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Analysis, Geometry and Topology of Positive Scalar Curvature Metrics

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Abstract. Riemannian metrics with positive scalar curvature play an important role in differential geometry and general relativity. To investigate these metrics, it is necessary to employ concepts and techniques from global analysis, geometric topology, metric geometry, index theory, and general relativity. This workshop brought together researchers from a variety of backgrounds to combine their expertise and promote cross-disciplinary exchange.

Mathematics Subject Classification (2020): Primary: 53C20. Secondary: 53C21, 53C27, 58D27, 53E10, 53E20, 58J20, 58J32, 19K56.

Introduction by the Organizers

The workshop *Analysis, Geometry and Topology of Positive Scalar Curvature Metrics*, organized by Bernd Ammann (Regensburg), Bernhard Hanke (Augsburg) and Anna Sakovich (Uppsala), was the fourth in a series of workshops with the same title, the previous ones taking place in 2014, 2017 and 2021. It was attended by 48 participants in person, mainly from Europe, the USA and Asia, and 13 virtual participants from Europe, the USA and China. Both numbers include some graduate students and postdoctoral researchers.

On Monday morning, Christian Bär (Potsdam) and Piotr Chruściel (Vienna) gave two extended 80-minute survey talks entitled *Rigidity for scalar curvature* and *Mathematical relativity, this and that*, highlighting two important pillars in scalar curvature geometry that were taken up in different variations by many of the talks that followed.

These included recent results on scalar curvature rigidity on convex polytopes, a Llarull-type comparison theorem for possibly non-spin oriented 4 manifolds, the relation of band width to the Rosenberg index, and scalar curvature rigidity of manifolds with conical singularities.

Continuing a theme highlighted in Piotr Chruściel's presentation, a number of talks focused on topics related to the positive mass conjecture in general relativity. In particular, we heard about a variant of the mass for asymptotically hyperbolic manifolds defined using renormalized volume, we listened to talks about the relation of positive mass theorem to non-uniqueness of Ricci flow, about the proof of Riemannian Penrose inequality using potential theory, about estimates for the Bartnik mass, and about new results on positivity of mass for spin initial data sets. Another talk discussed a common ingredient of all positive mass theorems, namely the dominant energy condition, from both a spacetime and initial data set perspective. In a different talk, a summary of recent results for spacetimes of low regularity featuring related curvature bounds was presented. The initial value problem in general relativity received further attention in subsequent talks, in particular a gluing construction yielding initial data that collapses to a black hole in finite time was presented, and drawbacks of the conformal method that aims to parametrise the set of all initial data were discussed.

A number of talks were devoted to the application of geometric flows to the recent solution of the Hamilton-Lott conjecture and its possible extension to higher dimensions, to scalar curvature extremality and rigidity with non-smooth metrics, to unstable Einstein 4-manifolds, and to the geometry of $(3+1)$ Minkowski spacetime.

Spaces and moduli spaces of Riemannian metrics with positive scalar curvature are classical topics that still attract a lot of attention. This was addressed by several talks using advanced bordism-theoretic techniques and family versions of Seiberg-Witten invariants, respectively. One talk presented an extension of the Lichnerowitz obstruction to a family of intermediate curvature conditions and studied their behavior under surgery, leading to new invariants of rational bordism. A gluing technique for manifolds with corners under positive scalar and mean curvature assumptions was the topic of another talk, with applications to minimal concordance.

Questions of extremality and rigidity are naturally related to questions of stability, identifying suitable topologies in which non-extremal situations converge to extremal or rigid ones. This topic has attracted a lot of attention recently, which was reflected in a survey lecture on tools, theorems and questions in scalar curvature stability, and a lecture on limits of sequences of manifolds with nonnegative scalar curvature. Somewhat related in spirit was another talk on the solution of Schoen's conjecture in the presence of a noncompact area-minimizing boundary, which arises as a limit of isoparametric surfaces.

As the thematic field of the conference was very active in the last years, with new ideas and important progress in several directions, we decided to have a total of 26 reaseach talks (in addition to the 2 survey talks), in order to reflect a large spectrum of new developments and also to let some younger researchers to present their work. Most of the research talks were 40 or 50 minutes long, but we also had 25 minute talks contributed by younger participants. Similar to the previous meetings in this series, we observed an intense interaction of scientists with different mathematical backgrounds throughout the workshop, integrating both face-to-face and online participants.

We could always rely on the perfect working conditions at the Oberwolfach institute and the great support by its staff. In particular we appreciated the possibility to invite a significant number of in-person participants who enjoyed the traditional, stimulating Oberwolfach atmosphere, which was indispensible for the success of our workshop.

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Workshop: Analysis, Geometry and Topology of Positive Scalar Curvature Metrics

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Abstracts

Scalar Curvature Stability: Tools, Theorems, and Questions

Brian Allen
(joint work with E. Bryden, D. Kazaras, R. Perales, and C. Sormani)

1. Introduction

Although scalar curvature is the simplest curvature invariant, our understanding of scalar curvature has not matured to the same level as Ricci or sectional curvature. Despite this fact, many rigidity phenomenon have been established which give some of the strongest insights into scalar curvature. Important examples include Geroch's conjecture, the positive mass theorem, and Llarull's theorem. In order to further understand scalar curvature we ask corresponding geometric stability questions, where the hypotheses of the rigidity phenomenon are relaxed, and one would like to show that Riemannian manifolds which satisfy the relaxed conditions are close to the rigid objects in some topology. In this talk we will survey what is known for scalar curvature stability, discuss what the questions are in this area, and introduce important tools which have been useful so far.

2. Scalar Curvature Phenomenon

When exploring sequences of Riemannian manifolds with scalar curvature lower bounds we see that splines, bubbles, and drawstrings, depicted in Figure 2, are persistent phenomenon which need to either be ruled out by making an assumption or addressed by the choice of topology one makes when choosing an appropriate notion of convergence. Examples with splines and bubbles are constructed using Gromov-Lawson tunnels with estimates. Careful constructions of this type have been carried out by Basilio, Dodziuk, and Sormani [12], Basilio and Sormani [13], Dodziuk [19], and Sweeney [29]. If one allows a sequence of manifolds which are non-diffeomorphic to the limit manifold then Basilio and Sormani [13] have also shown that sewing is possible. The existence of drawstrings in dimensions $n \geq 4$ with scalar curvature bounds was constructed by Lee, Naber, and Neumayer [33] and constructed in dimension $n = 3$ by Kazaras and Xu [30].

Due to the presence of splines in all scalar curvature stability problems we know that Gromov-Hausdorff stability is not appropriate since a sequence of manifolds with increasingly many splines cannot converge in the Gromov-Hausdorff sense by Gromov's compactness theorem [16]. Hence we are left to look for other notions of convergence for sequences of Riemannian manifolds. In this note we will give examples of three different choices of convergence one can use. We emphasize that there is no clear hierarchy between the three choices we have and hence it is the philosophy of the author that our goal should be to prove all three notions of convergence for most scalar curvature rigidity results.

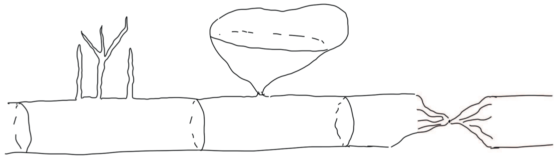

Figure 1. From left to right, splines, bubbling, and a drawstring where a circle has been pulled to almost a point. Along such a sequence the splines would become arbitrarily thin, the neck of the bubble would pinch to a point, and the circle would be pulled to a point by the drawstring.

3. Geroch Conjecture Stability

We start our exploration by considering the Geroch conjecture rigidity result.

Theorem 1 (Schoen and Yau [36], Gromov and Lawson [24])**.** *If $(\mathbb{T}^n, g)$ is a Riemannian metric with non-negative scalar curvature then $(\mathbb{T}^n, g)$ is isometric to a flat torus.*

Various special cases of Geroch stability have already been addressed. Single and double warped products were addressed by Allen, Vazquez, Parise, Payne, and Wang [3], the graph case was studied by Pacheco, Ketterer, and Perales [35], and conformal cases by Allen [2], and Chu and Lee [17]. In these special cases, Sormani-Wenger Intrinsic Flat (see Sormani and Wenger [39] for the definition) or Gromov-Hausdorff convergence was established. In general, since splines, bubbles, and drawstrings are possible in the case of Geroch stability, one needs to add a hypothesis to remove two of the three scalar curvature phenomenon to show stability. In the following stability result, the authors choose to assume an entropy bound, which rules out splines and bubbles, and show d_p stability to a flat torus. One can find the definition of d_p convergence in [33] where it is demonstrated that d_p convergence is resilient to drawstrings. See [8] for an explanation of the fact that d_p convergence is not well suited in the presence of splines and bubbles.

Theorem 2 (Lee, Naber, and Neumayer [33])**.** *Fix $n \geq 2$ and $p \geq n+1$. There exists a $\delta = \delta(n,p) > 0$ and $V_0 = v_0(n,p) > 0$ such that the following holds: For any $V > V_0$ and $(\mathbb{T}^n, g_j)$ a sequence of Riemannian tori such that*

$$R_{g_j} \geq -\frac{1}{j}, \qquad \mathrm{Vol}(\mathbb{T}^n, g_j) \leq V_0, \qquad \nu(\mathbb{T}^n, g_j) \geq -\delta, \tag{1}$$

a subsequence of $(\mathbb{T}^n, g_j)$ converges to a flat torus $(\mathbb{T}^n, g_F)$ in the d_p sense.

Question 1. *Given that the entropy bound of Theorem 2 removes the possibility of splines and bubbles forming along a sequence, can we formulate and prove Geroch stability where one removes the possibility of bubbles and drawstrings and shows volume preserving Sormani-Wenger Intrinsic Flat convergence to a flat torus?*

4. Positive Mass Theorem Stability

Theorem 3 (Schoen and Yau [8], Witten [40]). *If (M^3, g) is an asympotically flat manifold with non-negative scalar curvature then the ADM mass is non-negative. If the ADM mass is zero then (M^3, g) is isometric to Euclidean space.*

The stability of the positive mass theorem has been studied in many special cases. Assuming the existence of a smooth IMCF by Allen [4–7], in terms of the Brown-York Mass for graphs in Euclidean space by Alaee, Cabrera Pacheco, and McCormick [1], metrics conformal to Euclidean space by Corvino [18], Lee [31], axially symmetric metrics by Bryden [15], using spinors by Finster and Bray [14], Finster and Kath [22], and Finster [21], in the geometrostatic setting by Stavrov-Allen and Sormani [38], and the rotationally symmetric setting by Lee and Sormani [32]. In these special cases, many different notions of convergence were considered which generally do not imply Gromov-Hausdorff convergence due to the presences of splines.

In the following result, the authors address a conjecture by Huisken and Ilmanen [28] where the philosophy is to remove all of the scalar curvature phenomenon depicted in Figure 2 in a bad set and show that the remaining set converges to Euclidean space.

Theorem 4 (Dong and Song [20]). *Let (M_i^3, g_i) be a sequence of asymptotically flat Riemannian manifolds with non-negative scalar curvature and suppose that the ADM mass $m_{ADM}(M_i, g_i) \searrow 0$. Then for all $i \in \mathbb{N}$ and each end in M_i, there is a domain $Z_i \subset M_i$ with smooth boundary so that $|\partial Z_i|_{g_i} \searrow 0$, $M_i \setminus Z_i$ contains the given end, and*

$$(M_i \setminus Z_i, \hat{d}_{g_i}, p_i) \to (\mathbb{R}^3, d_{\mathbb{E}^3}, 0), \tag{2}$$

in the pointed measured Gromov-Hausdorff sense, where $p_i \in M_i \setminus Z_i$ is any choice of base point, and $\hat{d}_{g_i}$ is the restricted length metric on $M_i \setminus Z_i$ induced by g_i.

Question 2. *Can one formulate and prove a version of Geroch stability in the style of Theorem 4?*

Question 3. *In light of Theorem 4, one can ask to say more about what is happening on the bad set Z_i. For instance, is Z_i made up of splines, bubbles, and drawstrings only? One way of providing some evidence in this direction is to formulate a stability conjecture where one assumes a condition which rules out splines and bubbles and concludes d_p stability and another version of a stability conjecture where one rules out bubbles and drawstrings which concludes volume preserving Sormani-Wenger Intrinsic Flat convergence. In concert, these three stability results would be giving strong evidence that the bad sets Z_i are made up of only splines, bubbles, and drawstrings, reinforcing the belief that these are the only phenomenon which one needs to worry about when discussing sequences with scalar curvature lower bounds (note that sewing is also possible if one allows the topology of the sequence to vary).*

5. Llarull Stability

We now explore our last scalar rigidity theorem example. Here we state a slightly less general version of Llarull's rigidity theorem.

Theorem 5 (Llarull [11]). *If $(\mathbb{S}^n, g)$ is a Riemannian metric so that $g \geq g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the round sphere, whose scalar curvature $R_g \geq n(n-1)$ then $(\mathbb{S}^n, g)$ is isometric to $(\mathbb{S}^n, g_{\mathbb{S}^n})$.*

In this case, when we consider the stability of Theorem 5 we notice that drawstrings are not possible due to the metric lower bound assumption which is necessary in the rigidity theorem. Hence, volume preserving Sormani-Wenger Intrinsic Flat convergence is most appropriate for stability in this case. In fact, any time that rigidity is phrased in terms of a 1−Lipschitz map $F : (M, g) \to (N, h)$ we claim that volume preserving Sormani-Wenger Intrinsic Flat convergence will be the correct choice for proving the corresponding stability result.

Theorem 6 (Allen, Bryden, and Kazaras [9]). *Let $V, D, \overline{m}, \Lambda > 0$. If a sequence $\{(\mathbb{S}^3, g_i)\}_{i=1}^{\infty}$ of Riemannian 3-spheres satisfies*

$$g_i \geq g_{\mathbb{S}^3}, \qquad \mathrm{Vol}(\mathbb{S}^3, g_i) \leq V, \tag{3}$$

$$\mathrm{Diam}(\mathbb{S}^3, g_i) \leq D, \qquad \inf_{\Omega \subset \mathbb{S}^3} \frac{\mathrm{Area}(\partial\Omega, g_i)}{\min(\mathrm{Vol}(\Omega, g_i), \mathrm{Vol}(\mathbb{S}^3 \setminus \Omega, g_i))} \geq \Lambda, \tag{4}$$

and

$$\left\| (6 - R_{g_i})^+ \right\|_{L^2(g_i)}^{1/2} \to 0, \tag{5}$$

then it converges in the volume preserving Sormani-Wenger Intrinsic Flat sense:

$$d_{\mathcal{VF}}((\mathbb{S}^3, g_i), (\mathbb{S}^3, g_{\mathbb{S}^3})) \to 0. \tag{6}$$

The proof uses two main tools from the literature: spacetime harmonic functions developed by Hirsch, Kazaras, and Khuri [25] and used to give a new proof of Llarull's theorem by Hirsch, Kazaras, Khuri, and Zang [26] and the Volume Above Distance Below (VADB) theorem of Allen, Perales, and Sormani [11]. Shortly after Theorem 6 was established a more general result was established using the spinor formulas which Llarull used to establish stability and the VADB theorem.

Theorem 7 (Hirsch and Zhang [27]). *Let $V, D, \overline{m}, \Lambda > 0$. If a sequence $\{(\mathbb{S}^n, g_i)\}_{i=1}^{\infty}$, $n \geq 3$ of Riemannian n-spheres satisfies*

$$g_i \geq g_{\mathbb{S}^n}, \qquad \mathrm{Diam}(\mathbb{S}^n, g_i) \leq D, \qquad \inf_{u \in W^{1,2}(\mathbb{S}^n)} \frac{\|\nabla u\|^2_{L^2(\mathbb{S}^n)}}{\inf_{k\in\mathbb{R}} \|u-k\|^2_{L^2(\mathbb{S}^n)}} \geq \Lambda, \tag{7}$$

and

$$R_{g_i} \geq n(n-1) - \frac{1}{i} \tag{8}$$

then it converges in the volume preserving Sormani-Wenger Intrinsic Flat sense:

$$d_{\mathcal{VF}}((\mathbb{S}^n, g_i), (\mathbb{S}^n, g_{\mathbb{S}^n})) \to 0. \tag{9}$$

It should be noted that Hirsch and Zhang prove two other versions of Llarull stability, one with an L^p assumption of the negative part of the scalar curvature below $n(n-1)$ and another result with a good set bad set decomposition similar to Theorem 4.

Question 4. *Can one use spinors in order to establish Geroch stability and positive mass theorem stability in dimensions $n \geq 3$?*

Question 5. *Can one generalize Theorem 7 to the case of the results of Goette and Semmelmann* [23]*?*

6. Volume Above Distance Below Theorem

We end this note by stating and discussing the VADB theorem which has proved useful so far in proving stability of Llarull as well as many special cases of Geroch stability and positive mass thoerem stability.

Theorem 8 (Allen, Perales, and Sormani [11])**.** *Suppose we have a fixed compact, oriented, Riemannian manifold, (M^m, g_0), without boundary and a sequence of continuous Reimannian manifolds (M, g_j) such that*

$$g_j(v,v) \geq (1 - C(j))\, g_0(v,v), \quad \forall p \in M, v \in T_pM, \quad C(j) \searrow 0, \tag{10}$$

and a uniform upper bound on diameter, $\mathrm{Diam}(M_j) \leq D_0$*, and volume convergence*

$$\mathrm{Vol}(M, g_j) \to \mathrm{Vol}(M, g_0), \tag{11}$$

then we find volume preserving Sormani-Wenger Intrinsic flat convergence

$$d_{\mathcal{VF}}((M, g_j), (M, g_0)) \to 0. \tag{12}$$

It should also be noted that Allen and Perales [10] have proved a version of Theorem 8 for manifolds with boundary. Hence one can use the VADB theorem with boundary to discuss stability of scalar curvature rigidity results on manifolds with boundary.

Question 6. *Can one prove a version of Theorem 8 for sequences which are not diffeomorphic to the limit manifold? Addressing this question would allow us to extend Theorem 6 and Theorem 7 to $1-$Lipschitz maps from manifolds which are not diffeomorphic to the sphere.*

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Rigidity for scalar curvature

Christian Bär

(joint work with Simon Brendle, Bernhard Hanke, Yipeng Wang)

In this talk we survey some older and some recent results about rigidity phenomena for scalar curvature. There is no claim to completeness of the list of results.

The setup is as follows: By M and M_0 we will always denote manifolds which are connected, oriented, and of dimension n. We will fix the Riemannian manifold (M_0, g_0), and we want to know if each smooth spin map

$$\Phi\colon (M,g) \to (M_0,g_0)$$

with $\deg(\Phi) \neq 0$ and Lipschitz constant 1 must be an isometry, provided $S \geq S_0 \circ \Phi$. Here S stands for the scalar curvature of g and S_0 for the one of g_0. In this case, we say that (M_0, g_0) is rigid for scalar curvature.

Example. Let $M_0 = S^2$ equipped with any metric of positive scalar curvature S_0. In dimension 2, scalar curvature is the same as Gauss curvature, up to factor of 2. From $S \geq S_0 \circ \Phi$ we deduce that M also has positive curvature. By the Gauss-Bonnet theorem, M must be diffeomorphic to S^2 as well. W.l.o.g. let $\deg(\Phi) \geq 1$; otherwise reverse the orientation of M. Denoting the area 2-forms of M and M_0 by dA and dA_0, respectively, we can write $\Phi^* dA_0 = f dA$ for some function $f\colon M \to \mathbb{R}$. The Lipschitz property of Φ implies $|f| \leq 1$. Applying the Gauss-Bonnet theorem to M and to M_0 we find

$$\deg(\Phi)\cdot 8\pi = \deg(\Phi)\int_{M_0} S_0\, dA_0 = \int_M (S_0\circ\Phi)\Phi^* dA_0 \leq \int_M S\, dA = 8\pi.$$

This implies $\deg(\Phi) = 1$, $S = S_0 \circ \Phi$ and $\Phi^* dA_0 = dA$. The last fact, together with the Lipschitz-1-property, implies that Φ is a local isometry. Since its degree is 1, it is an isometry. This shows that S^2 with any metric of positive curvature is rigid for scalar curvature.

Llarull ([8, Thm. B]) extended this rigidity result to spheres of arbitrary dimension when equipped with their standard metric. The proof uses spin geometry and index theory for Dirac operators. Goette and Semmelmann ([3, Thm. 2.1]) proved rigidity for scalar curvature for M_0 with non-vanishing Euler number, nonnegative curvature operator and positive Ricci curvature.

If M_0 is the smooth boundary of a strictly convex domain $\Omega \subset \mathbb{R}^{n+1}$, then M_0 is diffeomorphic to S^n and its Weingarten map $W_{\partial\Omega}$ is definite. The Gauss equation implies that the curvature operator and the Ricci curvature of the induced metric are positive definite. Hence, for even n, the result by Goette and Semmelmann implies that S^n with such a metric is rigid for scalar curvature. For odd n, this has been shown by Li, Su, and Wang ([7, Thm. 1.2]).

Remark. It should be emphasized that there are many Riemannian manifolds which are *not* rigid for scalar curvature. To see this, we consider a closed manifold $M = M_0$ of dimension $n \geq 3$. Start with a Riemannian metric g and denote its scalar curvature by S. Assume $S > 0$.

Given $\varepsilon > 0$, there exists a Riemannian metric g_0 with $|S_0 - \frac{1}{2}S| < \varepsilon$ and $|g - g_0| < \varepsilon$, see [9, Thm. 1]. If ε is chosen small enough, we can rescale g to a metric $\tilde{g}$ such that $\tilde{g} > g_0$ and $\tilde{S} > S_0$. Putting $\Phi = \mathrm{id}$ we see that (M_0, g_0) is not rigid for scalar curvature.

The results for compact M and M_0 with empty boundary are summarized in Table 1.

	M_0	g_0	assumption	conclusion
Gauss-Bonnet	S^2	$S_0 > 0$	$S \geq S_0 \circ \Phi$	Φ isometry
[8]	S^n	g_{std}	$S \geq S_0 \circ \Phi$	Φ isometry
[3] for n even [7] for n odd	S^n	$g_{\partial\Omega}$ $\Omega \subset \mathbb{R}^{n+1}$ convex	$S \geq S_0 \circ \Phi$	Φ isometry
[3]	$\chi(M_0) \neq 0$	$(R_0 \colon \Lambda^2 \to \Lambda^2) \geq 0$ $\mathrm{ric}_0 > 0$	$S \geq S_0 \circ \Phi$	Φ isometry

Table 1. M and M_0 are compact with empty boundary

Lott ([10, Cor. 1.2]) generalized the results by Goette and Semmelmann to manifolds with boundary. In addition to the scalar curvature assumption, one has to require $\Phi(\partial M) \subset \partial M_0$ and to make an analogous assumption on the mean curvature of the boundary, $H_{\partial M} \geq H_0 \circ \Phi$.

