Mathematisches Forschungsinstitut Oberwolfach

Report No. 22/2024

DOI: 10.4171/OWR/2024/22

Non-commutative Function Theory and Free Probability

Organized by
Kelly Bickel, Lewisburg
Michael Hartz, Saarbrücken
John E. McCarthy, Saint Louis
Roland Speicher, Saarbrücken

28 April – 3 May, 2024

Abstract. The workshop brought together researchers in two fields, non-commutative function theory and free probability. There was a mini-course in each of these areas, and speakers in all the talks were encouraged to give expository talks that illuminated the broader reaches of their fields.

Mathematics Subject Classification (2020): 15B52, 32A70, 46L52, 46L54, 47A60, 60B20.

Introduction by the Organizers

The workshop *Non-commutative Function Theory and Free Probability*, organized by Kelly Bickel (Lewisburg), Michael Hartz (Saarbrücken), John E. McCarthy (Saint Louis), and Roland Speicher (Saarbrücken), was extremely successful. There were 44 participants from 12 different countries, with the largest contingents coming from the USA and Germany.

The goal of the workshop was to bring together researchers from two different fields, non-commutative function theory and free probability, to enhance understanding and establish links between them. The two fields have distinct cultures, but they are both part of non-commutative analysis. The organizers believed that in some sense, the two theories are the "complex analysis" and "real analysis" theories of non-commutative functions and that helping each group understand the other's field would reap benefits. Participants enthusiastically affirmed that the workshop succeeded in its goals.

Non-commutative function theory started with the work of J. Taylor in 1972 but really only took off in the twenty-first century. The first idea is that non-commutative polynomials in d variables can be better understood by evaluating them on d-tuples of $n \times n$ matrices, where n is allowed to vary. For example, Helton's theorem (2002) says that a non-commutative polynomial is a sum of squares if and only if it is positive on every tuple of self-adjoint matrices. The second idea is that limits of non-commutative polynomials should play the role in non-commutative theory that holomorphic functions play in the theory of several complex variables. This idea has been developed and extended in recent years.

Free probability started with the work of D. Voiculescu in the mid-1980s. He promoted the point of view of considering tuples of operators from a non-commutative probabilistic perspective. In particular, he defined the notion of "freeness" for operators as an analogue of the classical probabilistic notion of "independence." He proved that free variables model the limiting behavior of independently chosen random n-by-n matrices as n tends to infinity. Free probability has deep links to random matrix theory, von Neumann algebras, and combinatorics. Non-commutative functions play an important role in the analytic description of free variables.

There were two mini-courses of three lectures each—one in non-commutative function theory by Orr Shalit, the other in free probability by Hari Bercovici—to establish some common ground. Both mini-course lecturers provided their slides to all participants. In addition, there were 14 other long talks and 6 short talks, split between the two groups, talking about recent advances. There was also an illuminating personal history session, in which participants shared the mathematical problems that had inspired their interest in non-commutative analysis.

One of the workshop's highlights was the talk by Mireille Capitaine (Toulouse), who spoke of the solution, by her and S. Belinschi, of the Peterson–Thom conjecture. This was the 2011 conjecture in von Neumann algebras that any diffuse, amenable subalgebra of a free group factor is contained in a unique maximal amenable subalgebra. B. Hayes proved that it was equivalent to a conjecture in random matrices, which Belinschi and Capitaine proved.

Another highlight was the presentation by Michael Jury (Gainesville), who exhibited a surprising connection between random matrices and the Drury–Arveson space, also known as the symmetric Fock space. The Drury–Arveson space is a space of holomorphic functions that appears in operator theory and has strong ties to non-commutative function theory. Jury explained how the reproducing kernel of the Drury–Arveson space appears as the large-n limit of the expectation of a determinant involving random unitary $n \times n$ matrices. This talk sparked much discussion between the two different communities.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers Evangelos A. Nikitopoulos, Zachary Stier, and Jurij Volčič in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows." Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting David Jekel, Michael T. Jury, and Constanze Liaw in the "Simons Visiting Professors" program at the MFO.

Workshop: Non-commutative Function Theory and Free Probability

Table of Contents

Abstracts

Mini-course: Non-commutative function theory

Orr Moshe Shalit

The theory of non-commutative (henceforth, nc) functions was developed ahead of its time by J. Taylor in the early '70s but found no applications at the time [3, 4]. Several decades later, Voiculescu rediscovered these functions and proved some fundamental results about them in the context of free probability [5, 6]. Independently, researchers in systems theory and control theory or other areas related to multivariate operator theory have found a need to use nc rational functions and power series and, eventually, nc functions in full generality. The field has been methodologically developed and reorganized by many researchers, and to a certain extent, it has been standardized in the hands of Kaliuzhnyi-Verbovetskyi and Vinnikov in the monograph [2] that has become a sort of bible. In the decade that passed since the publication of The Bible, there have been many exciting developments in several directions; see, for example, Part II of the book [1].

I was invited to give a mini-course on nc function theory in Workshop 2418. The purpose of this series of three 50-minute lectures was to provide a solid basis for researchers who plan to start working in or making use of nc function theory.

In the **first lecture** of the series, we introduced the basic definitions and proved a few basic facts. Given an operator space $\mathcal{E} \subseteq B(H)$, we define the *nc universe over* $\mathcal{E}$ to be the disjoint union (over n) of the spaces of $n \times n$ matrices over $\mathcal{E}$:

$$\mathcal{E}_{nc} = \mathbb{M}(\mathcal{E}) = \bigsqcup_{n=1}^{\infty} M_n(\mathcal{E}).$$

An *nc set* is a subset $\Omega \subseteq \mathcal{E}_{nc}$ that is closed under direct sums. We write $\Omega_n = \mathcal{E}_{nc} \cap M_n(\mathcal{E})$ for the nth level of Ω. An *nc function* is a function $f\colon \Omega \to \mathcal{F}_{nc}$ from an nc set in $\mathcal{E}_{nc}$ into the nc universe over some other operator space $\mathcal{F}$ that is *graded* (i.e., $f(\Omega_n) \subseteq M_n(\mathcal{F})$) and respects direct sums ($f(X \otimes Y) = f(X) \oplus f(Y)$) and similarities ($f(S^{-1}XS) = S^{-1}f(X)S$). We showed the basic fact that a graded function is an nc function if and only if it respects intertwinings ($XT = TY \Rightarrow f(X)T = Tf(Y)$). We then defined an *nc holomorphic* function to be an nc function that is locally bounded and proved the remarkable fact that the local boundedness of an nc function implies that the function is continuous and, in turn, holomorphic. In fact, the derivative of f can be found by applying f to an upper triangular 2×2 block matrix:

$$f\left(\begin{pmatrix} X & Z \\ 0 & X \end{pmatrix}\right) = \begin{pmatrix} f(X) & Df(X)[Z] \\ 0 & f(X) \end{pmatrix}.$$

We discussed examples of the notions introduced, in particular polynomials and inverses.

In the **second lecture**, we took a deep look at the so-called *nc difference-differential calculus*. The key property is that the action of an nc function on an upper triangular block matrix is given by

$$f\left(\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}\right) = \begin{pmatrix} f(X) & \Delta f(X,Y)[Z] \\ 0 & f(Y) \end{pmatrix},$$

where $\Delta f(X, Y)$ is a linear map (which is completely bounded when f is nc holomorphic). As a function of X and Y, the creature $\Delta f(X, Y)$ is a so-called first-order nc function. We discussed nc functions of order n and how they can be "differentiated" to obtain nc functions of order $n+1$. This gives rise to the notion of nth order difference-differential operators $\Delta^n f(X_0, \ldots, X_n)[Z_1, \ldots, Z_n]$. We found that these higher-order "derivatives" can also be obtained by applying f to large upper triangular block matrices. The analysis culminated with the so-called Taylor–Taylor formula:

$$\begin{aligned} f(X) = &\sum_{k=0}^{n} \Delta^k f(Y, \ldots, Y)[X - Y, \ldots, X - Y] \\ &+ \Delta^{n+1} f(Y, \ldots, Y, X)[X - Y, \ldots, X - Y]. \end{aligned}$$

In the **third lecture**, we continued the analysis of the Taylor–Taylor formula, making it explicit in special cases and concluding that one can obtain a Taylor–Taylor series expansion around a point that converges absolutely and uniformly in certain balls:

$$f(X) = \sum_{k=0}^{\infty} \Delta^k f(Y, \ldots, Y)[X - Y, \ldots, X - Y].$$

We then concluded the lecture series by discussing some theorems about self-maps of the nc operator row unit ball (mentioning works of mine with Salomon and Shamovich and of Belinschi and Shamovich) and how these theorems can be applied to the classification of algebras of bounded nc functions on subvarieties of the nc operator row unit ball.

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Mini-course: The use of analyticity in free probability
Hari Bercovici

This is a summary of three introductory lectures on free probability theory, with emphasis on the use of analytic functions and non-commutative functions. The foundations of the subject were laid out by Dan Voiculescu.

The general context is that of an *algebraic probability space*. This consists of a complex unital algebra $\mathcal{A}$ endowed with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(1) = 1$. The elements of $\mathcal{A}$ are thought of as *random variables*, and φ plays the role of *expected value.*

Suppose that $(\mathcal{A}, \varphi)$ is such an object and that $(\mathcal{A}_i)_{i\in I}$ is a family of unital subalgebras of $\mathcal{A}$. These subalgebras are said to be (classically) *independent* (relative to φ) if the elements of $\mathcal{A}_i$ commute with the elements of $\mathcal{A}_j$ for $i \neq j$, and $\varphi(a_1a_2\cdots a_n) = 0$ whenever $a_j \in \mathcal{A}_{i_j}$, $\varphi(a_j) = 0$, and $i_j \neq i_k$ for all $j, k = 1, \dots n$. On the other hand, the algebras $(\mathcal{A}_i)_{i\in I}$ are said to be *free* (relative to φ) if $\varphi(a_1 \cdots a_n) = 0$ whenever $a_j \in \mathcal{A}_{i_j}$, $\varphi(a_j) = 0$, and $i_j \neq i_{j-1}$ for all $j = 2, \dots, n$.

Supposing that the algebras $(\mathcal{A}_i)_{i\in I}$ generate the algebra $\mathcal{A}$, the functional φ is uniquely determined by the restrictions $\varphi|_{\mathcal{A}_j}$ if the subalgebras are either independent or free. To illustrate the difference between the two notions, suppose that $J = \{1, 2\}$, $a_1, b_1 \in \mathcal{A}_1$, and $a_2, b_2 \in \mathcal{A}_2$. Then $\varphi(a_1a_2b_1b_2) = \varphi(a_1b_1)\varphi(a_2b_2)$ if $\mathcal{A}_1$ is independent from $\mathcal{A}_2$, but

$$\varphi(a_1b_1)\varphi(a_2b_2) - \varphi(a_1a_2b_1b_2) = (\varphi(a_1b_1) - \varphi(a_1)\varphi(b_1))\,(\varphi(a_2b_2) - \varphi(a_2)\varphi(b_2))$$

if $\mathcal{A}_1$ is free from $\mathcal{A}_2$. Thus, freeness is an analog of independence, rather than a generalization.

Simple examples of the two notions are obtained as follows. Suppose that G is a group, and denote by $\ell^2(G)$ the Hilbert space of square-summable functions defined on G. Then the operators $\lambda(g) : \ell^2(G) \to \ell^2(G)$ defined by $(\lambda(g)f)(h) = f(g^{-1}h)$ are unitary. The group algebra $\mathbb{C}G$ is generated by $\{\lambda(g) : g \in G\}$, and $\varphi : \mathbb{C}G \to \mathbb{C}$ is defined by $\varphi(T) = \langle T\chi_{\{e\}}, \chi_{\{e\}}\rangle$ or, equivalently, $\varphi(\lambda(g)) = 0$ for $g \neq e$. If G is the direct sum of a family $\{G_j\}_{j\in J}$ of groups, then $\{\mathbb{C}G_j\}_{j\in J}$ are independent. If G is a free product of a family $\{G_j\}_{j\in J}$ of groups, then $\{\mathbb{C}G_j\}_{j\in J}$ are free.

One denotes by LG the weak operator–closed algebra generated by $\mathbb{C}G$. It is an open problem whether $L\mathbb{F}_2$ is isomorphic to $L\mathbb{F}_3$, where $\mathbb{F}_n$ is the free non-commutative group with n generators. It is hoped that free probability theory will settle this question.

Another example of free algebras is provided by a similar construction starting from the free monoid W generated by $a_1, \dots, a_n$. The functions $e_w = \chi_{\{w\}}$ form an orthonormal basis in $\ell^2(W)$, and one sets $\varphi(T) = \langle Te_\varnothing, e_\varnothing\rangle$ for every bounded linear operator T on $\ell^2(W)$. The formula $L_je_w = e_{a_jw}$ defines an isometric operator on $\ell^2(W)$, and the algebras $\mathcal{A}_j$ generated by $\{L_j, L_j^*\}$ are free relative to φ. Moreover, the weak operator–closed algebra generated by $\{L_j + L_j^*\}_{j=1}^n$ is isomorphic to $L\mathbb{F}_n$.

There is a remarkable connection between freeness and classical independence provided by random matrices. Suppose that $U_1^{(N)}, \dots, U_n^{(N)}$ are random $N \times N$ unitary matrices, chosen independently and uniformly distributed on the unitary group $\mathrm{U}(N)$. For $A \in M_N(\mathbb{C})$, we write $\mathrm{tr}_N(A)$ for the normalized trace. For a random matrix A, we denote by $\varphi_N(A)$ the expected value of $\mathrm{tr}_N(A)$. Suppose that $\mathbb{F}_n$ is freely generated by $x_1, \dots, x_n$ and $\lambda(x_j) \in L\mathbb{F}_n$ are defined as above. Then Voiculescu proved that, given a monomial p in $2n$ non-commuting variables a_j,a_j^*, we have

$$\lim_{N\to\infty} \varphi_N(p(U_j^{(N)}, U_j^{*N})) = \varphi(p(\lambda(x_j), \lambda(x_j^*))).$$

In other words, $U_1^{(N)}, \dots, U_n^{(N)}$ are *asymptotically free*. This has generated fruitful interactions between free probability and the study of random matrices.

It is natural to extend the notions of (*joint*) *distribution* to random variables in an algebraic probability space $(\mathcal{A}, \varphi)$. If $\mathbb{C}[a]$ is the algebra generated by $a \in \mathcal{A}$, the distribution μ_a should encode the pair $(\mathbb{C}[a], \varphi|_{\mathbb{C}[a]})$. Similarly, if $a = (a_1, \dots, a_n) \in \mathcal{A}^n$, the distribution μ_a should encode $(\mathbb{C}[a_1, \dots, a_n], \varphi|_{\mathbb{C}[a_1,\dots,a_n]})$. In a classical probability space $A = L^\infty(P)$, μ_a is the pushforward of P, a probability measure on $\mathbb{C}^n$. In the general case, the distribution of a single variable a can be encoded in the sequence $\{\varphi(a^n) : n \in \mathbb{N}\}$ of *moments*. Similarly, if $a = (a_1, \dots, a_n)$, μ_a is encoded in the collection of joint moments $\varphi(p(a))$, $p \in W$ (non-commuting monomials). If $\mathcal{A}$ is a $*$-algebra, one includes monomials in a and a^* to obtain the $*$-*distribution* of a. The moments of a single variable can also be encoded in the formal series

$$G_a(z) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{z^{n+1}}.$$

When $\mathcal{A}$ is a Banach algebra and φ is continuous, we have $G_a(z) = \varphi((z-a)^{-1})$ for large $|z|$. The formal series $G_a(z) = \sum_{n=0}^{\infty} \tau(a^n)/z^{n+1}$ has a formal (convergent near ∞ in the Banach case) inverse $K_a(z) = (1/z) + \sum_{n=1}^{\infty} c_n z^{n-1}$ (that is, $G_a(K_a(z)) = z$); we set

$$R_a(z) = R_{\mu_a}(z) = \sum_{n=1}^{\infty} c_n z^{n-1}.$$

Voiculescu showed that, given free random variables a_1, a_2 in $(\mathcal{A}, \varphi)$, we have $R_{a_1+a_2} = R_{a_1} + R_{a_2}$. Equivalently, $K_{a_1+a_2}(z) = K_{a_1}(z) + K_{a_2}(z) - 1/z$. (A short proof of this result was presented in the first lecture.) This leads to the definition of the *free additive convolution* $\mu_1 \boxplus \mu_2$ of distributions that satisfies $R_{\mu_1 \boxplus \mu_2} = R_{\mu_1} + R_{\mu_2}$. If μ_1 and μ_2 are compactly supported probability measures on $\mathbb{R}$, then so is $\mu_1 \boxplus \mu_2$. This is seen by realizing μ_j as the distributions of free self-adjoint variables a_j in a C^*-algebra. In this context, G_{μ_1}, G_{μ_2}, and $G_{\mu_1 \boxplus \mu_2}$ are defined in the upper complex half-plane $\mathbb{H}$, and they are conformal at ∞.

Set $\omega_j = G_{\mu_j}^{-1} \circ G_\mu$ near ∞, where $\mu = \mu_1 \boxplus \mu_2$. The equation $\omega_1(z) + \omega_2(z) = z + 1/G_\mu(z)$ near ∞ is equivalent to $R_{\mu_1} + R_{\mu_2} = R_\mu$. In the special case $\mu_1 = \mu_2$, we see that

$$\omega_1(z) = \omega_2(z) = \frac{1}{2}\left(z + \frac{1}{G_\mu(z)}\right) = \frac{1}{2}\left(z + \frac{1}{G_{\mu_1}(\omega_1(z))}\right),$$

and thus, ω_j is defined on $\mathbb{H}$, not just near ∞.

The above observation provides a connection to classical function theory. Given $\mu = \mu_1 \boxplus \mu_1$ (on $\mathbb{R}$), we set $f(z) = 2z - (1/G_{\mu_1}(z))$ and conclude that $f(\omega_1(z)) = z$, $z \in \mathbb{H}$. This provides a free probability proof of the following statement:

Given an analytic $f : \mathbb{H} \to \mathbb{C}$ such that $\Im f(z) \leq \Im z$ and $\Im f(iy)/y \to 1$ as $y \uparrow \infty$, there exists $\omega : \mathbb{H} \to \mathbb{H}$ analytic such that $f(\omega(z)) \equiv z$. Moreover, ω extends continuously to $\mathbb{R}$.

