Mathematisches Forschungsinstitut Oberwolfach

Report No. 14/2024

DOI: 10.4171/OWR/2024/14

K-Stability, Birational Geometry and Mirror Symmetry

Organized by
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17 March – 22 March 2024

Abstract. The workshop *K-stability, Birational Geometry and Mirror Symmetry* presented recent advances in all three topics, in the form of research level mini-courses, research talks and lightning talk sessions. Deep interactions between the three topics were highlighted, together with applications (e.g. to K-moduli or subgroups of the Cremona groups) and new directions (such as non-Archimedean methods in K-stability and Mirror Symmetry).

Mathematics Subject Classification (2020): 14E07, 14E30, 14J33, 14J45, 14B07, 14N35, 14J10, 32Q20, 32Q26.

Introduction by the Organizers

The workshop *K-stability, Birational Geometry and Mirror Symmetry*, organised by Thibaut Delcroix (Montpellier), Liana Heuberger (Bath) and Susanna Zimmermann (Orsay) was well attended with 44 participants. The program of the workshop consisted in three mini-courses of 4 hours each, delivered by pairs of experts each morning from Monday to Thursday, 8 talks, and two sessions of 5 lightning talks of 10 minutes each. In what follows, we describe the main themes covered by the workshop.

The mini-course on birational geometry, with lectures by Julia Schneider and Jeremy Blanc, focused on recent major results on quotients of the Cremona group (the group of birational self-maps of projective space) in various dimensions and over various fields. One of the major means of understanding this classical group (or, rather, the groupoid of birational maps between Mori fibre spaces), expanded upon in detail during the mini-course, is its presentation in terms of Sarkisov links and elementary relations. Focusing first on surfaces with Julia Schneider's two lectures, and switching to higher dimensions in Jérémy Blanc's talks, surveyed the main ideas from groundbreaking results of Blanc, Lamy, Zimmermann, Schneider, Yasinsky et al. on the quotients of the Cremona group, as well as highlighting key ingredients of their proofs.

The mini-course on K-stability, by Ivan Cheltsov and Elena Denisova, centred around recent techniques in K-stability applied to the explicit study of K-stability for smooth Fano threefolds. They presented the vast program whose results are collected in the book "The Calabi problem for Fano threefolds" by Araujo, Cheltsov et al. The purpose of this study is to determine which smooth Fano threefolds are K-stable, as well as finding applications which study of their behaviour in families, i.e. to determine the properties of K-moduli spaces. Here, a key input is Fujita and Li's valuative criterion for the K-stability of Fano manifolds, together with Abban and Zhuang's method of finding a lower bound of the delta invariant from a geometrically relevant choice of flag of subvarieties. With these tools at hand, the problem essentially reduces to a difficult and subtle, but explicit, study of the birational geometry of Fano threefolds, and Sarkisov links appear again in this setting.

The third mini-course, by Giulia Gugiatti and Andrea Petracci, presented recent advances around the Fano/Landau-Ginzburg (F/LG) correspondence in mirror symmetry. Various incarnations of this correspondence were discussed, culminating in a construction of smoothings of affine singular toric threefolds by Corti-Hacking-Petracci. A nuanced discussion of the limits of currently applicable systematic methods crystallised in the study of Johnsson-Kollàr surfaces, whose anticanonical system is known to be empty. A discussion of the homological mirror symmetry (a stronger version of the mirror theorem than the F/LG correspondence) for del Pezzo surfaces, by Auroux-Katzarkov-Orlov was incorporated in this more general singular setting.

Some of the research talks shared striking similarities, highlighting the interconnectedness and broad impact of the chosen topics. The talks of Anne-Sophie Kaloghiros and Yuchen Liu focussed on the topic of K-moduli spaces, which was partly addressed in the mini-course, and expanded upon it in different directions. The result of Kaloghiros describes certain irreducible components of K-moduli space for Fano threefolds in the simplest possible non-trivial case (i.e. not reduced to a point), namely when it is of dimension one, by combining birational geometry with a technique from mirror symmetry which produces toric smoothings. Liu delivered, on the final day of the workshop, a talk on the moduli continuity method for K-stability. In connection to the mini-course, he explained how to apply this method to a family of smooth Fano threefolds (No. 2-15 in the Mori-Mukai list) to explicitly determine its irreducible component in the K-moduli space using its description as a GIT quotient.

In the last part of the workshop, the talks of Sébastien Boucksom and Enrica Mazzon were hinging toward non-Archimedean techniques in two of the main subjects of the workshop. Boucksom explained how non-Archimedean formalism allows for a valuative interpretation of K-stability beyond the Fano case, emphasizing that nonetheless explicit methods as presented in the mini-course are currently out of reach in the general polarization case and require further research. Mazzon's talk introduced the SYZ conjecture in mirror symmetry, and explained how the non-Archimedean approach first proposed by Kontsevich and Soibelman is giving rise to exciting new results appearing in work of Yang Li and Hultgren-Jonsson-Mazzon-McCleerey.

The remaining talks provide a wide overview of various recent advances and fields of investigation in algebraic geometry, which we briefly summarize. Hendrik Süß highlighted surprising relations between the notion of normalized volume of singularities, underlying the valuative criterion for K-stability, and convex geometry in the setting of toric singularities. Ana-Maria Castravet's talk introduced the notion of higher Fano manifolds, and presented an overview of known examples and non-examples. Carolina Araujo focused on a problem of Gizatullin, asking when automorphisms of a quartic surface in the projective space are induced by Cremona transformations of the projective space, and presented her results in this direction relating to the Sarkisov program presented in the mini-courses. Finally, Alex Duncan's talk focused, in contrast to the mini-course, on small, finite subgroups of Cremona groups, measuring the complexity of the Cremona group in terms of the representation dimension of its finite subgroups.

We ran two hour-long sessions of 10-minute lightning talks, so that participants could advertise a new result, a question or a key idea. Both junior and senior participants were eager to contribute, resulting in a widely representative, discussion-fuelling workshop. We also welcomed the input of an Oberwolfach Research Fellow which was not originally planned as a participant of the workshop. The lightning talks were given by Andrés Jaramillo Puentes, Jaroslaw Wisniewski, Roland Púĉek, Egor Yasinsky, King Leung Lee, Eduardo Alves da Silva, Ignacio Barros, Erroxe Etxabarri Alberdi, Tran Trung Nghiem and Jürgen Hausen.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Alexander R. Duncan in the "Simons Visiting Professors" program at the MFO.

Workshop: K-Stability, Birational Geometry and Mirror Symmetry

Table of Contents

Abstracts

Sarkisov links and Cremona groups

Jérémy Blanc and Julia Schneider

Let K be a field. The *Cremona group of rank n over K* is the group of birational transformations of the projective space of dimension n, written $\mathrm{Bir}_K(\mathbb{P}^n)$. Non-trivial normal subgroups of $\mathrm{Bir}_K(\mathbb{P}^2)$ have been constructed using geometric group theory (see [CL13] for algebraically closed fields and [Lon16] for arbitrary fields). The resulting quotients, however, are complicated and remain mysterious. In fact, if K is algebraically closed, every non-trivial quotient of $\mathrm{Bir}_K(\mathbb{P}^2)$ contains $\mathrm{PGL}_3(K)$; in particular, there is no finite and no abelian quotients, except the trivial group.

For fields that are not algebraically closed, this is no longer true, as was discovered by Zimmermann who showed that the abelianisation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ [Zim18]. This opened up the approach of studying Cremona groups via their group homomorphisms. For perfect fields K, quotients of $\mathrm{Bir}_K(\mathbb{P}^2)$ that are a free product of direct sums of $\mathbb{Z}/2\mathbb{Z}$ have been constructed in [LZ20, Sch22, Zim22]. For $n \geq 3$, quotients of $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^n)$ that are a free product of direct sums of $\mathbb{Z}/2\mathbb{Z}$ have been constructed in [BLZ21], showing that complex Cremona groups in higher dimensions are not simple. More quotients were constructed in this setting [BY20, Zik23]. Finally, using motivic invariants of birational maps, it was shown that $\mathbb{Z}$ is a quotient of $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^n)$ for $n \geq 4$, as well as of $\mathrm{Bir}_{\mathbb{Q}}(\mathbb{P}^n)$ for $n \geq 3$ [LS22]. Hence, these Cremona groups are not generated by elements of finite order, contrasting the case of plane Cremona groups over perfect fields which are generated by involutions [LS24]. Notably, the question whether $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^3)$ is generated by elements of finite order remains open.

1. The results

Together with Egor Yasinsky, we studied $\mathrm{Bir}_K(S)$, where S is a non-trivial Severi-Brauer surface over a perfect field K, that is, $S_{\bar{K}} \simeq \mathbb{P}^2_{\bar{K}}$ and $S(K) = \emptyset$.

Theorem 1. [BSY23] *Let S be a non-trivial Severi-Brauer surface. For $d \in \{3, 6\}$ we denote by $\mathcal{P}_d$ the set of degree d points of S up to the action of $\mathrm{Aut}_K(S)$. Then, $|\mathcal{P}_3| \geq 2$ and for each $p \in \mathcal{P}_3$, there is a surjective group homomorphism*

$$\Psi\colon \mathrm{Bir}_K(S) \to \bigoplus_{\mathcal{P}_3 \setminus \{p\}} \mathbb{Z}/3\mathbb{Z} * \left(\mathop{\ast}_{\mathcal{P}_6} \mathbb{Z} \right).$$

In particular, $\mathrm{Bir}_K(S)$ is not a perfect group (and is thus not simple). Moreover, if $\mathcal{P}_6 \neq \emptyset$, then $\mathrm{Bir}_K(S)$ is not generated by elements of finite order.

As an application, we study fibrations $X \to B$ whose generic fibre is a non-trivial Severi-Brauer surface S over $\mathbb{C}(B)$. Such S exist only if $\dim B \geq 2$, and hence $\dim X \geq 4$. For $n \geq 4$, we construct a surjective group homomorphism from $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^n)$ to $F(\mathbb{C}) = \ast_{\mathbb{C}} \mathbb{Z}$, the free group indexed by $\mathbb{C}$. This implies the following:

Theorem 2. [BSY23] *Let $n \geq 4$, and let G be any group of cardinality $|G| \leq |\mathbb{C}|$. Then G is a quotient of* $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^n)$.

2. The strategy: The Sarkisov program

Instead of just considering $\mathrm{Bir}(X)$ one can consider a larger class of birational maps, namely the *groupoid* $\mathrm{BirMori}(X)$ consisting of birational maps between *Mori fibre spaces* birational to X. The *Sarkisov program* states that $\mathrm{BirMori}(X)$ is generated by *Sarkisov links* and isomorphisms of Mori fibre spaces, and every relation between Sarkisov links is generated by *elementary relations* and trivial relations (see [Isk96, Cor95, HM13, Kal13, LZ20, BLZ21]). In order to define Mori fibre spaces, Sarkisov links and elementary relations in a uniform way, we introduce rank r fibrations:

Assume first that X is a smooth projective surface over a perfect field K. A surjective morphism $\pi\colon X \to B$ with connected fibres is a *rank r fibration* if B is smooth with $\dim(B) < \dim(X)$, relative Picard rank $\rho(X/B) = r$, and $-K_X$ is relatively ample. Note that $X/\mathrm{Spec}(K)$ is a rank r fibration exactly if X is a del Pezzo surface of Picard rank $\rho(X) = r$. The higher dimensional analogue of rank r fibrations tries to capture some of the nice properties of del Pezzo surfaces in terms of birational geometry. In particular (see [BLZ21] for the precise definition), one requires that X is terminal and $\mathbb{Q}$-factorial (mild singularities), X/B is a Mori Dream space (hence one can run any MMP from X over B, and there are only finitely many outputs), every output of an MMP from X over B is again $\mathbb{Q}$-factorial and terminal and $-K_X$ is big over B.

