placing indeterminates $x_1, \ldots, x_n$ in one row, the determinantal condition produces the cluster variables of a cluster algebra of Dynkin type A. By a result of Conway and Coxeter [2], frieze patterns over positive integers are in bijection with triangulations of polygons. In the talk we presented various directions in which Conway and Coxeter's theory can be generalized.

2. Frieze patterns from dissections [8] Let $p \geq 3$ be an integer. A frieze pattern is *of type* Λ_p if the quiddity cycle consists of positive integral multiples of $\lambda_p = 2\cos(\frac{\pi}{p})$. For $p = 3$ these are the classic Conway-Coxeter frieze patterns. We then have the following generalization of the classic result of Conway and Coxeter from triangulations to p-angulations [8].

Theorem 1. *There is a bijection between p-angulations of a regular $(n+3)$-gon and frieze patterns of type Λ_p of height n.*

3. Frieze patterns over algebraic number fields [6] We addressed the fundamental question whether for a subring R of algebraic numbers, there are finitely or infinitely many frieze patterns over $R \setminus \{0\}$ for any given height. For the case $R = \mathbb{Z}$ this is known to be finite by the classic result of Conway and Coxeter and a more recent result by Fontaine [7]. Our aim is to extend this to many more subrings of $\overline{\mathbb{Q}}$, the ring of algebraic numbers.

Let $R \leq \mathbb{C}$ be a subring. The *frieze subring* R° of R is generated by all entries of all frieze patterns over $R \setminus \{0\}$.

Theorem 2. [6] *Let $R \leq \mathbb{C}$ be a subring with $R^\circ \subseteq \overline{\mathbb{Q}}$. There are finitely many frieze patterns over $R \setminus \{0\}$ in each positive height if and only if*

(i) $R^\circ = \mathbb{Z}$ or
(ii) R° is an order in $\mathbb{Q}(\sqrt{d})$ with $d \in \{-1, -2, -3, -7, -11\}$.

For the proof of this main result we need a number-theoretic result which might be of independent interest.

Theorem 3. [6] *Let $R \leq \overline{\mathbb{Q}}$ be a subring with finitely many units. Then $R = \mathbb{Z}$ or there exists an integer $d \in \mathbb{Z}_{<0}$ such that R is an order in $\mathbb{Q}(\sqrt{d})$.*

4. Noncommutative frieze patterns (with coefficients) Frieze patterns of height n over R can be seen as maps assigning to each diagonal of an $(n+3)$-gon a value in R such that all Ptolemy relations are satisfied. Berenstein and Retakh [1] introduced the notion of *noncommutative polygons*. Based on this we presented in the talk a theory of *noncommutative frieze patterns* [5]. This yields a generalization of the theory of frieze patterns with coefficients in [4].

Let $\mathcal{P}$ an m-gon (where $m \geq 3$) and R a ring. Berenstein and Retakh [1] define a *noncommutative frieze* on $\mathcal{P}$ over R as a map from the set of directed diagonals of $\mathcal{P}$ to R^* such that all triangle relations and all exchange relations are satisfied. We show that it suffices to impose only local triangle relations (involving three consecutive vertices) and local exchange relations (involving two pairs of consecutive vertices), these local relations imply all relations [5]. A useful tool for the proof and for obtaining further results on noncommutative frieze patterns are propagation formulae for obtaining all entries in a noncommutative frieze pattern by successive multiplication with suitable 2×2-matrices [5]. We then explained in the talk how these formulae can be used to provide noncommutative generalizations of large parts of the classic theory of frieze patterns (with coefficients), e.g., to define noncommutative quiddity cycles and to obtain reduction formulae for noncommutative quiddity cycles (generalizing the removal/insertion of 1's underlying the classic Conway-Coxeter theory).

Classic frieze patterns can be categorified by cluster categories and cluster characters. A very interesting question is whether noncommutative frieze patterns can also be categorified, perhaps by noncommutative generalizations of cluster categories and cluster characters?

