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Mini-Workshop: Bridging Number Theory and Nichols Algebras via Deformations

Organized by
Giovanna Carnovale, Padova
István Heckenberger, Marburg
Leandro Vendramin, Brussels

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Abstract. Nichols algebras are graded Hopf algebra objects in braided tensor categories. They appeared first in a paper by Nichols in 1978 in the search for new examples of Hopf algebras. Rediscovered later several times, they also provide a conceptual explanation of the construction of quantum groups. The aim of the workshop is to review recent developments in the field, initiate collaborations, and discuss new approaches to open problems.

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Introduction by the Organizers

Nichols algebras are graded Hopf algebra objects in braided tensor categories. They appeared first in a paper by Nichols in 1978 in the search for new examples of Hopf algebras. Rediscovered later several times, they also provide a conceptual explanation of the construction of quantum groups (Woronowicz, Lusztig, Rosso, Schauenburg). Nichols algebras play a significant role in problems on Hopf algebras or quantum groups, and in mathematical physics.

In recent years, the interplay between Nichols algebras and Number Theory has become apparent. The aim of the workshop is to foster interaction of these different communities, review recent developments in the field and discuss new approaches to open problems.

A striking incarnation of Nichols algebras has been recently found by Kapranov and Schechtman, where an equivalence of categories between the category of connected and coconnected bialgebras in braided categories and the category of factorized perverse sheaves on the space $\mathrm{Sym}(\mathbb{C})$ of complex monic polynomials is given. This equivalence established a dictionary between the braided bialgebra world and the perverse sheaves world, through which Nichols algebras correspond to intermediate extensions of local systems corresponding to coherent families of representations of the braid groups. This dictionary opens new pathways in the understanding of Nichols algebras, but also in their application to the study of problems of geometric nature.

In a similar direction, Ellenberg, Tran and Westerland used Hurwitz spaces and Nichols algebras to prove Malle's conjecture on finite extensions of certain global fields with fixed Galois group.

Arithmetic aspects show up in the theory of Nichols algebras from the beginning and provide, from time to time, hairpin sharp tools to solve crucial problems.

The structure theory of pointed Hopf algebras depends crucially on a deep understanding of Nichols algebras over groups, in particular those of finite dimension or finite Gelfand–Kirillov dimension. Each Yetter–Drinfeld module over a group (or more generally a Hopf algebra) uniquely determines a Nichols algebra. Currently, the classification of Yetter–Drinfeld modules over groups providing finite-dimensional Nichols algebras is one of the most challenging open problems in the theory of (pointed) Hopf algebras.

This mini-workshop brought together experts from different fields related to Nichols algebras. Discussions spread into several directions, with special emphasis on open questions and conjectures. The meeting has been highly productive, allowing participants to keep up with the recent developments, initiate collaborations, and discuss new approaches to open problems.

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Table of Contents

Abstracts

Some applications of Nichols algebras

Nicolás Andruskiewitsch

1. Finite-dimensional Hopf algebras

The base field is $\mathbb{C}$. The classification of the finite-dimensional Hopf algebras is a wild problem since the classification of the finite groups is a wild problem; see [12]. Indeed the classification of the 2-step (nilpotent) p-groups with non-cyclic center is wild [14]. Still, one looks for a structural theory of finite-dimensional Hopf algebras that has finite groups (and more) as parameters. We are far from any proposal for such a theory; in fact, there are several kinds of finite-dimensional Hopf algebras treated by specific methods that are difficult to extend to other classes. Pointed Hopf algebras form the class best understood and Nichols algebras play an important role in their classification. Two basic invariants of a pointed Hopf algebra H are the group $G = G(H)$ of group-likes and the infinitesimal braiding V, see e.g. [6]. Our present knowledge leads to:

Conjecture 1. A finite-dimensional pointed Hopf algebra H is a cocycle twist of the bosonization of the Nichols algebra $\mathscr{B}(V)$ by the group algebra of G:

$$H \simeq (\mathscr{B}(V)\#\mathbb{C}G)_\sigma ,$$

where $\sigma : \mathscr{B}(V)\#\mathbb{C}G \otimes \mathscr{B}(V)\#\mathbb{C}G \to \mathbb{C}$ is an invertible 2-cocycle.

This conjecture contains two important subproblems. Consider a group G and $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ with $\dim \mathscr{B}(V) < \infty$.

Question 1. Is $\mathscr{B}(V)$ the only finite-dimensional post-Nichols algebra of V? *(generation in degree one).*

The answer is yes when G is abelian [7] and for any known V (but is no in positive characteristic). The proofs follow the strategy outlined in [5, Prop. 5.4] that requires the knowledge of the defining relations of $\mathscr{B}(V)$. There are alternative proofs in [13], [10] but (part of) the relations are still needed. It is intriguing whether a proof free of the knowledge of the relations could be obtained.

Question 2. Any H such that $\operatorname{gr} H \simeq \mathscr{B}(V)\#\mathbb{C}G$ is a cocycle twist as above *(any lifting is a cocycle deformation).*

This is true when G is abelian [8] and for any known V; again the proofs depend on the knowledge of the defining relations of $\mathscr{B}(V)$. A more abstract proof is desirable and perhaps more expectable. When G is abelian and $(|G|, 210) = 1$, the relations of any lifting H can de described explicitly [6]. Beyond that, liftings were computed in several cases but their relations seem to be too complicated.

Conjecture 1 makes sense for Hopf algebras whose coradical is a Hopf subalgebra; Questions 1 and 2 are valid replacing $\mathbb{C}G$ by a semisimple Hopf algebra. Finally, the previous considerations could be adapted to Hopf algebras with finite GK-dimension; see [1] and the talk by Angiono.

2. Racks

We do not know whether the following Question could be answered.

Question 3. Classify the pairs (G, V) where G is a finite group and $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ satisfies $\dim \mathscr{B}(V) < \infty$ and compute the relations of $\mathscr{B}(V)$.

If G is abelian, then both problems are solved by Heckenberger and Angiono respectively, see the talk of the latter. Next, one may focus on finite simple groups (see the talk of Vendramin), or, in the language of racks [4], on V's with $\operatorname{supp} V$ a simple rack [3]. Simple racks, classified in [4, 11], are of two kinds: those with p^n elements (p prime) and the rest (including conjugacy classes in finite simple groups that were studied intensively). Here is a recent progress for the former:

Theorem 1. [9] If V has $\dim \mathscr{B}(V) < \infty$ and $\operatorname{supp} V$ is a simple rack with p elements (p prime), then V belongs to a short list (and $p \leq 7$).

Based on unpublished calculations we dare to claim that the techniques of [9] could be adapted to simple racks with p^n elements (p prime).

3. Finite-generation of cohomology

A well-known conjecture by Etingof and Ostrik was positively answered in [2] for finite-dimensional pointed Hopf algebras with abelian group, using the structure as stated in Conjecture 1, that is a Theorem in this case, as already said. We shall outline the corresponding strategy.

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Nichols algebras of diagonal type and finite Gelfand-Kirillov dimension

Iván Angiono

Let $\Bbbk$ be an algebraically closed field of characteristic zero. The purpose of this expository article is to recall the last advances towards Nichols algebras of diagonal type and finite Gelfand-Kirillov dimension (GK-dim for short) over $\Bbbk$.

Recall that the GK-dim of an associative algebra A is a measure of the growth of products, mimicking the notion of growth for groups. In fact, given a group Γ, the growth of Γ coincides with GK-dim $\Bbbk\Gamma$.

The classification of finite-dimensional Hopf algebras has been successfully developed, mainly for the family of pointed ones. Using the so-called *Lifting method* by Andruskiewitsch and Schneider, this problem leads to the classification of finite-dimensional Nichols algebras over groups [6, 1]. One may wonder to extend this theory to a subclass of infinite-dimensional Hopf algebras (and Nichols algebras) with some kind of *non-commutative geometry* behind, and this leads to the analogous problem of Nichols algebras of finite GK-dim.

As a first source of examples we can consider Nichols algebras whose underlying braided vector space is of *diagonal type*; that is, there exists a matrix $\mathbf{q} = (q_{ij}) \in (\Bbbk^\times)^{\theta\times\theta}$ and a basis $\{x_i\}_{1\le i\le\theta}$ of V such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad \text{for all } 1 \le i, j \le \theta.$$

Although the braiding is simple, the structure of the corresponding Nichols algebra $\mathcal{B}_{\mathbf{q}}$ might be complicated. For example fix $\mathbf{q} = (q_{ij})$, $q_{ij} = q^{a_{ij}}$ for $q \neq 0, \pm 1$ and $A = (a_{ij})$ a finite Cartan matrix of a semisimple Lie algebra $\mathfrak{g}$. If q is not a root of unity, then the Nichols algebra is $U_q^+(\mathfrak{g})$, the positive part of the quantized enveloping algebra of $\mathfrak{g}$, while for q a root of unity of order N (up to mild restrictions on N), the Nichols algebra is $\mathfrak{u}_q^+(\mathfrak{g})$, the positive part of the small quantum group.

The examples above suggest that we can find a basis *á la* PBW, with restricted powers. Before to introduce this kind of bases, recall that $\mathcal{B}_{\mathbf{q}}$ is $\mathbb{N}_0^\theta$-graded, with each x_i in degree α_i (the θ-uple with 1 in the i-th entry and 0 otherwise). For each $\beta = (b_1, \cdots, b_\theta) \in \mathbb{N}_0^\theta$ set $q_\beta := \prod_{i,j=1}^\theta q_{ij}^{b_i b_j}$ and $N(\beta) := \operatorname{ord} q_\beta$. By [13] there exists a restricted PBW basis whose set of *letters* $\mathcal{L} \subset \mathcal{B}_{\mathbf{q}}$ is made of homogeneous elements, $\mathcal{L}$ has a total order $<$ and

$$\ell_1^{a_1} \cdots \ell_k^{a_k}, \qquad \ell_1 < \cdots < \ell_k \in \mathcal{L}, \qquad 0 < a_i < N(\deg(\ell_i)),$$

is a basis of $\mathcal{B}_{\mathbf{q}}$. In addition, $\Delta_+^{\mathbf{q}} := \{\deg \ell : \ell \in L\} \subseteq \mathbb{N}_0^\theta$ does not depend on the chosen PBW basis. This invariant set is called the *set of positive roots* of $\mathbf{q}$, while $\Delta^{\mathbf{q}} := \Delta_+^{\mathbf{q}} \cup (-\Delta_+^{\mathbf{q}})$ is the root system of $\mathbf{q}$. It is known that $\Delta^{\mathbf{q}}$ is a (generalized) root system in the sense of [12], a combinatorial datum generalizing the corresponding root system of contragredient Lie superalgebras. The main result of [11] gives the classification of all Nichols algebras $\mathcal{B}_{\mathbf{q}}$ of diagonal type such that $|\Delta^{\mathbf{q}}| < \infty$. The list is made of *(generalized) Dynkin diagrams*: a certain graph with θ vertices and labels on the edges attached to each braiding matrix $\mathbf{q}$.

One can see that every Nichols algebra in the list of [11] (i.e. with a PBW basis of finite generators) has finite GK-dim, since the dimension of the subspace of iterated products of PBW generators is less or equal than the corresponding one for a polynomial ring in as many variables as PBW generators. Hence we may wonder if there exists a relation between GK-dim $\mathcal{B}_{\mathbf{q}}$ and $\Delta^{\mathbf{q}}$. More precisely, based on the evidence in some cases as [3]:

Conjecture 1 ([2])**.** Let $\mathbf{q}$ be a braiding matrix. Then GK-dim $\mathcal{B}_{\mathbf{q}} < \infty$ if and only if $|\Delta^{\mathbf{q}}| < \infty$; i.e. $\mathbf{q}$ appears in the list of [11].

A positive answer to this conjecture was recently obtained. Indeed:

Theorem 1 ([9])**.** If GK-dim $\mathcal{B}_{\mathbf{q}} < \infty$, then $\mathbf{q}$ belongs to the list of [11].