Bär, Brendle, Hanke, and Wang ([1, Thm. A]) showed scalar curvature rigidity for a class of warped product spaces, generalizing earlier results by Cecchini and

Zeidler ([2, Thm. 10.2]). Here $M_0 = [\theta_-, \theta_+] \times S^{n-1}$ is equipped with a warped-product metric $g_0 = g_\rho = d\theta^2 + \rho(\theta)^2 g_{S^{n-1}}$. The crucial assumption is that ρ is strictly logarithmic concave, i.e. $(\log \circ \rho)'' < 0$. This comprises annular regions in Euclidean space ($\rho(\theta) = \theta$), in spheres ($\rho(\theta) = \sin(\theta)$), and in hyperbolic space ($\rho(\theta) = \sinh(\theta)$). So positivity of the scalar curvature is no longer required.

Table 2 summarizes the results if M and M_0 are compact with nonempty boundary.

	M_0	g_0	assumption	conclusion
[10]	$\chi(M_0) \neq 0$	$(R_0\colon \Lambda^2 \to \Lambda^2) \geq 0$ $\mathrm{ric}_0 > 0$ $\mathrm{II}_{\partial M_0} \geq 0$	$S \geq S_0 \circ \Phi$ $\Phi(\partial M) \subset \partial M_0$ $H_{\partial M} \geq H_0 \circ \Phi$	Φ local isometry
[2] n odd [1] $n \geq 3$	$[\theta_-, \theta_+] \times S^{n-1}$ $n \geq 3$	$g_0 = g_\rho$ $(\log \circ \rho)'' < 0$	$S \geq S_0 \circ \Phi$ $\Phi(\partial M) \subset \partial M_0$ $H_{\partial M} \geq H_0 \circ \Phi$	Φ isometry

TABLE 2. M and M_0 are compact with nonempty boundary

Remarkably, rigidity for such warped products fails if $n = 2$. Indeed, putting $M := M_0$ and $g = g_{\lambda\rho}$ for any constant $\lambda > 1$ we have that $\Phi = \mathrm{id}$ is Lipschitz with Lipschitz constant 1, has degree 1 but is not an isometry. Yet g_ρ and $g_{\lambda\rho}$ have the same scalar curvature and the same boundary mean curvature.

Scalar curvature rigidity is also of interest for noncompact, even for incomplete Riemannian manifolds. Gromov [4, Sec. 3.9] suggested considering the standard sphere with finitely many points removed as a test case. Even this is still open. But if we remove one point or a pair of antipodal points, the punctured sphere is known to be scalar curvature rigid, provided $n \geq 3$. For $n = 2$ we have the same counterexamples as mentioned above. For $n = 3$ and $\Phi = \mathrm{id}$ rigidity has been proved using μ-bubbles by Hu, Liu, and Shi, see [6, Thm. 1.6]. An independent proof for $n = 3$ and $\Phi = \mathrm{id}$ using spacetime harmonic functions is due to Hirsch, Kazaras, Khuri, and Zhang ([5, Main Thm. D]). The general case $n \geq 3$ has been resolved by Bär, Brendle, Hanke, and Wang ([1, Thm. B]) using Dirac operator methods.

We summarize this in Table 3.

	M_0	g_0	assumption	conclusion
[5], [6] for $n = 3$ and $\Phi = \mathrm{id}$ [1] for $n \geq 3$	$S^n \setminus \{p, -p\}$	g_{std}	$S \geq S_0 \circ \Phi$ Φ proper	Φ isometry

TABLE 3. M and M_0 are incomplete

The question of scalar curvature rigidity can be modified in many ways. The condition that Φ be Lipschitz with constant 1 can often be weakened to being contracting on 2-vectors. One can allow M to have higher dimension than M_0 and then has to replace the degree of Φ by its $\hat{A}$-degree. One can try to drop the assumption that Φ is a spin map. Finally, one can allow low regularity of the metrics and of Φ. Various talks at this workshop deal with these modifications.

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Curvature operators and rational cobordism

Renato G. Bettiol

(joint work with McFeely Jackson Goodman)

A natural way to generalize the celebrated Lichnerowicz obstruction to positive scalar curvature on spin manifolds is to find curvature conditions which imply that some twisted Dirac operators have vanishing index. To make these generalizations most interesting, the curvature conditions should be as weak as possible, easily computable, and, ideally, invariant under appropriate surgeries.

Following this scheme and inspired by recent works of Petersen and Wink [2,3], we determine a family of pointwise piecewise linear inequalities $C_p(R) > 0$ on the eigenvalues of curvature operators $R\colon \wedge^2 TM \to \wedge^2 TM$ that imply vanishing of the twisted $\hat{A}$-genus $\hat{A}(M,E) = \langle \hat{A}(TM) \cdot \operatorname{ch} E, [M] \rangle$ on a closed Riemannian spin manifold (M,g), where the twisting bundle $E \subseteq TM_{\mathbb{C}}^{\otimes p}$ is any prescribed parallel bundle of p-tensors. For instance, in dimension $n = 8$, Einstein metrics with 5-positive curvature operator have $C_1(R) > 0$. This shows that $M = \mathbb{H}P^2$ does not admit Einstein metrics with 5-positive curvature operator, since it has

$\hat{A}(M, TM_{\mathbb{C}}) \neq 0$. The curvature conditions $C_p(R) > 0$ determine spectrahedral cones of curvature operators and are hence "easily computable"; they are also stable under surgeries of sufficiently high codimension.

We show that every nontorsion cobordism class in Ω_n^{SO}, $n \geq 10$, has a manifold with $C_1(R) > 0$, that is, without the spin assumption, this curvature condition does not impose any restrictions on the cobordism class (hence it is "as weak as possible"). On the other hand, with the spin assumption, the vanishing of $\hat{A}(M)$ and $\hat{A}(M, TM_{\mathbb{C}})$ are the *only* restrictions on the cobordism class of manifolds with $C_1(R) > 0$. Namely, a closed spin manifold M^n, $n \geq 10$, with $\hat{A}(M) = 0$ and $\hat{A}(M, TM_{\mathbb{C}}) = 0$ is rationally spin cobordant to a manifold with $C_1(R) > 0$.

The above can be used to annihilate further rational cobordism invariants, such as the Witten genus, elliptic genus, signature, and even the rational cobordism class itself, by requiring $C_p(R) > 0$ for appropriate values of p.

For further details, please refer to the preprint [1].

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Attaching faces of positive scalar curvature manifolds with corners

Alessandro Carlotto

(joint work with Chao Li)

It is a natural task, also partly motivated by the study of *concordance* for suitable classes of Riemannian metrics on a compact manifold with boundary (see [2]), to develop techniques to smoothly attach two assigned manifolds with corners while preserving certain curvature conditions. In the context of this lecture we wish to investigate the case when such conditions are a lower bound on the scalar curvature (most notably the *positive scalar curvature* requirement) together with say the mean-convexity or minimality of the leftover boundary (namely: the portion of the boundary that remains after the attachment operation is performed). This can be regarded as a natural, albeit long-awaited generalization of the classical theorem by Miao [5] and - to some extent - of its much more recent refinements by Bär and Hanke [1].

In my talk I have tried to describe some of the things we have learnt, over the past few years, about these sorts of questions, in the special case when the designated faces to be attached are smooth compact manifolds with boundary meeting, in any of the two manifolds in question, their adjacent faces orthogonally.

1. A sample statement

A prototypical statement we obtained (see [3]) reads as follows:

Main Theorem. Let (M_-, g_-) and (M_+, g_+) be (smooth) compact Riemannian manifolds with corners, both having dimension $n+1 \geq 3$; assume $F_- \subset M_-$ (respectively: $F_+ \subset M_+$) are cylindrical faces and there exists $\phi : M_- \to M_+$ giving an isometry between $[g_-]_{|F_-}$ and $[g_+]_{|F_+}$. Let $M := M_- \sqcup_\phi M_+$, with its natural atlas of manifold with corners and let $\pi_\pm : M_\pm \to M$ be the corresponding projection maps (for either consistent choice of signs). We shall then introduce the following notation:

- X is the codimension-one submanifold that is the common image of F_+, F_- in M;
- $Y_\pm \subset \partial M_\pm$ is the disjoint union of all faces of $M_\pm$ having non-empty intersection with $F_\pm$, and Y the disjoint union of all faces of M having non-empty intersection with X.

Suppose that $R_{g_\pm} > 0$ on $M_\pm$, that $Y_\pm \subset \partial M_\pm$ are mean-convex and meet $F_\pm$ at a right angle, and in addition there holds for the mean curvature of the isometric faces

$$(\star) \quad H_{g_-,F_-} \geq f(x), \quad H_{g_+,F_+} \geq -f(x),$$
$$\text{where } \underline{\text{either}}\ f(x) \geq 0\ (\forall x \in X) \quad \underline{\text{or}}\ f(x) \leq 0\ (\forall x \in X).$$

Given a neighborhood U of X in M, such that $U \cap \partial M \subset Y$ (thus disjoint from $\mathrm{sing}(M)$), there exist a Riemannian metric g on M and an open set $\hat{U} \subset U$ such that the restriction of g to $M \setminus \hat{U}$ satisfies $\pi_\pm^* g = g_\pm$ on $M_\pm$, and in U the following two properties hold:

(1) (M, g) has positive scalar curvature;
(2) (M, g) has mean-convex boundary, and in fact minimal boundary if the same is assumed to be true for Y_-, Y_+ respectively in (M_-, g_-) and (M_+, g_+).

Remark. Some comments on the assumptions are appropriate:

- condition $(\star)$ coincides (given the different sign convention) with the jump condition (**H**) in [5], together with the additional technical requirement that at least one of the two functions H_{g_+,F_+}, H_{g_-,F_-} does not change sign;
- we stress the construction we present is local near the given interface, so the singularities of M away from X do not play any role; that said, an important special case occurs when

$$\mathrm{sing}(M_-) \setminus F_- = \mathrm{sing}(M_+) \setminus F_+ = \emptyset$$

  for in that case the output of the theorem is a smooth compact manifold with positive scalar curvature and mean-convex boundary (or minimal boundary under the same assumption on the input data).

We wish to stress that condition ($\star$) is always satisfied when the two faces to be glued are mean-convex in the standard geometric sense that an outward deformation will (weakly) *increase* area, to leading order, at each point; this includes the minimal case as a special instance.

2. Some applications to minimal concordance

Among the most direct application of our gluing result, we want to mention those that are directly connected to the (two) notions of concordance that we gave in [2]. So here they are. As a first application, we have that *"weak PSC min-concordance and weak PSC mc-concordance are transitive, therefore equivalence relations"*. Here is the precise statement:

Corollary 1. Let X be a compact manifold with boundary and let us assume that there exist PSC Riemannian metrics $g_{0,1}$ and $g_{1,2}$ on $X \times [0,1]$ such that, in both cases, $\partial X \times [0,1]$ is minimal (respectively: mean-convex), and the slices $X \times \{0\}$ and $X \times \{1\}$ are free boundary minimal surfaces; furthermore, $g_{0,1}$ restricts to h_0 on $X \times \{0\}$, and to h_1 along $X \times \{1\}$; similarly $g_{1,2}$ restricts to h_1 on $X \times \{0\}$, and to h_2 along $X \times \{1\}$.

Then there exists a PSC Riemannian metric $g_{0,2}$ on $X \times [0,1]$ that makes $\partial X \times [0,1]$ minimal (respectively: mean-convex), and both $X \times \{0\}$ and $X \times \{1\}$ free boundary minimal surfaces, and in addition $g_{0,2}$ restricts to h_0 on $X \times \{0\}$, and to h_2 along $X \times \{1\}$.

Furthermore, *"the relation of weak PSC min-concordance (or, respectively, weak PSC mc-concordance) is the same as its strong counterpart, if one restricts a priori to the subclass $\mathscr{R}_{R>0,H=0}(X)$ (respectively: $\mathscr{R}_{R>0,H\geq 0}(X)$)."* By that we mean what follows:

Corollary 2. Let X be a compact manifold with boundary. If $h_{-1}, h_1 \in \mathscr{R}_{R>0,H=0}(X)$ are weakly PSC min-concordant through a PSC metric g on $M = X \times [-1,1]$ (thus: a metric making the cylidrical boundary minimal, and both faces free boundary minimal surfaces) then they are also (strongly) PSC min-concordant through a PSC metric $\overline{g}$ (thus: making the cylindrical boundary minimal, and being a Riemannian product in a neighborhood of both faces).

Such a result ensures, in particular, that our definition of weak concordance is an honest generalization of the standard (strong) one, in that it reduces to the latter when working with the privilged subspsace of metrics having positive scalar curvature and minimal boundary.

3. Next steps and related open problems

We wish to end this report by mentioning three significant open problems that are directly related to our construction, and can be regarded as the next (desirable) steps in that same direction.

Question A. Is it possible to remove, in the statement of the Main Theorem above, the assumption that at least one of the mean curvature functions H_{g_-,F_+}, H_{g_+,F_+} does *not* change its sign?

It is indeed very natural to expect one should be able to prove the smoothing theorem under the very same condition singled out in Miao's work, [5].

Question B. Is it possible to prove an analogue of the Main Theorem above simply assuming that the dihedral angles at each point add up to π?

It was pointed out to us by Hanke that such a construction relates to the discussion by Gromov in [4, Section 2]. Moving even one step further, one may then wonder what follows:

Question C. What are the sharp assumptions to attach a pair of *not necessarily cylindrical* distiguished isometric faces of a given pair of manifolds with corners, under the geometric requirements above?

How to translate this last question into a well-posed mathematical problem is also not trivial and part of the question itself; however, in the special case when the requirement on the leftover boundary is to be mean-convex (or minimal) we do have natural weak notions, coming from the first variation of the area functional.

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Scalar curvature rigidity of the four-dimensional sphere

Simone Cecchini

(joint work with Jinmin Wang, Zhizhang Xie, Bo Zhu)

A cornerstone result in comparison geometry with scalar curvature is the rigidity of the round sphere in the spin setting, established by Llarull.

Theorem 1 ([11, Theorem B])**.** *Let (M, g) be an n-dimensional closed connected spin Riemannian manifold with $Sc_g \geq n(n-1)$. If $f\colon (M, g) \to (\mathbb{S}^n, g_{\mathbb{S}^n})$ is a smooth, distance non-increasing map of non-zero degree, then f is an isometry.*

Its proof relies on the Dirac operator method, requiring the hypothesis that M is spin. A big open question in the field is whether the spin assumption can be dispensed with in Theorem 1. We address this question affirmatively, at least in dimension four.

Theorem A (C.-Wang-Xie-Zhu)**.** *Let* (M, g) *be a four-dimensional closed connected oriented (possibly non-spin) Riemannian manifold and* $f\colon (M, g) \to (\mathbb{S}^4, g_{\mathbb{S}^4})$ *a smooth map of non-zero degree. If* f *is distance non-increasing and* $\mathrm{Sc}_g \geq 12$*, then* f *is an isometry.*

Our strategy to establish Theorem A is outlined below.

(1) We first rule out from Theorem A the case when all the inequalities are strict, following ideas of Gromov [1]. This involves utilizing μ-bubbles and a version with coefficients of Theorem 1 due to Listing, that applies to our μ-bubbles since all three-dimensional oriented manifolds are spin.

(2) We then employ the harmonic map heat flow coupled with the Ricci flow to demonstrate that the general case of Theorem A reduces to the situation where all the inequalities are strict, unless the metric g is Einstein with $\mathrm{Ric}_g = 3g$. Here, we make use of recent results of Lee and Tam [9], showing that the harmonic map heat flow coupled with the Ricci flow provides appropriate control of the Lipschitz constant with respect to the change of the scalar curvature under Ricci flow.

(3) Finally, we prove Theorem A for Einstein manifolds, which follows as a consequence of Bishop's volume comparison theorem.

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Mathematical Relativity, this and that

Piotr T. Chruściel

(joint work with Raphaela Wutte)

The language of mathematical general relativity is that of Lorentzian geometry, the methods are these of differential topology, of differential geometry, and of geometric analysis. The aim is to exhaustively understand the space of solutions of Einstein equations with physically relevant matter sources. The key question is to understand the dynamics of Einstein equations. When doing so one is, as a first step, led to the classification of stationary solutions; next, one faces the need for an exhaustive description of general relativistic initial data sets, and of their properties.

This talk is a double feature. In the first, "landscape" part, I will review some of the key open problems in the field, leaving aside these questions which will be addressed by other speakers in this meeting. In the second part, based on joint work with Raphaela Wutte [25], I will introduce the audience to the unusual world of mass of two-dimensional manifolds, with properties strikingly different compared to other dimensions.

1. The landscape

1.1. **The Belinski-Khalatnikov-Lifschitz (BKL) conjecture.** A bold conjecture of Belinski, Khalatnikov and Lifschitz [9] posits that generic singularities resulting by evolution by the vacuum Einstein equations behave in an oscillatory way, reminiscent of that of the spatially homogeneous Bianchi IX solutions. Here one should keep in mind that even the Bianchi IX solutions are not completely understood so far, in spite of substantial progress in [7, 8, 42, 47, 48]. The conjecture is more than challenging, without any real evidence supporting it. In addition to the Bianchi IX model there exists only one known family of metrics exhibiting this behaviour, constructed by Berger and Moncrief [10] by a solution generating technique starting from Bianchi IX metrics. The Berger-Moncrief family has only a few free parameters, with all metrics having one Killing vector. It would be of great interest to exhibit a family of metrics without symmetries, parameterised by a function space, which has the conjectured BKL behaviour, or to prove that there are no such families. A proof of the BKL conjecture would presumably also prove the Strong Cosmic Censorship conjecture, another challenging problem in the field; see [11, 16, 22, 34] and references therein.

1.2. **Black hole uniqueness.** A topic of significant interest is the classification of time-independent black-hole spacetimes. The expectation is that, in four spacetime dimensions, the Kerr black holes exhaust the family of well-behaved stationary solutions of the vacuum Einstein equations. So far we only have by now a satisfactory classification of *static* solutions: it has been established in full detail that the Schwarzschild metrics are the only vacuum and static solutions satisfying a few natural global regularity conditions [17, 19]. The current uniqueness theorem for Kerr black holes, where the hypothesis of staticity has been replaced by that of *stationarity*, requires the further assumptions of both connectedness of the black hole and of analyticity of the metric. (In the static case, these last two properties are *derived* and not *assumed*.) The undesirable assumption of analyticity has only been removed for near-Kerr solutions [1] so far. And we only have a proof of non-existence of well behaved vacuum *stationary and axisymmetric* solutions with two components [21, 31]; the general case remains open. Settling the connectedness-and-analyticity issue is the key problem of the mathematical theory of black holes.

1.3. **Initial data.** A key topic in mathematical general relativity is the construction of initial data. The "conformal method" (see [6, 12] and references therein) provides an exhaustive description of Cauchy data sets when the trace τ of the extrinsic curvature tensor is *constant* and satisfies

$$\tau^2 \geq 3\Lambda\,,$$

where Λ is the cosmological constant. There has *not* been any progress on removing these conditions since [29, 33, 46]. A pressing problem is to obtain a better understanding of such data.

Gluing techniques [13,20,27] provide an alternative method for constructing solutions of the constraint equations with useful properties. Nice progress on gluing *spacelike* Cauchy data has been achieved in [32,43,44]. A gluing method for *characteristic* initial data has been developed by Aretakis, Czimek and Rodnianski [1], with interesting applications in [2, 3, 28, 35, 36]; see also [18, 26] for the linearised problem.

2. Mass of two dimensional Riemannian manifolds

The definition of mass of initial data sets is well understood in space dimensions $n \geq 3$ (cf., e.g., [15, 23, 39]). It is, however, not widely known that the mass has rather different properties in dimension two.

2.1. Two-dimensional asymptotically locally Euclidean (ALE) manifolds. It has been suggested that a reasonable replacement for the notion of mass is provided by Shiohama's theorem [49]. which ties the integral of the Ricci scalar with a deficit angle at infinity. From this perspective the only geometrically finite, complete manifold, with nonnegative scalar curvature in L^1, is the Euclidean plane when the usual asymptotic flatness conditions are imposed. The remaining such manifolds are asymptotically locally Euclidean, with the geometry at infinity approaching that of a cone with strictly positive deficit angle. This can be thought of as a positive mass theorem in dimension two, and should be contrasted with the higher dimensional case, where the positive energy theorem is known to hold in various situations for asymptotically Euclidean (AE) initial data sets (cf., e.g., [14, 41] and references therein), but fails for ALE ones [38].

2.2. Two-dimensional asymptotically locally hyperbolic (ALH) manifolds. While for $\Lambda = 0$ the two-dimensional case needs a non-standard framework to start with (ALE as opposed to AE), in the ALH case (where $\Lambda < 0$) to define mass one can start with the usual Fefferman-Graham-type expansions. For constant negative scalar curvature metrics ("time-symmetric vacuum initial data") the large-r expansion is provided by an *exact* formula:

$$g = r^{-2}dr^2 + \big(r^2 + \frac{\mu(\varphi)}{2} + \frac{\mu(\varphi)^2}{8r^2}\big)d\varphi^2\,,$$

where φ is a coordinate on S^1, with r tending to infinity as the conformal boundary at infinity is approched. Here μ is an arbitrary function of φ, called the *mass aspect function.* (For non-vacuum or non-time-symmetric data there is no reason for the $1/r$-expansion to stop, but it does for constant negative scalar curvature metrics; this is specific to dimension two.) The Hamiltonian methods of [24, 37], or the Noether charge methods of [40], or the "holographic" approach of [30], all lead to the same global quantity, which we will refer to as *Hamiltonian mass*

$$H = \frac{1}{2\pi}\int_{S^1} \mu\, d\varphi\,,$$

and which is the direct specialisation to two dimensions of an expression valid in all dimensions $n \geq 2$. While this expression provides an invariant, or covariant, quantity for $n \geq 3$ [23, 50], when $n = 2$ we have:

(a) Under asymptotic symmetries, which in the current case are coordinate transformations of the form

$$\varphi = f(\hat{\varphi}) - \frac{f''(\hat{\varphi})}{2\hat{r}^2}\,, \qquad r = \frac{\hat{r}}{f'(\hat{\varphi})}\,,$$

the Hamiltonian mass transforms as

$$H \mapsto \hat{H} = \frac{1}{2\pi}\int_{S^1} \big(\mu(f(\hat{\varphi}))f'(\hat{\varphi})^2 - 2S(f)(\hat{\varphi})\big)\, d\hat{\varphi}\,,$$

where $S(f)$ denotes the Schwarzian derivative:

$$S(f)(\hat{\varphi}) = \frac{f^{(3)}(\hat{\varphi})}{f'(\hat{\varphi})} - \frac{3}{2}\left(\frac{f''(\hat{\varphi})}{f'(\hat{\varphi})}\right)^2 .$$

The first surprise is that $\hat{H}$ can be made as large as desired when varying f.

(b) The second surprise is that there exist mass aspect functions μ for which $\hat{H}$ can be made as negative as desired when varying f [5].

(c) However, when $\mu \geq -1$ the Hamiltonian mass is bounded from below, so that an invariant can be obtained by minimisation. In joint work with Raphaela Wutte [25] we conjecture that this happens in all physically relevant cases, namely for globally well behaved initial data sets with matter satisfying positivity conditions. The conjecture is supported by the existence of an identity which implies positivity, as well as a Penrose-type inequality [25], both established when there exists a solution without critical points of the equation

$$D_i\Big(\frac{D^i\rho}{\rho|D\rho|}\Big) = 0\,,$$

with suitable asymptotic and boundary conditions. (The function ρ is then used as a coordinate on the manifold.) However, because of the asymptotic-symmetries-dependence of the Hamiltonian mass, the relevance of these identities is not clear.

