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Geometric, Algebraic and Topological Combinatorics

Organized by
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10 December – 15 December 2023

Abstract. The 2023 Oberwolfach meeting "Geometric, Algebraic, and Topological Combinatorics" was organized by Gil Kalai (Jerusalem), Isabella Novik (Seattle), Francisco Santos (Santander), and Volkmar Welker (Marburg). It covered a wide variety of aspects of Discrete Geometry, Algebraic Combinatorics with geometric flavor, and Topological Combinatorics. Some of the highlights of the conference were (1) Federico Ardila and Tom Braden discussed recent exciting developments in the intersection theory of matroids; (2) Stavros Papadakis and Vasiliki Petrotou presented their proof of the Lefschetz property for spheres, and, more generally, for pseudomanifolds and cycles (this second part is joint with Karim Adiprasito); (3) Gaku Liu reported on his joint work with Spencer Backman that establishes the existence of a regular unimodular triangulation of an arbitrary matroid base polytope.

Mathematics Subject Classification (2020): 05Exx, 52Bxx, 13Dxx,13Fxx, 14Txx, 52Cxx, 57Qxx, 90Cxx.

Introduction by the Organizers

The 2023 Oberwolfach meeting "Geometric, Algebraic, and Topological Combinatorics" was organized by Gil Kalai (Hebrew University, Jerusalem), Isabella Novik (University of Washington, Seattle), Francisco Santos (University of Cantabria, Santander), and Volkmar Welker (Philipps-Universität Marburg, Marburg).

The conference featured three 1-hour talks by Federico Ardila on "Intersection theory of matroids", Eran Nevo on "Rigidity expander graphs", and by Tom Braden on "The intersection cohomology module of a matroid", two back-to-back 35-minute talks by Vasiliki Petrotou and Stavros Papadakis "Lefschetz properties via anisotropy", a 50-minute talk by Nati Linial on "Some stories about graphs and geometry", and 23 additional talks, ranging from 30 to 40 minutes. On Thursday evening we held a problem session. After and before the lectures many small groups embarked in discussions, some of which initiated new collaborations. All together it was a very productive, intense and enjoyable week.

The conference covered a broad spectrum of topics from Algebraic Combinatorics (intersection cohomology modules, Lefschetz theorems, Koszul duality), Topological Combinatorics (configuration spaces, envy-free partitions, random complexes), and Geometric Combinatorics (face enumeration, polytope theory, matroid polytopes, lattice polytopes, rigidity theory).

In the next paragraphs we summarize the richness and depth of the work and the presentations, concentrating on some of the highlights.

The first lecture on Monday, by Federico Ardila (based on his Clay lecture at the British Combinatorial Conference 2024, see F. Ardila-Mantilla, Intersection theory of matroids: Variations on a theme, in: Surveys in Combinatorics 2024, pp. 1-30, Cambridge University Press, 2024) discussed four different ways to define the Chow ring of a toric variety due to Billera, Brion, Fulton–Sturmfels, and Allermann–Rau. Federico also explained how the different representations of the Chow ring enable different proofs of recent spectacular combinatorial results such as unimodality of the coefficients of chromatic polynomials.

Gaku Liu's talk then presented an ingenious inductive proof that every matroid base polytope has a regular unimodular triangulation.

The rest of Monday was devoted to a variety of topics in algebraic and geometric combinatorics. For instance, Eran Nevo discussed a proof of the existence of an infinite family of k-regular d-rigidity-expander graphs for every $d \geq 2$ and $k \geq 2d + 1$.

Tuesday morning focused on topological combinatorics. Florian Frick talked about topological methods in zero-sum Ramsey theory. Pablo Soberon discussed high-dimensional envy-free partitions. Kevin Piterman talked about fixed-point-free actions of finite groups on contractible spaces. More specifically, Kevin presented a solution to a central problem about the existence of fixed points for every finite group acting on a compact 2-complex. Finally, Roy Meshulam's lecture on random balanced Cayley complexes was a very rich blend of combinatorial, topological, Fourier-theoretical, and algebraic methods.

On Tuesday afternoon we had several talks related to polytope theory and in particular to lattice polytopes.

Wednesday morning started with an hour lecture by Tom Braden. This lecture complemented Ardila's talk from Monday morning reporting on recent fascinating developments in the matroid theory; this time via the lens of Algebraic Geometry.

The second part of Wednesday morning consisted of two back-to-back talks by Stavros Papadakis and Vasiliki Petrotou. They discussed their notion of anisotropicity of simplicial spheres which led to their proof of the Lefschetz property for spheres, and, more generally, for pseudomanifolds and cycles (this second part is joint with Adiprasito). The Lefschetz property, in turn, leads to a simpler proof of the g-conjecture for spheres. Their talks were followed by Christos Athanasiadis' talk on face enumeration and real-rootedness.

On Thursday, the focus returned to topological questions, a highlight being Geva Yashfe's talk about the number of triangulations of homology 3-spheres.

Friday morning was devoted to a mixture of topics in polyhedral geometry and hyperplane arrangements. The final lecture of the conference was given by Nati Linial who discussed recent progress on geodetic and metrizable graphs.

It bears repeating that numerous breakthrough results were announced and presented during the conference.

We are extremely grateful to the Oberwolfach institute, its directorate and to all of its staff for providing a perfect setting for an inspiring, intensive week of “Geometric, Algebraic, and Topological Combinatorics”.

Gil Kalai, Isabella Novik, Francisco Santos, Volkmar Welker
Jerusalem/Seattle/Santander/Marburg, April 2024

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Workshop: Geometric, Algebraic and Topological Combinatorics

Table of Contents

Abstracts

Intersection theory of matroids: techniques and examples

Federico Ardila

Chow rings of toric varieties, which originate in intersection theory, feature a rich combinatorial structure of independent interest. We survey four different ways of computing in these rings, due to Billera, Brion, Fulton–Sturmfels, and Allermann–Rau. We illustrate the beauty and power of these methods by giving four proofs of Huh and Huh–Katz's formula $\mu^k(M) = deg_M(\alpha^{r-k}\beta^k)$ for the coefficients of the reduced characteristic polynomial of a matroid M as the mixed intersection numbers of the hyperplane and reciprocal hyperplane classes α and β in the Chow ring of M. Each of these proofs sheds light on a different aspect of matroid combinatorics, and provides a framework for further developments in the intersection theory of matroids. Our presentation is combinatorial, and does not assume previous knowledge of toric varieties, Chow rings, or intersection theory.

Face enumeration of order complexes and real-rootedness

Christos A. Athanasiadis
(joint work with Katerina Kalampogia-Evangelinou)

Given a finite poset P, the order complex $\Delta(P)$ is the abstract simplicial complex which consists of all chains in P. Order complexes of Cohen–Macaulay posets form a class of flag simplicial complexes with especially nice properties [12, Section III.4]. Somewhat unexpectedly, their face enumeration is far from being well understood. We aim to show that their f-polynomials (equivalently, their h-polynomials) tend to be real-rooted surprisingly often by discussing examples and methods that can be applied.

Let us denote by $c_k(P)$ the number of k-element chains in P. The f-polynomial and the h-polynomial of $\Delta(P)$ are then defined as

$$\begin{aligned} f(\Delta(P),x) &= \sum_{k=0}^{n} c_k(P)x^k, \\ h(\Delta(P),x) &= \sum_{k=0}^{n} c_k(P)x^k(1-x)^{n-k} = (1-x)^n f\left(\frac{x}{1-x}\right), \end{aligned}$$

where n is the largest cardinality of a chain in P. The polynomial $f(\Delta(P),x)$ is also called the chain polynomial of P. We recall that $h(\Delta(P),x)$ has nonnegative coefficients for every Cohen–Macaulay poset P (see [12, Chapter II]) and that $f(\Delta(P),x)$ is real-rooted (meaning, all its roots are real) if and only if so is $h(\Delta(P),x)$.

Our motivation comes from the following two conjectures. The first was posed as a question by Brenti–Welker [9] and claims that barycentric subdivisions of convex polytopes have real-rooted h-polynomials.

Conjecture 1. (cf. [9, Question 1]) *The polynomial $h(\Delta(P), x)$ is real-rooted if P is the face lattice of a convex polytope.*

Conjecture 2. ([3, Conjecture 1.2]) *The polynomial $h(\Delta(P), x)$ is real-rooted for every geometric lattice P (equivalently, if P is the lattice of flats of a matroid).*

The latter conjecture would imply the unimodality of the h-polynomials of order complexes of geometric lattices. These two conjectures naturally raise the following question.

Question 3. ([3, Question 1.1]) *For which finite Cohen–Macaulay posets P is $h(\Delta(P), x)$ real-rooted? Equivalently, which finite Cohen–Macaulay posets have a real-rooted chain polynomial?*

Let us briefly discuss some answers to Question 3 which are known in interesting special cases. For distributive lattices the question is known to be equivalent to the Neggers conjecture [10] (see also [5] [11, Conjecture 1]), which claims the real-rootedness of poset Eulerian polynomials. Thus, there exist distributive lattices which fail to have real-rooted chain polynomials [13]. On the other hand, classes of Cohen–Macaulay posets with real-rooted chains polynomials include some classes of distributive lattices [8, 14] and:

- Cohen–Macaulay simplicial posets [9] (in particular, face lattices of simplicial or simple polytopes) and all their rank-selected subposets [4];
- CL-shellable cubical posets [2] (in particular, face lattices of cubical polytopes);
- the face lattices of the pyramid and the prism over polytopes which have a face lattice with real-rooted chain polynomial [3];
- partition lattices of types A and B and subspace lattices [3];
- the lattices of flats of paving matroids [7] and those of near-pencils, uniform matroids and all matroids on at most nine elements [3];
- all noncrossing partition lattices associated to irreducible finite Coxeter groups [4].

A popular method of proof is to express $h(\Delta(P), x)$ as a nonnegative linear combination of real-rooted polynomials with positive leading coefficients which form an interlacing sequence (or, more generally, which have a common interleaver); see [6, Section 7.8] and references therein for the relevant background. We illustrate this method in two cases in which it has been successful, namely those of simplicial and cubical posets (see [12, Section II.6] and [1] for background on simplicial and cubical posets and their h-vectors).

Theorem 4. (cf. [9, Theorems 1 and 2]) *For every positive integer n, there exists an interlacing sequence $(p_{n,k}(x))_{0 \le k \le n}$ of real-rooted polynomials with nonnegative coefficients such that*

$$h(\Delta(P), x) = \sum_{k=0}^{n} h_k(P) p_{n,k}(x)$$

for every simplicial poset P of rank n, where $(h_k(P))_{0\le k\le n}$ is the simplicial h-vector of P. In particular, $h(\Delta(P),x)$ is real-rooted for every simplicial poset P with nonnegative simplicial h-vector.

The polynomial $p_{n,k}(x)$ can be defined by the formula

$$\sum_{m\ge 0} m^k(1+m)^{n-k}x^m = \frac{p_{n,k}(x)}{(1-x)^{n+1}},$$

or as the descent enumerator of permutations w of $\{1,2,\dots,n+1\}$ such that $w(1)=k+1$.

Theorem 5. ([2]) *For every nonnegative integer n, there exists an interlacing sequence $(p^B_{n,k}(x))_{0\le k\le n+1}$ of real-rooted polynomials with nonnegative coefficients such that*

$$h(\Delta(Q),x) = \sum_{k=0}^{n+1} h_k(Q)p^B_{n,k}(x)$$

for every cubical poset Q of rank $n+1$, where $(h_k(Q))_{0\le k\le n+1}$ is the cubical h-vector of Q. In particular, $h(\Delta(Q),x)$ is real-rooted for every cubical poset Q which has a nonnegative cubical h-vector.

The polynomials $p^B_{n,k}(x)$ can be defined by the formula

$$\frac{p^B_{n,k}(x)}{(1-x)^{n+1}} = \begin{cases} \sum_{m\ge 0}(2m+1)^n x^m, & \text{if } k=0,\\ \sum_{m\ge 0}(4m)(2m-1)^{k-1}(2m+1)^{n-k}x^m, & \text{if } 1\le k\le n,\\ \sum_{m\ge 1}(2m-1)^n x^m, & \text{if } k=n+1.\end{cases}$$

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Convex partitions in a slice

Pavle V. M. Blagojević
(joint work with Michael C. Crabb)

1. Convex partitions of Euclidean spaces

Problems related to the existence of convex partitions of a Euclidean space of a desired type have a long and rich history, starting with the 1930's ham-sandwich theorem of Steinhaus and Borsuk as the most famous example. The ham-sandwich theorem claims that for every collection of d proper convex bodies $C_1, \dots, C_d$ in $\mathbb{R}^d$ there exists a convex partition of $\mathbb{R}^d$ into 2 pieces A_1 and A_2 such that

$$\mathrm{vol}_d(C_1 \cap A_1) = \mathrm{vol}_d(C_1 \cap A_2)\,,\ \dots\ ,\ \mathrm{vol}_d(C_d \cap A_1) = \mathrm{vol}_d(C_d \cap A_2).$$

Here, the closed convex sets A_1 and A_2 with non-empty interior form a convex partition of $\mathbb{R}^d$ if $A_1 \cup A_2 = \mathbb{R}^d$ and $\mathrm{int}(A_1) \cap \mathrm{int}(A_2) = \emptyset$, (hence $\mathrm{vol}_d(A_1 \cap A_2) = 0$).

A natural extension is the question: For given integers $d, k, j \geq 1$ and an arbitrary collection $\mathcal{C}$ of j proper convex bodies in $\mathbb{R}^d$ is it possible to find k affine hyperplanes such that every orthant Ω determined by them contains the same piece of each convex body in $\mathcal{C}$, that is $\mathrm{vol}_d(C \cap \Omega) = \frac{1}{2^k}\mathrm{vol}_d(C)$ for every $C \in \mathcal{C}$. The work on this generalisation of the ham-sandwich theorem, the so called Grünbaum–Hadwiger–Ramos problem, was pioneered by Grünbaum [11], Hadwiger [12] and Avis [2], and a bit later continued by Edgar Ramos [16]. Topological challenges emerging in the process of solving this problem were recently discussed in [7].

In 2006 Nandakumar & Ramana-Rao asked for a solution of the following intriguing problem: Is it true that for every integer $n \geq 2$ and every proper convex body C in the plane there is a convex partition of the plane into n pieces $A_1, \dots, A_n$ having equal area and equal perimeter, that is

$$\mathrm{vol}_2(C \cap A_1) = \cdots = \mathrm{vol}_2(C \cap A_n) \quad \text{and} \quad \mathrm{per}(C \cap A_1) = \cdots = \mathrm{per}(C \cap A_n),$$

where "per" denotes the plane perimeter function. This naive-looking question caught a lot of attention and many authors contributed to its better understanding. For more details see the work of Bárány, Blagojević & Szűcs [3], Soberón [17], Karasev, Hubard & Aronov [14], Blagojević & Ziegler [10], and Blagojević & Sadovek [8]. Recently a promising work of Akopyan, Avvakumov & Karasev offered a new insight into a complete solution of the original, plane, Nandakumar & Ramana-Rao problem [1]. The work on a solution of this problem brought into

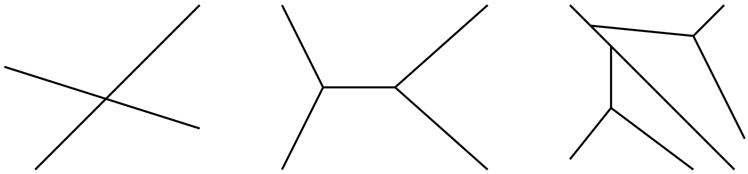

Figure 1. Convex partitions of the plain by orthants into 4 pieces, by generalised Voronoi diagram into 6 pieces, and by iteration of depth 2 into $6 = 2 \cdot 3$ pieces.

focus convex partitions generated by generalised Voronoi diagrams of $\mathbb{R}^d$ which in addition equipart a fixed convex body into pieces of equal volume. Surprisingly such convex partitions can be completely parametrised by the configuration space of pairwise distinct points in $\mathbb{R}^d$. Each point in the space corresponds to a collection of the so-called "sites" of a generalised Voronoi diagram.

Iterated convex partitions appeared for the first time in context of the Gromov's waist of the sphere theorem. Gromov worked with the partitions into 2^i pieces, which can be parametrised by the wreath products of spheres. For a different waist of the sphere result, Palić with Blagojević and Karasev in [15] considered iterated convex partitions into p^k pieces indexed by the kth wreath product of the configuration spaces. Iterated partitions appeared also in the work of Blagojević & Soberón [9], where they were parametrised by the join of the configuration space. The most general iterated convex partition in the context of the Nandakumar & Ramana-Rao problems were recently considered by Blagojević & Sadovek in [8].

A general convex equipartition problem can be formulated as follows.

Problem (Convex partitions of a Euclidean space) Let $d, j, n \geq 1$ be fixed integers, $\mathcal{C}$ an arbitrary collection of j proper convex bodies in $\mathbb{R}^d$ and let $\mathcal{P}$ be a predetermined class of convex partitions of $\mathbb{R}^d$, like partitions by orthants, by (generalised) Voronoi diagrams, or by iterated convex partition (see Figure 1 for an illustration). *Is there a partition $(A_1, \dots, A_n)$ of $\mathbb{R}^d$ from the class $\mathcal{P}$ with the property that* $\mathrm{vol}_d(C \cap A_1) = \cdots = \mathrm{vol}_d(C \cap A_n)$ *for every convex body $C \in \mathcal{C}$.*

2. Convex partitions of Euclidean vector bundles

Motivated by the classical problems of convex partitions of a Euclidean space we ask whether a similar result can be obtained if instead of one (ambient) Euclidean space we consider a (parametrised) family of Euclidean spaces and look for a convex partition of at least one of these spaces satisfying the desired property. A prototype of the problems we want to address can be phrased in the following way.

Problem (Convex partitions of tautological vector bundles) Let $d, j, n, k \geq 1$ be fixed integers, $\mathcal{C}$ a collection of j proper convex bodies in $\mathbb{R}^d$ with the origin in their interiors and let $\mathcal{P}$ be a predetermined class of convex partitions of $\mathbb{R}^k$. *Is there an ℓ-dimensional linear subspace L of $\mathbb{R}^d$ and a partition $(A_1, \dots, A_n)$ of L*

from the class $\mathcal{P}$ *with the property that* $\mathrm{vol}_d(C \cap L \cap A_1) = \cdots = \mathrm{vol}_d(C \cap L \cap A_n)$ *for every convex body* $C \in \mathcal{C}$.

In the case of partitions by orthants in a tautological vector bundle Blagojević, Calles Loperena, Crabb & Dimitrijević Blagojević, using the parametrised Fadell–Husseini index theory and delicate spectral sequence computations, proved the following result [4, Thm. 1.5].