The proof in [9] can be summarized as follows:

- Arguing recursively on the rank θ (the answer for rank two was obtained in [3]), we get a list of diagrams of rank $\theta + 1$ whose underlying subdiagrams belong to the list. Some diagrams depend on a parameter q (as for quantum groups) and we prove that either the diagram of rank $\theta + 1$ belongs to list or else the parameter of each subdiagram of rank θ is evaluated in a root of unity of order $N \leq 20$. Thus we have a finite list of diagrams to be glued, and this is the first instance where we use of the software `GAP` [10].
- Using a technique introduced in [3] we construct Nichols algebras $\mathcal{B}_{\mathbf{p}}$ obtained as subquotients of the initial Nichols algebra $\mathcal{B}_{\mathbf{q}}$. Thus GK-dim $\mathcal{B}_{\mathbf{p}} \leq$ GK-dim $\mathcal{B}_{\mathbf{q}}$. They are related with hyperplanes $H \subseteq \mathbb{Z}^\theta$, in such a way that the root system of$\mathcal{B}_{\mathbf{p}}$ is $\Delta^{\mathbf{q}} \cap H$. Now we apply criteria for appropriate hyperplanes in order to discard all the diagrams in rank $\theta + 1$ which do not belong to the list.

Although some Nichols algebras of finite GK-dim over abelian groups are not of diagonal type, Theorem 1 above has deep consequences towards the classification, as shown in [2, 4]. There is a related problem towards the classification of pointed Hopf algebras of finite GK-dim, which is the classification of pre-Nichols algebras of finite GK-dim. Again, Theorem 1 is a key step, see e.g. [5, 7, 8].

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Prehomogeneous vector spaces and Nichols algebras over symmetric groups

Kevin Chang

This talk was about a mysterious connection between certain prehomogeneous vector spaces and Nichols algebras over symmetric groups $\mathbf{S}_d$. Here are the relevant objects:

- Prehomogeneous vector spaces: A **prehomogenous vector space** is a faithful representation $G \to \mathrm{GL}(V)$ such that G is a connected linear algebraic group and V has a Zariski open G-orbit. Over an algebraically closed field of characteristic 0, prehomogeneous vector spaces where V is an irreducible G-representation are classified up to "castling transforms" by Sato-Kimura [7]. For any prehomogeneous vector space (G, V), we automatically know that $\dim G \geq \dim V$. The case $\dim G = \dim V$ where the generic stabilizer is finite is very special, and the classification yields only the following examples:
  - $(G_2, V_2) \coloneqq (\mathbb{G}_m, \mathrm{std}_1)$, with trivial generic stabilizer. If we don't require the representation to be faithful, we can get any cyclic group (e.g. $\mathbf{S}_2$) as a generic stabilizer by considering the nth power representations.
  - $(G_3, V_3) \coloneqq (\mathrm{GL}_2, \mathrm{Sym}^3(\mathrm{std}_2) \otimes \det \mathrm{std}_2^\vee)$, with generic stabilizer $\mathbf{S}_3$.
  - $(G_4, V_4) \coloneqq (\{(g_3, g_2) \in \mathrm{GL}_3 \times \mathrm{GL}_2 \mid \det g_3 = \det g_2\}, \mathrm{Sym}^2(\mathrm{std}_3) \otimes \mathrm{std}_2^\vee)$, with generic stabilizer $\mathbf{S}_4$.
  - $(G_5, V_5) \coloneqq (\{(g_4, g_5) \in \mathrm{GL}_4 \times \mathrm{GL}_5 \mid (\det g_4)^2 = \det g_5\}, \mathrm{std}_4 \otimes \det \mathrm{std}_4^\vee \otimes \Lambda^2 \mathrm{std}_5)$, with generic stabilizer $\mathbf{S}_5$.

In particular, any $\mathbf{S}_d$ with $d \geq 6$ cannot appear as the generic stabilizer of a prehomogeneous vector space.

- Nichols algebras: For $d \geq 2$, we use $\mathfrak{B}_d$ to denote the Nichols algebra $\mathfrak{B}(\mathbb{Q} \cdot \tau_d, -1)$, where $\tau_d \subset \mathbf{S}_d$ is the conjugacy class of transpositions. For $d \geq 5$, $\mathfrak{B}_d$ is known to be finite-dimensional and quadratic. For $d \geq 6$, $\mathfrak{B}_d$ is expected to be infinite-dimensional and quadratic, but neither property has been proven for any d.

One immediate connection between these two objects lies in their numerology. For $d = 2, 3, 4, 5$, $\dim G_d$ (which is the same as $\dim V_d$) is $1, 4, 12, 40$, respectively. Similarly, the top nonzero degree of $\mathfrak{B}_d$ for $d = 2, 3, 4, 5$ is $1, 4, 12, 40$, respectively. Moreover, things seem to go wrong in both settings when $d \geq 6$.

A more substantial connection, studied in [3], comes from the geometry and arithmetic of Hurwitz spaces. For any $d \geq 2$ and $n \geq 0$, let $\mathcal{H}_n^d$ denote the **Hurwitz space** whose points correspond to isomorphism classes of degree d simply branched covers of $\mathbb{A}^1$ with n branch points. There is a natural branch map $\mathcal{H}_n^d \to \mathrm{Conf}^n(\mathbb{A}^1)$ to the configuration space of n unordered points in $\mathbb{A}^1$, which takes a cover to its set of branch points. A theorem of Kapranov-Schechtman [5, Corollary 3.3.5] implies that the $\mathbf{S}_d$-invariant cohomology $\mathrm{Ext}^{n-j,n}_{\mathfrak{B}_d}(\mathbb{Q}, \mathbb{Q})^{\mathbf{S}_d}$ is equal to the homology $H_j(\overline{\mathcal{H}}_n^d, \mathbb{Q})$, where $\overline{\mathcal{H}}_n^d$ is a smooth partial compactification of $\mathcal{H}_n^d$ admitting a proper and small extension of the branch map to the symmetric power $\mathcal{H}_n^d \to \mathrm{Conf}^n(\mathbb{A}^1) \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^1)$.

In [3], we construct $\overline{\mathcal{H}}_n^d$ for $d \leq 4$ and a candidate for $\overline{\mathcal{H}}_n^5$. The k-points $\overline{\mathcal{H}}_n^d(k)$ correspond to $G_d(k[t])$-orbits in $V_d(k[t])$ with degree n discriminant (for $d = 3$, this is the usual discriminant of binary cubic forms), and the map $\overline{\mathcal{H}}_n^d \to \mathrm{Sym}^n(\mathbb{A}^1)$ sends an orbit to the vanishing locus of its discriminant. The fact that these orbit spaces compactify Hurwitz spaces is due to the parametrizations of low degree algebras by prehomogeneous vector spaces, which are used extensively in arithmetic statistics [1, 2, 4, 6, 8]. We can compute the homology of $\overline{\mathcal{H}}_n^d$ in terms of counts of $G_d(k[[t]])$-orbits in $V_d(k[[t]])$ with valuation n discriminant as k ranges over finite fields. As n ranges over nonnegative integers, these counts can be packaged into an integral over $V_d(k[[t]])$ called an **Igusa zeta function**. Thus, we obtain an identity [3, Theorem 11.14] relating the Igusa zeta function for (G_d, V_d) and the generating function for the $\mathbf{S}_d$-invariant cohomology of $\mathfrak{B}_d$.

Despite what we know, the precise relationship between prehomogeneous vector spaces and Nichols algebras remains very mysterious. We have no idea how the numerology and the Igusa zeta function/Nichols algebra cohomology connection relate to one another, as it is unclear how the dimension of V_d can be read from its Igusa zeta function or how the top degree of $\mathfrak{B}_d$ can be read from its cohomology. On the prehomogeneous side, it is also unclear how to interpret the finite-dimensionality of $\mathfrak{B}_d$ or other numbers associated to $\mathfrak{B}_d$ such as its dimension or the dimensions of its graded pieces. Furthermore, we point out that the objects (G_d, V_d) and $\mathfrak{B}_d$ are fundamentally different types of objects. (G_d, V_d) falls out of the classification of prehomogeneous vector spaces, and it seems like a coincidence that we get $\mathbf{S}_d$ as a stabilizer. On the other hand, the Nichols algebra $\mathfrak{B}_d$ is intrinsic to the Coxeter group $\mathbf{S}_d$. It would be very interesting to resolve some of these mysteries.

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Geometric problems related to approximations of Nichols algebras

Francesco Esposito
(joint work with Giovanna Carnovale, Lleonard Rubio y Degrassi)

Let (V, c) be a braided vector space. Understanding the structure of the Nichols algebra $\mathcal{B}(V, c)$ attached to (V, c) is an interesting algebraic problem that in general is very hard to settle. For example, in [1] a quadratic algebra with applications to Schubert calculus is defined, which is shown in [2] to be a Nichols algebra in small cases. The general case is an open problem.

In the important special case in which (V, c) is obtained from a rack consisting in a conjugation-invariant subset C of a finite group G and the one-cocycle identically equal to -1 on C, we translate questions regarding $\mathcal{B}(V, c)$ in geometric terms.

Let $\mathrm{Sym}^n_{\neq}(\mathbb{C}) := \mathrm{Conf}_n(\mathbb{C})$ be the space of configurations of n distinct points on the complex affine line. The fundamental group of $\mathrm{Conf}_n(\mathbb{C})$ is the group of braids $\mathcal{B}_n$ on n strings. The set C^n admits an action of $\mathcal{B}_n$. Let

$$\mathrm{Hur}_{C,n} \to \mathrm{Conf}_n(\mathbb{C})$$

be the finite étale covering of $\mathrm{Conf}_n(\mathbb{C})$ corresponding to the $\mathcal{B}_n$-set C^n; it is the so-called Hurwitz space attached to the data (C, n). The Hurwitz space $\mathrm{Hur}_{C,n}$ is smooth, being an étale cover of the smooth space $\mathrm{Conf}_n(\mathbb{C})$; but it extends by normalization to a ramified covering $\widetilde{\mathrm{Hur}}_{C,n}$ of $\mathrm{Sym}^n(\mathbb{C})$, the space of degree n monic polynomials with complex coefficients.

In general, the spaces $\widetilde{\mathrm{Hur}}_{C,n}$, for varying n and fixed C, are singular. In fact, the singularities of these spaces are closely connected to structural properties of the Nichols algebra $\mathcal{B}(V,c)$. Furthermore, one has a product

$$\widetilde{\mathrm{Hur}}_{C,n} = \mathbb{C} \times \ {}^0\widetilde{\mathrm{Hur}}_{C,n}$$

where ${}^0\widetilde{\mathrm{Hur}}_{C,n}$ is a $n-1$-dimensional variety, smoothly equivalent to $\widetilde{\mathrm{Hur}}_{C,n}$. Moreover, the stratification on ${}^0\widetilde{\mathrm{Hur}}_{C,n}$ has a single closed stratum, which is reduced to a point o. Thus to study the singularities of $\widetilde{\mathrm{Hur}}_{C,n}$ one is reduced to studying the singularities of ${}^0\widetilde{\mathrm{Hur}}_{C,n}$, in particular at its point o.

As an example, the fact that the Nichols algebra is quadratic translates into the following geometric statement about ${}^0\widetilde{\mathrm{Hur}}_{C,n}$:

For every n,

$$IH^{n-1}({}^0\widetilde{\mathrm{Hur}}_{C,n} \setminus \{o\}, \mathbb{Q}) = 0$$

where IH denotes global intersection cohomology, and $n-1$ is the middle degree.

The proof uses the equivalence between connected bialgebras and factorizable perverse sheaves on $\mathrm{Sym}(\mathbb{C})$ established in [3], translating geometrically the the approximation of a Nichols algebra, identifying the local system involved as the push-forward of the constant sheaf on the Hurwitz space, and finally using smallness of the finite map from $\widetilde{\mathrm{Hur}}_{C,n}$ to $\mathrm{Sym}^n(\mathbb{C})$.