For example, note that the Hirzebruch surface $\mathbb{F}_2$ is a rank 1 fibration over $\mathbb{P}^1$ but not a rank 2 fibration over $\mathrm{Spec}(K)$: $-K_{\mathbb{F}_2}$ is big but the contraction of the section with self-intersection -2 produces a singular surface (not terminal).

The observant reader might have realised by now that rank 1 fibrations are exactly Mori fibre spaces. A rank 2 fibration Y/B dominates exactly two rank 1 fibrations X_1/B_1 and X_2/B_2, giving rise to a diagram as follows, called *Sarkisov diagram*, where the dotted arrow denotes a pseudo-isomorphism:

$$\begin{array}{ccccc} Y = Y_1 & & \cdots\cdots\cdots\rightarrow & & Y_2 \\ \downarrow & & & & \downarrow \\ X_1 & & \dashrightarrow & & X_2 \\ \downarrow & & & & \downarrow \\ B_1 & \longrightarrow & B & \longleftarrow & B_2 \end{array}$$

A birational map $\chi\colon X_1 \dashrightarrow X_2$ between two Mori fibre spaces is called *Sarkisov link* exactly if it can be put into such a diagram. Depending on whether the birational morphisms $Y_i \to X_i$ are isomorphisms or divisorial contractions one obtains four types of links; if both of them are divisorial the link is said to be a Sarkisov link of type II.

A rank 3 fibration gives rise to a relation between Sarkisov links. These are the *elementary relations* that generate relations, as explained before.

This allows us to define *groupoid* homomorphisms $\mathrm{BirMori}(X) \to G$ as follows. We choose a certain type of Sarkisov links, typically a link of type II, such that we have a good control over the elementary relations in which they appear. For a Sarkisov link $\chi\colon X_1 \dashrightarrow X_2$ of type II, denote by $\Gamma \subset X_1$ the exceptional locus of χ, and $Y \to X_1$ the blow-up at Γ. In the case of surfaces, Bertini involutions where Γ is a point of degree 8 were used in [LZ20]: They do not appear in any elementary relation (because Y is a del Pezzo surface of degree 1) but they do satisfy the trivial relation $\chi = \chi^{-1}$, giving a quotient of the form $\ast_{\mathcal{P}_8} \mathbb{Z}/2\mathbb{Z}$.

For Theorem 1 we use the fact that the degree of any point on a non-trivial Severi-Brauer surface S is divisible by 3, and that any Sarkisov link χ from S goes to the opposite Severi-Brauer surface S^{op}. The construction of the groupoid homomorphism goes as follows, depending on the degree d of the base-point Γ of χ. If $d = 6$, then χ does not appear in any elementary relation, so we send χ onto $1_{[\Gamma]} \in \ast_{\mathcal{P}_6} \mathbb{Z}$. If $d = 3$, there is an elementary relation involving χ; we send χ onto $1_{[\Gamma]} \in \bigoplus_{\mathcal{P}_3} \mathbb{Z}/3\mathbb{Z}$.

In higher dimensions, [BLZ21] used links between conic bundles where Γ has large covering gonality. To prove Theorem 2, we use Sarkisov links χ between Mori fibre spaces whose generic fibre is a non-trivial Severi-Brauer surface, such that χ induces a Sarkisov link $\widehat{\chi}$ between the corresponding Severi-Brauer surfaces, and, moreover, such that Γ has large covering genus.

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Calabi problem for smooth Fano threefolds

Ivan Cheltsov and Elena Denisova
(joint work with many people)

This is an abstract for four lectures given at the workshop. The goal of these lectures were to explain how to prove K-stability of smooth Fano threefolds.

1. Inductive approach to K-stability

Let X be a smooth Fano threefold, let $f\colon W \to X$ be a birational morphism such that W is normal, and let E be a prime divisor in W. We say that E is a divisor over X, and we denote this as E/X. We can relate several numbers to E. One is log-discrepancy $A_X(E) = 1+\mathrm{ord}_E\big(K_{W/X}\big)$, which is easy to compute. The second one is the pseudo-effective threshold:

$$\tau(E) = \sup\Big\{u \in \mathbb{R}_{>0} \ \Big|\ f^*(-K_X) - uE \text{ is pseudo-effective}\Big\},$$

which is not so easy to compute. The third number is the most difficult one:

$$S_X(E) = \frac{1}{(-K_X)^n}\int\limits_0^{\infty} \mathrm{vol}\big(-K_X - uE\big)du = \frac{1}{(-K_X)^n}\int\limits_0^{\tau(E)} \mathrm{vol}\big(f^*(-K_X) - uE\big)du.$$

Let $\beta(E) = A_X(E) - S_X(E)$. Then the Fujita–Li valuation criterion [6, 7] says that X is X is K-stable $\Leftrightarrow$ $\beta(E) > 0$ for every prime divisor E over X. This criterion can be restated as follows. For every point $P \in X$, let

$$\delta_P(X) = \inf_{\substack{E/X \\ P\in C_X(E)}} \frac{A_X(E)}{S_X(E)}$$

where the infimum is taken by all prime divisors over X whose center on X contains the point P. Then X is K-stable if $\delta_P(X) > 1$ for every $P \in X$.

To show that $\delta_P(X) > 1$, we can use Abban–Zhuang theory [1] and Fujita's formula derived from this theory [2]. Namely, let us do the following:

(1) choose a surface $S \subset X$ such that $P \in S$ and S has Du Val singularities;
(2) compute $\tau = \tau(S) = \sup\big\{u \in \mathbb{Q}_{>0}\ \big|\ -K_X - uS \text{ is pseudo-effective}\big\}$
(3) for $u \in [0, \tau]$, compute
- $P(u) =$ the positive part of the Zariski decomposition of $-K_X - uS$,
- $N(u) =$ the negative part of the Zariski decomposition of $-K_X - uS$.

Then $S_X(S) = \frac{1}{(-K_X)^3}\int\limits_0^\tau P(u)^3 du$. If $S_X(S) \geqslant 1$, then $\delta_P(X) \leqslant 1$. Thus, we may assume $S_X(S) < 1$. Then Abban–Zhuang theory and the Fujita's formula give:

$$\delta_P(X) \geqslant \min\left\{\frac{1}{S_X(S)}, \delta_P\big(S, W^S_{\bullet,\bullet}\big)\right\}$$

for

$$\delta_P\big(S, W^S_{\bullet,\bullet}\big) = \inf_{\substack{F/S\\ P\in C_S(F)}} \frac{A_S(F)}{S\big(W^S_{\bullet,\bullet}; F\big)},$$

where the infimum is taken by all prime divisors F over S such that $P \in C_S(F)$ and

$$S\big(W^S_{\bullet,\bullet}; F\big) = \frac{3}{(-K_X)^3}\int\limits_0^\tau \big(P(u)^2\cdot S\big)\cdot \mathrm{ord}_F\Big(N(u)\big|_S\Big)du \\ + \frac{3}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^\infty \mathrm{vol}\big(P(u)\big|_S - vF\big)dvdu.$$

Note that the number $\delta_P(S, W^S_{\bullet,\bullet})$ has the same nature as $\delta_P(X)$, but it is always easier to compute, because S is a surface. Moreover, to estimate $\delta_P(S, W^S_{\bullet,\bullet})$, we can apply Abban–Zhuang theory again.

2. δ-invariants of polarized surfaces

Abban–Zhuang theory can be applied to estimate δ-invariants of any polarized variety. To illustrate this, let S be a smooth surface, and let D be a big and nef $\mathbb{R}$-divisor D on the surface S. For every prime divisor F over S, we set

$$S_D(F) = \frac{1}{D^2}\int\limits_0^\infty \mathrm{vol}\big(D - vF\big)dv$$

similar to what we did for smooth Fano threefold in Section 1. Then we let

$$\delta_P\big(S, D\big) = \inf_{\substack{F/S\\ P\in C_S(F)}} \frac{A_S(F)}{S_D\big(F\big)}$$

where the infimum is taken by all prime divisors F over S whose support contains the point P.

To apply Abban–Zhuang theory to estimate $\delta_P(S, D)$, we do the following:

(1) choose a smooth curve $C \subset S$ that passes through P;
(2) compute $\tau = \sup\Big\{v \in \mathbb{R}_{\geqslant 0} \;\Big|\; \text{the divisor } D - vC \text{ is pseudo-effective}\Big\}$;
(3) for $v \in [0, \tau]$, compute
   - $P(v)$ = the positive part of the Zariski decomposition of $D - vC$,
   - $N(v)$ = the negative part of the Zariski decomposition of $D - vC$.

Then $S_D(C)=\frac{1}{D^2}\int\limits_0^\infty \mathrm{vol}\big(D-vC\big)dv=\frac{1}{D^2}\int\limits_0^\tau P(v)^2dv$. Thus, we have

$$\delta_P(S,D)\leqslant\frac{A_S(C)}{S_D(C)}=\frac{1}{S_D(C)}.$$

Set

$$\begin{aligned}S\big(W^C_{\bullet,\bullet};P\big)&=\frac{2}{D^2}\int\limits_0^\tau \mathrm{ord}_P\big(N(v)|_C\big)\big(P(v)\cdot C\big)dv+\frac{1}{D^2}\int\limits_0^\tau \big(P(v)\cdot C\big)^2dv\\&=\frac{2}{D^2}\int\limits_0^\tau h(v)dv,\end{aligned}$$

where

$$h(v)=\big(P(v)\cdot C\big)\times\big(N(v)\cdot C\big)_P+\frac{\big(P(v)\cdot C\big)^2}{2}.$$

Then it follows from Abban–Zhuang theory and Fujita's formula that

$$\delta_P\big(S,D\big)\geqslant\min\left\{\frac{1}{S_D(C)},\frac{1}{S\big(W^C_{\bullet,\bullet};P\big)}\right\}.$$

We can apply Abban–Zhuang theory to the exceptional curve of a weighted blow up of S at P. For simplicity, let us show how to do this for the usual blow up. Namely, let $f\colon\widetilde{S}\to S$ be the blow up of the surface S at the point P, and let E be the f-exceptional curve. Set

$$\widetilde{\tau}=\sup\Big\{v\in\ \mathbb{R}_{\geqslant 0}\ \Big|\ \text{the divisor } f^*(D)-vE \text{ is pseudo-effective}\Big\}.$$