(d) Last but not least, there exist mass aspect functions μ which *cannot* be transformed to a constant by an asymptotic symmetry. For such μ's global invariants are obtained by an analysis of the solutions of the associated *Hill equation*, namely

$$\psi'' = \frac{\mu}{4}\psi\,,$$

Counting zeros of solutions of this equation, and analysing the behaviour of the solutions under shifts by 2π, provides a complete classification of the mass aspect functions μ [5]; see also [25, 45].

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A volume-renormalized mass for asymptotically hyperbolic manifolds

Mattias Dahl
(joint work with Klaus Kröncke, Stephen McCormick)

We introduce a new mass-like quantity for the class of asymptotically hyperbolic (AH) manifolds that are asymptotically Poincaré–Einstein (APE), which means that $\mathrm{Ric} + (n-1)g$ decays at an appropriate rate. By AH we mean that the manifolds are conformally compact with sectional curvature tending to -1 towards the conformal boundary.

The definition of the mass-like quantity utilizes the fact that a linear combination of the ADM boundary expression and the renormalized volume of the manifold is well-defined even in cases where neither is well-defined independently. Given two AH manifolds (M^n, g) and $(\widehat{M}^n, \widehat{g})$ with diffeomorphic conformal infinities, the *volume-renormalized mass* of g with respect to $\widehat{g}$ is defined as

$$\begin{aligned}\mathfrak{m}_{\mathrm{VR},\widehat{g}}(g) = &\int_{\partial M} (\mathrm{div}_{\widehat{g}}(\varphi_* g) - d\mathrm{tr}_{\widehat{g}}(\varphi_* g))(\nu) dA \\ &+ 2(n-1)\left(\int_M dV_g - \int_{\widehat{M}} dV_{\widehat{g}}\right),\end{aligned}$$

where φ is a diffeomorphism between neighborhoods of the conformal infinities such that $\varphi_* g - \widehat{g}$ decays suitably, and the integrals should be understood as appropriate limits.

When the asymptotic fall-off is so fast that the boundary integral vanishes, the volume-renormalized mass is simply proportional to the renormalized volume. In this sense, we can also view the quantity as a generalization of the renormalized volume. Positivity of the renormalized volume has been proven for metrics on $\mathbb{R}^3$ asymptotic to the standard hyperbolic metric by Brendle and Chodosh [1], which can be viewed as a positive mass theorem for the volume-renormalized mass under strong decay conditions.

The first main result is

Theorem 1. *Let (M^n, g) and $(\widehat{M}^n, \widehat{g})$ be APE manifolds with isometric conformal boundaries that both satisfy* $\mathrm{scal} + n(n-1) \in L^1$*. Then $\mathfrak{m}_{\mathrm{VR},\widehat{g}}(g)$ is well defined and finite.*

The theorem follows from the observation that the volume-renormalized mass contributes to a renormalized version of the Einstein–Hilbert action for AH manifolds.

A priori, the definition of the volume-renormalized mass depends on the choice of φ. From a physical perspective however, a mass should be a coordinate-invariant object and therefore not depend on the choice of diffeomorphism φ. We are indeed able to show that this is the case for the volume-renormalized mass, provided that an additional condition holds.

Theorem 2. *Let (M^n, g) and $(\widehat{M}^n, \widehat{g})$ be APE manifolds with isometric conformal boundaries that both satisfy $\mathrm{scal} + n(n-1) \in L^1$. If the conformal boundaries are proper, $\mathfrak{m}_{\mathrm{VR},\widehat{\mathrm{g}}}(g)$ does not depend on the choice of φ.*

Here, we call a conformal class proper, if it is the conformal boundary of a Poincaré–Einstein manifold $(\overline{M}, \overline{g})$ such that every isometry of the conformal boundary extends to an isometry of $(\overline{M}, \overline{g})$.

We prove some positive mass theorems for the volume-renormalized mass. The first is for two-dimensional manifolds.

Theorem 3. *Consider a surface (M^2, g) asymptotic to $\mathbb{R}^2$ with the metric $\widehat{g} = dr^2 + \sinh^2(r)\left(\frac{\omega}{2\pi}\right)^2 d\theta^2$, which is the hyperbolic metric with angular defect ω. Under the assumption that $\mathrm{scal}_g + 2$ is nonnegative and integrable we have*

$$\mathfrak{m}_{\mathrm{VR},\widehat{\mathrm{g}}}(g) + 2(2\pi - \omega) \geq 0,$$

where equality holds if and only if (M^2, g) is isometric to $(\widehat{M}, \widehat{g})$.

For three-dimensional manifolds we prove the following.

Theorem 4. *Let g be a complete APE metric on $\mathbb{R}^3$ whose conformal boundary is the round 2-sphere. Assume furthermore that $\mathrm{scal}_g + 6$ is nonnegative and integrable. Then $\mathfrak{m}_{\mathrm{VR},\mathrm{g}_{\mathrm{hyp}}}(g)$ is nonnegative and vanishes if and only if g is isometric to g_{hyp}.*

The proof of this theorem uses the aforementioned positivity result for the renormalized volume by Brendle and Chodosh, combined with the following conformal positive mass theorem.

Theorem 5. *Let $(M^n, \widehat{g})$ be a complete APE manifold with $\mathrm{scal}_{\widehat{g}} = -n(n-1)$, and let (M^n, g) be a complete APE manifold conformal to $(M^n, \widehat{g})$. Then if $\mathrm{scal}_g + n(n-1)$ is nonnegative and integrable, we have $\mathfrak{m}_{\mathrm{VR},\widehat{\mathrm{g}}}(g) \geq 0$ with equality only if $g = \widehat{g}$.*

The results presented are from [2]

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On the Hamilton-Lott conjecture in higher dimensions

Alix Deruelle
(joint work with Felix Schulze and Miles Simon)

In this report, we consider smooth, complete solutions $(M^n, g(t))_{t\in(0,T)}$ to the Ricci flow defined on smooth, connected manifolds satisfying for $t \in (0,T)$,

$$\text{(1)}\qquad \mathrm{Ric}(g(t)) \geq 0 \quad \text{and} \quad |\mathrm{Rm}(g(t))| \leq \frac{D_0}{t},$$

where D_0 is a positive constant. The curvature conditions (1) are particularly relevant since they are invariant under parabolic rescaling. Due to [10] it is known that (1) ensures the existence of an initial metric d_0 on M (interpreted as a metric space) such that the flow converges back to it in the distance sense.

This setting has been shown to occur in many situations, a prominent one being that of self-similar solutions (also known as expanding solitons) with non-negative curvature operator coming out of cones with non-negative curvature operator: see for example [9], [2], [10], [1].

The first result of this talk concerns solutions to Ricci flow satisfying (1) under the assumption that the scalar curvature controls the whole curvature tensor pointwise, starting from a sufficiently regular metric cone. It quantifies locally (in space) how far such a solution is from being self-similarly expanding: we refer the reader to [4, Theorem 1.1] for a statement.

The second main result of this talk is motivated by the recent resolution of the Hamilton-Lott conjecture on the rigidity of 3-dimensional Ricci-pinched metrics by the authors [3] and Lee-Topping [7]. See also [5] for a proof using inverse mean curvature flow. Recall that a Riemannian manifold (M^n, g) is Ricci-pinched if $\mathrm{Ric}(g) \geq 0$ and if there exists a positive constant c such that $\mathrm{Ric}(g) \geq c\,\mathrm{R}_g\, g$ in the sense of symmetric 2-tensors. The Hamilton-Lott conjecture states that 3-dimensional Ricci pinched Riemannian manifolds are either flat or compact. In [3, Question 1.5], we asked whether such a conjecture holds in higher dimensions when the metric is not only Ricci-pinched but also 2-pinched i.e. if there exists a constant $c > 0$ such that the sum of the two lowest eigenvalues $\lambda_i(g)$, $i = 1, 2$, of the curvature operator satisfies $\lambda_1(g) + \lambda_2(g) \geq c\,\mathrm{R}_g$ on M.

We are able to answer [3, Question 1.5] (and even more) under an additional non-collapsing assumption:

Theorem 1. *Let (M^n, g) be a smooth, complete, connected Riemannian manifold that is PIC1 pinched. Assume it is non-collapsed at all scales:* $\mathrm{AVR}(g) := \lim_{r\to+\infty} r^{-n}\,\mathrm{vol}_g\, B_g(p,r) > 0$. *Then (M^n, g) is isometric to Euclidean space.*

Theorem 1 also yields a new proof of Hamilton-Lott conjecture in dimension 3.

The starting point of the proof of this conjecture for $n = 3$, in case the metric has bounded curvature given in [3], are the following existence (E) and non-collapsing (NC) results of [8] for starting metrics (M^3, g_0) which are non-flat, complete, connected with non-negative Ricci curvature and bounded curvature:

(E) there exists a smooth solution $(M^3, g(t))_{t\in[0,\infty)}$ to Ricci flow for all time and the solution remains uniformly Ricci pinched, $\mathrm{Ric}(g(t)) \geq \alpha\, \mathrm{R}_{g(t)}\, g(t) > 0$ for some $\alpha > 0$, and $|\mathrm{Rm}(g(t))|_{g(t)} \leq c/t$ for $t \in (0,\infty)$.

(NC) the solution is non-collapsed at all scales uniformly in time. More precisely, it has constant positive asymptotic volume ratio: $\mathrm{AVR}(g(t)) = V_0 > 0$ for all $t \in [0,\infty)$.

Assuming the initial metric to be Ricci-pinched, the existence part (E) was extended by [7] allowing the initial metric to have unbounded curvature. Important ingredients in the proof of [3] are a local-in-time stability theorem for the Ricci flow (see [3, Theorem 1.2]), existence results for self-similar solutions coming out of non-negatively curved 3-dimensional Alexandrov metric cones and a number of non-trivial results from the theory of RCD spaces. The proof of Theorem 1 in this talk does not require any of these ingredients.

In the proof of Theorem 1 we require an existence result of the type given in (E). Since we assume that the initial metric is PIC1 pinched, this is provided by [6, Theorem 4.4].

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The positive scalar curvature cobordism category

Johannes Ebert
(joint work with Boris Botvinnik, Oscar Randal–Williams)

The Dirac operator on a closed spin manifold yields a secondary index invariant for Riemannian metrics of positive scalar curvature, as first introduced by Gromov–Lawson [9] and Hitchin [10]. This secondary invariant is a map of spaces

$$\mathrm{inddiff} : \mathcal{R}^+(M) \times \mathcal{R}^+(M) \to \Omega^{\infty+d+1}\mathbb{KO}$$

where $d = \dim(M)$ and $\mathbb{KO}$ is the real K-theory spectrum. Fixing the first variable, this map induces on homotopy groups a map

$$\mathrm{inddiff}(g_0, \_)_* : \pi_k(\mathcal{R}^+(M)) \to KO_{d+k+1}(*).$$

It was shown in [1] that when $d \geq 6$, the last map is surjective after tensoring with $\mathbb{Q}$, for all k, and in the talk, we sketched the streamlined proof of this result from [5].

Both, the construction of the map inddiff and the surjectivity result, have suitable variants which take the fundamental group of M into account; see [4] for more details.

The proof uses cobordism categories. The cobordism category $\mathrm{Cob}_d^{\mathrm{Spin}}$ of $(d-1)$-dimensional closed spin manifolds and d-dimensional spin cobordisms is a well-known object; it is a classical result [6] that its classifying space $B\mathrm{Cob}_d^{\mathrm{Spin}}$ is weakly homotopy equivalent to the infinite loop space $\Omega^{\infty-1}\mathrm{MTSpin}(d)$ of the relevant Madsen–Tillmann spectrum; at least rationally, the homotopy groups of the latter space are easily computed.

A variant $\mathrm{PCob}_d^{\mathrm{Spin}}$ where all manifolds and cobordisms are equipped with Riemannian metrics of positive scalar curvature is straightforward to define; the homotopy type of $B\mathrm{PCob}_d^{\mathrm{Spin}}$ is unknown. We use a subcategory $\mathrm{Cob}_d^{\mathrm{Spin},2,1} \subset \mathrm{Cob}_d^{\mathrm{Spin}}$, where the meaning of the decorations is as follows: we only take 1-connected manifolds as objects, and allow only those morphisms $W : M_0 \to M_1$ with 2-connected inclusion map $M_1 \to W$. Such subcategories have been introduced in [7], together with a surgery technique that proves that $B\mathrm{Cob}_d^{\mathrm{Spin},2,1} \to B\mathrm{Cob}_d^{\mathrm{Spin}}$ is a weak equivalence if $d \geq 6$.

We define $\mathrm{PCob}_d^{\mathrm{Spin},2,1,\mathrm{st}} \subset \mathrm{PCob}_d^{\mathrm{Spin}}$ by taking only manifolds and cobordisms in $\mathrm{Cob}_d^{\mathrm{Spin},2,1}$ and psc metrics which are *stable* in the sense of [4, §3]; this notion isolates an important property that metrics on elementary cobordisms coming from a Gromov–Lawson surgery construction have. With some homotopy-theoretic techniques and a use of Chernysh's extension [2] of the Gromov–Lawson surgery theorem [8], one can prove that the homotopy fibre of the forgetful map $B\mathrm{PCob}_d^{\mathrm{Spin},2,1,\mathrm{st}} \to B\mathrm{Cob}_d^{\mathrm{Spin}}$ is a delooping of the space $\mathcal{R}^+(S^{d-1}\times[0,1])^{\mathrm{st}}_{g_{S^{d-1}},g_{S^{d-1}}}$.

To get an index-theoretic conclusion, one uses a variant of the Atiyah–Singer family index theorem developed in [3]. More precisely, one establishes a commutative diagram

$$\begin{array}{ccc} B\mathrm{PCob}_d^{\mathrm{Spin},2,1,\mathrm{st}} & \longrightarrow & \simeq * \\ \downarrow & & \downarrow \\ B\mathrm{Cob}_d^{\mathrm{Spin}} & \longrightarrow & \Omega^{\infty+d-1}\mathbb{KO}; \end{array}$$

the bottom map is a delooped version of the usual family index of the Dirac operator. The fact that the composition with the forgetful map $B\mathrm{PCob}_d^{\mathrm{Spin},2,1,\mathrm{st}} \to B\mathrm{Cob}_d^{\mathrm{Spin}}$ is nullhomotopic is a consequence of the Lichnerowicz formula. We get an induced map on (vertical) homotopy fibres; after taking loop spaces once more, this map might be identified with $\mathrm{inddiff}(g_{S^d}, \_)$. A variant of the Atiyah–Singer index theorem and standard characteristic class calculations prove that the bottom map is surjective on rational homotopy groups, which finishes the proof.

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From primary to secondary obstructions to positive scalar curvature

Georg Frenck

Given a manifold M, we will explain how to construct distinguished *concordance classes* of psc-metrics.

Definition 1 (Concordance)**.** *Let M be a manifold and g_0, g_1 psc-metrics on M. We say that g_0 is concordant to g_1 if there exists a psc-metric G on $M \times [0,1]$ such that the restriction of G to a neighbourhood of $M \times \{i\}$ equals $g_i + \mathrm{dt}^2$ for $i = 0, 1$. We denote by $\pi_0\widetilde{\mathcal{R}^+}(M)$ the set of concordance classes of psc-metrics on M.*

Now, if two psc-metrics are concordant, this yields an obstruction to positive scalar curvature on the cylinder $M \times [0,1]$ with given cylindrical boundary restrictions g_0 and g_1. Therefore, it is natural to ask, if obstructions to positive scalar curvature on closed manifolds can also be employed to distinguish concordance classes of psc-metrics, hence turning *primary obstructions into secondary obstructions.*

Before doing so, we need to introduce the notion of tangential types. A *tangential structure* is defined to be a map $\theta\colon B \to B\mathrm{O}$ for $B\mathrm{O} := \mathrm{colim}_{d\to\infty} B\mathrm{O}(d)$. The tangent bundle of a given manifold M has a classifying map $\tau\colon M \to B\mathrm{O}(d)$ and hence also a map to $B\mathrm{O}$, and we define a *stable θ-structure on M* to be a lift $\ell\colon M \to B$ of τ.

The reason for tangential structures being relevant in the study of positive scalar curvature is the following reformulation of the Gromov–Lawson–Schoen–Yau surgery theorem:

Theorem 1 ([2],[3]). *Let θ be a 2-coconnected tangential structure, that is the induced map is injective on π_2 and bijective on $\pi_{\geq 3}$. Let M_0 and M_1 be manifolds of dimension at least 5 such that*

(1) *M_0 admits positive scalar curvature and a θ-structure,*
(2) *M_1 admits a 2-connected θ-structure ℓ, that is ℓ is bijective on $\pi_{\leq 1}$ and surjective on π_2,*
(3) *there is a manifold W^{d+1} with a θ-structure, $\partial W = M_0 \amalg M_1$ and the θ-structure on W restricts to the given ones on M_i.*

Then M_1 admits positive scalar curvature.

Since θ depends on M_1, we call it the *tangential 2-type of M_1*.This theorem can be even improved as follows: There is another cobordism W^+, θ-cobordant to W relative to the boundary that admits a psc-metric G of product type near the boundary that extends the given one on M_0.

In order to reformulate the surgery theorem, we define

$$\Omega_n^\theta := \frac{\{n\text{-dimensional closed } \theta\text{-manifold}\}}{\theta\text{-cobordism}}$$
$$\Omega_n^{\theta,+} := \{x \in \Omega_n^\theta \mid \exists N \text{ representing } x \text{ with psc}\}$$

Then, given a manifold M of dimension $d \geq 5$ with tangential 2-type θ we have

$$M \text{ admits psc} \iff [M] \in \Omega_d^{\theta,+}$$

Having this notation at hand, we can now define the *surgery map*: Let M be a manifold with a psc-metric g and tangential 2-type θ and assume we are given a self-cobordism $W\colon M \rightsquigarrow M$ with a θ-structure. We modify W as in the above remark to obtain another self-cobordism W^+ with a psc-metric G extending g on the incoming boundary and we define

$$\mathcal{S}_W(g) := \iota^* G$$

for ι the inclusion of the outgoing boundary. The following lemma states that this is well-defined:

Lemma 1.

(1) *The concordance class of* $\mathcal{S}_W(g)$ *is independent of* W^+ *and* G.
(2) *If* g *and* g' *are concordant, then so are* $\mathcal{S}_W(g)$ *and* $\mathcal{S}_W(g')$.

In particular, we obtain a well-defined action

$$\mathcal{S}\colon \Omega^{\theta}_{d+1} \times \pi_0\widetilde{\mathcal{R}^+}(M) \to \pi_0\widetilde{\mathcal{R}^+}(M)$$

defined by $(X, g) \mapsto \mathcal{S}_{M\times[0,1]\amalg X}(g)$. We will abbreviate $\mathcal{S}_X := \mathcal{S}_{M\times[0,1]\amalg X}$. The driving feature of this action is that it is precisely possible to determine the stabilizer subgroup:

Proposition 1. *Let* X *be a closed* $(d+1)$*-manifold with the same tangential* 2*-type* θ *as* M. *Then the following are equivalent:*

(1) $\mathcal{S}_X = \mathrm{id}$.
(2) $\mathcal{S}_X(g) = g$ *for some* g.
(3) X *admits a psc-metric, that is* $[X] \in \Omega^{\theta,+}_{d+1}$.

Therefore, we get a free action of the quotient $\Omega^{\theta,-}_{d+1} := \Omega^{\theta}_{d+1}/\Omega^{\theta,+}_{d+1}$ on $\pi_0\widetilde{\mathcal{R}^+}(M)$. In particular, the orbit map $\Omega^{\theta,-}_{d+1} \to \pi_0\widetilde{\mathcal{R}^+}(M)$, $g \mapsto \mathcal{S}_X(g)$ is injective for every g.

Remark 1. The idea behind this construction resembles the ones from [4]. Here, Stolz constructs for every tangential 2-type, he constructs a group that acts freely and transitively on $\pi_0\widetilde{\mathcal{R}^+}(M)$. Transitivity of the action however comes at the cost of computability: Even in the simplest cases, his groups are not computable. On the other hand, $\Omega^{\theta,-}_n$ can be fully computed in many cases: If $\theta = spin\colon B\mathrm{Spin} \to BO$, then

$$\Omega^{spin,-}_n \cong \mathrm{KO}^{-d-1}(*) \cong \begin{cases} \mathbb{Z}/2 & \text{if } d \equiv 0, 1(8) \\ \mathbb{Z} & \text{if } d \equiv 3(4) \\ 0 & \text{else} \end{cases}$$

and if $\theta = \mathrm{SO}\colon B\mathrm{SO} \to BO$, then $\Omega^{\mathrm{SO},-}_n = 0$.

We end this note with the following observation: If M is an orientable manifold of dimension $d \geq 5$ whose fundamental group split-surjects onto $\mathbb{Z}^{d+1}$, then the $(d+1)$-torus $\mathbb{T}^{d+1}$ admits a θ-structure such that every manifold in its class is orientable and admits a map of non-zero degree to $\mathbb{T}^{d+1}$. It is known, that no such manifold admits positive scalar curvature if $d + 1 \leq 10$, see [1, 3]. Therefore, we obtain the following result:

Theorem 2. *Let* M *be an orientable manifold of dimension* $5 \leq d \leq 9$ *such that there is a split-surjection* $\pi_1 M \to \mathbb{Z}^{d+1}$. *Then* $\pi_0\widetilde{\mathcal{R}^+}(M)$ *is infinite.*

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The conformal method is not conformal

Romain Gicquaud

Constructing and parameterizing physically relevant initial data in general relativity is a long standing problem having its roots in the proof by Y. Choquet-Bruhat that the Cauchy problem for the Einstein equations is well-posed.

As the Einstein equations are covariant under diffeomorphism, it turns out that they are not a priori hyperbolic and part of them dictate restrictions on the allowed initial data. These restrictions are called the constraint equations. So, if one is willing to get a solution to the Einstein equations, the first task is to find a solution to the constraint equations before letting it evolve according to the remaining Einstein equations.

In more precise terms, the initial data for the Cauchy problem in general relativity is given as a triple $(M, \widehat{g}, \widehat{K})$, where $(M, \widehat{g})$ is a Riemannian manifold of dimension $n \geq 3$ and $\widehat{K}$ is a symmetric 2-tensor on M. If we think of M as an embedded hypersurface in the spacetime $(\mathcal{M}, h)$ solving Einstein's equations, $\widehat{g}$ is the metric induced by h on M and $\widehat{K}$ is the second fundamental form of M.