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Dualities in abelian monoidal functor categories

Sebastian Halbig

(joint work with Tony Zorman)

Questions concerning the existence and kind of dualisability for a given structure are ubiquitous in modern mathematics. The Ext and Tor functors of homological algebra are derived from the closed monoidal structure expressed by the tensor-hom adjunction. Pontryagin duality can be interpreted in the framework of $*$-autonomy, [Bar]. And rigidity, which corresponds to the existence of a dual basis with a compatible evaluation, is an essential ingredient in the reconstruction of Hopf algebras [Ulb90].

The starting point for our investigation are the following observations about representations of a commutative algebra.

(i) If the algebra is Frobenius its regular representation is a dualising object for its finitely-generated modules.

(ii) A module is finitely-generated projective if and only if it is rigidly dualisable.

Using the perspective that modules over an algebra can be identified with linear functors from an one-object category to vector spaces, we are going to discuss in this talk generalisations of the above statements to linear presheaves. Our results have on the one hand applications in modular representation theory as Mackey functors, which abstract the operations of induction, restriction and conjugation, admit a concise definition in terms of presheaves, see [Lin76]. On the other hand, we obtain a transparent way to construct $*$-autonomous categories, which in turn are connected to modular functors and topological quantum field theories, see [MW23].

Throughout, $\mathtt{k}$ is an arbitrary but fixed field and all categories, functors and natural transformations are linear, i.e. enriched over $\mathcal{V}$:=$\mathtt{k}\text{-}\mathtt{Vect}$. The presheaves $[\mathcal{X},\mathcal{V}]$:=Fun$(\mathcal{X},\mathcal{V})$ of $\mathcal{X}^{\mathrm{op}}$ inherit the abelian structure of $\mathcal{V}$. Furthermore, if $\mathcal{X}^{\mathrm{op}}$ is monoidal, *Day convolution* allows us to extend the tensor product along the Yoneda embedding $\mathcal{X}^{\mathrm{op}} \hookrightarrow [\mathcal{X},\mathcal{V}]$ to the presheaves of $\mathcal{X}^{\mathrm{op}}$. In case $\mathcal{X}$ has only one object, it can be identified with a commutative algebra and Day convolution recovers tensor product of its modules.

The presheaves $[\mathcal{X},\mathcal{V}]$ are closed monoidal. That is, tensoring with a fixed object has a right adjoint. In order to address the first of the two statements given above, we need to explain the notion of $*$-autonomous categories, which are also called *Grothendieck–Verdier categories.*

Definition: A $*$-*autonomous category* comprises a monoidal category $\mathcal{X}$ and an object $d \in \mathcal{X}$, called its *dualising object*, such that there exists an anti-equivalence $D\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{X}$ and a natural isomorphism $\mathcal{X}(a \otimes b, d) \cong \mathcal{X}(b, Da)$.

Recently, it was shown, see [FSSW23], that finite-dimensional modules over a finite-dimensional algebra are $*$-autonomous. The language of functor categories provides us with a conceptual approach to this result and a method to obtain new $*$-autonomous categories.

Theorem ([HZ24, Lemma 5.6]). Let $(\mathcal{X}, d)$ be $*$-autonomous and assume that the subcategory $[\mathcal{X},\mathcal{V}_{\mathrm{fin}}] \subset [\mathcal{X},\mathcal{V}]$ of object-wise finite-dimensional functors is closed monoidal. The presheaves $[\mathcal{X},\mathcal{V}_{\mathrm{fin}}]$ are $*$-autonomous with $\mathcal{X}(-,d)^*$ as their dualising object.

One can transfer the notion of finitely-generated projectiveness from modules over algebras to objects in general abelian categories by characterising it in terms of commutation relations with certain colimits. This allows us to obtain a variant of the second observation.

Theorem ([HZ24, Proposition 5.17]). Let $\mathcal{X}$ be a small rigid monoidal category. A functor $F \in [\mathcal{X},\mathcal{V}]$ is finitely-generated projective if and only if it is rigidly dualisable.

One consequence is that (quantum) traces for Mackey functors exist if and only if these are finitely-generated projective. This was suggested by Bouc, see [Bou05], who gave a detailed proof-sketch based on a different approach.

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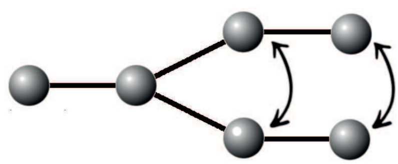
Folding Nichols algebras

Simon Lentner

In this talk I review an older construction of mine [Len14], give modern (categorical) perspectives on some of the ideas and then present recent results with I. Angiono and G. Sanmarco in [ALS23]. Ultimately, we settle the classification of finite-dimensional pointed Hopf algebras whose coradical are given by nonabelian groups, up to certain small-rank exceptional examples.

Folding means in our context in the most concrete form: Let M be a Yetter-Drinfeld module over a group Γ and let Z be an abelian group of automorphisms (in a twisted sense) of the Dynkin diagram associated to M in the theory of Nichols algebras [AHS08], then we assign a Yetter-Drinfeld module $\tilde{M}$ over a group G, which is a central extension of Γ by the dual group $\hat{Z}$. The associated Nichols algebras $\mathcal{B}(M)$, $\mathcal{B}(\tilde{M})$ are isomorphic as vector spaces, and the root system is folded, in a sense known from Satake diagrams. For example, E_6 with the obvious involution on the Dynkin diagram is folded to F_4:



The roots in one orbit merge into a single root in the folded root system (in this example, orbits of 1 resp. 2 roots in E_6 merge into a long resp. short root of F_4). On the level of Yetter-Drinfeld modules, this means that multiple irreducible Γ-Yetter-Drinfeld modules combine into a new irreducible G-Yetter-Drinfeld module, as the conjugacy classes increase due to the central extension.

We now discuss converse statements: It is quite clear that conversely any Hopf algebra containing a central group algebra $\mathbb{C}[Z]$ is a folding. However, we may consider a Doi-twist by another 2-cocycle, now over G, to quickly produce from folding closely related Hopf algebras where Z is not central.

In a remarkable series of works culminating in [HV17], Heckenberger and Vendramin have classified all finite-dimensional Nichols algebras of rank > 2 over nonabelian groups. They find exceptional cases in rank $2, 3$ and several large families, which seem to correspond to the different possible foldings of diagonal Nichols algebras, which are all of Lie type. In [ALS23] we now prove:

Lemma 1. For any family in [HV17], all Nichols algebras in this family are a Doi twist of a corresponding folding of a Nichols algebra of diagonal type.

This result allows us to talk about these families uniformly and with notions from the diagonal Nichols algebras over Γ and their root system. For example, we obtain a special PBW-type basis, which is not homogeneous over G but otherwise much easier. Using this, we now settle for these families all steps of the *Andruskiewitsch-Schneider program* for classifying pointed Hopf algebras:

Theorem 2. Any finite-dimensional pointed Hopf algebra, whose coradical is a nonabelian group ring and whose infinitesimal braiding has rank > 3, is a Hopf cocycle twist of a folding $\mathbb{C}[G]\#\mathcal{B}(\tilde{M})$.

A more general version of folding starts with a Hopf algebra H and a group Z of H-H-bigalois objects. Then their direct sum can be given the structure of a Hopf algebra. This contains the previous notion of folding, if $H = \mathbb{C}[\Gamma]\#\mathcal{B}(M)$ and the bigalois objects are given by 2-cocycles σ_z on $\mathbb{C}[\Gamma]$ and by coalgebra automorphisms f_z on $\mathcal{B}(M)$. The direct sums of the respective coradicals is a direct sum of twisted group rings, which altogether forms the group ring of the centrally extended group G

$$\mathbb{C}[G] = \bigoplus_{z\in Z} \mathbb{C}_{\sigma_z}[\Gamma]$$

An even more general notion, from today's perspective, is to consider the extension of tensor categories classified in [ENOM09], depending on a categorical homomorphism of the group Z to th Brauer-Picard group of $\mathrm{Rep}(H)$. Note that Bigalois objects give an equivalence of tensor categories $\mathrm{Rep}(H^*)$ and thus via α-induction elements in the Brauer-Picard group. This already gives a hint how to modify the construction discussed above in order to produce a *braided* tensor category, which will however involve nontrivial associativity constraints.

From a physics perspective, folding should appear in the context of a generalized logarithmic Kazhdan-Lusztig conjecture [CLR23] (more precisely: kernels of screenings in lattice orbifold vertex algebras). In this context, the Nichols algebra appears as algebras of screening operators, associated to the monodromy of certain bundles over the configuration space. For usual quantum groups, this appearance is a rather old story. But the author was very excited to learn about Nichols algebras over nonabelian groups in several talks of the MFO miniworkshop. It is conceivable that these appearances might not be unrelated.

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Non-existence of Hopf orders for certain Hopf algebras

Elisabetta Masut

(joint work with Giovanna Carnovale and Juan Cuadra)

The study of the (non)-existence of Hopf orders for finite-dimensional Hopf algebras was initially motivated by Kaplansky's sixth conjecture, which states that given a complex finite-dimensional semisimple Hopf algebra H, the dimension of every irreducible representation of H divides the dimension of H.

Larson proved a weaker version of this conjecture. Specifically, he proved that if the Hopf algebra admits a Hopf order over a number ring, then the conjecture holds.

It is also worth to mention that the property of admitting a Hopf order could be useful for classifying semisimple Hopf algebras.

Let H be a finite-dimensional Hopf algebra over a field K and let R be a subring of K. Roughly speaking, a Hopf order is a Hopf algebra over R, such that the extension by scalars to K is isomorphic to H.

The goal of this talk is to present some results on the non-existence of Hopf orders for certain Hopf algebras. In particular, the Hopf algebras we consider are obtained by deforming group algebras, by means of a twist.

Given a Hopf algebra H, we say that an invertible element $J \in H \otimes_K H$ is a twist if it satisfies the following equations

$$(1_H \otimes J)(\mathrm{Id} \otimes \Delta)(J) = (J \otimes 1_H)(\Delta \otimes \mathrm{Id})(J);$$

$$(\varepsilon \otimes \mathrm{Id})(J) = (\mathrm{Id} \otimes \varepsilon)(J) = 1_H.$$

Using such an element J, we can define the Drinfeld twist H_J of H as follows: $H_J = H$ as an algebra, the counit is that of H, and the coproduct and antipode differ from those in H in the following way:

$$\Delta_J(h) = J\Delta(h)J^{-1}, \qquad S_J(h) = u_J S(h) u_J^{-1} \qquad \text{for all } h \in H.$$

Here u_J is an invertible element of H constructed by means of J.

Since we are interested in twisting group algebras, we specify the above construction to this case, using the so called Movshev's strategy.

Let G be a group and let M be an abelian subgroup of G. Movshev in [4] proved that the element

$$J = \sum_{\psi,\phi\in\widehat{M}} \omega(\psi,\phi)e_\phi \otimes e_\psi$$

is a twist for KM and in consequence for KG. Here, $\widehat{M}$ is the group of characters of M, ω is a 2-cocycle on $\widehat{M}$ and $e_\phi = \frac{1}{|M|}\sum_{m\in M}\phi(m)m^{-1}$ is the idempotent, primitive central element associated to an element ϕ in $\widehat{M}$.

In [2] and in [3] Cuadra and Meir proved the non-existence of Hopf orders for several families of twisted group algebras, which turn out to be simple as Hopf algebras. The authors hypothesized a relationship between the simplicity of the twisted group algebras and the non-existence of Hopf orders. Since a twisted group algebra of a simple group is simple, we tried to understand whether it admits a Hopf order. In particular, in [1] we prove the following.

Theorem 1. Let K be a number field and let G be a finite non-abelian simple group. Then, there is a twist J for KG, arising from a 2-cocycle on an abelian subgroup of G, such that $(KG)_J$ does not admit a Hopf order over $\mathcal{O}_K$.

For the Janko group, the Suzuki groups and $\mathbf{PSL}_2(q)$, where q is a prime power, we show that the non-existence result holds for any twist arising from a 2-cocycle on an abelian subgroup of G.