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Semistable torsion classes and canonical decompositions in Grothendieck groups

Osamu Iyama
(joint work with Sota Asai)

This talk is based on [AI]. Let A be a finite dimensional algebra over a field k. We study the class of torsion classes determined by stability conditions, i.e. elements θ in the real Grothendieck group $K_0(\mathrm{proj}A)_{\mathbb{R}} := K_0(\mathrm{proj}A) \otimes_{\mathbb{Z}} \mathbb{R}$ of the category $\mathrm{proj}A$ of finitely generated projective A-modules. The notion of θ-semistable modules naturally appears in geometric invariant theory of quiver representations [Ki]. Each θ gives two torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ [B, BKT], which we call *semistable torsion pairs*. The intersection $\mathcal{W}_\theta := \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$ is the wide subcategory of θ-semistable modules. The semistable torsion classes $\overline{\mathcal{T}}_\theta, \mathcal{T}_\theta$ of $\theta = [U]$ for a 2-term presilting complex U are well-studied in tilting theory.

We call $\theta, \eta \in K_0(\mathrm{proj}A)_{\mathbb{R}}$ *TF equivalent* if $\overline{\mathcal{T}}_\theta = \overline{\mathcal{T}}_\eta$ and $\mathcal{T}_\theta = \mathcal{T}_\eta$ [A]. We denote by $[\theta]_{\mathrm{TF}}$ the TF equivalence class of θ. It is an important problem to give explicit descriptions of TF equivalence classes of elements of $K_0(\mathrm{proj}A)$. For a subset X of $K_0(\mathrm{proj}A)_{\mathbb{R}}$, let

$$\mathrm{cone}\, X := \sum_{\theta \in X} \mathbb{R}_{\geq 0}\theta \supseteq \mathrm{cone}^\circ\, X := \sum_{\theta \in X} \mathbb{R}_{>0}\theta.$$

For example, for a 2-term presilting complex $U = U_1 \oplus \cdots \oplus U_\ell$ with indecomposable U_i and $\theta = [U]$, we have $[\theta]_{\mathrm{TF}} = \mathrm{cone}^\circ\{[U_1], \ldots, [U_\ell]\}$.

As an analogue of the canonical decompositions of quiver representations [Ka, CS], the *canonical decomposition* of $\theta = [P_0] - [P_1] \in K_0(\mathrm{proj}A)$ with $P_0, P_1 \in \mathrm{proj}A$ describes Krull-Schmidt decompositions of general points in $\mathrm{Hom}_A(P_1, P_0)$ as 2-term complexes [DF]. It played an important role in categorification of cluster algebras [P]. For example, if $U = U_1 \oplus \cdots \oplus U_\ell$ is a 2-term presilting complex with indecomposable U_i, then $[U] = [U_1] \oplus \cdots \oplus [U_\ell]$ is a canonical decomposition. For $\theta \in K_0(\mathrm{proj}A)$, we take a canonical decomposition $\theta = \theta_1 \oplus \cdots \oplus \theta_\ell$, and set

$$\mathrm{ind}\,\theta := \{\theta_1, \ldots, \theta_\ell\} \quad \text{and} \quad \mathrm{ind}\,\mathbb{N}\theta := \bigcup_{\ell \geq 1} \mathrm{ind}\,\ell\theta.$$

Our first main result shows that all elements in the cone given by a canonical decomposition are TF equivalent.

Theorem 1. *Let A be a finite dimensional algebra over an algebraically closed field k. For each $\theta \in K_0(\mathrm{proj}A)$, we have*

$$[\theta]_{\mathrm{TF}} \supseteq \mathrm{cone}^\circ(\mathrm{ind}\,\theta).$$

Notice that $\mathrm{cone}(\mathrm{ind}\,\mathbb{N}\theta) \supseteq \mathrm{cone}(\mathrm{ind}\,\theta)$ holds clearly, but the equality does not necessarily hold, see Theorem 5 below. Since Theorem 1 implies $[\theta]_{\mathrm{TF}} \supseteq \mathrm{cone}^\circ(\mathrm{ind}\,\mathbb{N}\theta)$, it is natural to pose the following.