For $v\in[0,\widetilde{\tau}]$, let $\widetilde{P}(v)$ and $\widetilde{N}(v)$ be the positive and negative part of the Zariski decomposition of the divisor $f^*(D)-vE$, respectively. Then $A_S(E)=2$ and $S_D(E)=\frac{1}{D^2}\int\limits_0^{\widetilde{\tau}}\widetilde{P}(v)^2dv$. Thus, we have

$$\delta_P(S,D)\leqslant\frac{2}{S_D(E)}$$

Now, we for every point $O\in E$, we set

$$\begin{aligned}S\big(W^E_{\bullet,\bullet};O\big)&=\frac{2}{D^2}\int\limits_0^{\widetilde{\tau}} \mathrm{ord}_O\big(\widetilde{N}(v)|_E\big)\big(\widetilde{P}(v)|_E\big)dv+\frac{1}{D^2}\int\limits_0^{\widetilde{\tau}} \big(\widetilde{P}(v)\cdot E\big)^2dv\\&=\frac{1}{D^2}\int\limits_0^{\widetilde{\tau}} h(v)dv,\end{aligned}$$

where

$$h(v)=\big(\widetilde{P}(v)\cdot E\big)\times\big(\widetilde{N}(v)\cdot E\big)_P+\frac{\big(\widetilde{P}(v)\cdot E\big)^2}{2}.$$

Then Abban–Zhuang theory gives

$$\delta_P\big(S,D\big) \geqslant \min\left\{\frac{2}{S_D(E)}, \inf_{O\in E}\frac{1}{S\big(W^E_{\bullet,\bullet};O\big)}\right\}.$$

3. Nemurro lemma

Now, we go back to smooth Fano threefolds. Let us use all assumptions and notations of Section 1. Set

$$D = D(u) = P(u)|_S.$$

for every $u\in[0,\tau]$. Consider the polarized pair (S,D). Here, the divisor D is nef by construction. However, it may not be big in general (especially for $u=\tau$), but everything we described in Section 2 still works. Thus, arguing as in Section 2, we can find an estimate

$$\delta_P\big(S,D\big)\geqslant q(u)$$

for $u\in[0,\tau]$, where $q(u)\colon[0,\tau]\to\mathbb{R}_{>0}$ is a continuous non-negative function. Following [3], let us show how to use this estimate for $\delta_P(S,D)$ to estimate $\delta_P(X)$. Namely, observe than

$$\begin{aligned}
S\big(W^S_{\bullet,\bullet};F\big) &= \frac{3}{(-K_X)^3}\int\limits_0^\tau \big(P(u)^2\cdot S\big)\cdot\mathrm{ord}_F\Big(N(u)\big|_S\Big)du\\
&\quad+\frac{3}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^\infty \mathrm{vol}\big(P(u)\big|_S-vF\big)dvdu\\
&\leqslant \frac{3}{(-K_X)^3}\int\limits_0^\tau \big(P(u)^2\cdot S\big)\cdot\mathrm{ord}_F\Big(N(u)\big|_S\Big)du\\
&\quad+\left(\frac{3}{(-K_X)^3}\int\limits_0^\tau\frac{D^2}{q(u)}du\right)A_S(F)
\end{aligned}$$

for every prime divisor F over S. Thus, if $P\notin\mathrm{Supp}(N(u))$ for $u\in[0,\tau]$, then

$$\delta_P\big(S,W^S_{\bullet,\bullet}\big)\geqslant\frac{1}{\frac{3}{(-K_X)^3}\int\limits_0^\tau\frac{D^2}{q(u)}du}.$$

What if $P\in\mathrm{Supp}(N(u))$ for some $u\in[0,\tau]$? In this case, we want to find a $K>0$ such that

$$\frac{3}{(-K_X)^3}\int\limits_0^\tau \big(P(u)^2\cdot S\big)\cdot\mathrm{ord}_F\Big(N(u)\big|_S\Big)du\leqslant K\cdot A_S(F)$$

for every prime divisor F over X. How to do this?

Example 1. Suppose that there is $a \in (0,\tau)$ such that

$$N(u)=\begin{cases} 0 & \text{for } u\in[0,a],\\ (u-a)E & \text{for } u\in[a,\tau],\end{cases}$$

where E is a prime divisor in X such that $E\neq S$. If $(S,E|_S)$ is log canonical, then

$$\frac{3}{(-K_X)^3}\int\limits_0^{\tau}\big(P(u)^2\cdot S\big)\cdot\mathrm{ord}_F\Big(N(u)\big|_S\Big)du \leqslant \left(\frac{3}{(-K_X)^3}\int\limits_a^{\tau}\big(P(u)^2\cdot S\big)(u-a)du\right)A_S(F)$$

for every prime divisor F over S. This holds if S is smooth, and $E|_S$ is a smooth curve.

4. Second Fujita's formula

Let us use all assumptions and notations of Section 1. Suppose, in addition, that S is smooth. Let us show a simple way how to estimate $\delta_P(X)$ using the second Fujita's formula found in [2]. To do this, fix a smooth curve $C\subset S$ such that $P\in C$. For every $u\in[0,\tau]$, write

$$N(u)\big|_S=d(u)C+N'(u),$$

where $N'(u)$ is effective $\mathbb{R}$-divisor such that $C\not\subset\mathrm{Supp}(N'(u))$, and

$$d(u)=\mathrm{ord}_C\big(N(u)|_S\big).$$

Then, for every $u\in[0,\tau]$, compute

$$t(u)=\sup\Big\{v\in\mathbb{R}_{\geqslant 0}\ \Big|\ \text{the divisor } P(u)\big|_S-vC \text{ is pseudo-effective}\Big\}.$$

After this, for every $v\in[0,t(u)]$, compute

- $P(u,v)$ = positive part of Zariski decomposition of $P(u)|_S-vC$,
- $N(u,v)$ = negative part of Zariski decomposition of $P(u)|_S-vC$.

Finally, compute

$$S\big(W^S_{\bullet,\bullet};C\big)=\frac{3}{(-K_X)^3}\int\limits_0^{\tau}d(u)\big(P(u,0)\big)^2du+\frac{3}{(-K_X)^3}\int\limits_0^{\tau}\int\limits_0^{t(u)}\big(P(u,v)\big)^2dvdu$$

compute

$$F_P\big(W^{S,C}_{\bullet,\bullet,\bullet}\big)=\frac{6}{(-K_X)^3}\int\limits_0^{\tau}\int\limits_0^{t(u)}\big(P(u,v)\cdot C\big)\cdot\mathrm{ord}_P\big(N'(u)\big|_C+N(u,v)\big|_C\big)dvdu,$$

and compute

$$S\big(W^{S,C}_{\bullet,\bullet,\bullet};P\big)=\frac{3}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^{t(u)}\big(P(u,v)\cdot C\big)^2dvdu+F_P\big(W^{S,C}_{\bullet,\bullet,\bullet}\big).$$

Then the second Fujita's formula derived from Abban–Zhuang theory gives

$$\delta_P(X)\geqslant\min\left\{\frac{1}{S_X(S)},\frac{1}{S\big(W^S_{\bullet,\bullet};C\big)},\frac{1}{S\big(W^{S,C}_{\bullet,\bullet,\bullet};P\big)}\right\}.$$

In many cases, this gives us the desired estimate $\delta_P(X)>1$.

However, if this approach does not work, we can blow up the surface S and apply a similar formula to the exceptional curve [2]. Namely, let $f\colon\widetilde{S}\to S$ be the blow up of the point P, and let F be the f-exceptional curve. Write

$$f^*\big(N(u)\big|_S\big)=\widetilde{d}(u)F+\widetilde{N}'(u),$$

where $\widetilde{N}'(u)$ is the strict transform on $\widetilde{S}$ of $N(u)|_S$, and $\widetilde{d}(u)=\mathrm{mult}_P(N(u)|_S)$. For $u\in[0,\tau]$, compute

$$\widetilde{t}(u)=\sup\Big\{v\in\mathbb{R}_{\geqslant 0}\;\Big|\;\text{the divisor }f^*\big(P(u)\big|_S\big)-vF\text{ is big}\Big\}.$$

Then, for every $v\in[0,\widetilde{t}(u)]$, compute

- $\widetilde{P}(u,v)=$ positive part of Zariski decomposition of $f^*(P(u)|_S)-vF$,
- $\widetilde{N}(u,v)=$ negative part of Zariski decomposition of $f^*(P(u)|_S)-vF$.

After this, we compute

$$S\big(W^S_{\bullet,\bullet};F\big)=\frac{3}{(-K_X)^3}\int\limits_0^\tau\widetilde{d}(u)\big(P(u,0)\big)^2du+\frac{3}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\big)^2dvdu.$$

Then, for every point $O\in F$, compute

$$F_O\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet}\big)=\frac{6}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\cdot F\big)\cdot\mathrm{ord}_O\big(\widetilde{N}'(u)\big|_F+\widetilde{N}(u,v)\big|_F\big)dvdu$$

and

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)=\frac{3}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\cdot F\big)^2dvdu+F_O\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet}\big).$$

Then we have

$$\delta_P(X)\geqslant\min\left\{\frac{1}{S_X(S)},\frac{2}{S\big(W^S_{\bullet,\bullet};F\big)},\inf_{O\in F}\frac{1}{S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)}\right\}.$$

5. One application

Let $Y=\mathbb{P}^1\times\mathbb{P}^1$, and let Z be a smooth curve in Y of degree $(5,1)$. Then, choosing appropriate coordinates $([u:v],[x:y])$ on the surface Y, we may assume that the curve Z is given by

$$u\big(x^5+a_1x^4y+a_2x^3y^2+a_3x^2y^3\big)=v\big(y^5+b_1xy^4+b_2x^2y^3+b_3x^3y^2\big)$$

for some a_1, a_2, a_3, b_1, b_2, b_3. Consider the embedding $Y\hookrightarrow\mathbb{P}^1\times\mathbb{P}^2$ given by

$$\big([u:v],[x:y]\big)\mapsto\big([u:v],[x^2:xy:y^2]\big).$$

Identify Y and Z with their images in $\mathbb{P}^1\times\mathbb{P}^2$. Let $\pi\colon X\to\mathbb{P}^1\times\mathbb{P}^2$ be the blow up of the curve Z. Then X is a Fano threefold of degree $-K_X^3=20$.

Let $\mathrm{pr}_1\colon\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^1$ be the projection to the first factor. Set $\phi_1=\mathrm{pr}_1\circ\pi$. Then ϕ_1 is a fibration into del Pezzo surfaces of degree 4.

Theorem 1 ([4])**.** *Suppose that every singular fiber of the fibration ϕ_1 has singular points of type $\mathbb{A}_1$. Then X is K-stable.*

Let us briefly explain how to prove this result. Suppose that every singular fiber of the del Pezzo fibration ϕ_1 has singular points of type $\mathbb{A}_1$. Fix a point $P\in X$. To prove Theorem 1, it is enough to show that $\delta_P(X)>1$. Let $\widetilde{Y}$ be the strict transform on X of the surface Y.

Lemma 1 ([2, Lemma 5.68])**.** Suppose that $P\in\widetilde{Y}$. Then $\delta_P(X)>1$.

Proof. Apply results described in Section 4 with $S=\widetilde{Y}$ and C being one of the rulings of the smooth surface $\widetilde{Y}\simeq Y\simeq\mathbb{P}^1\times\mathbb{P}^1$ that passes through P. □

Thus, we may assume that $P\notin\widetilde{Y}$. Let us apply results of Section 1 with S being the fiber of ϕ_1 that contains P. Then $\tau=2$. Moreover, we compute

$$P(u)=\begin{cases}-K_X-uS \text{ for } u\in[0,1],\\ -K_X-uS-(u-1)\widetilde{Y} \ \text{ for } u\in[1,2]\end{cases}$$

and

$$N(u)=\begin{cases}0 \ \text{ for } u\in[0,1],\\ (u-1)\widetilde{Y} \ \text{ for } u\in[1,2].\end{cases}$$

This gives $S_X(S)=\frac{1}{20}\int\limits_0^2 P(u)^3du=\frac{69}{80}<1$.