The next natural step is to focus on non-simple groups. In this regard, we show the following.

Theorem 2. Let K be a number field and let $G = \prod_{i=1}^n D_i \rtimes_{\varphi_i} M_i$, where $D_i \rtimes_{\varphi_i} M_i$ is a Frobenius group for every $i \in \{1,\dots,n\}$ and M_i is an abelian group of prime power order for every $i \in \{1,\dots,n\}$. Assume also that $M = \prod_{i=1}^n M_i$ is of square order. Then, for the twist J arising from a 2-cocycle on $\widehat{M}$, the Hopf algebra $(KG)_J$ does not admit a Hopf order over $\mathcal{O}_K$.

The deformed group algebras considered in the above theorem are simple when $n = 2$, while for $n \geq 3$ the simplicity is still unclear.

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Geometry of braided vector spaces and Nichols algebras

Ehud Meir

A braided vector space is a vector space V equipped with a braiding $c : V \otimes V \to V \otimes V$ that satisfies the Yang-Baxter (or Braid) Equation.

Such a structure defines a Nichols algebra $\mathcal{B}(V, c)$.

The goal of this talk is to present geometric and characteristic p methods in the study of braided vector spaces and their Nichols algebras.

The main idea is the following: instead of looking at a specific braided vector space, we can also consider the set of all possible braidings on a given vector space V of some fixed dimension d.

This set has a structure of an affine variety Z. There is an action of GL_d on Z, and the different orbits correspond to isomorphism types of braidings.

It then becomes natural to ask what orbits are contained in the closure of other orbits.

Intuitively, if one orbit $\mathcal{O}_1$ is contained in the closure of another orbit $\mathcal{O}_2$, then it is, in an appropriate sense, simpler. We say in this case that $\mathcal{O}_2$ degenerates to $\mathcal{O}_1$. It holds that if $c \in \mathcal{O}_2$ has a finite dimensional Nichols algebras, then the same holds for the braidings in $\mathcal{O}_1$.

In this talk we use this geometric point of view to show that many Nichols algebras are infinite dimensional. Specifically, we consider Nichols algebras that arise from affine racks.

We do not have degenerations of these braidings in characteristic zero, but when we reduce everything modulo some prime p we do. We then show that apart from a list of 5 specific braidings we do not have any finite dimensional Nichols algebras.

This talk is based on a joint work with Istvan Heceknberger and Leandro Vendramin (arXiv:2306.02989).

Cohomology, Representation theory and tt-geometry

Julia Pevtsova

Let A be a finite dimensional Hopf algebra over (a field) k. We consider the derived and stable categories of A which are related by a localization sequence of tensor triangulated categories (tt-categories):

$$\mathbb{D}^{\mathrm{perf}}(A) \to \mathbb{D}^{\mathrm{b}}(A) \to \mathrm{stab}(A).$$

Here, $\mathbb{D}^{\mathrm{b}}A$ is the bounded derived category of A-modules; $\mathbb{D}^{\mathrm{perf}}(A)$ is the derived category of perfect complexes and $\mathrm{stab}\, A$ is the stable category. It was an observation of Rickard and Buchweitz independently [4], [11] that $\mathrm{stab}(A)$ can be described explicitly in a different way.

Namely, the objects of $\mathrm{stab}\, A$ are finite dimensional A-modules whereas for morphisms we quotient out A-module homomorphisms by the subset of maps factoring through a projective module.

Effectively, we kill projective A-modules, taking out injectives on the way since projectives and injectives are the same by the self-injectivity of A.

We study these triangulated categories associated to A using the techniques of *tt-geometry* (tensor triangular geometry), a field introduced by Balmer in 2005 [1]. To an essentially small tt-category T one associates its *spectrum* $\operatorname{Spec} T$, together with a universal support function:

$$\operatorname{supp}: T \to \text{ subsets of } \operatorname{Spec} T.$$

By a theorem of Balmer [1], $\operatorname{Spec}(\operatorname{stab} A)$ (or of $\mathbb{D}^{\mathrm{b}}(A)$) classifies A-modules up to "homological operations", thus capturing the global structure of $\operatorname{stab} A$ via this topological invariant. Explicitly, there is one-to-one correspondence

$$\left\{ \begin{array}{c} \text{thick tensor ideals} \\ \text{of } \operatorname{stab} A \end{array} \right\} \sim \left\{ \begin{array}{c} \text{specialization closed subsets} \\ \text{of } \operatorname{Spec} \operatorname{stab} A \end{array} \right\}$$

One can reformulate this correspondence by saying that a module M can be "constructed" out of module N using homological operations (such as shifts, cones, sums, summands and tensoring with a simple module) if and only if $\operatorname{supp} M \subset \operatorname{supp} N$.

During the talk due to the limited time I was vague about the following point: the "classical" tt-theory requires the tt-category T to be symmetric, hence a priori applies only to cocommutative Hopf algebras A. Nonetheless, the generalizations, leading to *non-commutative tt-geometry* have been developed which allow to consider the more general situations, see [5], [10], [9].

If one accepts that line of approach to understanding the representation category of A, then a natural question to ask is how to compute the spectrum? Another theorem of Balmer [2] constructs a natural continuous map from the spectrum $\operatorname{Spec} T$ to the spectrum of homogeneous prime ideals of the ring of endomorphisms of the unit object in T. Applied to our situation, it has the following form:

$$\Psi: \operatorname{Spec}(\operatorname{stab} A) \to \operatorname{Proj} H^*(A,k)$$

where $H^*(A,k) = \operatorname{Ext}^*_A(k,k)$. This observation brings one in line with at least one of the subjects of the mini-workshop.

The map Ψ, together with known results for finite groups and finite group schemes over a field (see [4], [7]), suggest that studying $H^*(A,k)$ brings one closer to understanding the spectrum, and, hence, the global structure, of $\operatorname{stab} A$. Moreover, when $H^*(A,k)$ is Noetherian, the map Ψ is known to be surjective.

In the talk I aimed to describe our current knowledge on the following three questions, with the second being also touched upon in N. Andruskiewitsch's talk:

(1) When is $\operatorname{Proj} H^*(A,k)$ a scheme of finite type (when is $H^*(A,k)$ Noetherian)?
(2) When is the map Ψ a homeomorphism?
(3) Can we calculate $\operatorname{Proj} H^*(A,k)$?

When A is not cocommutative, the results on (2) and (3) are scarce. On the other hand, for a finite projective cocommutative Hopf algebra over any commutative Noetherian ring R one has a complete answer to questions (1) and (2), due to a recent work of van der Kallen [12], Barthel, Benson, Iyengar, Krause, Pevtsova [3]; see also Lau [8].

Theorem 1 (W. van der Kallen, 2023)**.** *Let R be a commutative Noetherian ring, and A be a finite projective cocommutative Hopf algebra over R. Then A satisfies "cohomological finite generation" property; in particular, $H^*(A,R)$ is Noetherian.*

Theorem 2 (Barthel, Benson, Iyengar, Krause, Pevtsova, 2023)**.** *Let R be a commutative Noetherian ring, and A be a finite projective cocommutative Hopf algebra over R. Then the tt-category $\mathrm{Stab}(A,R)$ (the stable module category of lattices of A-modules over R) is stratified by $\operatorname{Proj} H^*(A,R)$.*

In particular, the map $\Psi : \mathrm{Spec}(\mathrm{stab}(A,R)) \to \operatorname{Proj} H^*(A,R)$ is a homeomorphism.

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A geometric approach to Nichols algebras

Lleonard Rubio y Degrassi
(joint work with Giovanna Carnovale and Francesco Esposito)

1. Introduction

Let $\mathcal{V}$ be an abelian braided monoidal category and let $\mathcal{CB}(\mathcal{V})$ be the category of connected graded bialgebras (also called primitive bialgebras). An object $A = \oplus_{n\geq 0} A_n$ in $\mathcal{CB}(\mathcal{V})$ is a bialgebra which is $\mathbb{N}$-graded as an algebra and a coalgebra and satisfies $A_0 = k$. For a given object V in $\mathcal{V}$, notable examples of objects in $\mathcal{CB}(\mathcal{V})$ are: the tensor algebra $T_!(V)$, the cotensor algebra $T_*(V)$ and the Nichols algebra $\mathscr{B}(V)$ which is the image of a graded bialgebra morphism $\Omega : T_!(V) \mapsto T_*(V)$ which extends the identity on V.

A fundamental question in this context is to understand when $\mathscr{B}(V)$ is finitely presented, or equivalently, if there is a $d \geq 2$ such that

$$\mathscr{B}(V) \cong \widehat{\mathscr{B}(V)}_d := T_!(V)/(\ker(\Omega) \cap \oplus_{j=2}^{d} V^{\otimes j}).$$

In other words, if $\mathscr{B}(V)$ is isomorphic with its d-th approximation $\widehat{\mathscr{B}(V)}_d$ for some $d \geq 2$.

Example 1. The Fomin-Kirillov algebras FK_n for $n \geq 3$ are quadratic algebras introduced in [1]. The degree 1 term V_n in FK_n is a Yetter-Drinfeld module over the symmetric group $\mathbb{S}_n$ and by [3] the FK_n is isomorphic to the quadratic approximation $\widehat{\mathscr{B}(V)}_2$ of $\mathscr{B}(V)$. For $n = 3, 4, 5$, it has been shown that $FK_n \cong \mathscr{B}(V_n)$ ([4, 2], with the contribution of Graña), and for such values of n these algebras are finite-dimensional [5, 1, 2]. For $n \geq 6$, both the dimensions of these algebras and the isomorphism $\mathscr{B}(V) \cong \widehat{\mathscr{B}(V)}_2$ are still widely open problems.

2. Primitive bialgebras

We define the category $\mathcal{CB}_{\leq d}(\mathcal{V})$ of bialgebras $\mod d+1$ in $\mathcal{V}$ as follows: objects are graded objects of the form $A = \bigoplus_{l=0}^{d} A_l$, non-trivial only in degree $\leq d$, together with a coalgebra and algebra structure that are graded and connected and satisfy the compatibility condition

$$\Delta_A(m_A(x \otimes y)) - m_{A\otimes A}(\Delta_A(x) \otimes \Delta_A(y)) \in (A \otimes A)_{\geq d+1}.$$

Morphisms are graded coalgebra maps preserving the multiplication.

We consider the truncation functor $\Theta_d \colon \mathcal{CB}(\mathcal{V}) \to \mathcal{CB}_{\leq d}(\mathcal{V})$ defined as follows: If $A = \bigoplus_{l\geq 0} A_l$ is an object in $\mathcal{CB}(\mathcal{V})$, we set $\Theta_d(A)$ to be the object $\bigoplus_{l=0}^{d} A_l$, endowed with the graded algebra structure coming from the quotient by the graded ideal $A_{>d} := \oplus_{j>d} A_j$ via the identification $A/A_{>d} \simeq \bigoplus_{l=0}^{d} A_l$ and the graded coalgebra structure coming from the coalgebra inclusion $\bigoplus_{l=0}^{d} A_l \to A$.

Let $A = \bigoplus_{l=0}^{d} A_l$ be an object in $\mathcal{CB}_{\leq d}(\mathcal{V})$. We consider the tensor algebra $T(A)$, with grading induced from the grading of A. Let J_A be the algebra ideal generated by the elements:

$$a \otimes b - ab, a \in A_i, b \in A_j,\ 0 \leq i + j \leq d; \quad 1_k - 1_A,$$

The extension functor $\Psi_d \colon \mathcal{CB}_{\leq d}(\mathcal{V}) \to \mathcal{CB}(\mathcal{V})$ sends an object $A = \bigoplus_{l=0}^{d} A_l$ to $T(A)/J_A$.