If one is only interested in the gravitational field (i.e. the so called vacuum case), which is the case we will restrict ourselves to, the constraint equations then read:

$$0 = \mathrm{scal}^{\widehat{g}} + (\mathrm{tr}_{\widehat{g}}\, \widehat{K})^2 - \left|\widehat{K}\right|^2_{\widehat{g}}, \tag{1}$$

$$0 = \mathrm{div}_{\widehat{g}} \widehat{K} - d(\mathrm{tr}_{\widehat{g}}\, \widehat{K}). \tag{2}$$

Equation (1) is a scalar equation called the Hamiltonian constraint, while Equation (2) is a vector equation called the momentum constraint. A loose counting of the degrees of freedom shows that $\widehat{g}$ and $\widehat{K}$ are each locally given by $\frac{n(n+1)}{2}$ functions providing a total of $n(n+1)$ local degrees of freedom while the Hamiltonian and the momentum constraints only form a set of $n+1$ equations. This makes the system formed by (1) and (2) underdetermined.

A natural strategy is then to decompose $\widehat{g}$ and $\widehat{K}$ into parameters (also called "seed data") that can be chosen arbitrarily and dependent variables. Several such splitting have been explored in the litterature. The major ones being the gluing techniques introduced by J. Corvino and R. Schoen and the "conformal-like" ones that we will study.

The original conformal method was introduced by A. Lichnerowicz and by J. W. York. It consists in choosing $\widehat{g}$ in the conformal class of a given metric g: $\widehat{g} = \phi^\kappa g$ for some positive function ϕ and with $\kappa = \frac{4}{n-2}$. The decomposition for

$\widehat{K}$ is more complex. We set

$$\widehat{K} = \frac{\tau}{n}\widehat{g} + \phi^{-2}(\sigma + \mathbb{L}W),$$

where $\mathbb{L}$ is the Ahlfors operator, i.e. $\mathbb{L}W = \mathring{\mathcal{L}}_W g$ is the traceless part of the Lie derivative of g with respect to the vector field W, σ is a trace-free symmetric tensor which is also divergence-free, i.e. $\operatorname{div}_g \sigma \equiv 0$. And τ is a function that corresponds to the mean curvature of the embedding $M \hookrightarrow \mathcal{M}$.

The seed data then consist in the metric g, the mean curvature τ and the TT-tensor σ and the unknowns are ϕ and W. The equations of the conformal method are then

$$\begin{aligned} -\frac{4(n-1)}{n-2}\Delta\phi + \operatorname{scal}\phi &= -\frac{n-1}{n}\phi^{\kappa+1} + \frac{|\sigma + \mathbb{L}W|^2}{\phi^{\kappa+3}}, \\ \operatorname{div}(\mathbb{L}W) &= \frac{n-1}{n}\phi^{\kappa+2}d\tau. \end{aligned}$$

Despite technical difficulties to solve these equations, the conformal method has been highly successful in constructing large families of initial data.

However, the choice of the metric g in a given conformal class is arbitrary. The consequence of this fact is that several choices for the seed data (g, τ, σ) will lead to the same solution to the constraint equations and it is a priori very difficult to tell whether two such choices will lead to the same initial data or not.

The point of this talk is to show that there cannot be any way to do this apart from solving the equations of the conformal method.

The strategy to prove this fact is to find a situation where the equations of the conformal method has two solutions. In a previous work, I showed that such a situation occurs when g has vanishing Yamabe invariant with τ and σ carefully chosen by numerical methods. We then get two solutions $(\widehat{g}_1, \widehat{K}_1)$ and $(\widehat{g}_2, \widehat{K}_2)$ to the constraint equations and we now wonder how $\widehat{K}_1$ and $\widehat{K}_2$ decompose according to a different choice of a metric $\widetilde{g}$ in the conformal class of g. The claim is then that their TT-tensor part $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ differ.

This shows that there cannot be any map $\sigma \mapsto \widetilde{\sigma}$ such that the set of solutions to the conformal constraint equations with seed data (g, τ, σ) coincide with that associated to $(\widetilde{g}, \tau, \widetilde{\sigma})$.

How are the dominant energy conditions for Lorentzian spacetimes and initial data sets related?

Jonathan Glöckle

This short talk, based on the article [1], had the goal to discuss the relationship between the following two notions of dominant energy condition (=dec):

Definition 1. *A spacetime $(\overline{M}, \overline{g})$, i. e. a time-oriented Lorentzian manifold, is said to satisfy the* dominant energy condition *if* $\operatorname{Ein}^{\overline{g}}(V, W) \geq 0$ *for all future-causal vectors V, W in the same tangent space.*

Here, the Einstein curvature is given by $\mathrm{Ein}^{\overline{g}} := \mathrm{ric}^{\overline{g}} - \frac{1}{2}\mathrm{scal}^{\overline{g}}\overline{g}$ and we work with the signature convention $(-,+,\dots,+)$.

Definition 2. *An initial data set* (M,g,k)*, i. e. a manifold* M *equipped with a Riemannian metric* g *and a symmetric* 2*-tensor* k*, is said to satisfy the* dominant energy condition *if* $\rho \geq |j|_g$ *for*

$$\rho = \frac{1}{2}\left(\mathrm{scal}^g + \mathrm{tr}^g(k)^2 - |k|_g^2\right) \tag{1}$$
$$j = \mathrm{div}^g(k) - \mathrm{d}\,\mathrm{tr}^g(k).$$

There is a well-known way to pass from the first to the second notion of dec. Any spacelike hypersurface M of a spacetime $(\overline{M},\overline{g})$ carries an induced Riemannian metric g and an induced second fundamental form k (w. r. t. the future unit normal e_0), giving rise to an initial data set (M,g,k). In this case the so-called constraint equations yield the decomposition

$$\mathrm{Ein}^{\overline{g}}_{|M} = \rho \mathrm{d}t^2 + j \otimes \mathrm{d}t + \mathrm{d}t \otimes j + S, \tag{2}$$

where $\mathrm{d}t := -e_0^\flat$ and S is a symmetric 2-tensor on M. The dominant energy condition for $(\overline{M},\overline{g})$ implies that $\mathrm{Ein}^{\overline{g}}(e_0,-)^\sharp = -\rho e_0 + j$ is either past-causal or zero and hence $\rho \geq |j|_g$. So: The induced initial data set on any spacelike hypersurface of a dec spacetime satisfies dec.

We ask for a converse:

Question 1. *Does there for any dec initial data set* (M,g,k) *exist a dec spacetime* $(\overline{M},\overline{g})$ *such that* M *is contained in* $(\overline{M},\overline{g})$ *as spacelike hypersurface and* (M,g,k) *is the induced initial data set?*

This question came up in [2], where the authors explain how methods developed to study positive scalar curvature can be used to study the space of dec initial data sets. The goal, however, would be to eventually say something about dec spacetimes, and the authors suspect the answer to the question to be yes, providing a part of the missing link. This expectation was based on the following special cases:

- All vacuum initial data sets (i. e. $\rho = 0$, $j = 0$) are contained in a dec spacetime. In fact, by the celebrated solution of the vacuum Cauchy problem due to Yvonne Choquet-Bruhat [3], they are contained in a vacuum spacetime (i. e. $\mathrm{Ein}^{\overline{g}} = 0$).
- All initial data sets satisfying the strict dec $\rho > |j|_g$ admit a dec spacetime extension, cf. [4, Prop 1.10].

The first point could be generalized by considering the Cauchy problem with some form of matter. Given an initial data set (M,g,k) the procedure would be to first find suitable initial data for the matter fields. This is not easy, since typically some constraints need to satisfied in order to be able to carry through the second step: solving the Cauchy problem. Nevertheless, one might hope that with the help of a matter model with sufficiently many degrees of freedom, e. g. Vlasov matter [5, eq. (7.13)-(7.15)], it is possible to construct such matter initial data for all initial data sets (M, g, k). In the smooth world, however, this hope is destroyed by the negative answer to the above question.

Theorem 1 ([1, Main Theorem]). *For every manifold M of dimension $n \geq 3$ there is a smooth dec initial data set (M, g, k) for which there is no smooth dec spacetime $(\overline{M}, \overline{g})$ with spacelike hypersurface M such that (M, g, k) is the induced initial data set.*

The proof of the theorem relies on the following observation: If, in some point $p \in M$, the equation $\rho_p = |j_p|_g$ holds and $\mathrm{Ein}^{\overline{g}}(V, W) \geq 0$ for all future-causal vectors $V, W \in T_p\overline{M}$, then the tensor S from the decomposition (2) necessarily has to satisfy

$$S_p = \begin{cases} \frac{j_p \otimes j_p}{\rho_p} & \rho_p \neq 0 \\ 0 & \rho_p = 0. \end{cases} \tag{3}$$

The point is now that the function $p \mapsto S_p$ defined by (3) does not need be smooth even if ρ and j are smooth and $\rho = |j|_g$. In fact, one can cook up examples of such ρ and j where S is not even C^2. With this in mind, the task is to find an initial data set (M, g, k) that satisfies dec and such that on an open subset ρ and j (calculated by (1)) coincide with an example of this kind. Since this is essentially a local construction, the topology of the manifold M does not play a role. For details we refer to the aforementioned article.

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Rough metrics and curvature bounds from a Lorentzian perspective

Melanie Graf

A recurring theme of the present workshop has been how to deal with scalar curvature and more specifically scalar curvature bounds in the context of Riemannian metrics of low regularity. Similar issues with curvature bounds for rough (i.e. low regularity) metrics have a long history in the Lorentzian setting as well. There are several well-known key similarities and differences between these two settings:[1]

[1] References for all classical results may be found in either the standard textbook by O'Neill, *Semi-Riemannian Geometry*, or by Beem, Ehrlich and Easley, *Global Lorentzian Geometry*.

Riemannian: Smooth manifold M w. positive definite metric tensor g	Lorentzian: Non-degenerate metric tensor g with one negative eigenvalue & (M, g) *time-oriented*[2] $0 \neq v \in TM$ with $g(v, v) < 0$ (resp. ≤ 0) are called timelike (resp. causal) Causal Structure: $J^{\pm}, \leq$ and $I^{\pm}, \ll$[3]
$d_g(p, q) := \inf\{L(\gamma)\}$ where γ is a C^1 curve from p to q	$\tau(p, q) := \sup\{\int_a^b \sqrt{-g(\dot{\gamma}, \dot{\gamma})}\}$ where $\gamma : [a, b] \to M$ timelike from p to q if $p \ll q$ and $\tau(p, q) := 0$ otherwise
Volume measure $d\mathrm{vol}_g$	analogous
Curvature: Riem, Ric, Scal	analogous
Completeness and Hopf-Rionw	No Hopf-Rinow; Global hyperbolicity[4] acts as a replacement for completeness in *some* contexts
Curvature intimately linked to local and global properties of d_g and $d\mathrm{vol}_g \rightsquigarrow$ Metric Geometry: (M, d_g)	Lorentzian Length Spaces [6]: $(\underbrace{M, d_{\text{background}}}_{\text{metric space}}, \tau, \ll, \leq)$
Toponogov triangle comparison & Alexandrov Geometry	Lorentzian triangle comparison [1, 5] & synthetic timelike sectional curvature bounds [6]
Ricci curvature bounds via optimal transport & (R)CD spaces	Lorentzian optimal transport [7, 8] & synthetic Ricci curvature bounds [3]

Since "metric" (or "synthetic") Lorentzian geometry has only started to emerge much more recently than the Riemannian counterparts, analytic tools have always been important for dealing with rough metrics in Lorentzian geometry. To illustrate one of my favorites among these, we looked at the following

Theorem 1 (Lorentzian Myers Theorem, smooth version)**.** *Let (M, g) be a time-oriented n-dimensional Lorentzian manifold. Assume that (M, g) is globally hyperbolic and that there exists $k > 0$ such that $\mathrm{Ric}(v, v) \geq (n-1)\,k$ for all $v \in TM$ with $g(v, v) = -1$. Then $\sup\{\tau(p, q) : p, q \in M\} \leq \frac{\pi}{\sqrt{k}}$.*

This bound on the *timelike diameter* is analogous to the diameter bound one obtains in the Riemannian setting and the proof is similar as well, using that global hyperbolicity guarantees the existence of τ-realizing timelike geodesics between any $p, q \in M$ with $p \ll q$. One should, however, remark upon a key difference in the meaning of the theorems: The Riemannian theorem assumes geodesic completeness

[2]I.e. there exists a C^0 timelike vector field X allowing to continuously define future pointing causal vectors ($g(v, X) < 0$) and past pointing ones ($g(v, X) > 0$).

[3]We say $q \in I^+(p)$, or equivalently $p \ll q$, if there exists a smooth future directed timelike curve from p to q. J^+ and $\leq$ are defined analogously, but with causal curves (and $p \leq p$ by definition). I^- and J^- are defined via past directed curves.

[4]We say (M, g) is globally hyperbolic if there are no closed future directed causal curves and $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$.

to be able to use Hopf-Rinow, whereas (timelike) geodesic incompleteness is a consequence of the Lorentzian Myers theorem. The latter thus is the simplest example of a so-called "singularity theorem", predicting singular behavior in the sense of geodesic incompleteness from causality and curvature assumptions.

Non-smooth versions include a version for C^1-metrics where the Ricci curvature bound is interpreted distributionally, cf. [4], as well as versions for Lorentzian length spaces satisfying appropriate synthetic timelike curvature bounds, cf. [2, 3].

The proofs for the distributional versions all rely on estimates on smoothings via convolution summarized very informally as

$$(ab) \star \rho_\varepsilon - (a \star \rho_\varepsilon)(b \star \rho_\varepsilon) \to 0 \qquad \text{better than expected,}$$

where ρ_ε is a standard mollifier, for functions $a, b : \mathbb{R} \to \mathbb{R}$. For example, if $a \in C^0, b \in C^1$ then the difference above converges to zero locally uniformly in C^1 and not merely in C^0 as one would expect from the individual convergence of the factors. This is rooted in the same principle as commutator estimates for (linear) PDOs and convolution operators commonly referred to as Friedrichs Lemma and is very widely applicable.

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Moduli Spaces of Positive Curvature Metrics

Thorsten Hertl

Many (global) differential geometric quantities, like the diameter, the volume, or the average scalar curvature of a Riemannian manifold, are invariant under the pull-back action of the diffeomorphism group $\mathrm{Diff}(M)$ on the space of Riemannian metrics $\mathrm{Riem}(M)$. If one wishes to study the dependency of these quantities on those Riemannian metrics that satisfy a curvature condition C, like positive scalar curvature $\mathrm{scal} > 0$ or positive sectional curvature $\sec > 0$, it is thus enough to study these quantities as functions on the *moduli space* $\mathcal{M}^C(M) := \mathrm{Riem}^C(M)/\mathrm{Diff}(M)$. As the stabilisers of the diffeomorphism group, which are by definition the isometry groups, may vary in dimension with the underlying metric, we cannot expect that these moduli spaces are *not* manifolds in general, but rather (infinite dimensional) stratifolds.

To get rid of the singularities, one can consider the *observer moduli space* $\mathcal{M}^C_{x_0}(M) := \mathrm{Riem}^C(M)/\mathrm{Diff}_{x_0}(M)$ instead, where $\mathrm{Diff}_{x_0}(M)$ is the subgroup of *observer diffeomorphisms* consisting of those diffeomorphisms φ that satisfy $D_{x_0}\varphi = \mathrm{id}$ at the initially given point x_0. It is not hard to prove that $\mathrm{Diff}_{x_0}(M)$ acts freely on $\mathrm{Riem}(M)$ if the underlying manifold is connected.

Since these (observer) moduli spaces carry a lot of information about the geometries the underlying manifold M can carry, topologists try to answer the following question:

What can we say about the global topological properties of the observer moduli spaces? In particular, how rich or complicated are their homotopy groups?

Early results indicating that such moduli spaces can be disconnected were established in the early 90's by Kreck and Stolz in [5]. The first examples that path components of observer moduli spaces of positive scalar curvature metrics can be non-contractible were found in the pioneering work [1], in which the authors prove that $\pi_{4k}(\mathcal{M}^{\mathrm{scal}>0}_{x_0}(S^n)) \otimes \mathbb{Q} \neq 0$ provided n is odd and $k \ll n$. These results were later subsequently refined in [2] and in the recent preprint [6] to prove similar statements for positive Ricci curvature and certain intermediate curvatures respectively. For a comprehensive overview of this field, we refer to [8].

The main result of this talk based on [4] provides, to the authors knowledge, the first examples for non-trivial elements in the higher homotopy groups of the observer moduli space of positive sectional curvature metrics. Furthermore, in contrast to earlier resulst, the degree of the homotopy groups that host these non-trivial examples, can be close to the dimension of the manifold under consideration.

Main Theorem.

(1) $\pi_2(\mathcal{M}^{\mathrm{scal}>0}_{x_0}(M^4 \sharp \mathbb{C}P^2)) \neq 0$ *if* M^4 *has positive scalar curvature,*
(2) $\pi_2(\mathcal{M}^{\mathrm{sec}>0}_{x_0}(\mathbb{C}P^n)) \neq 0$,
(3) $\pi_{2k}(\mathcal{M}^{\mathrm{sec}>0}_{x_0}(\mathbb{C}P^n)) \otimes \mathbb{Q} \neq 0$ *if* $n \geq 4$ *and* $k \in \{3, \ldots, n-1\}$ *is odd.*

1. From Bundles to Elements in Homotopy Groups

How does one construct a non-contractible map $S^k \to \mathcal{M}^C_{x_0}(M)$? The naive approach would be to construct a promising map $S^k \to \mathrm{Riem}^C(M)$, compose it with the canonical projection, and hope that the result remains non-trivial. This turns out to be quite hard, as one needs highly non-trivial maps. The following ansatz has a higher chance to succeed.

Assume that we are given a smooth fibre bundle $M \hookrightarrow E \xrightarrow{p} B$, whose structure group reduces to $\mathrm{Diff}_{x_0}(M)$, together with fibre metric $\{g_b\}_{b\in B}$, formally a smooth family of inner products on the vertical tangent bundle $T^{\mathrm{vert}}E := \ker Tp$, satisfying our favourite curvature condition C. This datum defines a map *classifying map*

$$f_{E,g}\colon S^k \to \mathcal{M}^C_{x_0}(M) \qquad \text{given by} \qquad b \mapsto [F^* g_b],$$

where F is an appropriate diffeomorphism that identifies M and the fibre $p^{-1}(\{b\})$. (Here, we use the $\mathrm{Diff}_{x_0}(M)$ reduction to guarantee that compatible choices can be made.)

The name *classifying map* was not chosen by accident: Since the space of Riemannian metrics $\mathrm{Riem}(M)$ without any curvature condition is convex, in particular contractible, the observer moduli space $\mathcal{M}_{x_0}(M) \simeq B\mathrm{Diff}_{x_0}(M)$ is a model for the classifying space of M-fibre bundles with structure group $\mathrm{Diff}_{x_0}(M)$. Thus, if we compose the classifying map $f_{E,g}$ with the canonical inclusion $\mathcal{M}^C_{x_0}(M) \hookrightarrow \mathcal{M}_{x_0}(M)$, we get the classifying map f_E for the bundle $E \to S^k$, which is homotopic to the constant map if and only if the bundle is trivial.

Upshot: To construct non-trivial elements in $\pi_k(\mathcal{M}^C_{x_0}(M))$, it suffices to construct non-trivial bundles with structure group $\mathrm{Diff}_{x_0}(M)$ and a fibre metric satisfying C.

2. Anti Blow-Up Families

In order to find promising $M^4\sharp\mathbb{C}P^2$-fibre bundles, we first observe that $S^3 = \partial D^4 = \partial D\mathcal{O}(1)$, where latter is the disc bundle associated to the dual tautological line bundle over $\mathbb{C}P^1$. The projection to the second component turns

$$\mathcal{DO}(1) := (S^3 \times \mathbb{C}P^1 \times D^2)/\sim, \qquad (p,[q],\lambda) \sim (q^{-1}\mathrm{e}^{\mathrm{i}\theta}qp,[q],\mathrm{e}^{\mathrm{i}\theta}\lambda)$$

into a fibre bundle $D\mathcal{O}(1) \hookrightarrow \mathcal{DO}(1) \to \mathbb{C}P^1$ with trivial boundary $\partial\mathcal{DO}(1) \cong S^3 \times \mathbb{C}P^1$. The zero section provides a canonical embedding $\mathbb{C}P^1 \times \mathbb{C}P^1 \hookrightarrow \mathcal{DO}(1)$. Furthermore, one can put a positive scalar curvaure fibre metric $g_{\mathcal{DO}(1)}$ on this bundle that is the product metric $g_{S^3} \oplus \mathrm{d}\,t^2$ of the standard round sphere and the Euclidean metric near the boundary.

If M^4 is a manifold that carries a positive scalar curvature metric, we can use the surgery theorem [3] to deform any positive scalar curvature metric on M^4 so that it has product structure near the boundary of $M^4 \setminus D^4$. Thus, we can equip the fibre-connected sum

$$E_M := \big(M \setminus D^4\big) \times \mathbb{C}P^1 \cup_{S^3 \times \mathbb{C}P^1} \mathcal{DO}(1),$$

which is the total space of a bundle $M\sharp\mathbb{C}P^2 \hookrightarrow E_M \to \mathbb{C}P^1$, with a fibre metric that has positive scalar curvature.

One can prove that

$$\langle p_1(T^{\mathrm{vert}}E_M);[\mathbb{C}P^1 \times \mathbb{C}P^1]\rangle = \int_{\mathbb{C}P^1 \times \mathbb{C}P^1} p_1(T^{\mathrm{vert}}E_M) = -2,$$

which would be divisible by 3 if E_M were fibre-diffeomorphic to $(M\sharp\mathbb{C}P^2) \times \mathbb{C}P^1$ due to Hirzebruch's signature theorem.

3. Positive Sectional Curvature and Homotopy Theory

Since E_{S^4} is a $\mathbb{C}P^2$-fibre bundle and all constructions of the previous section can be carried out explicitly, one could ask whether it is possible to deduce non-triviality results for $\mathcal{M}^{\sec>0}_{x_0}(\mathbb{C}P^2)$? The answer is positive, but we cannot rely on the gluing result of the previous section. Instead, we will use the isometry group of the Fubini-Study metric and homotopy theory, which further allows us to produce results for higher dimensional complex projective spaces.

Since the unitary group $\mathrm{U}(n+1)$ acts isometrically on $\mathbb{C}P^n$, we get the following commutative diagram of fibre bundle maps

$$\begin{array}{ccccc}
\mathrm{U}(n+1) & \longrightarrow & E\mathrm{U}(n+1) & \longrightarrow & B\mathrm{U}(n+1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fr}^+(\mathbb{C}P^n) & \longrightarrow & \mathrm{Riem}^{\sec>0}(\mathbb{C}P^n)/\!/\mathrm{Diff}_{x_0}(\mathbb{C}P^n) & \longrightarrow & \mathrm{Riem}^{\sec>0}(\mathbb{C}P^n)/\!/\mathrm{Diff}(\mathbb{C}P^n) \\
 & & \downarrow \simeq & & \downarrow \\
 & & \mathcal{M}^{\sec>0}_{x_0}(\mathbb{C}P^n) & & B\mathrm{hAut}(\mathbb{C}P^n),
\end{array}$$

in which $\mathrm{hAut}(\mathbb{C}P^n)$ denotes the topological monoid of all continuous maps homotopic to the identity, $\mathrm{Fr}^+(\mathbb{C}P^n)$ the $\mathrm{GL}_{2n}(\mathbb{R})$-principal bundle of positively oriented frames, and $\mathrm{Riem}^{\sec>0}(\mathbb{C}P^n)/\!/\mathrm{Diff}_{(x_0)}(M)$ the homotopy quotients, one of which, in this particular case, is homotopy equivalent to the actual quotient, the observer moduli space $\mathcal{M}^{\sec>0}_{x_0}(\mathbb{C}P^n)$.