Set $Z = \widetilde{Y}|_S$. For every prime divisor F over S whose support on S contains the point P, we have

$$\begin{aligned}
S\big(W^S_{\bullet,\bullet};F\big) &= \frac{3}{20}\Bigg(\int\limits_0^1\int\limits_0^\infty \mathrm{vol}\big(-K_S - vF\big)dvdu \\
&\quad + \int\limits_1^2\int\limits_0^\infty \mathrm{vol}\big(-K_S-(u-1)Z - vF\big)dvdu\Bigg) \\
&\leqslant \frac{3}{20}\Bigg(\int\limits_0^\infty \mathrm{vol}\big(-K_S - vF\big)dv + \int\limits_0^\infty \mathrm{vol}\big(-K_S - vF\big)dv\Bigg) = \\
&= \frac{3}{10}\Bigg(\int\limits_0^\infty \mathrm{vol}\big(-K_S - vF\big)dv\Bigg) = \frac{6}{5}\Bigg(\frac{1}{4}\int\limits_0^\infty \mathrm{vol}\big(-K_S - vF\big)dv\Bigg) \\
&= \frac{6}{5}S_S(F) \leqslant \frac{6}{5}\cdot\frac{A_S(F)}{\delta_P(S)}
\end{aligned}$$

Thus, if $\delta_P(S) > \frac{6}{5}$, then $\delta_P(X) > 1$. If S is smooth, then $\delta_P(S) > \frac{6}{5}$ [2], which implies that $\delta_P(X) > 1$. Thus, we may assume that S is singular.

As in Section 3, set $D = P(u)|_S$ for every $u \in [0,2]$. Then, applying results described in Section 2 to the minimal resolution of singularities of the surface S, and arguing as in [5], we see that $\delta_P(S,D) \geqslant q(u)$ for

$$q(u) = \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } u\in[1,a], \\ \dfrac{15-3u^2}{11-u^3} \text{ for } u \in [a,2], \end{cases}$$

where $a \in [1,2]$ such that $3a^3 - 9a^2 + 3a + 5 = 0$. Thus, for every prime divisor F over the del Pezzo surface S whose support on S contains P, we have

$$\begin{aligned}
S\big(W^S_{\bullet,\bullet};F\big) &= \frac{3}{20}\int\limits_0^2\int\limits_0^\infty \mathrm{vol}\big(D - vF\big)dvdu \\
&\leqslant \frac{3}{20}\cdot\frac{4A_S(F)}{\delta_P(S)} + \Bigg(\frac{3}{20}\int\limits_1^2 \frac{(5-u^2)}{q(u)}du\Bigg)A_S(F) \leqslant \frac{99}{100}A_S(F)
\end{aligned}$$

which implies that $\delta_P(S, W^S_{\bullet,\bullet}) \geqslant \frac{100}{99}$. Then $\delta_P(X) > 1$ by the inequality described in Section 1. This shows that X is K-stable.

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Minicourse on Mirror Symmetry for Fano varieties

Giulia Gugiatti and Andrea Petracci

Mirror Symmetry for Fano varieties takes the shape of the Fano/Landau–Ginzburg (LG) correspondence: conjecturally, the mirror of a Fano orbifold of dimension n is an n-dimensional LG model (Y, w), i.e. a pair formed by a non compact manifold Y and a complex-valued function w on Y called superpotential. Broadly speaking, the correspondence interchanges the symplectic/complex geometry of X with the complex/symplectic geometry of the critical points of w.

In this series of lectures we discussed certain aspects of the Fano/LG correspondence.

Lecture 1 (AP). In this lecture we provided an introduction to the original formulation of the Fano/LG correspondence, due to Batyrev, Givental, Hori, Vafa, and others. This formulation predicts an identity between two cohomological invariants: the regularised quantum period of X, which is a generating function for certain Gromov–Witten invariants of X, and a period of (Y, w), encoding information on the variation of cohomology of the smooth fibres of w.

The lecture primarily focused on a series of conjectures, first laid out by Coates, Corti, Galkin, Golyshev, and Kasprzyk [17, 18, 9], according to which $\mathbb{Q}$-Gorenstein (qG) deformation families of Fano orbifolds of dimension n should be mirror (in the sense of the above-mentioned formulation) to mutation equivalence classes of maximally mutable Laurent polynomials [12] in n variables. This conjectural framework agrees with classical mirror constructions [15, 22] and finds theoretical ground in the intrisic mirror symmetry program by Gross– Siebert [19]. However, it notably excludes Fano varieties with empty anticanonical linear system [14, Remark 2.7].

Part of the lecture was devoted to define the quantum period of a Fano orbifold, and to sketch some of the available techniques for its computation [16, 11, 13, 10, 6, 8, 26, 28]. The lecture was supported by some running examples in dimension two.

Lecture 2 (GG). The lecture was structured in two parts.

In the first part, we expanded on the complex geometry of LG models. We revisited the notion of periods of a LG model and its specialisation to that of classical period of a Laurent polynomial [9, 29]. We discussed an equivalent formulation of the Fano/LG correspondence in terms of polynomial ordinary differential equations/complex local systems underlying a one-dimensional variation of (pure) Hodge Structure. Throughout the exposition, we revisited the 2-dimensional examples encountered in the first lecture and computed the relative periods and differential operators.

The focus of the second part of the lecture was on Fano varieties with empty anticanonical linear system, currently lying beyond the context of any systematic mirror construction. The simplest instance of such varieties is the series of log del Pezzo surfaces $X_{8k+4} \subset \mathbb{P}(2, 2k{+}1, 2k{+}1, 4k{+}1)$, $k \in \mathbb{N}_{>0}$, first studied by Johnson and Kollár [24]. The work of GG with Corti [14] builds the only known mirror series to this series. In the lecture we sketched the main ideas of our mirror construction, which builds upon the hypergeometric nature of the regularised quantum period of the surfaces and the motivic origin of hypergeometric functions [7, 31].

Lecture 3 (AP). Since qG-deformations of del Pezzo surfaces are unobstructed [20], it is clear what the general qG-deformation of a toric del Pezzo surface is. In dimension ≥ 3 this is no longer true, and so one needs to study deformation theory of Fano varieties more carefully. This has also applications to the moduli theory of Fano varieties.

In this lecture we presented some features of deformation theory of toric Fano varieties. In particular, we showed that there exist toric Fano 3-folds which deform to different smooth Fano 3-folds [23, 30, 25]. This is reflected on the mirror side by the fact that on the same polytope there might be different maximally mutable Laurent polynomials. These examples and results on the deformation theory of toric Fano 3-folds builds on the deformation theory of toric singularities studied by Altmann [4, 3, 2, 1].

Lecture 4 (GG). This lecture approached the Fano/LG correspondence from the point of view of Kontsevich's Homological Mirror Symmetry (HMS) conjecture [27]. One formulation of HMS predicts an equivalence between the bounded derived category of coherent sheaves of X and the analog of the Fukaya category for a symplectic fibration, namely the bounded derived category of Lagrangian vanishing cycles of (Y, w). A rigorous definition of this category was proposed by Seidel [32] in the case where w is a symplectic Lefschetz fibration.

After briefly sketching the construction of the category of Lagrangian vanishing cycles, we focused on the implication of the above formulation of HMS at the numerical level, i.e. at the level of the numerical Grothendieck groups of the two categories [33, 21]. For smooth del Pezzo surfaces we reviewed the homological mirror construction given in [5], and we explained how to recover the outputs of this construction from the Laurent polynomial mirrors to the surfaces. A special focus was placed on the case of smooth del Pezzo surfaces of degree two. These surfaces appear as the degenerate case ($k = 0$) of the series of log del Pezzo surfaces constructed by Johnson–Kollár [24], for which HMS has not yet been established.

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On the normalised volume of toric singularities

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(joint work with Joaquín Moraga)

The notion of normalised volume for log terminal singularities was introduced by Chi Li in [2]. In the special case of the anticanonical cone over a K-semistable Fano variety it coincides with the usual anticanonical volume (aka Fano degree) of the Fano variety.

In general the normalised volume is hard to calculate, but in the case of Gorenstein toric singularities we obtain a nice formula in terms of convex geometry. Given a full-dimensional polytope $P \subset \mathbb{R}^{d-1}$, we consider the Gorenstein toric variety $X = \operatorname{Spec}(k[\sigma^\vee \cap \mathbb{Z}^d])$ corresponding to the cone $\sigma = \mathbb{R}_{\geq 0} \cdot (P \times \{1\}) \subset \mathbb{R}^d$. We choose $x \in X$ to be the (unique) torus fixed point. Then by [3] we have

$$\widehat{\operatorname{vol}}(X, x) = \min \{\operatorname{vol}((P - v)^*) \mid v \in \operatorname{int}(P)\}.$$

Here, $(P - v)^*$ denotes the polar dual of the polytope P after translation by $-v$. In convex geometry the (unique) point $v \in \operatorname{int}(P)$ where the minimum is attained, is known as the *Santaló point* of P. Moreover, for a convex body P the product

$$M(P) = \operatorname{vol}(P) \cdot \min \{\operatorname{vol}((P - v)^*) \mid v \in \operatorname{int}(P)\}$$

is known as the *Mahler volume* of P, which is an affine invariant of the polytope. The Mahler volume appears in two remarkable inequalities. The first one is known as the *Blaschke-Santaló inequality* and states that the Mahler volume of convex bodies of fixed dimension is maximised by the unit ball. In [3] we utilised this inequality to show that for a fixed dimension and fixed $\epsilon > 0$ there are only finitely many toric singularieties with normalised volume being at least ϵ.

A second, but to this point only conjectural, inequality states that for $\dim(P) = d-1$ one has

$$M(P) \geq d^d,$$

where the equality is achieved for simplices. The latter inequality is known as the *non-symmetric Mahler conjecture*. It is natural to ask whether there is a reasonable interpretation of this inequality in terms of algebraic geometry. Hence, we are looking for an interpretation of the volume of P in terms of algebraic geometry. But this volume is known to coincide with the Euler characteristic of a crepant resolution $\widetilde{X}$ of X (at least if such a resolution exists). However, even for non-abelian quotient singularities does the corresponding inequality

$$\widehat{\mathrm{vol}}(X,x)\chi(\widetilde{X}) \geq d^d \tag{1}$$

not longer hold. In [1] Gulotta suggests to replace $\chi(\widetilde{X})$ by another quantity associated to *non-commutative crepant resolutions* of X in the sense of [4]. With this adjustment the corresponding inequality of the type (1) does again hold (with equality) for finite quotient singularities and we can also verify it for a range of other examples, such as anticanonical cones over del Pezzo surfaces and cA_n singularities.