This result does have a classical proof as well: $\omega(z)$ is the Denjoy–Wolff point of $w \mapsto z + w - f(w)$ (viewed as a function on $\mathbb{H}$). The existence of ω leads to faster calculations of certain free convolutions, illustrated as follows. Suppose that $\nu = \mu \boxplus \mu$ and $\mu = (\delta_1 + \delta_{-1})/2$. Then $G_\nu = G_\mu \circ \omega$, where ω is a right inverse of $2z - 1/G_\mu(z) = z + (1/z)$. Thus

$$\omega(z) = \frac{z + \sqrt{z^2 - 4}}{2}, \quad G_\nu(z) = \frac{1}{\sqrt{z^2 - 4}},$$

and $-\Im G_\mu/\pi = \text{Poisson}(\mu)$ gives the density $1/(\pi\sqrt{4 - t^2})$ on $(-2, 2)$. Another example yields the free analog of normal distributions. Suppose we want variables x_1, x_2, x_3 identically distributed so that x_1, x_2 are free, $\varphi(x_1) = 0$, $\varphi(x_1^2) = 1$, and $x_1 + x_2 = \sqrt{2}x_3$. If μ is the common distribution, it follows that $K_\mu(z) = z + 1/z$, so

$$G_\mu(z) = \frac{z + \sqrt{z^2 - 4}}{2}, \quad d\mu = \frac{\sqrt{4 - t^2}}{2\pi}\chi_{(-2,2)}\, dt.$$

This is the distribution that appears as the weak limit in Voiculescu's free central limit theorem.

The fact that ω_j extends to $\mathbb{H}$ is not unique to convolutions of identical measures, as first proved by Voiculescu and Biane (with subsequent improvements by Belinschi). Thus, given free self-adjoint a_1 and a_2, we have $G_{a_1 + a_2} = G_{a_j} \circ \omega_j$, where the functions ω_j map $\mathbb{H}$ to itself and extend continuously to $\overline{\mathbb{H}}$. The function ω_j also satisfies $E_j(z - (a_1 + a_2))^{-1} = (\omega_j(z) - a_j)^{-1}$, where E_j is the conditional expectation onto the (closure of) $\mathbb{C}[a_j]$.

The functions ω_j are usually called *subordination functions,* and their existence has important consequences. For instance, $\mu_1 \boxplus \mu_2$ is absolutely continuous except for a few point masses. The only exception to this rule arises when either μ_1 or μ_2 is a point mass. In addition, $\mu_1 \boxplus \mu_2$ has a point mass only for unavoidable reasons, namely $\mu_1(\{a_1\}) + \mu_2(\{a_2\}) > 1$.

For certain matrix models, say $X_N = A_N + U_N B_N U_N^*$, with U_N uniform random in $\mathrm{U}(N)$ and A_N, B_N non-random, $\mathbb{E}[(z - X_N)^{-1}]$ is a function of A_N (in the sense of functional calculus), close to $(\omega_1(z) - A_N)^{-1}$. This provides information about eigenvalue distribution and spikes.

One can also use the subordination functions to deduce locally uniform convergence in weak limit theorems, to study the connectedness of supports, and to prove local limit theorems for some random matrices.

One extension of free probability arises when considering *operator-valued* random variables. In this context, the basic setup consists of a unital complex algebra $\mathcal{A}$, a unital subalgebra $\mathcal{B}$ of $\mathcal{A}$, and a *conditional expectation* $E : \mathcal{A} \to \mathcal{B}$. That is, E is linear, $E1 = 1$, and $E(b_1 a b_2) = b_1 E(a) b_2$ for $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$. The elements of $\mathcal{A}$ are thought of as $\mathcal{B}$-valued random variables. A family $\{\mathcal{A}_j\}_{j\in J}$ of subalgebras of $\mathcal{A}$ such that $\mathcal{B} \subset \mathcal{A}_j$ is said to be $\mathcal{B}$-free if $E(a_1 \cdots a_n) = 0$ whenever $E(a_i) = 0$ and $a_i \in \mathcal{A}_{j_i}$ with $i_j \neq i_{j-1}$, $j = 2, \dots, n$.

The distribution of a $\mathcal{B}$-valued random variable $a \in \mathcal{A}$ must encode the pair $(\mathcal{B}[a], E|_{\mathcal{B}[a]})$, where $\mathcal{B}[a]$ is the algebra generated by $\mathcal{B} \cup \{a\}$. One must consider moments of order n, namely $E(b_0 a b_1 \cdots a b_n)$ with $b_0, \dots, b_n \in \mathcal{B}$. (More economically, $E(a b_1 a \cdots b_{n-1} a)$.). The formal series

$$G_a(b) = E[(b-a)^{-1}] = \sum_{k=0}^{\infty} E[b^{-1}(ab^{-1})^k]$$

collects only symmetric moments. One can still define $R_a(b) = K_a(b) - b^{-1}$, where K_a is inverse to G_a, and additivity holds: $R_{a_1+a_2} = R_{a_1} + R_{a_2}$ if a_1 is $\mathcal{B}$-free from a_2.

If $\mathcal{A}$ is a C^*-algebra, $\mathcal{B}$ a C^*-subalgebra, E is continuous, and $a_1, a_2 \in \mathcal{A}$ are $*$-free and self-adjoint, then $G_{a_j}(b)$ is defined in

$$\mathbb{H}(\mathcal{B}) = \{x + iy : x, y \in \mathcal{B},\ x = x^*,\ y = y^* > 0\}.$$

Subordination persists in this context. There exist $\Omega_j : \mathbb{H}(\mathcal{B}) \to \mathbb{H}(\mathcal{B})$ analytic so that

$$G_{a_1+a_2}(b) = G_{a_j}(\Omega_j(b)), \quad j = 1, 2,$$
$$\Omega_1(b) + \Omega_2(b) = b + [G_{a_j}(\Omega_j(b))]^{-1}.$$

In the W^* case, we also have

$$E_{\mathcal{B}(a_j)}[(b - (a_1 + a_2))^{-1}] = (\Omega_j(b) - a_j)^{-1},$$

and there is a natural proof based on a coalgebra structure that makes $E_{\mathcal{B}(a_j)}$ homomorphisms. Provided a is algebraically free from $\mathcal{B}$, one defines a derivation

$$\partial = \partial_a : \mathcal{B}[a] \to \mathcal{B}[a] \otimes \mathcal{B}[a]$$

such that $\delta(a) = 1 \otimes 1$. Elements of the form $f = (b-a)^{-1}$ “$\in$ ”$\mathcal{B}[a]$ satisfy the equation $\partial_a f = f \otimes f$. Conversely, every *invertible* solution f of this equation is a resolvent of a.

Suppose a_1, a_2 are $\mathcal{B}$-free; ∂_{a_1} is defined on $\mathcal{B}[a_1]$, $\partial_{a_1+a_2}$ on $\mathcal{B}[a_1 + a_2]$. Then the conditional expectation $E_1 : \mathcal{B}[a_1, a_2] \to \mathcal{B}[a_1]$ is a coalgebra homomorphism. If $f \in \mathcal{B}[a_1 + a_2]$ satisfies $\partial_{a_1+a_2} f = f \otimes f$ then $g = E_1 f$ satisfies $\partial_{a_1} g = g \otimes g$. In the self-adjoint case, $b \in \mathbb{H}(\mathcal{B})$, consider $f = (b - (a_1 + a_2))^{-1}$. Then $g = E_1 f$ is invertible, hence $g = (b' - a_1)^{-1}$ with $b' \in \mathbb{H}(\mathcal{B})$. Define Ω_1 by $\Omega_1 : b \mapsto b'$.

Returning to the distribution of a $\mathcal{B}$-valued random variable, we can encode all moments if we use the matricial resolvent

$$G_a(B) = E_n[(B - 1_n \otimes a)^{-1}],\ B \in M_n(\mathcal{B}),$$

where $E_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is simply E applied entrywise. In the C^* context (with $a_j = a_j^*$), Ω_1 and Ω_2 also exist on $\mathbb{H}(M_n(\mathcal{B}))$, and they are non-commutative functions. It is not known whether these functions to extend to $\overline{\mathbb{H}(M_n(\mathcal{B}))}$ or even to the self-adjoint matrices.

There are good reasons why operator-valued variables are useful. Suppose $\{a_1, b_1\}$ is free from $\{a_2, b_2\}$ relative to φ (scalar-valued expected value). Then X_1 is $\mathcal{B}$-free from X_2 if

$$X_1 = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix},\ X_2 = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix},\ \mathcal{B} = M_2(\mathbb{C}) \subset M_2(\mathcal{A}).$$

This trades joint distributions relative to φ for single $\mathcal{B}$-valued distributions and also allows one to consider operations other than addition. For instance, suppose that a_1 is free from a_2 (relative to φ) and p is a polynomial in two non-commuting variables. Then the invertibility of $p(a_1, a_2)$ is equivalent to the invertibility of a linear expression

$$\alpha_1 \otimes a_1 + \alpha_2 \otimes a_2 + \beta \in M_n(\mathcal{A}).$$

Here $\alpha_1, \alpha_2, \beta \in M_n(\mathbb{C})$, and the summands are $\mathcal{B}$-free, $\mathcal{B} = M_n(\mathbb{C})$.

This observation allows one to treat, among other things, $\mathcal{B}$-valued atoms of a polynomial (or even rational function) $p(a_1, a_2)$.

The bibliography below contains three books that describe various aspects of free probability, as well as two works of Voiculescu in which the theory of non-commutative functions appears explicitly in the subject.

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An overview of non-commutative Choquet theory

Eli Shamovich

(joint work with Matthew Kennedy)

The classical theorem of Choquet, which was later extended by Bishop and de Leeuw to the non-metrizable setting, can be viewed as an improvement on the celebrated Krein–Milman theorem. To be more precise, if $\mathcal{E}$ is a locally convex topological vector space and $K \subset \mathcal{E}$ is a metrizable compact convex set, then for every $x \in K$, there exists a probability measure supported on $\partial_e K$, the extreme boundary of K, such that for every continuous functional $f \in \mathcal{E}'$, $\int_{\partial_e K} f \, d\mu = f(x)$. In this case, we say that μ represents x.

A probability measure is a limit in the weak* topology of convex combinations of point masses. In this sense, a probability measure is a generalization of a convex combination. Therefore, one can view the Choquet theorem as saying that every point in K is a generalized convex combination of the extreme points. In the non-metrizable setting, an example of Mokobodski shows that $\partial_e K$ may not be a Borel set. Hence, Bishop and de Leeuw refined the statement using the Choquet order. For two probability measures μ and ν on K, $\mu \leq_c \nu$ if for every convex function $f \in C(K)$, $\mu(f) \leq \nu(f)$. The full statement of the theorem is now:

Theorem 1 (Choquet, Bishop–de Leeuw)**.** *For every $x \in K$, there exists a probability measure μ on K, maximal in the Choquet order, that represents x.*

In the metrizable case, maximality is equivalent to having its support on the extreme boundary. In the non-metrizable case, it is equivalent to annihilating every Baire set that is disjoint from the boundary.

There are numerous applications of the Choquet theorem in analysis. There are also quite important applications in dynamics. To this end, we need the notion of a Choquet simplex. A compact convex set K is called a (Choquet) simplex if every point $x \in K$ admits a *unique* maximal representing measure. This definition is a straightforward generalization of the definition of a simplex in $\mathbb{R}^n$. In dynamics, if Γ is a discrete group that acts on a compact topological space, then the collection of all invariant probability measures is a simplex. The extreme points are the ergodic measures. Thus, Choquet's theorem yields the ergodic decomposition theorem. In infinite dimensions, simplices come in many forms. The two extremes are Bauer simplices and the Poulsen simplex. A Bauer simplex K is a simplex such that $\partial_e K$ is closed. By a result of Bauer, this simplex is affine homeomorphic to the space of probability measures on $\partial_e K$. On the other extreme lies the Poulsen simplex, a simplex K such that $\overline{\partial_e K} = K$. The article "the" is justified by a result of Lindenstrauss, Olsen, and Sternfeld that says the Poulsen simplex is unique up to affine homeomorphism.

The ideas of non-commutative Choquet theory were introduced first by Arveson in his seminal work [1]. The original point of view of Arveson was a dual one. Classically, Kadison's representation theorem states that there is a duality between the category of compact convex sets with affine continuous maps between them and the category of function systems with unital positive maps as morphisms.

We recall that a function system is a closed and conjugation-invariant subspace $F \subset C(X)$ containing 1, where X is a compact Hausdorff space. Every function system inherits an order from $C(X)$, and 1 is an Archimedean order unit. The duality sends a compact convex set K to $A(K)$, the function system of affine continuous functions on K. The other side of the duality is $F \mapsto \text{state}(F)$, where $\text{state}(F)$ is the compact convex set of all states, i.e., unital positive functionals, on F. Arveson considered operator systems—namely, closed and $*$-invariant subspaces $S \subset B(\mathcal{H})$ containing 1. The natural maps between operator systems are unital, completely positive maps. Arveson introduced the boundary of an operator system. If A is a C^*-algebra that contains S and is generated by S, an irreducible representation π of A is called boundary if $\pi|_S$ has a unique UCP extension to A. Every boundary representation factors through $C^*_e(S)$, the minimal C^*-algebra that admits a completely isometric embedding of S. In fact, by results of Dritschel and McCullough, Arveson, and Davidson and Kennedy, the C^*-envelope is the C^*-algebra generated by S in its image under the direct sum of all boundary representations. Wittstock first considered the non-commutative version of compact convex sets. Webster and Winkler first proved the duality between the categories. We shall present a different approach tailored specifically to Choquet theory and introduced by Davidson and Kennedy in [2]. To each operator system, we can associate its non-commutative state space

$$\text{ncState}(S) = \bigsqcup_{n \leq \kappa} \text{UCP}(S, B(\mathcal{H}_n)).$$

Here, κ is some sufficiently large cardinal, and $\mathcal{H}_n$ is a fixed n-dimensional Hilbert space. We note that the direct sum of two UCPs is a UCP (with a certain identification of the Hilbert spaces), and a compression of a UCP by an isometry is also a UCP. Moreover, each level is compact with respect to the point weak* topology. This leads us to a definition of an nc compact convex set. Roughly speaking, it is a graded set as above that is invariant under taking direct sums and under compression by isometries and level-wise compact. Davidson and Kennedy [2] show that every nc compact convex set is determined by finite levels. Moreover, the Webster–Winkler duality provides a duality of the category of nc compact convex sets with nc affine maps as morphisms and the category of operator systems. An affine map is a map that is graded and respects direct sums and compressions by isometries. In [2], the authors extensively develop non-commutative Choquet theory. The role of the continuous functions is played by $C^*_{max}(S)$, the maximal C^*-cover of an operator system constructed by Kirchberg and Wassermann. Probability measures are replaced by UCPs on $C^*_{max}(S)$, and the Choquet order is replaced by its nc version that has two equivalent formulations, one using non-commutative convex functions and the other using dilations. It is quite surprising how far the analogy carries through.

Non-commutative Choquet simplices were studied in [3]. An nc Choquet simplex is an nc compact convex set, such that every point admits a unique maximal representing UCP on $C^*_{max}(S)$. It turns out that the corresponding operator systems are precisely the C^*-systems studied by Kirchberg and Wassermann.

Kirchberg and Wassermann constructed nc Poulsen simplices. Those are operator systems S such that $C_e^*(S) = C_{max}^*(S)$. Lupini proved that the nuclear nc Poulsen simplex is unique. However, the construction of Kirchberg and Wassermann can yield non-nuclear Poulsen simplices. On the other side of the spectrum, there are the nc Bauer simplices. We were able to show that an nc compact convex set is a Bauer simplex if and only if it is the nc state space of a C^*-algebra. Moreover, if a discrete group Γ acts on a C^*-algebra, then the invariant UCPs in the nc state space (if not empty) form an nc simplex. These observations allowed us to extend a classical result of Glasner and Weiss in dynamics.

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Regularity in free probability: An overview and some recent results on graph products

Ian Charlesworth

Regularity questions in free probability are inspired by information theory and entropy and can be thought of as measuring how close to freely independent a tuple of (non-commutative) variables is. The goal of this talk is to give an overview of the development of these ideas, as well as some recent results they lead to. The theory of free entropy was initiated by Voiculescu in a series of papers in the 1990s, but see [2] for an overview. One of the main motivations is to find von Neumann algebraic consequences of probability-flavoured information about the generators of an algebra. Broadly speaking, there are two approaches to the theory: the microstates approach and the non-microstates approach. With the announced resolution of the Connes Embedding Problem of Ji, Natarajan, Vidick, Wright, and Yuen, it follows that the two approaches are not always the same, although there currently are no explicit examples where the two versions of entropy disagree.

The non-microstates approach to free entropy, due to Voiculescu, was the second chronologically. It begins with an analogy to de Bruijn's identity in the classical setting: Under sufficiently strong hypotheses, if p is a probability density and p_v is the result of convolving p with a Gaussian of variance v, then the derivative of the entropy of p_v with respect to v is one half the Fisher information of p [3]. The goal then becomes finding the right notion of free Fisher information from which to compute free entropy. This is done using the free difference quotients: Given a tuple of self-adjoint variables $x_1, \dots, x_n$ in some tracial von Neumann algebra $(\mathcal{M}, \tau)$, one defines the free difference quotients ∂_i as densely defined unbounded operators $\mathbb{C}\langle x_1, \dots, x_n\rangle \to L^2(\mathcal{M}) \otimes L^2(\mathcal{M}^{\mathrm{op}})$ via linearity, the Leibniz rule, and the condition that $\partial_i(x_j) = \delta_{i,j} 1 \otimes 1$. Their adjoints are unbounded operators $\partial_j^* : L^2(\mathcal{M}) \otimes L^2(\mathcal{M}^{\mathrm{op}}) \to L^2(\mathcal{M})$; if $1 \otimes 1$ happens to be in the domain of the adjoint of ∂_j^* for each j, these become the free score functions, and one defines the free Fisher information as $\Phi^*(x_1, \ldots, x_n) = \sum_{j=1}^n \|\partial_j^* 1 \otimes 1\|_2^2$. One can then integrate this quantity for perturbations $x_j(t) = x_j + \sqrt{t} s_j$ of the xs by free semicircular elements and arrive at a notion of free entropy. (This must be done more carefully if the xs have algebraic relations, but the $x_j(t)$s never will for $t > 0$, so this subtlety can be avoided.) Assumptions on the non-microstates free entropy can have strong consequences: For example, if one assumes (something weaker than) that $x_1, \ldots, x_n$ have finite entropy in this sense, then Mai, Speicher, and Yin have shown that the variables have no "non-commutative rational relations."