Applying homotopy groups to that diagram yields maps between two long exact sequences. It was proven in [7] that the map $B\mathrm{PU}(n+1) \to B\mathrm{hAut}(\mathbb{C}P^n)$ is injective on π_2 and rationally an isomorphism. A diagram chase now implies part (2) and (3) of the main theorem.

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Schoen's conjecture for limits of isoperimetric surfaces

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(joint work with Michael Eichmair)

Let (M, g) be a connected complete Riemannian manifold of dimension $3 \le n \le 7$ with integrable scalar curvature R. We assume that (M, g) is asymptotically flat in the sense that there is a number $n-3 < \tau \le n-2$ and a nonempty compact set whose complement is diffeomorphic to $\{x \in \mathbb{R}^n : |x| > 1\}$ such that, as $x \to \infty$,

$$|g_{ij} - \delta_{ij}| + |x|\,|\partial_k g_{ij}| + |x|^2\,|\partial^2_{k\ell} g_{ij}| = O(1)\,|x|^{-\tau}.$$

Recall that the mass of (M, g) is given by

$$m = \frac{1}{2\,(n-1)\,n\,\omega_n} \lim_{\lambda \to \infty} \lambda^{-1} \int_{S^{n-1}_\lambda(0)} \sum_{i,\,j=1}^{n} x^i\,(\partial_j g_{ij} - \partial_i g_{jj}). \tag{1}$$

Here, ω_n is the Euclidean volume of an n-dimensional unit ball.

The positive mass theorem proven by R. Schoen and S.-T. Yau [8] confirms the basic heuristic that, if (M, g) has nonnegative scalar curvature, then the mass is nonnegative and zero if and only if (M, g) is flat $\mathbb{R}^n$. Its proof is based on the following two observations. First, if (M, g) has negative mass, then there is a noncompact area-minimizing boundary $\Sigma \subset M$ that is asymptotic to a coordinate hyperplane and stable with respect to asymptotically constant variations, i.e., for every $f \in C^\infty_c(\Sigma)$,

$$\int_\Sigma (|h|^2 + \mathrm{Ric}(\nu, \nu))\,(1 + f)^2 \le \int_\Sigma |\nabla f|^2. \tag{2}$$

Here, h is the second fundamental form of Σ with respect to the normal ν and Ric the Ricci curvature of (M, g). Geometrically, (2) means that Σ passes the second derivative test for area among variations that are asymptotic to a vertical translation. Second, if the scalar curvature of (M, g) is positive, then (M, g) does not contain a noncompact area-minimizing boundary that is asymptotic to a coordinate hyperplane and satisfies (2). In view of this argument, R. Schoen has made the following conjecture.

Conjecture 1. *Let (M, g) be an asymptotically flat Riemannian manifold of dimension $3 \le n \le 7$ with nonnegative scalar curvature. Suppose that there exists a noncompact area-minimizing boundary Σ. Then (M, g) is isometric to flat $\mathbb{R}^n$.*

Conjecture 1 has been confirmed in the case where $n = 3$ and in the case where $3 \le n \le 7$ under the additional assumption that (M, g) is asymptotic to spatial Schwarzschild and Σ satisfies (2); see [1, 2]. In [4], we have resolved Conjecture 1 as follows. It turns out that, quite surprisingly, Conjecture 1 fails in the case where $3 < n \le 7$.

Theorem 1. *Let (M, g) be spatial Schwarzschild of dimension $3 < n \le 7$. There exist infinitely many mutually disjoint noncompact area-minimizing boundaries in (M, g).*

It can be seen that the noncompact area-minimizing boundaries constructed in the proof of Theorem 1 do not satisfy (2). Our next result shows that Conjecture 1 fails even with the additional assumption that Σ satisfies (2).

Theorem 2. *Let $3 < n \leq 7$ and $n-3 < \tau < n-2$. There exists a Riemannian manifold (M, g) of dimension n that is asymptotically flat of rate τ with nonnegative scalar curvature and positive mass that contains infinitely many mutually disjoint noncompact area-minimizing boundaries all of which are stable with respect to asymptotically constant variations.*

Theorem 2 shows that Conjecture 1 cannot possibly hold unless Σ captures additional global information on the geometry of (M, g). A natural such situation is when Σ arises as the limit of isoperimetric surfaces. As the main result of the presented paper, we have settled Conjecture 1 in this case.

Theorem 3. *Let (M, g) be an asymptotically flat Riemannian manifold of dimension $3 < n \leq 7$ with nonnegative scalar curvature. Suppose that there exist a noncompact area-minimizing boundary $\Sigma = \partial\Omega$ and isoperimetric regions $\Omega_1, \Omega_2, \ldots \subset M$ with $\Omega_k \to \Omega$ locally smoothly. Then (M, g) is isometric to flat $\mathbb{R}^n$.*

In asymptotically flat Riemannian three-manifolds with nonnegative scalar curvature and positive mass, there is a unique isoperimetric region for every given sufficiently large amount of volume and these large isoperimetric regions are close to centered coordinate balls; see [3]. An important consequence of Theorem 3 and a step toward the characterization of large isoperimetric regions in asymptotically flat Riemannian manifolds of dimension $3 < n \leq 7$ is that the (unique) large components of the boundaries of such regions necessarily diverge as their volume tends to infinity; see [5] for the corresponding result in the case where $n = 3$.

Below, I will outline our proof of Theorem 3. Let (M, g) be an asymptotically flat Riemannian manifold of dimension $3 < n \leq 7$ with nonnegative scalar curvature. We assume that $\Sigma = \partial\Omega$ is a noncompact area-minimizing boundary. Suppose for a contradiction that $m > 0$. A first difficulty not present in the case where $n = 3$ is to show that Σ is asymptotic to a coordinate hyperplane. This is complicated by the fact that Σ is not known to satisfy (2) at this point. To remedy this, we prove explicit estimates for the density ratio of Σ in large coordinate balls. The proof is based on the monotonicity formula applied to carefully chosen, off-centered balls. We then use these estimates to establish a precise asymptotic expansion for Σ. Using that $\tau > n-3$, it follows that Σ is asymptotically flat with mass zero.

Next, we assume that $\Sigma = \partial\Omega$ where Ω is the limit of large isoperimetric regions $\Omega_1, \Omega_2, \ldots$ with $|\Omega_k| \to \infty$ and prove that Σ is stable with respect to asymptotically constant variations. To this end, we consider the second variation of area of Ω_k with respect to a suitable Euclidean translation that is corrected to be volume-preserving. The stability with respect to asymptotically constant variations then follows by passing to the limit $k \to \infty$. Revisiting the proof of the positive mass theorem, we see that Σ is isometric to flat $\mathbb{R}^{n-1}$ and totally geodesic and that both R and $\mathrm{Ric}(\nu,\nu)$ vanish along Σ; see [7].

Then, given any $p \in M$, we construct a noncompact area-minimizing boundary $\Sigma_p \subset M$ with $p \in \Sigma_p$. In view of Theorem 1 and different from the case where $n = 3$, we need to ensure that Σ_p again satisfies (2). To this end, we construct suitable local perturbations of the metric g and obtain Σ_p as the limit of large isoperimetric regions with respect to these perturbations. A crucial ingredient in this construction is that asymptotically flat Riemannian manifolds of positive mass admit isoperimetric regions of every sufficiently large volume; see [2].

Finally, we show that the Riemann curvature tensor Rm of (M, g) vanishes, contradicting that $m > 0$. To this end, we first observe that, since Σ is flat and totally geodesic, the Gauss equation and Codazzi equation imply that Rm vanishes on Σ along four tangential directions and three tangential directions and one normal direction. We then pick a point $p \in \Sigma$ and choose a sequence Σ_1, $\Sigma_2, \dots$ of flat and totally geodesic noncompact area-minimizing boundaries that converges locally smoothly to Σ. We locally write Σ_k as a graph over Σ and linearize the second fundamental form of Σ along this approximation. This gives a positive function f that satisfies

$$\nabla^2 f + f\ \mathrm{Rm}(\,\cdot\,,\nu,\nu,\,\cdot\,) = 0 \tag{3}$$

and is defined on all of Σ, a half-space of Σ, or a slab within Σ, depending on as to whether $\Sigma_k \cap \Sigma$ has zero, one, or at least two components. Tracing (3) and using that $\mathrm{Ric}(\nu,\nu) = 0$ along Σ, we see that f is harmonic. In the first two cases, by the Liouville theorem, f equals an affine function. By contrast, we show that the third scenario does not occur. Revisiting (3), we see that $\mathrm{Rm} = 0$ at p.

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Space of positive scalar curvature metrics on 4-manifolds

Hokuto Konno

Given a closed smooth manifold X, let $\mathcal{R}^+(X)$ denote the space of positive scalar curvature metrics. For $\mathcal{R}^+(X) \neq \emptyset$ and $\dim X \neq 4$, the topology of $\mathcal{R}^+(X)$ has been extensively studied. For $\dim X < 4$, the space $\mathcal{R}^+(X)$ turns out to be contractible [1,3]. For $\dim X > 4$, in contrast, many authors proved that $\mathcal{R}^+(X)$ can have quite non-trivial topology. The main technique to prove such a result in higher dimensions is a combination of surgery and index theory.

However, because of the failure of surgery techniques, little is known about the topology of $\mathcal{R}^+(X)$ for 4-manifolds X. In dimension 4, certain family versions of the Seiberg–Witten invariant sometimes tell non-trivial information about the topology of $\mathcal{R}^+(X)$. We shall summarize what is known in this direction in the literature so far.

To understand general strategy, we start by recalling the Seiberg–Witten invariant as an obstruction to the existence of positive scalar curvature metric. Let $(X, \mathfrak{s})$ be a closed smooth spin^c 4-manifold. Fix a Riemannian metric g on X. The *Seiberg–Witten equations* are of the form

$$(SW)_{(X,\mathfrak{s},g)} : \begin{cases} F_A^+ = \sigma(\Phi, \Phi), \\ D_A\Phi = 0. \end{cases}$$

Here A is a $U(1)$-connection on the determinant line bundle for $\mathfrak{s}$, F_A^+ is the self-dual part of the curvature of A, Φ is a positive spinor for $\mathfrak{s}$, $\sigma(-,-)$ is a certain quadratic form, and D_A is the spin^c Dirac operator. The Seiberg–Witten equations are a non-linear partial differential equations, which are elliptic if one takes into account the gauge symmetry. The gauge symmetry is given by an infinite-dimensional group $\mathrm{Map}(X, U(1))$, which naturally acts on the space of $U(1)$-connections and spinors, and the Seiberg–Witten equations are invariant under the action of $\mathrm{Map}(X, U(1))$. The space of solutions divided by $\mathrm{Map}(X, U(1))$

$$\mathcal{M}(X, \mathfrak{s}, g) := \{(A, \Phi) \mid (A, \Phi) \text{ satisfies } (SW)_{(X,\mathfrak{s},g)}\}/\mathrm{Map}(X, U(1)).$$

is called the *moduli space* of solutions to the Seiberg–Witten equations.

In practice, it is convenient to consider a perturbation (which we shall omit from our notation) of the equations by adding a self-dual 2-form $\mu \in i\Omega_g^+(X)$ to F_A^+ to achieve the transversality. Under a generic choice of (g, μ), the moduli space $\mathcal{M}(X, \mathfrak{s}, g)$ is a smooth manifold of dimension

$$d(\mathfrak{s}) = \frac{1}{4}(c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)).$$

Most notably, the moduli space $\mathcal{M}(X, \mathfrak{s}, g)$ is always compact. Further, the moduli space can be oriented by picking a topological data called homology orientation. Thus, when $d(\mathfrak{s}) = 0$, we can count points in the moduli space and get an integer.

Let $b^+(X)$ be the maximal dimension of positive-definite subspaces of $H^2(X;\mathbb{R})$ with respect to the intersection form. If $b^+(X) \geq 2$, it turns out that the count $\#\mathcal{M}(X, \mathfrak{s}, g) \in \mathbb{Z}$ is independent of choice of g. This count is called the *Seiberg–Witten invariant*, denoted by $SW(X, \mathfrak{s})$.

In his paper introducing the Seiberg–Witten equations to mathematicians, Witten [5] proved that the moduli space $\mathcal{M}(X,\mathfrak{s},g)$ is empty if g is a positive scalar curvature metric. In particular, if $b^+(X)\geq 2$, we have $SW(X,\mathfrak{s})=0$.

Now we turn to a family version of this story. Let $(X,\mathfrak{s})\to E\to B$ be a smooth fiber bundle whose fiber is a closed spin^c 4-manifold $(X,\mathfrak{s})$ and whose base is a closed manifold B. Suppose that $d(\mathfrak{s})<0$, and $d(\mathfrak{s})=-\dim B$. Picking a fiberwise metric $g_E=\{g_b\}_{b\in B}$ on E, we can form the *parameterized moduli space*

$$\mathcal{M}(E,g_E):=\bigcup_{b\in B}\mathcal{M}(E_b,\mathfrak{s}_b,g).$$

Generically, $\mathcal{M}(E,g_E)$ is a closed manifold of dimension 0. Thus we can count $\#\mathcal{M}(E,g_E)$ at least over $\mathbb{Z}/2$ (which can be upgraded to a count over $\mathbb{Z}$ under a certain hypothesis), and if $b^+(X)\geq \dim B+2$, this count is independent of g_E. Thus we can get a topological invariant $SW(E):=\#\mathcal{M}(E,g_E)\in\mathbb{Z}/2$ of the fiber bundle E, which we call the *families Seiberg–Witten invariant.* Just as in the unparameterised case, if E admits a fiberwise positive scalar curvature metric g_E, we get $\mathcal{M}(E,g_E)=\emptyset$, and hence $SW(E)=0$. Thus, if $SW(E)\neq 0$, one can conclude that $\mathcal{R}^+(X)$ has non-trivial topology.

Ruberman [4] gave the first example of a 4-manifold X with $\mathcal{R}^+(X)\neq\emptyset$ such that $\mathcal{R}^+(X)$ is shown to have non-trivial topology, based on the above idea applied to a 1-dimensional base space (but with a more elaborate version of the families Seiberg–Witten invariant):

Theorem 1 (Ruberman [4]). *There are 4-manifolds X such that $\mathcal{R}^+(X)\neq\emptyset$ and $\mathcal{R}^+(X)$ have infinitely many components. More concretely, $X=\#_{2n}\mathbb{CP}^2\#_k\overline{\mathbb{CP}}^2$ for $n\geq 2$ and $k\geq 10n+1$ are such 4-manifolds.*

Note that $b^+(X)$ is even for examples in Theorem 1, which is an essential restriction in the proof of this result. However, using the above idea of the families Seiberg–Witten invariant applied to a family over the 2-dimensional torus T^2, we can prove that $\mathcal{R}^+(X)$ can have non-trivial topology even when $b^+(X)$ is odd. More precisely, we have:

Theorem 2 ([2]). *Let $X=\#_{2n+1}\mathbb{CP}^2\#_k\overline{\mathbb{CP}}^2$ for $n\geq 2$ and $k\geq 10n+3$. Then $\mathcal{R}^+(X)$ is not 1-connected, i.e. at least one of $\pi_0(\mathcal{R}^+(X))$ and $\pi_1(\mathcal{R}^+(X))$ is non-trivial.*

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A version of Ilmanen's conjecture for asymptotically hyperbolic manifolds

Klaus Kröncke
(joint work with Mattias Dahl, Stephen McCormick and Louis Yudowitz)

In 2008, Ilmanen conjectured that there should be a relation for ALE manifolds and Ricci-flat cones between

(i) λ not a local maximum,
(ii) failure of positive mass.

It suggests a deep relation between Ricci flow and the positive mass theorem.

Recall that for a metric g on a compact manifold, $\lambda(g)$ is defined to be the smallest eigenvalue of the operator $4\Delta_g + \mathrm{scal}_g$. The importance of the λ-entropy lies in the fact that is is monotonically increasing under the Ricci flow. The right analogue of the λ-entropy on noncompact ALE-manifold was found by Deruelle-Ozuch in [2] (building up on Haslhofer in [4]) and is

$$\lambda_{ALE}(g) = \inf_{f \in C_c^\infty} \int_M (|\nabla f|_g^2 + \mathrm{scal}_g) e^{-f} \, dV - m_{ADM}(g),$$

where $m_{ADM}(g)$ is the ADM-mass. The functional λ_{ALE} is shown to be monotonically increasing under the Ricci flow on ALE manifolds. By combining results by Deruelle-Ozuch [2] and Hall-Haslhofer-Siepmann [3], one gets

Theorem 1. *Let $(M, \widehat{g})$ be a Ricci-flat ALE-manifold. Then the following are equivalent:*

(i) *$\widehat{g}$ is a local maximizer of λ_{ALE}*
(ii) *$m_{ADM}(g) \geq m_{ADM}(\widehat{g}) = 0$ for all metrics g near $\widehat{g}$ with* scal_g *being integrable and nonnegative.*

This solves the above conjecture by Ilmanen. In our own work, we will focus on the asymptotically hyperbolic (AH) setting. This geometric setting requires different notions of mass and entropy which we defined in [1]. The expander entropy (relative to a reference metric $\widehat{g}$) is

$$\mu_{AH,\hat{g}}(g) = \inf_{f \in C_c^\infty} \int_M (|\nabla f|_g^2 + \mathrm{scal}_g + n(n-1) - 2(n-1)(f+1)) e^{-f} \, dV_g + 2(n-1) \int_M dV_{\widehat{g}} - m_{ADM,\widehat{g}}(g),$$

and it is nondecreasing under the normalized Ricci flow in the AH setting. The volume-renormalized mass is

$$m_{VR,\widehat{g}}(g) = m_{ADM,\widehat{g}}(g) + 2(n-1) \int_M (\, dV_g - \, dV_{\widehat{g}}).$$

If $g - \widehat{g} \in H^k$, $k > n/2+2$, then $\mu_{AH,\widehat{g}}(g) \in \mathbb{R}$. If in addition, $\mathrm{scal}_g + n(n-1) \in L^1$, $m_{VR,\widehat{g}}(g) \in \mathbb{R}$. If furthermore, $g - \widehat{g} = O(e^{-(n-1+\epsilon)r})$, the boundary integral of the ADM-mass drops and we get $m_{VR,\widehat{g}}(g) = RV_{\widehat{g}}(g) = \mathrm{vol}(g) - \mathrm{vol}(\widehat{g})$, which is the renormalized volume.

Recall that an AH Einstein manifold is called Poincaré-Einstein. Our analogue of Theorem 1, established in joint work with Dahl-McCormick [1] and Yudowitz [5], is

Theorem 2. [1,5] *Let $(M, \hat{g})$ be a Poincaré-Einstein manifold. Then the following are equivalent:*

(i) *$\hat{g}$ is a local maximizer of $\mu_{AH,\hat{g}}$*
(ii) *$\mu_{AH,\hat{g}}(g) \geq \mu_{AH,\hat{g}}(\hat{g}) = 0$ for all metrics g near $\widehat{g}$ with $\mathrm{scal}_g + n(n-1)$ being integrable and nonnegative.*
(iii) *$RV_{\widehat{g}}(g) \geq 0$ for all metrics g near $\widehat{g}$ with $g - \widehat{g} = O(e^{-(n-1+\epsilon)r})$ and $\mathrm{scal}_g + n(n-1)$ being integrable and nonnegative.*

Due to monotonicity of the entropies along (the respective notion of) Ricci flow, it is natural to relate Theorems 1 and 2 to dynamical stability of Ricci-flat ALE manifolds, resp. Poincaré-Einstein manifolds. While in the ALE setting, this is done under the integrability condition, we can establish this relation without further conditions in the AH setting.

Theorem 3. [5] *(Dynamical stability) Let $(M, \hat{g})$ be a Poincaré-Einstein manifold. If $\hat{g}$ is a local maximizer of $\mu_{AH,\hat{g}}$, then for every $L^2 \cap L^\infty$-neighborhood $\mathcal{U}$ of $\hat{g}$, there exists a $L^2 \cap L^\infty$-neighborhood $\mathcal{V} \subset \mathcal{U}$ such that the Ricci flow starting at any metric in $\mathcal{V}$ exists for all times and converges (modulo diffeomorphisms) to a Poincaré-Einstein metric in $\mathcal{U}$.*

Theorem 4. [5] *(Dynamical instability) Let $(M, \hat{g})$ be a Poincaré-Einstein manifold. If $\hat{g}$ is not a local maximizer of $\mu_{AH,\hat{g}}$, then there exists a nontrivial ancient Ricci flow $\{g(t)\}_{t\in(-\infty,0]}$ that converges (modulo diffeomorphisms) to $\hat{g}$ for $t \to -\infty$.*

Observe also that the converse implications do also hold due to the monotonicity of $\mu_{AH,\hat{g}}$ along the Ricci flow. In particular, each Poincaré-Einstein manifold is either dynamically stable or dynamically unstable and this entirely depends on the local behaviour of $\mu_{AH,\hat{g}}$. Summarizing and combining these results and the discussion, we establish the following equivalences for Poincaré-Einstein manifolds:

dynamical stability ⇔ positive mass theorem for nearby metrics

dynamical instability ⇔ failure of positive mass for nearby metrics

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Band width and the Rosenberg index

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The Rosenberg index $\mathrm{Ind}_{\pi_1(M)}(D_{\widetilde{M}})$, the C*-algebraic $\pi_1(M)$-equivariant higher index of the Dirac operator of the universal covering $\widetilde{M}$ of a closed spin manifold M, is defined by using (Real) C*-algebra K-theory and coarse geometry as a vast generalization of the Fredholm index. It has been one of the most powerful obstructions for a closed spin manifold to admit a positive scalar curvature metric. Indeed, under the assumption of the Baum–Connes injectivity of $\pi_1(M)$, the Rosenberg index is a complete obstruction to the existence of a PSC metric on $M \times B^n$ for some $n \in \mathbb{N}$, where B is an 8-dimensional closed spin manifold B with $\mathrm{Sgn}(B) = 0$ and $\hat{A}(B) = 1$ (the Rosenberg–Stolz theorem [5]). On the other hand, Schick used the Schoen–Yau minimal surface method to construct a closed spin manifold in dimensions 5, 6, 7 which does not admit any PSC metric but its Rosenberg index vanishes. This leads us to explore a PSC obstruction beyond the Rosenberg index.

Recently, Gromov shed new lights on this problem [2]. One of the remarkable ideas is a quantitative refinement of the non-existence of a complete PSC metric on $M \times \mathbb{R}$ when the PSC metric on M is obstructed. A (proper) compact *Riemannian band* V is a compact Riemannian manifold with inward and outward boundaries $\partial_\pm V$. The distance of $\partial_+ V$ and $\partial_- V$ is called its *width*. Gromov proved that, if $\partial_+ V$ does not admit any PSC metric by the reason coming from the minimal surface method, then the width of V is bounded by a constant depending on the infimum of the scalar curvature and the dimension. Moreover, this inequality is also used to define a quantity for a closed Riemannian manifold that is related to the lower bound of the scalar curvature.