Theorem 1. *For any $d > 0$ the functor $\Psi_d \circ \Theta_d : \mathcal{CB}(\mathcal{V}) \to \mathcal{CB}(\mathcal{V})$ extends the d-th approximation construction from primitive bialgebras generated in degree 1 to all primitive bialgebras.*

3. Factorizable perverse sheaves

Kapranov and Schechtman [3] have established an equivalence L between the category $\mathcal{CB}(\mathcal{V})$ and the category $\mathcal{FP}(\mathrm{Sym}(\mathbb{C}); \mathcal{V})$ of factorizable systems of perverse sheaves on all symmetric products $\mathrm{Sym}^n(\mathbb{C})$ with values in $\mathcal{V}$.

The space $\mathrm{Sym}(\mathbb{C})$ is the space of monic polynomials, which is the coproduct of infinitely many connected components $\mathrm{Sym}^n(\mathbb{C})$, for $n \geq 0$ given by monic polynomials of degree n. Each space $\mathrm{Sym}^n(\mathbb{C})$ is stratified in terms of multiplicities of the roots. The open stratum $\mathrm{Sym}^n_{\neq}(\mathbb{C})$ consists of multiplicity-free polynomials of degree n. Note that the fundamental group of each $\mathrm{Sym}^n_{\neq}(\mathbb{C})$ is the braid group $\mathbb{B}_n$.

A sequence of perverse sheaves on each $\mathrm{Sym}^n(\mathbb{C})$ is factorizable if it satisfies a condition ensuring compatibility with the monoid structure on $\mathrm{Sym}^n(\mathbb{C})$, see [3, Definition 3.2.5]. The restriction of a factorizable perverse sheaf on $\mathrm{Sym}^n(\mathbb{C})$ to $\mathrm{Sym}^n_{\neq}(\mathbb{C})$ is a sequence of local systems on each $\mathrm{Sym}^n_{\neq}(\mathbb{C})$, that is, a representation of each braid group $\mathbb{B}_n$. Factorizability makes these representations to be compatible, i.e., they are the representations $V^{\otimes n}$ for V an object in $\mathcal{V}$, with action coming from the braiding. We denote this collection of local systems by $\mathcal{L}(V)$. For $d > 0$, consider the dense open subsets $\mathrm{Sym}^n_{\leq d}(\mathbb{C})$ of $\mathrm{Sym}^n_{\neq}(\mathbb{C})$ consisting of polynomials with root multiplicities not exceeding d, and the corresponding open inclusions:

$$\alpha_d : \mathrm{Sym}_{\leq d}(\mathbb{C}) \to \mathrm{Sym}(\mathbb{C}); \quad j : \mathrm{Sym}_{\neq}(\mathbb{C}) \to \mathrm{Sym}(\mathbb{C}) :$$

Following 6-functors formalism, under Kapranov and Schechtman's equivalence, one obtains:

$$L(T_!(V)) = j_!\mathcal{L}(V); \quad L(T_*(V)) = j_*\mathcal{L}(V); \quad L(\mathscr{B}(V)) = j_{!*}\mathcal{L}(V)$$

Let's consider the category $\mathcal{FP}(\mathrm{Sym}_{\leq d}(\mathbb{C}); \mathcal{V})$ of factorizable perverse on $\mathrm{Sym}_{\leq d}(\mathbb{C})$. Let ${}^p\tau_{\geq 0}$ be the perverse truncation functor. The (geometric) truncation and extension functors are defined as $\alpha_d^* : \mathcal{FP}(\mathrm{Sym}(\mathbb{C}); \mathcal{V}) \to \mathcal{FP}(\mathrm{Sym}_{\leq d}(\mathbb{C}); \mathcal{V})$, and ${}^p\tau_{\geq 0}\alpha_{d!} : \mathcal{FP}(\mathrm{Sym}_{\leq d}(\mathbb{C}); \mathcal{V}) \to \mathcal{FP}(\mathrm{Sym}(\mathbb{C}); \mathcal{V})$, respectively. Then we define the (geometric) d-th approximation functor as the composition ${}^p\tau_{\geq 0}\alpha_{d!} \circ \alpha_d^*$.

One of our main results establishes an equivalence between the category $\mathcal{CB}_{\le d}(\mathcal{V})$ and the category $\mathcal{FP}(\mathrm{Sym}_{\le d}(\mathbb{C}); \mathcal{V})$ compatible with the d-th approximation functors. We also rephrase the condition that a Nichols algebra coincides with its d-th approximation in geometric terms. More precisely:

Theorem 2. There is an equivalence $L_{\le d} : \mathcal{CB}_{\le d}(\mathcal{V}) \to \mathcal{FP}(\mathrm{Sym}_{\le d}(\mathbb{C}); \mathcal{V})$ compatible with the d-th approximation functors. In addition, for an object $V \in \mathcal{V}$, we have that $\mathscr{B}(V) \cong \widehat{\mathscr{B}(V)}_d$ if and only if

$$j_{!*}\mathcal{L}(V) \;=\; {}^{p}\tau_{\ge 0}\alpha_{d!} \circ \alpha_d^* \circ j_{!*}\mathcal{L}(V).$$

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Shuffle algebras, quantum groups and Hall algebras

Olivier Schiffmann

The talk presented two results concerned with shuffle algebras, understood in a broad sense:

The first is a joint work with A. Negut and F. Sala, and deals with the structure of certain shuffle algebras Sh_Q associated to quivers. From a geometric point of view, this shuffle algebra arises as the so-called cohomological Hall algebra of the cotangent stack to the stack of representations of a quiver Q (which may be arbitrary). From a representation-theoretic point of view, this shuffle algebra is a quantum affinization of the (Borcherds) Kac-Moody $\mathfrak{g}_Q$ algebra attached to Q. One motivation for the study of these algebras is that they act naturally on the equivariant K-theory of Nakajima quiver varieties. Our main theorem provides a complete description, by generators and relations of Sh_Q, when the set of equivariant parameters is large enough (or for a generic enough specialization of the equivariant parameters). One first main application is the description, again by generators and relations of the spherical Hall algebra $\mathbf{H}_X$ of a generic curve X (of arbitrary genus) defined over a finite field, which encodes the operators of Eisenstein series and Constant Term for (everywhere unramified) automorphic functions for the groups GL_n over the function field of X.. This comes from a Langlands-type isomorphism between the Hall algebra $\mathbf{H}_X$ and Sh_Q where Q is a quiver with one vertex and g loops, g being the genus of X.

The second work, which is still in a very preliminary form, is a joint project with M. Kapranov, V. Schechtman and J. Yuan. Motivated again by the theory of Eisenstein series for automorphic forms over function fields, but this time for an arbitrary reductive group G and parabolic subgroup P, we define a 'reductive' version of shuffle algebras, the latter corresponding to the case of the general linear groups GL_n. We describe a set of axioms for such a structure, based on the geometry of Coxeter complexes and double cosets in Weyl groups.

An Example of a Braided Monoidal 2-category

Catharina Stroppel

(joint work with Y. L. Liu, A. Mazel-Gee, D. Reutter, P. Wedrich)

Braided monoidal categories play an important role in the construction and study of Nichols algebras. Starting with a braided vector space, the Nicholas algebra can be defined using the tensor algebra given by this vector space.

On the other hand, tensor products of a vector space with its obvious symmetric group action is Schur Weyl dual to the general linear group over this vector space. This duality quantizes to a duality between the Hecke algebra H_n attached to the symmetric group and the quantum group of the general linear Lie algebra which underlies the construction of all quantum link group invariants type A.

Hecke algebras. The Hecke algebras for all symmetric groups taken together form a braided monoidal category that controls these quantum link invariants and, by extension, the standard canon of topological quantum field theories in dimension 3 and 4. More precisely, there is a monoidal category $\mathcal{H}$ with

- Objects: the natural numbers $n \in \mathbb{N}_0$.
- Morphisms: the endomorphism algebra of each object $n \in H$ is H_n. All other hom-sets are trivial.
- The monoidal structure is given on objects by addition, i.e. $m \otimes n = m + n$, and on morphism as the map $H_m \times H_n \to H_{m+n}$ corresponding to the parabolic subgroup $S_m \times S_n \hookrightarrow S_{m+n}$.
- The braiding—a natural isomorphism $m \otimes n \cong n \otimes m$ for each $m, n \in H$—is given by the image in $\mathrm{End}_H(m+n) = H_{m+n}$ of the positive (m, n)-shuffle braid in the braid group Br_{m+n}.

Motivated by categorified knot invariants and by the desire to construct higher TQFTs one would like to categorify this structure, see [9] for some overview.

Braided monoidal 2-categories. The result of such a construction should be a version of a *braided monoidal 2-category.* The construction itself includes in particular a precise definition of the term *braided monoidal 2-category.* The notion of braided monoidal 2-categories has a long history, see e.g. [4, 5, 2, 1]. One difficulty is the precise formulation and packaging of the required data and higher coherences. The modern formulation is operadic and in terms of $\mathbb{E}_2$-algebras: an $\mathbb{E}_2$-algebra in categories is a braided monoidal category. The goal can be therefore reformulated as

*C*onstruct an $\mathbb{E}_2$-algebra in an appropriate higher category.

Background: Soergel's categorification Hecke algebra. The main result is based on the categorification theorem of Soergel, [8]: The additive monoidal category of (graded) *Soergel bimodules* categorifies the Hecke algebra in the sense that its split Grothendieck ring is isomorphic to the Hecke algebra. The indecomposable bimodules correspond hereby to the Kazhdan–Lusztig basis. The standard basis can be categorified by passing to complexes of Soergel bimodules. *Rouquier complexes*, [7], (as objects in the homotopy category of Soergel bimodules) can be used to categorify the standard basis.

The main result is now an upgrade of $\mathcal{H}$, where the morphism spaces are categorified. The Hecke algebras H_n are replaced by the homotopy categories $\mathcal{K}(\mathrm{SBim})$ of Soergel bimodules. This provides a monoidal 2-category. The main problem is

Construct a homotopy coherent of a braiding.

Formulation of the main result. We let st_k denote the ∞-category of small stable idempotent-complete k-linear ∞-categories. (Such ∞-categories can be modelled by small pretriangulated idempotent-complete k-linear dg-categories.) Day convolution induces a symmetric monoidal structure induces a symmetric monoidal structure on st_k (and even on a graded version of it) and also on the ∞-category $\mathrm{Cat}[\mathrm{st}_k]$ of small ∞-categories enriched in st_k in the sense of Gepner-Haugseng [3].

Theorem 1.

(1) There is a monoidal $(\infty, 2)$-category $\mathcal{K}(\mathrm{SBim})$ with objects labelled by natural numbers $n \in \mathbb{N}_0$ and whose endomorphim ∞-categories are the k-linear, stable, idempotent-complete ∞-categories $\mathcal{K}^b(\mathrm{SBim}_n)$ of chain complexes of Soergel bimodules, with a $\mathbb{Z}$-action by grading shift. More precisely, $\mathcal{K}(\mathrm{SBim})$ defines an $\mathbb{E}_1$-algebra in the symmetric monoidal ∞-category $\mathrm{Cat}[\mathrm{st}_k]$.

(2) There exists a unique braided monoidal, i.e., $\mathbb{E}_2$-algebra, structure on $\mathcal{K}(\mathrm{SBim}) \in \mathrm{Cat}[\mathrm{st}_k]$ that enhances its monoidal structure and satisfies
   (a) The fiber functor to st_k is braided monoidal.
   (b) The braiding $1 \otimes 1 \xrightarrow{\sim} 1 \otimes 1$ admits an equivalence with the Rouquier complex corresponding to the braid group generator $\sigma \in \mathrm{Br}_2$.

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Linkage principle for small quantum groups

Cristian Vay

We call "small quantum group" the Drinfeld double $\mathcal{D}_{\mathfrak{q}}$ of the bosonization of a finite-dimensional Nichols algebra $\mathfrak{B}_{\mathfrak{q}}$ of diagonal type over $\mathbf{k}\mathbb{Z}^\theta$; here $\mathbf{k}$ is a field and $\mathfrak{q} \in \mathbf{k}^{\theta\times\theta}$ gives the braiding of the Nichols algebra. For instance, if $\mathfrak{q} = (q^{d_i c_{ij}})_{i,j}$, where $C = (c_{ij})_{i,j}$ is a Cartan matrix symmetrizable by $D = (d_i)_i$ and q a root of unity of odd order, then $\mathfrak{B}_{\mathfrak{q}}$ is the positive part of the small quantum group introduced by Lusztig and associated to C. Moreover, the proper Lusztig's small quantum group is a quotient of $\mathcal{D}_{\mathfrak{q}}$ by a central group subalgebra.