One way to extend this idea is the free Stein dimension, which is joint work with Nelson. Writing $\mathcal{M} = W^*(x_1, \ldots, x_n)$, one considers the space of derivations $\mathbb{C}\langle x_1, \ldots, x_n \rangle \to L^2(\mathcal{M}) \otimes L^2(\mathcal{M}^{\mathrm{op}})$ with $1 \otimes 1$ in the domain of their adjoints, which can be equipped with the structure of an $\mathcal{M} \bar{\otimes} \mathcal{M}^{\mathrm{op}}$-module; the dimension of this module is then the free Stein dimension. Again, this leads to structural conditions on $\mathcal{M}$.

On the microstates side, one seeks to understand non-commutative random variables by seeing how easily they can be approximated by matrices. (Starting now, we shall make the standing assumption that all von Neumann algebras considered are embeddable in an ultrapower of the hyperfinite II_1 factor $\mathcal{R}$, so at least *some* matricial approximations always exist.) If the variables $x_1, \ldots, x_n$ are freely independent and $X_1^{(N)}, \ldots, X_n^{(N)}$ are $N \times N$ matrices whose laws individually converge to those of the xs and if the Xs are conjugated by independent uniformly distributed unitary matrices, then these conjugates have (joint) law converging to that of the tuple $(x_1, \ldots, x_n)$. Hence, when variables are actually free, matricial approximations are easy to find. Meanwhile, if the xs instead satisfy some relation (for example, if $W^*(x_1, \ldots, x_n)$ has a trace-1/2 central projection), then this leads to restrictions on the microstates (in this example, that they must be almost simultaneously block diagonalizable) that cause them to be relatively scarce; Voiculescu showed, for example, that the existence of a Cartan subalgebra in $W^*(x_1, \ldots, x_n)$ dramatically limits (in a technical sense) the availability of microstates. This allowed Voiculescu to show that the free group factors do not admit Cartan subalgebras, as they can be generated by standard semicircular systems with ample microstates.

The technical definition is as follows: for a tuples of variables x and y, the (microstates) free entropy of x in the presence of y is given by

$$\chi(x : y) = \sup_{R>0} \inf_{\mathcal{U}} \limsup_{k \to \infty} \left[\frac{n \log k}{2} + \frac{1}{k^2} \log \lambda \left\{ X \in M_k(\mathbb{C})_{\mathrm{sa}}^n \;\middle|\; \substack{\exists Y \in M_k(\mathbb{C})_{\mathrm{sa}}^m \\ \mu_{X,Y} \in \mathcal{U} \\ \|X\|, \|Y\| < R} \right\} \right],$$

where the infimum is over all weakly open neighbourhoods of the law $\mu_{x,y}$ of x and y in $\mathbb{C}\langle T_1, \ldots, T_n, S_1, \ldots, S_m \rangle^*$. It turns out that it is often interesting to study the behaviour of free entropy of a tuple x under small perturbations by freely independent semicirculars; along these lines, Jung introduced the following: A tuple x is strongly 1-bounded if, when s is a standard free semicircular family free from x and $x_1(t) = x_1 + \sqrt{t} s_1, \ldots, x_n(t) = x_n + \sqrt{t} s_n$, one has, for some constant C, that $\chi(x(t):s) \leq (n-1)\log\sqrt{t} + C$ [1]. Strong 1-boundedness is an extreme restriction on the space of microstates; if $W^*(x_1,\dots,x_n)$ is diffuse, it is saying that there are, in some sense, as few microstates as possible. Significantly, under very minor assumptions, this is a property of the von Neumann algebra rather than the tuple of generators. The result of Voiculescu mentioned above can be strengthened to show that any von Neumann algebra that admits a Cartan subalgebra must be strongly 1-bounded.

Strong 1-boundedness and the related 1-bounded entropy (implicit in Jung's work) developed by Hayes form one of the promising directions of recent study in free regularity. For example, recent joint work with de Santiago, Hayes, Jekel, Kunnawalkam Elayavalli, and Nelson shows that in the setting of graph products of matrix algebras, strong 1-boundedness can be deduced from the vanishing of the first ℓ^2 Betti number of the algebra formed by the generators within the larger von Neumann algebra.

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Asymptotic inference in quantum statistical models

Anna Skripka

(joint work with Michael Nussbaum)

Asymptotic equivalence of statistical models (experiments) allows one to utilize methods developed for simpler models in the analysis of more complex models. For instance, finding an optimal estimator or inferring properties of an unknown parameter of a quantum model can become feasible after establishing an asymptotic equivalence of the latter to a manageable classical experiment.

A quantum statistical experiment is a collection of normal states on a von Neumann algebra indexed by a parameter set,

$$\{\varphi_\theta : \mathcal{A} \to \mathbb{C} \mid \theta \in \Theta\},$$

which generalizes a classical experiment of a parametric set of probability measures on a measurable space (Ω, Σ).

A sequence of statistical experiments $\mathcal{E}_n = \{\varphi_\theta : \mathcal{A}_n \to \mathbb{C} \mid \theta \in \Theta\}$ is said to be asymptotically more informative than a sequence of statistical experiments $\mathcal{F}_n = \{\psi_\theta : \mathcal{B}_n \to \mathbb{C} \mid \theta \in \Theta\}$ if

$$\delta(\mathcal{E}_n, \mathcal{F}_n) := \inf_{\alpha_n} \sup_{\theta\in\Theta} \|\psi_\theta - \varphi_\theta \circ \alpha_n\| \to 0 \text{ as } n \to \infty,$$

where the infimum is taken over all quantum channels $\alpha_n : \mathcal{B}_n \to \mathcal{A}_n$. The sequences of experiments $\{\mathcal{E}_n\}_n$ and $\{\mathcal{F}_n\}_n$ are said to be equivalent if

$$\Delta(\mathcal{E}_n, \mathcal{F}_n) := \max\{\delta(\mathcal{E}_n, \mathcal{F}_n),\, \delta(\mathcal{F}_n, \mathcal{E}_n)\} \to 0 \text{ as } n \to \infty.$$

One of the fundamental notions in asymptotic statistics is local asymptotic normality introduced by L. Le Cam in 1960. A typical result on local asymptotic normality is stated below.

Theorem 1 ([1, Thm. 3.1]). *Let Θ be an open subset of $\mathbb{R}^d$, (Ω, Σ, μ) be a probability measure space, and $\{p_\theta\}_{\theta\in\Theta}$ be a family of probability densities on Ω satisfying*

$$\lim_{u\to 0} \int_\Omega \left(\sqrt{p_{\theta+u}} - \sqrt{p_\theta}(1 - \langle \ell_\theta, u\rangle)\right)^2 d\mu = 0$$

for some measurable function $\ell_\theta : \Omega \to \mathbb{R}^d$. Let $dP_\theta = p_\theta\, d\mu$, and let P_θ^n be the product measure of n copies of P_θ. Given $\theta_0 \in \Theta$ and $C > 0$, define the experiments

$$\mathcal{E}_n = \{P^n_{\theta_0 + u/\sqrt{n}} \mid \|u\| \le C\} \quad \textit{and} \quad \mathcal{F} = \{N(u, I_{\theta_0}^{-1}) \mid \|u\| \le C\},$$

where $N(u, I_{\theta_0}^{-1})$ is a multivariate normal distribution and I_{θ_0} is the Fisher information matrix of $\{p_\theta\}_{\theta\in\Theta}$. Then,

$$\Delta(\mathcal{E}_n, \mathcal{F}) \to 0 \textit{ as } n \to \infty.$$

Local asymptotic normality has been generalized to the quantum setting (see, e.g., [1]), while, to the best of our knowledge, global asymptotic equivalence results were known only in the classical setting. The works in progress [2, 3] establish an asymptotic equivalence of *quantum time series models* to some classical experiments and use the latter equivalence to construct optimal estimators of state parameters. Those quantum models pertain to certain gauge invariant quasifree states with Toeplitz symbols on algebras of bounded linear operators on Fock spaces.

Let $\Gamma_s(\mathbb{C}^n)$ be a symmetric (bosonic) and $\Gamma_a(\mathbb{C}^n)$ an antisymmetric (fermionic) Fock spaces over $\mathbb{C}^n$, respectively, that is,

$$\Gamma_s(\mathbb{C}^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_n^{ⓢ^k} \quad \text{and} \quad \Gamma_a(\mathbb{C}^n) = \bigoplus_{k=0}^{n} \mathcal{H}_n^{ⓐ^k},$$

where

$$\begin{aligned} \mathcal{H}_n^{ⓢ^0} &= \mathbb{C}, \quad \mathcal{H}_n^{ⓢ^k} = \{u \in (\mathbb{C}^n)^{\otimes k} \mid u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)} = u \quad \forall\, \sigma \in S_k\}, \quad k \in \mathbb{N}, \\ \mathcal{H}_n^{ⓐ^0} &= \mathbb{C}, \quad \mathcal{H}_n^{ⓐ^k} = \{u \in (\mathbb{C}^n)^{\otimes k} \mid u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)} = \operatorname{sign}(\sigma) u \quad \forall\, \sigma \in S_k\}. \end{aligned}$$

To define a state, we start with a measurable function f on $[-\pi, \pi]$, set f_k to be its kth Fourier coefficient, $f_{j,k} = f_{k-j}$, and $Q_n(f) = (f_{j,k})_{j,k=1}^n$. Let $d \in \mathbb{N}$ and

$\vartheta > \frac{1}{2}$, and set

$$\Theta_d := \Big\{ f \mid f_j = 0 \ \ \forall \, |j| > d, \ \sum_{j=-d}^{d} |f_j|^2 < M, \ M_l < f < M_u \Big\},$$

$$\Theta := \Big\{ f \mid f_0^2 + \sum_{j=-\infty}^{\infty} |j|^{2\vartheta} \, |f_j|^2 < M, \ M_l < f < M_u \Big\},$$

where ϑ, M, M_l, M_u are some constants different in the bosonic and fermionic settings. Define

$$\omega_Q := \frac{2^n}{\det(Q+I)} \bigoplus_{k=0}^{\infty} \big((Q-I)(Q+I)^{-1}\big)^{\otimes k}\big|_{\mathcal{H}_n^{\circledS k}} \quad \text{if } I < Q,$$

$$\omega_Q := \det(I-Q) \bigoplus_{k=0}^{n} \big(Q(I-Q)^{-1}\big)^{\otimes k}\big|_{\mathcal{H}_n^{ⓐ k}} \quad \text{if } 0 < Q < I.$$

For $f \in \Theta_d$, let

$$\varphi_f(B) := \operatorname{tr}\big(\omega_{Q_n(f)} B\big)$$

be a state with the density matrix $\omega_{Q_n(f)}$ defined on the algebra of bounded linear operators on $\Gamma_s(\mathbb{C}^n)$ or $\Gamma_a(\mathbb{C}^n)$, respectively. The symbol matrix $Q_n(f)$ is Toeplitz, and it is an analog of the covariance matrix of a stationary sequence of dependent random variables, which is called a time series.

It is proved in [2] that the sequence of quantum models

$$\mathcal{E}_n = \{\varphi_f : \mathcal{B}(\Gamma_s(\mathbb{C}^n)) \to \mathbb{C} \mid f \in \Theta_d\}$$

is asymptotically equivalent to a sequence of classical models of observing n independent random variables with geometric distributions determined by the parameter f.

A sequence of classical experiments that is asymptotically more informative than the sequence of quantum models

$$\mathcal{E}_n = \{\varphi_f : \mathcal{B}(\Gamma_a(\mathbb{C}^n)) \to \mathbb{C} \mid f \in \Theta\}$$

is constructed in [3]. The respective classical experiment consists of observing n independent random variables having Bernoulli distributions determined by the parameter f. An asymptotic equivalence is expected when the set of parameters is restricted to the set of trigonometric polynomials Θ_d.

The aforementioned asymptotic results established in [2, 3] are applied in the construction of an asymptotically normal optimal estimator of the parameter f.

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Non-commutative rational multipliers of the full Fock space

Robert Martin
(joint work with Michael T. Jury, Eli Shamovich)

The full Fock space over $\mathbb{C}^d$ can be interpreted as the multivariate and *non-commutative* (NC) or *free Hardy space*, $\mathbb{H}^2_d$, of square-summable power series in several non-commuting variables, $\mathfrak{z} = (\mathfrak{z}_1, \cdots, \mathfrak{z}_d)$. Namely, the classical Hardy space, $H^2 = H^2(\mathbb{D})$, is the Hilbert space consisting of all analytic functions in the unit disk, $\mathbb{D}$, with square-summable Taylor series coefficients at 0, equipped with the ℓ^2-inner product of these coefficients. By working in analogy with classical Hardy space theory, we obtain analogs of classical results characterizing (a) the inclusion of NC rational functions in the Fock space, (b) the inner-outer factorizations of NC rational functions in the Fock space, and (c) the NC "Clark measures" of contractive, and in particular, inner (isometric) NC rational left multipliers of the free Hardy space.

Classically, a rational function, $\mathfrak{r}$, in one variable belongs to the Hardy space of the unit disk if and only if its poles lie outside of the closed unit disk. It follows that the following are equivalent: (i) $\mathfrak{r} \in H^2$; (ii) $\mathfrak{r}$ belongs to the unital Banach algebra of uniformly bounded holomorphic functions in the open disk equipped with the supremum norm, i.e., $\mathfrak{r} \in H^\infty$, the Hardy algebra; (iii) $\mathfrak{r}$ belongs to the *disk algebra*, $\mathbb{A}(\mathbb{D})$, of uniformly bounded analytic functions in $\mathbb{D}$ which extend continuously to the circle; and (iv) the radius of convergence of the Taylor series of $\mathfrak{r}$ is greater than 1. We show that all of these equivalencies persist in several NC variables for characterizing the inclusion of NC rational functions in the free Hardy space. Any NC rational function, $\mathfrak{r}$, that is defined at the origin, $0 = (0, \cdots, 0)$, of the complex, d-dimensional *NC universe* of all row d-tuples of square complex matrices of any fixed size, n, has a finite *realization*, $\mathfrak{r} \sim (A, b, c)$. This is a triple consisting of a d-tuple, $A = (A_1, \cdots, A_d)$, of square matrices, $A_j \in \mathbb{C}^{m\times m}$, and vectors $b, c \in \mathbb{C}^m$, so that for any point in the NC universe, $X = (X_1, \cdots, X_d)$, $X_j \in \mathbb{C}^{n\times n}$, at which $\mathfrak{r}$ is defined, the evaluation of $\mathfrak{r}$ at X is given by the realization formula: $\mathfrak{r}(X) = I_n \otimes b^*(I_n \otimes I_m - \sum X_j \otimes A_j)^{-1} I_n \otimes c$. We apply this to develop new realization-theoretic characterizations of NC rational functions in the Fock space. In particular, we show that $\mathfrak{r}$ belongs to the full Fock space if and only if it is an *NC Szegő kernel* or "matrix-entry" point evaluation vector at a d-tuple of matrices in the *NC unit row ball* consisting of all row d-tuples of $n\times n$ matrices that define strictly contractive linear maps from d copies of $\mathbb{C}^n$ into one copy. This last characterization is further equivalent, by G. Popescu's multivariate Rota–Strang theorem, to $\mathfrak{r}$'s having a (minimal) realization, (A, b, c), with $A = (A_1, \cdots, A_d)$'s having joint spectral radius strictly less than 1.

By classical results of G. Herglotz, F. Riesz, and A. Beurling, any element of the classical Hardy space, $h \in H^2$, has a unique *inner-outer factorization*, $h = \theta \cdot f$, where $\theta \in H^\infty$ is *inner*, i.e., θ is an isometric multiplier of H^2, and $f \in H^2$ is *outer*, i.e., cyclic for the isometric *shift* operator, $S = M_z$, of multiplication by the independent variable on H^2. This factorization can be refined further. Any outer function is non-vanishing in the disk so that all the vanishing information of h in the disk is contained in its inner factor, θ. This inner θ then further factors as the product $\theta = B \cdot \sigma$, where B is a *Blaschke inner*, completely determined by the variety of h in the disk and σ is a *singular inner function* whose variety is empty. The classical inner-outer factorization of any $h \in H^2$ was extended to the NC and multivariate setting of the full Fock space by Popescu and Davidson–Pitts, and the Blaschke-singular-outer factorization was also later extended to the free Hardy space by Jury, Martin, and Shamovich using a suitable definition of "free variety" of any $h \in \mathbb{H}^2_d$ in the NC unit row ball. Applying the classical inner-outer factorization to any rational $\mathfrak{r} \in H^2$ yields $\mathfrak{r} = \mathfrak{b} \cdot \mathfrak{f}$, where $\mathfrak{b}$ is rational and Blaschke (in fact, a finite Blaschke product) and $\mathfrak{f}$ is rational and outer. That is, $\mathfrak{r}$ has no singular inner factor. We show that an exact analog of this factorization also holds for NC rational functions in the full Fock space.

Contractive analytic functions in the complex unit disk, i.e., elements in the closed unit ball of H^∞, are (essentially) in bijective correspondence with positive, finite, regular Borel measures on the complex unit circle. If such a positive measure, μ, corresponds to a contractive analytic $b \in H^\infty$, one writes $\mu = \mu_b$, and μ_b is called the *Clark* or *Aleksandrov–Clark measure* of b. Classically, many fine properties of contractive analytic functions are reflected in corresponding properties of their Clark measures. For example, by the Radon–Nikodym formula of Fatou's theorem, a contractive analytic function in the disk is *inner*, i.e., defines an isometric multiplier of the Hardy space, H^2, if and only if its Clark measure is singular with respect to the Lebesgue measure on the unit circle. Moreover, using Szegő's theorem, one can show that a contractive analytic b is an extreme point of the closed unit ball of H^∞ if and only if the norm-closure of the analytic polynomials on the circle in $L^2(\mu_b)$, $H^2(\mu_b)$, is equal to $L^2(\mu_b)$. This latter condition is, in turn, equivalent to $M_z|_{H^2(\mu_b)}$'s being a surjective isometry, i.e., a unitary.