Theorem 1. *Let* $d, j, n, k, \ell \geq 1$ *be fixed integers,* $\mathcal{C}$ *a collection of* j *proper convex bodies in* $\mathbb{R}^d$ *with the origin in theirs interiors such that* $1 \leq k \leq \ell$ *and* $d \geq 2^{\lfloor \log_2 j \rfloor}(2^{k-1} - 1) + j$. *There exists an* ℓ*-dimensional linear subspace* L *of* $\mathbb{R}^d$ *and* k *affine hyperplanes in* L *such that* $\mathrm{vol}_\ell(C \cap L \cap \Omega) = \frac{1}{2^k}\mathrm{vol}_\ell(C \cap L)$ *for every* $C \in \mathcal{C}$ *and every orthant* $\Omega \subseteq L$ *determined by the affine hyperplanes.*

In the followup work, Blagojević & Crabb [5] gave the complete treatment of a problem of convex partitions by orthants on Euclidean vector bundles. Using a new insight they reprove known results and extend them to arbitrary Euclidean vector bundles putting various types of constraints on the solutions. Furthermore, the developed methods allowed them to give new proofs and extend results of Guth & Katz, Schnider and Soberón & Takahashi.

Levinson, in collaboration with Blagojević & Crabb, considered the problem of convex partitions in Euclidean vector bundle by generalised Voronoi diagrams [6, 13]. An example of the results they obtained is the following theorem.

Theorem 2. *Let* $d \geq 2$, $j \geq 1$, *and* $1 \leq \ell \leq d$ *be integers and let* n *an odd prime. Consider an arbitrary collection* $\mathcal{C}$ *of* j *proper convex bodies in* $\mathbb{R}^d$ *with the origin in theirs interiors. If* $j \leq d - 2$, *then there exists an* ℓ*-dimensional linear subspace* L *of* $\mathbb{R}^d$ *and a convex partition* $A_1, \ldots, A_n$ *of* L *by a generalised Voronoi diagram such that* $\mathrm{vol}_\ell(C \cap L \cap A_1) = \cdots = \mathrm{vol}_\ell(C \cap L \cap A_n)$, *for every convex body* $C \in \mathcal{C}$.

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The intersection cohomology module of a matroid

Tom Braden

(joint work with June Huh, Jacob Matherne, Nicholas Proudfoot, Botong Wang)

Let M be a matroid of rank d on the ground set $[n]$, and let $\mathcal{L} = \mathcal{L}(\mathrm{M})$ be its lattice of flats. The number of flats of rank k is $W_k = W_k(\mathrm{M})$, the k^{th} *Whitney number* of the second kind of M. In the 1975 paper [8], Dowling and Wilson proved that if $k \leq d/2$ then

$$W_0 + W_1 + \cdots + W_k \leq W_{d-k} + W_{d-k+1} + \cdots + W_d.$$

They also made the stronger conjecture that

$$W_k \leq W_{d-k},$$

which has become known as the Dowling–Wilson or "top-heavy" conjecture for matroids. It was proved for $d = 3$ by de Bruijn and Erdős [6], and for $k = 1$ by Basterfield and Kelly [1]. When M is realizable it was proved by Huh and Wang [9] using the intersection cohomology of an associated algebraic variety, and it is proved in general in [4], by defining the intersection cohomology combinatorially for an arbitrary matroid, and showing that it has the required properties.

If M is realized by vectors $v_1, v_2, \ldots, v_n$ spanning a vector space V over $\mathbb{C}$, then

$$\xi \mapsto (\xi(v_1), \ldots, \xi(v_n))$$

gives an injection $V^* \hookrightarrow \mathbb{C}^n$. Huh and Wang considered the singular variety Y which is the closure of the image of V^* inside $(\mathbb{P}^1)^n$. It has a decomposition into affine spaces indexed by elements of $\mathcal{L}$, which implies that its odd cohomology vanishes, and that $\dim_{\mathbb{Q}} H^{2k}(Y;\mathbb{Q}) = W_k$. Its intersection cohomology $IH^*(Y;\mathbb{Q})$ is a module over the cohomology ring, and the hard Lefschetz theorem says that for an ample class $\ell \in H^2(Y;\mathbb{Q})$ and $k \leq 2d$, the multiplication

$$\ell^{d-2k}\colon IH^{2k}(Y;\mathbb{Q}) \to IH^{2d-2k}(Y;\mathbb{Q})$$

is an isomorphism. Huh and Wang then appeal to a theorem of Björner and Ekedahl [2], which says that the cohomology of Y injects into the intersection cohomology $IH^*(Y;\mathbb{Q})$ as $H^*(Y;\mathbb{Q})$-modules. This implies that the multiplication

$$\ell^{d-2k}\colon H^{2k}(Y;\mathbb{Q}) \to H^{2d-2k}(Y;\mathbb{Q})$$

is an injection, proving the Dowling–Wilson conjecture in this case.

In [4] we consider combinatorial avatars of the cohomology ring and intersection cohomology module which make sense for any matroid M. The cohomology ring is replaced by the *graded Möbius algebra* H(M), which has a $\mathbb{Q}$-basis the symbols y_F, with multiplication

$$y_F y_G = \begin{cases} y_{F\vee G} & \text{if}\,\text{rank}(F\vee G) = \text{rank}\,F + \text{rank}\,G \\ 0 & \text{otherwise.}\end{cases}$$

It is the associated graded of the usual Möbius algebra under the filtration by rank. If M is realized by vectors as above, then H(M) is isomorphic to the cohomology ring of Y, with degrees halved.

The main result of [4] is the construction of a graded H(M)-module IH(M), the *intersection cohomology module* of M. It satisfies the following properties:

(1) There is an element $1 \in \mathrm{IH}^0(\mathrm{M})$ so that $y \mapsto y\cdot 1$ defines an injection of H(M) into IH(M),
(2) its graded dual $\mathrm{IH}(\mathrm{M})^*$ is isomorphic to $\mathrm{IH}(\mathrm{M})[d]$,
(3) it satisfies hard Lefschetz: if $\ell = \sum_{\text{rank}\,F=1} c_F y_F$ where all $c_F > 0$, then

$$\ell^{d-2k}\cdot\, : \mathrm{IH}^k(\mathrm{M}) \to \mathrm{IH}^{d-k}(\mathrm{M})$$

is an isomorphism for $k \le d/2$,
(4) the Hodge–Riemann bilinear relations: the restriction of the pairing

$$(a,b) \mapsto (-1)^k \langle \ell^{d-2k} a, b\rangle$$

to the kernel of multiplication by ℓ^{d-2k+1} in $\mathrm{IH}^k(\mathrm{M})$ is positive definite, where $\langle , \rangle$ is the pairing on IH(M) induced by an isomorphism as in (2), normalized so that $\langle y_{[n]}\cdot 1, 1\rangle = 1$.

Properties (1) and (3) are enough to deduce the Dowling–Wilson conjecture, but the proof of (3) involves a complicated induction in which all four statements are needed for all matroids on smaller ground sets.

The module IH(M) is constructed as a direct summand of the *augmented Chow ring* CH(M), which was defined in [3]. It is the graded algebra generated over $\mathbb{Q}$ in degree 1 by x_F, $F \in \mathcal{L}(\mathrm{M}) \setminus \{[n]\}$ and y_i, $i \in [n]$, subject to the relations

- $x_F x_G = 0$ if F, G are not comparable,
- $y_i = \sum_{i\notin F} x_F$, and
- $y_i x_F = 0$ if $i \notin F$.

There is an injection $\mathrm{H}(\mathrm{M}) \hookrightarrow \mathrm{CH}(\mathrm{M})$ which sends y_F to $\prod_{i\in B} y_i$, where B is any basis of F. By Krull-Schmidt, the direct summands of CH(M) as a graded H(M)-module are unique up to isomorphism and permutation. Up to isomorphism, IH(M) is the unique direct summand which contains 1. In [4], a particular summand representing IH(M) is defined by a complicated inductive procedure which does not depend on any choices.

A more intrinsic characterization of IH(M) is given by the following forthcoming result. For an upwardly closed subset $\Sigma \subset \mathcal{L}(\mathrm{M})$, $\Upsilon_\Sigma := \mathrm{span}\{y_F \mid F \in \Sigma\}$ is an ideal of H(M).

Theorem ([5]). *Up to isomorphism,* IH(M) *is the unique graded* H(M)*-module satisfying:*

(1) IH(M) *is indecomposable and* $y_{[n]}\mathrm{IH}(\mathrm{M}) \neq 0$*,*
(2) $\mathrm{IH}(\mathrm{M})^* \cong \mathrm{IH}(\mathrm{M})[d]$*, and*
(3) *for any upwardly closed sets* $\Sigma_1, \Sigma_2 \subset \mathcal{L}(\mathrm{M})$*,*

$$\Upsilon_{\Sigma_1}\mathrm{IH}(\mathrm{M}) \cap \Upsilon_{\Sigma_2}\mathrm{IH}(\mathrm{M}) = \Upsilon_{\Sigma_1\cap\Sigma_2}\mathrm{IH}(\mathrm{M}).$$

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Local h^*-polynomials for one-row Hermite normal form simplices

Benjamin Braun

(joint work with Esme Bajo, Giulia Codenotti, Johannes Hofscheier, Andrés R. Vindas-Meléndez)

This talk is based on the preprint [2]. The local h^*-polynomial of a lattice polytope is an important invariant arising in Ehrhart theory. When the polytope S is a simplex, the local h^*-polynomial is often called the *box polynomial* and denoted $B(S; z)$. Our focus in this work is the study of $B(S; z)$ for lattice simplices presented in Hermite normal form with a single non-trivial row, i.e., simplices S such that the vertices of S are the rows of a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & \cdots & a_{d-2} & a_{d-1} & N \end{bmatrix}$$

with $0 \leq a_i < N$ for all i. We prove that when the off-diagonal entries are fixed, the distribution of coefficients for the local h^*-polynomial of these simplices has a limit as the normalized volume N goes to infinity. More precisely, we prove the following:

Theorem 1. *Fix $a_1, \ldots, a_{d-1} \in \mathbb{Z}_{\geq 1}$ and let*

$$M := \operatorname{lcm}\big(a_1, \ldots, a_{d-1}, -1 + \sum_{i=1}^{d-1} a_i \big).$$

Let S_N denote the simplex defined by the matrix above, where the values of a_i are held constant for varying N. Let k be a positive integer and $0 \leq r \leq M-1$. Then we have that

$$\lim_{k\to\infty} B(S_{kM+r}; z)/B(S_{kM+r}; 1) = B(S_{M+1}; z)/B(S_{M+1}; 1)\,.$$

It follows that if $B(S_{M+1}; z)$ is strictly unimodal, i.e., if the coefficients are unimodal with strict increases and strict decreases, then $B(S_{kM+r}; z)$ is strictly unimodal for all sufficiently large k.

It is known by work of Adiprasito, Papadakis, Petrotou, and Steinmeyer [1] that if a lattice polytope P has the integer decomposition property, then the local h^*-polynomial has unimodal coefficients. One notable aspect of Theorem 1 is that experiments with random simplices in one-row Hermite normal form suggests that unimodality is frequently present even when S does not have the integer decomposition property. It would thus be interesting to further investigate unimodality of local h^*-polynomials for S_{M+1} for various sequences $a_1, \ldots, a_{d-1}$.

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Poincaré-extended ab-index

Galen Dorpalen-Barry
(joint work with Joshua Maglione, Christian Stump)

Grunewald, Segal, and Smith introduced the subgroup zeta function of finitely-generated groups [8], and Du Sautoy and Grunewald gave a general method to compute such zeta functions using p-adic integration and resolution of singularities [6]. This motivated Voll and the second author to examine the setting where the multivariate polynomials factor linearly. They found that the p-adic integrals are specializations of multivariate rational functions depending only on the combinatorics of the corresponding hyperplane arrangement [10]. After a natural specialization, its denominator greatly simplifies, and they conjecture that the numerator polynomial has nonnegative coefficients.

In this work, we prove their conjecture, which is related to the poles of these zeta functions. Specifically, we reinterpret these numerator polynomials by introducing and studying the *(Poincaré-)extended* **ab**-*index*, a polynomial generalizing both the *Poincaré polynomial* and **ab**-*index* of the *intersection poset* of the arrangement. These polynomials have been studied extensively in combinatorics, although from different perspectives. The coefficients of the Poincaré polynomial have interpretations in terms of the combinatorics and the topology of the arrangement [5, Section 2.5]. The **ab**-index, on the other hand, carries information about the order complex of the poset and is particularly well-understood in the case of face posets of oriented matroids—or, more generally, Eulerian posets. In those settings, the **ab**-index encodes topological data via the *flag f-vector* [1].

We study the extended **ab**-index in the generality of graded posets admitting R-labelings. This class of posets includes intersection posets of hyperplane arrangements and, more generally, geometric lattices and geometric semilattices. We show that the extended **ab**-index has nonnegative coefficients by interpreting them in terms of a combinatorial statistic. This generalizes statistics given for the **ab**-index by Billera, Ehrenborg, and Readdy [3] and for the pullback **ab**-index (defined below) by Bergeron, Mykytiuk, Sottile and van Willigenburg [2]. This interpretation proves the aforementioned conjecture [10], as well as a related conjecture from Kühne and the second author [9].

Motivated by the proofs of these conjectures, we describe a close relationship between the Poincaré polynomial and the **ab**-index by showing that the extended **ab**-index can be obtained from the **ab**-index by a suitable substitution. This recovers, generalizes and unifies several results in the literature. Concretely, special cases of this substitution were observed by Billera, Ehrenborg and Readdy for lattices of flats of *oriented matroids* [3], by Saliola and Thomas for lattices of flats of *oriented interval greedoids* [11], and by Ehrenborg for *distributive lattices* [7].

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Discrete homotopy theory

Daniel Carranza

(joint work with Chris Kapulkin)

Discrete homotopy theory, introduced by H. Barcelo and collaborators, is a homotopy theory of (simple) graphs. Homotopy invariants of graphs have found numerous applications, for instance, in the theory of matroids, hyperplane arrangements, topological data analysis, and combinatorial time series analysis. Discrete homotopy theory is also a special instance of a homotopy theory of simplicial complexes, developed by R. Atkin, to study social and technological networks.

I will report on joint work with C. Kapulkin on developing a new foundation for discrete homotopy theory, based on the homotopy theory of cubical sets. To demonstrate the robustness of this foundation, we use it to prove a conjecture of Babson, Barcelo, de Longueville, and Laubenbacher from 2006 relating homotopy groups of a graph to the homotopy groups of a certain cubical complex associated to it.

Topological methods in zero-sum Ramsey theory

Florian Frick
(joint work with Jacob Lehmann Duke, Meenakshi McNamara, Hannah Park-Kaufmann, Steven Raanes, Steven Simon, Darrion Thornburgh, and Zoe Wellner)

A 1961 result of Erdős, Ginzburg, and Ziv [4] guarantees that any sequence $a_1, \dots, a_{2n-1} \in \mathbb{Z}/n$ of length $2n-1$ of integers modulo n has a subsequence of length n that sums to zero. Algebraic techniques, such as the Chevalley–Warning theorem, have proven fruitful in deriving numerous variants and extensions of the original Erdős–Ginzburg–Ziv theorem; see Caro [3] for a survey of these results, which are collectively known as zero-sum Ramsey theory. We develop an equivariant-topological framework to derive zero-sum results in combinatorial number theory; see [5] for full details.

Observe that general zero-sum Ramsey results may be phrased as follows: Let H be an n-uniform hypergraph on ground set V and let $c\colon V \to \mathbb{Z}/n$; decide whether there is a $\sigma \in H$ with $\sum_{v\in\sigma} c(v) = 0$. We refer to any function $c\colon V \to \mathbb{Z}/n$ as a $\mathbb{Z}/n$-*coloring* of H and call $\sigma \in H$ with $\sum_{v\in\sigma} c(v) = 0$ a *zero-sum hyperedge*. The original Erdős–Ginzburg–Ziv theorem in this language states that any $\mathbb{Z}/n$-coloring of the complete n-uniform hypergraph on ground set $\{1, 2, \dots, 2n-1\}$ has a zero-sum hyperedge.

Comparing the setup above to that of classical hypergraph colorings, where one is interested in the existence of a coloring that avoids monochromatic hyperedges (that is, hyperedges where c is constant), observe that avoiding zero-sum hyperedges is a stronger condition. Equivariant-topological techniques, as first developed in this context by Alon, Frankl, and Lovász [1] and Kríž [6], provide strong obstructions for the existence of colorings without monochromatic hyperedges. It is thus natural to ask, whether these methods may also be applied in the more restrictive setting of obstructing colorings without zero-sum hyperedges. Our work shows that this is indeed possible.

To each n-uniform hypergraph H on ground set V associate a topological space that is symmetric with respect to a natural action by $\mathbb{Z}/n$, and in fact by the symmetric group, although we will not make use of this generality. This symmetric space is built as a simplicial complex, the *box complex* $B(H)$: For pairwise disjoint $A_0, \dots, A_{n-1} \subseteq V$ let $A_0 \times \{0\} \cup \dots \cup A_{n-1} \times \{n-1\}$ be in $B(H)$ if for all $a_0 \in A_0, \dots, a_{n-1} \in A_{n-1}$ we have that $\{a_0, \dots, a_{n-1}\} \in H$. Thus $B(H)$ is a simplicial complex on $V \times \mathbb{Z}/n$ that is symmetric with respect to the natural $\mathbb{Z}/n$-action on the second factor. Denote the d-dimensional sphere by S^d. For odd d we fix a free action by the cyclic group $\mathbb{Z}/n$ on S^d. The following gives a topological criterion for existence of zero-sum hyperedges for any $\mathbb{Z}/p$-coloring of a hypergraph for p a prime:

Theorem 1. *Let $p \geq 2$ be a prime, and let H be a p-uniform hypergraph. If there is no $\mathbb{Z}/p$-equivariant map $B(H) \to S^{2p-3}$, then any $\mathbb{Z}/p$-coloring of H has a zero-sum hyperedge.*

In particular, if $B(H)$ is homotopically $(2p-3)$-connected then any $\mathbb{Z}/p$-coloring of H has a zero-sum hyperedge. If H is the complete p-uniform hypergraph on $\{1,\dots,2p-1\}$ then the faces of $B(H)$ consists of p-tuples of pairwise disjoint sets in $\{1,\dots,2p-1\}$. This simplical complex is $(2p-3)$-connected, which recovers the result of Erdős, Ginzburg, and Ziv for p a prime. In the same way as for the standard algebraic proofs of this theorem, the general case then follows by a simple induction on prime divisors.

A $\mathbb{Z}/p$-coloring of H without zero-sum hyperedge induces a simplex-wise linear $\mathbb{Z}/p$-equivariant map $B(H) \to \mathbb{R}^{2p-2} \setminus \{0\}$. Using this same approach now for an arbitrary finite group G instead of $\mathbb{Z}/p$ and convex-geometric results to ascertain the existence of zeros of G-equivariant simplex-wise linear maps, yields Olson's generalization [7] of the Erdős–Ginzburg–Ziv theorem to arbitrary finite groups G.