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Higher Fano Manifolds

Ana-Maria Castravet

(joint work with Carolina Araujo, Roya Beheshti, Kelly Jabbusch, Enrica Mazzon, Svetlana Makarova, Will Reynolds, Libby Taylor, Nivedita Viswanathan)

A Fano manifold X is a complex projective manifold with ample first Chern class $c_1(T_X)$. This condition has far reaching geometric implications. For instance, any Fano manifold is rationally connected, i.e., there are rational curves connecting any two points [6, 11]. A celebrated result of Graber, Harris and Starr states that proper families of rationally connected complex projective manifolds over smooth curves always admit sections [10]. This generalizes Tsen's theorem in the case of function fields of curves. A theorem of Tsen and Lang states that a family

$\pi : \mathcal{X} \to B$ of degree d hypersurfaces over a k-dimensional base B admit (rational) sections if $d^k \leq n$. For hypersurfaces of degree d in $\mathbb{P}^n$, being Fano or rationally connected is equivalent to the numerical condition $d \leq n$. Hence, for $k = 1$, the result of Graber, Harris and Starr replaces the condition of being a hypersurface of degree $d \leq n$ with the condition of being rationally connected.

There has been quite some effort towards finding suitable geometric conditions on the fibers of $\pi : \mathcal{X} \to B$ that generalize the Tsen-Lang theorem for higher dimensional bases B. More generally one can consider the following problem: Find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- Hypersurfaces of degree d in $\mathbb{P}^n$ satisfy $\mathcal{F}_k$ iff $d^k \leq n$,
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds,
- The Tsen–Lang Theorem holds for a family $\pi : \mathcal{X} \to B$ over a k-dimensional base B if the fibers of π satisfy $\mathcal{F}_k$ (modulo the Brauer obstruction).

For $k = 1$, the condition $\mathcal{F}_1$ can hence be taken as either the condition of being Fano or the condition of being rationally connected.

In a series of papers [7, 8, 9], de Jong and Starr asked if for $k = 2$ the condition $\mathcal{F}_2$ can be taken to be *rationally simply connected*, a technical condition taking inspiration from topology which requires (at the very least) that a suitable irreducible component of the space of rational curves through two general points is itself rationally connected. De Jong, He and Starr proved that rational homogeneous spaces and hypersurfaces of degree d in $\mathbb{P}^n$ with $d^2 \leq n$ are rationally simply connected and that they satisfy the Tsen-Lang theorem. However, the condition of being rationally simply connected is hard to verify in practice and it is desirable to have natural geometric conditions that imply it. In this context, 2-manifolds were introduced by de Jong and Starr in [8, 9]. More generally, consider the following:

Definition. [9, 1] A complex projective manifold X is *k-Fano* if X is Fano and the i-th graded piece of the Chern character $ch_i(T_X) \in H^{2i}(X, \mathbb{Z})$ is positive for all $i \in \{1, \ldots, k\}$, i.e., $ch_i(T_X) \cdot Z > 0$ for all $Z \subset X$ with $\dim(Z) = i$.

The following may be easily verified:

(1) The n-dimensional projective space $\mathbb{P}^n$ is n-Fano.

(2) Hypersurfaces of degree d in $\mathbb{P}^n$ are k-Fano if and only if $d^k \leq n$. In particular, hypersurfaces of degree d in $\mathbb{P}^n$ are 2-Fano if and only if $d^2 \leq n$.

Question. [1] *Can one take the condition $\mathcal{F}_k$ to be "X is k-Fano"?*

Some evidence towards an affirmative answer has been given in [9, 1]. Roughly speaking, we know that 2-Fano manifolds are covered by rational surfaces and 3-Fano manifolds are covered by rational threefolds. The approach taken in [1] is to consider the implications of the k-Fano condition on the *spaces of minimal rational curves* H_x through a general point $x \in X$. For example, one may compute all the Chern characters of H_x in terms of the Chern characters of X and a canonical polarization on H_x coming from the map $H_x \to \mathbb{P}(T_x(X))$ that associates to a rational curve through x its tangent line at x. One can prove that there is an inductive structure: If X is 2-Fano, then H_x is Fano [9, 1], and similarly, if X is

3-Fano, then H_x is 2-Fano [1] (under certain technical assumptions). Results in the same spirit have been obtained in the case when X is k-Fano by Nagaoka [12] and Suzuki [13].

There are however few examples of higher Fano manifolds. What is known:
(1) Complete intersections $X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0,\dots,a_n)$ are k-Fano iff $\sum d_i^k < \sum a_i^k$.
(2) The only 2-Fano surface is $\mathbb{P}^2$.
(3) The only 2-Fano threefolds are $\mathbb{P}^3$ and the quadric hypersurface $Q \subset \mathbb{P}^4$ [1].
(4) The only 3-Fano threefold is $\mathbb{P}^3$ [1].
(5) The only known examples of 3-Fano manifolds are complete intersections

$$X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0,\dots,a_n), \quad \sum d_i^3 < \sum a_i^3.$$

(6) A classification of 2-Fano manifolds X of index $\geq \dim(X) - 2$ was given in [3]: they are either complete intersections $X_{d_1,\dots,d_c}$ in weighted projective spaces $\mathbb{P}(a_0,\dots,a_n)$ or certain rational homogeneous spaces of Picard number $\rho = 1$ or certain complete intersections in them [3]. A classification of 2-Fano manifolds among rational homogeneous spaces with $\rho = 1$ was also given in [3]:

Theorem 1. [3] *Classification of 2-Fano rational homogeneous spaces with $\rho = 1$:*
- A_n/P^k, $k = 1, \frac{n}{2}, \frac{n+1}{2}, n$
- B_n/P^k, $k = 1, \frac{2n-1}{3}, n$
- C_n/P^k, $k = 1, \frac{2n+2}{3}, n$
- D_n/P^k, $k = 1, \frac{2n+2}{3}, n-1, n$
- E_6/P^1,E_6/P^2,E_6/P^3,E_7/P^1,E_7/P^2,E_7/P^7,E_8/P^1,E_8/P^2,E_8/P^8
- F_4/P^4
- G_2/P^1, G_2/P^2

(Here we use the Bourbaki labeling of vertices in a Dynkin diagram.)

(7) Products $X \times Y$ with $\dim(X), \dim(Y) \geq 1$ are not 2-Fano.
(8) Projectivizations $\mathbb{P}(E)$ of vector bundles E of rank ≥ 2 over a positive dimensional base are not 2-Fano.
(9) No blow-up of a projective manifold along a smooth subvariety of codimension at least 2 is known to be 2-Fano. One is hence lead to the following:

Question. *Do all 2-Fano manifolds have Picard number one?*

Problem. Find examples of 3-Fano manifolds other than complete intersections in weighted projective spaces.

Conjecture. [3] *Let X be a k-Fano manifold of dimension n. If $k \geq \lceil \log_2(n+1) \rceil$, then $X \cong \mathbb{P}^n$.*

We now concentrate on the **toric** case:

Conjecture. *If X a toric 2-Fano manifold, then $X \cong \mathbb{P}^n$.*

Assume X a smooth projective toric variety with lattice N and fan Σ. We denote by $G(\Sigma) \subset N$ set of primitive vectors generating the rays of Σ.

Definition. A set $P = \{v_0, \ldots, v_r\} \subseteq G(\Sigma)$ is called a *primitive collection* if

– $\langle v_0, \ldots, v_r \rangle \notin \Sigma$, but
– $\langle v_0, \ldots, \hat{v_i}, \ldots, v_r \rangle \in \Sigma$ for all $i = 0, \ldots, r$

If $\sigma(P) = \langle w_1, \ldots, w_s \rangle$ is the smallest cone in Σ containing $v_0 + \ldots + v_r$, then $v_0 + \ldots + v_r = \mu_1 w_1 + \ldots + \mu_s w_s$, for some $\mu_1, \ldots \mu_s \in \mathbb{Z}_{>0}$. We say that the primitive collection $P = \{v_0, \ldots, v_r\}$ is *centered* if $\sigma(P) = 0$, i.e.,

$$v_0 + \ldots + v_r = 0.$$

Furthermore, as relations between the primitive vectors in $G(\Sigma)$ correspond to numerical equivalence classes of 1-cycles on X, via this correspondence primitive relations correspond to effective 1-cycle classes that generate the Mori cone of X.

A theorem of Batyrev [5] asserts that centered primitive relations always exist. In [4] it is proved that for every centered primitive relation $v_0 + \ldots + v_r = 0$ one may associate a torus invariant open set $U \subseteq X$ which comes with a $\mathbb{P}^r$-bundle structure $\pi : U \to W$ and the centered primitive relation corresponds to the classes of lines in the fibers of π. A theorem of Chen-Fu-Hwang states that all minimal covering families of rational curves on X may be described in this way.

Definition. [4] The minimal $\mathbb{P}$-dimension $\mathrm{m}(X)$ is the smallest $r > 0$ such that Σ has a centered primitive collection $\{v_0, \ldots, v_r\}$.

If X is a toric projective manifold of dimension n, it is not difficult to see that $m(X) = n$ implies that $X \cong \mathbb{P}^n$. Furthermore, if X is Fano and $m(X) = n - 1$, one can prove that X is the blow-up of $\mathbb{P}^n$ along a linear subspace of codimension 2 (which is not 2-Fano). We prove:

Theorem 2. [4] *Assume X is a Fano toric projective manifold of dimension n.*

(1) If $m(X) = n - 2$, then $\rho \geq 3$ and X belong to 8 explicit isomorphism classes. In particular X is not 2-Fano.
(2) If $m(X) = 1$, then X is not 2-Fano.

One can see from the classification of Fano toric manifolds in small dimensions that the toric Fano varieties X with $m(X) = 1$ form the largest class:

dim(X)	# Fanos	m=1	m=2	m=3	m=4	m=5	m=6
4	124	107	15	1	1		
5	866	744	112	8	1	1	
6	7622	6333	1174	105	8	1	1

Table 1. The minimal $\mathbb{P}$-dimension of toric Fano manifolds of low dimension.

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On K-moduli spaces of Fano 3-folds

Anne-Sophie Kaloghiros

The notion of K-polystability was introduced to characterise the existence of Kähler–Einstein metrics on Fano manifolds. More precisely, the Yau–Tian–Donaldson conjecture (now a theorem, due to Chen–Donaldson and Sun) states that a Fano manifold is Kähler–Einstein precisely when it is K-polystable.

Recent advances in the theory of K-stability have shown that this notion also allows one to construct moduli spaces for Fano varieties. More precisely:

Theorem 1. [1] *There is a projective good moduli space* $\mathrm{M}^{\mathrm{Kps}}_{n,V}$ *whose points parametrise K-polystable* $\mathbb{Q}$*-Fano varieties of dimension* n *and volume* V*.*

In each dimension, there are finitely many families of smooth manifolds, which have been classified in dimension up to 3. There are 10 families of smooth del Pezzo surfaces and 105 families of smooth Fano 3-folds. For each of these families, we can ask the following questions:

(A) Is the general member of the family K-polystable? (In other words, is the associated component of $\mathrm{M}^{\mathrm{Kps}}_{n,V}$ non-empty?)
(B) Is every (smooth) member of the family K-polystable?
(C) What is the compactification of the associated component of $\mathrm{M}^{\mathrm{Kps}}_{n,V}$? In particular, what are the K-polystable limits of Fano manifolds in the family?

In dimension 2, (A) and (B) were answered by Tian [2] and (C) was answereed in degree 4 by Mabuchi and Mukai and in general by Odaka-Spotti-Sun [3].