By the Riesz–Markov theorem, any positive, finite, regular Borel measure on the unit circle, $\partial\mathbb{D}$, can be viewed as a positive linear functional on the commutative C^*-algebra of continuous functions, $\mathscr{C}(\partial\mathbb{D})$. By Weierstraß approximation, $\mathscr{C}(\partial\mathbb{D})$ is the supremum norm–closure of the operator system generated by the disk algebra and its conjugates. The disk algebra can further be identified as the unital, norm-closed operator algebra generated by the shift, $S = M_z$. It follows that an immediate NC analog of a positive measure on the circle is a positive linear functional on the *free disk system*, the unital, norm-closed operator system generated by Popescu's *free disk algebra*, $\mathbb{A}_d$. Here, $\mathbb{A}_d$ is the unital, norm-closed operator algebra generated by the *left free shifts*, $L_j := M_{\mathfrak{z}_j}$, the isometries of left multiplications by any of the d independent NC variables on the NC Hardy space.

As in classical Hardy space theory, we show that any contractive analytic left multiplier, $b \in \mathbb{H}^\infty_d$, of the free Hardy space, $\mathbb{H}^2_d$, corresponds uniquely to its *NC Clark measure*, μ_b, a positive linear functional on the free disk system. Focusing on contractive and NC rational multipliers, we show that a contractive left multiplier, $\mathfrak{b}$, is NC rational if and only if its NC Clark measure is a *finitely correlated* functional as introduced by Bratteli and Jørgensen and studied by Davidson, Kribs, and Shpigel. Moreover, we show that $\mu_{\mathfrak{b}}$ is singular with respect to the NC Lebesgue measure in the sense of the NC Lebesgue decomposition of Jury–Martin if and only if $\mathfrak{b}$ is an NC rational inner function, in syzygy with the classical theory. We further apply this "NC measure theory" to develop explicit formulas for the minimal realizations of contractive and, in particular, inner NC rational multipliers of the free Hardy space. Finally, and again in analogy with Hardy space theory, we show that any contractive NC rational left multiplier, $\mathfrak{b}$, of the full Fock space enjoys the following dichotomy: Either $\mathfrak{b}$ is inner or it is not column-extreme. Here, applying a Gelfand–Naimark–Segal construction to the NC Clark measure, $\mu_{\mathfrak{b}}$, yields a GNS Hilbert space, $\mathbb{H}^2_d(\mu_{\mathfrak{b}})$, spanned by equivalence classes of free polynomials, and a GNS row isometry, $\Pi_j^{(\mathfrak{b})} = M^L_{\mathfrak{z}_j}$, defined by left multiplications by the d independent NC variables. This property of *column-extreme* can then be defined by the condition that $\Pi^{(\mathfrak{b})} = (\Pi_1^{(\mathfrak{b})}, \cdots, \Pi_d^{(\mathfrak{b})})$ is a surjective or *Cuntz* row isometry. Hence, in one variable, this notion of *column-extreme* is equivalent to being an extreme point of the closed unit ball of H^∞.

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Operator-valued semicircular elements: From free analysis to the free field . . . and back

Tobias Mai

(joint work with Johannes Hoffmann, Roland Speicher, Sheng Yin)

Operator-valued semicircular elements are among the most important non-commutative random variables that are studied in operator-valued free probability theory, as they are located at the crossroads of operator algebras, random matrix theory, and non-commutative algebra.

A prototypical (matrix-valued) example, which also makes the connection with non-commutative algebra, is constructed as follows. Let us fix a family $s_1, \dots, s_n$ of freely independent standard semicircular elements in some tracial W^*-probability space $(\mathcal{M}, \tau)$; recall that a *tracial W^*-probability space* $(\mathcal{M}, \tau)$ consists of a von Neumann algebra $\mathcal{M}$ and a faithful, normal, tracial state τ on $\mathcal{M}$. To any collection $b_0, b_1, \dots, b_n$ of self-adjoint matrices in $M_m(\mathbb{C})$, we can associate the operator

$$S := b_0 \otimes \mathbf{1} + b_1 \otimes s_1 + \dots + b_n \otimes s_n.$$

At first instance, S is a non-commutative random variable in the "augmented" tracial W^*-probability space $(M_m(\mathbb{C}) \otimes \mathcal{M}, \mathrm{tr}_m \otimes \tau)$, where tr_m denotes the normalized trace on $M_m(\mathbb{C})$. However, when $M_m(\mathbb{C}) \otimes \mathcal{M}$ is seen as an operator-valued W^*-probability space over $M_m(\mathbb{C}) \hookrightarrow M_m(\mathbb{C}) \otimes \mathcal{M}$ with respect to the conditional expectation $\mathbb{E} := \mathrm{id}_{M_m(\mathbb{C})} \otimes \tau$, then S reveals its actual structure as an *operator-valued semicircular element* with *mean* $\mathbb{E}[S] = b_0$ and *covariance map*

$$\eta : \ M_m(\mathbb{C}) \to M_m(\mathbb{C}), \quad b \mapsto \mathbb{E}\big[(S - b_0 \otimes \mathbf{1})b(S - b_0 \otimes \mathbf{1})\big] = \sum_{j=1}^{n} b_j b b_j.$$

As such, S enjoys the important feature that its *operator-valued Cauchy transform*

$$G_S : \ \mathbb{H}^+(M_m(\mathbb{C})) \to \mathbb{H}^-(M_m(\mathbb{C})), \quad b \mapsto \mathbb{E}\big[(b \otimes \mathbf{1} - S)^{-1}\big],$$

where $\mathbb{H}^{\pm}(M_m(\mathbb{C})) := \{b \in M_m(\mathbb{C}) \mid \pm\mathrm{Im}(b) > 0\}$ are the upper/lower half-plane in $M_m(\mathbb{C})$, is completely determined by the so-called *Dyson equation*

$$(b - b_0)G_S(b) = \mathbf{1}_m + \eta(G_S(b))G_S(b) \qquad \text{for all } b \in \mathbb{H}^+(M_m(\mathbb{C})).$$

Therefore, the operator-valued nature of S gives access to powerful analytic tools that allow an elegant treatment of S, even for questions concerning only its scalar-valued facet. The *distribution* $\mu_S \in \mathrm{Prob}(\mathbb{R})$ of S, for instance, which is of particular interest in many applications, is uniquely determined by the condition that $\mathcal{G}_{\mu_S}(z) = \mathrm{tr}_m(G_S(z\mathbf{1}_m))$ for all $z \in \mathbb{C}^+$. Here, $\mathrm{Prob}(\mathbb{R})$ stands for the set of all Borel probability measures on the real line $\mathbb{R}$, and $\mathcal{G}_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ denotes the (*scalar-valued*) *Cauchy transform* of $\mu \in \mathrm{Prob}(\mathbb{R})$ defined by $\mathcal{G}_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t)$. As a consequence, μ_S depends through the Dyson equation only on the self-adjoint matrix $b_0 \in M_m(\mathbb{C})$ and the completely positive map $\eta : M_m(\mathbb{C}) \to M_m(\mathbb{C})$. This insight allows for an interesting change of perspective and a vast generalization of the construction presented above. While the covariance maps η associated with operator-valued semicircular elements are necessarily completely positive, the Dyson equation itself can be considered without this restriction. We are not even limited to the matrix-valued case but can replace $(M_m(\mathbb{C}), \mathrm{tr}_m)$ by any other *C^*-probability space* (B, φ), that is, a unital C^*-algebra B endowed with a distinguished state $\varphi : B \to \mathbb{C}$. To any pair $\rho = (b_0, \eta)$ consisting of a self-adjoint element $b_0 \in B$ and a positive linear map $\eta : B \to B$, we can associate the so-called *density of states* $\mu_\rho \in \mathrm{Prob}(\mathbb{R})$, which is defined through the condition $\mathcal{G}_{\mu_\rho}(z) = \varphi(G_\eta(z\mathbf{1} - b_0))$ for all $z \in \mathbb{C}^+$, where $G_\eta : \mathbb{H}^+(B) \to \mathbb{H}^-(B)$ is uniquely determined by the Dyson equation $bG_\eta(b) = \mathbf{1} + \eta(G_\eta(b))G_\eta(b)$ for $b \in \mathbb{H}^+(B)$.

In [2], motivated by applications in random matrix theory and by questions that arose in the context of [1], we studied continuity properties of the map $\mu : B_{\mathrm{sa}} \times \mathcal{P}(B) \to \mathrm{Prob}(\mathbb{R})$, $\rho \mapsto \mu_\rho$, restricted to $B_{\mathrm{sa}} \times \mathcal{P}_2(B)$, where $\mathcal{P}_2(B)$ denotes the subset of $\mathcal{P}(B)$ consisting of 2-positive maps. We found that if $\mathrm{Prob}(\mathbb{R})$ is endowed with the *Lévy metric*, then μ is separately Hölder continuous and jointly uniformly continuous on $B_{\mathrm{sa}} \times \mathcal{P}_2(B)$. Our approach borrows tools from non-commutative function theory, which serve as a substitute for the lacking realization by operator-valued semicircular elements. The crucial insight is the following: If $\eta : B \to B$ is completely positive, then one can set up the Dyson equation for all amplifications

$\eta^{(k)} : M_k(B) \to M_k(B)$ of η, and its solutions $G_{\eta^{(k)}} : \mathbb{H}^+(M_k(B)) \to \mathbb{H}^-(M_k(B))$ glue to a non-commutative function. Under weaker assumptions on the positivity of η, one can build in this way only some "truncated" non-commutative function, but already 2-positivity is enough to control the Fréchet derivative DG_η of G_η. This implies various strong analytic properties of solutions of the Dyson equation; for instance, we obtain an operator-valued version of the *inviscid Burgers equation*, which is at the heart of our approach as it allows a comparison of G_{η_1} and G_{η_0} for $\eta_0, \eta_1 \in \mathcal{P}_2(B)$ by means of interpolation.

In [1], the link between operator-valued semicircular elements and non-commutative algebra discovered in [3] was explored further. The starting point is the ring of non-commutative polynomials $\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle$ in n formal non-commuting variables $\mathbf{x}_1, \dots, \mathbf{x}_n$. The elements of the *free field* $\mathbb{C}\langle\!\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle\!\rangle$ or, more formally, the *universal field of fractions* for $\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle$ are interpreted as non-commutative rational functions in the variables $\mathbf{x}_1, \dots, \mathbf{x}_n$. While the construction of the free field is quite involved, it has some very natural structure and enjoys various pleasant features. One of these peculiarities is that the question of whether a given square matrix over $\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle$ becomes invertible over $\mathbb{C}\langle\!\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle\!\rangle$ can be decided without reference to $\mathbb{C}\langle\!\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle\!\rangle$. The crucial notion in that respect is the *inner rank* $\operatorname{rank}(A)$ for matrices $A \in M_m(\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle)$; it is defined as 0 if $A = 0$ and otherwise as the least integer $k \geq 1$ for which A can be written as $A = R_1 R_2$ with rectangular matrices $R_1 \in M_{m\times k}(\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle)$ and $R_2 \in M_{k\times m}(\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle)$. Precisely those matrices $A \in M_m(\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle)$ are invertible in $M_m(\mathbb{C}\langle\!\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle\!\rangle)$ that are *full* in the sense that $\operatorname{rank}(A) = m$.

Non-commutative Edmond's problem asks now for deciding fullness or, more generally, for the inner rank of matrices $A \in M_m(\mathbb{C}\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle)$ of the form $A = a_1\mathbf{x}_1 + \ldots + a_n\mathbf{x}_n$. As a supplement to the work of A. Garg, L. Gurvits, R. Oliveira, and A. Wigderson (2016, 2020), we present an analytic approach to that problem based on free probability techniques. The initial observation is that to any such A, where we may assume, without loss of generality, that the coefficient matrices $a_1, \dots, a_n \in M_m(\mathbb{C})$ are self-adjoint, a matrix-valued semicircular element $S = a_1 \otimes s_1 + \ldots + a_n \otimes s_n$ can be associated. Remarkably, S serves an an operator-algebraic "avatar" of A that in particular stores the information about the rank of A as $\mu_S(\{0\})$; more precisely, as shown in [3], we have the relationship $\operatorname{rank}(A) = m(1 - \mu_S(\{0\}))$. For the computation of $\mu_S(\{0\})$, we can use the function $\theta_{\mu_S} : \mathbb{R}^+ \to \mathbb{R}^+, y \mapsto -y\operatorname{Im}(\mathcal{G}_{\mu_S}(iy))$ on $\mathbb{R}^+ := (0, \infty)$ since we have $\lim_{y\to 0} \theta_{\mu_S}(y) = \mu_S(\{0\})$; furthermore, θ_{μ_S} is monotonically increasing and satisfies $\lim_{y\to\infty} \theta_{\mu_S}(y) = 1$. Since θ_{μ_S} is determined by $\mathcal{G}_{\mu_S}$ and so by G_S, we can use that the Dyson equation can be solved numerically by means of a fixed point iteration; for this, we provide a detailed error analysis. In that respect, it is beneficial that the possible values of $\mu_S(\{0\})$ are limited to $\{k/m \mid k = 0, 1, \dots, m\}$.

However, there is the obstacle that the point $y \in \mathbb{R}^+$, for which $\theta_{\mu_S}(y)$ is sufficiently close to $\mu_S(\{0\})$, depends on the behavior of μ_S in a neighborhood of 0. Although D. Shlyakhtenko and P. Skoufranis (2015) have shown via the *Novikov–Shubin invariant* of S that μ_S has the correct type of regularity, this is not quite sufficient for our purpose as their results are only qualitative in nature. A preprint with R. Speicher, which is currently under preparation, proposes as an expedient to work instead with the *Fuglede–Kadison determinant* $\Delta(S) := \exp(\int_0^\infty \log(t)\, d\mu_{|S|}(t)) \in [0,\infty)$ of $S = a_1 \otimes s_1 + \ldots + a_n \otimes s_n$, where $\mu_{|S|}$ is the distribution of $|S| := (S^*S)^{1/2}$; note that a restriction to self-adjoint coefficient matrices $a_1, \ldots, a_n$ is not needed in this part of the discussion. Our main result is that $\Delta(S) = \operatorname{cap}(\eta)^{1/(2m)} e^{-1/2}$, whereby we connect $\Delta(S)$ with Gurvit's *capacity*

$$\operatorname{cap}(\eta) := \inf\{\det(\eta(b)) \mid b \in M_m(\mathbb{C}) : b > 0,\ \det(b) = 1\}$$

for the completely positive map $\eta : M_m(\mathbb{C}) \to M_m(\mathbb{C}), b \mapsto \sum_{j=1}^n a_j b a_j^*$. While the computation of $\operatorname{cap}(\eta)$ is again a non-trivial task, one can give explicit lower bounds. As a strengthening of corresponding results of Garg, Gurvits, Oliveira, and Wigderson, we provide one that is dimension independent: If $\operatorname{cap}(\eta) > 0$, which is equivalent to the fullness of $A = a_1\mathbf{x}_1 + \ldots + a_n\mathbf{x}_n$, and if all the coefficient matrices $a_1, \ldots, a_n$ are chosen from $M_m(\mathbb{Z})$, then $\operatorname{cap}(\eta) \geq 1$. Taken all together, this yields an analytic algorithm by which the fullness of $A = a_1\mathbf{x}_1 + \ldots + a_n\mathbf{x}_n$ for coefficient matrices $a_1, \ldots, a_n \in M_m(\mathbb{Z})$ can be decided. It is under investigation whether this approach can be extended to solve the more general rank problem.

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Completely positive noncommutative kernels

Victor Vinnikov

(joint work with Joseph A. Ball, Gregory Marx, Motke Porat)

The purpose of this talk was to give a brief introduction to the theory of completely positive (cp) free non-commutative (nc) kernels [1, 2] that are the analogue of the classical positive kernels (in the sense of Aronszajn) in nc function theory [3]. We also discussed some nc reproducing kernel Hilbert spaces that are defined by asymptotic integration [4] in the spirit of the random matrix theory view of free probability. We refer to these papers for both details and references.

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$, let $\Omega \subseteq \mathcal{V}_{\mathrm{nc}} := \coprod_{n=1}^\infty \mathcal{V}^{n\times n}$, write $\Omega_n = \Omega \cap \mathcal{V}^{n\times n}$, and let $\mathcal{W}_1$ and $\mathcal{W}_0$ be operator systems. A function

$$K\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{W}_{1,\mathrm{nc}}, \mathcal{W}_{0,\mathrm{nc}}) := \coprod_{n,m=1}^{\infty} \mathcal{L}(\mathcal{W}_1^{n\times m}, \mathcal{W}_0^{n\times m})$$

is called an nc kernel if it is graded ($K(\Omega_n, \Omega_m) \subseteq \mathcal{L}(\mathcal{W}_1^{n\times m}, \mathcal{W}_0^{n\times m})$) and it respects intertwining ($Z \in \Omega_n$, $\widetilde{Z} \in \Omega_{\widetilde{n}}$, $\alpha Z = \widetilde{Z}\alpha$, $\alpha \in \mathbb{C}^{n\times\widetilde{n}}$, $W \in \Omega_m$, $\widetilde{W} \in \Omega_{\widetilde{m}}$,

$\beta W = \widetilde{W}\beta$, $\beta \in \mathbb{C}^{m\times\widetilde{m}} \Rightarrow \alpha K(Z,W)(P)\beta = K(\widetilde{Z},\widetilde{W})(\alpha P\beta^*)$ for all $P \in \mathcal{W}_1^{n\times m}$). An nc kernel is essentially the same as an nc function of order 1 (an element in the tensor product of the space of nc functions with itself) except for the conjugation in the second variable (much like classical kernels are analytic in the first variable and conjugate analytic in the second variable).

An nc kernel is called hermitian if $K(W,Z)(P) = K(Z,W)(P^*)^*$ for all matrices of appropriate sizes. A hermitian nc kernel is called a cp nc kernel if for all $Z_1,\ldots,Z_k \in \Omega$, $Z_i \in \Omega_{n_i}$, $n = n_1+\cdots+n_k$, the mapping $[K(Z_i,Z_j]_{i,j=1}^k : \mathcal{W}_1^{n\times n} \to \mathcal{W}_0^{n\times n}$ is positive (which is equivalent in this case to being completely positive by repeating the sequence of matrices $Z_1,\ldots,Z_k$ arbitrary many times).