The topological criterion above has a sufficient condition that may be easily phrased in purely combinatorial terms. Let $\mathcal{F}$ be a set family on ground set X. The *n-colorability defect* $\mathrm{cd}^n(\mathcal{F})$ is $\min|X \setminus \bigcup_{i=1}^n A_i|$, where the minimum is taken over all n-tuples of sets $A_1,\dots,A_n$ that each have no subset in $\mathcal{F}$. The *Kneser hypergraph* $\mathrm{KG}^n(\mathcal{F})$ has $\mathcal{F}$ as its ground set and a hyperedge $\{A_1,\dots,A_n\} \in \mathrm{KG}^n(\mathcal{F})$ if the A_i are pairwise disjoint.

Theorem 2. *Let $n \geq 2$ be an integer, and let $\mathcal{F}$ be a set family with $\mathrm{cd}^n(\mathcal{F}) \geq 2n-1$. Then any $\mathbb{Z}/n$-coloring of $\mathrm{KG}^n(\mathcal{F})$ has a zero-sum hyperedge.*

For example, if $\mathcal{F}$ is the family of all k-element subsets of $\{1,2,\dots,(k+1)n-1\}$ then $\mathrm{cd}^n(\mathcal{F}) = 2n-1$. Thus the theorem above recovers that for any $f\colon \mathcal{F} \to \mathbb{Z}/n$ there are n pairwise disjoint $A_1,\dots,A_n \in \mathcal{F}$ with $\sum f(A_i) = 0$; see Bialostocki and Dierker [2]. The colorability defect bound gives a more general criterion for the existence of zero-sum matchings.

The version of box complex introduced above differs from that of Kríž and provides stronger obstructions. We refer to [5] for proofs and further consequences of the topological approach to zero-sum Ramsey results.

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Random order types

Xavier Goaoc
(joint work with Emo Welzl)

Labeled order types are geometric models of realizable uniform acyclic oriented matroids of rank 3 of particular relevance in discrete and computational geometry. The typical number of extreme points in a simple labeled order type can be determined exactly, and this number reveals bias in the labeled order types of several standard models of random point sets. This analysis can be extended to *unlabeled* simple order types, that is, to relabeling classes of simple realizable uniform acyclic oriented matroids of rank 3, via a combinatorial analogue of Klein's classification of the finite subgroups of $SO(3)$. We refer to the full paper [1] for details.

1. Labeled order types

The *orientation* $\chi(p, q, r)$ of an ordered triple (p, q, r) of points in $\mathbb{R}^2$ is defined as 1 (resp. -1, 0) if r is to the left of (resp. to the right of, on) the line through p and q, oriented from p to q. Two point *sequences* $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n)$ *have the same labeled order type* if

$$\forall 1 \leq i, j, k \leq n, \quad \chi(p_i, p_j, p_k) = \chi(q_i, q_j, q_k). \tag{1}$$

This is an equivalence relation, and a *labeled order type* is an equivalence class for that relation. A labeled order type is *simple* if no three points are aligned in a member of that class. We denote by LOT_n the set of simple labeled order types of size n.

2. A combinatorial version of Sylvester's problem

A famous question of Sylvester asked for the average number of extreme points in a "random" planar point set. Since the notion of extreme point can be defined at the level of labeled order type, Sylvester's question makes sense in the combinatorial setting. We prove:

Theorem 1. *For $n \geq 3$, the number of extreme points in a random simple labeled order type chosen equiprobably in LOT_n has average $4 - \frac{8}{n^2-n+2}$ and variance less than 3.*

Our approach is to divide up the simple planar labeled order types into projectively equivalent classes, and average the number of extreme points within each class.

3. Labeled order types of random point sets

Before we elaborate on the proof of Theorem 1, let us mention that it reveals that the labeled order types of several models of random point sets are rather biased. Formally, a family $\{\mu_n\}_{n\in\mathbb{N}}$, where μ_n is a probability measure on LOT_n, *exhibits concentration* if there exist subsets $A_n \subseteq \mathsf{LOT}_n$, $n \in \mathbb{N}$, such that $\mu_n(A_n) \to 1$ and $|A_n|/|\mathsf{LOT}_n| \to 0$.

Theorem 2. *Let μ be a probability distribution on $\mathbb{R}^2$ that is Gaussian or uniform on a compact convex set K, with K smooth or polygonal. The family of probabilities on LOT_n induced by the labeled order type of n random points chosen independently from μ exhibits concentration.*

We prove Theorem 2 by comparing the typical number of extreme points given by Theorem 1 to the typical number of extreme points in random point sets established in stochastic geometry.

4. Projective classes of labeled order types

To divide up labeled order types into classes under projective equivalence, it is convenient to identify $\mathbb{R}^2$ with an open hemisphere of $\mathbb{S}^2$, the unit sphere of $\mathbb{R}^3$. Let S be a point sequence, labeled from 1 to n, in an open hemisphere of $\mathbb{S}^2$, and let ω denote its labeled order type. We let $P = S \cup -S$, where antipodal points have the same labels, and we define an *affine hemiset* of P as an intersection of size n between P and a closed hemisphere of $\mathbb{S}^2$. Like S, every affine hemiset of P contains exactly one point from each antipodal pair. For a labeled order type τ, the following statements are equivalent:

(i) there exist projectively equivalent point sequences that realize τ and ω,
(ii) there exists a point sequence projectively equivalent to S that realizes τ,
(iii) there exists an affine hemiset of P that realizes τ.

It turns out that for $n \geq 4$, any two affine hemisets of P have distinct labeled order types. The affine hemisets of P are therefore in bijection with the labeled order types projectively equivalent to ω.

5. Averaging via duality

For any point $p \in \mathbb{S}^2$ let $p^* = \{u \in \mathbb{S}^2 : p \cdot u = 0\}$ denote the great circle orthogonal to p. Note that a hemisphere of $\mathbb{S}^2$ centered in x intersects P in an affine hemiset if and only if x lies in a 2-dimensional cells of the arrangement of $P^* = \{p^* : p \in P\}$. This in fact defines a bijection between the affine hemisets of P and the 2-cells of the arrangement of P^*. A key observation is that in this bijection, the number of extreme points of the affine hemiset equals the number of edges of the 2-cell. Among the labeled order type projectively equivalent to ω, the average number of extreme points is therefore the average number of edges in a 2-cell of P^*. For every ω, this average is equal to

$$\frac{8\binom{n}{2}}{2\binom{n}{2}+2} = 4 - \frac{8}{n^2-n+2},$$

so the average is the same over LOT_n. The upper bound on the variance follows from the *zone theorem*.

6. Unlabeling

A coarser classification of n-point *sets* identifies P and Q when there exists a bijection $f : P \to Q$ that preserves orientations. An equivalence class for this coarser relation is called an *order type*. Again, any point set S in an open hemisphere of $\mathbb{S}^2$ gives rise to a set $P = S \cup -S$ that is *projective* in the sense that $P = -P$. Again, the order types of the affine hemisets of P are exactly the order types τ that are projectively equivalent to the order type ω of S (in the sense that τ and ω admit projectively equivalent realizations). In the unlabeled setting, however, several affine hemisets of P may have the same order type...

7. Symmetries

... and how many is a matter of symmetries. Formally, a *symmetry* of a point set $S \subseteq \mathbb{S}^2$ is a bijection $S \to S$ that preserves orientations. Any symmetry of a projective point set P maps every affine hemiset of P to an affine hemiset of P. In the action of the symmetry group of P on its affine hemisets, the *orbit* of an affine hemiset A is exactly the set of affine hemisets of P with the same order type as A, and the *stabilizer* of A is isomorphic to the symmetry group of A. By the orbit-stabilizer theorem, the number of affine hemisets of P with order type ω is therefore

$$\frac{\#\text{symmetries of } P}{\#\text{symmetries of } \omega}$$

To control these ratios, and establish an analogue of Theorem 1 for *unlabeled* order type, we actually characterize the possible symmetry groups of affine and projective subsets of $\mathbb{S}^2$.

8. Classifying symmetry groups

The symmetry group of an affine point set acts on its convex hull (and, actually, on any layer of its "convex peeling") by a circular permutation. This readily implies that every affine point set has a cyclic symmetry group. The key insight to analyze the symmetries of projective point sets is the following analogue of the fact that any rotation $\rho \in SO(3)$ leaves exactly two hemispheres of $\mathbb{S}^2$ globally invariant.

Proposition 3. *For $n \geq 3$, every non-trivial symmetry of a $2n$-point projective point set P leaves exactly two affine hemisets of P globally invariant.*

With Proposition 3, Klein's approach to classifying the finite groups of rotations can be implemented and it yields that the symmetry group of any projective set of $2n$ points in general position is a finite subgroup of $SO(3)$.

Theorem 4. *The symmetry group of any projective set of $2n$ points in general position is $\mathbb{Z}_1$ (trivial group), $\mathbb{Z}_m$ (cyclic group) or D_m (dihedral) with m dividing n or $n-1$, S_4 (octahedral = cubical), A_4 (tetrahedral), or A_5 (icosahedral).*

Each of these groups occurs as the symmetry group of some projective order type.

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Generalized recursive atom ordering and equivalence to CL-shellability

Patricia Hersh
(joint work with Grace Stadnyk)

1. Introduction

This abstract describes joint work with Grace Stadnyk. We introduce a new technique for studying the topological structure of order complexes of finite partially ordered sets (posets), namely we introduce generalized recursive atom orderings. This is a relaxation of the fundamental and widely used technique known as recursive atom ordering that was introduced several decades ago by Björner and Wachs in [BW83].

We establish a number of fundamental properties of these generalized recursive atom orderings (GRAOs), including the property that any generalized recursive atom ordering may be transformed into a traditional recursive atom ordering (RAO) by a process we call the atom reordering process. Since GRAOs are easier to construct than RAOs, this may give a useful new pathway to proving a poset is CL-shellable. These generalized recursive atom orderings further allow us to prove that several different forms of lexicographic shellability (in the not necessarily graded case) are all equivalent to each other, by which we mean that a finite bounded poset admits any one of these types of lexicographic shelling if and only if it admits each of the others. One might expect this to imply the stronger statement that any instance of any one of these types of lexicographic shelling is also an instance of any other of these types of lexicographic shelling, but this is not always true. For instance, one may deduce that not every "self consistent CC-shelling" is a CL-shelling from the fact that not every generalized recursive atom ordering is a recursive atom ordering.

We prove that a finite bounded poset admits a recursive atom ordering (RAO) if and only if it admits a generalized recursive atom ordering (GRAO).

A **chain-atom ordering** Ω of a finite bounded poset P is a choice of ordering on the atoms of each rooted interval $[u, \hat{1}]_r$ of P. Now we are ready to state our main new definition.

Definition 1. *A finite bounded poset P admits a **generalized recursive atom ordering** (GRAO) if the length of P in in P) is 1 or if the length of P is greater than 1 and there is an ordering $a_1, a_2, \dots a_t$ on the atoms of P satisfying:*

- (i) (1) *For $1 \leq j \leq t$, $[a_j, \hat{1}]$ admits a GRAO. sive atom ordering*
  - (ii) *For any atom a_j and any $x, w \in P$ satisfying $a_j \lessdot x \lessdot w$, the following property holds when the chain-atom ordering given by the GRAO from (i)(a) is restricted to $[a_j, w]$: either the first atom of $[a_j, w]$ is above an atom a_i with $i < j$, or no atom of $[a_j, w]$ is above any atom a_i with $i < j$.*
- (iii) *For any $y \in P$ and any atoms a_i, a_j satisfying $a_i < y$ and $a_j < y$ with $i < j$, there exists an element $z \in P$ with $z \leq y$ and an atom a_k with $k < j$ such that $a_j \lessdot z$ and $a_k < z$.*

The statement about cover relations in condition (i)(b) in the definition of GRAO can be strengthened to a corresponding statement about all order relations:

Lemma 2. *Let P be a finite bounded poset, and let Λ be a GRAO for P with atom ordering $a_1, a_2, \ldots a_t$. For each $\hat{0} \lessdot a_j < v$, restricting $\Lambda|_{[a_j,\hat{1}]}$ $[a_j, v]$ yields a GRAO, denoted $\Lambda|_{[a_j,v]}$, for $[a_j, v]$ with the following property: either (a) the first atom of $[a_j, v]$ is greater than some atom a_i satisfying $i < j$ or (b) no atom of $[a_j, v]$ is greater than any atom a_i satisfying $i < j$.*

Our atom reordering process will take any chain-atom ordering and output a chain-atom ordering that will satisfy condition (i)(b) from the defini tion of recursive atom ordering; moreover, it is set up to do so in such a way that when applied to a GRAO, it preserves the property of being a GRAO. Broadly, the algorithm starts at the bottom of the poset P and works its way to the top, reordering the atoms of each rooted interval in a way that takes into account the reordering that has already occurred lower in the poset.

Proposition 3. *Let P be a finite bounded poset with a chain-atom ordering Λ. Let $\Lambda|_{[\hat{0},v]}$ (resp. $\Lambda^{re}|_{[\hat{0},v]}$) be the chain-atom ordering for $[\hat{0}, v]$ obtained by restricting Λ (resp. Λ^{re}) to $[\hat{0}, v]$. Then $\Lambda^{re}|_{[\hat{0},v]}$ equals the chain-atom ordering for $[\hat{0}, v]$ obtained by applying the atom reordering process to $\Lambda|_{[\hat{0},v]}$.*

Lemma 4. *Let P be a finite, bounded poset with Λ a GRAO for P. Then for any $u < v$ in P and any root r for $[u, v]$, the first atom of $[u, v]_r$ in Λ is the first atom of $[u, v]_r$ in Λ^{re}, namely in the atom reordering of Λ.*

These results allow us to prove:

Theorem 5. *A finite bounded poset admits a generalized recursive atom ordering (GRAO) if and only if it admits a recursive atom ordering (RAO).*

Definition 6. *Consider a chain-edge labeling λ such that each rooted interval has a unique lexicographically earliest saturated chain. We define such λ to be **self-consistent** if for any rooted interval $[u, v]_r$ we have the following condition: if a is the atom in the lexicographically first saturated chain of $[u, v]_r$ and $b \neq a$ is also an atom of $[u, v]_r$, then for any $[u, v']_r$ containing a and b all saturated chains of $[u, v']_r$ containing b come lexicographically later than all saturated chains of $[u, v']_r$ containing a.*

The following condition implies self-consistency and is more readily checkable :

Definition 7. *A chain-edge labeling λ of a finite bounded poset P has the **unique earliest (UE) property** if for each rooted interval $[u, v]_r$ in P, the smallest label occurring on any cover relation upward from u only occurs on one such cover relation.*

Equipped with these definitions, we are ready to state one of our main results:

Theorem 8. *Let P be a finite, bounded poset. Then the following are equivalent:*

(1) *P admits a recursive atom ordering*
(2) *P admits a generalized recursive atom ordering*
(3) *P admits a CL-labeling*
(4) *P admits a CL-labeling with the UE property*
(5) *P admits a self-consistent CC-labeling.*
(6) *P admits a CC-labeling with the UE property*
(7) *P admits a self-consistent topological CL-labeling*
(8) *P admits a topological CL-labeling with the UE property*

Moreover, all of these implications are proven constructively. That is, for each implication either it is shown how to construct the latter type of object from the former or else the former type of object is proven also to be the latter type of object.

We apply our results to deduce that a class of posets previously shown to be CC-shellable in [HK] is in fact CL-shellable. That is, we prove that the dual posets to the uncrossing orders (conjectured to be lexicographically shellable by Lam in [La14a]) are CL-shellable. These uncrossing orders arise naturally as face posets of stratified spaces of planar electrical networks (see e.g. [La14a], and references therein). The fact that they are shellable posets combines with Lam's result from [La14a] that they are Eulerian posets to imply that they are CW posets, i.e. face posets of regular CW complexes with finitely many cells. Thus, the shellability of uncrossing orders provides an important step in understanding the topological structure of these spaces of planar electrical networks.

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Flies and regular subdivisions

Michael Joswig
(joint work with Holger Eble, Lisa Lamberti, Will Ludington)

Genetic epistasis is a biological concept for an interaction between two genetic loci as the degree of non-additivity in their phenotypes. This idea goes back as far as 1909, when Bateson analyzed the landmark results by Mendel [1]. If there are more than two loci, things get considerably more complicated. Beerenwinkel, Pachter and Sturmfels proposed to read a suitable regular subdivisions of some convex polytope, called the *genotope*, as a *fitness landscape* [2]; see Figure 1 for an example. In their framework genetic populations which are fittest correspond to points in that polytope, and fitness is expressed in terms of linear programs. The monograph [3] is recommended for background on the relevant concepts from polyhedral geometry.

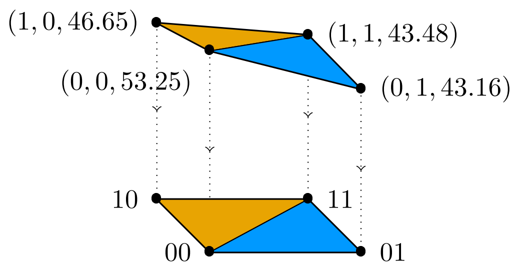

Figure 1. Biallelic genetic system with two loci. The genotope is the unit square $[0,1]$. The phenotype maps each vertex in $\{0,1\}^2$ to a real number; this induces a regular subdivision of the square.

Our contribution is a general method for processing such fitness landscapes, taking statistical aspects into account. For conciseness, we sketch the procedure for an n-biallelic system, where the genotype is the unit cube $[0,1]^n$. Our input are samples of measurements for each genotype, i.e., vertex in $\{0,1\}^n$, and we assume that this input is generic.

(1) Their average values are read as the phenotypes which give rise to a regular subdivision $\mathcal{S}$, which is computed via the convex hull. Due to genericity, $\mathcal{S}$ is a triangulation of $[0,1]^n$.
(2) Let Γ be the dual graph of $\mathcal{S}$. For each edge we compute an *epistatic weight*. Sorting these real numbers gives rise to a filtration of Γ into a sequence of subgraphs, the *epistatic filtration*.
(3) To take the empirical distribution of measurements for each genotype into account, we devised a one-sided significance test for each edge of Γ.
(4) The epistatic filtration with the epistatic weights and their significance form the output.