Definition 1. *Let $\mathcal{V}$ be a class of compact Riemannian bands. The $\mathcal{V}$-width of a closed Riemannian manifold (M,g) is defined by* $\mathrm{width}_{\mathcal{V}}(M,g) := \sup \mathrm{width}(V, g_V)$, *where (V, g_V) runs over all bands in $\mathcal{V}$ that is isometrically immersed to M.*

A point is that the infiniteness of the $\mathcal{V}$-width independent of g. If the band width inequality holds for any $\mathcal{V}$-band, then $\mathrm{width}_{\mathcal{V}}(M,g) = \infty$ implies the non-existence of a PSC metric on M.

Following this line, Zeidler, Cecchini, and Guo–Xie–Yu [1, 3, 7] imported this idea to the Dirac operator method. In these papers, a band width inequality for bands in the class $\mathcal{KO}$ consisting of (V, g_V) such that V is equipped with a spin structure and the higher index of the inward boundary $\mathrm{Ind}_{\pi_1(V)}(D_{\partial_+ \widetilde{V}})$ does not vanish. Our main result is to compare the PSC obstruction coming from $\mathcal{KO}$-band width with the Rosenberg index, which answers to a conjecture by Zeidler [7, Conjecture 4.12].

Theorem 1 ([4, Theorem 1.3])**.** *Let (M,g) be a closed spin manifold. If M has infinite $\mathcal{KO}$-width, then* $\mathrm{Ind}_{\pi_1(M)}(D_{\widetilde{M}}) \neq 0$.

Since the infiniteness of the $\mathcal{KO}$-width is stable under the direct product with the manifold B, the theorem follows from the Rosenberg–Stolz theorem if $\pi_1(M)$ satisfies the Baum–Connes injectivity.

The (coarse) higher index $\mathrm{Ind}_\Gamma(D_X)$ is defined for a complete Riemannian spin manifold X on which a discrete group Γ acts properly. More generally, a relative version of the higher index $\mathrm{Ind}_{\mathrm{rel},\pi}(D_W)$ is defined for a complete Riemannian manifold W with boundary on which a discrete group π acts properly. It takes value in the (Real) K-theory of a certain quotient C*-algebra, and can be nontrivial only if $\partial W \subset W$ is not coarsely equivalent.

(1) If Y is a Γ-equivariant submanifold-with-boundary of X with the same dimension, the 'restriction map' sends $\mathrm{Ind}_\Gamma(D_X)$ to $\mathrm{Ind}_{\mathrm{rel},\Gamma}(D_Y)$.
(2) The 'K-theory boundary map' sends $\mathrm{Ind}_{\mathrm{rel},\pi}(D_W)$ to $\mathrm{Ind}_\pi(D_{\partial W})$ (this fact is called the 'boundary of Dirac is Dirac' principle).

A plausible strategy to the proof of Theorem 2 is to relate $\mathrm{Ind}_{\pi_1(M)}(D_{\widetilde{M}})$ with $\mathrm{Ind}_{\pi_1(V)}(D_{\partial_+\widetilde{V}})$ by using the above (1) and (2). However, there are some stumbling blocks to execute it.

(i) Even after taking the universal coverings, $\widetilde{V} \to \widetilde{M}$ may not be injective.
(ii) The inclusion $\partial\widetilde{V} \subset \widetilde{V}$ is a coarse equivalence, thus $\mathrm{Ind}_{\mathrm{rel},\pi_1(V)}(D_{\widetilde{V}})$ is trivial and has no information.
(ii) A $\mathcal{KO}$-band V has two boundary components ∂_+V and ∂_-V. We need to separate one to another to get $\mathrm{Ind}(\partial_+V)$.

To solve the problems (ii) and (iii), we take a sequence of $\mathcal{KO}$-bands $\{V_n\}$ and immersions $V_n \to M$ such that the width of V_n with respect to the induced metric goes to infinity, consider the box space $\mathbf{V} := \bigsqcup V_n$, and focus on the asymptotic behavior of (relative) higher indices by considering a suitable quotient C*-algebra.

To solve (i), we construct a canonical lifting of a finite propagation operator on $\widetilde{M}$ to $\widetilde{V}_n$ in the way that $\mathrm{Ind}_{\pi_1(M)}(D_{\widetilde{M}})$ is lifted to $\mathrm{Ind}_{\mathrm{rel},\pi_1(V_n)}(D_{\widetilde{V_n}})$. If there is a subspace $Z \subset X$ any short loop of X is contained in the R-neighborhood of Z for some $R > 0$, then any finite propagation operator on X lifts to $\widetilde{X}$ modulo the subspace $\widetilde{Z} \subset \widetilde{X}$. A typical example is $X = \mathbb{R}^2 \setminus \mathrm{int}\,\mathbb{D}^2$ and $Z = \partial\mathbb{D}^2$, in which case $\widetilde{X}$ is the helical surface. This idea fits well with immersions of bands which we are considering now. A key observation is that $\widetilde{M}$ is not only 1-connected but also uniformly 1-connected, that is, there is an increasing function $\varphi\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\varphi(t) \to \infty$ as $t \to \infty$ such that any loop in $B_R(x)$ is trivial in $B_{\varphi(R)}(x)$.

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Scalar curvature rigidity using Ricci flows

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(joint work with Luen-Fai Tam)

In scalar curvature geoemetry, there has been interest in developing the compactness theory in constrast with the Ricci geometry where the structure of Ricci limit space has been studied extensively. At the same time, metrics with low-regularity arise naturally also in the study of Brown-York quasi-local mass. Hence, understand the notion of scalar curvature lower bound for metrics with weaker regularity has been one of the important part in this problem. To extend the concept of scalar curvature lower bound to non-smooth metrics in a meaningful way, it is important to ensure singular metrics with $\mathrm{scal} \geq \kappa$ still satisfies the rigidity Theorem in scalar curvature geometry. For instances, the classicial torus rigidity Theorem states that metrics with $\mathrm{scal} \geq 0$ on $\mathbb{T}^n$ must be flat. When $3 \leq n \leq 7$, this was proved by Schoen-Yau [13, 14] using method of minimal surface while the general case was proved by Gromov-Lawson [4] using spin method.

This question was first took up by Gromov in [5]. By reformulating the scalar curvature lower bound using local C^0 structure of metric, Gromov showed that on a fixed manifold M, if a sequence of smooth metric g_i satisfies $\mathrm{scal}(g_i) \geq \kappa$ for some continuous function $\kappa \in C^0(M)$ and converges to a smooth metric g_∞ in C^0_{loc} sense, then the scalar curvature of g_∞ also satisfies $\mathrm{scal}(g_\infty) \geq \kappa$. This opened the door of understanding scalar curvature in a general framework. In particular, this semi-lower continuity nature suggests the following definition of scalar curvature lower bound for C^0 metric on a given manifold:

Definition 1. *Given a closed manifold M and a continuous metric g on M, we say that* $\mathrm{scal}(g) \geq \kappa$ *for some $\kappa \in C^0(M)$ if there exists a sequence of smooth metrics g_i on M such that* $\mathrm{scal}(g_i) \geq \kappa - o(1)$ *on M and $g_i \to g$ in C^0 sense as $i \to +\infty$.*

In this way, we see that the definition is a direct generalization of the notion in smooth case. In [1], Bamler gave an alternative approach to Gromov's Theorem using Ricci flow which turns out to be very powerful in studying problem in C^0 category. The idea of Bamler is based on a result of Koch-Lamm [7] on the C^0-stability of Ricci-Deturck flow on $\mathbb{R}^n$. The Ricci-Deturck flow is a one parameter family of metric $g(t)$ which solves

$$(1) \qquad \begin{cases} \partial_t g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i; \\ W^k = g^{pq}(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}). \end{cases}$$

This is a strictly parabolic system and is known to be diffeomorphic to the Ricci flow in the smooth case. By localizing the scalar curvature lower bound uniformly, Bamler [1] was able to show that the (unique!) Ricci-Deturck flow smoothing satisfies the desired scalar curvature lower bound as $t \to 0^+$. The circle of idea was generalized by Burkhardt-Guim [3] to general compact manifolds and was used to define a notion of $\mathrm{scal} \geq k$ for continuous metrics.

In studying the class of C^0 metrics, the Ricci flow method turns out to be more flexible and powerful. In [6], Huang and the author extended the idea of Bamler to show that if a g_∞ is a C^0 limit of a sequence of smooth metric g_i such that $(\mathrm{scal}(g_i) - \kappa)_-$ converges to 0 in $L^{n/2}_{loc}$, then the Ricci-Deturck flow smoothing $g_\infty(t)$ from g_∞ satisfies $\mathrm{scal}(g_\infty(t)) - \kappa - o(1)$ as $t \to 0$. This integral flexibility of scalar curvature lower bound is related to Miao's positive mass Theorem with corner [12]. Indeed if κ is a constant, we might drop the buffer term $o(1)$ thanks to the improved approximation from Ricci flow smoothing.

As discussed, the notion of scalar curvature lower bound for singular metrics is related to the positive mass Theorem with corners. To ease the discussion, we consider metric g on $\mathbb{T}^n$ such that g is smooth outside a singular set $\mathcal{S}$ with $\mathrm{scal}(g) \geq 0$ on $\mathbb{T}^n \setminus \mathcal{S}$. The basic question is to ask under what conditions on $\mathcal{S}$ and g, we can conclude the singularity $\mathcal{S}$ is removable. Restricting ourselves to C^0 metric on M, we might also ask if g satisfies $\mathrm{scal} \geq 0$ in the sense of definition 1 under those conditions on $\mathcal{S}$. Likewise, we might formulate the question in the setting of positive mass Theorem where $\mathbb{T}^n$ is replaced by asymptotically flat manifold with suitable (weighted) regularity at infinity. When $\mathcal{S}$ is a sub-manifold with co-dimension 1 or 2, it is now well-known that singularity $\mathcal{S}$ might contribute some scalar curvature in distributional sense, [10, 12]. When the co-dimension ≥ 3, in a joint work with Tam we showed that $\mathcal{S}$ is in some sense invisible for C^0 metrics. More precisely,

Theorem 1 (Corollary 4.2 and Theorem 1.1 in [8])**.** *If g is a metric in $C^\infty_{loc}(M \setminus \mathcal{S}) \cap C^0(M)$ such that $\mathrm{scal}(g) \geq 0$ outside $\mathcal{S}$ and the upper Minkowski dimension $\leq n-3$, then $\mathrm{scal}(g) \geq 0$ on M in the sense of definition 1. Moreover if $M = \mathbb{T}^n$, then (M, g) is distance-isometric to a flat metric.*

What about rigidity in comparison with sphere? The classical Llarull Theorem [11] states that a smooth map f from a closed spin manifold (M^n, g) with $\mathrm{scal} \geq n(n-1)$ to $\mathbb{S}^n$ with $f^* g_{\mathbb{S}^n} \leq g$ and $\deg(f) \neq 0$ must be an isometry. It was asked by Gromov if one can extend this to Lipschitz framework in order to understand the singular scalar curvature world. The problem was first took up by Cecchini-Hanke-Schick [2] by developing the singular spin method where distance non-increasing Lipschitz maps $f : M \to \mathbb{S}^n$ was considered. In [9], the author with Tam instead considered the smoothing method. Even if the underlying metric g is smooth, the major difficulty lies in regularizing the rough map f in a way that is related to scalar curvature and distance non-increasing. To this end, the harmonic map heat flow coupled with the Ricci flow (or Ricci-Deturck flow when g is non-smooth) was found to be useful in smoothing out the map while preserving all necessary geometric properties. Together with classical smoothing method of Greene-Wu and Llarull Theorem, we showed:

Theorem 2 (Theorem 1.2 in [9]). *Let M^n be a compact Riemannian spin manifold and g_0 is a C^0 metric on M with* $\mathrm{scal}(g) \geq n(n-1)$ *in the sense of definition 1. Suppose there is 1-Lipschitz continuous map $f : M \to \mathbb{S}^n$ with non-zero degree, then f is a distance isometry.*

The underlying principle is to construct a smooth map $F_i : M \to \mathbb{S}^n$ where $F_i \to f$ while $F_i^* g_{\mathbb{S}^n} \leq g$ when g is smooth. This was done by deforming f along the direction: $\partial_t F = \Delta_{g, g_{\mathbb{S}^n}} F$ with initial data $F(0) = f$ in suitable C^0 sense. The method is purely a smoothing approach which is different from the approach taken in [2]. Indeed, the Theorem of Llarull also holds if distance non-increasing is relaxed to area non-increasing. Using the singular spin method in [2], Cecchini-Hanke-Schick were also able to prove the singular rigidity under certain area non-increasing assumption. It is at the same time unclear if such condition is natural in smoothing procedure. More precisely, we ask the following:

Question 1. *In general, suppose $f : M \to \mathbb{S}^n$ is a Lipschitz map which satisfies $\Lambda^2 f^* g_{\mathbb{S}^n} \leq \Lambda^2 g$ almost everywhere, can we find a suitable smoothing F of f such that $\Lambda^2 F^* g_{\mathbb{S}^n} \leq \Lambda^2 g$ smoothly on M?*

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ADM Mass and Potential Theory

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(joint work with V. Agostiniani, C. Mantegazza, L. Mazzieri.)

Asymptotically flat Riemannian 3-manifolds arise physically as spacelike hypersurfaces of spacetimes modeling isolated gravitational systems.

Every end of an asymptotically flat Riemannian manifold has the remarkable property of being equipped with a well-defined notion of mass, called ADM mass, introduced in [4] by Arnowitt, Deser and Misner and denoted by m_{ADM}. It is a geometric invariant, furthermore, it satisfies the following crucial property in any 3-dimensional complete asymptotically flat Riemannian manifold (M, g) with nonnegative scalar curvature and with a single end.

- **Positive mass theorem (with a single end)** [9, 10]**:** If the boundary ∂M of M is empty, then $m_{\mathrm{ADM}} \geq 0$, and the equality is fulfilled if and only if (M, g) is isometric to the Euclidean space.
- **Riemannian Penrose Inequality (with a single black hole)** [8]**:** If the boundary ∂M of M is compact and connected, and it is the unique minimal closed surface in (M, g), then $m_{\mathrm{ADM}} \geq \sqrt{|\partial M|/(16\pi)}$, and the equality holds if and only if (M, g) is isometric to a Schwarzschild manifold of positive mass.

There are different generalizations of the previous theorems, moreover, different approaches/tools were being used to prove them.

In [3], an alternative proof of the positive mass inequality, $m_{\mathrm{ADM}} \geq 0$, is obtained via a monotonicity formula holding along the regular level sets of an appropriate harmonic function related to the minimal positive Green's function with pole.

Sketch of the proof, [3]: By [6], it is sufficient to show that the positive mass inequality is true in the class of 3-dimensional, complete, one-ended asymptotically flat, Riemannian manifolds (M, g), with nonnegative scalar curvature and satisfying the following two properties:

- M is diffeomorphic to $\mathbb{R}^3$ (*topological simplification*);
- there exists a distinguished asymptotically flat chart Φ such that

$$\Phi_* g = \left(1 + \frac{m_{\mathrm{ADM}}}{2|x|}\right)^4 g_{\mathbb{R}^3}$$

(*simplification of the metric near infinity*).

Let us consider the function

$$u := 1 - 4\pi \mathcal{G}_o,$$

where $\mathcal{G}_o$ is the minimal positive Green's function for the Laplacian operator Δ_g with pole at some point $o \in M$. Notice that $\mathcal{G}_o$ vanishes at infinity. We define

$$F(t) \;:=\; 4\pi t \;-\; t^2 \int\limits_{\{u=1-1/t\}} |\nabla u| \, \mathrm{H} \, d\sigma \;+\; t^3 \int\limits_{\{u=1-1/t\}} |\nabla u|^2 \, d\sigma,$$

for every $t \in (0, +\infty)$, where H is the mean curvature computed with respect to the ∞–pointing unit normal vector field $\nu = \nabla u/|\nabla u|$. The function F is well–posed and in the absence of critical points, is everywhere continuously differentiable with $F'(t) \geq 0$ for every $t \in (0, +\infty)$. Indeed, in this case, one has

$$F'(t) \,=\, 4\pi - \int\limits_{\Sigma_t} \frac{\mathrm{R}^{\Sigma_t}}{2}\, d\sigma \,+ \int\limits_{\Sigma_t} \left[\frac{|\,\nabla^{\Sigma_t} |\nabla u|\,|^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{2|\nabla u|}{1-u} - \mathrm{H}\right)^2 \right] d\sigma,$$

and all of the level sets of u are regular and diffeomorphic to the 2-sphere. Thus, the Gauss–Bonnet theorem and the fact that the scalar curvature R is nonnegative imply $F'(t) \geq 0$ for all $t \in (0, +\infty)$. In presence of critical points, by using the topological simplification, a suitable smooth vector field, an appropriate family of cut-off functions and the Sard's theorem with the Gauss–Bonnet theorem, one can anyway show that the function F is nondecreasing on the open set $\{t \in (0, +\infty) : 1 - 1/t$ is a regular value of $u\}$. As a consequence, one gets

$$8\pi m_{\mathrm{ADM}} \,=\, \lim_{t\to+\infty} F(t) \,\geq\, \lim_{t\to 0^+} F(t) \,=\, 0,$$

where the first limit follows from the simplification of the metric near infinity and the second one from the behavior of the minimal positive Green's function $\mathcal{G}_o$ near the pole o. Thus, $m_{\mathrm{ADM}} \geq 0$.

In [1, 2], an alternative proof the Riemannian Penrose inequality (with a single black hole) is exposed. In this case, the Riemannian Penrose inequality is a consequence of a monotonicity formula holding along the regular level sets of appropriate p–harmonic functions.

Sketch of the proof, [1, 2]: In this case, we consider the (weak) solution of the following problem

$$\text{(1)} \qquad \begin{cases} \Delta_p u = 0 & \text{in } M \\ u = 0 & \text{on } \partial M \\ u \to 1 & \text{at } \infty \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the *p–Laplacian* operator, for $p \in (1, 3)$, and we define the function

$$F_p(t) \,=\, 4\pi t \,-\, \frac{t^{\frac{2}{p-1}}}{c_p} \int\limits_{\{u=\alpha_p(t)\}} |\nabla u|\, \mathrm{H}\, d\sigma \,+\, \frac{t^{\frac{5-p}{p-1}}}{c_p^2} \int\limits_{\{u=\alpha_p(t)\}} |\nabla u|^2\, d\sigma$$

whenever $\alpha_p(t)$ is a regular value of u, where

$$\alpha_p(t) = 1 - \frac{p-1}{3-p}\frac{c_p}{t^{\frac{3-p}{p-1}}} \quad \text{and} \quad c_p^{p-1} \,=\, \frac{\mathrm{Cap}_p(\partial M)}{4\pi}\,.$$

Above, H is the mean curvature computed with respect to the ∞–pointing unit normal vector field $\nu = \nabla u/|\nabla u|$ and

$$\mathrm{Cap}_p(\partial M) \,=\, \inf\left\{ \int\limits_M |\nabla v|^p\, d\mu :\, v \in \mathrm{C}_c^\infty(M),\, v = 1 \text{ on } \partial M \right\}.$$

In absence of critical points, the function F is everywhere continuously differentiable, with

$$F_p'(t) = \int_{\{u=\alpha_p(t)\}} \left[\frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{5-p}{p-1}\left(\frac{|\nabla u|}{\frac{3-p}{p-1}(1-u)} - \frac{\mathrm{H}}{2} \right)^2 \right] d\sigma$$
$$+ 4\pi - \int_{\{u=\alpha_p(t)\}} \frac{\mathrm{R}^{\Sigma_t}}{2}\, d\sigma\,,$$

and all of the level sets of u are regular and diffeomorphic to the boundary ∂M. Then, by the Gauss–Bonnet theorem and the assumption $\mathrm{R} \geq 0$, one has $F'(t) \geq 0$ for all $t \in [t_p, +\infty)$, where

$$t_p = \left(\frac{p-1}{3-p}\, c_p \right)^{\frac{p-1}{3-p}}.$$

In presence of critical points, we cannot proceed similarly to the proof the positive mass inequality. The reason lies in the fact, being the p–harmonic functions in general only of class C^1, the Sard's theorem cannot be applied. In order to obtain the monotonicity, then we consider the solutions $u^{\varepsilon,T}$ (in the classical sense) of the following "perturbed" problem

$$\begin{cases} \mathrm{div}\left(|\nabla u^{\varepsilon,T}|_\varepsilon^{p-2}\nabla u^{\varepsilon,T}\right) = 0 & \text{in } M_T = \{0 < u < T\}\,, \\ u^{\varepsilon,T} = 0 & \text{on } \partial M\,, \\ u^{\varepsilon,T} = T & \text{on } \{u = T\}\,, \end{cases} \tag{2}$$

where $|\nabla u^{\varepsilon,T}|_\varepsilon = \sqrt{|\nabla u^{\varepsilon,T}|^2 + \varepsilon^2}$ and T is a fairly large regular value of u. The functions $u^{\varepsilon,T}$ are smooth (so Sard theorem can be applied) and C^k_{loc}–converge to the function u outside $\{|\nabla u| = 0\}$, for every $k \in \mathbb{N}$, as $\varepsilon \to 0$. We can consider analogous functions F_p^ε, pointwise converging to F_p, as $\varepsilon \to 0$, which are "almost" nondecreasing, up to an "error term" going to zero as $\varepsilon \to 0$. Hence, sending $\varepsilon \to 0$, we obtain the monotonicity of the original function F_p. Thus, as before, for every $p \in (1,3)$, one gets

$$(4\pi)^{\frac{2-p}{3-p}} \left(\frac{p-1}{3-p} \right)^{\frac{p-1}{3-p}} \mathrm{Cap}_p(\partial M)^{\frac{1}{3-p}} \leq F_p(t_p) \leq \lim_{t\to+\infty} F_p(t) \leq 8\pi m_{\mathrm{ADM}}$$

where the first inequality is a consequence of the fact that $\mathrm{H} = 0$ on $\partial M = \{u = \alpha_p(t_p) = 0\}$ and the last one follows from the behavior of the p–harmonics function u near infinity, proved in [5], and by doing some computations in the same spirit of [8]. Then, the Riemannian Penrose inequality follows by sending $p \to 1^+$, as $\lim_{p\to 1^+} \mathrm{Cap}_p(\partial M) = |\partial M|$, by [7].

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Instabilities of Einstein 4-manifolds and selfduality along Ricci flow

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(joint work with Olivier Biquard)

Einstein metrics and Ricci solitons are the fixed points of Ricci flow and model the singularities forming. They are also critical points of natural functionals in physics. Their stability in both contexts is a crucial question, since one should be able to perturb away from unstable models.

In physics and in the formation of singularities of Ricci flow, the most important Einstein metrics are either compact with positive scalar curvature, or noncompact and Ricci-flat with finite L^2-norm of curvature (gravitational instantons). In both contexts, optimistic conjectures say that in dimension 4, the only *stable* such metrics should be the round metric on $\mathbb{S}^4$ and hyperkähler metrics. The notions of stability (at second perturbative order) are equivalent in physics, in the context of Yamabe metrics, and in the dynamical sense for Ricci flow.