A (cp/hermitian) nc kernel K extends uniquely to a (cp/hermitian) nc kernel on the nc envelope $[\Omega]_{\rm nc}$ of Ω, the smallest nc subset of $\mathcal{V}_{\rm nc}$ containing Ω—equivalently, the set of direct sums of elements of Ω. If Ω is an nc set, then K respects intertwinings if and only if K respects direct sums and similarities; here, respecting of direct sums means that $K(Z_1 \oplus Z_2, W_1 \oplus W_2)\left(\left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{smallmatrix}\right]\right) = [K(Z_i,W_j)(P_{ij})]_{i,j=1,2}$. In particular, if Ω is an nc set, then an nc kernel K is a cp nc kernel iff $K(Z,Z)(P) \geq 0$ for all $Z \in \Omega_n$, $P \in \mathcal{W}_1^{n\times n}$, and $n \in \mathbb{N}$.

Example 1. Let $\Omega_1 \subseteq \mathcal{V}$, and let $k\colon \Omega\times\Omega \to \mathbb{C}$ be a positive kernel in the sense of Aronszajn, i.e., $[k(z_i,z_j)]_{i,j=1,\ldots,k} \geq 0$ for all $z_1,\ldots,z_k \in \Omega_1$. Then $K(z,w)(p) = k(z,w)p$, $z,w \in \Omega_1$, $p \in \mathbb{C}$, is a cp nc kernel on Ω_1 with values in $\mathcal{L}(\mathbb{C}_{\rm nc},\mathbb{C}_{\rm nc})$. Notice that in this case, $[\Omega_1]_{\rm nc}$ consists of diagonal matrices with diagonal entries in Ω_1.

Example 2. Let $\Omega_1 = \{v_0\}$ for some $v_0 \in \mathcal{V}$, and let $\varphi\colon \mathcal{W}_1 \to \mathcal{W}_0$ be a cp map of operator systems. Then $K(v_0,v_0) = \varphi$ is a cp nc kernel on Ω_1 with values in $\mathcal{L}(\mathcal{W}_{1,\rm nc},\mathcal{W}_{0,\rm nc})$. Notice that in this case, $[\Omega_1]_{\rm nc} = \{I_n \otimes v_0\}_{n=1}^\infty$.

There is a common generalization of Examples 1 and 2 due to Barreto–Bhat–Liebscher–Skeide.

Example 3. Of course, both of the previous examples are essentially commutative since the nc kernel is defined at level 1. Perhaps the simplest genuinely non-commutative example is the nc Szegő kernel $K_{\rm Sz}$. Let $\mathcal{V} = \mathbb{C}^d$, and consider the nc row ball (the unit ball of $(\mathbb{C}^d)_{\rm nc}$ with respect to the row operator space structure)

$$(\mathbb{B}^d)_{\rm nc} = \coprod_{n=1}^\infty \left\{ Z = (Z_1,\ldots,Z_d) \in \left(\mathbb{C}^{n\times n}\right)^d : Z_1Z_1^* + \cdots + Z_dZ_d^* < I_n \right\}.$$

Define a cp nc kernel on $(\mathbb{B}^d)_{\rm nc}$ with values in $\mathcal{L}(\mathbb{C}_{\rm nc},\mathbb{C}_{\rm nc})$ by $K_{\rm Sz}(Z,W)(P) = \sum_{w\in\mathbf{G}_d} Z^w P(W^w)^*$, where $\mathbf{G}_d$ denotes the free monoid on d generators $1,\ldots,d$, and we use the nc multipower notation $Z^{i_1\cdots i_\ell} = Z_{i_1}\cdots Z_{i_\ell}$.

Returning to the general theory, we may assume without loss of generality that $\mathcal{W}_0 = \mathcal{L}(\mathcal{Y})$ for a Hilbert space $\mathcal{Y}$.

Theorem 4 (Arveson extension theorem for CP NC kernels)**.** *Let $K\colon \Omega\times\Omega \to \mathcal{L}(\mathcal{W}_{1,\rm nc},\mathcal{L}(\mathcal{Y})_{\rm nc})$ be a cp nc kernel, and assume that $\mathcal{W}_1 \subseteq \mathcal{A}$, where $\mathcal{A}$ is a C^**

algebra. Then there exists a cp nc kernel $\widetilde{K}\colon \Omega\times\Omega \to \mathcal{L}(\mathcal{A}_{\rm nc}, \mathcal{L}(\mathcal{Y})_{\rm nc})$ *such that* $\widetilde{K}(Z,W)|_{\mathcal{W}_1^{n\times m}} = K(Z,W)$ *for all* $Z\in\Omega_n$*,* $W\in\Omega_m$*, and* $n,m\in\mathbb{N}$*.*

For the proof, we first consider the case $\Omega=\{Z\}$, $Z\in\mathcal{V}^{n\times n}$. Then a cp nc kernel $K\colon \Omega\times\Omega\to\mathcal{L}(\mathcal{W}_{1,\rm nc},\mathcal{W}_{0,\rm nc})$ is simply a cp map $\varphi=K(Z,Z)\colon \mathcal{W}_1^{n\times n}\to\mathcal{W}_0^{n\times n}$ that is a $(\mathcal{S},\mathcal{S}^*)$-bimodule map: $\varphi(\alpha P\beta^*)=\alpha\varphi(P)\beta^*$ for all $P\in\mathcal{W}_1^{n\times n}$ and all $\alpha,\beta\in\mathcal{S}$, where $\mathcal{S}=Z'$ (the commutator of Z in in $\mathbb{C}^{n\times n}$). The theorem then can be established in this case by "upgrading" the usual Arveson's extension theorem for cp maps. The case of an nc set generated by finitely many points reduces to the case of a singleton set by taking the direct sum of the generators. The general case is finally established by a compactness argument—more specifically, the theorem of Kurosh that the limit of an inverse system of nonempty compacta is a nonempty compactum.

Theorem 5 (Kolmogorov decomposition for CP NC kernels)**.** $K\colon\Omega\times\Omega\to\mathcal{L}(\mathcal{A}_{\rm nc},\mathcal{L}(\mathcal{Y})_{\rm nc})$ *is a cp nc kernel if and only if*

$$K(Z,W)(P)=H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}}\otimes\sigma)(P)H(w)^*$$

for some nc function $H\colon\Omega\to\mathcal{L}(\mathcal{X},\mathcal{Y})_{\rm nc}$ *and some* $*$*-representation* $\sigma\colon\mathcal{A}\to\mathcal{X}$ *for some Hilbert space* $\mathcal{X}$ *(*$Z\in\Omega_n$*,* $W\in\Omega_m$*,* $P\in\mathcal{A}^{n\times m}$*).*

The proof of the theorem proceeds via the nc reproducing kernel Hilbert space (rkHs) construction. Let $\mathcal{H}$ be a Hilbert space of nc functions $f\colon\Omega\to\mathcal{L}(\mathcal{A},\mathcal{Y})_{\rm nc}$. Assume that (a) for all $Z\in\Omega_n$ and all $n\in\mathbb{N}$, $\mathcal{H}\ni f\mapsto f(Z)\in\mathcal{L}(\mathcal{A},\mathcal{Y}^{n\times n})$ is bounded; and (b) $\sigma\colon\mathcal{A}\to\mathcal{L}(\mathcal{H})$ defined by $(\sigma(a)f)(Z)(u)=f(Z)(ua)$ ($a\in\mathcal{A}$, $f\in\mathcal{H}$, $Z\in\Omega_n$, $u\in\mathcal{A}^n$, $a\in\mathcal{A}$) is a $*$-representation of $\mathcal{A}$. It follows that for all $Z\in\Omega_n$, $u\in\mathcal{A}^{1\times n}$, $x\in\mathcal{Y}^n$, and $n\in\mathbb{N}$, there exists $K_{Z,u,x}\in\mathcal{H}$ such that $\langle f(Z)(u^*),x\rangle_{\mathcal{Y}^n}=\langle f,K_{Z,u,x}\rangle_{\mathcal{H}}$ for all $f\in\mathcal{H}$. One then shows that $\langle K(Z,W)(v^*u)x,y\rangle_{\mathcal{Y}^m}:=\langle K_{Z,u,x},K_{W,v,y}\rangle_{\mathcal{H}}$ ($W\in\Omega_m$, $v\in\mathcal{A}^{1\times m}$, $y\in\mathcal{Y}^m$, $m\in\mathbb{N}$) defines a cp nc kernel $K\colon\Omega\times\Omega\to\mathcal{L}(\mathcal{A}_{\rm nc},\mathcal{L}(\mathcal{Y})_{\rm nc})$. Conversely, one can show by an appropriate nc variation of the classical Aronszajn construction that given a cp nc kernel $K\colon\Omega\times\Omega\to\mathcal{L}(\mathcal{A}_{\rm nc},\mathcal{L}(\mathcal{Y})_{\rm nc})$, one can construct the corresponding nc rkHs $\mathcal{H}(K)$ of nc functions $f\colon\Omega\to\mathcal{L}(\mathcal{A},\mathcal{Y})_{\rm nc}$ with a natural $*$-representation $\sigma\colon\mathcal{A}\to\mathcal{L}(\mathcal{H}(K))$.

For the nc Szegő kernel of Example 3, we obtain the so-called nc Hardy space on the nc row ball (the realization of the full Fock space in nc function theory),

$$\mathcal{H}(K_{\rm Sz})=\left\{f\colon(\mathbb{B}^d)_{\rm nc}\to\mathbb{C}_{\rm nc}\colon f=\sum_{w\in\mathbf{G}_d}Z^wf_w,\ \|f\|^2_{\mathcal{H}(K_{\rm Sz})}=\sum_{w\in\mathbf{G}_d}|f_w|^2<\infty\right\}.$$

By analogy with the usual integral definition of the classical Hardy spaces, it is then natural to try to define this space by asymptotic integration. Notice that $\big((\mathbb{B}^d)_{\rm nc}\big)_n$ is a bounded symmetric domain in $\mathbb{C}^{n^2d}$ with distinguished boundary

$$\big(\partial(\mathbb{B}^d)_{\rm nc}\big)_n:=\left\{Z=(Z_1,\ldots,Z_d)\in\left(\mathbb{C}^{n\times n}\right)^d: Z_1Z_1^*+\cdots+Z_dZ_d^*=I_n\right\}.$$

The unitary group $U(dn)$ acts on $\big(\partial(\mathbb{B}^d)_{\rm nc}\big)_n$ by $(Z_1,\ldots,Z_d)\mapsto(Z_1,\ldots,Z_d)U$; let ν_n be the unique normalized invariant measure. For $p,q\in\mathbb{C}\langle z_1,\ldots,z_d\rangle$ (the free algebra), we define $\langle p, q\rangle = \lim_{n\to\infty} \int_{(\partial(\mathbb{B}^d)_{\mathrm{nc}})_n} \mathrm{tr}_n\, q(Z)^* p(Z)\, d\nu_n(Z)$, where tr_n is the normalized trace. This is a well-defined inner product on $\mathbb{C}\langle z_1, \ldots, z_d\rangle$, and the completion is

$$\left\{ f\colon \frac{1}{\sqrt{d}}(\mathbb{B}^d)_{\mathrm{nc}} \to \mathbb{C}_{\mathrm{nc}}\colon f = \sum_{w\in\mathbf{G}_d} Z^w f_w,\ \|f\|^2 = \sum_{w\in\mathbf{G}_d} d^{|w|}|f_w|^2 < \infty \right\}.$$

The proof uses a Fubini-type theorem to convert the integral into one on $U(dn)$ followed by an application of asymptotic integration on the unitary group and asymptotic freeness. Notice that, in stark contrast to the classical case, the resulting space of analytic nc functions is on a smaller domain $d^{-1/2}(\mathbb{B}^d)_{\mathrm{nc}}$. It is also possible to define the usual nc Hardy space on the full nc row ball as defined above; however, we then have to use asymptotic integration on the distinguished boundary of the nc polydisc. These results should be the first steps toward a general theory of nc bounded symmetric domains.

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Short talk: Interpolating sequences for the NC Hardy space

Alberto Dayan

Let H^2_d be the Drury–Arveson space on the unit ball $\mathbb{B}_d$. Given a sequence $\Lambda = (\lambda_n)_n$ in $\mathbb{B}_d$, one has that for all $f \in H^2_d$,

$$|f(\lambda_n)| \le \frac{\|f\|}{\sqrt{1-\|\lambda_n\|^2}}, \qquad n \in \mathbb{N},$$

since point evaluations are bounded in the Drury–Arveson space. Hence, the restriction operator

$$R_\Lambda\colon f \mapsto \left(f(\lambda_n)\sqrt{1-\|\lambda_n\|^2}\right)_n$$

maps H^2_d into ℓ^∞. One says Λ is interpolating for H^2_d if $R_\Lambda(H^2_d) \supseteq \ell^2$. Interpolating sequences have been intensively studied in the last decades in the setting of various distinct reproducing kernel Hilbert spaces. In particular, the Drury–Arveson space on the (possibly infinite-dimensional) unit ball plays a fundamental role in the theory of complete Pick spaces. With this as a motivation, interpolating sequences for the Drury–Arveson space have been recently completely understood [1].

The goal of this talk is to introduce a similar notion of interpolating sequences in the setting of the non-commutative (NC) Hardy space, the natural NC analog of the classical Drury–Arveson space. The case $d = 1$ was studied in [2], where a complete characterization that is reminiscent of the celebrated theorems of Carleson in the scalar case was given. On the other hand, very little is known about the case $d > 1$.

We shall discuss some possible approaches to the solution of the problem. We shall notice how the positive solution of the Feichtinger conjecture, which was used in the proof of the main result in [1], cannot be used when studying matrix node interpolation problems, since it fails in this setting [2]. The characterization for the case $d = 1$ in [2] mostly uses Blaschke products in the unit disc. In [3], such Blaschke products were defined in the non-commutative setting, though it is not known whether they enjoy some of the properties of classical Blaschke products that would make them useful in understanding interpolating sequences.

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Short talk: NC Majorization

Paul Skoufranis

(joint work with Matt Kennedy, Laurent Marcoux)

Choquet theory and the Choquet order appear in many places in mathematics.

Definition 1. Let μ and ν be probability measures on a compact convex set K. It is said that μ is *dominated by ν in the Choquet order*, written $\mu \prec_c \nu$, if

$$\int_K f\, d\mu \leq \int_K f\, d\nu$$

for all continuous, convex functions $f : K \to \mathbb{R}$.

In operator algebras, the Choquet order is directly related to the notion of matrix majorization.

Theorem 2 ([1])**.** *Let $A, B \in M_n$ be self-adjoint with eigenvalues*

$$a_1 \geq a_2 \geq \cdots \geq a_n \quad \textit{and} \quad b_1 \geq b_2 \geq \cdots \geq b_n.$$

A is said to be majorized *by B if one of the following equivalent conditions holds:*

(1) *There exists a doubly stochastic matrix $X \in M_n$ such that $X\vec{b} = \vec{a}$.*
(2) $\sum_{k=1}^m a_k \leq \sum_{k=1}^m b_k$ *for all $m \in \{1, \ldots, n\}$ with equality at $m = n$.*
(3) $\mathrm{Tr}(f(A)) \leq \mathrm{Tr}(f(B))$ *for every continuous, convex $f : \mathbb{R} \to \mathbb{R}$.*
(4) $\mu_A \prec_c \mu_B$.

(5) (Schur–Horn) *There exist unitaries* $U, V \in M_n$ *such that*

$$U^*AU = E_D(V^*BV),$$

where $E_D : M_n \to M_n$ *is the expectation onto the diagonal.*

(6) $A \in \operatorname{conv}(\{U^*BU \mid U \in M_n \text{ a unitary}\})$.

(7) *There exists a unital quantum channel (i.e., a trace-preserving cp map)* $\Phi : M_n \to M_n$ *such that* $\Phi(B) = A$.

The above result has many generalizations. Specifically, it holds when the matrices are replaced with self-adjoint elements in a II_1 factor and when the matrices are replaced with m-tuples of commuting self-adjoint operators. As we want to remove the condition of commutativity, we pondered the following question.

Question 3. *Given* $(A_1, \ldots, A_m), (B_1, \ldots, B_m) \in (M_n)^m_{sa}$, *when does there exist a unital quantum channel* $\Phi : M_n \to M_n$ *such that* $\Phi(B_k) = A_k$ *for all* k*?*

The correct condition to generalize the spectral characterizations of matrix majorization comes from non-commutative (NC) Choquet theory.

Definition 4 ([2])**.** Let K be a compact NC convex set, and let $\mu, \nu : C_{nc}(K) \to M_m$ be unital completely positive linear maps. It is said that μ is *dominated by* ν *in the NC Choquet order*, written $\mu \prec_{nc} \nu$, if $\mu(f) \le \nu(f)$ for all self-adjoint, NC convex $f \in M_m(C_{nc}(K))$.

Using the NC Choquet order, we were able to prove many results, including the following.

Theorem 5 ([3])**.** *Let* $(A_1, \ldots, A_m), (B_1, \ldots, B_m) \in (M_n)^m_{sa}$. *There exists a unital quantum channel* $\Phi : M_n \to M_n$ *such that* $\Phi(B_k) = A_k$ *for all* k *if and only if*

$$\operatorname{Tr}(f(A_1, \ldots, A_m)) \le \operatorname{Tr}(f(B_1, \ldots, B_m))$$

for all NC convex $f \in C_{nc}(K)$.

Theorem 6 ([3])**.** *Let* $\mathfrak{N} = W^*(A_1, \ldots, A_m)$ *and* $\mathfrak{M} = W^*(B_1, \ldots, B_m)$ *be tracial von Neumann algebras. There exists a unital, normal quantum channel*

$$\Phi : \mathfrak{M} \to \mathfrak{N}$$

such that $\Phi(B_k) = A_k$ *for all* k *if and only if*

$$\tau_{\mathfrak{N}}(f(A_1, \ldots, A_m)) \le \tau_{\mathfrak{M}}(f(B_1, \ldots, B_m))$$

for all NC convex $f \in C_{nc}(K)$.