The theoretical underpinnings have been worked out in [4]. That reference also features a synthetic experiment to explain why our method works. In our new article we report on processing actual data sets from biology [5]. This includes the analysis of classical data, where we can confirm previous findings by other researchers. This also includes the analysis of one new data set, which was obtained in the lab of Will Ludington at Carnegie Science. Those data are concerned with the microbiome of *Drosophila*. We consider $n = 5$ different species of bacteria which may or may not exist in the gut of any fly. So the genotype is the unit cube $[0,1]^5$. It turns out that the fitness landscape for the lifespan of these flies changes dramatically when certain bacteria are there or not. In biological terms, our results suggest that the co-evolution in this experiment is considerably more complicated than in a simple antagonistic scenario.

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Some Stories of Geometry and Graphs

Nati Linial

(joint work with Daniel Cizma and Maria Chudnovsky)

A *consistent path system* in a graph G is an intersection-closed collection of paths, with exactly one path between any two vertices in G. We call G *metrizable* if every consistent path system in it is the system of geodesic paths defined by assigning some positive lengths to its edges. Our work shows that metrizable graphs are, in essence, subdivisions of a small family of basic graphs with additional *compliant edges*. In particular, every metrizable graph with 11 vertices or more is outerplanar plus one vertex.

Let $G = (V, E)$ be a connected graph, and let $w : E \to \mathbb{R}_{>0}$ be a positive weight function on its edges. This induces a metric on V, where the distance between any two vertices is the least w-length of a path between them. What can be said about such a system of geodesics? E.g., what does the collection of w-geodesics tell us about w? Is it possibly true that *every* collection of paths in a graph constitute the system of geodesics corresponding to some graph metric? To simplify matters, suppose that w is such that the shortest path between any two vertices is unique. Clearly, any subpath of a geodesic in G is itself a geodesic. This leads us to define the notion of a *consistent path system* $\mathcal{P}$ in G - a collection of paths that is closed under taking subpaths, with a unique uv path in $\mathcal{P}$ for each pair $u, v \in V$. So, we ask if every consistent path system coincides with the set of geodesics that corresponds to some positive weight function on the edges. Our first paper on this subject [1] showed that this is far from the truth, and that metrizable graphs are in fact quite rare. E.g., all large metrizable graphs are planar and not 3-connected. On the other hand, that paper also showed that all outerplanar graphs are metrizable. Still, [1] did not provide a satisfactory description of *metrizable* graphs, and in [3] we made further progress on this question.

Call a path in G *flat* if every internal vertex in it has degree 2 in G, and call an edge xy *compliant* if x and y are also connected by a flat path. We show in particular that every large 2-connected metrizable graph can be obtained starting from one of some basic graphs, and iteratively subdividing edges and adding a compliant edge between its end vertices. This, in particular, implies that every large metrizable graph can be made outerplanar by removing at most one vertex.

Here are some of the main ingredients of these studies.

Proposition 1 ([1]).

- *The family of metrizable graphs is closed under topological minors.*
- *If e is a compliant edge in G, then G is metrizable if and only if $G \setminus e$ is metrizable.*

Consider a consistent path system $\mathcal{P}$ in a graph $G = (V, E)$. Associated with $\mathcal{P}$ is a system of linear inequalities, and $\mathcal{P}$ is metric iff this system is feasible. So if the chosen $\mathcal{P}$ is non-metric, we can use LP-duality to create a *hand-checkable certificates* of this. Thus, using a computer, we created a "zoo" of 16 non-metrizable graphs along with such path systems and the corresponding certificates. The basic methodology developed in [1] is to prove that a graph at hand is non-metrizable by showing that it contains a subdivision of some graph from the zoo.

Theorem 2 ([3]). *If a 2-connected metrizable graph G with at least 11 vertices has no compliant edges, then it is either $K_{2,n}$ for some $n \geq 4$ or a subdivision of one of the following: $K_{2,3}$, K_4, W_4 or $K_5 \setminus e$.*

Consequently

Theorem 3 ([3]). *If a graph G with at least 11 vertices is (i) 2-connected, (ii) has no compliant edges, (iii) has at least two disjoint cycles, then G is non-metrizable.*

Corollary 4. *Every 2-connected metrizable graph with at least 11 vertices can be made outerplanar by removing at most one vertex.*

Many open questions are mentioned in [1, 3], e.g., the notion of *irreducible* path systems introduced in [2].

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A regular unimodular triangulation of the matroid base polytope

Gaku Liu

(joint work with Spencer Backman, Gaku Liu)

A lattice triangulation of a lattice polytope is *unimodular* if all of its simplices have minimal volume. A triangulation is *regular* if there is a convex, piecewise linear function whose regions of linearity are exactly given by the triangulation. We give the first construction of regular unimodular triangulations for matroid base polytopes. This construction extends to integral generalized permutahdera. Previously, it was not known whether matroid polytopes admitted covers by unimodular simplices.

The construction is motivated by a set of conjectures collectively known as *White's conjecture* in matroid theory. Given a matroid M with ground set E and set of bases $\mathcal{B}$, define the *toric ideal* of M to be the kernel of the $\mathbb{R}$-algebra homomorphism

$$\mathbb{R}[x_B : B \in \mathcal{B}] \to \mathbb{R}[x_e : e \in E]$$

sending x_B to $\prod_{e \in B} x_e$. The weakest version of White's conjecture states that the toric ideal of a matroid is generated by quadratic binomials. A stronger version of this conjecture is that the toric ideal of a matroid has a quadratic Gröbner basis. The latter conjecture is equivalent to the statement that the matroid base polytope has a flag, regular unimodular triangulation. (A triangulation is *flag* is its minimal non-faces have size 2.) We hope our construction may shed light on this conjecture and lead to future work in this direction.

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Polyhedral Geometry of ReLU Neural Networks

Georg Loho

(joint work with C. Haase, C. Hertrich; M. Brandenburg, G. Montúfar, H. Tseran)

We show new insights in the structure of ReLU Neural Networks based on polyhedral geometry. On one hand, we describe natural subdivisions of the space of piecewise-linear classifiers represented by a ReLU neural network. On the other hand, we show lower bounds on the number of layers for representing integral piecewise-linear functions. The advances involve (generalizations of) oriented matroids, Newton polytopes of tropical polynomials and the use of geometric invariants, in particular normalized volume of lattice polytopes.

First, we give an introduction of two basic concepts, linear classification and tropical rational functions. The geometric point of view on linear classification is captured by the oriented matroid of the hyperplane dual to the vector arrangement associated with data points. This idea is generalized to classifiers arising from continuous piecewise-linear functions. These are exactly the functions represented by ReLU Neural Networks, or equivalently, the functions represented by tropical rational functions (with real 'tropical' exponents). Grouping linear classifiers by the dichotomy imposed on the data points leads to a subdivision of their parameter spaces. This subdivision equals the normal fan of the zonotope given by the Minkowski sum of the line segments associated to the data points. We generalize this to the setting of tropical rational functions with a fixed number of terms in the numerator and denominator [3]. Here, subdividing by the classification pattern yields the normal fan of a sum of simplices, one for each data point.

Second, we look at the expressivity of ReLU neural networks depending on their depth. We recall the known duality between neural networks and Newton polytopes via tropical geometry [1]. Imposing an integrality assumption on the weights in the network implies that these Newton polytopes are lattice polytopes. Using a parity argument on the normalized volume of faces of such polytopes, we show that $\lceil \log_2(n) \rceil$ hidden layers are indeed necessary to compute the maximum of n numbers, matching known upper bounds. This implies that the set of functions representable by ReLU neural networks with integer weights strictly increases with the network depth while allowing arbitrary width [4].

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Random Balanced Cayley Complexes

Roy Meshulam

The Laplacian $L(\mathcal{C})$ of a graph $\mathcal{C} = (V, E)$ is the $V \times V$ positive semidefinite matrix whose (u, v) entry is given by

$$L(\mathcal{C})_{uv} = \begin{cases} \deg_C(u) & u = v, \\ -1 & \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $0 = \lambda_1(\mathcal{C}) \leq \lambda_2(\mathcal{C}) \leq \cdots \leq \lambda_{|V|}(\mathcal{C})$ be the eigenvalues of $L(\mathcal{C})$. The second smallest eigenvalue $\lambda_2(\mathcal{C})$, called the *spectral gap* of $\mathcal{C}$, is a parameter of central importance in a variety of problems. In particular it controls the expansion properties of $\mathcal{C}$ and the convergence rate of a random walk on $\mathcal{C}$ (see e.g., Chapters XIII and IX in [3]).

Let G be a finite group of order n and let $T \subset G$ be symmetric subset, i.e. $T = T^{-1}$. The *Cayley graph* $\mathcal{C}(G,T)$ of G with respect to T is the graph on the vertex set G with edge set $\{\{g, gt\} : g \in G, t \in T\}$. The seminal Alon-Roichman theorem [1] is concerned with the expansion of Cayley graphs with respect to random sets of generators.

Theorem 1 (Alon-Roichman). *For any $\epsilon > 0$ there exists a constant $c(\epsilon) > 0$ such that for any group G, if S is a random subset of G of size $\lceil c(\epsilon) \log |G| \rceil$ and $m = |S \cup S^{-1}|$, then $\lambda_2(\mathcal{C}\left(G, S \cup S^{-1}\right))$ is asymptotically almost surely (a.a.s.) at least $(1-\epsilon)m$.*

This talk is based on [4] and concerns higher dimensional counterparts of Theorem 1. We briefly recall some terminology. For a simplicial complex X and $k \geq -1$ let $X^{(k)}$ denote the k-dimensional skeleton of X. For $k \geq -1$ let $C^k(X)$ denote the space of real valued simplicial k-cochains of X and let $d_k : C^k(X) \to C^{k+1}(X)$ denote the coboundary operator. For $k \geq 0$ define the reduced k-th Laplacian of X by $L_k(X) = d_{k-1}d_{k-1}^* + d_k^* d_k$. The minimal eigenvalue of $L_k(X)$, denoted by $\mu_k(X)$, is the *k-th spectral gap* of X.

Let $k \geq 1$. For $1 \leq i \leq k+1$ let $V_i = \{i\} \times G$. Let $Y_{G,k}$ denote the simplicial join $V_1 * \cdots * V_{k+1}$, where each V_i is viewed as 0-dimensional complex. Thus $Y_{G,k}$ is homotopy equivalent to an N-fold wedge $\bigvee^N \mathrm{S}^k$ of k-dimensional spheres, where $N = (n-1)^{k+1}$. The *balanced k-dimensional Cayley Complex* associated with a subset $\emptyset \neq A \subset G$ is the subcomplex $Y_{A,k} \subset Y_{G,k}$ whose k-simplices are all $\{(1,y_1), \ldots, (k+1, y_{k+1})\} \in Y_{G,k}$ such that $y_1 \cdots y_{k+1} \in A$. Note that $Y_{A,k} \supset Y_{G,k}^{(k-1)}$.

Let 1_A denote the indicator function of $A \subset G$. Let $\widehat{G} = \{\rho\}$ be the set of irreducible unitary representations of G, where $\rho : G \to U(d_\rho)$. Let $D(G) = \sum_{\rho \in \widehat{G}} d_\rho$. Let $\mathbf{1} \in \widehat{\mathbf{G}}$ denote the trivial representation of G and let $\widehat{G}_+ = \widehat{G} \setminus \{\mathbf{1}\}$. For $\rho \in \widehat{G}$ let $\widehat{1_A}(\rho) = \sum_{x \in A} \rho(x) \in M_d(\mathbb{C})$ be the Fourier transform of 1_A at ρ. For a matrix $T \in M_d(\mathbb{C})$ let $\|T\| = \max_{\|v\|=1} \|Tv\|$ denote the spectral norm of T. Let $\nu(A) = \max_{\rho \in \widehat{G}_+} \|\widehat{1_A}(\rho)\|$. Our first result is a lower bound on $\mu_{k-1}(Y_{A,k})$ in terms of $\nu(A)$.

Theorem 2.

$$\mu_{k-1}(Y_{A,k}) \geq |A| - k \cdot \nu(A).$$

Our main result is the following k-dimensional analogue of the Alon-Roichman Theorem.

Theorem 3. *Let k and $\epsilon > 0$ be fixed. Let G be a finite group of order n and fix an integer m such that $\frac{9k^2 \log D(G)}{\epsilon^2} \leq m \leq \sqrt{n}$. Let A be a random subset of G of size m. Then*

$$\Pr\left[\ \mu_{k-1}(Y_{A,k}) < (1-\epsilon)m\ \right] < \frac{6}{n}.$$

Remark 4. *It is straightforward to check that* $\mu_{k-1}(Y_{A,k}) \leq |A|+k$ *for any* $A \subset G$ *(see Eq. (2) in [2]). Theorem 3 thus implies that if* A *is a random subset of* G *and* $\log|G| = o(|A|)$*, then* $Y_{A,k}$ *is a.a.s. a near optimal spectral expander.*

Our final result concerns the homotopy type of $Y_{A,k}$ when A is a subgroup of G. For $1 \leq m$ let

$$\gamma_0(m,k) = (n-m)n^k + \left(\frac{n}{m}\right)^k (m-1)^{k+1} - (n-1)^{k+1},$$
$$\gamma_1(m,k) = \left(\frac{n}{m}\right)^k (m-1)^{k+1}.$$

Theorem 5. *Let* A *be a subgroup of* G *of order* $|A| = m$*. Then*
(i)

$$Y_{A,1} \simeq \coprod^{n/m} \bigvee^{(m-1)^2} \mathrm{S}^1. \tag{1}$$

(ii) For $k \geq 2$

$$Y_{A,k} \simeq \bigvee^{\gamma_0(m,k)} \mathrm{S}^{k-1} \vee \bigvee^{\gamma_1(m,k)} \mathrm{S}^k. \tag{2}$$

Remark 6. *As* $\gamma_0(m,k) > 0$ *for all* $m < n$*, it follows from Theorem 5 that if* $A \subset G$ *generates a subgroup* $\langle A \rangle$ *of order* $m < n$ *then*

$$\tilde{\beta}_{k-1}(Y_{A,k}) \geq \tilde{\beta}_{k-1}(Y_{\langle A \rangle,k}) = \gamma_0(m,k) > 0$$

and therefore $\mu_{k-1}(Y_{A,k}) = 0$*. As there are families of groups* G *(e.g. elementary abelian groups of fixed exponent) that cannot be generated by* $o(\log|G|)$ *elements, this implies that the* $\log D(G) = \Theta(\log n)$ *factor in Theorem 3 cannot in general be improved.*

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Rigidity expander graphs

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(joint work with Alan Lew, Yoval Peled, Orit Raz)

Jordán and Tanigawa recently introduced the d-dimensional algebraic connecti vity $a_d(G)$ of a graph G. This is a quantitative measure of the d-dimensio nal rigidity of G which generalizes the well-studied notion of spectral expans ion of graphs. We present a new lower bound for $a_d(G)$ defined in terms of the spectral expansion of certain subgraphs of G associated with a partition of i ts vertices into d parts. In particular, we obtain a new sufficient condition for the rigidity of a graph G. As a first application, we prove the existence of an infinite family of k-regu lar d-rigidity-expander graphs for every $d \geq 2$ and $k \geq 2d+1$. Conjectural ly, no such family of $2d$-regular graphs exists. Second, we show that $a_d(K_n) \geq \frac{1}{2}\left\lfloor \frac{n}{d} \right\rfloor$, which we conjecture to be essentially tight. In addition, we study the extremal values $a_d(G)$ attains i f G is a minimally d-rigid graph.

Context. Graph expansion is one of the most influential concepts in mod ern graph theory, with numerous applications in discrete mathematics and compute r science (see [4, 7]). Intuitively speaking, an expander is a "highly-connected" graph, and a standard way to quantitatively measure the connectivity, or expansion, of a graph uses t he spectral gap in its Laplacian matrix. A main theme in the study of expander g raphs deals with the construction of sparse expanders. In particular, bounded-de gree regular expander graphs have been studied extensively in various areas of m athematics. This paper studies a generalization of spectral graph expansion that was recentl y introduced by Jordán and Tanigawa via the theory of graph rigidity [5].

A d-dimensional framework is a pair (G, p) consisting of a graph $G = (V, E)$ an d a map $p : V \to \mathbb{R}^d$. The framework is called d-rigid if every contin uous motion of the vertices starting from p that preserves the distance betwee n every two adjacent vertices in G, also preserves the distance between *every pair* of vertices; see e.g. [2, 3] for background on framework rigidity. Asimow and Roth showed in [1] that if the map p is generic (e.g. if the $d|V|$ coordinates of p are algebraically independent over the rationales), t hen the d-rigidity of (G, p) does not depend on the map p. Moreover, they s howed that for a generic p, rigidity coincides with the following stronger lin ear-algebraic notion of infinitesimal rigidity.

Definitions. For every $u, v \in V$ we define $d_{uv} \in \mathbb{R}^d$ by

$$d_{uv} = \begin{cases} \frac{p(u)-p(v)}{\|p(u)-p(v)\|} & \text{if } p(u) \neq p(v), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathbf{b}_{u,v}^T = \begin{bmatrix} 0 & \dots & 0 & \overset{u}{d_{uv}^T} & 0 & \dots & 0 & \overset{v}{d_{vu}^T} & 0 & \dots & 0 \end{bmatrix}.$$

The (normalized) *rigidity matrix* $R(G, p) \in \mathbb{R}^{d|V| \times |E|}$ is the matrix whose columns are the vectors $\mathbf{b}_{u,v}$ for all $\{u, v\} \in E$. $\mathbb{R}^d$. For p generic and $|V| \geq d+1$,

$\text{rank}(R(G,p)) \leq d|V| - \binom{d+1}{2}$; see [1]. The framework (G,p) is called *infinitesimally rigid* if this bound is attained, that is, if $\text{rank}(R(G,p)) = d|V| - \binom{d+1}{2}$.

A graph G is called *d-rigid*, if it is infinitesimally rigid with resp ect to some map p (or, equivalently, if it is infinitesimally rigid for all ge neric maps [1]).

For $d = 1$ and an injective map $p : V \to \mathbb{R}$, the rigidity matrix $R(G,p)$ is equal to the incidence matrix of G, hence both notions of rigidity coincid e with graph connectivity. One can extend this analogy and define a higher dimen sional version of the graph's Laplacian matrix, that is called the *stiffne ss matrix* of (G,p), and is defined by

$$L(G,p) = R(G,p)R(G,p)^T \in \mathbb{R}^{d|V|\times d|V|}.$$

We denote by $\lambda_i(A)$ the i-th smallest eigenvalue of a symmetric matrix A. Since $\text{rank}(L(G,p)) = \text{rank}(R(G,p)) \leq d|V| - \binom{d+1}{2}$, the kernel of $L(G,p)$ is of dimension at least $\binom{d+1}{2}$. Therefore, $\lambda_{\binom{d+1}{2}+1}(L(G,p)) \neq 0$ if and only if (G,p) is infinitesimally rigid.