Following Fujita and Li's valuative criterion for K-polystability [6, 7], a purely algebraic theory of (K-poly)stability was formulated, and this has lead to much progress in recent years. Notably, Abban and Zhuang [4] developed techniques to determine K-stability using flags and Zhuang showed how to exploit symmetries effectively [5]; these have yielded many results in explicit K-stability of Fano 3-folds. We now know:

Theorem 2. [9] *Let X be the general member of one of the 105 deformation families of Fano 3-folds. Then one of:*

- *X belongs to family MM_{2-26}, or*
- *X belongs to one of the 26 deformation families of K-divisorially unstable Fano 3-folds classified by Fujita [8], or*
- *X is K-polystable*

This answers (A) above in dimension 3; 78 families have smooth K-polystable members, and in some cases the families contain both K-polystable and non-K-polystable smooth Fano 3-folds. The answer to (B) is known for 58 out of 78 families with K-polystable members.

Relatively few known examples of K-moduli spaces of Fano 3-folds are known: Liu and Xu have shown that the K-moduli space of cubic threefolds coincides with the GIT moduli space [10], and a number of recent works have considered specific families.

We could also investigate specifically K-moduli spaces of small dimension. 44 of the 105 deformation families of Fano 3-folds have 0-dimensional moduli, and out of these, 21 yield a non-empty component of the associated K-moduli $\mathrm{M}^{\mathrm{Kps}}_{3,V}$. There are 8 families with 1-dimensional moduli, and 6 of these yield a non-empty component of the associated K-moduli $\mathrm{M}^{\mathrm{Kps}}_{3,V}$. My collaborators and I show:

Theorem 3. [11] *All one dimensional components of $\mathrm{M}^{\mathrm{Kps}}_{3,V}$ associated to families of smoothable Fano 3-folds are isomorphic to $\mathbb{P}^1$.*

As a by-product, we obtain:

Corollary 1. [11] *All singular K-polystable limits of Fano 3-folds in families MM_{2-22}, MM_{2-24}, MM_{2-25}, MM_{3-12}, MM_{3-13} and MM_{4-13} are constructed explicitly.*

Finally, I present a construction of the 3-dimensional component of $\mathrm{M}^{\mathrm{Kps}}_{3,24}$ associated to the deformation family MM_{4-1}. Smooth members of MM_{4-1} are divisors of multidegree $(1, 1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 4 ([12]). *A K-polystable limit of members of* MM_{4-1}

- *either an irreducible divisor* $(1,1,1,1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *of the form*

$$(*) \quad X = \{a(x_1x_2x_3x_4 + y_1y_2y_3y_4) + b(x_1x_2y_3y_4 + x_3x_4y_1y_2) + \\ + c(x_1x_3y_2y_4 + x_2x_4y_1y_3) + d(x_1x_4y_2y_3 + x_2x_3y_1y_4) = 0\}$$

  for $(a : b : c : d) \in \mathbb{P}^3$.
- *or an irreducible* $(2,2) \subset \mathbb{P}(1,1,2) \times \mathbb{P}(1,1,2)$ *of the form*

$$(**) \qquad X = \{w_1w_2 + \alpha s_1t_1s_2t_2 + \beta(s_1^2s_2^2 + t_1^2t_2^2) + \gamma(s_1^2t_2^2 + t_1^2s_2^2) = 0\}$$

  for $(\alpha : \beta : \gamma) \in \mathbb{P}^2$.

We show:

Theorem 5. *The component of K-moduli space* $\mathrm{M}^{\mathrm{Kps}}_{3,24}$ *associated to family* MM_{4-1} *is the blowup of* $\mathbb{P}(1,3,4,6)$ *at the smooth point* $\{[2:2:0:0]\}$ *with weights* $(1,2,3)$.

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On a problem of Gizatullin

Carolina Araujo

1. Overview

Our work is motivated by the general problem of describing automorphisms of smooth hypersurfaces in projective spaces. Let $X_d \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface of degree d. Apart from the two exceptional cases $(n,d) = (1,3)$ and $(n,d) = (2,4)$, every automorphism of X_d is the restriction of a linear automorphism of the ambient space $\mathbb{P}^{n+1}$ ([8], [4]). It is natural to wonder where the automorphisms of X_d come from in the two exceptional cases.

For $(n,d) = (1,3)$, that is, when $C = X_3 \subset \mathbb{P}^2$ is a smooth cubic curve, the automorphism group of C is well known:

$$\mathrm{Aut}(C) \;=\; C \rtimes \mathbb{Z}_m, \;\text{ for some } m \in \{2,4,6\},$$

where C is viewed as an elliptic curve acting on itself by translation. In this case, every automorphism of C is the restriction of a Cremona transformation of $\mathbb{P}^2$ ([10, §2]). To see this, first make a linear change of coordinates to write C in Weierstrass form. Then write down the expression for the translation by a point on the curve in these coordinates, and check that this expression gives a Cremona transformation of $\mathbb{P}^2$ inducing the given translation on C. The automorphisms in the finite factor $\mathbb{Z}_m$ are easily seen to be restrictions of linear automorphisms of $\mathbb{P}^2$. The group of Cremona transformations of $\mathbb{P}^2$ stabilizing the curve C is called the *decomposition group* of C and is denoted by $\mathrm{Dec}(\mathbb{P}^2, C)$, while the group of Cremona transformations fixing the curve C pointwise is called the *inertia group* of C and is denoted by $\mathrm{In}(\mathbb{P}^2, C)$. So we have an exact sequence

$$1 \;\to\; \mathrm{In}(\mathbb{P}^2, C) \;\to\; \mathrm{Dec}(\mathbb{P}^2, C) \;\to\; \mathrm{Aut}(C) \;\to\; 1\;.$$

Generators of the decomposition group $\mathrm{Dec}(\mathbb{P}^2, C)$ were given in [11], while the inertia group $\mathrm{In}(\mathbb{P}^2, C)$ was investigated in [3].

For $(n,d) = (2,4)$, $S = X_4 \subset \mathbb{P}^3$ is a K3 surface, and $\mathrm{Aut}(S)$ can be infinite and fairly complicated. The following question is attributed to Gizatullin:

Problem (Gizatullin)**.** Which automorphisms of a smooth quartic surface $S \subset \mathbb{P}^3$ are restrictions of Cremona transformations of $\mathbb{P}^3$?

In [9], Oguiso constructed a smooth quartic surface $S \subset \mathbb{P}^3$ with Picard rank $\rho(S) = 2$, $\mathrm{Aut}(S) \cong \mathbb{Z}$ and trivial decomposition group, $\mathrm{Dec}(\mathbb{P}^3, S) = \{1\}$. So no nontrivial automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$. In [10], Oguiso constructed a smooth quartic surface $S \subset \mathbb{P}^3$ with $\rho(S) = 3$, $\mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, and such that every automorphism of S is the restriction of a Cremona transformation of $\mathbb{P}^3$. Inspired by these examples, Oguiso asked whether every automorphism of finite order of a smooth quartic surface is the restriction of a Cremona transformation of $\mathbb{P}^3$. This question was negatively answered by Paiva and Quedo in [12], with the construction of a smooth quartic surface $S \subset \mathbb{P}^3$ with

$\rho(S) = 2$, $\mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ (and hence $\mathrm{Aut}(S)$ is generated by involutions), while $\mathrm{Dec}(\mathbb{P}^3, S) = \{1\}$.

In this talk, we present a general framework that can be used to study Gizatullin's problem and to investigate the decomposition and inertia groups of a quartic surface $S \subset \mathbb{P}^3$, namely, the birational theory of *Calabi-Yau pairs*. We also report on recent progress on these problems in collaboration with Alessio Corti and Alex Massarenti, and with Daniela Paiva and Sokratis Zikas.

2. Birational geometry of Calabi-Yau pairs

Let $S \subset \mathbb{P}^3$ be a smooth quartic surface. The pair $(\mathbb{P}^3, S)$ is an example of a *Calabi-Yau pair*, and an element of its decomposition group $\mathrm{Dec}(\mathbb{P}^3, S)$ is an example of a *volume preserving* birational self-map of $\mathbb{P}^3$ with respect to the pair $(\mathbb{P}^3, S)$. In this section, we introduce these notions in a more general context, and explain how they allow us to use tools from the Minimal Model Program (MMP) in order to investigate Gizatullin's problem.

Definition. A *Calabi-Yau (CY) pair* is a pair (X, D) consisting of a terminal projective variety X and an effective Weil divisor D on X such that $K_X + D \sim 0$ and (X, D) has klt singularities.

Let (X, D_X) and (Y, D_Y) be CY pairs, and $f : X \dashrightarrow Y$ a birational map. We say that f is *volume preserving* if, for every geometric valuation E with center on both X and Y, the discrepancies of E with respect to the pairs (X, D_X) and (Y, D_Y) are equal: $a(E, K_X + D_X) = a(E, K_Y + D_Y)$.

The *birational group of a CY pair* (X, D) is the group $\mathrm{Bir}(X, D)$ of birational self-maps of X which are volume preserving with respect to (X, D).

Remark. The terminology is explained by the following interpretation. Given a CY pair (X, D), there is a rational volume form ω on X, unique up to scaling, such that $D + \mathrm{div}(\omega) = 0$. A birational self-map of X is volume preserving with respect to (X, D) if and only if it preserves the volume form ω up to scaling.

When the CY pair (X, D) has canonical singularities, a birational self-map of X is volume preserving if and only if it restricts to a birational self-map of D. This is the case with the pair $(\mathbb{P}^3, S)$, where $S \subset \mathbb{P}^3$ is a quartic surface with at worst rational double points as singularities. In this case, the decomposition group $\mathrm{Dec}(\mathbb{P}^3, S)$ coincides with the birational group $\mathrm{Bir}(\mathbb{P}^3, S)$ of the pair $(\mathbb{P}^3, S)$, which can be studied with tools of the MMP, as we now explain.

Given a uniruled variety, the MMP produces a *Mori fiber space* that is birationally equivalent to it. In general, there might be several different Mori fiber spaces in the same birational equivalence class. The *Sarkisov program* provides a factorization theorem for birational maps between Mori fiber spaces in terms of simpler birational maps, called *Sarkisov links*. It was established in dimension 3 in [5], and in higher dimensions in [7]. The Sarkisov program has become a powerful tool to investigate the Cremona group, as it allows one to factorize any birational self-map of $\mathbb{P}^n$ as a composition of Sarkisov links between Mori fiber spaces. In [6], Corti and Kaloghiros established the following volume preserving version of the Sarkisov program:

Theorem 1. *A volume preserving birational map between Mori fibered CY pairs is a composition of volume preserving Sarkisov links.*

3. The decomposition group of quartic surfaces

We end this report by discussing a few consequences of Theorem 1. In [1], in collaboration with Alessio Corti and Alex Massarenti, we developed a framework to study the birational geometry of CY pairs. By our first main result ([1, Theorem A]), if the quartic surface $S \subset \mathbb{P}^3$ is very general, then the group $\mathrm{Bir}(\mathbb{P}^3, S)$ is trivial. More generaly:

Theorem 2. *Let $D \subset \mathbb{P}^{n+1}$ be a hypersurface of degree $n+1$. Suppose that D is terminal and $\mathrm{Cl}(D) = \mathbb{Z} \cdot \left[\mathcal{O}_{\mathbb{P}^{n+1}}(1)_{|D}\right]$. Then $\mathrm{Bir}(\mathbb{P}^{n+1}, D) = \{1\}$.*

Therefore, if we want to produce interesting subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$ using CY pairs $(\mathbb{P}^3, S)$, then the quartic surface $S \subset \mathbb{P}^3$ must be chosen special. Namely, either S should be singular, or it must have Picard rank $\rho(S) \geq 2$. The singular case was treated in [1, Theorem B]:

Theorem 3. *Let $S \subset \mathbb{P}^3$ be a general singular quartic surface, so that S has a unique rational double point of type A_1, and $\mathrm{Cl}(S) = \mathbb{Z} \cdot \left[\mathcal{O}_{\mathbb{P}^3}(1)_{|S}\right]$. Then we have a split exact sequence*

$$1 \to \mathrm{In}(\mathbb{P}^3, S) \to \mathrm{Bir}(\mathbb{P}^3, S) \curvearrowleft\!\!\!\!\longrightarrow \mathrm{Bir}(S) \cong \mathbb{Z}_2 \to 1,$$

and the inertia group $\mathrm{In}(\mathbb{P}^3, S)$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$, i.e., $\mathrm{In}(\mathbb{P}^3, S)$ is an algebraic group over the field $\mathbb{C}(x, y)$ which is isomorphic to $\mathbb{G}_m$ over the algebraic closure $\overline{\mathbb{C}(x, y)}$.