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Short talk: Almost versus near q-commuting unitaries

Malte Gerhold
(joint work with Orr Moshe Shalit)

It is well known that not all pairs of unitary matrices (u_1, u_2) whose commutator $[u_2, u_1] = u_2u_1 - u_1u_2$ is small can be well approximated by commuting unitary matrices. More precisely, there is a sequence of pairs of unitaries such that the norm of the commutators tends to zero, but the distance to commuting pairs of matrices stays bounded from below. Voiculescu constructed such a sequence from the standard q-commuting matrices (which can be reduced to finite matrices if and only if q is a root of unity)

$$S = \begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ & & & & \ddots & \ddots \end{pmatrix}, \quad \sum_k q^k p_k = \begin{pmatrix} \ddots & & & & \\ & q^{-k} & & & \\ & & \ddots & & \\ & & & q^k & \\ & & & & \ddots \end{pmatrix},$$

i.e., S is the bilateral shift on $\ell^2(\mathbb{Z})$ and $\sum_k q^k p_k$ is the diagonal matrix of powers of q (p_k the kth coordinate projection). Nevertheless, Lin showed that the infinite ampliations of almost commuting u_1 and u_2 are indeed always close in operator norm to a pair of commuting unitary operators (or *near commuting* for short). In the talk, we consider the analogous statement for pairs of almost q-commuting unitaries: We show that if q is not a root of unity, then the infinite ampliation of an almost q-commuting unitary pair is near q-commuting.

By a concrete dilation construction, which we shall sketch in this paragraph, we find that an almost q-commuting unitary pair (u_1, u_2) is necessarily close to the compression of a q-commuting pair $(\tilde{u}_1, \tilde{u}_2)$. Let $[u_2, u_1]_q := u_2u_1 - qu_1u_2$ be the q-commutator, and assume that $\|[u_2, u_1]_q\| < \delta$. Denote by $\alpha(\cdot) := u_1 \cdot u_1^*$ the automorphism of $B(H)$ given by conjugation with u_1. Note that $\alpha(u_1) = u_1$ and $\|q\alpha(u_2) - u_2\| < \delta$, so, by the triangle inequality, $\|q^k\alpha^k(u_2) - u_2\| < |k|\delta$. Now, we define

$$\tilde{u}_1 = \begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & u_1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & u_1 & \\ & & & & \ddots & \ddots \end{pmatrix}, \quad \tilde{u}_2 = \begin{pmatrix} \ddots & & & & \\ & q^{-k}\alpha^{-k}(u_2) & & & \\ & & \ddots & & \\ & & & q^k\alpha^k(u_2) & \\ & & & & \ddots \end{pmatrix},$$

i.e., $\tilde{u}_1 = u_1 \otimes S$ and $\tilde{u}_2 = \sum \alpha^k(u_2) \otimes q^k p_k$. A simple calculation shows that $\tilde{u}_1$ and $\tilde{u}_2$ commute. Now, we claim that there is an isometry $\iota \colon H \hookrightarrow H \otimes \ell^2(\mathbb{Z})$ such that $\|\iota^*\tilde{u}_i\iota - u_i\| < \sqrt{\delta}$. (We refer to $\iota^*\tilde{u}_i\iota$ as the compression of $\tilde{u}_i$ with respect to ι.) Indeed, consider the maps

$$\iota_N \colon h \mapsto h \otimes \xi_N, \quad \xi_N := \frac{1}{\sqrt{2N+1}} \sum_{k=-N}^{N} e_k,$$

which are obviously isometries. One can observe that the compression of $\tilde{u}_1$ with respect to ι_N is close to u_1 if N is large, while the compression of $\tilde{u}_2$ with respect to ι_N is close to u_2 if N is small. One can find a balanced choice for N (namely, $N+1 < \delta^{-1/2} < 2N+1$) such that both compressions are not more than $\sqrt{\delta}$ away from the original u_1, u_2.

The construction can in some weak sense be inverted: With a very similar construction, one finds a pair $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2)$ that almost compresses to $(\tilde{u}_1, \tilde{u}_2)$ such that the pair $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2)$ is unitarily equivalent to the direct sum

$$\bigoplus_{k \in \mathbb{Z}} \big(u_1 \otimes S, u_2 \otimes q^{-k}\mathrm{id}\big).$$

With a refinement of the techniques from [1]—in particular, Theorem 2.6—in which it is shown that unitary pairs that are close in "dilation distance" can be faithfully represented by pairs of unitaries that are close in operator norm, one can now prove the following: If (u_1, u_2) is a unitary pair with $\|[u_2, u_1]_q\| < \delta$ that is furthermore *gauge invariant*, i.e., for all $(\lambda_1, \lambda_2) \in \mathbb{T}^2$, there is a $*$-isomorphism sending (u_1, u_2) to $(\lambda_1 u_1, \lambda_2 u_2)$, then the infinite ampliation of (u_1, u_2) is close in operator norm to a q-commuting pair. However, if q is not a root of unity, then an almost q-commuting unitary pair is automatically *almost* gauge invariant because conjugation with a product of u_is is an automorphism that maps (u_1, u_2) almost to $(q^k u_1, q^\ell u_2)$ for some $k, \ell \in \mathbb{Z}$. Combining the estimates, this allows us to draw the conclusion that the infinite ampliation of an almost q-commuting unitary pair must be close in operator norm to a q-commuting unitary pair whenever q is not a root of unity.

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Short talk: Clark measures for rational inner functions

Alan A. Sola

(joint work with John T. Anderson, Linus Bergqvist, Kelly Bickel, Joseph A. Cima)

A bounded holomorphic function φ on the unit polydisk $\mathbb{D}^d$ is said to be inner if its non-tangential boundary values are unimodular at almost every point on the d-torus. For each parameter $\alpha \in \mathbb{T}$, we can associate a measure σ_α to a given inner function via the relation

$$\mathrm{Re}\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right) = P[\sigma_\alpha](z), \quad z \in \mathbb{D}^d,$$

where $P[\mu]$ denotes the Poisson integral of a measure on $\mathbb{T}^d$. Since the left-hand side in this identity is the positive real part of a holomorphic function in the polydisk, each σ_α is a positive pluriharmonic measure. In addition, each σ_α is singular with respect to the Lebesgue measure on the d-torus because of the assumption on the non-tangential boundary values of φ.

For the class of rational inner functions in the bidisk, we present structure formulas giving a detailed description of the support sets of the associated Clark measures. The pairing of a continuous function f on $\mathbb{T}^2$ with σ_α can, in this case, be represented as a sum of integrals of the restriction of f to the union of smooth one-dimensional curves in the 2-torus, weighted by non-negative functions W_α that can be expressed using partial derivatives of the inner function φ. This can be viewed as a generalization of a well-known description of Clark measures arising from finite Blaschke products as sums of point masses at elements of the finite set $\{\zeta \in \mathbb{T} \colon \varphi(\zeta) = \alpha\}$.

As an application of this description of Clark measures σ_α for a rational inner function φ, we give a characterization of parameter values $\alpha \in \mathbb{T}$ for which the associated Clark operator mapping the model space $(\varphi H^2(\mathbb{D}^2))^\perp$ to $L^2(\sigma_\alpha)$ is unitary. This explains, for the class of rational inner functions, an earlier observation of E. Doubtsov [3] that contrasts with the one-variable setting: In polydisks with $d \geq 2$, Clark embeddings need not be surjective.

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Short talk: Counterexamples of von Neumann's inequality on the polydisc

Yi Wang

Given a contraction T on a Hilbert space $\mathcal{H}$ and an analytic polynomial p of one variable, von Neumann showed that its functional calculus $p(T)$ has operator norm less than or equal to the supreme norm of p on the unit disc. For two commuting contractions, Andô showed in [1] that an analogous result holds on the bidisc. Surprisingly, Varopoulos showed in [2] that for more than two commuting contractions, the analogous result does not hold on the polydisc. Later, more counterexamples were constructed. We give a method to understand the counterexamples in one framework and provide a method to construct more counterexamples. The method is based on a connection between Hilbert space operators and linear functionals on the polynomial ring $\mathbb{C}[z, \bar{z}]$, which was previously used to study subnormal and hyponormal operators.

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Strong convergence for GUE matrices

Mireille Capitaine
(joint work with Serban Belinschi)

In the fifties, Wigner proposed in quantum mechanics to replace the self-adjoint Hamiltonian operator in an infinite-dimensional Hilbert space by an ensemble of very large Hermitian matrices. Let us present the most frequently used Gaussian matrix ensemble: GUE (Gaussian unitary ensemble, so named because its law is invariant under conjugation by unitary matrices). A GUE matrix W_N of size N is a self-adjoint matrix such that the entries $(W_N)_{i,j}$ of W_N are centered Gaussian random variables, $\{(W_N)_{i,i}\colon 1 \leq i \leq N\} \cup \{\Re(W_N)_{i,j}, \Im(W_N)_{i,j}\colon 1 \leq j < i \leq N\}$ are independent, and $(W_N)_{i,i}, 1 \leq i \leq N$, and $\sqrt{2}\Re(W_N)_{i,j}, \sqrt{2}\Im(W_N)_{i,j}$, $1 \leq j < i \leq N$ all have variance equal to 1. We call $X_N = N^{-1/2}W_N$ a standard *normalized* GUE. The famous theorem of Wigner states that the empirical distribution of the eigenvalues of X_N converges weakly, in probability, to the standard semicircular distribution μ_{sc}, where $\mathrm{d}\mu_{sc}(t) = (2\pi)^{-1}\sqrt{4-t^2}\mathbf{1}_{[-2,2]}(t)\,\mathrm{d}t$. The semicircular distribution appeared also as the central limit distribution in Voiculescu's free probability theory, developed in the eighties. This occurrence hinted at a closer relationship between free probability and random matrices. In the early nineties, Voiculescu made this concrete by showing that freeness shows also up asymptotically in the random matrix world. Indeed, one of his results states that an r-tuple of independent standard normalized GUE-distributed matrices $X_N^{(1)}, \ldots, X_N^{(r)}$ are asymptotically free, that is, for every non-commutative polynomial P in r variables, one has

$$\lim_{N\to+\infty} \mathbb{E}\big[\mathrm{tr}_N P\big(X_N^{(1)},\dots,X_N^{(r)}\big)\big] = \tau(P(s_1,\dots,s_r)),$$

where tr_N stands for the normalized trace on $M_N(\mathbb{C})$ and $s_1,\dots,s_r$ are free standard semicircular variables in a C^*-probability space $(\mathcal{A},\tau)$. (A non-commutative random variable x in $(\mathcal{A},\tau)$ is called a standard semicircular variable if its distribution with respect to τ is μ_{sc}, that is, if $x=x^*$ (i.e., x is self-adjoint) and for any $k\in\mathbb{N}$, $\tau(x^k)=\int_{\mathbb{R}} t^k\,\mathrm{d}\mu_{sc}(t)$.) Actually, this result holds almost surely according to the results of Hiai–Petz and Thorbjørnsen. Later, Haagerup and Thorbjørnsen [2] proved a strong version of asymptotic freeness in the GUE case, namely a convergence for the operator norm: Almost surely, for every non-commutative polynomial P in r variables, one has

$$\lim_{N\to+\infty} \big\|P\big(X_N^{(1)},\dots,X_N^{(r)}\big)\big\| = \|P(s_1,\dots,s_r)\|\,. \tag{1}$$

Now, let $X_N=\{X_N^{(i)} : i=1,\dots r_1\}$ and $Y_N=\{Y_N^{(j)} : i=1,\dots r_2\}$ be independent tuples of independent standard normalized GUE $N\times N$ matrices. Let $(\mathcal{A},\tau)$ be a C^*-probability space equipped with a faithful tracial state and $\mathbf{s}=\{\mathbf{s}_i : i=1,\dots,r_1\}$ and $\mathbf{t}=\{\mathbf{t}_i : i=1,\dots,r_2\}$ be (possibly different) free semicircular systems, that is, tuples of free standard semicircular variables, in $(\mathcal{A},\tau)$. Denote by I_N the $N\times N$ identity matrix and by $1_{\mathbf{s}}$ the unit of $C^*(\mathbf{s}_1,\dots,\mathbf{s}_r)$, the unital C^*-algebra generated by the free semicirculars $\mathbf{s}_1,\dots,\mathbf{s}_r$. It is straightforward to deduce from the asymptotic freeness of the $X_N^{(i)}$s on the one hand and the $Y_N^{(i)}$s on the other that almost surely, for any non-commutative polynomial P in r_1+r_2 variables, one has

$$\begin{aligned}\lim_{N\to+\infty} (\mathrm{tr}_N\otimes\mathrm{tr}_N)\Big[P\big(X_N^{(1)}\otimes I_N,...,X_N^{(r_1)}\otimes I_N, I_N\otimes Y_N^{(1)},...,I_N\otimes Y_N^{(r_2)}\big)\Big]\\ = \quad (\tau\otimes\tau)[P(\mathbf{s}_1\otimes 1_{\mathbf{t}},...,\mathbf{s}_{r_1}\otimes 1_{\mathbf{t}},1_{\mathbf{s}}\otimes\mathbf{t}_1,...,1_{\mathbf{t}}\otimes\mathbf{t}_{r_2})]\,.\end{aligned}\tag{2}$$

In [1], we prove the following strong convergence.

Theorem 3. *Almost surely, for any non-commutative polynomial P in r_1+r_2 variables, one has*

$$\begin{aligned}\lim_{N\to+\infty} \big\|P\big(X_N^{(1)}\otimes I_N,\dots,X_N^{(r_1)}\otimes I_N, I_N\otimes Y_N^{(1)},\dots,I_N\otimes Y_N^{(r_2)}\big)\big\|\\ = \quad \|P(\mathbf{s}_1\otimes 1_{\mathbf{t}},\dots,\mathbf{s}_{r_1}\otimes 1_{\mathbf{t}},1_{\mathbf{s}}\otimes\mathbf{t}_1,\dots,1_{\mathbf{s}}\otimes\mathbf{t}_{r_2})\|_{\min}\,.\end{aligned}\tag{4}$$

In the wake of Voiculescu's discovery, random matrix theory became a powerful tool in the study of operator algebras. The option of modeling operator algebras asymptotically by random matrices led to new results on von Neumann algebras, in particular on the free group factors. Our investigation was motivated by a result of Hayes showing that a conjecture about the structure of certain finite von Neumann algebras is implied by a strong convergence result for tuples of random matrices. Specifically, the conjecture is the following. Assume $r\in\mathbb{N}$, $r>1$, is given. Denote by $\mathbb{F}_r$ the free group with r free generators and by $L(\mathbb{F}_r)$ the free group von Neumann algebra, that is, the von Neumann algebra generated by the left regular representation of $\mathbb{F}_r$ in the space $B(\ell^2(\mathbb{F}_r))$ of bounded linear operators on the Hilbert space $\ell^2(\mathbb{F}_r)$. Assume that Q is a von Neumann subalgebra of $L(\mathbb{F}_r)$ that is diffuse (meaning that it contains no minimal projections) and amenable (meaning that there exists a conditional expectation $E\colon B(\ell^2(\mathbb{F}_r)) \to Q$). Then there exists a unique maximal amenable von Neumann subalgebra P of $L(\mathbb{F}_r)$ such that $Q \subseteq P$. This conjecture is known as the Peterson–Thom conjecture. Hayes proved that if (4) holds with $r_1 = r_2$, then the Peterson–Thom conjecture is true as well. Note that previous works established the strong convergence of matrices $X_N^{(1)} \otimes I_M, \ldots, X_N^{(r)} \otimes I_M, I_N \otimes Y_M^{(1)}, \ldots, I_N \otimes Y_M^{(r)}$, where the dimension of the GUE matrices $Y_M^{(i)}$s is M and $M = o(N^{1/4})$ (by Pisier), $M = o(N^{1/3})$ (by Collins, Guionnet, and Parraud), and $M = o(N/(\log N)^3)$ (by Bandeira, Boedihardjo, and van Handel). This did not suffice for the purpose of Hayes. Note that after the first version of our paper appeared on arXiv, another proof of the Peterson–Thom conjecture was provided by Bordenave and Collins, by establishing a reformulation of Hayes's conjecture dealing with Haar-distributed random unitary matrices (instead of GUE matrices).

Our approach to proving (4) is very similar to that of [2, 3], based on matrix-valued Stieltjes–Cauchy transforms. Therefore, in this talk, we gave a gradual presentation of the ideas of Haagerup–Thorbjørnsen and Schultz that we use. First, we described how to use the Stieltjes–Cauchy inversion formula to prove the convergence of the largest eigenvalue of a GUE matrix. Later, we presented the so-called linearization trick to deal with several matrices. The method of Haagerup–Thorbjørnsen and Schultz requires sharp estimates on Stieltjes–Cauchy transforms. By necessity, our method to prove such sharp estimates, now with the tensorized GUE matrices in Theorem 3, is based on a series expansion of resolvents viewed as non-commutative rational functions successively in each variable. Thus, first, thanks to results from Yin, we re-phrase Theorem 3 in terms of the Cayley transforms of the self-adjoint variables involved in order to deal with a bounded sequence of matrices. Then a series expansion around infinity of the resolvents and an expansion in $1/N^2$ of the expectations of normalized traces of polynomials in Cayley transforms of tuples of independent GUE matrices allow us to obtain a precise formula for large $|z|$. Since this formula we obtained for large $|z|$ involves functions that can be analytically extended to $\mathbb{C}\setminus\mathbb{R}$, we can deduce that it holds on the whole $\mathbb{C} \setminus \mathbb{R}$. Our proof relies on the explicit asymptotic expansion of smooth functions in polynomials in independent GUE matrices obtained by Parraud. We use Parraud's formulas in an essential way in our work but with the free difference quotient replaced by the difference-differential operator in non-commutative function theory via the natural identification between the two operations.

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Indices for quadratic programs for fun and profit

James E. Pascoe
(joint work with Geoffrey Hutinet)

Let $\mathcal{H}$ be a real reproducing kernel Hilbert space on a space Ω. Given a compactly supported probability distribution μ, we can view it as an element of $\mathcal{H}$ via the Riesz representation theorem, that is, $\int f \, \mathrm{d}\mu = \langle f, \mu \rangle$. Such correspondence is known as the **kernel embedding of distributions**. Given a compact set K, we let $M_K \subseteq \mathcal{H}$ denote the set of distributions on K viewed as elements of $\mathcal{H}$.