A small quantum group admits a natural triangular decomposition:

$$\mathcal{D}_{\mathfrak{q}} = \mathcal{D}_{\mathfrak{q}}^- \otimes \mathcal{D}_{\mathfrak{q}}^0 \otimes \mathcal{D}_{\mathfrak{q}}^+ = \mathfrak{B}_{\mathfrak{q}^t} \otimes \mathbf{k}(\mathbb{Z}^\theta \times \mathbb{Z}^\theta) \otimes \mathfrak{B}_{\mathfrak{q}}.$$

Using this decomposition we can construct Verma modules as usual in Lie theory, and prove that the simple $\mathcal{D}_{\mathfrak{q}}$-modules are classified by the head of them. The "linkage principle" we alluded in the title provides us with information about the composition factors of the Verma modules. This is the main result of [3].

In the talk we exposed this principle and discuss how all the machinery of Nichols algebras of diagonal type (PBW basis, root system, Weyl groupoid, Lusztig isomorphisms) is used to prove it. It is worth noting that the ideas and techniques behind the proofs are borrowed from the work of Andersen, Jantzen and Soergel in the context of Lusztig's small quantum groups [1].

In order to state the linkage principle we need to introduce some notation. We fix a $\mathbb{Z}$-basis $\{\alpha_1, ..., \alpha_\theta\}$ of $\mathbb{Z}^\theta$. The matrix $\mathfrak{q}$ defines a bicharacter $\mathbb{Z}^\theta \times \mathbb{Z}^\theta \longrightarrow \mathbf{k}^\times$ which we denote also $\mathfrak{q}$. We denote $K_1, ..., K_\theta$ and $L_1, ..., L_\theta$ the generators of $\mathcal{D}_{\mathfrak{q}}^0$ which is the group algebra of $\mathbb{Z}^\theta \times \mathbb{Z}^\theta$. We think of it as a multiplicative group and write $K_\alpha = K_1^{n_1} \cdots K_\theta^{n_\theta}$ and $L_\alpha = L_1^{n_1} \cdots L_\theta^{n_\theta}$ if $\alpha = n_1\alpha_1 + \cdots + n_\theta\alpha_\theta \in \mathbb{Z}^\theta$.

We now recall the construction of the Verma modules and their simple quotients. Let $\pi : \mathcal{D}_{\mathfrak{q}}^0 \longrightarrow \mathbf{k}$ be an algebra map and $\mu \in \mathbb{Z}^\theta$. We define the algebra map $\pi\widetilde{\mu} : \mathcal{D}_{\mathfrak{q}}^0 \longrightarrow \mathbf{k}$ by $\pi\widetilde{\mu}(K_\alpha L_\beta) = \frac{\mathfrak{q}(\alpha,\mu)}{\mathfrak{q}(\mu,\beta)}\pi(K_\alpha L_\beta)$, $\alpha, \beta \in \mathbb{Z}^\theta$. We denote $\mathbf{k}^\mu = \mathbf{k}\,|\mu\rangle$ the one-dimensional $\mathbb{Z}^\theta$-graded $\mathcal{D}_{\mathfrak{q}}^0\mathcal{D}_{\mathfrak{q}}^+$-module concentrated in degree μ with action $su \cdot |\mu\rangle = \varepsilon(u)\,\pi\widetilde{\mu}(s)\,|\mu\rangle$ for all $s \in \mathcal{D}_{\mathfrak{q}}^0$ and $u \in \mathcal{D}_{\mathfrak{q}}^+$ where ε denotes the counit.

The Verma module associated to μ is the induced $\mathbb{Z}^\theta$-graded $\mathcal{D}_\mathfrak{q}$-module

$$Z_\mathbf{k}(\mu) = \mathcal{D}_\mathfrak{q} \otimes_{\mathcal{D}_\mathfrak{q}^0 \mathcal{D}_\mathfrak{q}^+} \mathbf{k}^\mu.$$

It has a unique simple quotient which is denoted $L_\mathbf{k}(\mu)$. Moreover, any simple module can be constructed in this way.

The linkage principle states the following:

If $L_\mathbf{k}(\lambda)$ is a composition factor of $Z_\mathbf{k}(\mu)$, then $\lambda = \mu$ or there exist $\beta_1, ..., \beta_r \in \Delta_+^\mathfrak{q}$ such that $\lambda = \beta_r \downarrow \cdots \beta_1 \downarrow \mu$.

Here, $\Delta_+^\mathfrak{q} \subset \mathbb{Z}_{\geq 0}^\theta$ denotes the root system of the Nichols algebra $\mathfrak{B}_\mathfrak{q}$ and the operation $\downarrow$ is defined as follows. Let $\rho^\mathfrak{q} : \mathbb{Z}^\theta \longrightarrow \mathbf{k}^\times$ be the group homomorphism such that $\rho^\mathfrak{q}(\alpha_i) = q_{ii}$. Thus, for $\beta \in \Delta_+^\mathfrak{q}$, we set $b^\mathfrak{q}(\beta) = \operatorname{ord} \mathfrak{q}(\beta, \beta)$ and define

$$\beta \downarrow \mu = \mu - n_\beta^\pi(\mu)\beta$$

where $1 \leq n_\beta^\pi(\mu) < b^\mathfrak{q}(\beta)$ is the unique natural number satisfying $\mathfrak{q}(\beta, \beta)^{n_\beta^\pi(\mu)} = \rho^\mathfrak{q}(\beta)\, \pi\widetilde{\mu}(K_\beta L_\beta^{-1})$, if it exists, and otherwise $n_\beta^\pi(\mu) = 0$.

A first consequence of this principle is that it gives rise to an equivalence relation in $\mathbb{Z}^\theta$ characterizing the blocks of the category of $\mathbb{Z}^\theta$-graded $\mathcal{D}_\mathfrak{q}$-modules. It is also the starting point to imagine character formulas for the simple modules. In this direction, we found a notion of (a)typicality analogous to the one in the representation theory of Lie superalgebras. The typical simple modules turn out to be the simple and projective Verma modules, and for 1-atypical simple modules we deduce a character formula.

Under certain assumption the linkage principle can be reformulated by replacing the operation $\downarrow$ with a true action, commonly called "dot action". For this purpose we have to consider the set $\Delta_{+,\mathrm{car}}^\mathfrak{q}$ of Cartan roots of $\mathfrak{B}_\mathfrak{q}$ and the associated reflection s_β, cf. [2]. We set $\varrho^\mathfrak{q} = \frac{1}{2}\sum_{\beta \in \Delta_+^\mathfrak{q}} (b^\mathfrak{q}(\beta) - 1)\beta$; notice this element is similar to the semi-sum of the positive roots in Lie theory but here the roots are scaled by their height in the PBW basis. Now, every $\beta \in \Delta_{+,\mathrm{car}}^\mathfrak{q}$ and $m \in \mathbb{Z}$ define an affine reflections $s_{\beta,m}$ determined by its (dot) action on $\mathbb{Z}^\theta$:

$$s_{\beta,m} \bullet \mu = s_\beta(\mu + mb^\mathfrak{q}(\beta)\beta - \varrho^\mathfrak{q}) + \varrho^\mathfrak{q}, \quad \mu \in \mathbb{Z}^\theta.$$

Let $\mathcal{W}_{\mathrm{link}}^\mathfrak{q}$ denote the group generated by all the affine reflections.

We recall that $\mathfrak{q}$ is of Cartan type if $\Delta_+^\mathfrak{q} = \Delta_{+,\mathrm{car}}^\mathfrak{q}$, and $\mathfrak{q}$ is of super type if its root system is isomorphic to the root system of a finite-dimensional contragredient Lie superalgebra in characteristic 0. If $\mathfrak{q}$ is of super type, then $\Delta_{+,\mathrm{odd}}^\mathfrak{q} := \Delta_+^\mathfrak{q} \setminus \Delta_{+,\mathrm{car}}^\mathfrak{q}$ is not empty and $\operatorname{ord} \mathfrak{q}(\beta, \beta) = 2$ if $\beta \in \Delta_{+,\mathrm{odd}}^\mathfrak{q}$; in this case $\beta \downarrow \mu = \mu$ or $\mu - \beta$.

Here it is the other formulation of the linkage principle:

If $\pi = \varepsilon$ and $L_\mathbf{k}(\lambda)$ is a composition factor of $Z_\mathbf{k}(\mu)$, then

(1) $\lambda \in \mathcal{W}_{\mathrm{link}}^\mathfrak{q} \bullet \mu$ if $\mathfrak{q}$ is of Cartan type.

(2) $\lambda \in \mathcal{W}_{\mathrm{link}}^\mathfrak{q} \bullet (\mu + \mathbb{Z}\Delta_{+,\mathrm{odd}}^\mathfrak{q})$ if $\mathfrak{q}$ is of super type.

We remark that (1) extends a similar result of [1] since here we can consider matrices of the form $(q^{d_i c_{ij}})_{i,j}$ with q of any order. Finally, we observe that we find in the setting of Lie superalgebras a phenomenon similar to (2), but here the Cartan roots play the role of the even roots.

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Nichols algebras: an overview

Leandro Vendramin

My presentation focuses on the *classification of finite-dimensional Nichols algebras, specifically, those over non-abelian groups.*

We begin by examining a fundamental concept in this classification: braided vector spaces. A *braided vector space* consists of a vector space (over a field K) equipped with a map $c \in \mathrm{GL}(V \otimes V)$ that satisfies the *braid equation*:

$$c_1c_2c_1 = c_2c_1c_2.$$

Here, $c_1 = c \otimes \mathrm{id}$ and $c_2 = \mathrm{id} \otimes c$.

The braided vector spaces under consideration are the Yetter–Drinfeld modules over the group algebra KG of a group G.

Given a braided vector space V, we can construct an algebra $\mathcal{B}(V)$, known as the *Nichols algebra* of V. It is defined as follows:

$$\mathcal{B}(V) = K \oplus V \oplus \bigoplus_{n\geq 2} V^{\otimes n} / \ker \mathcal{S}_n,$$

where

$$\mathcal{S}_{n+1} = (\mathrm{id} \otimes \mathcal{S}_n)(\mathrm{id} + c_1 + c_1c_2 + \cdots + c_1c_2\cdots c_n).$$

for all $n \geq 1$.

Complex finite-dimensional Nichols algebras over abelian groups were classified by Heckenberger in [7]. Consequently, my focus here shifts towards the case of non-abelian groups. In this context, the classification relies on a certain notion of reducibility of Yetter–Drinfeld modules. The exact notion of reducibility used will become apparent later.

Finite-dimensional Nichols algebras of "reducible" Yetter–Drinfeld modules with two irreducible summands was completely classified in [9].

Theorem 1 (with Heckenberger)**.** Let K be a field, G be a non-abelian group and V and W be absolutely irreducible Yetter-Drinfeld modules over KG. Assume that the support of $V \oplus W$ generates G, that $c_{W,V}c_{V,W} \neq \mathrm{id}$ and $\dim \mathcal{B}(V \oplus W) < \infty$. Then G is a quotient of a certain central extension of $\mathrm{SL}_2(3)$ or a quotient of

$$\langle g, h, \epsilon : hg = \epsilon gh, g\epsilon = \epsilon^{-1}g, h\epsilon = \epsilon h, \epsilon^n = 1\rangle$$

for $n \in \{2, 3, 4\}$.

In cases where $c_{W,V}c_{V,W} = \mathrm{id}$, it follows that $\mathcal{B}(V \oplus W) \simeq \mathcal{B}(V) \otimes \mathcal{B}(W)$ as vector spaces. Consequently, Theorem 1 explicitly assumes $c_{W,V}c_{V,W} \neq \mathrm{id}$ to ensure being in a proper "reducible" case

Theorem 1 holds independently of the characteristic of the field. Furthermore, it provides an explicit list of the finite-dimensional Nichols algebras relevant to this context.