Conjecture 2. *For each $\theta \in K_0(\mathrm{proj}A)$, we have*

$$[\theta]_{\mathrm{TF}} = \mathrm{cone}^\circ(\mathrm{ind}\,\mathbb{N}\theta).$$

Our second main result shows that Conjecture 2 is true for two classes of algebras. For $\theta, \eta \in K_0(\mathrm{proj}A)$, let

$$E(\eta, \theta) := \min\{\dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)}(P_f, P_g[1]) \mid (f,g) \in \mathrm{Hom}(\eta) \times \mathrm{Hom}(\theta)\}.$$

An algebra A is called *E-tame* if $E(\theta,\theta)$ is zero for all $\theta \in K_0(\mathrm{proj}A)$. This class contains all g-finite algebras as well as representation-tame algebras [PY].

Theorem 3. *Let A be a finite dimensional algebra over an algebraically closed field k, and $\theta \in K_0(\mathrm{proj}A)$. If A is either hereditary or E-tame, then*

$$[\theta]_{\mathrm{TF}} = \mathrm{cone}^{\circ}(\mathrm{ind}\,\theta).$$

In the proof of Theorem 3 for E-tame algebras, we prove the following characterization of E-tame algebras, which is interesting by itself.

Theorem 4. *For a finite dimensional algebra A over an algebraically closed field k, the following conditions are equivalent.*

(a) *A is E-tame.*
(b) *Let $\eta, \theta \in K_0(\mathrm{proj}A)$. Then η and θ are TF equivalent if and only if $\mathrm{ind}\,\eta = \mathrm{ind}\,\theta$.*

Each morphism f in the category $\mathrm{proj}A$ gives torsion pairs $(\mathcal{T}_f, \overline{\mathcal{F}}_f)$ and $(\overline{\mathcal{T}}_f, \mathcal{F}_f)$, which we call *morphism torsion pairs*. If f is presilting as a 2-term complex, then the morphism torsion classes are well studied in tilting theory. The semistable torsion classes can be described by using morphism torsion classes, that is,

$$\mathcal{T}_\theta = \bigcap_{\ell \geq 1} \mathcal{T}^{\mathrm{h}}_{\ell\theta}, \quad \mathcal{F}_\theta = \bigcap_{\ell \geq 1} \mathcal{F}^{\mathrm{h}}_{\ell\theta}, \quad \overline{\mathcal{T}}_\theta = \bigcup_{\ell \geq 1} \overline{\mathcal{T}}^{\mathrm{h}}_{\ell\theta}, \quad \overline{\mathcal{F}}_\theta = \bigcup_{\ell \geq 1} \overline{\mathcal{F}}^{\mathrm{h}}_{\ell\theta}, \quad \mathcal{W}_\theta = \bigcup_{\ell \geq 1} \mathcal{W}^{\mathrm{h}}_{\ell\theta},$$

where torsion pairs $(\mathcal{T}^{\mathrm{h}}_\theta, \overline{\mathcal{F}}^{\mathrm{h}}_\theta)$ and $(\overline{\mathcal{T}}^{\mathrm{h}}_\theta, \mathcal{F}^{\mathrm{h}}_\theta)$ are defined by unifying morphism torsion pairs of each morphism f in $\mathrm{proj}A$ satisfying $[f] = \theta$. They play a key role in the proof of Theorem 3.

We apply our results to study the behavior of canonical decomposition under multiplication by a positive integer. We say that an algebra A satisfies the *ray condition* if for each indecomposable wild element θ and $\ell \geq 1$, the element $\ell\theta$ is indecomposable. The ray condition is satisfied by E-tame algebras and hereditary algebras. The following result answers a question [DF, Question 4.7] negatively.