With Olivier Biquard, we motivate these conjectures by proving that indeed, all of the known examples are unstable unless they are the round metric on $\mathbb{S}^4$ or hyperkähler. The key point is that these other known examples are conformal to Kähler metrics with positive scalar curvature.

The proof relies on three main ingredients which are of independent interest.

- A *Weitzenböck formula* [BL81,BR15,NO24] states that in dimension 4, the stability at second order is equivalent to the nonnegativity of the spectrum of the following operator on traceless 2-tensors:

(1) $$P := d_- d_-^* - \mathbf{R}^+,$$

  where $d_- : \Omega^1 \otimes \Omega^+ \to \Omega^- \otimes \Omega^+$ is the covariant exterior derivative on the bundle of Ω^+-valued 1-forms composed with the projection on anti-selfdual 2-forms, and where $\mathbf{R}^+$ is the natural action of the selfdual (only!)

part of curvature. This is based on a classical 4-dimensional identification between $\Omega^- \otimes \Omega^+$ and traceless symmetric 2-tensors.

Note that (1), easily recovers that hyperkähler metrics with $\mathbf{R}^+ = 0$, and metrics satisfying the condition $\mathbf{R}^+ < 0$ of are **stable**.

- We then show that negative directions for $L := d_- d_-^* - \frac{2}{3}\delta_0^*\delta - \mathbf{W}^+$, for $\mathbf{W}^+$ the traceless part of $\mathbf{R}^+$, also detect instabilities of the metric with nonnegative scalar curvature, and that L is *conformally covariant*. To prove instability, it is therefore sufficient to show that the operator L has a negative eigenvalue for *one* metric in the conformal class of g. This can easily be found on Kähler metrics with positive scalar curvature and $b_- \geq 1$.
- One then proves that all of the considered metrics must have $b_- \geq 1$ by constructing decaying anti-selfdual harmonic 2-forms. This uses the fact that being Einstein and conformally Kähler forces the metric to carry a Killing vector field, and that Killing vector fields can be used to construct anti-selfdual 2-forms on Ricci-flat 4-manifolds.

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Estimates on the Bartnik mass and their geometric implications.

Annachiara Piubello

(joint work with Pengzi Miao)

A triple (Σ, γ, H) where Σ is a closed, connected orientable surface, γ is a metric on Σ and H is a function on Σ is called Bartnik data. For such a triple, we can define the Bartnik mass $\mathfrak{m}_B(\Sigma, \gamma, H)$ as

$$\mathfrak{m}_B(\Sigma, \gamma, H) := \inf\{\mathfrak{m}(g) | (M, g) \text{ is an admissible extension of } (\Sigma, \gamma, H)\},$$

where $\mathfrak{m}(g)$ is the mass of an admissible extension (M, g) of (Σ, γ, H). This is an asymptotically flat Riemannian 3-manifold with $R_g \geq 0$ whose boundary $(\partial M, g|_{\partial M})$ is isometric to (Σ, γ) and under this isometry the mean curvature of ∂M in (M, g) is H. We also assume that (M, g) satisfies some non-degeneracy conditions that prevent the mass $\mathfrak{m}(g)$ from being made arbitrarely small.

From its definition, we see that the mass of an admissible extension of the given Bartnik data will provide an upper bound for the Bartnik mass, as done for instance in [2, 4–6, 8, 11–13]. Our main result [10] is a generalization of Miao's estimate [9], for Bartnik data with arbitrary metric, positive mean curvature and nonnegative Gauss curvature K_γ.

Theorem 1 ([10]). *Let (Σ, γ, H) be Bartnik data with $H > 0$ and $K_\gamma \geq 0$. If $r_\gamma = \sqrt{\frac{|\Sigma|_\gamma}{4\pi}}$ is the area radius, then the Bartnik mass $\mathfrak{m}_B(\Sigma, \gamma, H)$ satisfies*

$$\mathfrak{m}_B(\Sigma, \gamma, H) \leq \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \left[\left(1 + \frac{\zeta(\gamma)}{8\pi r_\gamma} \int_\Sigma H \, d\gamma_0\right)^2 - \left(\frac{1}{8\pi r_\gamma} \int_\Sigma H \, d\gamma_0\right)^2 \right] \tag{1}$$

where $\zeta(\gamma)$ is a scaling invariant quantity so that

$$\zeta(\gamma_0) = 0 \quad \text{and} \quad \zeta(\gamma) \to 0 \quad \text{as} \quad \frac{\max_\Sigma K_\gamma}{\min_\Sigma K_\gamma} \to 1. \tag{2}$$

The estimate is sharp in the extreme cases of $H \to 0$ or $\gamma = \gamma_0$. However, for metrics away from the known cases, we do not know. If $K_\gamma > 0$, then the quantity $\zeta(\gamma) < C_K$, a constant depending on the min and max of K_γ.

To achieve this estimate, inspired by the constructions [8, 12], we considered $\{\gamma(t)\}_{t \in [0,1]}$ to be a smooth path of metrics connecting γ to a round metric, so that $\gamma(t)$ has positive Gauss curvature for $t > 0$ and $\mathrm{tr}_\gamma \gamma'(t) = 0$ for all $t \geq 0$. We reparametrized the path using a fuction $t : [1, \infty) \to [0, 1]$ with $t(1) = 0$, and $t(s) = 1$ $\forall s \geq \tilde{s}$ for some $\tilde{s}$. Using results by Bartnik [1] and Shi–Tam [13], we constructed a metric g on $M = [1, \infty) \times \Sigma$ that is scalar flat, asymptotically flat and whose mean curvature H_1 of $\Sigma_1 = \Sigma \times \{1\}$ agrees with H. Our main estimate for the Bartnik mass was obtained by deforming the total mean curvature of large spheres in (M, g). Set $\mathcal{H}_s = \frac{1}{8\pi} \int_{\Sigma_s} H_s d\mu_s$, where H_s is the mean curvature of $\Sigma_s = \{s\} \times \Sigma$ in (M, g). We have

$$\frac{d\mathcal{H}_s^2}{ds} \geq \left(\frac{1}{s} - \alpha_s |t'(s)|^2 s\right) \mathcal{H}_s^2 + s\beta_s, \tag{3}$$

where $\alpha_s = \alpha(t(s))$ and $\beta_s = \beta(t(s))$ are quantities that measure how far the metric along the path is different from a round metric. We note that in deriving (3), $\gamma(1)$ need not be a round metric; neither does Σ need to be a sphere. Combining (3) with the asymptotic behavior of $\mathcal{H}_s$ for large s as in [13], we obtained the following estimate.

Proposition 1 ([10]). *For a path of metrics $\{\gamma(t)\}$ as above, given any $C^1([0, 1])$ function $s(t)$ satisfying $s(0) = 1$ and $s'(t) > 0$, we have*

$$\mathfrak{m}_B(\Sigma, \gamma, H) \leq \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \left[s(1) - \int_0^1 \frac{\beta(t) s'(t)}{e^{\int_t^1 \alpha(t) \frac{s(t)}{s'(t)} dt}} dt - \frac{(\int H \, d\gamma)^2}{16\pi |\Sigma|_\gamma e^{\int_0^1 \alpha(t) \frac{s(t)}{s'(t)} dt}} \right]. \tag{4}$$

The main estimate (1) follows from some ad hoc choice of $s(t)$ and by setting

$$\zeta(\gamma) = \inf_{\{\gamma(t)\}} \int_0^1 \sqrt{\frac{\alpha(t)}{4\beta(t)}} \, dt. \tag{5}$$

To achieve the limit in (2), we constructed a more explicit path of metrics using the Uniformization Theorem, appropriately deformed with a family of diffeomorphisms to have vanishing trace. We can then estimate $\alpha(t)$ and $\beta(t)$ in terms of the conformal factor φ arising from the Uniformization Theorem. Moreover, from conformal geometry we know that $\Delta\varphi = 1 - e^{2\varphi}K_\gamma$. We can then use PDE estimates to get a bound of $\zeta(\gamma)$ in terms of the Gauss curvature K_γ. Assuming that $K_\gamma > 0$ and K_γ close to 1 in the C^1-norm, using estimates in [3, 14] we obtain

$$\zeta(\gamma) \le C \left| \frac{\max_\Sigma K_\gamma}{\min_\Sigma K_\gamma} - 1 \right|. \tag{6}$$

Our main construction, and in particular (3) has applications in the context of nonnegative scalar curvature fill-ins. Let (Σ, γ) be a closed $(n-1)$ dimensional manifold, $n \ge 3$ equipped with a metric with positive scalar curvature. Following [7], we define

$$\Lambda(\Sigma,\gamma) = \sup\left\{ \frac{1}{(n-1)\omega_{n-1}} \int_{\partial\Omega} H\, d\mu \;\; \Big| \, (\Omega, g_\Omega) \in \mathcal{F}_+(\Sigma,\gamma) \right\}, \tag{7}$$

where $\mathcal{F}_+(\Sigma,\gamma)$ consists of n dimensional, compact, connected Riemannian manifolds (Ω, g_Ω) of NNSC with boundary $\partial\Omega$, whose induced metric is isometric to (Σ,γ), and whose mean curvature $H > 0$. We obtain the following upper bound for the $\Lambda(\Sigma,\gamma)$.

Theorem 2 ([10]). *Let γ be a metric with positive scalar curvature on Σ. If $\mathcal{F}_+(\Sigma,\gamma) \neq \emptyset$, then*

$$\Lambda(\Sigma,\gamma) \ge r_\gamma^{n-1} \left(\frac{\min_\Sigma R_\gamma}{(n-1)(n-2)} \right)^{\frac{1}{2}}. \tag{8}$$

Here r_γ is the volume radius of (Σ,γ), i.e. $|\Sigma|_\gamma = \omega_{n-1} r_\gamma^{n-1}$.

If $n = 3$, (8) becomes $\Lambda(\Sigma,\gamma) \ge r_\gamma^2 \left(\min_\Sigma K_\gamma\right)^{\frac{1}{2}}$. This can be alternatively derived by isometrically embedding (Σ,γ) in $\mathbb{R}^3$, making use of the classic Minkowski inequality and the Gauss-Bonnet theorem.

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Rigidity of positive scalar curvature and index theory on manifolds with cone singularities

Thomas Schick
(joint work with Simone Cecchini, Bernhard Hanke, Lukas Schönlinner)

Llarull proved an influential rigidity result for the standard metric on the sphere [6], as already discussed in the introductory talk by Christian Bär during this conference.

We show the following generalization to low regularity maps and metrics:

Theorem 1. *Let M be a smooth compact spin manifold without boundary and g a metric tensor on M of regularity $W^{1,p}$ for $p > n := \dim(M)$. This means, when written in smooth local coordinates, the coefficient functions g_{ij} lie in the Sobolev space $W^{1,p}$.*

In this context, scalar curvature $scal_g$ of (M, g) is defined as a distribution (compare [4]*, and we assume that $scal_g \geq n(n-1)$, where the right hand side is the scalar curvature of the round metric g_{std} on the n-dimensional sphere S^n.*

Assume furthermore that $f\colon (M, g) \to (S^n, g_{std})$ is a Lipschitz map of non-zero degree. Recall by Rademacher's theorem that f is almost everywhere differentiable and assume in addition that f is area non-expanding*, i.e. that*

$$||\Lambda^2 D_x f\colon \Lambda^2 T_x M \to \Lambda^2 T_{f(x)} S^n|| \leq 1$$

for almost all $x \in M$.

Then it follows that f is a metric isometry.

The case for n even has been proven a while ago by a subset of the authors in [3]. The proof is based on a careful extension of the methods of Llarull (using the index theory of twisted Dirac operators) to Dirac operators in the context of low regularity Riemannian metrics and twist bundles. This relies on fundamental analytic work of Bartnick and Crusciel [1] and uses the general idea of using Dirac operator methods proposed in [7], but combined with a crucial new input, combining the information coming from the twisted Dirac operator with the theory of quasi-regular mappings.

The passage to odd dimensional spheres is more intricate than one could expect. The argument given in [6] not being quite complete, only recently [2] give a full argument in the smooth case.

An additional obstacle occurs in the non-smooth case when trying to adapt the tools from the theory of quasi-regular mappings.

To overcome this, we use a suspension construction leading a priori to an even dimensional space SM which is a manifold with two conical singularities. We then develop the necessary index theory on such manifolds with isolated conical singularities, where the operator is also singular/low regular due to the low regularity of the metric g and the pulled back twist bundle via the suspension of S.

For the case that f satisfies the stronger condition to be 1-Lipschitz, [5] gives an alternative proof of the main theorem by reduction to [6] via Ricci flow combined with harmonic map heat flow, as described in the talk of Man-Chu Lee during the conference.

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Limits of sequences of manifolds with nonnegative scalar curvature and other hypotheses

Christina Sormani

(joint work with Wenchuan Tian, Changliang Wang)

In 2014, Gromov suggested that one should formulate and prove a compactness theorem for sequences of compact manifolds, M_j^m, with $Scal \geq 0$ and develop a notion of generalized $Scal \geq 0$ on the class of limit spaces. He suggested that perhaps Sormani-Wenger's Intrinsic Flat (SWIF) convergence would work well. In fact Wenger had already proven a compactness theorem:

$$Vol(M_j^m) \leq V_0 \text{ and } Diam(M_j^m) \leq D_0 \implies \exists M_{j_k} \to M_\infty \text{ in SWIF sense,}$$

where M_∞ is an integral current space, possibly the 0 space. Note that collapsing sequences of round spheres SWIF converge to the 0 space, so a compactness theorem for manifolds with $Scal \geq 0$ would need a noncollapsing hypothesis to avoid such a limit.

In 2018, at an IAS Emerging Topics meeting, a conjecture was formulated in dimension 3 using the two hypotheses of Wenger's Compactness Theorem and a very weak noncollapsing hypothesis, $MinA(M_j) \geq A_0 > 0$, where

$$MinA(M) = min\{Area(\Sigma) : \text{ closed min surface } \Sigma \subset M^3\}.$$

Note that this hypothesis immediately rules out collapsing spheres (whose equators are unstable minimal surfaces) as well as bubbles (which have necks with stable minimal surfaces). This was a natural assume in light of Schoen-Yau theorems regarding stable minimal surfaces, Marques-Neves Theorems regarding unstable minimal surfaces, and the Penrose Inequality. See the 2023 survey by Sormani.

The $MinA$ hypothesis rules out sequences of M_j^3 with $Scal \geq 0$ which are built using increasingly thin Schoen-Yau or Gromov-Lawson tunnels. This includes the sewing examples of Basilio-Dodziuk-Sormani which GH and SWIF converge to a pulled string space and examples of Basilio-Sormani which GH and SWIF converge to a space where a set, $K \in \mathbb{S}^3$, has been identified to a single point. The key idea in the latter paper is that one can take M_j^3 to be $\mathbb{S}^3$ with an increasingly dense collection of balls located in $K \subset \mathbb{S}^3$ that are replaced by a collection of increasingly small tunnels that run between each pair of balls.

In the Basilio-Sormani paper these tunnels were taken to be increasingly short and it is proven via a scruching lemma that the region, K, contracts to a single point or if $K = \mathbb{S}^3$ the sequence SWIF converges to the 0 space. Dodziuk proved the tunnels could be taken arbitrarily thin and approaching any length. More recently Sweeney has proven tunnels can be built with various bounds on Scalar curvature including scalar curvature arbitrarily close to that of a sphere. Basilio-Kazaras-Sormani took the tunnels developed by Dodziuk running between a ball $B_p(\epsilon_j)$ and a ball $B_q(\epsilon_j)$ to have length close to the Euclidean distance between them as points $p, q \subset \mathbb{S}^3 \subset \mathbb{E}^4$ and proved this sequence of M_j SWIF converges to $(\mathbb{S}^3, d_{\mathbb{E}^4})$ which is a metric space with no midpoints and no geodesics. The $MinA$ hypothesis prevents all these bad examples involving tunneling.

Open Question 1: Suppose a sequence, (M_j^3, g_j), with $Scal \geq 0$ converge in some sense to a smooth Riemannian manifold, (M_0^3, g_0). Does (M_0^3, g_0) have $Scal \geq 0$? Gromov and Bamler proved this is true for C^0 convergence of the metric tensors when M_j^3 are diffeomorphic to one another and to the limit space. SWIF convergence allows one to study sequences which are not diffeomorphic. Does this work for SWIF convergence? I believe the answer should be no if we allow tunneling. If we take $(M_0^3, g_0) = (\mathbb{S}^3, d_g)$ where g is any Riemannian metric on the sphere such that $g \leq g_{\mathbb{S}}^3$, one can construct an example of a sequence of (M_j^3, g_j) with $Scal \geq 0$ using tunneling similar to the Basilio-Kazaras-Sormani sequence with tunnels running between a ball $B_p(\epsilon_j)$ and a ball $B_q(\epsilon_j)$ that have length close the Riemannian distance d_{g_0}. It may be possible to prove this sequence converges in the SWIF sense to (M_0^3, g_0) which does not have $Scal \geq 0$.

Open Question 2: If (M_j^3, g_j) have $Scalar \geq 0$ and satisfy the $MinA$ hypothesis and converge in the SWIF sense to smooth (M_0, g_0), does (M_0, g_0) have $Scal \geq 0$? The $MinA$ hypothesis prevents the use of tunneling to construct counterexamples. If a counter example is found, then one needs to consider a stronger hypothesis than the $MinA$ in the compactness conjecture. Proving this open question in the affirmative would be very challenging. An easier case to check would be to assume that the sequence converges in the VADB sense defined by Allen-Perales-Sormani (where it was proven that VADB implies SWIF convergence).

There has been some interesting progress on the IAS $MinA$ Compactness Conjecture described above by Park-Tian-Wang, Tian-Wang, and Kazaras-Xu. All three teams consider fixed manifolds, M^3, with varying metric tensors, g_j, of the following forms respectively:

$$(\mathbb{S}^3, dt^2 + f_j(t)^2 g_{\mathbb{S}^2}),\ (\mathbb{S}^2 \times \mathbb{S}^1, g_{\mathbb{S}^2} + f_j(u)^2 d\theta^2),\ \text{and}\ (\mathbb{T}^2 \times \mathbb{S}^1, h_j + f_j(u)^2 d\theta^2).$$

All three achieve $W^{1,p}$ convergence for $p < 2$ to a limit metric tensor, g_∞, of the same form satisfying some distributional notion of $Scal \geq 0$. Park-Tian-Wang prove their f_∞ are continuous and take values in $[0, \infty)$. Although $S = f_\infty^{-1}(0)$ may be an open set, they prove SWIF convergence of their (M, g_j) to the metric completion of $(M \setminus S, g_\infty)$ because the disappearing regions are wells.

Open Question 3: Recall that we already know there is a SWIF limit space, (M^3, d_∞), which is a rectifiable metric space (possibly the zero space) in the setting of the IAS $MinA$ Compactness Conjecture. If $(M^3, g_j) \to (M^3, g_\infty)$ have $g_j \to g_\infty$ in the $W^{1,p}$ sense can one prove the SWIF limit (M^3, d_∞) is isometric to the metric completion of $(M^3 \setminus S, g_\infty)$ where S is the singular set where g_∞ is infinite valued or degenerate? See work of Allen-Sormani and Allen-Bryden demonstrating how different these can be without $Scal \geq 0$ and the $MinA$ hypotheses. With these hypotheses, Sormani-Tian-Wang have an extreme limit space and Kazaras-Xu have announced an example with a pulled thread limit which should be studied closely. Are there additional hypothesis that can guarantee they are isometric? If these spaces are not isometric, how are they related? Can one use distributional $Scal \geq 0$ on the $W^{1,p}$ limit to say something about the SWIF limit?

The extreme limit space found by Sormani-Tian-Wang is a limit space achieved in the Tian-Wang setting where $g_j \leq g_{j+1}$ converges in the $W^{1,p}$ sense for $p \in [1, 2)$ to a limit tensor, g_∞, that is warped by a function, f_∞, which is ∞ along two circular fibres of infinite length. They prove this sequence satisfies the hypotheses of the IAS $MinA$ conjecture. In upcoming work of Sormani-Tian we will prove the SWIF limit (M^3, d_∞) is isometric to the metric completion of $(M^3 \setminus S, g_\infty)$ and also isometric to the GH limit. So this is an interesting space to study possible notions of generalized $Scal \geq 0$. Open questions in this direction will appear in this upcoming paper including possible generalizations of various such notions that might be tested on this extreme space. Tian-Wang have already generalized the Lee-LeFloch distributional $Scal \geq 0$ to the limits achieved in their paper.

It is possible that some very natural notions of generalized $Scal \geq 0$ may fail to hold on some of these $W^{1,p}$ and SWIF limit spaces. It is possible that a stronger hypothesis than the $MinA$ hypothesis is needed in order to obtain better control on the limit space and stronger convergence than simply SWIF convergence.

Open Question 4: We could consider $MinL(M)$, which is the length of shortest closed geodesic in a closed min surface in M. By a theorem of Croke, $MinL(M) \geq L_0 > 0$ is a stronger hypothesis than $MinA(M) \geq A_0 > 0$. This

$MinL$ hypothesis also provides a lower bound on the Gromov filling volumes of the minimal surfaces in M. Filling volumes are closely related to SWIF distances and can be used to prevent convergence to 0 and disappearing regions as seen in work of Sormani-Wenger and Portegies-Sormani. One might also consider bounding the systole, $Sys(M)$, which is the length of shortest closed geodesic in M. See related work of Nabutovsky, Rotman, and Sabourou.

Open Question 5: We could consider a noncollapsing condition defined using constant mean curvature surfaces or isoperimetric regions. Portegies and Jauregui-Lee have theorems about SWIF converging sequences of manfolds whose volumes converge which may be useful to apply in combination with such an hypothesis.

Open Question 6: Is there a notion of convergence for sequences of distinct Riemannian manifolds, (M_j, g_j), which implies this volume preserving SWIF convergence and also captures the information encoded in a $W^{1,p}$ limit? Perhaps a notion where the convergence on good diffeomorphic regons is in the $W^{1,p}$ sense and the bad regions have volume decreasing to 0. Can one define a distance between Riemannian manifolds which captures this notion? Exploration in this direction involves a deeper understanding of these examples mentioned above and other possible limit spaces with even more badly behaved singular sets.

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Event horizon gluing and black hole formation in vacuum: the very slowly rotating case

Ryan Unger

(joint work with Christoph Kehle)

One of the central principles of modern high energy and gravitational physics is the belief that black holes are *thermodynamic objects.* Namely, despite their enormous size and complexity, black holes should obey "laws" which are strikingly similar to the everyday laws of thermodynamics. This principle was first formulated in a landmark paper by Bardeen, Carter, and Hawking in 1973, as the celebrated *four laws of black hole thermodynamics* [4]. These laws are precise mathematical statements about solutions to Einstein's field equations

$$\mathrm{Ric}(g) - \tfrac{1}{2}Rg = 2T \tag{1}$$

and proofs of the zeroth, first, and second laws are taught in graduate courses on general relativity [9].