The classification of Nichols algebras in the case of at least three irreducible summands was accomplished in [9], again without imposing any restrictions on the characteristic of the base field. In this scenario, the finite-dimensional Nichols algebras are classified using Dynkin diagrams of finite type. Similar to the case with two summands, explicit descriptions of the algebras are provided.

What can be said about Nichols algebras over irreducible Yetter-Drinfeld modules? Let us explore some results in the context of complex numbers.

The following result appears in [3] and [4].

Theorem 2 (with Andruskiewitsch, Fantino and Graña)**.** *Let G be a finite alternating simple group or an sporadic simple group different from Fi_{22}, B and M. If $0 \neq V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, then $\dim \mathcal{B}(V) = \infty$.*

The theorem suggests a challenging question: Are Nichols algebras over finite non-abelian simple groups infinite-dimensional?

Theorem 3 (Andruskiewitsch–Carnovale–García)**.** *Let G be one of the following groups: $\mathrm{PSL}_n(q)$ for $n \geq 4$, $\mathrm{PSL}_3(q)$ for $q > 2$, $\mathrm{PSp}_{2n}(q)$ for $n \geq 3$, or for q even one of the following groups: $P\Omega^+_{4n}(q)$, $P\Omega^-_{4n}(q)$, ${}^3D_4(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$. If $0 \neq V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, then $\dim \mathcal{B}(V) = \infty$.*

Theorem 3 was proved in [1, 2].

Theorem 4 (Carnovale–Costantini)**.** *Complex Nichols algebras over the Suzuki–Ree groups are infinite-dimensional.*

Theorem 4 was proved in [6].

Theorem 5 (with Heckenberger and Meir)**.** *Let G be a group and V be an irreducible Yetter-Drinfeld module over $\mathbb{C}G$ of prime dimension p. Assume that the support of V generates G. Then $\dim \mathcal{B}(V) < \infty$ if and only if $p \in \{3, 5, 7\}$.*

The finite-dimensional Nichols algebras appearing in the context of Theorem 5 are documented in literature; see for example [5].

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Arithmetic statistics via the cohomology of Nichols algebras

Craig Westerland
(joint work with Jordan Ellenberg, TriThang Tran)

This was an overview of the paper [ETW23] which employs algebro-topological and Hopf-algebraic methods to address a function field version of a conjecture of Malle on the distribution of Galois groups.

Let G be a transitive subgroup of S_m, and let $\overline{K}$ be a separable closure of K. Define a function which counts degree n extensions of K inside of $\overline{K}$ with Galois group G:

$$N_{G,K}(X) := \#\{L \leq \overline{K} \mid \deg(L/K) = m, \, \mathrm{Gal}(L/K) \cong G, \, |\Delta_{L/K}| \leq X\}. \tag{1}$$

Here the isomorphism $\mathrm{Gal}(L/K) \cong G$ is required to be one of groups acting on the set of m embeddings of L into $\overline{K}$. In [Mal04], Malle conjectures an asymptotic for this quantity:

$$N_{G,K}(X) \sim cX^a \log(X)^{b-1},$$

where the constants a and b are given in terms of the group theory of $G \leq S_m$ and the action of $\mathrm{Gal}(\overline{K}/K)$ on $G = \mathrm{Hom}(\widehat{\mathbb{Z}}, G)$ through the cyclotomic character. In joint work with Ellenberg and Tran, we proved an upper bound when K is taken to be the function field $\mathbb{F}_q(t)$:

Theorem 1. For each integer m and each transitive $G \leq S_m$, there are constants $C(G), Q(G)$, and $e(G)$ such that, for all $q > Q(G)$ coprime to $\#G$ and all $X > 0$,

$$N_{G,\mathbb{F}_q(t)}(X) \leq C(G)X^{a(G)} \log(X)^{e(G)}$$

Here the exponent $e(G)$ is always at least as large as Malle's $b - 1$.

The main distinction between the number field and function field settings is geometric. By definition, function fields are tied to the geometry of curves, and their extensions correspond to ramified covering maps between curves. Specifically, extensions $L/k(t)$ correspond to curves $\Sigma = \mathrm{Spec}(\mathcal{O}_L)$ defined over k, and maps $\Sigma \to \mathbb{A}^1_k = \mathrm{Spec}\, k[t]$. When $K = \mathbb{F}_q[t]$, the set being counted in (1) can be reinterpreted as the set of isomorphism classes of such branched covers. This, in turn, may be identified as the set of $\mathbb{F}_q$-points of a moduli space of branched covers.

Explicitly, we make the following definition: for $n \in \mathbb{Z}_{\geq 0}$, the *Hurwitz moduli space* $\mathcal{H}_{G,n}$ is a Deligne-Mumford stack whose k points (with $\mathrm{car}(k)$ coprime to $\#G$) parameterize isomorphism classes of the following data:

- Σ is a smooth projective curve, and $\pi : \Sigma \to \mathbb{P}^1_k$ is a branched cover:
- Away from a reduced divisor $D \subseteq \mathbb{P}^1_k$, π is a G-Galois cover.
- $\deg(D \cap \mathbb{A}^1) = n$.
- π is tamely ramified at D.

By replacing a G-Galois cover Σ with the degree m cover $\Sigma \times_G [m]$, we may show

$$N_{G,\mathbb{F}_q(t)}(X) = \sum_{|\Delta| \leq X} \#\mathcal{H}_{G,n}(\mathbb{F}_q).$$

where the sum is over those components of $\mathcal{H}_{G,n}$ whose associated extensions have discriminant less than X.

To prove Theorem 1, then, we must bound the quantity $\#\mathcal{H}_{G,n}(\mathbb{F}_q)$ as a function of n. To do this, we employ the Grothendieck-Lefschetz trace formula. In our setting, this is the statement that

$$\#\mathcal{H}_{G,n}(\mathbb{F}_q) = q^n \sum_{j=0}^{2n} (-1)^j \operatorname{tr}(\mathrm{Frob}_q \circlearrowleft H^i_{et}(\mathcal{H}_{G,n}|_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell))$$

By way of the Artin comparison theorem and Deligne's bounds on the eigenvalues of Frobenius, Theorem 1 follows if we can control the Betti numbers $r(j,n) = \mathrm{rk}_{\mathbb{Q}}(H^j_{sing}(\mathcal{H}_{G,n}(\mathbb{C}), \mathbb{Q}))$. Specifically, we must show that these grow at worst exponentially in j and polynomially in n.

This is now a purely algebro-topological problem. The function $\mathcal{H}_{G,n}(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C})$ which carries a branched covering to its branch locus is a covering space. Further, the configuration space $\mathrm{Conf}_n(\mathbb{C}) = K(B_n, 1)$ is an Eilenberg-MacLane space for the n^{th} *braid group* B_n. Thus the computation of $H^j_{sing}(\mathrm{Hur}_{G,n}(\mathbb{C}), \mathbb{Q})$ may be rephrased as a group cohomology computation for B_n with coefficients in the *Hurwitz representation.*

Using the Fox-Neuwirth/Fuks cellular stratification of $\mathrm{Conf}_n(\mathbb{C})$, we reformulate this computation in terms of the cohomology of the *quantum shuffle algebra* $T^{co}(V^*[1])$ associated to the G-Yetter-Drinfeld module $V := \mathbb{Q}[G \setminus \{1\}]$:

(2) $$r(j,n) \leq \mathrm{rk}\,\mathrm{Ext}^{n-j,n}_{T^{co}(V^*[1])}(\mathbb{Q}, \mathbb{Q}).$$

In the filtration of the shuffle algebra by powers of the augmentation ideal, the associated graded Hopf algebra decomposes as a twisted tensor product of the Nichols algebra $\mathfrak{B}(V^*[1])$ with a complementary subalgebra $\mathfrak{C}$. The associated

May and Cartan-Eilenberg spectral sequences allow us to reduce the computation in (2) to bounding the ranks of $\mathrm{Ext}^{n-j,n}_{\mathfrak{B}(V^*[1])}(\mathbb{Q},\mathbb{Q})$. The requisite bounds are obtained through study of the Koszul complex for the Nichols algebra, and the Conway-Parker result which controls the number of braid orbits in the Hurwitz representation.

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Tensor product decompositions for the smallest quantum coideal and Jones–Wenzl projectors

Zbigniew Wojciechowski

(joint work with Catharina Stroppel)

1. Motivation

Consider the Lie subalgebra $\mathfrak{g} := \left\{\left(\begin{smallmatrix} 0 & \lambda \\ \lambda & 0 \end{smallmatrix}\right)\right\} \subset \mathfrak{sl}_2(\mathbb{C})$. The standard vector representation $V = \mathbb{C}^2$ decomposes into two 1-dimensional subspaces $L_1 := \operatorname{span}\{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\}$, $L_{-1} := \operatorname{span}\left\{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\right\}$, which are eigenspaces for the generator $b := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{g}$ with eigenvalues 1 and -1. Let $n \in \mathbb{N}$ and consider $V^{\otimes n}$ instead. The previous decomposition implies that b acts semisimply on $V^{\otimes n}$ with eigendecomposition $V^{\otimes n} = \bigoplus_{k=0}^{n} \binom{n}{k} L_{n-2k}$: one possible eigenbasis is n-fold tensors of the above vectors $v = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$. This simple starting situation becomes intriguing when replacing $\mathfrak{sl}_2$ by $\mathrm{U}_q(\mathfrak{sl}_2)$. The Hopf algebra embedding $\mathrm{U}(\mathfrak{g}) \subset \mathrm{U}(\mathfrak{sl}_2)$ has two quantizations as left respectively right coideal subalgebras

$$'\,\mathrm{U}_q(\mathfrak{g}) \subset \mathrm{U}_q(\mathfrak{sl}_2) \xleftarrow{\text{left coideal}} \mathrm{U}(\mathfrak{g}) \subset \mathrm{U}(\mathfrak{sl}_2) \xrightarrow{\text{right coideal}} \mathrm{U}_q'(\mathfrak{g}) \subset \mathrm{U}_q(\mathfrak{sl}_2)$$

generated by $B = E + qKF$ respectively $B = q^{-1}EK^{-1} + F$. Since these are not Hopf algebra embeddings the argument giving the decomposition of $V^{\otimes n}$ does not carry over. Instead of being a monoidal category, representations of $'\,\mathrm{U}_q(\mathfrak{g})$ and $\mathrm{U}_q'(\mathfrak{g})$ have a non-trivial structure as left respectively right module categories over $\mathrm{U}_q(\mathfrak{sl}_2)$. Our result is the diagrammatic description of those categories by generalized string calculus, where the decomposition is governed by idempotents which are type B/D analogues of Jones–Wenzl projectors.

2. Results

We work with the right coideal $\mathrm{U}_q'(\mathfrak{g})$. Let $V = \mathbb{C}(q)^2$ be the type 1 quantized standard representation. The first theorem gives the quantum coideal version of the decomposition of $V^{\otimes n}$ and is the combinatorial starting point:

Theorem 1 ([3], Thm 2.6)**.** Consider the family of quantizations of v, w given by

$$v_n = \begin{pmatrix} 1 \\ q^n \end{pmatrix}, \quad w_n = \begin{pmatrix} 1 \\ -q^{-n} \end{pmatrix}, \text{ where } n \in \mathbb{Z}.$$

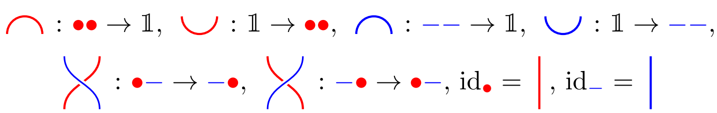
The representation V decomposes over $\mathrm{U}_q'(\mathfrak{g})$ into summands

$$V = L_{[1]} \oplus L_{-[1]} := \operatorname{span}\{v_0\} \oplus \operatorname{span}\{w_0\}.$$

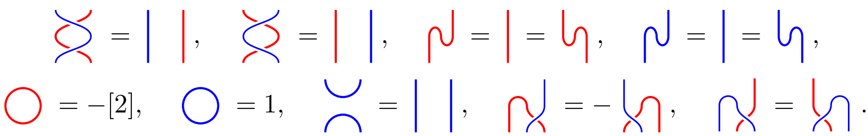
Let z be an eigenvector for B with eigenvalue $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. Then $z \otimes v_{n,\varepsilon}$ and $z \otimes w_{n,\varepsilon}$ are eigenvectors for B with eigenvalue $[n+1]$ and $[n-1]$ respectively. In particular we have the decomposition $V^{\otimes n} = \bigoplus_{k=0}^{n} \binom{n}{k} L_{[n-2k]}$.