Let ψ be a continuous function on Ω. Consider minimizing $\|\psi - \mu\|$ over distributions μ supported on K. Note, formally,

$$\|\psi - \mu\|^2 = \|\psi\|^2 - 2\langle \psi, \mu \rangle + \|\mu\|^2.$$

So, as the inner product is an integral and $\|\psi\|^2$ is constant, such a task is equivalent to maximizing the **aesthetic objective**

$$\mathcal{O}(\mu) = \int \psi \, \mathrm{d}\mu - \frac{\|\mu\|^2}{2}.$$

We note the latter formulation need not require $\psi \in \mathcal{H}$. We call the embedded maximizer μ the **topiary** of K with respect to ψ (so called because of a connection to hedging). We define the **aesthetic margin** to be

$$\iota(\mu)(x) = D\mathcal{O}(\mu)[\delta_x - \mu] = \psi(x) - \mu(x) - \int_K (\psi - \mu) \, \mathrm{d}\mu.$$

We note that if μ is the topiary of K with respect to ψ, then $\iota(\mu)|_K \leq 0$, where equality holds on the support of μ. We call K a **topiaric index** with respect to ψ if the aesthetic margin of the topiary of K with respect to ψ is identically equal to 0.

The **green frontier** of a set K with respect to ψ is defined to be

$$\mathit{Green}_\psi(K) = \bigcap_{U \subseteq M_K \text{ open, dense}} \left\{ x \in K \mid \psi(x) - \mu(x) = \sup_K (\psi - \mu), \ \mu \in U \right\}.$$

Theorem 1 (Green topiary theorem)**.** *The topiary of K with respect to ψ is equal to the topiary of $\mathit{Green}_\psi(K)$ with respect to ψ.*

The following reconciles the sparsity arising from the green topiary theorem with the philosophy that an optimal portfolio is a broadly supported index fund weighted according to the real economy.

Theorem 2 (Invisible index theorem)**.** *Suppose $K \subseteq K_0$, both compact, and K_0 is a topiaric index. The topiary of K with respect to ψ is equal to the topiary of K with respect to the topiary of K_0 with respect to ψ.*

Define $\beta_K(x) = \mu(x)/\|\mu\|^2$ and $r_K = \int(\psi - \mu)\,\mathrm{d}\mu$. (We call r_K the **topiaric rate**.) We have the following analog of the capital asset pricing model.

Theorem 3 (Captal asset pricing inequality)**.** *Let μ be the topiary of K with respect to ψ. Then*

$$\psi(x) - r_K \leq \beta_K(x)\left(\int \psi\,\mathrm{d}\mu - r_K\right).$$

Equality holds on the support of K.

Considering such over the real Hardy space, given M a compact subset of the disk with analytic boundary and connected complement not containing 0, the capital asset pricing inequality gives that the topiary of M with respect to 0 (which must be finitely supported) is greater than 1 on M, and thus, gradient descent from the origin gives a path to the boundary not intersecting M. One can view this as analogously comparable to physical maze solvers relying on insulated walls but where strategy replaces insulation.

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Models for operator-valued free convolution powers

David Jekel

(joint work with Ian Charlesworth)

A measure ν is said to be the tth free convolution power of μ if the R-transforms of these measures satisfy $R_\nu = tR_\mu$. For any probability measure on $\mathbb{R}$, the tth free convolution power exists for all $t \geq 1$; analytic arguments for this were given by Bercovici and Voiculescu [3] under additional hypotheses and then in full generality by Belinschi and Bercovici [2]. Nica and Speicher [4] showed with a combinatorial argument that if X is an operator with law μ and P is a projection of trace $1/t$ free from X, then PXP has law $\mu^{\boxplus t}$.

In the setting of $\mathcal{B}$-valued free probability (described in Hari Bercovici's lecture series at the workshop), if $\eta : \mathcal{B} \to \mathcal{B}$ is a completely positive map, then $\nu = \mu^{\boxplus \eta}$ means that $R_\nu = \eta \circ R_\mu$. Anshelevich, Belinschi, Fevrier, and Nica [1] showed that $\mu^{\boxplus \eta}$ always exists when $\eta \geq \mathrm{id}$. In fact, Shlyakhtenko [5] showed that if X is an operator with law μ, then one can construct explicitly an operator V such that V^*XV has law $\mu^{\boxplus \eta}$, thus showing the existence of such a convolution power by an explicit operator realization of it. The operator V is something like an isometry but certainly never exists in a tracial von Neumann algebra.

My joint work in progress with Ian Charlesworth gives a unification of the construction of free convolution powers through compression by a projection and Shlyakhtenko's partial isometry construction. In particular, we showed that if X and V are operators in a $\mathcal{B}$-valued non-commutative probability space $(\mathcal{A}, E)$ such that X and V are freely independent, X has law μ, $Vb_1V^*b_2V = Vb_1\eta(b_1)$ for all $b_1, b_2 \in \mathcal{B}$, and $E[VbV^*] = b$ for all $b \in \mathcal{B}$, then the law of V^*XV is $\mu^{\boxplus\eta}$. Our proof is analogous to the argument for the additivity of the R-transform for additive free convolution presented in Bercovici's lectures at the workshop, a method due to Haagerup and Lehner.

We also gave an analog for the setting of convolution powers of Biane's conditional expectation formula for resolvents of an additive free convolution, showing that if $\mathcal{E}$ is the conditional expectation from the algebra generated by $\mathcal{B}$, X, V to the algebra generated by $\mathcal{B}$ and X and if z is in the upper half-plane of $\mathcal{B}$, then

$$\mathcal{E}[V(z - V^*XV)V^*] = (F(z) - X)^{-1},$$

where F is the subordination function associated to the convolution power, namely the unique non-commutative analytic self-map of the upper half-plane satisfying $G_\nu = G_\mu \circ F$.

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Functional calculus calculus

Evangelos A. Nikitopoulos

Let $\mathcal{B}$ be a unital Banach algebra, let $U \subseteq \mathbb{C}$ be a non-empty open set, and let

$$\mathcal{B}_U \coloneqq \{a \in \mathcal{B} : \sigma(a) \subseteq U\}.$$

Given $a \in \mathcal{B}_U$, recall that the *holomorphic functional calculus* for a is defined by

$$f(a) \coloneqq \frac{1}{2\pi i}\int_\Gamma f(z)\,(z-a)^{-1}\,\mathrm{d}z \in \mathcal{B}, \qquad f \in \mathrm{Hol}(U),$$

where Γ is a cycle surrounding $\sigma(a)$ in U. Given $f \in \mathrm{Hol}(U)$, it is natural to ask whether the map $f_{\mathcal{B}} \colon \mathcal{B}_U \to \mathcal{B}$ defined by $a \mapsto f(a)$ is holomorphic. Indeed, as I explained in my talk, this is true and can be proven by differentiating under the (contour) integral in the definition of the holomorphic functional calculus.

Theorem 1. *If $f \in \mathrm{Hol}(U)$, then $f_{\mathcal{B}} \in \mathrm{Hol}(\mathcal{B}_U;\mathcal{B})$, and*

$$\partial_{b_k}\cdots\partial_{b_1} f_{\mathcal{B}}(a) = \frac{1}{2\pi i}\sum_{\pi\in S_k}\int_\Gamma f(z)\,(z-a)^{-1}b_{\pi(1)}\cdots(z-a)^{-1}b_{\pi(k)}(z-a)^{-1}\,\mathrm{d}z$$

for all $a \in \mathcal{B}_U$ and $b_1,\dots,b_k \in \mathcal{B}$, where S_k is the symmetric group on k letters and Γ is any cycle surrounding $\sigma(a)$ in U.

Let us see how this derivative formula can be rewritten in terms of the kth "divided difference" of f. If $S \subseteq \mathbb{C}$ and $f\colon S \to \mathbb{C}$ is any function, then we recursively define $f^{[0]} \coloneqq f$ and

$$f^{[k]}(\boldsymbol{\lambda}) \coloneqq \frac{f^{[k-1]}(\lambda_1,\dots,\lambda_k) - f^{[k-1]}(\lambda_1,\dots,\lambda_{k-1},\lambda_{k+1})}{\lambda_k-\lambda_{k+1}}$$

for all $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_{k+1}) \in S^{k+1}$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. The function $f^{[k]}$ is the *kth divided difference* of f. If $f \in \mathrm{Hol}(U)$, then $f^{[k]} \in \mathrm{Hol}(U^{k+1})$. If $V \subseteq \mathbb{R}$ is open and $f \in C^k(V)$, then $f^{[k]} \in C(V^{k+1})$. Next, if $b = (b_1,\dots,b_k) \in \mathcal{B}^k$, write $\mathcal{B}^{\hat{\otimes}_\pi(k+1)} \ni u \mapsto u\sharp_k b \in \mathcal{B}$ for the bounded linear map determined, via the universal property of the Banach space projective tensor product $\hat{\otimes}_\pi$, by

$$(a_1\otimes\cdots\otimes a_{k+1})\sharp_k b = a_1b_1\cdots a_kb_ka_{k+1}, \qquad a_i \in \mathcal{B}.$$

Finally, let $f \in \mathrm{Hol}(U)$. If $a_1,\dots,a_{k+1} \in \mathcal{B}_U$ and

$$\tilde{a}_i \coloneqq 1^{\otimes(i-1)}\otimes a_i \otimes 1^{k+1-i} \in \mathcal{B}^{\hat{\otimes}_\pi(k+1)}, \qquad i = 1,\dots,k+1,$$

then we write

$$f_\otimes^{[k]}(a_1,\dots,a_{k+1}) \coloneqq f^{[k]}(\tilde{a}_1,\dots,\tilde{a}_{k+1}) \in \mathcal{B}^{\hat{\otimes}_\pi(k+1)}, \tag{2}$$

where the right-hand side is defined via the multivariate (i.e., Taylor or Shilov–Arens–Calderón–Waelbroeck) holomorphic functional calculus in the Banach algebra $\mathcal{B}^{\hat{\otimes}_\pi(k+1)}$.

Theorem 3. *Retain the setting of Theorem 1. Then*

$$\partial_{b_k}\cdots\partial_{b_1} f_{\mathcal{B}}(a) = \sum_{\pi\in S_k} f_\otimes^{[k]}(\underbrace{a,\dots,a}_{k+1\text{ times}})\sharp_k[b_{\pi(1)},\dots,b_{\pi(k)}] \tag{4}$$

for all $a \in \mathcal{B}_U$ and $b_1,\dots,b_k \in \mathcal{B}$.

In the "real C^k case," things are not as nice. Let $\mathcal{A}$ be a unital C^*-algebra, and write $\mathcal{A}_{\mathrm{sa}} \coloneqq \{a \in \mathcal{A} : a^* = a\}$. If $f \in C(\mathbb{R})$, then it is elementary to show that the map $f_{\mathcal{A}}\colon \mathcal{A}_{\mathrm{sa}} \to \mathcal{A}$ defined, via the continuous functional calculus, by $a \mapsto f(a)$ is continuous. However, it is not generally true that $f \in C^k(\mathbb{R})$ implies $f_{\mathcal{A}} \in C^k(\mathcal{A}_{\mathrm{sa}};\mathcal{A})$ whenever $k \in \mathbb{N}$. There are counterexamples (due to Farforovskaya) in the $k=1$ case dating back to the 1970s. Equation (4) hints at a reason; it does not make sense in the "real C^k" setting. Specifically, if $f \in C^k(\mathbb{R})$, $a_i \in \mathcal{A}_{\mathrm{sa}}$, and $\mathcal{A} = \mathcal{B}$, then the natural setting for Equation (2) would be the minimal C^*-tensor product $\mathcal{A}^{\otimes_{\min}(k+1)}$, but the $\sharp_k$ operation is not defined even on the maximal C^*-tensor product $\mathcal{A}^{\otimes_{\max}(k+1)}$.

Historically, this issue has been overcome using a tool called a "multiple operator integral" (MOI). Consider the case $\mathcal{A} = B(H)$, where H is a complex Hilbert space. Then the functional calculus may be written as

$$f(a) = \int_{\sigma(a)} f(\lambda)\, P^a(\mathrm{d}\lambda),$$

where P^a is the projection-valued spectral measure of $a \in B(H)_{\mathrm{sa}}$ given by the spectral theorem. The correct *formal* interpretation of the right-hand side of Equation (4) in terms of P^a is

$$\sum_{\pi \in S_k} \underbrace{\int_{\sigma(a)} \cdots \int_{\sigma(a)}}_{k+1 \text{ times}} f^{[k]}(\boldsymbol{\lambda})\, P^a(\mathrm{d}\lambda_1)\, b_{\pi(1)} \cdots P^a(\mathrm{d}\lambda_k)\, b_{\pi(k)}\, P^a(\mathrm{d}\lambda_{k+1}). \tag{5}$$

However, even this expression poses a problem because standard theory only allows the P^a-integration of *scalar-valued* functions. This is precisely the problem MOIs were invented to solve, starting with pioneering papers of Daletskii–Krein (1956) and Birman–Solomyak (1960s–1980s). The literature on MOIs and their applications is surveyed nicely in [3]. Using this theory, one can make rigorous sense of iterated integral expressions as above if f is nice enough.

Theorem 6 ([2])**.** *If $f\colon \mathbb{R} \to \mathbb{C}$ belongs (locally) to the homogeneous Besov space $\dot{B}_1^{k,\infty}(\mathbb{R})$* [2, Def. 3.3.1]*, then $f_{\mathcal{A}} \in C^k(\mathcal{A}_{\mathrm{sa}}; \mathcal{A})$, and $\partial_{b_k} \cdots \partial_{b_1} f_{\mathcal{A}}(a)$ may be written as the MOI* (5) *whenever $a, b_1, \ldots, b_k \in \mathcal{A}_{\mathrm{sa}}$.*

This theorem generalizes a result of V. V. Peller (2006) in the case $\mathcal{A} = B(H)$ with H separable. In my talk, I discussed the most common approach to proving such results: the "method of perturbation formulas." One may also try, as Peller did, to differentiate the map $B(H)_{\mathrm{sa}} \ni b \mapsto f(a+b) - f(a) \in B(H)$, where a is an *unbounded* self-adjoint operator on H. More generally, if $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra, $\mathcal{I}$ is some normed ideal of $\mathcal{M}$, and a is an unbounded self-adjoint operator on H affiliated with $\mathcal{M}$, then one may try to differentiate the map $\mathcal{I}_{\mathrm{sa}} \ni b \mapsto f(a+b) - f(a) \in \mathcal{I}$. For examples of such results, please see [1] and [3, §5.3].

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Short talk: Singularity degrees for matrices of zeros and free semicircular elements

Torben Krüger
(joint work with David Renfrew)

We consider the empirical spectral distribution of structured random matrices $H = (h_{ij}^{lk}) \in \mathbb{C}^{K\times K} \otimes \mathbb{C}^{N\times N}$ that have independent entries above the diagonal and a variance profile. Here, $k, l = 1, \ldots, K$ are the block indices, and $i, j = 1, \ldots, N$ are the internal indices. The matrices are centered (i.e., $\mathbb{E}h_{ij}^{lk} = 0$) and Hermitian (i.e., $h_{ij}^{lk} = \overline{h_{ji}^{kl}}$), and the variance profile $S = (s_{lk})$ only depends on the block indices (i.e., $s_{lk} = \mathbb{E}|h_{ij}^{lk}|^2$). As the dimension N of the blocks tends to infinity, the associated eigenvalue distribution converges to the distribution of an operator-valued semicircular element of the form $C = \sum_{l,k} \sqrt{s_{lk}} E_{lk} \otimes c_{lk} \in \mathbb{C}^{K\times K} \otimes \mathcal{A}$, where $c_{lk} = c_{kl}$ are free semicircular element realised in a non-commutative probability space $\mathcal{A}$ and (E_{lk}) denotes the canonical basis of $\mathbb{C}^{K\times K}$. The spectral distribution of C is a probability measure on the real line of the form $\rho(d\tau) = \kappa\, \delta_0(d\tau) + \rho(\tau)\, d\tau$, where the density $\rho(\tau)$ is bounded for τ away from the origin $\tau = 0$ and the potential atom at zero has mass $\kappa \geq 0$. When S is bounded from below by a positive multiple of a permutation matrix entrywise, we say that S has support. The property of S having support is equivalent to $\kappa = 0$.

In case S has support, the density $\rho(\tau)$ may still diverge as $\tau \to 0$. We provide a complete classification of this divergence in terms of the location of the zero-entries of S. We show that the divergence follows a power law $\rho(\tau) \sim |\tau|^{-\gamma}$ and provide an algorithm for the computation of the singularity degree $\gamma \in [0, 1)$ and therefore the associated Novikov–Shubin invariants

$$\lim_{\tau \downarrow 0} \frac{\log \int_{-\tau}^{\tau} \rho(\sigma)\, d\sigma}{\log \tau} = 1 - \gamma\,.$$

As a consequence of our algorithm, the singularity degree is of the form $\gamma = \ell/(\ell + 2)$, where $\ell \in \mathbb{N}_0$ is the length of the longest path in a graph $\Gamma(S)$ with vertex set $\{1, \ldots, K\}$ and edges determined in terms of the location of the zero-entries of S.

The proof requires solving the discrete boundary value problem

$$f(i) = \frac{1}{2}\Big(\min_{j : i \lhd j} f(j) + \max_{j : j \lhd i} f(j) \Big)$$

for a function $f : \Gamma(S) \cup \{0, \infty\} \to \mathbb{R}$ with $f(0) = -1$ and $f(\infty) = 1$, where we write $i \lhd j$ if the graph contains an edge from i to j and $0 \lhd i \lhd \infty$ holds for all $i = 1, \ldots, K$.