Theorem 5. *There exists a finite dimensional algebra A and an indecomposable wild element $\theta \in K_0(A)$ such that* $\mathrm{cone}(\mathrm{ind}\,\theta) \neq \mathrm{cone}(\mathrm{ind}\,\mathbb{N}\theta)$. *In particular, θ does not satisfy the ray condition.*

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Donaldson–Thomas invariants for the Bridgeland–Smith correspondence

Nicholas J. Williams
(joint work with Omar Kidwai)

In the celebrated paper [1], Bridgeland and Smith show that there is a correspondence between certain quadratic differentials on a Riemann surface X and stability conditions on a particular 3-Calabi–Yau triangulated category. This gives a mathematical interpretation of part of the work of the physicists Gaiotto, Moore, and Neitzke describing BPS states in certain $\mathcal{N} = 2$, $d = 4$ gauge theories [3].

A quadratic differential φ on a Riemann surface X has a distinguished local coordinate w for which $\varphi = dw^{\otimes 2}$. One then obtains a foliation on X given by lines where $\operatorname{Im} w$ is constant, and a metric determined by pulling back the metric of the complex plane via w. An important part of the work of Bridgeland and Smith is that trajectories of this foliation which have finite length in the metric correspond to stable objects of phase 1 of the stability condition resulting from φ. These stable objects correspond to the BPS states in the work of Gaiotto, Moore, and Neitzke, and have associated Donaldson–Thomas (DT) invariants, which count them, in some sense. DT invariants were originally introduced by Thomas in [11], and were subsequently extended and generalised by Joyce and Song [7], and Kontsevich and Soibelman [9].

Work of Iwaki and Kidwai [4] gives predictions for the values of these DT invariants, according to the type of finite-length trajectory. These predictions are based on computations of Iwaki, Koike, and Takei using topological recursion [5, 6]. There are five different types of finite-length trajectories for our purposes, which are

- type I saddle trajectories, which connect two distinct simple zeros;
- type II saddle trajectories, which connect a simple zero with a simple pole;
- type III saddle trajectories, which connect two distinct simple poles;
- degenerate ring domains, which are familes of closed trajectories surrounding a double pole;

- non-degenerate ring domains, which are families of closed trajectories bounded by trajectories of positive length.

The predictions from [4] for the DT invariants are 1, 2, 4, -1, and -2 respectively.

The main result of our recent paper [8] is that the category used in [2] to extend the results of Bridgeland and Smith gives DT invariants agreeing with the predictions from [4]. We in fact compute the refined DT invariants, from which the numerical invariants are obtained by setting $q^{1/2} = -1$, as shown in the table.

Table 1. DT invariants for different types of finite-length trajectories

Finite-length trajectories	Refined	Numerical
Type I saddle	1	1
Type II saddle	2	2
Type III saddle	4	4
Degenerate ring domain	$q^{-1/2}$	-1
Non-degenerate ring domain	$q^{1/2} + q^{-1/2}$	-2

The original category from [1] produces a DT invariant of 0 for the degenerate ring domains. This is a consequence of the fact that the potential used in [1] is the usual one from [10], which contains cycles around the punctures, whereas the potential of [2, 8] does not.

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Universal F-polynomials for finite-dimensional algebras

Hugh Thomas

(joint work with Nima Arkani-Hamed, Hadleigh Frost, Pierre-Guy Plamondon, and Giulio Salvatori)

0. Introduction

Let $\mathcal{A}_Q$ be a finite type cluster algebra associated to a quiver Q, with cluster variables indexed by some finite index set Γ. Introduce another set of variables u_γ for $\gamma \in \Gamma$, and consider the system of equations, for each $\gamma \in \Gamma$,

$$u_\gamma + \prod_{\delta \in \Gamma} u_\delta^{c(\delta,\gamma)} = 1 \tag{1}$$

where $c(\delta, \gamma)$ is the compatibility degree in the usual sense for cluster algebras.

The u-equations for type A_n were written down in 1969 by physicists Koba and Nielsen for reasons which I shan't go into. We now understand that one should also think about equations like the u-equations above, but for other algebras.