In [5], Jordán and Tanigawa defined the *d-dimens ional algebraic connectivity* of G, $a_d(G)$, as

$$a_d(G) = \sup\left\{\lambda_{\binom{d+1}{2}+1}(L(G,p)) \,\middle|\, p : V \to \mathbb{R}^d\right\}.$$

For $d = 1$, $L(G,p)$ coincides with the graph Laplacian matrix $L(G)$, and $a_1(G) = a(G)$ is the usual algebraic connectivity, or Laplacian spectral gap, of G. For every $d \geq 1$, $a_d(G) \geq 0$ since $L(G,p)$ is positive semi-definite, and $a_d(G) > 0$ if and only if G is d-rigid.

The following notion of *rigidity expander graphs* extends the classical no tion of (spectral) expander graphs, corresponding to the $d = 1$ case: Let $d \geq 1$. A family of graphs $\{G_i\}_{i\in\mathbb{N}}$ of increasing size is called a *family of d-rigidity expander graphs* if there exists $\epsilon > 0$ such that $a_d(G_i) \geq \epsilon$ for all $i \in \mathbb{N}$.

Results. It is well known that, for every $k \geq 3$, there exist families of k-regular ($d = 1$-rigid) expander graphs (see e.g. [4]). Our main result is an extension of this fact to general d:

Theorem 1. *Let $d \geq 1$ and $k \geq 2d + 1$. Then, there exists an infinite family of k-regular d-rigidity expander graphs.*

It was conjectured by Jordán and Tanigawa that families of $2d$-regular d-ri gidity expanders do not exist (see [5, Conj. 2] for the statement in the $d = 2$ case, and see [6, Conj. 6.2] for the general case), and clearly families of k-regular d-rigidity expanders do not exist for $k < 2d$ since, for n large e nough, such graphs have less than $dn - \binom{d+1}{2}$ edges, and are therefore n ot even d-rigid. Therefore, assuming this conjecture, our result is sharp.

Our main tool for the proof of Theorem 1 is a new low er bound on $a_d(G)$, given in terms of the (1-dimensional) algebraic connecti vity of certain subgraphs of G associated with a partition of its vertex set i nto d parts. For convenience, we let $a(G) = \infty$ if G consists of a single vertex.

Let $G = (V, E)$ be a graph, and let $A, B \subset V$ be two disjoint sets. We denote by $G[A]$ the subgraph of G induced on A, and by $G(A, B)$ the subgraph of G with vertex set $A \cup B$ and edge set $E(A, B) = \{e \in E : |e \cap A| = |e \cap B| = 1\}$. Recall that a *partition* of V is a set $\{A_1, \dots, A_d\}$ of n on-empty subsets of V such that $V = A_1 \cup \cdots \cup A_d$ is a disjoint union .

Theorem 2. *Let $d \geq 2$. For every graph $G = (V, E)$ and a partition $\{A_1, \dots, A_d\}$ o f V there holds*

$$a_d(G) \geq \min\left(\left\{a(G[A_i])\right\}_{1\leq i\leq d} \bigcup \left\{\frac{1}{2}a(G(A_i, A_j))\right\}_{1\leq i<j\leq d}\right).$$

In particular, if $G[A_i]$ is connected for all $i \in [d]$ and $G(A_i, A_j)$ is co nnected for all $1 \leq i < j \leq d$, then G is d-rigid.

Remark 3. *In the $d = 2$ dimensional case, the statement in Theorem 2 can be slightly improved (by removing the constant $1/2$) to*

$$a_2(G) \geq \min\{a(G[A_1]), a(G[A_2]), a(G(A_1, A_2))\},$$

for every partition A_1, A_2 of V.

For another application of Theorem 2, we derive a slight improvement of the previously known lower bound for $a_d(K_n)$ from [6, Theorem 1.5].

Corollary 4. *Let $d \geq 3$ and $n \geq d + 1$. Then*

$$a_d(K_n) \geq \frac{1}{2}\left\lfloor \frac{n}{d} \right\rfloor.$$

Conjecturally, under these conditions, $a_d(K_n) \leq \frac{n}{d}$. The upper boun d given in [Thm.1.6][6] is $a_d(K_n) \leq \frac{2n}{3(d-1)} + \frac{1}{3}$.

Problems and comments. Many parallels to classical graph expansion are sill missing: find optimal d-rigidity expander graphs. For $d = 1$ these were co nstructed, known as Ramanujan graphs. The Alon-Boppana bound is valid also for $a_d(G)$, as we prove that:

Theorem 5. *Let $d \geq 2$, and let G be a graph. Then,*

$$a_d(G) \leq a(G).$$

Jordán and Tanigawa [5, Theorem 4.2] proved Theorem 5 for $d = 2$, and in [8] it was proved indep endently for all d. Our proof is different, using the probabilistic method, an d we believe it to be of independent interest.

Regarding the Cheeger inequality, it remains a challenge to find a lower bound o n $a_d(G)$ in terms of combinatorial invariants of G.

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Framed polytopes and higher cellular strings

Arnau Padrol
(joint work with Guillaume Laplante-Anfossi and Anibal M. Medina-Mardones)

Higher categories offer a framework for systematizing complex hierarchies in mathematics, physics, and computer science. To illustrate this, we mention Grothendieck's homotopy hypothesis in topology, Baez–Dolan's cobordism hypothesis in quantum field theory, and the extensive applications of higher category theory in computer science, particularly in language semantics, concurrency calculus, and type theory.

Polytopes in higher category theory were first introduced to organize coherence relations. Kapranov and Voevodsky significantly expanded the connection between convex geometry and higher category theory announcing several intriguing results in [1], including the following insightful idea. Consider a convex d-polytope $P \subseteq \mathbb{R}^d$ and a generic ordered basis B of $\mathbb{R}^d$, which we refer to as a *frame*. Using the frame we define, for each face F, two distinct subsets of its k-faces: its k*-source* $\mathrm{s}_k(F)$ and k*-target* $\mathrm{t}_k(F)$. Kapranov and Voevodsky conjectured [1, Thm. 2.3] that the data consisting of all sources and targets, referred to as the *globular structure* of (P, B), defines a d-dimensional *pasting diagram*, a special and important type of d-dimensional categories. Using ideas of Steiner [3], we show that this claim holds if and only if the framed polytope has no cellular loops, a notion we now define. A *cellular* k*-string* in a framed polytope is a sequence $F_1, \ldots, F_\ell$ of faces such that two consecutive faces F_i and F_{i+1} share a k-face G with $\mathrm{t}_k(F_i) \cap \mathrm{s}_k(F_{i+1}) = G$. We say it is a *cellular loop* if and $F_i = F_j$ for some $i \neq j$.

The first contribution we discuss are counterexamples to [1, Thm. 2.3]. More precisely, we provide examples showing the following.

Theorem 1. *Starting in dimension 4 there exist framed polytopes with cellular loops.*

We also considered whether the following weaker version of their claim could be true: For any polytope there is a frame making it into a pasting diagram. However, this weaker version also fails since we provide a construction establishing the following.

Theorem 2. *Starting in dimension 4 there exist polytopes for which all frames lead to cellular loops.*

An important infinite family of framed polytopes, which was studied by Kapranov and Voevodsky, is given by the *canonically framed cyclic simplices* $(C(d), \{e_k\})$, where $\{e_k\}$ is the canonical frame of $\mathbb{R}^d$ and $C(d)$ is the convex closure of $d+1$ distinct points in the *moment curve* $t \mapsto (t, t^2, \dots, t^d)$. In an insightful observation [1, Thm. 2.5], they announced that $(C(d), \{e_k\})$ has no cellular loops and recover Street's free d-category on the d-simplex, a fundamental object in higher category theory [4]. We were able to verify this claim after replacing the canonical frame by $\{e_1, -e_2, e_3, -e_4, \dots\}$.

These framed simplices are rare and special in the following probabilistic sense. A *Gaussian d-simplex* is the convex hull of $d+1$ independent random points in $\mathbb{R}^d$, each chosen according to a d-dimensional standard normal distribution. We prove the following.

Theorem 3. *The probability that a canonically framed Gaussian d-simplex has a cellular loop tends to* 1 *as d tends to* ∞.

We next turn our attention to the moduli of frames of a simplex Δ_d under the equivalence relation induced by globular structures. Our aim is to quantify the complexity of the *realization space* of a globular structure on Δ_d, that is, the set of all frames of Δ_d inducing it. Using a celebrated result of N. E. Mnëv [2], we show the following.

Theorem 4. *For every open primary basic semi-algebraic set S defined over $\mathbb{Z}$ there is a globular structure on some simplex Δ_d whose realization space is stably equivalent to S.*

A key step in the proof of this result is the following theorem, which we consider noteworthy in its own right.

Theorem 5. *Globular structures of framed simplices are in bijection with uniform acyclic realizable full flag chirotopes.*

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Lefschetz properties via anisotropy for simplicial spheres and cycles

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(joint work with Karim Adiprasito)

1. Introduction

An important recent breakthrough in Discrete Geometry was the 2018 proof of McMullen's g-conjecture for simplicial spheres by Karim Adiprasito [1]. Two years later, the paper [3] appeared, which gave a substantially different second proof of the conjecture based on the notion of generic anisotropy and certain characteristic 2 differential identities. Finally, the 2021 paper [2] proved Lefschetz type properties in the setting of pseudomanifolds and simplicial cycles and gave an application to 2-Cohen-Macaulay simplicial complexes.

2. Generic Artinian Reduction

Assume $m \geq 1$ and k is a field. We consider the polynomial ring $k[x_1, \dots, x_m]$, where the degree of the variable x_i is equal to 1, for all $1 \leq i \leq m$. Assume $I \subset k[x_1, \dots, x_m]$ is a homogeneous ideal. We denote by d the Krull dimension of the quotient ring $k[x_1, \dots, x_m]/I$. We assume $d \geq 1$, and denote by E the field of fractions of the polynomial ring

$$k[a_{i,j} : 1 \leq i \leq d,\ 1 \leq j \leq m].$$

For $1 \leq i \leq d$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j.$$

Definition 1. We define the generic Artinian reduction of $k[x_1, \dots, x_m]/I$ to be the Artinian E-algebra

$$E[x_1, \dots, x_m]/((I) + (f_1, \dots, f_d)),$$

where (I) denotes the ideal of $E[x_1, \dots, x_m]$ generated by I.

3. Generic Anisotropy of simplicial spheres

Assume k is a field and D is a simplicial sphere of dimension $d-1$ with vertex set $\{1, \dots, m\}$. We denote by $k[D] = k[x_1, \dots, x_m]/I_D$ the Stanley-Reisner ring of D over k and by A the generic Artinian reduction of $k[D]$ defined above. We remark that A is an Artinian Gorenstein standard graded E-algebra with socle degree d, where E as above.

Definition 2. We call D generically anisotropic over k, if for all integers j with $1 \leq 2j \leq d$ and all nonzero elements $u \in A_j$ we have $u^2 \neq 0$.

Three of the main results of [3] are the following:

Theorem 3 ([3]). Assume that k is any field and D is a simplicial sphere of dimension 1. Then D is generically anisotropic over k.

Theorem 4 ([3]). Assume that k is any field of characteristic 2 and D is any simplicial sphere. Then D is generically anisotropic over k.

Remark 5. It is easy to see that by clearing denominators the previous theorem implies that any simplicial sphere D is generically anisotropic over the field of rationals $\mathbb{Q}$.

Theorem 6 ([3]). Assume k is any field and D is a simplicial sphere.

(1) If the suspension $S(D)$ of D is generically anisotropic over k, then $E[D]$ has the Weak Lefschetz property.
(2) If both D and the suspension $S(D)$ of D are generically anisotropic over k, then $E[D]$ has the Strong Lefschetz property.

Question 7. Is any simplicial sphere generically anisotropic over any field?

Question 8. Identify classes of Gorenstein standard graded algebras which have the generic anisotropy property.

4. Lefschetz properties for cycles

As mentioned above, the paper [2] contains Lefschetz type theorems for pseudomanifolds and simplicial cycles. An interesting application of them is the following:

A simplicial complex D of dimension $d-1$ is called 2-Cohen-Macaulay over the field k if $k[D]$ is Cohen-Macaulay, and for any vertex v of D the following hold for the simplicial complex

$$C = D \setminus \{v\}.$$

It has dimension $d-1$ and the Stanley-Reisner ring $k[C]$ is Cohen-Macaulay.

Theorem 9 ([2]). Assume D is a 2-Cohen-Macaulay simplicial complex of dimension $d-1$ over an infinite field k and denote by A a sufficiently general Artinian reduction of $k[D]$. Then, there exists $\omega \in A_1$ such that the multiplication by ω^{d-2i} from A_i to A_{d-i} is injective for all $0 \leq i \leq d/2$.

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Acyclonestohedra

Vincent Pilaud
(joint work with Chiara Mantovani and Arnau Padrol)

We use classical terminology on building sets and their nested complexes [FK04, FS05, Pos09] and on oriented matroids [BLS$^+$99].

Motivated by recent work of P. Galashin on poset associahedra [Gal21], we consider the acyclic part of a given nested complex with respect to a given oriented matroid in the following sense.

Definition 1. *An* ***oriented building set*** *on a ground set S is a pair $(\mathcal{B}, \mathcal{M})$ where $\mathcal{B}$ is a building set on S and $\mathcal{M}$ is an oriented matroid on S such that $\mathcal{B}$ contains the support of any circuit of $\mathcal{M}$.*

Definition 2. *A nested set $\mathcal{N}$ on $\mathcal{B}$ is* ***acyclic*** *if $\mathcal{M}_{/\bigcup \mathcal{N}'}$ is acyclic for any $\mathcal{N}' \subseteq \mathcal{N}$. The* ***acyclic nested complex*** *$\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is the simplicial complex of acyclic nested sets on $\mathcal{B}$.*

Our main results concern realizations (as boundary complexes of oriented matroids or polytopes) of these acyclic nested complexes.

Theorem 3. *The acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ of any oriented building set $(\mathcal{B}, \mathcal{M})$ is the boundary complex of the positive tope of an oriented matroid obtained by stellar subdivisions of $\mathcal{M}$.*

Theorem 4. *The acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ of any* ***realizable*** *oriented building set $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ is the boundary complex of the* ***acyclonestohedron****, a polytope obtained as the section of a nestohedron of $\mathcal{B}$ with the evaluation space of the vector configuration $\boldsymbol{A}$.*

Our original motivation was the following graphical situation.

Definition 5. *The* ***graphical oriented building set*** *of a directed graph D with edge set S is given by*

- *the graphical building set of the line graph of D, and*
- *the graphical oriented matroid of D.*

Proposition 6. *The acyclic nested complex of the graphical oriented building set of D is isomorphic to the* ***piping complex*** *of the transitive closure of D, defined by P. Galashin in its his work on poset associahedra* [Gal21].

Corollary 7. *The piping complex of a poset P is isomorphic to the boundary complex of the* ***graphical acyclonestohedron****, obtained as a section of a graph associahedron of the line graph of the Hasse diagram of P.*

This corollary is illustrated in Figure 1 and answers a question open by P. Galashin in [Gal21], and independently settled by A. Sack in [Sac23] with a more specific method.

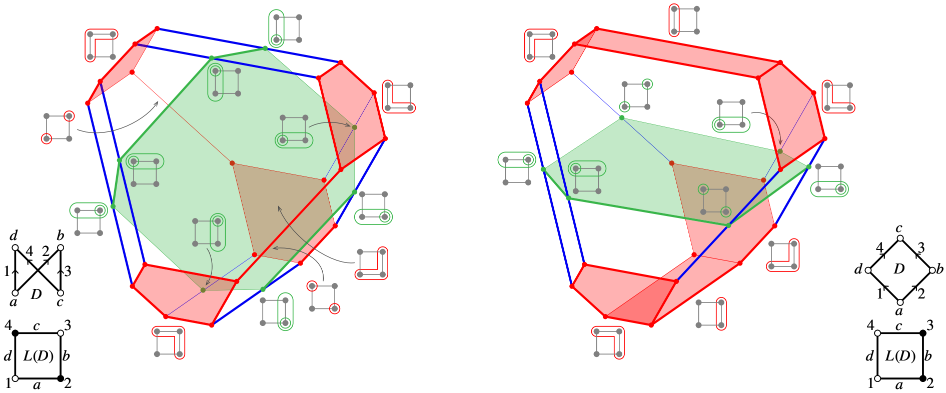

FIGURE 1. The poset associahedra of two posets, obtained as sections of the graph associahedra of their line graphs with their cycle spaces.

Finally, we show that acyclic nested complexes of oriented building sets essentially correspond to $\mathcal{F}(\mathcal{M})$-nested complexes of $\mathcal{F}(\mathcal{M})$-building sets in the sense of E.-M. Feichtner and D. Kozlov [FK04], where $\mathcal{F}(\mathcal{M})$ is the Las Vergnas face lattice of the oriented matroid $\mathcal{M}$.

We use this observation for two further applications:

- type B nestohedra, starting from the oriented matroid whose positive tope is a cross-polytope,
- iterated nestohedra, recovering in particular the permuto-permutahedra.

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Fixed points on contractible spaces

Kevin I. Piterman

For a group G and a G-contractible G-complex X, both X and its fixed point set X^G are contractible. In particular, there is a fixed point by the action of G. We ask then for suitable topological conditions that also imply the existence of a fixed point in a wide family of spaces and groups.

For example, by Smith theory, a finite p-group acting on a mod p acyclic finite-dimensional *regular* simplicial complex has a fixed point. We also know by Brouwer's fixed point theorem that a cyclic group acting on a disc has a fixed point. Hence, p-groups and cyclic groups always act with fixed points on discs. In the seventies, B. Oliver classified the finite groups that can act without fixed points on discs. In fact, he showed that a finite group G acts without fixed points on a disc if and only if G does not contain subgroups $P \leq H \leq G$ such P is a p-group normal in H, H/P is cyclic, and H is a normal subgroup of G such that G/H is a q-group, where p, q are some primes.

On the other hand, a famous theorem by J.P. Serre in the eighties states that a finite group acting on a tree has a fixed point. However, it is known that finite groups can act without fixed points on contractible complexes of dimension at least 3. The first example of this nature was constructed by E. Floyd and R.W. Richardson [5]. That is, there is an action of the alternating group A_5 on the 2-skeleton X_P of the Poincaré homology 3-sphere, and X_P is acyclic and fixed point free. Then the join $X = A_5 * X_P$ is a 3-dimensional compact and contractible complex with $X^{A_5} = \emptyset$. In dimension 2, it was conjectured by C. Casacuberta and W. Dicks that a finite group acting on a contractible 2-complex has a fixed point, and they proved this for solvable groups by using Smith theory [4]. Independently and at the same time, M. Aschbacher and Y. Segev raised this question but only for compact complexes [2]. Moreover, they proved that if a finite group G acts without fixed points on a compact acyclic 2-complex then G has a composition factor isomorphic to the Janko group J_1 or to one of the simple group of Lie type and Lie rank 1. In 2002, B. Oliver and Y. Segev achieved substantial progress on this problem by classifying finite groups acting without fixed points on finite acyclic 2-complexes [6]. One of their main theorems states that a finite group G has an *essential* action without fixed points on a finite acyclic 2-complex if and only if G is one of the simple groups $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(2^{2k+1})$, with some restrictions on q and k. We refer to the beautiful exposition by A. Ádem [1] for more details on these theorems.