Gizatullin's problem for smooth quartic surfaces $S \subset \mathbb{P}^3$ with $\rho(S) = 2$ was addressed in [12] and in the recent paper [2], in collaboration with Daniela Paiva and Sokratis Zikas. Let $S \subset \mathbb{P}^3$ be a general smooth quartic surface with $\rho(S) = 2$. In a suitable basis for $\mathrm{Pic}(S) \cong \mathbb{Z}^2$, the intersection product is given by a matrix of the form

$$\begin{pmatrix} 4 & b \\ b & 2c \end{pmatrix},$$

with $b, c \in \mathbb{Z}$. Denote by $r = b^2 - 8c$ the discriminant of S. It follows from [12] that $\mathrm{Dec}(\mathbb{P}^3, S) = \{1\}$ whenever $r > 233$. In [2], we determined the image of the restriction homomorphism $\mathrm{Dec}(\mathbb{P}^3, S) \to \mathrm{Aut}(S)$ for each value of $r \leq 233$. In particular, there are examples of smooth quartic surfaces S with $\rho(S) = 2$, $\mathrm{Aut}(S) \cong \mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$, and such that every automorphism of S is the restriction of a Cremona transformation of $\mathbb{P}^3$.

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Finite subgroups of Cremona groups and representation dimension

Alexander Duncan

(joint work with Jason Bailey Heath, Christian Urech)

For a field k and a positive integer n, the *Cremona group of degree n over k*, denoted $\mathrm{Cr}_n(k)$, is the group of birational automorphisms of the projective space $\mathbb{P}^n_k$. We are interested in understanding how complicated its finite subgroups can be.

We have $\mathrm{Cr}_1(k) \cong \mathrm{PGL}_2(k)$, so the finite subgroups are completely classified for all fields when $n = 1$. The finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ were (almost) completely classified by Dolgachev and Iskovskikh in [2], building on work going back more than a century. Over other fields there has been abundant progress towards full classifications of finite subgroups, but much work still needs to be done. In higher dimensions, there is some hope for $\mathrm{Cr}_3(k)$, but a full classification in higher degrees seems out of reach of current techniques.

Rather than a full classification, a coarser approach is to merely bound the complexity of finite subgroups of $\mathrm{Cr}_n(k)$. For some "small" fields, such as number fields, one can bound the order of finite subgroup groups as was done by Serre for $\mathrm{Cr}_2(k)$ in [4]. However, there is no finite bound on order for $k = \mathbb{C}$ — even for $n = 1$. Another alternative is to study the *Jordan constant* of $\mathrm{Cr}_n(k)$, which bounds the index of normal abelian subgroups of the finite subgroups. As a consequence of

Birkar's proof of the BAB Conjecture [1], Prokhorov and Shramov [3] have shown that the Jordan constant is finite for $\mathrm{Cr}_k(n)$ for all n and all fields of characteristic 0.

Here we consider another measure of complexity. The *representation dimension* of a finite group G over k, denoted $\mathrm{rdim}_k(G)$, is the minimal N such that there is an embedding $G \hookrightarrow \mathrm{GL}_N(k)$. For a field k and a positive integer n, define

$$c_n(k) := \sup\{\mathrm{rdim}_k(G) \mid G \text{ finite group such that } G \subseteq \mathrm{Cr}_n(k)\} \ .$$

We are able to compute $c_n(k)$ exactly for small n and arbitrary fields.

$$c_1(k) = \begin{cases} 2 & \text{if } \mathrm{char}(k) = 2, \\ 3 & \text{if } \mathrm{char}(k) \geq 3, \\ 3 & \text{if } \mathrm{char}(k) = 0 \text{ and } -1 \text{ is a sum of two squares,} \\ 2 & \text{otherwise.} \end{cases}$$

$$c_2(k) = \begin{cases} \infty & \text{if } \mathrm{char}(k) \neq 0, \\ 8 & \text{if } \mathrm{char}(k) = 0 \text{ and } \sqrt{-3} \in k, \\ 6 & \text{otherwise.} \end{cases}$$

In the important special case of the complex space Cremona group, we have the following bounds

$$15 \leq c_3(\mathbb{C}) \leq 62208$$

where we do not believe that the upper bound is sharp.

We also prove that $c_n(k)$ is infinite for all $n \geq 2$ and all fields k of positive characteristic. As a consequence of the finiteness of the Jordan constant, we can show that $c_n(k)$ is finite for all n when $k = \mathbb{C}$ or k is a number field. Indeed, it is likely that $c_n(k)$ is finite for all fields of characteristic 0.

Finally, by investigating the automorphism groups of (possibly non-split) toric varieties, we obtain lower bounds for all dimensions n and all fields k

n	1	2	3	4	5	6	≥ 7
$c_n(k) \geq$	2	6	12	24	40	72	2^n

While explicit *upper* bounds for $c_n(k)$ for large n is difficult with current techniques, we do think it is worth trying to find new lower bounds and encourage the community to try to improve the numbers above.

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A valuative criterion for K-stability: beyond the Fano case

Sébastien Boucksom
(joint work with Mattias Jonsson)

The notion of K-stability was introduced in complex differential geometry as a conjectural—and now partially confirmed—algebro-geometric criterion for the existence of special Kähler metrics. Lately, it has also become a subject in its own respect. The main purpose of this talk is to review a series of two joint papers with Mattias Jonsson [BJ21, BJ22b], in which we show how global pluripotential theory over a trivially valued field, as developed in [BoJ22a], can be used to study K-stability.

Let X be a projective variety (reduced and irreducible) of dimension $n \geq 1$ over an algebraically closed field k of characteristic 0, and L an ample $\mathbb{Q}$-line bundle on X. The definition of K-stability of the polarized variety (X, L), as given by Donaldson, involves the sign of an invariant attached to (ample) test configurations for (X, L), which can be interpreted as a non-Archimedean version of the Mabuchi K-energy functional. Filtrations of the section ring of (X, L) provide another, widely used description of test configurations; more precisely, the latter correspond to $\mathbb{Z}$-filtrations of finite type, as first pointed out by Witt Nyström. Chi Li's recent breakthrough on the Yau–Tian–Donaldson conjecture for cscK metrics shows that a stronger form of uniform K-stability, formulated in terms of filtrations, indeed implies the existence of a cscK metric.

To describe this, note that each filtration χ can be canonically approximated by a sequence χ_d of finitely generated $\mathbb{Z}$-filtrations, i.e. test configurations. The proper definition of K-stability for filtrations relies on a detailed study of the non-Archimedean Mabuchi K-energy functional $\mathrm{M}(\chi)$. For a test configuration, the latter decomposes into 'energy' terms $\mathrm{E}(\chi), \mathrm{E}^{K_X}(\chi)$, and an 'entropy' term $\mathrm{Ent}(\chi) := \int A_X \, \mathrm{MA}(\chi)$, where A_X is the log dicrepancy function, defined on the set X^{div} of divisorial valuations, and the *Monge–Ampère measure* $\mathrm{MA}(\chi)$ is a *divisorial measure*, i.e. a probability measure with finite support in X^{div}. Our first main result is as follows:

Theorem 1. *There exists a unique extension of* $\mathrm{E}(\chi)$, $\mathrm{E}^{K_X}(\chi)$, $\mathrm{MA}(\chi)$ *to arbitrary filtrations* χ, *obtained as the limits of the corresponding quantities for the canonical approximants* χ_d.

Here $\mathrm{MA}(\chi)$ is a positive measure on the Berkovich space X^{an}, a natural compatification of X^{div}. Since A_X extends to X^{an} [BFJ08], this provides a natural extension of $\mathrm{M}(\chi)$ to all filtrations. Our second main result is then:

Theorem 2. *The Monge–Ampère operator induces a 1–1 correspondence between the set of divisorial filtrations (up to translation by a constant) and that of divisorial measures. Furthermore, K-stability for filtrations can be tested on the subset of divisorial filtrations.*

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Moduli Continuity method in K-stability

Yuchen Liu

(joint work with Chenyang Xu, Junyan Zhao)

This talk is a report on the paper [12] joint with Chenyang Xu and the preprint [14] joint with Junyan Zhao.

It is remarkable that K-stability, a notion introduced by Tian [19] and Donaldson [6] to characterize Kähler–Einstein metrics on Fano varieties, provides the correct condition to construct compact moduli spaces of Fano varieties. As a consequence of about a dozen important recent papers, the K-moduli theorem states that for every fixed dimension n and volume V, there exists a projective scheme $M^{\mathrm{K}}_{n,V}$, known as the K-moduli space, parameterizes all n-dimensional K-polystable Fano varieties X with $(-K_X)^n = V$.

Despite the general theory being completed, a natural question remains: can we determine the K-moduli space for explicit Fano varieties? More precisely, the question often asks to describe a certain irreducible component M^{K} of $M^{\mathrm{K}}_{n,V}$ that compactifies a given family of smooth Fano manifolds. A notable approach to this question, called the *moduli continuity method*, is based on the study of singularities and volumes and crucially uses the compactness of K-moduli spaces. The moduli continuity method first appeared in Tian's solution of the Kähler–Einstein problem for smooth del Pezzo surfaces [18]. Later, Mabuchi–Mukai [15] and Odaka–Spotti–Sun [16] used this method to successfully describe K-moduli spaces of del Pezzo surfaces. In this note, we shall focus on two families of Fano threefolds: cubic threefolds and blow-ups of $\mathbb{P}^3$ along a genus 4 curve. For further families in higher dimensions, see e.g. [17, 11, 2].

Theorem 1 ([12])**.** *The K-moduli space of cubic threefolds is isomorphic to the corresponding GIT moduli space.*

Theorem 2 ([14])**.** *The K-moduli space of blow-ups of $\mathbb{P}^3$ along a genus 4 curve is isomorphic to a variation of GIT moduli space of slope $\frac{22}{51}$ of $(2,3)$-complete intersections in $\mathbb{P}^3$.*

Below, we sketch the proofs of these results using the moduli continuity method. Note that the a priori estimate for singularities is good enough to determine the

K-moduli space in the first family. Nonetheless, some new ingredients such as moduli of K3 surfaces and Sarkisov links are needed in the second family.