In the talk, I discussed two points:

1. Solutions to the u-equation in Dynkin type yield what are called "cluster configuration spaces," as worked out by Arkani-Hamed, He, and Lam [1].
2. A similar approach allows us to write down and solve u-equations for any algebra of finite representation type.

1. Cluster configuations spaces, following [1]

[1] consider the cluster algebra $\mathcal{A}_Q^{univ}$ with universal coefficients u_γ for $\gamma \in \Gamma$. Setting all its cluster variables $\widehat{x}_\gamma$ to 1 imposes conditions on the u_γ, and [1] shows that it imposes exactly the u-equations (1).

For example, in the A_1 case, the exchange relation in $\mathcal{A}_{A_1}^{univ}$ is $\widehat{x}_\alpha \cdot \widehat{x}_{-\alpha} = u_\alpha + u_{-\alpha}$. If we set $\widehat{x}_\alpha = 1 = \widehat{x}_{-\alpha}$, the exchange relation becomes $1 = u_\alpha + u_{-\alpha}$, which is exactly the u-equation of type A_1. In general, what happens is the following:

$$V(u\text{-equations}) = V(\{\widehat{x}_\gamma = 1\}) = \Big(\mathbb{C}^{|\Gamma|} \setminus \bigcup_{\gamma\in\Gamma} V(\widehat{x}_\gamma = 0)\Big)/(\mathbb{C}^*)^{|\Gamma|-n}$$

where the torus quotient on the righthand side arises from the grading corresponding to the coefficients. This viewpoint shows that the solutions form an irreducible variety, which is otherwise not visible.

2. The u-equations in finite representation type

Let A be a finite dimensional algebra over $\mathbb{C}$, of finite representation type with n isomorphism classes of simples. We introduce a variable u_N for each $N \in \mathcal{I} =$

ind $A \cup \{P_i[1]\}$, and we consider the equations, for each $M \in \mathcal{I}$,

$$u_M + \prod_{N \in \mathcal{I}} u_N^{c(M,N)} = 1,$$

where $c(M,N) = \dim \operatorname{Hom}(M, \tau N) + \dim \operatorname{Hom}(N, \tau M)$.

To imitate [1], and inspired by ideas from cluster algbera theory, we replace cluster variables by F-polynomials, and pass from F-polynomials (morally, "principal coefficients") to F-polynomials with universal coefficients, which we denote $\widetilde{F}_M$, by a monomial substitution and multiplication by a monomial factor.

In analogy with the Dynkin case discussed above, we have:

$$V(u\text{-equations}) \supseteq V(\{\widetilde{F}_M = 1\}) = \Big(\mathbb{C}^{|\mathcal{I}|} / \bigcup_{M \in \mathcal{I}} V(\widetilde{F}_M = 0)\Big) / (\mathbb{C}^*)^{|\mathcal{I}|-n}$$

and the space on the right is manifestly irreducible.

We don't know how to prove that the first containment is an equality. However, we can show that the positive real solutions to the u-equations lie in the locus cut out by $\{\widetilde{F}_M = 1\}$. So in fact, the part that we really want is contained in the locus we understand well via the universal F-polynomials.

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From deformations of the Farey graph to faithfulness of Burau representations

Sophie Morier-Genoud

(joint work with Valentin Ovsienko, Alexander Veselov)

In 1936 Werner Burau introduced representations of the Artin braid groups by matrices depending on a parameter t. Since then, the Burau representation is an important tool to study invariants of knots and links, e.g. to compute Alexander's polynomials. The reduced Burau representation is a group homomorphism from $\mathcal{B}_n$ to $\mathrm{GL}(n-1, \mathbb{Z}[t, t^{-1}])$, where t is a formal parameter [2].

A fundamental problem extensively studied by many authors is to determine whether the Burau representation is faithful, i.e. has trivial kernel. For $n = 3$ the representation is known to be faithful since the 1960's, whereas for $n \geq 5$ the representations are known to be unfaithful since the 1990's. The case $n = 4$ is still undetermined and it is a major open problem in the area.