In this talk, we review some of these results on fixed points. We also take a look at the case of finite T_0 topological spaces, where a result by R.E. Stong shows that a contractible finite T_0-space always has a fixed point. This relates to a conjecture raised by D. Quillen [10]: for a fixed prime p and a finite group G, the poset of nontrivial p-subgroups is contractible if and only if it is contractible as a finite space. This conjecture remains open, and we briefly comment on recent developments [3, 7, 9, 10]. We also present a sketch of the proof of the Casacuberta-Dicks conjecture for compact complexes, a joint work with Sadofschi Costa [8, 11].

This work is based on a previous article [12] which establishes the case $G = A_5$ of this conjecture. In this work, Sadofschi Costa reduced the study of the conjecture to the simple groups G in the theorem of Oliver-Segev (namely, the 2-dimensional finite linear groups and Suzuki groups), and also to a very particular family of 2-complexes related to the examples constructed in [6]. Once we have these reductions for the group G and a fixed point free finite acyclic 2-dimensional G-complex X, we construct a manifold M encoding representations of the group extension $\pi_1(X) : G$, obtained by lifting the maps $g \in G$ to the universal cover of X. For the rest of the proof, we show that there is a differential map $f : M \to N$ between orientable connected and compact manifolds of the same dimension and conclude by a degree argument that at least one of the points in a preimage $f^{-1}(x_0)$, for a particular point $x_0 \in N$, must correspond to a representation of $\pi_1(X) : G$ that does not factor through G. This implies that $\pi_1(X)$ is nontrivial, that is, X is not contractible.

Finally, we mention that the non-compact case of the Casacuberta-Dicks conjecture remains open.

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Stirling numbers and Koszul algebras with symmetry

Victor Reiner
(joint work with Ayah Almousa, Sheila Sundaram)

Stirling numbers $c(n,k), S(n,k)$ of the first and second kind give the answers to two basic counting problems:

- How many permutations of $\{1,2,...,n\}$ have k cycles?
- How many set partitions of $\{1,2,...,n\}$ have k blocks?

Although they have no simple product formulas, they do have triangular recursions

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k), \tag{1}$$

$$S(n,k) = S(n-1,k-1) + k\cdot c(n-1,k), \tag{2}$$

and closely related generating functions

$$\sum_{k=0}^{n-1} c(n,n-i)t^i = (1+t)(1+2t)\cdots(1+(n-1)t), \tag{3}$$

$$\sum_{k=0}^{\infty} S((n-1)+i,n-1)t^i = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}. \tag{4}$$

We re-interpret $c(n,k)S(n,k)$ as Hilbert functions for certain well-studied *Koszul algebras* A and their less-studied *Koszul duals* $A^!$, in the sense of Priddy [4].

The algebras A are the cohomology rings H^*X for the configuration space

$$X = \mathrm{Conf}_n(\mathbb{R}^d) = \{(x_1,\ldots,x_n) \in \mathbb{R}^d)^n : x_i \neq x_j \text{ for } 1 \leq i < j \leq n\}$$

of n labeled points in $\mathbb{R}^d$, where $d = 2,3,4,5,\ldots$. For $d = 2,4,6,\ldots$ even, this cohomology algebra A is isomorphic to the usual *Orlik-Solomon algebra* of the type A_{n-1} reflection hyperplane arrangement, also known as the *braid arrangement.* For $d = 3,5,7,\ldots$ odd, A is isomorphic to the *associated graded Varchenko-Gelfand ring* of the same hyperplane arrangement. Both rings have simple presentations, either as quotients of an exterior algebra or a commutative polynomial algebra on generators $\{x_{ij}\}_{1\leq i<j\leq n}$, with simple quadratic relations found by V.I Arnold (for $d = 2$) and F. Cohen (for general $d \geq 2$).

These quadratic presentations actually form quadratic Groebner bases for the defining ideals, showing that these algebra A are Koszul, and that the Hilbert series $\mathrm{Hilb}(A,t)$ is given by the generating function in (3). This implies also that their Koszul dual algebras $A^!$ have Hilbert series $\mathrm{Hilb}(A^!,t)$ given by the generating function in (4), related by

$$\mathrm{Hilb}(A^!,t) = \frac{1}{\mathrm{Hilb}(A,-t)}. \tag{5}$$

It is also known that $A^!$ is the homology ring $H_*(\Omega X)$ of the loop space ΩX.

All of these algebra $A, A^!$ carry actions of the symmetric group $\mathfrak{S}_n$ via graded automorphisms. We are interested in the describing and decomposing the actions on each graded component $A_i, A^!_i$, or equivarlant versions of the above Hilbert series. For the original algebras A, good descriptions of the $\mathfrak{S}_n$-characters on A_i

in terms of generating functions are known via work of Sundaram and Welker [5]. The characters of the Koszul duals $\{A_i^!\}$ can be computed recursively in terms of the $\{A_i\}$, but we currently lack simple generating function descriptions for $\{A_i^!\}$.

Nevertheless, they enjoy nice properties, considered as families $\{A(n)\}, \{A(n)^!\}$ depending on n. For example, there are branching rules that restrict $A(n)$ or $A(n)^!$ from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$, giving representaiton-theoretic lifts of the recursions (1), (2). As another example, when one fixes some $i = 0, 1, 2, , \ldots$, the sequences of $\mathfrak{S}_n$-representations $\{A(n)_i\}, \{A^!(n)_i\}$ both turn out to be *representation stable* in the sense of Church and Farb [2].

Many of their properties and the results come from general facts about Koszul algebras, and generalize from the type A reflection arrangement to all *supersolvable* hyperplane arrangements.

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Tropical Ideals

Felipe Rincón

Tropical ideals are combinatorial objects introduced in [3] with the aim of giving tropical geometry a solid algebraic foundation. They can be thought of as combinatorial generalizations of the possible collections of subsets arising as the supports of all polynomials in an ideal. In general, their structure is dictated by an infinite sequence of ‘compatible’ matroids. In this talk I will introduce and motivate the notion of tropical ideals, and I will discuss work over the last decade studying some of their main algebraic properties, the structure of their associated varieties, and the tropical Nullstellensatz.

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From pivot rules to colliding particles

Raman Sanyal
(joint work with A. Benjes, A. Black, J. De Loera, N. Lütjeharms, and G. Poullot)

Geometrically, a linear program can be viewed as a convex polytope $P \subset \mathbb{R}^d$ together with a unique-sink orientation of its graph that induced by a linear function $x \mapsto \langle c, x\rangle$ for some fixed $c \in \mathbb{R}^d$. For a given starting vertex $v \in V(P)$, the simplex algorithm follows a directed path from v to the unique sink v_{opt}. Which path is taken is dictated by the pivot rule adopted by the simplex algorithm. A pivot rule is *memory-less*, if it chooses the next vertex on the path utilizing only v and its c-improving neighbors $N_+(v) \subset V(P)$. The behaviour of a memory-less pivot rule is completely determined by an *arborescence* (or rooted tree), that is, a map $A : V(P) \to V(P)$ with $A(v_{\text{opt}}) = v_{\text{opt}}$ and $A(v) \in N_+(v)$ for $v \neq v_{\text{opt}}$.

In [2] we introduced the *max-slope* pivot rule that for a given generic $\omega \in \mathbb{R}^d$ corresponds to the arborescence

$$A^\omega(v) \;=\; \operatorname{argmax}\left\{ \frac{\langle \omega, u-v\rangle}{\langle c, u-v\rangle} : u \in N_+(v) \right\} \tag{1}$$

for $v \neq v_{\text{opt}}$. The max-slope pivot rule generalizes the well-known *shadow vertex* simplex algorithm: if r is the vertex of P that maximizes ω, then $r, A^\omega(r)$, $(A^\omega)^2(r), \dots, v_{\text{opt}}$ is precisely the shadow path associated to ω. It is straightforward to see that the collection of ω that give rise to the same max-slope arborescence form an open polyhedral cone and the closures of these cones yield a complete fan in $\mathbb{R}^d$.

In [2] we associate to every arborescence A of the linear program (P, c) a point $\psi(A) \in \mathbb{R}^d$ and define the *max-slope pivot rule polytope* $\Pi(P, c)$ as the convex hull of these points for all A.

Theorem 1 ([2])**.** *$\Pi(P, c)$ is a polytope of dimension $\dim P - 1$ with the following property: for any generic $\omega \in \mathbb{R}^d$, $\psi(A^\omega)$ is the unique maximizer of ω over $\Pi(P, c)$.*

Our pivot rule polytopes are related to certain fiber polytopes [1]: the *monotone path polytope* $\Sigma(P, c)$ that parametrizes coherent monotone paths on (P, c) is a weak Minkowski summand of $\Pi(P, c)$. Note that the construction of pivot rule polytopes works for the more general class of *normalized weight* pivot rules as explained in [2].

While our constructions where motivated by studying 'spaces of pivot rules', it turns out that max-slope pivot rule polytopes have fascinating and surprising applications to geometric combinatorics.

Let $\Delta_n \subset \mathbb{R}^{n+1}$ be the standard n-simplex with vertices $e_1, \dots, e_{n+1}$ equipped with a generic objective function $c = (c_1 < c_2 < \cdots < c_{n+1})$. An arborescence can be viewed as a map $A : [n+1] \to [n+1]$ with $A(n+1) = n+1$ and $A(i) > i$ for $i < n+1$. Of the $n!$ many arborescences, it turns out that exactly C_n are max-slope arborescences, where C_n is the n-th Catalan number. The max-slope arborescences can be characterized as the *non-crossing* arborescences, where a *crossing* is a pair $i, j \in [n+1]$ with $i < j < A(i) < A(j)$. It is not too difficult to see that non-crossing arborescences satisfy the same recurrence as the Catalan numbers. Stasheff's *associahedron* Asso_{n-1}, which is the poset of partial parenthesizations of a product of $n+1$ letters and which is the face poset of a $(n-1)$-dimensional polytope, famously embodies the Catalan numbers.

Theorem 2 (Black, Lütjeharms, Sanyal'23+)**.** *If P is an n-simplex and c a generic objective function, then $\Pi(P,c)$ is combinatorially isomorphic to* Asso_{n-1}.

This result is quite fascinating in that simplices are trivial from an optimization point of view. However, the simplex method is a sophisticated algorithm that can exhibit complex behaviour even on trivial instances.

If ρ is a linear projection for which (P, c) and $(P' := \rho(P), c' := \rho(c))$ have the same directed graph, then $\Pi(P', c') = \rho(\Pi(P, c))$. Thus, if P is a simplex and P' has a complete graph, then $\Pi(P', c')$ is a projection of an associahedron. A particularly nice case is when $P = \mathrm{Cyc}_n(t_1, \ldots, t_{n+1})$ is a cyclic polytope, $c = e_1$, and ρ is the projection onto the first d coordinates. The projection P' is again a cyclic polytope and $\Pi(P', e_1)$ is a (generic) projection of an associahedron parametrized by $t_1, \ldots, t_{n+1}$. In joint work with Aenne Benjes and Germain Poullot, we are currently investigating these polytopes that we call *cyclic associahedra*.

If $P = \mathrm{prism}(\Delta_n) = \Delta_n \times \Delta_1$ is the prism over the simplex and c is a generic objective function, then $\Pi(P, c)$ turns out to be combinatorially isomorphic to the *multiplihedron* Mul_n. The multiplihedron was also described by Stasheff. It encodes the evaluations of $f(a_1 a_2 \cdots a_{n+1})$, where f is a morphism between two non-associative structures. For example for $n = 1$, Mul_n is a segment with endpoints labelled by $f(a_1 a_2)$ and $f(a_1)f(a_2)$. For $n = 2$, Mul_n is a hexagon whose vertices are labelled by the evaluations of $f(a_1 a_2 a_3)$. A generalization to more morphisms was introduced by Chapoton and Pilaud [4] under the name (n, k)-multiplihedron.

Theorem 3 (Black, Lütjeharms, Sanyal'23+)**.** *If $P = \Delta_n \times \Delta_1^k$ is the k-fold prism over Δ_n and c is a generic objective function, then $\Pi(P, c)$ is combinatorially isomorphic to the (n, k)-multiplihedron.*

Finally we consider products of simplices $P = \Delta_m \times \Delta_n$.

Theorem 4 (Black, Lütjeharms, Sanyal'23+)**.** *For $m, n \geq 1$, $\Pi(\Delta_m \times \Delta_n, c)$ is combinatorially isomorphic to the (m, n)-constrainahedron.*

The (m, n)-constrainahedron was introduced by Bottman and Poliakova [3] to capture the collisions of mn particles that sit at the intersections of m horizontal and n vertical lines in the plane. The $(1, n)$-constrainahedra are associahedra, the $(2, n)$-constrainahedra are multiplihedra.

In order to show the stated combinatorial isomorphism to the associahedron, we make a connection between max-slope arborescences and particles with locations and velocities. We consider n particles at locations $-\omega_1 \leq -\omega_2 \leq \cdots \leq -\omega_n$ at time $t = 0$. For $t > 0$, the particles travel at constant velocities $-c$, where

$0 < c_1 < \cdots < c_n$. In this model one can interpret $A^\omega(i)$ as the *earliest* particle that will collide (and then absorb) with particle i.

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Hardness of linearly ordered 4-coloring of 3-colorable 3-uniform hypergraphs

Uli Wagner

(joint work with Marek Filakovský, Tamio-Vesa Nakajima, Jakub Opršal, and Gianluca Tasinato)

Deciding whether a given finite graph is 3-colorable (or, more generally, k-colorable, for a fixed $k \geq 3$) was one of the first problems shown to be NP-complete [7]. Since then, the complexity of *approximating* the chromatic number of a graph has been studied extensively, and it is known that, in general, the chromatic number cannot be approximated in polynomial time within a factor of $n^{1-\varepsilon}$, for any fixed $\varepsilon > 0$, unless P = NP [14]. However, this hardness result only applies to graphs whose chromatic number grows with the number of vertices, and the case of graphs with *bounded* chromatic number is much less well understood. The *approximate graph coloring problem* concerns the computational complexity of the following problem: Given an input graph that is either k-colorable or not ℓ-colorable, for some integers $\ell \geq k \geq 3$, how hard is it to distinguish between the two cases? Khanna, Linial, and Safra [8] showed that this problem is NP-hard for $(k, \ell) = (3, 4)$, and it is a long-standing conjecture that the problem is NP-hard[1] for all constants $\ell \geq k \geq 3$; to date, this is known for $\ell \leq 2k - 1$ for all $k \geq 3$ [3], and for $k \geq 6$, the bound on ℓ has been improved to $\ell \leq \binom{k}{\lfloor k/2 \rfloor}$ [13].

For hypergraphs, it is known [5] that given a c-uniform hypergraph that is either k-colorable or not ℓ-colordable, it is NP-hard to distinguish between the two cases, for all constants $c \geq 3$ and $\ell \geq k \geq 2$. Here, we consider the following variant of hypergraph coloring, focusing on 3-uniform hypergraphs. A *linearly ordered k-coloring* ($\mathbf{LO}_k$*-coloring*, for short) of a (3-uniform) hypergraph H is an assignment of elements ("colors") in $[k] = \{1, \dots, k\}$ to the vertices of H such that, for every hyperedge, the maximal color assigned to elements of that hyperedge occurs exactly once in the hyperedge. Linearly ordered hypergraph coloring

[1] There are conditional hardness results (assuming different variants of Khot's *Unique Games Conjecture*) for approximate graph coloring for all $\ell \geq k \geq 3$, see [4].

generalizes both classical graph coloring and certain versions of the boolean satisfiability problem and has recently received a lot of attention [1, 11, 12]. Our main result (see [6] for more details) is the following:

Theorem 1. *The following problem is* NP*-hard: Given a* 3*-uniform hypergraph* H*, distinguish between the case that* H *is* $\mathbf{LO}_3$*-colorable and the case that* H *is not* $\mathbf{LO}_4$*-colorable.*

More generally, it is conjectured [1, Conjecture 27] that distinguishing between $\mathbf{LO}_k$-colorable hypergraphs and not $\mathbf{LO}_\ell$-colorable hypergraphs is NP-hard for all constants $\ell \geq k \geq 2$. (We remark that, for $k \geq 4$, an easy reduction shows that this conjecture is true whenever the approximate graph coloring problem with parameters $(k-1, \ell-1)$ is NP-hard, but our hardness result cannot be obtained this way.)

The proof of Theorem 1 builds on and extends a topological approach for studying approximate graph colouring introduced by Krokhin, Opršal, Wrochna, and Živný [9] and has two main parts. $\mathbf{LO}_k$-colorability of a hypergraph H is equivalent to the existence of a homomorphism from H to a certain relational structure $\mathbf{LO}_k$. For a natural number n, let $(\mathbf{LO}_3)^n$ be the n-fold power of the relational structure $\mathbf{LO}_3$. In the first part of the proof, we use topological methods to show that with every homomorphism $f\colon (\mathbf{LO}_3)^n \to \mathbf{LO}_4$, we can associate an *affine map* $\chi(f)\colon \mathbb{Z}_3^n \to \mathbb{Z}_3$ (i.e., a map of the form $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n \alpha_i x_i$, for some $\alpha_i \in \mathbb{Z}_3$ and $\sum_{i=1}^n \alpha_i \equiv 1 \pmod 3$); moreover, the assignment $f \mapsto \chi(f)$ preserves natural so-called *minor relations* that arise from maps $\pi\colon [n] \to [m]$, i.e., χ is a so-called *minion homomorphism*.

In the second part of the proof, we show by combinatorial arguments that the maps $\chi(f)\colon \mathbb{Z}_3^n \to \mathbb{Z}_3$ form a very restricted subclass of affine maps: They are projections, i.e., maps of the form $\mathbb{Z}_3^n \to \mathbb{Z}_3$, $(x_1, \ldots, x_n) \mapsto x_i$. Theorem 1 then follows from a hardness criterion obtained as part of a general algebraic theory so-called *promis constraint satisfaction problems* [2].

In a nutshell, topology enters in the first part of the proof as follows. First, with every homomorphism $f\colon (\mathbf{LO}_3)^n \to \mathbf{LO}_4$ we associate a continuous map $f_*\colon T^n \to P^2$, where T^n is the n-dimensional torus (the n-fold power of the circle S^1) and P^2 is a suitable target space; moreover, the cyclic group $\mathbb{Z}_3$ naturally acts on both T^n and P^2, and the map f_* preserves these symmetries (it is *equivariant*). This first step uses *homomorphism complexes* (a well-known construction in topological combinatorics that goes back to the work of Lovász [10]). Second, using equivariant obstruction theory, we show that equivariant continuous maps $T^n \to P^2$, when considered up to a natural equivalence relation of symmetry-preserving continuous deformation (*equivariant homotopy*), are in bijection with affine maps $\mathbb{Z}_3^n \to \mathbb{Z}_3$.