Type 1 representations of $\mathrm{U}_q(\mathfrak{sl}_2)$ are described by the Temperley–Lieb category. The following theorem is a slight generalization that allows us to work with all finite dimensional representations (both type 1 and -1):

Theorem 2 ([3], Thm 4.6)**.** Let $\mathrm{Kar}(\mathrm{Add}(\mathscr{V}))$ be the Karoubian envelope of the additive envelope of the k-linear monoidal category $(\mathscr{V}, \otimes, \mathbb{1})$ generated by two objects $\bullet$ and $-$, and morphisms



subject to the relations:

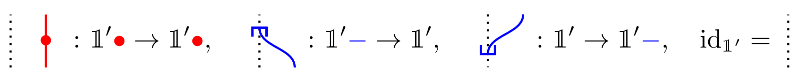


We have an equivalence of monoidal categories

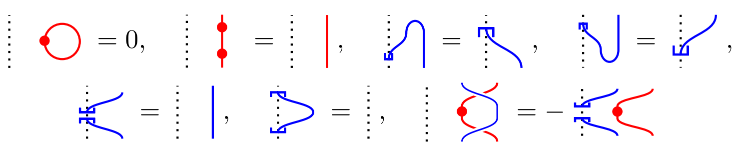
$$(\mathbf{Rep}^{\text{f.d.}}(\mathrm{U}_q(\mathfrak{sl}_2)), \otimes_k, k) \simeq (\mathrm{Kar}(\mathrm{Add}(\mathscr{V})), \otimes, \mathbb{1}).$$

Denote by $\mathbf{Rep}^{\text{f.d.}}(\mathrm{U}_q'(\mathfrak{g}), \mathrm{U}_q(\mathfrak{sl}_2)) \subset \mathbf{Rep}^{\text{f.d.}}(\mathrm{U}_q'(\mathfrak{g}))$ the subcategory of all objects which appear as direct summand of a restriced $\mathrm{U}_q(\mathfrak{sl}_2)$-representation. This subcategory is naturally a right module category over $(\mathbf{Rep}^{\text{f.d.}}(\mathrm{U}_q(\mathfrak{sl}_2)), \otimes_k)$. The following theorem describes it under the equivalence in Theorem 2.

Theorem 3 ([3], Thm 4.11)**.** Let $(\mathscr{M}, \otimes\colon \mathscr{M} \times \mathscr{V} \to \mathscr{M})$ be the k-linear right module category over $\mathscr{V}$ generated by one object $\mathbb{1}'$ and morphisms



subject to the relations:



We have an equivalence of right module categories

$$(\mathbf{Rep}^{\text{f.d.}}(\mathrm{U}_q'(\mathfrak{g}), \mathrm{U}_q(\mathfrak{sl}_2)), \otimes_k) \simeq (\mathrm{Kar}(\mathrm{Add}(\mathscr{M})), \otimes).$$

Our work relies on the isomorphism $\mathrm{End}_{\mathrm{U}_q(\mathfrak{g})}(V^{\otimes n}) \cong \mathrm{TL}(B_n)$ and the diagrammatics for $\mathrm{TL}(B_n)$ in [1]. In [3], Theorem 5.12 we connect Theorem 1 and Theorem 3 via type B/D analogues of Jones–Wenzl projectors. These idempotents appear in a seperate topological context in [2]. The study of their diagrammatic properties including counting formulas for morphism spaces are the topic of the rest of our paper ([3], §5 and §6).

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Poisson orders on large quantum groups

Milen Yakimov
(joint work with Nicolás Andruskiewitsch, Iván Angiono)

We report on the results in [2] where we develop a Poisson geometric framework for studying the representation theory of all contragredient quantum super groups at roots of unity. This is done in a uniform fashion by treating the larger class of quantum doubles of bozonizations of all distinguished pre-Nichols algebras [1] associated to the ones in the celebrated classification of Heckenberger [9, 10] and belonging to a one-parameter family. We call these algebras *large* quantum groups. For a corresponding braiding matrix $\mathfrak{q}$, subject to a mild technical hypothesis, we have the chain of algebras

$$U_{\mathfrak{q}} \supset U_{\mathfrak{q}}^{\geqslant} \supset U_{\mathfrak{q}}^{+},$$

a large quantum group, and corresponding quantum Borel subgroup and quantum unipotent subgroup. There is a canonical chain of central subalgebras

$$Z_{\mathfrak{q}} \supset Z_{\mathfrak{q}}^{\geqslant} \supset Z_{\mathfrak{q}}^{+},$$

constructed by Angiono [1], where in the super A case we extend those in [1] by an extra generator to ensure module finiteness of the corresponding noncommutative algebra over the central subalgebra.

The following notion was axiomatized by Brown and Gordon [5] following the fundamental work of De Concini, Kac and Procesi [7].

Definition. A pair of $\mathbb{C}$-algebras (R, Z) is called a *Poisson order* if Z is a central subalgebra of R, R is a finitely generated Z-module and the following conditions hold:

(a) Z is equipped a structure of Poisson algebra $\{\cdot,\cdot\}$;
(b) There exists a linear map $D : Z \to \mathrm{Der}_{\mathbb{C}}(R)$ such that $D_z|_Z = \{z, -\}$ for all $z \in Z$.

The key application of Poisson orders is the following theorem of Brown and Gordon [5], which extended related results of De Concini, Kac and Procesi [6, 7, 8] on quantum groups at roots of unity [7].

Theorem. [5, Theorem 4.1] *Let (R, Z) be a Poisson order and $M := \operatorname{Maxspec} Z$. Given $x \in M$ with maximal ideal $\mathfrak{M}_x$, let $R_x := R/\mathfrak{M}_x R$, a finite dimensional algebra. If x and y belong to the same symplectic core, then $R_x \simeq R_y$ as algebras.*

Our main results are summarized as follows:

Main Theorem. *For all large quantum groups $U_\mathfrak{q}$, the following hold:*

(1) *The pairs $(U_\mathfrak{q}, Z_\mathfrak{q})$, $(U_\mathfrak{q}^{\geqslant}, Z_\mathfrak{q}^{\geqslant})$ and $(U_\mathfrak{q}^{+}, Z_\mathfrak{q}^{+})$ have Poisson order structures in the sense of* [5] *obtained from specialization.*

(2) *We have*

$$\operatorname{MaxSpec} U_\mathfrak{q} \cong B_\mathfrak{q}^{+} \times B_\mathfrak{q}^{-}, \quad \operatorname{MaxSpec} U_\mathfrak{q}^{\geq} \cong B_\mathfrak{q}^{+}, \quad \operatorname{MaxSpec} U_\mathfrak{q}^{+} \cong G_\mathfrak{q}/B_\mathfrak{q}^{+},$$

where $B_\mathfrak{q}^{\pm}$ are opposite Borel subgroups of an explicit semisimple algebraic group $G_\mathfrak{q}$ of adjoint type.

(3) *The Poisson structures on the first two Poisson algebraic groups are the dual ones of the Poisson algebraic groups of $G_\mathfrak{q}$ and $B_\mathfrak{q}^{-}$ with (nonstandard) Poisson structures from the Belavin–Drinfeld classification* [3] *with "empty BD triples." The Poisson structure on the third one is the push forward to the flag variety of the Poisson structure on $G_\mathfrak{q}$.*

(4) *The symplectic foliations in the three cases are given in terms of twisted conjugacy classes, double Bruhat cells and open Richardoson varieties. The corresponding fiber algebras are isomorphic to each other across those sets.*

The above theorem presents new results already in the case of big quantum groups considered by De Concini, Kac and Procesi [6, 7, 8] because we do not place any conditions on the root of unity. Interestingly, depending on the order of the root of unity one often needs to go to the Langlands dual group. Besides all (multiparameter) big quantum groups of De Concini–Kac–Procesi and big quantum super groups at roots of unity, our framework also contains the quantizations in characteristic 0 of the 34-dimensional Kac-Weisfeiler Lie algebras [11] in characteristic 2 and the 10-dimensional Brown Lie algebras [4] in characteristic 3. The previous approaches to the above problems relied on reductions to rank two cases and direct calculations of Poisson brackets, which is not possible in the super case since there are 13 kinds of additional Serre relations on up to 4 generators. We use a new approach that relies on perfect pairings between restricted and nonrestricted integral forms.

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Reporter: Thomas Letourmy

Participants

Prof. Dr. Nicolás Andruskiewitsch
FAMAF–CIEM (CONICET)
Universidad Nacional de Córdoba
Medina Allende s/n
5000 Córdoba
ARGENTINA

Prof. Iván E. Angiono
FAMAF
Universidad Nacional de Córdoba
Medina Allende s/n
5000 Córdoba
ARGENTINA

Dr. Giovanna Carnovale
Dipartimento di Matematica
Tullio Levi-Civita
Universita di Padova
Via Trieste, 63
35121 Padova
ITALY

Kevin Yaolin Chang
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Francesco Esposito
Dipartimento di Matematica
Universita di Padova
Via Trieste, 63
35121 Padova
ITALY

Dr. Sebastian Halbig
Fachbereich Mathematik und Informatik
Philipps-Universität Marburg
Hans-Meerwein-Straße
35032 Marburg
GERMANY

Prof. Dr. István Heckenberger
Fachbereich Mathematik und Informatik
Philipps-Universität Marburg
Hans-Meerwein-Straße
35032 Marburg
GERMANY

Prof. Dr. Simon David Lentner
Fachbereich Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Thomas Letourmy
Dept. de Mathématiques
Université Libre de Bruxelles
CP 216 Campus Plaine
Bd. du Triomphe
1050 Bruxelles
BELGIUM

Fengchang Li
Fachbereich Mathematik
Universität Marburg
Hans-Meerwein-Str.
35043 Marburg
GERMANY

Dr. Elisabetta Masut
Dipartimento di Matematica
Universita di Padova
Via Trieste, 63
35121 Padova
ITALY

Dr. Ehud Meir
Department of Mathematics
University of Aberdeen
Aberdeen AB24 3UE
UNITED KINGDOM

Dr. Julia Pevtsova
Department of Mathematics
University of Washington
P.O. Box 354350
Seattle, WA 98195-4350
UNITED STATES

Dr. Lleonard Rubio y Degrassi
Matematiska Institutionen
Uppsala Universitet
Box 480
751 06 Uppsala
SWEDEN

Prof. Dr. Olivier Schiffmann
Département de Mathématiques
Université de Paris-Saclay
Bâtiment 307
91405 Orsay Cedex
FRANCE

Prof. Dr. Catharina Stroppel
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Cristian Vay
FAMAF–CIEM (CONICET)
Universidad Nacional de Córdoba
Medina Allende s/n
5000 Córdoba
ARGENTINA

Prof. Dr. Leandro Vendramin
Department of Mathematics (WIDS)
Vrije Universiteit Brussel
Pleinlaan 2
1050 Brussels
BELGIUM

Prof. Dr. Craig Westerland
School of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis, MN 55455-0436
UNITED STATES

Zbigniew Wojciechowski
Institut für Geometrie
Technische Universität Dresden
Zellescher Weg 25
01217 Dresden
GERMANY

Prof. Dr. Milen Yakimov
Northeastern University
Department of Mathematics
Boston, MA 02115
UNITED STATES