In the simple case $n = 3$ the braid group $\mathcal{B}_3$ is generated by two elements σ_1, σ_2 subject to the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. In that case the (reduced) Burau representation is given by

$$\rho_3(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho_3(\sigma_2) = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}.$$

A complex specialisations of ρ_3 is the representation $\rho_3^t : \mathcal{B}_3 \to \mathrm{GL}(2, \mathbb{C})$, defined as the Burau representation ρ_3 with fixed value of $t \in \mathbb{C}^*$.

The Open Problem 1 formulated in [1] is: *"At which complex specialisations of t is the Burau representation ρ_3^t faithful?"*

We give the following partial answer.

Theorem ([5])**.** *The specialised Burau representation ρ_3^t is faithful for all $t \in \mathbb{C}^*$ outside the annulus* $\{3 - 2\sqrt{2} \le |z| \le 3 + 2\sqrt{2}\}$.

The result is obtained by using the notion of q-rational numbers introduced earlier in [4]. The q-rationals can be defined recursively using a deformation of the Farey graph (see Figure 1). They are associated to a natural deformation of matrices of $\mathrm{SL}(2, \mathbb{Z})$ defined by deformations of the standard generators

$$R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \qquad L_q = \begin{pmatrix} 1 & 0 \\ 1 & q^{-1} \end{pmatrix}.$$

The numerators and denominators of q-rationals can be found in the entries of any matrices M_q belonging to the group generated by R_q and L_q. In other words, q-rationals belong to the orbit of the point 0, under the linear-fractional action of the group generated by R_q and L_q.

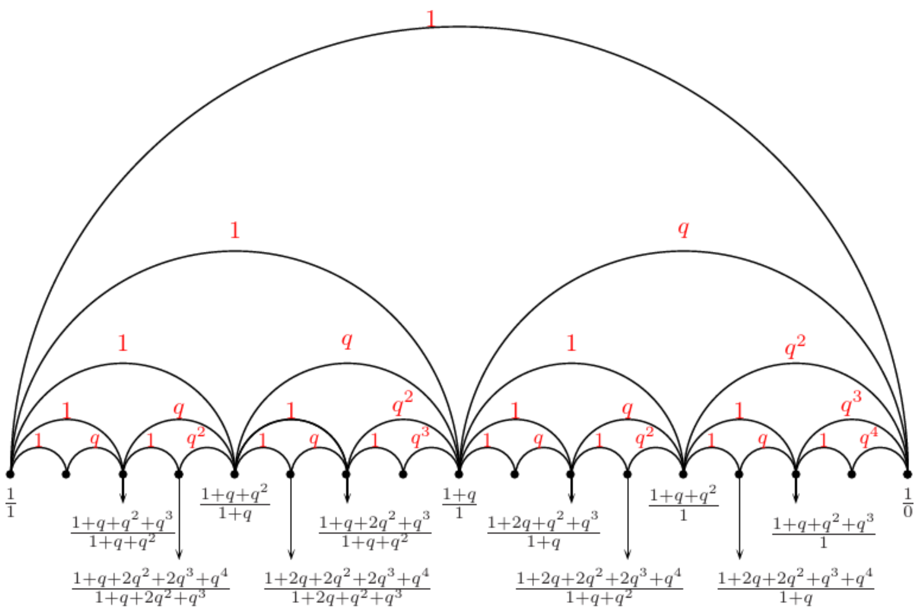

FIGURE 1. Upper part of the q-deformed Farey graph between $\frac{1}{1}$ and $\frac{1}{0}$.

As one can easily check, the matrices R_q and L_q^{-1} evaluated at $q = -t$ correspond exactly to the images of the braid elements σ_1 and σ_2 under the Burau representation ρ_3. Theorem is deduced from the results of [3] about zeros and poles of q-rationals.

In addition we formulate the conjecture that the annulus could be reduced to $\{\frac{3-\sqrt{5}}{2} \leq |z| \leq \frac{3+\sqrt{5}}{2}\}$. This conjectural result would agree with known result in the real case [6].

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