Step 0. We show that the K-moduli space M^{K} is non-empty. In other words, this is saying that there exists at least one K-stable member of the given family of Fano varieties. Usually, one finds a member (sometimes singular) with large symmetry and apply the equivariant K-stability criterion [22]. Then, by the openness of K-(semi)stability [3, 20], a general member of the given family is K-stable. For cubic threefolds, we choose the Fermat cubic threefold. For blow-ups of $\mathbb{P}^3$ along a genus 4 curve, we choose a specific curve with large symmetry; see [1, Proposition 4.33].

Step 1. We establish an a priori estimate for singularities that can appear in $X \in M^{\mathrm{K}}$. The key estimate is the following local-to-global volume comparison from [10] (after [7]):

$$\frac{(-K_X)^n}{(-K_{\mathbb{P}^n})^n} \leq \frac{\widehat{\mathrm{vol}}(x,X)}{\widehat{\mathrm{vol}}(p,\mathbb{P}^n)}. \tag{1}$$

Here $n = \dim X$, $p \in \mathbb{P}^n$ is a point, and $\widehat{\mathrm{vol}}$ stands for the local volume (also known as normalized volume) of a klt singularity introduced by C. Li in [9]. Let us restrict to the case of Fano threefolds. Together with the finite degree formula for local volumes [21] and the ODP Gap Theorem in dimension 3 [12], the inequality (1) implies that as long as $(-K_X)^3 \geq 20$, every smoothable $\mathbb{Q}$-Cartier Weil divisor L on X is Cartier; in particular, X is Gorenstein canonical.

Step 2. We describe the geometry of $X \in M^{\mathrm{K}}$ by investigating the linear system $|L|$ for a suitable divisor L. Let $\mathcal{X} \to T$ be a $\mathbb{Q}$-Gorenstein smoothing of $X \cong X_0$ over a pointed smooth curve $0 \in T$.

If X_t is a cubic threefold for $t \neq 0$, we take L to be the degeneration of $\mathcal{O}_{X_t}(1)$ as a Weil divisor. Then L is $\mathbb{Q}$-Cartier ample as $-K_X \sim 2L$. From Step 1, we conclude that L is Cartier. Thus [8] implies that $|L|$ is very ample and induces a closed embedding $X \hookrightarrow \mathbb{P}^4$ as a (possibly singular) cubic hypersurface.

If $\pi_t : X_t \to \mathbb{P}^3$ is the blow-up along a genus 4 curve C_t for $t \neq 0$, we take $\mathcal{L}$ to be the divisor on the total space $\mathcal{X}$ such that $\mathcal{L}|_{X_t} = \pi_t^* \mathcal{O}_{\mathbb{P}^3}(1)$, and let $L = \mathcal{L}|_{X_0}$. Then we encounter a major issue as $\mathcal{L}$ may not be $\mathbb{Q}$-Cartier since it is no longer proportional to $-K_{\mathcal{X}/T}$. To resolve this, we take a small $\mathbb{Q}$-Cartierization $\widetilde{\mathcal{X}} \to \mathcal{X}$ so that the strict transform $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ on $\widetilde{\mathcal{X}}$ is $\mathbb{Q}$-Cartier and ample over $\mathcal{X}$. Then we prove that $\widetilde{\mathcal{L}}$ is indeed big and semiample over T, and $X \cong \widetilde{X}_0$ is still a blow-up of $\mathbb{P}^3$. This is achieved by Reid's technique of general elephants, a delicate analysis of moduli of lattice-polarized K3 surfaces, and the Sarkisov link structure on these Fano threefolds as blow-ups of singular cubic threefolds.

Step 3. We show that the K-moduli space M^{K} is isomorphic to a suitable GIT moduli space M^{GIT}. In summary, the previous steps show that every $X \in M^{\mathrm{K}}$ belongs to a suitable parameter space W with an action of a reductive group G. If the above estimates are strong enough, then we often have that the Picard rank of W is small, and the CM line bundle λ_{CM} on W is ample. Thus we can take the GIT quotient $M^{\mathrm{GIT}} = W /\!\!/_{\lambda_{\mathrm{CM}}} G$. By the Paul–Tian criterion, we have an injective birational morphism

$$\phi : M^{\mathrm{K}} \to M^{\mathrm{GIT}}.$$

Since M^{K} is proper [4, 13], we conclude that ϕ is an isomorphism by Zariski's main theorem. For cubic threefolds, we take $W = \mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}(3))$ and $G = \mathrm{PGL}(5)$, and Theorem 1 follows. For blow-ups of $\mathbb{P}^3$ along genus 4 curves, we take W to be the projective bundle over $\mathbb{P}^9 = \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(2))$ parameterizing $(2,3)$-intersections in $\mathbb{P}^3$, and $G = \mathrm{PGL}(4)$. Then computations of the CM line bundle show that λ_{CM} has slope $\frac{22}{51}$ in the Picard group of W, which has rank 2. Thus Theorem 2 follows.

Finally, we note that the variation of GIT in Theorem 2 was studied in detail in [5], where it was shown that the VGIT moduli spaces provide models for the Hassett–Keel program of genus 4 curves. Therefore, our K-moduli space M^{K} for the second family of Fano threefolds appears as a specific model in the Hassett–Keel program.

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Non-archimedean approach to SYZ conjecture

Enrica Mazzon

Mirror symmetry is a fast-moving research area at the boundary between mathematics and theoretical physics. Originated from observations in string theory, it suggests the existence of a duality between Calabi–Yau (CY) manifolds, complex manifolds with a nowhere vanishing holomorphic form of maximal degree. It predicts that every CY manifold X has a mirror partner $\check{X}$, such that the complex geometry of $\check{X}$ is equivalent to the symplectic geometry of X, in some appropriate sense, and vice versa.

Various approaches have been developed to find a rigorous definition of a mirror pair $(X, \check{X})$, and methods to construct mirror partners; a geometric explanation was proposed by Strominger, Yau and Zaslow (SYZ) in [SYZ96]. In its current formulation, the SYZ conjecture concerns CY manifolds in certain degenerating families rather than individual manifolds. More precisely, consider a projective family $(X_t)_t$ of CY varieties of dimension n over a punctured disk, such that the family is maximally degenerate, i.e. the monodromy operator on the degree n cohomology of X_t has a Jordan block of maximal size, that is $n+1$.

Conjecture 1 (SYZ conjecture)**.** For all sufficiently small t, X_t admits a fibration $\pi : X_t \to B$, whose fibres are special Lagrangian tori, away from a locus Δ of codimension 2 in B. Moreover, the mirror partner $\check{X}_t$ of X_t is obtained by dualizing the special Lagrangian toric fibres of π and by suitably compactifying the resulting space.

While some examples of special Lagrangian torus fibrations can be produced, dealing with the general case seems very difficult. The insight of Kontsevich and Soibelman is to replace the above conjecture by an analogous one in the non-archimedean world, and to interpret the latter as an asymptotic limit of the complex phenomenon when $t \to 0$.

More precisely, one can associate to the degenerating family $X = (X_t)_t$ the Berkovich non-archimedean space X^{an}, whose points are valuations defined locally on X. Given a degeneration $\mathcal{X}$ of X, we say that $\mathcal{X}$ is snc (respectively dlt) if the pair $(\mathcal{X}, \mathcal{X}_0)$ is strict normal crossing (respectively divisorially log terminal), where $\mathcal{X}_0$ is fiber over $t = 0$; see [Kol13] for more details. Given any snc or dlt degeneration, the dual intersection complex $\mathcal{D}(\mathcal{X}_0)$ is a simplicial complex encoding the combinatorics of the multiple intersections of the components of $\mathcal{X}_0$. It admits a canonical embedding in X^{an}, whose image is called the skeleton of $\mathcal{X}$ and denoted $\mathrm{Sk}(\mathcal{X})$, and a retraction $\rho_{\mathcal{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathcal{X})$. Among various degenerations, minimal (in the sense of MMP) dlt models $\mathcal{X}$ of X determine a canonical skeleton $\mathrm{Sk}(X) = \mathrm{Sk}(\mathcal{X})$, called the essential skeleton of X and independent of the choice of the minimal model; see [MN15, NX16] for more details. The essential skeleton and the retractions $\rho_{\mathcal{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathcal{X})$ are of particular relevance in the non-archimedean approach to the SYZ conjecture, as we will see in the sequel.

By the celebrated Yau theorem, X_t carries a unique Kähler form ω_t such that $[\omega_t] \in c_1(L_t)$ and $\omega_t^n = C_t \Omega_t \wedge \overline{\Omega}_t$ for a constant C_t. Finding such form ω_t boils down to solving an equation, called complex Monge–Ampère equation. By the works [CL06, BFJ15], non-archimedean Monge–Ampère equations can be defined on X^{an} as well, and solved for any measure supported on a subset $\mathcal{D}(\mathcal{X}_0)$ of X^{an}. In particular, let Ψ be the solution to

$$\mathrm{MA}_{\mathrm{NA}}(\Psi) = d\mu_{\mathrm{Sk}(X)},$$

where $\mathrm{MA}_{\mathrm{NA}}$ denotes the non-archimedean Monge–Ampère operator, and $d\mu_{\mathrm{Sk}(X)}$ the Lebesgue measure on $\mathrm{Sk}(X)$. Up to fixing a reference metric, we can think of Ψ as a function on X^{an}. In [Li23] Li reduced the SYZ conjecture to a conjecture in non-archimedean geometry about Monge–Ampère metrics

Theorem 1 ([Li23]). *Let X be a maximally degenerate family of Calabi–Yau varieties. If there exists a degeneration $\mathcal{X}$ of X such that the solution Ψ is invariant with respect to the retraction $\rho_{\mathcal{X}}$, i.e.*

$$\Psi = \Psi \circ \rho_{\mathcal{X}} \quad \textit{on} \quad \rho_{\mathcal{X}}^{-1}(\mathrm{Int}(\tau)),$$

over the interior of any maximal face τ of $\mathcal{D}(\mathcal{X}_0)$, then an SYZ fibration exists on a large region $U_t \subseteq X_t$.

This approach reduces the construction of SYZ fibrations to a property in non-archimedean geometry. In [HJMM24], Hultgren, Jonsson, McCleerey and I provided first evidence for such conjecture, when X is not one-dimensional or an abelian variety. More precisely, let $X = \{z_0 z_1 \ldots z_{n+1} + tf(z) = 0\} \subset \mathbb{P}^{n+1}$ be a family of Calabi–Yau hypersurfaces where f is a generic polynomial of degree $n+2$. In this case, the essential skeleton $\mathrm{Sk}(X)$ is a sphere and can be identified with the boundary of the standard unit simplex in $\mathbb{R}^{n+1}$.

Theorem 2 ([HJMM24]). *If ν is a symmetric measure on $\mathrm{Sk}(X)$, then the solution to $\mathrm{MA}_{\mathrm{NA}}(\cdot) = \nu$ is the restriction of a symmetric toric metric on $\mathcal{O}_{\mathbb{P}^{n+1}}(n+2)^{\mathrm{an}}$, thus is determined by the restriction to $\mathrm{Sk}(X)$ of a convex function on $\mathbb{R}^{n+1}$.*

Applying Theorem 2 to $\nu = d\mu_{\mathrm{Sk}(X)}$, we show that the characterization of the solution Ψ provided by the theorem is sufficient to prove the invariance property of Theorem 1. We conclude therefore that SYZ fibrations exist on large regions of CY hypersurfaces. See

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