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Counting triangulations of homology 3-spheres

Geva Yashfe

(joint work with Karim Adiprasito, Marc Lackenby, Juan Souto, and (separately) with Yuval Peled)

M. Gromov popularized the following problem.

Problem 1. Let t_N be the number of (combinatorial isomorphism types of) triangulations of S^3 with N facets. Is t_N exponential or superexponential in N?

This problem remains unsolved.

Known results and related work.

- For triangulations of S^2 Tutte [9] proved that there are exponentially many triangulations with N triangles.

- It is known that $\exp(cN) \leq t_N \leq \exp(c'N \log N)$ for some[1] constants $c, c' > 0$.

  These bounds are relatively straightforward to prove. The upper bound $\exp(c'N \log N)$ actually holds for triangulations of d-manifolds for any fixed d (with an appropriate constant c' depending on d).

  It seems there is no hope for a very precise answer, so rough asymptotic results are all we aim for.
- Benedetti–Ziegler [3] showed that shellable spheres are at most exponentially many in N (in any fixed dimension). They also did this for a larger class spheres called "locally-constructible" (or LC). Benedetti–Pavelka [2] later extended this to a significantly larger class called 2-LC, but only in dimension 3.
- If we parametrize sphere triangulations by the number of vertices (call it M) instead, the problem has a different character. Some main results are:
  - Alon and Goodman–Pollack [1, 4] showed that there are relatively few (approximately $\exp(cM \log M)$) polytopes in M.
  - Kalai [6] showed that most triangulations (in terms of M) are not polytopal, and Lee [7] showed that Kalai's triangulations are shellable.
  - Nevo–Santos–Wilson [8] found still more triangulations than Kalai, and Yang [10] showed the families they produced are constructible.

Here we mainly consider the following relaxation of Problem 1.

Problem. Let t_N be the number of triangulations of 3-dimensional homology spheres with N facets. Is t_N exponential or superexponential in N?

For this we consider homology with coefficients in a fixed field $\mathbb{F}$. This problem also remains unsolved, with the best bounds remaining of the form $\exp(cN) \leq t_N \leq \exp(c'N \log N)$ for some $c, c' > 0$. This talk is about very modest progress and currently still-unsuccessful attempts.

1. Dual graphs and short graphs

Gromov and Nabutovski suggested reducing the problem to a problem about graphs. Gromov explains roughly what the result should be without describing the reduction in [5]. This section is based on joint work with K. Adiprasito, M. Lackenby, and J. Souto, and contains a sketch of our implementation of (part of) this idea and of two applications.

Dual graphs and enumeration. Suppose X is a triangulated 3-manifold with N facets, but we only have access to its dual graph G consisting of one vertex per facet, with an edge for every two facets that intersect in a triangle. Then there are at most $\exp(cN)$ possibilities for X given G: we have to put one tetrahedron in place of every vertex of G, and the only information missing is the manner in which adjacent tetrahedra are glued to each other. This leaves us with constantly many possibilities per facet of X, of which there are N.

[1] We don't care very much about the constants, and different occurences of "c" in this abstract do not refer to the same number.

Given the bounds we have on t_N, we may consider just the dual graphs of triangulated homology 3-spheres rather than the entire triangulations: asymptotics remain essentially unchanged (exponential factors only change our constants).

Remark. This discussion explains the upper bound $\exp(c'N \log N)$ for the number of triangulated d-manifolds: the dual graphs of triangulated d-manifold have constant degree $d+1$, and there are only $\exp(c'N \log N)$ constant-degree graphs on N vertices (with c' depending on the degree).

Properties of dual graphs. The family of dual graphs G of triangulated homology d-spheres over $\mathbb{F}$ has the following pleasant properties:

(1) It has bounded degree.
(2) For each G in the family there exists an $\mathbb{F}$-homology basis (given by taking a maximal independent subset of the dual cells of $(d-2)$-faces of the triangulated homology sphere) such that:
   (a) The average cycle length (in terms of the number of edges) in the basis is bounded by a constant. Equivalently, if G has N vertices then the total length of cycles in the basis is bounded by cN for some c.
   (b) Each edge of G participates in boundedly many of the cycles in the basis.

Definition. A class of graphs satisfying properties 1, 2(a), and 2(b) with some fixed constants is called *a class of short graphs.* These classes are parametrized by the field $\mathbb{F}$, the degree bound, and the constants in conditions 2(a) and 2(b) above.

Basically, one can think of classes of short graphs as a "soft / approximate" versions of dual graphs of triangulated homology spheres.

Theorem (Adiprasito, Lackenby, Souto, Yashfe)**.** *For each class $\mathcal{C}$ of short graphs there is a "machine"*

$$\mathcal{C} \hookrightarrow (\textit{triangulated homology } 3\textit{-spheres over } \mathbb{F})$$

taking each N-vertex graph in $\mathcal{C}$ to a triangulated homology 3-sphere with at most $c \cdot N$ tetrahedra (for c depending on $\mathcal{C}$).

Corollary. *If there exists $d > 3$ such that there are superexponentially many N-facet triangulated homology d-spheres over $\mathbb{F}$, then the same holds for $d = 3$.*

Proof. Take the dual graphs of these superexponentially many homology d-spheres to obtain a family of graphs contained in a class $\mathcal{C}$ of short graphs, and apply the machine to this class. □

The machines of the theorem preserve some of the geometry of the input graphs. This can also be applied to prove the following.

Theorem (Originally proved in unpublished work of M. Lackenby and J. Souto by slightly different methods.)**.** *There is a family of triangulations of S^3 for which the dual graphs form an expander family.*

(Getting triangulations of S^3 and not just some homology spheres requires an additional idea and a special family of short graphs.)

A sketch of the machine. Given a graph G together with a "short" homology basis over $\mathbb{F}$, construct a 2-complex by pasting cells along basis elements. Then thicken this complex to a triangulated 4-manifold in a geometrically controlled way (so as not to increase degrees or face numbers by too large a factor; this process is not canonical). Finally, pass to the boundary, which is a homology 3-sphere over $\mathbb{F}$. Injectivity of this process is not automatic and requires that we locally encode some combinatorial data in the resulting triangulation.

In the talk some additional ideas were sketched, mainly on the relation between problems here and problems about subgroup growth.

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Stress spaces, reconstruction problems and lower bound problems

Hailun Zheng

(joint work with Satoshi Murai and Isabella Novik)

What partial information about a simplicial d-polytope P allows one to determine P up to certain equivalences? Specifically, consider the following two equivalences: Given two polytopes P and P', we say that P and P' are *combinatorially equivalent* if they have isomorphic face lattices, and they are *affinely equivalent* if there is an affine map that sends P to P'. Perles (unpublished) and Dancis [3] proved that to determine the combinatorial type of a simplicial d-polytope P, it suffices to know the $\lfloor d/2 \rfloor$-skeleton of P. This result is optimal in the sense that distinct simplicial d-polytopes may have isomorphic $(\lfloor d/2 \rfloor - 1)$-skeleta. (For example, it is known that there are $2^{\Theta(n \log n)}$ combinatorial types of neighborly d-polytopes with n vertices [12].) On the other hand, the space of affine dependencies among the vertices of P determines the affine type of P (and hence also the combinatorial type of P). This observation is at the heart of the theory of Gale diagrams developed by Perles [15, Chapter 6].

A common ground of the above two results lies in the theory of stress spaces developed by Lee [6]. To see it, note that for a simplicial d-polytope P, the space of affine dependencies of vertices of P is equivalent to the space of affine 1-stresses on P, while the space of affine k-stresses is trivial for any $k \geq \lfloor d/2 \rfloor + 1$. Hence these two results are precisely the $k = \lfloor d/2 \rfloor + 1$ and $k = 1$ cases of the following conjecture of Kalai [4].

Conjecture 1. *Let P be a simplicial d-polytope and let $1 \leq k \leq \lfloor d/2 \rfloor + 1$. Then the $(k-1)$-skeleton of P and the space of affine k-stresses of P uniquely determine the combinatorial type of P.*

Another conjecture concerning the affine types of polytopes is the following

Conjecture 2. *Let $d \geq 2k \geq 4$ and let P be a simplicial d-polytope with the natural embedding and with no missing faces of dimension $\geq d - k + 1$. Then the space of affine k-stresses uniquely determines P up to affine equivalence.*

We present two partial results of the above two conjectures. The first result verifies the case of $k = 2$ of Conjecture 1, namely

Theorem 3 ([10])**.** *Let $d \geq 3$. The graph of a simplicial d-polytope P together with the space of affine 2-stresses on P uniquely determine the combinatorial type of P.*

The proof is geometric-combinatorial. The idea is to use the rigidity theory of frameworks to show that the missing faces of P can be identified by the sign patterns of the coefficients of the squarefree terms in certain affine 2-stresses on P.

The second result deals with Conjecture 2 and more generally the structures of affine stress spaces of polytopes with no large missing faces.

Theorem 4 ([8])**.** *Let $1 \leq j < k \leq \frac{d-1}{2}$ and let P be a simplicial d-polytope with no missing faces of dimension $\geq d - k + 1$. Then the space of affine k-stresses on P determines the space of affine j-stresses on P.*

In particular, Theorem 4 verifies the $k \leq \frac{d-1}{2}$ case of Conjecture 2 (by letting $j = 1$). At the moment, the $d = 2k \geq 4$ case remains open.

The proof of Theorem 4 is algebraic. In particular, it relies on identifying the space of affine stresses on P with the Matlis dual N of the Stanley-Reisner ring of P modulo the linear system of parameters and the Lefschetz element. (Hence, the condition on the missing faces translates into a condition on the degrees of generators of N.)

Three comments are in order. First, prior to Theorem 4, Conjecture 2 was proved by Cruickshank, Jackson and Tanigawa [2] in the case that P is a simplicial polytope whose vertices have generic coordinates, and by Novik and Zheng [11] for all simplicial d-polytopes that have no missing faces of dimension $\geq d-2k+2$. Second, with the proof of the g-theorem for simplicial spheres [1, 13, 5], Theorem 4 not only applies to simplicial d-polytopes with natural embeddings but also simplicial $(d-1)$-spheres with generic embeddings. Finally, Theorem 4 leads to two corollaries on the g-numbers of simplicial $(d-1)$-spheres that are interesting in their own right. For a simplicial complex Δ, denote by $m_i(\Delta)$ the number of missing i-faces of Δ. Recall that the g-theorem [14, 1, 13, 5] states that the g-numbers of a simplicial $(d-1)$-sphere form an M-sequence, i.e., $0 \leq g_{k+1} \leq g_k^{<k>}$ holds for all $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$. The following corollary is a strengthening of the g-theorem; part of the inequality appeared first in [9].

Corollary 5. *Let $d \geq 4$ and let Δ be a simplicial $(d-1)$-sphere. Then for all $1 \leq k \leq \lceil d/2 \rceil - 1$, $g_k(\Delta) \geq m_{d-k}(\Delta)$. Furthermore, $0 \leq g_{k+1}(\Delta) \leq \big(g_k(\Delta) - m_{d-k}(\Delta)\big)^{<k>}$.*

Recall also that the Generalized Lower Bound Theorem [7] asserts that for $2 \leq k \leq \lfloor d/2 \rfloor$, a simplicial $(d-1)$-sphere has $g_{k+1} = 0$ if and only if it is k-stacked. The following corollary gives a second characterization of spheres attaining a minimal g-number.

Corollary 6. *Let Δ be a simplicial $(d-1)$-sphere. Then for $1 \leq k \leq \lfloor d/2 \rfloor - 1$, Δ is k-stacked if and only if $g_k(\Delta) = m_{d-k}(\Delta)$. Moreover, if d is odd and Δ is $\frac{d-1}{2}$-stacked, then $g_{\frac{d-1}{2}}(\Delta) = m_{\frac{d+1}{2}}(\Delta)$.*

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Open Problems in Geometric, Algebraic and Topological Combinatorics

Collected by Edward Swartz

PROBLEM 1 (Nati Linial, joint with Jordan Smith). Computing certain invariants of topological spaces of dimension three

Say that a graph is *geodesic* if between every two vertices there is a unique shortest path. Ore (1960) defined this class of graphs and asked for a characterization, but this quest seems way out of hand. It suffices, of course to consider only 2-connected graphs. There is a known infinite family of (i) geodesic, (ii) 2-connected graphs (iii) in which all vertex degrees are at least 3 and have diameter 5, but nothing beyond. Question: Can such graphs have arbitrarily large diameter?

PROBLEM 2 (Benjamin Braun, joint with Kaitlin Bruegge). Bounding facet numbers for symmetric edge polytopes

Let G be a finite simple graph and let $P_G = \text{conv}\{e_i - e_j, e_j - e_i : ij \in E(G)\}$ be the *symmetric edge polytope* of G. Determining properties of the facets of symmetric edge polytopes is of interest both in combinatorics and in applications. To this end, the authors made the following conjecture regarding bounds on the number of facets for symmetric edge polytopes of connected graphs on a fixed number of vertices.

Conjecture. (1) (Braun and Bruegge [1]). Let $n \geq 3$.

(1) For $n = 2k+1$, the maximum number of facets for P_G for a connected graph G on n vertices is 6^k, which is attained by a wedge of k cycles of length three.
(2) For $n = 2k$, the maximum number of facets for P_G for a connected graph G on n vertices is $14 \cdot 6^{k-2}$, which is attained by a wedge of K_4 with $k-2$ cycles of length three.
(3) For $n = 2k+1$, the minimum number of facets for P_G for a connected graph G on n vertices is $3 \cdot 2^k - 2$, which is attained by $K_{k,k+1}$.
(4) For $n = 2k$, the minimum number of facets for P_G for a connected graph G on n vertices is $2^{k+1} - 2$, which is attained by $K_{k,k}$.

Partial progress on this conjecture was announced in a preprint by Mori, Mori, and Ohsugi [2].

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PROBLEM 3 (Benjamin Braun, joint with Matias von Bell, Derek Hanely, Khrystyna Serhiyenko, Julianne Vega, Andrés Vindas-Meléndez and Martha Yip). ENUMERATING REGULAR TRIANGULATIONS OF ORDER POLYTOPES FOR SNAKE POSETS

For $n \in \mathbb{Z}_{\geq 0}$, a *generalized snake word* is a word of the form $\mathbf{w} = w_0 w_1 \cdots w_n$ where $w_0 = \varepsilon$ is the empty letter and w_i is in the alphabet $\{L, R\}$ for $i = 1, \ldots, n$. The *length* of the word is n, which is the number of letters in $\{L, R\}$. Given a generalized snake word $\mathbf{w} = w_0 w_1 \cdots w_n$, we define the *generalized snake poset* $P(\mathbf{w})$ recursively in the following way:

- $P(w_0) = P(\varepsilon)$ is the poset on elements $\{0, 1, 2, 3\}$ with cover relations $1 \prec 0$, $2 \prec 0$, $3 \prec 1$ and $3 \prec 2$.
- $P(w_0 w_1 \cdots w_n)$ is the poset $P(w_0 w_1 \cdots w_{n-1}) \cup \{2n+2, 2n+3\}$ with the added cover relations $2n+3 \prec 2n+1$, $2n+3 \prec 2n+2$, and

$$\begin{cases} 2n+2 \prec 2n-1, & \text{if } n = 1 \text{ and } w_n = L, \text{ or } n \geq 2 \text{ and } w_{n-1}w_n \in \{RL, LR\}, \\ 2n+2 \prec 2n, & \text{if } n = 1 \text{ and } w_n = R, \text{ or } n \geq 2 \text{ and } w_{n-1}w_n \in \{LL, RR\}. \end{cases}$$

In this definition, the minimal element of the poset $P(\mathbf{w})$ is $\widehat{0} = 2n+3$, and the maximal element of the poset is $\widehat{1} = 0$.

As part of a more extensive investigation of triangulations of order polytopes related to generalized snake posets, the authors made the following conjecture regarding the order polytope of the following poset: for the length n word $\varepsilon LRLR \cdots$, $S_n := P(\varepsilon LRLR \cdots)$ is the *snake poset*.

Conjecture. (2) (von Bell, Braun, Hanely, Serhiyenko, Vega, Vindas-Meléndez, Yip [1]). The number of regular triangulations of the order polytope of S_n is $2^{n+1}\mathrm{Cat}(2n+1)$, where $\mathrm{Cat}(2n+1)$ denotes the $2n+1$-st Catalan number.

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PROBLEM 4 (Raman Sanyal, joint with Sebastian Manecke). STRONGLY INSCRIBABLE ARRANGEMENTS AND REFLECTION ARRANGEMENTS

Consider an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ of n hyperplanes in $\mathbb{R}^d$, all passing through the origin. Choosing a normal vector z_i for each hyperplane H_i gives rise to an associated zonotope

$$Z(\mathcal{A}) = [-z_1, z_1] + \cdots + [-z_n, z_n] = \{\mu_1 z_1 + \cdots + \mu_n z_n : -1 \leq \mu_1, \ldots, \mu_n \leq 1\}.$$

$Z(\mathcal{A})$ is a convex polytope whose combinatorics faithfully represents that of $\mathcal{A}$. We call $\mathcal{A}$ *strongly inscribable* if there is a choice of normal vectors such that $Z(\mathcal{A})$ is inscribed, that is, has all vertices on the unit sphere. For example, reflection arrangements obtained from finite reflection groups are strongly inscribable.

In [4], we showed that the restriction of a strongly inscribable arrangement to any of its hyperplanes is again strongly inscribable. Thus, further examples are provided by restrictions of reflection arrangements, which generally are not reflection arrangements themselves.

Conjecture. (3) ([4, Conj. 1.7]). An arrangement of hyperplanes in $\mathbb{R}^d$ with $d \geq 3$ is strongly inscribable if and only if it is linearly isomorphic to the restriction of a reflection arrangement.

An important structural property that we observe in [4] is that every strongly inscribable arrangement is *simplicial*, that is, every connected component of $\mathbb{R}^d \setminus \bigcup \mathcal{A}$ is linearly isomorphic to $\mathbb{R}^d_{>0}$. Simplicial arrangements are fascinating but rare. There is a conjecturally complete catalog of simplicial arrangements in $\mathbb{R}^3$ due to Grünbaum and Cuntz [1, 2, 3]. We verify that the only strongly inscribable arrangements in this catalog are restrictions of reflection arrangements. Assuming the completeness of the Grünbaum–Cuntz catalog, this proves the conjecture in dimension 3.