According to black hole thermodynamics, a black hole may be assigned a *temperature* which is proportional to its *surface gravity*. Black holes sitting at absolute zero are called *extremal* and black holes with a positive temperature are called *sub*extremal. The two most important families of explicit black hole solutions of Einstein's equations, the *Reissner–Nordström* and *Kerr* families, have extremal and subextremal variants. Reissner–Nordström is electrically charged and increasing charge at fixed mass decreases temperature, while Kerr is rotating and increasing angular momentum at fixed mass decreases temperature. The famous Schwarzschild solution is contained in both of these families and is subextremal.

In analogy with Nernst's "unattainability" formulation of the third law of classical thermodynamics, Bardeen, Carter, and Hawking proposed [4]:

Conjecture 1 (The third law of black hole thermodynamics)**.** *A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.*

This version is distilled from the literature, particularly from the work of Israel [6]. The first result discussed in my talk shows that the third law is fundamentally flawed:

Theorem 1 ([7])**.** *The "third law of black hole thermodynamics" is false: A self-gravitating charged scalar field can collapse to form a subextremal Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Reissner–Nordström event horizon at a later advanced time.*

Our construction in the proof of Theorem 1 is fundamentally "teleological" and is a variant of *characteristic gluing* for the Einstein–Maxwell-charged scalar field system in spherical symmetry. Characteristic gluing is a powerful new method for constructing solutions of Einstein's equations by gluing together two existing solutions along a null hypersurface. The technique of characteristic gluing was recently pioneered by Aretakis, Czimek, and Rodnianski in a series of works on perturbative characteristic gluing near Minkowski space [1–3]. In contrast, the gluing in Theorem 1 is performed *on the event horizon*, which is a fundamentally nonperturbative regime. The usefulness of characteristic gluing is twofold:

- The global causal structure of the glued spacetime can be immediately read off from the construction.
- The free data for the characteristic initial value problem of Einstein's equations are more transparent than for the Cauchy problem.

Physically, the gluing in Theorem 1 can be interpreted as firing multiple oscillating pulses of an electrically charged scalar field into an uncharged Schwarzschild black hole, which initially has positive temperature. These pulses cause the black hole to become charged, which decreases its temperature. By carefully tuning the amplitudes of the pulses with the help of a topological argument, we ensure that:

- The scalar field is entirely swallowed by the black hole in finite time.
- At the very moment when the scalar field is completely absorbed by the black hole, the black hole's temperature hits zero.

The solutions constructed in [7] contain charged black holes. It is natural (and very relevant to astrophysics) to ask if the extremal threshold can also be accessed via rotating black holes. In the second part of my talk, I discussed the following conjecture:

Conjecture 2. *The "third law of black hole thermodynamics" is already false in vacuum: Gravitational waves can collapse to form a subextremal Kerr apparent horizon, only for the spacetime to form an exactly extremal Kerr event horizon at a later advanced time.*

Proving this conjecture will require significantly more work than was required to prove Theorem 1 because no symmetry reductions are possible and no mechanism for transferring large amounts of angular momentum to a black hole via gravitational waves is known. So far, we have obtained the following partial result, which also provides an alternative and fundamentally different proof of Christodoulou's seminal theorem of black hole formation in vacuum [5]:

Theorem 2 ([8])**.** *There exists a number* $0 < \mathfrak{a}_0 \ll 1$ *such that for any* M *and* a *satisfying* $0 \le |a|/M \le \mathfrak{a}_0$*, there exists a solution of the Einstein vacuum equations describing gravitational collapse to a Kerr black hole with mass* M *and specific angular momentum* a *in finite time.*

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Scalar Curvature Rigidity of Polytopes

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(joint work with Simon Brendle)

The first significant result concerning the rigidity phenomenon of scalar curvature can be traced back to theorems by Gromov-Lawson [9,10] and Schoen-Yau [14,15]. These theorems establish that any Riemannian metric on the torus with non-negative scalar curvature must be isometric to the flat metric. Schoen and Yau's proof employs the method of minimal hypersurfaces, applicable up to dimension 7, due to the emergence of singularities in minimal surfaces at higher dimensions. While Gromov-Lawson utilized Dirac operator techniques and the index theorem which highly relies on the spin condition. The spinor approach usually involves solving a Dirac equation on a twisted spinor bundle and applying the Weitzenböck formula to the Dirac operator, leading to the conclusion that any solution must be parallel. The existence of such a parallel spinor underpins the rigidity result.

Remarkably, these techniques have been extended to manifolds with boundaries. Bär and Ballmann [1] developed a powerful index theory for boundary value problem of first order elliptic operator. When integrating the Weitzenböck formula, it becomes necessary to consider both the scalar curvature of the manifold and the mean curvature of the boundary. A pivotal theorem in this context, due to Shi and Tam [16], concerns the positivity of quasi-local mass in general relativity, highlighting the interplay between scalar and mean curvature on manifiold with boundary.

Theorem 1 (Shi-Tam [16])**.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth convex domain with a Riemannian metric g, if*

- $R_g \geq 0$ *in* Ω.
- $H_g \geq H_{\mathbb{R}^n}$ *on* $\partial\Omega$.

Then g is isometric to the Euclidean metric.

A question in this flavor, which concerns scalar curvature rigidity of convex polytope is the *dihedral rigidity conjecture*, first considered by Gromov [6]. We remark that this conjecture can be considered as a singular version of Theorem 1.

Conjecture 1 (Dihedral Rigidity Conjecture)**.** *Let $\Omega \subset \mathbb{R}^n$ be a convex polytope with a Riemannian metric g, suppose*

- $R_g \geq 0$ *in* Ω.
- $H_g \geq 0$ *on* $\partial\Omega$.
- *The dihedral angle with respect to g is not greater than the Euclidean angle.*

Then g is isometric to the Euclidean metric.

The Dihedral Rigidity Conjecture is crucial for understanding the geometry of scalar curvature for several reasons. For example, from the perspective of comparison geometry, this conjecture can be regarded as an analogue to the Topogonov comparison theorem in the context of scalar curvature.

Partial results of the dihedral rigidity problem are obtained by Gromov in his seminal work [6–8], when the polytope is a cube. For cases where the dimension is at most 7, Li [11, 12] has utilized capillary surface techniques to demonstrate that the conjecture holds true for a large class of polytopes. Wang-Xie-Yu [17] have also explored this problem, basing their studies on index theories for manifolds with corners.

Last year, Brendle [2] established the conjecture in all dimensions under the assumption that the dihedral angles are equal. The approach involves solving a boundary value problem for the Dirac equation on a smooth domain. Additionally, Brendle employed a smoothing technique to approximate the polytope with smooth domains and used a profound result by Fefferman-Phong to take the limit [5]. This method was later generalized by Brendle-Chow [3] to study the scalar curvature rigidity of convex polytopes under the dominant energy condition.

By modifying the smoothing technique developed by Brendle, we have managed to remove the matching angle hypothesis in Brendle's theorem, provided that the dihedral angle is at most $\frac{\pi}{2}$.

Theorem 2 ([2] [4])**.** *Let $\Omega \subset \mathbb{R}^n$ be a convex polytope, let g be a Riemannian metric on Ω such that $R_g \geq 0$ in Ω and $H_g \geq 0$ on $\partial\Omega$. Suppose that either*

(1) *The dihedral angle with respect to g equals to the dihedral angle with respect to g_{Eucl} along the intersection of any two faces.*
(2) *Or, dihedral angle with respect to $g \leq$ to the dihedral angle with respect to $g_{Eucl} \leq \frac{\pi}{2}$ along the intersection of any two faces.*

Then g is isometric to the Euclidean metric.

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Ricci flow on surfaces along the standard lightcone in the 3 + 1-Minkowski spacetime

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A version of null mean curvature flow along a null hypersurface was first studied by Roesch–Scheuer [11] to detect marginally outer trapped surfaces (MOTS). For a given null hypersurface $\mathcal{N}$ in an ambient spacetime (M,g), they say a smooth family of spacelike cross sections $x\colon [0,T)\to\mathcal{N}$ evolves under what we will call *null mean curvature flow* if

$$\frac{d}{dt}x=\frac{1}{2}g\left(\vec{\mathcal{H}},L\right)\underline{L}, \tag{1}$$

where $\vec{\mathcal{H}}$ is the codimension-2 mean curvature vector of the cross sections Σ_t in (M,g), $\underline{L}$ is a choice of null generator of $\mathcal{N}$, and L is the unique null vector field normal to Σ_t such that $g(\underline{L},L)=2$.

Recall that a null hypersurface $\mathcal{N}$ in a spacetime (M,g) is an orientable hypersurface with degenerate induced metric. Thus, there exists a tangent null vector field $\underline{L}\in\Gamma(T\mathcal{N})$ that is normal to all tangent directions. $\underline{L}$ is called a *null generator* of $\mathcal{N}$ as $\mathcal{N}$ is ruled by the integral curves of $\underline{L}$, and one can always reparametrize the integral curves to be null geodesics ruling $\mathcal{N}$. In particular, $a\underline{L}$ is also a choice of null generator for any non-vanishing function $a\in C^\infty(\mathcal{N})$.

A spacelike cross section (Σ,γ) of $\mathcal{N}$ is a spacelike (γ Riemannian), codimension-2 surface in (M,g) with $\Sigma\subseteq\mathcal{N}$ such that any integral curve of $\underline{L}$ intersects Σ exactly once. As $\underline{L}$ is normal to all tangent directions along $\mathcal{N}$, $\underline{L}$ is normal to Σ and there exists a unique choice of null vector field L normal to Σ such that $g(\underline{L},L)=2$, and $\{\underline{L},L\}$ form a null frame of the normal bundle of Σ. One defines the *null second fundamental forms* of (Σ,γ) (in (M,g)) as

$$\underline{\chi}(V,W):=-g\left(\overline{\nabla}_V W,\underline{L}\right),\qquad \chi(V,W):=-g\left(\overline{\nabla}_V W,L\right),$$

for $X,Y\in\Gamma(T\Sigma)$. Note that the mean curvature vector $\vec{\mathcal{H}}$ satisfies

$$\vec{\mathcal{H}}=-\frac{1}{2}\theta\underline{L}-\frac{1}{2}\underline{\theta}L,$$

where $\underline{\theta} := \operatorname{tr}_\gamma \underline{\chi}$, $\theta := \operatorname{tr}_\gamma \chi$ are the *null expansions* of (Σ, γ). Note that the decomposition $\underline{\chi}$, χ, $\underline{\theta}$, θ, ζ depends on the choice of null frame, i.e. the choice of null generator. Due to this gauge freedom, it is important to study the (extrinsic) geometry of (Σ, γ) with respect to quantities that are independent of the choice of null frame. For example, the above decomposition gives that

$$\mathcal{H}^2 := g(\vec{\mathcal{H}}, \vec{\mathcal{H}}) = \underline{\theta}\theta,$$

so the product of the two null expansions will always agree with the (Lorentzian) length of the mean curvature vector, which is manifestly independent of the choice of null frame. We call the function $\mathcal{H}^2$ the *spacetime mean curvature* of Σ, and say Σ is a *surface of constant spacetime mean curvature* (STCMC surface) if $\mathcal{H}^2$ is constant along Σ. Although such surfaces have already been studied more broadly, see e.g. [5], the term spacetime mean curvature was first coined by Cederbaum–Sakovich [3], where they construct asymptotic foliations by STCMC surfaces in the asymptotically Euclidean setting to define a notion of center of mass. Note that we do not impose any asymptotic assumptions on the spacetime (M, g) here, and since g is Lorentzian, $\mathcal{H}^2$ can be a (at least locally) negative function. In a similar way, one can check that null mean curvature flow (1) is independent of the choice of $\underline{L}$. Additionally, the right-hand side of (1) is precisely the projection of the mean curvature vector $\vec{\mathcal{H}}$ in direction of $\underline{L}$. This greatly reduces the complexity of the codimension-2 problem, and one can show that (at least locally) the flow is always equivalent to single, scalar parabolic equation.

In the setting of Roesch–Scheuer [11] the presence of appropriate barriers ensures existence for all times and smooth convergence to a MOTS. On the other hand, when no MOTS exists, one heuristically expects the flow to develop finite time singularities. In the explicit case of the round Minkowski lightcone, this is made precise by the following result:

Theorem 1 (W. '23, [13])**.** *Let (Σ_0, γ_0) be a spacelike cross section of the future-pointing standard lightcone $\mathcal{N}$ in the $3+1$-Minkowski spacetime. Then the solution of null mean curvature flow starting from Σ_0 extinguishes in the tip of the cone in finite time and the renormalization by area converges to a surface of constant spacetime mean curvature, which exactly arise as the image of a round sphere of a Lorentz transformation in $\mathrm{SO}^+(3,1)$.*

The result follows directly once it is shown that in this setting null mean curvature flow is in fact equivalent to $2d$-Ricci flow in the conformal class of the round sphere, and using a classical result first proven by Hamilton [10]:

Theorem 2. *For any Riemannian metric on a compact manifold, Hamilton's Ricci flow exists for all times and converges to a metric of constant (scalar) curvature.*

Here, Hamilton's Ricci flow denotes the rescaled Ricci flow equation preserving area. Theorem 2 was initially proven by Hamilton [10] in the conformally round case only for metrics of strictly positive curvature. This restriction was later removed by Chow [6], and their methods yield an independent proof of the uniformization theorem [4]. There are several independent proofs of Theorem 2 in the conformally round setting, cf. [1, 2, 12].

The round (future-pointing) Minkowski lightcone $\mathcal{N}$ is given as the lvel-set $\{v = 0\}$ with respect to the the double null coordinates $u := r + t$, $v := r - t$. In these coordinates, the Minkowski metric η takes the form

$$\eta = \frac{1}{2}\left(dudv + dvdu\right) + r^2 d\Omega^2,$$

where $r = \frac{u+v}{2}$, and $d\Omega^2$ denotes the round metric on $\mathbb{S}^2$. It is easy to check that the induced metric on $\mathcal{N}$ is indeed degenerate and that $\underline{L} = 2\partial_u$ is a geodesic null generator of $\mathcal{N}$. As $\mathcal{N}$ is a shear-free null hypersurface, one finds $\overset{\circ}{\underline{\chi}} = 0$ for any spacelike cross section (Σ, γ). Thus, as the ambient spacetime is flat, the codimension-2 Gauss equaton simply states that

$$\mathcal{H}^2 = 2\,\mathrm{R}_\Sigma \tag{2}$$

for any spacelike cross section (Σ, γ), where R_Σ denotes the scalar curvature of (Σ, γ). Moreover, one can uniquely identify any cross section (Σ, γ) with a conformally round metric $\gamma = \omega^2 d\Omega^2$. In particular, an STCMC surface carries a conformally round metric of constant scalar curvature, and it is a well-known fact that such a metric is generated by the action of a suitable Möbius transformation on the round metric (up to scaling). As the Möbius group is isometric to the restricted Lorentz group $\mathrm{SO}^+(1,3)$, one can also understand this from an extrinsic viewpoint as a Lorentz transformation represents a change of observer that leaves $\mathcal{N}$ invariant but distortes the sphere.

Equation (2) additionally yields the equivalence between the two flows. Up to a constant depending on the dimension (2) remains true for conformally round metrics in all dimensions, which yields an equivalence between null mean curvature flow and Yamabe flow in any dimension. In fact, whenever a construction of Fefferman–Graham [9] is possible for a Riemannian metric g_0, which again identifies any metric conformal to g_0 with a unique spacelike cross section of a hypersurface in the constructed spacetime, (2) holds and null mean curvature flow and Yamabe flow are equivalent.

This viewpoint now allows one to study $2d$-Ricci flow (at least in the conformal class of the round sphere) as an extrinsic curvature flow. As $\overset{\circ}{\mathrm{Ric}} = 0$, the scalar curvature is the only intrinsic curvature quantity. By defining a *scalar second fundamental form* $A := \underline{\theta}\chi$, which is again independent of the choice of frame, we obtain additional (extrinsic) curvature information. In particular, A satisfies a Codazzi euqation

$$\nabla_i A_{jk} = \nabla_j A_{ik}, \tag{3}$$

which yields that $\mathring{A} = 0$ if and only if Σ is an STCMC surface. Studying the evolution equations

$$\frac{d}{dt}\mathcal{H}^2 = \Delta\mathcal{H}^2 + \frac{1}{2}\left(\mathcal{H}^2\right)^2,$$
$$\frac{d}{dt}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + \frac{1}{2}\left(\mathcal{H}^2\right)^3$$

under the flow, one can prove the following gradient estimate simply by applying the maximum principle:

Theorem 3 (W. '23, [13])**.** *Let $\mathcal{H}^2 > 0$ initially, $p > \frac{1}{2}$, $\eta > 0$. Then there exists a constant $C = C(\eta, p, \Sigma_0)$ such that*

$$|\nabla\mathcal{H}^2| \leq \eta^2(\mathcal{H}^2)^p + C.$$

From this, one can give an independent prove of Hamilton's classical result (under Hamiltons initial assumption of positive scalar curvature).

Studying a suitable quantity that is monotone under the flow, one obtains a quantitative statement of the consequence of the Codazzi Equation (3) for metrics on non-negative scalar curvature:

Theorem 4. *If $\mathcal{H}^2 \geq 0$, then*

$$\left|\left|\mathcal{H}^2 - \frac{1}{|\Sigma|}\int \mathcal{H}^2\right|\right|^2_{L^2(\Sigma)} \leq 4||\mathring{A}||^2_{L^2(\Sigma)},$$

where equality holds if and only if Σ is an STCMC surface.

A corresponding estimate in Euclidean space was first proven by De Lellis–Müller [7]. Additionally, one can view Theorem 4 as a generalization of an almost-Schur lemma by De Lellis–Topping [8] to two dimensions, and the two estimates are in fact equivalent in higher dimensions in the conformal class of the round sphere.

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Dominant energy shields on spin initial data sets

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(joint work with Simone Cecchini, Martin Lesourd)

The Riemannian positive mass theorem states that in a complete asymptotically flat (AF) manifold of nonnegative scalar curvature, the ADM-mass of each AF end is non-negative. The (more general) spacetime positive mass theorem yields a similar inequality for AF *initial data sets* (M, g, k), where (M, g) is an AF Riemannian manifold endowed with a suitable symmetric 2-tensor k. In this setting, non-negativity of the scalar curvature becomes the *dominant energy condition* $\mu \geq |J|$, where

$$\mu = \frac{1}{2}(\mathrm{scal}_g + (\mathrm{tr}_g(k))^2 - |k|_g^2), \quad J = \mathrm{div}(k) - \mathrm{d}\,\mathrm{tr}_g(k).$$

Moreover, the ADM mass is replaced by the *ADM energy* E and the *ADM linear momentum* P at each asymptotically flat end. The positive mass theorem for initial data sets [3, 5–7] states that if $(M^{n\geq 3}, g, k)$ is a complete AF initial data set that satisfies the dominant energy condition $\mu - |J| \geq 0$, then $\mathrm{E} \geq |\mathrm{P}|$ for each asymptotically flat end, provided that $n \leq 7$ or M^n is spin.

In our recent work [1], extending previous work of Cecchini–Zeidler [2] to the spacetime case, we establish several results which allow to relax the asymptotic flatness condition by focussing on a single end at a time. The first main result is a "shielded" variant of the spacetime positive mass theorem in the following sense.

Definition 1. *Let (M^n, g, k) be a Riemannian manifold, not assumed to be complete. We say that (M, g, k)* contains a dominant energy shield *$U_0 \supset U_1 \supset U_2$ if U_0, U_1, and U_2 are open subsets of M such that $U_0 \supset \overline{U}_1$, $U_1 \supset \overline{U}_2$, the closure of U_0 in (M, g) is a complete manifold with compact boundary, and we have the following:*

(1) $\mu - |J| \geq 0$ *on* U_0,
(2) $\mu - |J| \geq \sigma n(n-1)$ *on* $U_1 \setminus U_2$,

(3) *the mean curvature* $\mathrm{H}_{\partial\bar{U}_0}$ *on* $\partial\bar{U}_0$ *and the symmetric two tensor* k *satisfy*

$$\mathrm{H}_{\partial\bar{U}_0} - \tfrac{2}{n-1}\left|k(\nu,-)|_{\mathrm{T}\partial\bar{U}_0}\right| > -\Psi(d,l).$$

Here, $\Psi(d,l)$ *is the constant defined as*

$$\Psi(d,l) := \begin{cases} \frac{2}{n}\frac{\lambda(d)}{1-l\lambda(d)} & \text{if } d < \frac{\pi}{\sqrt{\sigma}n} \text{ and } l < \frac{1}{\lambda(d)}, \\ \infty & \text{otherwise,} \end{cases}$$

where $d := \mathrm{dist}_g(\partial U_2, \partial U_1)$*,* $l := \mathrm{dist}_g(\partial U_1, \partial U_0)$*, and*

$$\lambda(d) := \frac{\sqrt{\sigma}n}{2}\tan\left(\frac{\sqrt{\sigma}nd}{2}\right).$$

Notably, $\Psi(d,l)$ tends to ∞ as d approaches a certain fixed number and something similar happens for l after fixed $d > 0$. This means that the boundary condition (3) eventually becomes empty and can be dropped provided that the combination of distances is large enough.

Theorem 1 ([1, Theorem A])**.** *Let* $(M^{n\geq 3}, g, k)$ *be an initial data set, not necessarily complete, that contains an asymptotically flat end* $\mathcal{E}$ *and a dominant energy shield as in Definition 1 with* $\mathcal{E} \subset U_2$*. Assume that* U_0 *is spin and that* $\overline{\mathcal{E}\backslash U_0}$ *is compact. Then* $\mathrm{E}_{\mathcal{E}} > |\mathrm{P}_{\mathcal{E}}|$*.*

The proof of Theorem 1 for $k = 0$ can be reduced to the original spacetime positive mass theorem by introducing an artificial tensor k that makes use of the strictly dominant energy condition on the shield, and this idea underlies the previous approaches [2, 4]. However, for $k \neq 0$, a new conceptual ingredient is required: We need to introduce another 'time' direction in the spin bundle which allows us to introduce an additional term in the Dirac operator (independent of the given tensor k).

Using this idea we also obtain the following result showing that embedding an end, which violates the positive mass theorem, into a complete initial data set is obstructed in a quantitative way:

Theorem 2 ([1, Theorem B])**.** *Let* $(\mathcal{E}, g, k)$ *be an asymptotically flat initial data end of dimension* $n \geq 3$ *such that* $\mathrm{E}_{\mathcal{E}} < |\mathrm{P}_{\mathcal{E}}|$*. Then there exists a constant* $R = R(\mathcal{E}, g, k)$ *such that the following holds: If* (M, g, k) *is an* n*-dimensional initial data set (without boundary) that contains* $(\mathcal{E}, g, k)$ *as an open subset and* $\mathcal{N} = \mathcal{N}_R(\mathcal{E}) \subseteq M$ *denotes the open neighborhood of radius* R *around* $\mathcal{E}$ *in* M*, then at least one of the following conditions must be violated:*

(1) $\overline{\mathcal{N}}$ *(metrically) complete,*
(2) $\mu - |J| \geq 0$ *on* $\mathcal{U}$*,*
(3) $\mathcal{N}$ *is spin.*

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