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PROBLEM 5 (Georg Loho). Realizable M-convex functions

A *generalized permutahedron* is a polytope whose edge directions are of the form $e_i - e_j$ for standard unit vectors e_i, e_j. An *integral generalized permutahedron* is a generalized permutahedron that is also a lattice polytope. An *M-convex set* is the set of lattice points in an integral generalized permutahedron. Let $f\colon S \to \mathbb{R}$ be a function for some finite subset $S \subseteq \mathbb{Z}^n$. It is *M-convex* if $\operatorname{argmin}_{x\in S}(f(x) - \langle c, x\rangle)$ is an M-convex set for each $c \in \mathbb{R}^n$.

Matroids are special M-convex sets, namely those contained in the unit cube $\{0,1\}^n$. Furthermore, *valuated matroids* are special M-convex functions, namely those with a matroid as domain. Matroids are *realizable* if they arise from the independence structure of a matrix. Valuated matroids are *realizable* if they arise as tropicalization of a Pluecker vector of a linear space. M-convex sets are *realizable* if their defining integral generalized permutahedron can be described by a submodular function arising from a subspace arrangement.

However, it is not clear when an M-convex function should be called *realizable.* There are some potential candidates. One could argue that M-convex functions arising from realizable valuated matroids by induction through a directed graph should be called realizable. Furthermore, M-convex functions arise by tropicalization from Lorentzian polynomials. For the latter, one could argue that those arising as volume polynomials should be realizable, giving rise to another notion of realizability for M-convex functions. Still, the notion does not seem clear compared to the nice picture for (valuated) matroids and M-convex sets, leaving the following questions open.

What is a 'realizable' M-convex function?
What is a 'Pluecker vector' of a subspace arrangement?

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PROBLEM 6 (Germain Poullot)**.** Is log-concavity arising through matrix recursion?

The problem presented here arises when studying the monotone path polytope of the hypersimplex $\Delta(n,2)$. After numerous pages of tedious proofs, one can count the number of coherent monotone paths of $\Delta(n,2)$ thanks to the following matrix recursion. Similar problems give rises to similar question, but we present here a very concrete occurrence, hoping from someone to develop tools to address the general setting.

Let's define the sequences of polynomials T_n, Q_n and C_n satisfying the following recursive formula:

$$\forall n \geq 4, \quad \begin{pmatrix} T_{n+1} \\ Q_{n+1} \\ C_{n+1} \end{pmatrix} = \mathcal{M} \begin{pmatrix} T_n \\ Q_n \\ C_n \end{pmatrix}$$

$$\text{with } \mathcal{M} = \begin{pmatrix} z & 1+z & 1+z \\ 0 & 1+z & z \\ z+z^2 & 0 & 1+z \end{pmatrix}, \quad \begin{pmatrix} T_4 \\ Q_4 \\ C_4 \end{pmatrix} = \begin{pmatrix} z^4+2z^3 \\ z^4 \\ 2z^4+2z^3 \end{pmatrix}$$

and the polynomial $V_n = T_n + Q_n + C_n = \sum_k v_{n,k} z^k$.

Conjecture: For all $n \geq 4$, the sequence $(v_{n,k})_n$ is (ultra-)log-concave.

The value $v_{n,k}$ counts the number of coherent monotone paths of $\Delta(n,2)$, and a good combinatorial model allows to exhibit this recursion, but I haven't able to extract log-concavity from this combinatorial interpretation (yet).

This conjecture have been checked for all $n \leq 300$ (and one can easily go further, but where to stop?), please prove it!

Obviously, the question can be posed more generally: given a starting vector $X_0 \in \mathbb{N}[z]^m$ and a matrix $\mathcal{M} \in \mathrm{Mat}_{m\times m}\big(\mathbb{N}[z]\big)$, what kind of tools can we develop to address (ultra-)log-concavity and unimodality questions for (the polynomials of the vector) $X_n = \mathcal{M}^n X_0$ and the polynomial $\sum_{r=1}^m X_{n,r}$?

Note that, for $n \leq 300$:

- T_n, Q_n, C_n and V_n are ultra-log-concave (so homogenizing each of them gives lorentzian polynomials).
- (in general) T_n, Q_n, C_n and V_n **are not** symmetric.
- (in general) T_n, Q_n, C_n and V_n **are not** real-rooted.

- Properties of $\mathcal{M}$ (left- or right-kernel and eigenvectors) seems not helpful: one computes $\mathcal{M}^n$, but then extracting log-concavity it out of my reach...
- Starting with different polynomials for T_4, Q_4 and C_4, its seems that V_n becomes ultra-log-concave after a certain rank.

One can mathematically prove that:

- $\deg V_n = d = \left\lfloor \frac{3}{2}(n-1) \right\rfloor$ (with $v_{n,d} = 1$ if n odd ; $v_{n,d} = d$ if n even).
- the "constant coefficient" of V_n is 4 (i.e. $v_{n,4} = 4$).
- $V_n(1) = \sum_k v_{n,k} = \frac{1}{3}(25 \times 4^{n-4} - 1)$
- for a fixed k, the value of $v_{n,k}$ is a polynomial in n (of degree $k-3$), mimicking slightly the behavior of binomial coefficients.

PROBLEM 7 (Bruno Benedetti, joint with Matteo Varbaro)**.** The dual graph of Cohen–Macaulay algebras

In the following, $S = \mathbb{K}[x_1, \ldots, x_n]$ is the polynomial ring with n variables over some field $\mathbb{K}$. Let I be any ideal of S. Let $\wp_1, \ldots, \wp_s$ be the minimal primes of I that have height equal to the height of I. We define the *dual graph* $G(I)$ on the vertex set $[s] = \{1, \ldots, s\}$ as follows: there is an edge $[i, j]$ if and only if

$$\text{height}(P_i + P_j) = 1 + \text{height}\, I.$$

There are two motivations for this definition:

(1) When I is radical and height-unmixed, $I = \wp_1 \cap \ldots \cap \wp_s$. Passing to the Zariski sets, this means that $Z(I) = Z(\wp_1) \cup \ldots \cup Z(\wp_s)$. Thus $G(I)$ coincides with the dual graph of $Z(I)$, where an edge connects two irreducible components iff their intersection has codimension one.

(2) When $I = I_\Delta$ is the Stanley-Reisner ideal of some simplicial complex Δ on n vertices, then $G(I)$ coincides with the dual graph of Δ. In this case $\text{height}\, I = n - \dim \Delta - 1$.

Recall that in a connected graph G, the distance between two vertices is the number of edges in a shortest path connecting them, and $\text{diam}\, G$ is the maximum distance between any two of its vertices.

Conjecture. [Benedetti–Varbaro [2], 2014]: Let $I \subseteq S$ be an ideal generated in degree two. If S/I is Cohen–Macaulay, then $\text{diam}\, G(I) \leq \text{height}(I)$.

In the meantime, the conjecture has been proven true for many interesting cases, cf. e.g. [3], [4], [5], [6], [7]. It holds for squarefree monomial ideals: This follows from the result by Adiprasito–Benedetti that "flag normal complexes satisfy the Hirsch conjecture" [1]. In fact, when $I = I_\Delta$, the upper bound $\text{height}\, I = n - \dim \Delta - 1$ reflects exactly the Hirsch bound.

A final comment: The condition "generated in degree two" is not really restrictive. Via Veronese embeddings, if the Conjecture above is true, one automatically gets a polynomial bound of the type

$$\text{diam}\, G(I) \leq \text{height}(I)^{\lceil k/2 \rceil}$$

for all ideals I generated in degree $\leq k$ and such that S/I is Cohen–Macaulay [2]. Thus in particular the Conjecture above would imply the following:

Conjecture. If Δ is a normal simplicial complex of dimension d, with n vertices, and no missing face of dimension $\leq k$, then the dual graph of Δ has diameter at most $P(n)$, where P is a polynomial in n of degree $\lceil k/2 \rceil$.

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PROBLEM 8 (Felipe Rincón). Tensor products of matroids

If M is a matroid on the ground set E and N is a matroid on the ground set F, a *quasi-product* of M and N is a matroid T on the ground set $E \times F$ satisfying: for any $f \in F$, the natural bijection between E and $E \times \{f\}$ induces a matroid isomorphism between M and the restriction $T|_{E\times\{f\}}$, and similarly, for any $e \in E$, the natural bijection between F and $\{e\} \times F$ induces a matroid isomorphism between N and the restriction $T|_{\{e\}\times F}$.

It is a simple exercise to show that the rank of any quasi-product T of M and N has rank at most $\operatorname{rank}(M) \cdot \operatorname{rank}(N)$. The quasi-product T is called a *tensor product* if in fact we have the equality $\operatorname{rank}(T) = \operatorname{rank}(M) \cdot \operatorname{rank}(N)$.

To my knowledge, there are only very few things that we know about tensor products:

- If M and N are realizable over the same field K then M and N admit a tensor product. This is due to the fact that, for subspaces $L_M \subset K^E$ and $L_N \subset K^F$ that represent M and N, respectively, we can construct the tensor product $L_M \otimes L_N \subset K^E \otimes K^F$, which then represents a tensor product T of M and N. The resulting tensor product T might depend on the realizations chosen, though.
- The matroids V_8 and $U_{2,3}$ do not admit a tensor product! This is one of the main results of [2].
- If M and N admit a tensor product, M' is a minor of M, and N' is a minor of N, then M' and N' admit a tensor product. This is not too difficult – for details, you can see, for instance, [1].

The research problem I am proposing is to study further the class of matroids M, N that admit a tensor product. For instance, can you construct more matroids that admit a tensor product? Can you describe tensor products combinatorially for particular classes of matroids? Can you say something about forbidden minors for the existence of tensor products (see [1])?

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PROBLEM 9 (Pablo Soberón)**.** A PROBLEM IN THE PLANE

Given a family $\mathcal{F}$ of lines in the plane such that no two are parallel, we can determine the size of a set $A \subset \mathbb{R}^2$ as follows:

$$\mu(A) = \max\{k \in \mathbb{N} : \text{ there exists a set of } k \text{ lines of } \mathcal{F} \text{ whose pairwise intersections are all in } A.\}$$

If A does not contain any intersection of lines of $\mathcal{F}$, we define $\mu(A) := 1$.

Let (A, B, C) be a convex partition of the plane into three parts. In other words, each of A, B, C is a closed convex set in $\mathbb{R}^2$, their interiors are pairwise disjoint, and their union is $\mathbb{R}^2$.

Show that

$$\mu(A)\mu(B)\mu(C) \geq |\mathcal{F}|.$$

The case when one of A, B, C is a half-plane is easy to prove. The best known bound is [1]:

$$\mu(A)\mu(B)\mu(C) \geq \left(\frac{2}{3}\right) |\mathcal{F}|.$$

[1] A. Xue, P. Soberón, *Balanced convex partitions of lines in the plane*, Discrete & Computational Geometry **66** (2021), 1150–1167.

The homogenized Linial arrangement and Genocchi numbers

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(joint work with Alexader Lazar)

The *braid arrangement* (or *type A Coxeter arrangement*) is the hyperplane arrangement in $\mathbb{R}^n$ defined by

$$\mathcal{A}_{n-1} := \{x_i - x_j = 0 : 1 \leq i < j \leq n\}.$$

Note that the hyperplanes of $\mathcal{A}_{n-1}$ divide $\mathbb{R}^n$ into open cones of the form

$$R_\sigma := \{\mathbf{x} \in \mathbb{R}^n : x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}\},$$

where σ is a permutation in the symmetric group $\mathfrak{S}_n$. Hence the braid arrangement $\mathcal{A}_{n-1}$ has $|\mathfrak{S}_n| = n!$ regions.

A classical formula of Zaslavsky [11] gives the number of regions of any real hyperplane arrangement $\mathcal{A}$ in terms of the Möbius function of its intersection (semi)lattice $\mathcal{L}(\mathcal{A})$. Indeed, given any finite, ranked poset P of length ℓ, with a minimum element $\hat{0}$, the *characteristic polynomial* of P is defined to be

$$\chi_P(t) := \sum_{x \in P} \mu_P(\hat{0}, x) t^{\ell - \mathrm{rk}(x)}, \tag{1}$$

where μ_P is the Möbius function of P and $\mathrm{rk}(x)$ is the rank of x. Zaslavsky's formula for the number of regions $r(\mathcal{A})$ of $\mathcal{A}$ is

$$r(\mathcal{A}) = (-1)^{\ell}\chi_{\mathcal{L}(\mathcal{A})}(-1). \tag{2}$$

It is well known and easy to see that the lattice of intersections of the braid arrangement $\mathcal{A}_{n-1}$ is isomorphic to the lattice Π_n of partitions of the set $[n] := \{1, 2, \ldots, n\}$. It is also well known that the characteristic polynomial of Π_n is given by

$$\chi_{\Pi_n}(t) = \sum_{k=1}^{n} s(n,k)t^{k-1}, \tag{3}$$

where $s(n, k)$ is the Stirling number of the first kind, which is equal to $(-1)^{n-k}$ times the number of permutations in $\mathfrak{S}_n$ with exactly k cycles; see [10, Example 3.10.4]. Hence $\chi_{\Pi_n}(-1) = (-1)^{n-1}|\mathfrak{S}_n|$. Therefore, from (2), we recover the result observed above that the number of regions of $\mathcal{A}_{n-1}$ is $n!$.

In this talk, we consider a hyperplane arrangement introduced by Hetyei [5]. The *homogenized Linial arrangement* is the hyperplane arrangement in $\mathbb{R}^{2n}$ defined, for $n \geq 2$, by

$$\mathcal{H}_{2n-3} := \{x_i - x_j = y_i : 1 \leq i < j \leq n\}.$$

Note that by intersecting $\mathcal{H}_{2n-3}$ with the subspace $y_1 = y_2 = \cdots = y_n = 0$ one gets the braid arrangement $\mathcal{A}_{n-1}$. Similarly by intersecting $\mathcal{H}_{2n-3}$ with the subspace $y_1 = y_2 = \cdots = y_n = 1$, one gets the Linial arrangement in $\mathbb{R}^n$,

$$\{x_i - x_j = 1 : 1 \leq i < j \leq n\}.$$

Postnikov and Stanley [8] show that the number of regions of the Linial arrangement is equal to the number of alternating trees on node set $[n + 1]$.

Hetyei [5] shows that the number of regions of the homogenized Linial arrangement is equal to a number known as the median Genocchi number. He uses the finite field method of Athanasiadis [1] to obtain a recurrence for $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1)$ and shows that the recurrence reduces to a known formula for the median Genocchi number h_n. The result then follows from Zaslavsky's formula (2). The Genocchi numbers g_n and the median Genocchi numbers h_n can be characterized by the Barsky and Dumont [2] generating function formulas:

$$\sum_{n\geq 1} g_n x^n = \sum_{n\geq 1} \frac{(n-1)!\, n!\, x^n}{\prod_{k=1}^{n}(1+k^2 z)} \tag{4}$$

$$\sum_{n\geq 0} h_n z^n = \sum_{n\geq 0} \frac{n!\,(n+1)!\, z^n}{\prod_{k=1}^{n}(1+k(k+1)z)}. \tag{5}$$

In [6] we further study the intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1})$ and its characteristic polynomial $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)$ using an approach quite different from Hetyei's. We prove

$$\sum_{n\geq 1} \chi_{L(\mathcal{H}_{2n-1})}(t)\, x^n = \sum_{n\geq 1} \frac{(t-1)_{n-1}(t-1)_n\, x^n}{\prod_{k=1}^{n}(1-k(t-k)x)}, \tag{6}$$

where $(a)_n$ denotes the falling factorial $a(a-1)\cdots(a-n+1)$. By setting $t=-1$, we recover Hetyei's result. Moreover, by setting $t=0$, we relate the (non-median) Genocchi numbers to the homogenized Lineal arrangement. Indeed, since $\chi_{L(\mathcal{H}_{2n-1})}(0)$ is the Möbius invariant of $L(\mathcal{H}_{2n-1})$, we have

$$\mu_{L(\mathcal{H}_{2n-1})}(\hat{0},\hat{1}) = -g_n. \tag{7}$$

Our proof of (6) has the following steps.

1. Show that $t\chi_{L(\mathcal{H}_{2n-1})}(t)$ equals the chromatic polynomial $\mathrm{ch}_{\Gamma_n}(t)$ of a certain graph Γ_n.
2. Using the Rota-Whitney NBC theorem, show that the coefficients of $\mathrm{ch}_{\Gamma_n}(t)$ can be described in terms of a certain class of alternating forests.
3. Construct a bijection from this class of alternating forests to a new class of permutations similar to those introduced by Dumont [3] to study Genocchi numbers. This yields a result analogous to (3) involving cycles of Dumont-like permutations.
4. Construct a bijection from the Dumont-like permutations to a certain class of objects called surjective staircases. Results of Randrianarivony [9] and Zeng [12] on generating functions for surjective staircases are used to complete the proof.

We also introduce a Dowling analog of the homogenized Linial arrangement. Let $\omega = e^{2\pi i/m}$ be a primitive mth root of unity. The *homogenized Linial-Dowling arrangement* is the complex hyperplane arrangement in $\mathbb{C}^{2n}$, defined by

$$\mathcal{H}^m_{2n-1} = \{x_i - \omega^\ell x_j = y_i : 1 \le i < j \le n,\ 0 \le \ell < m\} \cup \{x_i = y_i : 1 \le i \le n\}.$$

Note that when $m=2$, the arrangement $\mathcal{H}^m_{2n-1}$ is a complexified version of the type B homogenized Linial arrangement. When $m=1$, the arrangement $\mathcal{H}^m_{2n-1}$ is the complexified version of the arrangement obtained by intersecting $\mathcal{H}_{2n-1}$ with the coordinate hyperplane $x_{n+1}=0$. The resulting arrangement on the coordinate hyperplane has the same intersection lattice as $\mathcal{H}_{2n-1}$.

Using similar techniques as for the homogenized Linial arrangement, we prove in [7] the following generalization of (6):

$$\sum_{n\ge1} \chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(t)\, x^n = \sum_{n\ge1} \frac{(t-1)_{n,m}(t-m)_{n-1,m}\, x^n}{\prod_{k=1}^n (1-mk(t-mk)x)}. \tag{8}$$

where $(a)_{n,m} = a(a-m)(a-2m)\cdots(a-(n-1)m)$.

There is a well-studied polynomial analog of the Genocchi numbers known as the Gandhi polynomials $G_n(x)$; see [4, Section 3]. We obtain the following m-analog of (7):

$$\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(0) = \mu_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(\hat{0},\hat{1}) = -m^{2n-1}G_n(m^{-1}).$$

The polynomials $\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(0)$ and $\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(-1)$ can be viewed as m-analogs of the Genocchi and median Genocchi numbers, respectively. It would be interesting to generalize known relationships between the Genocchi numbers and median Genocchi numbers to these m-analogs.

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