Determinants of pencils of random unitaries

Michael T. Jury
(joint work with George Roman)

Let $\mathcal{U}(N) \subset M_N(\mathbb{C})$ denote the $N \times N$ unitary group, equipped with the Haar measure dU, normalized to have total mass 1. For a fixed $U \in \mathcal{U}(N)$, we form the characteristic polynomial $\det(1 - zU^*)$, which has its zeroes at the eigenvalues of U and is normalized to be 1 at the origin. It is known that for all N and all complex numbers z, w,

$$(1) \quad \int_{\mathcal{U}(N)} \det(1 - zU^*)\overline{\det(1 - wU^*)}\, dU = 1 + z\overline{w} + \cdots + (z\overline{w})^N = \frac{1 - (z\overline{w})^{N+1}}{1 - z\overline{w}}$$

and hence that

$$(2) \qquad \lim_{N\to\infty} \int_{\mathcal{U}(N)} \det(1 - zU^*)\overline{\det(1 - wU^*)}\, dU = \frac{1}{1 - z\overline{w}}$$

whenever $|z|, |w| < 1$. (By the invariance of the Haar measure, these integrals are unchanged if we make the change of variable $U \to U^*$, which will be more convenient for what follows.) More generally if p, q are polynomials that are *stable* (that is, have no zeroes in the closed unit disk $|z| \leq 1$) and normalized to have $p(0) = q(0) = 1$, then we can write $p(t) = \prod_{i=1}^n (1 - a_i t)$, $q(t) = \prod_{j=1}^m (1 - b_j t)$ with all $|a_i|, |b_j| < 1$, and we have

$$(3) \qquad \lim_{N\to\infty} \int_{\mathcal{U}(N)} \det(p(U)) \det(q(U)^*)\, dU = \prod_{i,j=1}^{m,n} \frac{1}{1 - a_i\overline{b_j}}.$$

(See, for example, [1].) If we let A, B be the diagonal matrices with eigenvalues $\{a_i\}$, $\{b_j\}$, respectively, and note that $\det(p(U)) = \det(I_n \otimes I_N - A \otimes U)$ and $\det(q(U)) = \det(I_m \otimes I_N - B \otimes U)$, this last expression may be rewritten as

$$(4) \quad \lim_{N\to\infty} \int_{\mathcal{U}(N)} \det(I\otimes I - A\otimes U) \det((I\otimes I - B\otimes U)^*)\, dU = \det((I\otimes I - A\otimes \overline{B})^{-1}).$$

We are concerned with multivariate generalizations of these expressions. We are able to prove multivariate analogs of (1) and (2), and conjecture multivariate analogs of (3) and (4) for an appropriate family of "stable" non-commutative polynomials.

Theorem 5. *Let $d \geq 1$ be an integer, and let $U_1, \ldots, U_d$ denote $N \times N$ unitary matrices sampled independently with respect to the Haar measure on $\mathcal{U}(N)$. We write dU^d for the d-fold product of the Haar measure. Then for each integer $N \geq 1$ and all complex numbers $z_1, \ldots, z_d, w_1, \ldots, w_d$, we have*

$$(6) \quad \int_{\mathcal{U}(N)^d} \det\left(1 - \sum_{j=1}^d z_j U_j\right) \det\left(1 - \sum_{j=1}^d \overline{w_j} U_j^*\right) dU^d = \sum_{|\alpha|\leq N} c_\alpha(N) z^\alpha \overline{w}^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ *is the usual multi-index notation and* $c_\alpha(N)$ *is expressed in terms of binomial and multinomial coefficients as*

$$c_\alpha(N) = \binom{N}{|\alpha|}\binom{|\alpha|}{\alpha_1, \alpha_2, \ldots, \alpha_d}^2 \prod_{j=1}^{d} \binom{N}{\alpha_j}^{-1}. \tag{7}$$

In particular, if $\sum |z_j|^2, \sum |w_j|^2 < 1$, *then*

$$\lim_{N\to\infty} \int_{\mathcal{U}(N)^d} \det\left(1 - \sum_{j=1}^{d} z_j U_j\right) \det\left(1 - \sum_{j=1}^{d} \overline{w_j} U_j^*\right) dU^d = \frac{1}{1 - \langle z, w\rangle}, \tag{8}$$

where $\langle z, w\rangle = \sum z_j \overline{w_j}$ *is the standard inner product on* $\mathbb{C}^d$.

The statement about the limit (8) is a straightforward calculation from the closed-form expression (6). While the analogous single-variable expression (1) can be proved by several means (direct computation using the Weyl integration formulas, or in terms of Toeplitz determinants via the Heine–Szegő identity, or via representation theory and Schur–Weyl duality), none of these approaches seems to generalize adequately to several variables. The proof therefore proceeds by a brute force expansion of the determinants into minors and ultimately reduces to a calculation of covariances of minors of a Haar unitary, which may be evaluated explicitly via the Weingarten calculus. It is remarkable that what appears on the right-hand side in (8) is the so-called *Drury–Arveson kernel*, which frequently appears in the context of multivariable operator theory (see, for example, the survey [3]), but to our knowledge, this is the first appearance of this object in connection with random matrix theory.

To state the conjectured generalizations of (3) and (4), we first introduce (what we conjecture is) the correct notion of stability, which is compatible with an appropriate determinantal representation, via the following lemma. To state it, define $\|X\|_{row}^2 = \|X_1X_1^* + \cdots + X_dX_d^*\|$ for a d-tuple of square matrices $X = (X_1, \ldots, X_d)$ (all of the same size).

Lemma 9. *Let* $p(x_1, \ldots, x_d)$ *be a polynomial in non-commuting indeterminates* $x_1, \ldots, x_d$ *satisfying* $p(0) = 1$. *The following are equivalent:*

1) $\det p(X_1, \ldots, X_d) \neq 0$ *for all* n *and all* d*-tuples* X *of* $n \times n$ *matrices satisfying* $\|X\|_{row} \leq 1$.
2) *There exists a* d*-tuple of square matrices* $A_1, \ldots, A_d$ *of some size* $m \times m$, *with* $\|A\|_{row} < 1$, *such that for all* n *and all* d*-tuples of* $n \times n$ *square matrices* X, *we have* $\det p(X) = \det(I_m \otimes I_n - \sum A_j \otimes X_j)$.

The lemma is a consequence of [2, Thm. A] applied to the non-commutative rational function $p(x)^{-1}$. Polynomials p satisfying the equivalent conditions of the lemma we call *stable*, and we refer to the expression in the second item as a *determinantal representation* of p. We make the following conjecture, using the abbreviated notation $A \otimes X = \sum_{j=1}^{d} A_j \otimes X_j$, etc.

Conjecture 10. *Let p and q be polynomials in non-commuting indeterminates $x_1, \dots, x_d$ with $p(0) = q(0) = 1$, and suppose p and q are stable with determinantal representations* $\det p(X) = \det(I - A \otimes X)$, $\det q(X) = \det(I - B \otimes X)$*. Then*

$$\lim_{N\to\infty} \int_{\mathcal{U}(N)^d} \det(p(U))\overline{\det q(U)}\, dU^d = \det((I - A \otimes \overline{B})^{-1}). \tag{11}$$

One can give a formal argument for the conjecture in the spirit of the proof of the strong Szegő limit theorem given by Bump and Diaconis [1]. The formal argument can be made rigorous if the following weaker conjecture holds:

Conjecture 12. *For each $0 \leq r < 1$ and each integer $k \geq 1$, there is a constant $C = C(r,k)$ such that for all d-tuples of $k \times k$ matrices $A = (A_1, \dots, A_d)$ satisfying $\|A\|_{row} \leq r$, we have*

$$\sup_N \int_{\mathcal{U}(N)^d} |\det(I - A \otimes U)|^2\, dU^d \leq C(r,k). \tag{13}$$

From above, the conjecture holds for all d when $k = 1$ and for all k when $d = 1$. It is plausible that $C(r,k) = (1 - r^2)^{-k^2}$ should work and that the integrals in (13) should increase monotonically with N, but we have not been able to prove this. (Both are true when $k = 1$.)

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Regularity questions in free probability

Dimitri Shlyakhtenko

Let $X_1, \dots, X_n$ be non-commutative random variables in a non-commutative probability space (M, τ) (that is, M is a von Neumann algebra; τ is a faithful, normal, tracial state; and $X_1, \dots, X_n$ are self-adjoint elements of M). The notion of their joint law in general does not have a ready measure-theoretic interpretation. Nonetheless, Voiculescu (1998) introduced the *free conjugate variables*, which are a free probability analog of the gradient of the logarithm of the density of a classical probability measure.

Voiculescu's definition involves free difference quotient derivations, which are close to the subject of the workshop. Let $A = \mathbb{C}\langle t_1, \dots, t_n \rangle$ be the algebra of non-commutative polynomials in n indeterminates $t_1, \dots, t_n$, and view $A \otimes A$ as a bimodule over A using the action $a \cdot (b \otimes c) \cdot d = ab \otimes cd$. The difference quotient derivations ∂_j are determined by $\partial_j t_i = \delta_{i=j} 1 \otimes 1$ and the Leibnitz rule $\partial(ab) = a \cdot \partial(b) + \partial(a) \cdot b$.

Let $ev : A \to M$ (respectively, $ev \otimes ev : A \otimes A \to M \otimes M$) be the evaluation map, obtained by substituting X_1 for t_1, X_2 for t_2, and so on. Voiculescu defined free conjugate variables $\xi_1, \ldots, \xi_n$ to be vectors in $L^2(M, \tau)$ with the property that the equation

$$\tau(\xi_j\, ev(P)) = \tau \otimes \tau(ev \otimes ev(\partial_j(P)))$$

holds whenever $P \in A$.

It turns out that the existence of $\xi_1, \ldots, \xi_n$ implies the absence of any algebraic relations between the X_js, so the evaluation map ev has to be injective; in this case, we can identify A with its image inside M and view ∂_j as a densely defined operator from $L^2(M)$ to $L^2(M) \otimes L^2(M)$. Closely related to ∂_j is the operator

$$D_j = (1 \otimes \tau)\partial_j.$$

In important examples (such as a free semicircular n-tuple), it is actually a bounded operator from $L^2(M)$ to $L^2(M)$ and satisfies

$$\partial_j(Q)\#P_1 = [D_j, Q],$$

where P_1 is the rank-one projection onto $1 \in L^2(M)$ and $(a \otimes b)\#T = aTb$. The n-tuple $D_1, \ldots, D_n$ was called a dual system by Voiculescu (1998).

In the case $n = 1$, one has a single variable X with associated probability measure μ (given by the trace applied to the spectral measure of X); in this case, the free conjugate variable ξ turns out to be the Hilbert transform of the density of μ restricted to the spectrum of X, and the operator D is closely related to the Hilbert transform, given on $L^2(\mu)$ by the "integral kernel" $(x - y)^{-1}$.

So what kind of regularity can one obtain if one makes various regularity assumptions on the free conjugate variables?

Let us assume that the dual system $D_1, \ldots, D_n$ consists of bounded operators. In our work [2] on L^2-homology, we considered the following definition of a 1-cycle $T_1, \ldots, T_n$: the T_js must be operators satisfying

$$\sum [T_i, X_i] = 0.$$

Central to L^2-homology is the question of when there are such 1-cycles with T_j belonging to finite-rank operators versus Hilbert–Schmidt operators. Let now Q be some non-commutative polynomial in indeterminates $t_1, \ldots, t_n$. Then the non-commutative difference quotients satisfy

$$Q \otimes 1 - 1 \otimes Q = \sum_i (t_i \partial_i Q - \partial_i Q t_i).$$

Evaluating both sides in $X_1, \ldots, X_n$ gives

$$ev(Q) \otimes 1 - 1 \otimes ev(Q) = \sum_i [X_i, ev \otimes ev(\partial_i(Q))\#P_1].$$

In particular, if an algebraic relation holds (so we can choose Q so that $ev(Q) = 0$) we get $\sum_i [X_i, R_i] = 0$ with $R_i = ev \otimes ev(\partial_i(Q))\#P_1$ finite-rank operators. On the other hand,

$$\operatorname{Tr}\left(D_j \sum_i [X_i, R_i]\right) = \sum_i \operatorname{Tr}(R_i[D_j, X_i]) = \operatorname{Tr}(R_j P_1).$$

Replacing D_j by $\sum a_k D_j b_k$ with a_k, b_k in the commutant of $W^*(X_1, \ldots, X_n)$ gives $R_j = 0$, which then easily implies that Q must be a constant polynomial. This gives the absence of algebraic relations.

However, one can also write down a quantitative version of this proof. It shows that $Y = Q(X_1, \ldots, X_n)$ cannot have a singular spectral measure, since otherwise, it is possible to find a sequence of positive, trace-class contractions increasing to the identity that asymptotically commute with Y (in the sense that the trace-class norm of the commutator goes to zero). The details can be found in [1].

On the other hand, there is a wonderful result of Mai, Speicher, and Yin (2023), which shows that the condition that $\sum [T_i, X_i] = 0$ has no finite-rank solution is sufficient to guarantee that the variables $X_1, \ldots, X_n$ generate, in the algebra of operators affiliated to $W^*(X_1, \ldots, X_n)$, a division algebra that is isomorphic to the so-called non-commutative free field.

If one now assumes more regularity by putting conditions on the conjugate variables $\xi_1, \ldots, \xi_n$, further results become available. For a semicircular system, $\xi_j = X_j$; if one instead requires that $\xi_j = X_j + Q(X_1, \ldots, X_n)$, where Q_j are power series "sufficiently close to zero," then [5] shows the existence of power series F_i that are invertible and have the property that $(F_1(X_1, \ldots, X_n), \ldots, F_n(X_1, \ldots, X_n))$ forms a free semicircular system. In fact, as was shown later in [6], the resulting transformation

$$(X_1, \ldots, X_n) \mapsto (F_1(X_1, \ldots, X_n), \ldots, F_n(X_1, \ldots, X_n))$$

gives an "optimal transport map," in the sense that the coupling

$$((X_1, \ldots, X_n), (F_1(X_1, \ldots, X_n), \ldots, F_n(X_1, \ldots, X_n))$$

achieves the Wasserstein distance (Biane–Voiculescu (2001)). In particular, the holomorphic closure, the C^*-algebra, and the von Neumann algebra generated by $X_1, \ldots, X_n$ are, in this way, isomorphic to the corresponding algebras generated by a free semicircular system.

It is possible to further relax these assumptions to the situation where ξ_j is a so-called cyclic gradient of a convex function V; see [3, 4] for several notions of convexity. In this case, one can construct, using an ODE, the functions $F_1, \ldots, F_n$, which realize isomorphisms between the C^*- and W^*-algebras generated by $X_1, \ldots, X_n$ and ones generated by a free semicircular system.

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On operator-valued circular and Haar unitary elements

Ken Dykema

(joint work with March Boedihardjo, John Griffin)

Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a normal, faithful, tracial state τ. Thus, $(\mathcal{M},\tau)$ is a non-commutative probability space. An element $z \in \mathcal{M}$ is a *circular element* if its real and imaginary parts are free, centered semicircular elements having the same second moment. Voiculescu proved that the polar decomposition of z is $z = u|z|$, where u is a *Haar unitary* element (namely, is a unitary satisfying $\tau(u^n) = 0$ for all nonzero integers n) that is free from the positive element $|z|$.

After the appearance of Speicher's free cumulants, it became natural to generalize circular elements to the operator-valued case. Here, we let B be a von Neumann algebra and work in a B-valued non-commutative probability space $(\mathcal{M}, E)$. This entails that $\mathcal{M}$ is a von Neumann algebra containing a unital copy of B, and $E : \mathcal{M} \to B$ is a faithful, normal conditional expectation. In fact, we always work in *tracial* B-valued non-commutative probability space, which means that we also have a faithful, normal, tracial state τ_B on B such that $\tau_B \circ E$ is a trace on $\mathcal{M}$. A *B-valued circular* element is $z \in \mathcal{M}$ whose B-valued free cumulants all vanish, except for those of second order given by the completely positive maps $\alpha, \beta : B \to B$,

$$\alpha(b) = E(zbz^*), \qquad \beta(b) = E(z^*bz).$$

In practice, this means that all B-valued $*$-moments of z can be computed from these two maps.

As was shown in [1], given two completely positive maps α and β from B to B, they arise as the cumulant maps of a B-valued circular element in a tracial setting precisely when we have

$$\tau_B(\alpha(b_1)b_2) = \tau_B(b_1\beta(b_2)), \qquad b_1, b_2 \in B.$$

We consider the polar decompositions $z = u|z|$ of such B-valued circular elements z and suppose u is unitary (namely, $\ker(z) = \{0\}$). We ask: (1) When must u and $|z|$ be $*$-free (over B) with respect to E (in such case, we say that z has a *free polar decomposition*), and (2) must u be a Haar unitary? In fact, there are several possible notions of Haar unitary, and we distinguish them with examples. We know that u must be an R-diagonal Haar unitary. (R-diagonality is a notion introduced by Nica and Speicher, and in the B-valued context by Speicher and Śniady, it can be defined in terms of B-valued free cumulants and has other nice characterizations.) We might ask for a lot: Must u normalize B and satisfy $E(u^n) = 0$ for all nonzero integers n? (Such a unitary we call a *normalizing Haar unitary.*) From [1], we have a nontrivial example of a B-valued circular element z when B is the two-dimensional algebra $\mathbb{C}^2$, having $\ker(z) = \{0\}$ with polar decomposition $z = u|z|$ that is not free and where u is not normalizing. We would like to understand better the B-valued distribution of this unitary u or of other unitaries that are polar parts of B-valued circulars and that are not normalizing.

This prompted us to ask, when z is B-valued circular and has free polar decomposition, must the unitary be normalizing? Unfortunately, from the determining maps α and β, we are not able to decide, in general, whether the polar decomposition is free and normalizing. We are only able to decide this for *bipolar decompositions.*

A *bipolar decomposition* of $z \in \mathcal{M}$ is a pair (v, x) in some B-valued noncommutative probability space where $x = x^*$ and v is a partial isometry, so that the B-valued $*$-moments of vx agree with those of z. Bipolar decompositions are not unique. We say that the bipolar decomposition (v, x) is *free* if v and x are $*$-free over B and is *normalizing* if v is unitary and normalizes B; then the *associated automorphism* is $b \mapsto v^*bv$. From [1], we know that a B-valued circular element z with associated maps α and β has a free, normalizing bipolar decomposition with associated automorphism θ if and only if

$$\alpha(b) = \theta(\beta(\theta(b)))$$

holds for all $b \in B$.

In [2], we show that when $B = \mathbb{C}^2$, if z is a B-valued circular element with $\ker(z) = \{0\}$ and a free bipolar decomposition, then it has a free bipolar decomposition that is normalizing. The proof is, unfortunately, devoid of intuition and relies on algebraic computations using Mathematica.

Natural questions are: (Q1) Does the above assertion hold for more general B? (Q2) If a B-valued circular element z has a free bipolar decomposition, must it have a free polar decomposition? The answer is "no," but the counterexample is somewhat degenerate. If we assume the nondegeneracy condition that $\mathrm{span}\{E((z^*z)^k) \mid k \geq 0\}$ is dense in B, the question is open.

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Infinitesimal operators

James A. Mingo
(joint work with Pei-Lun Tseng)

We introduce the concept of infinitesimal operators in an infinitesimal probability space and show how recent results of Fujie and Hasebe can be interpreted in terms of infinitesimal operators.

The Markov transform. We begin recalling some ideas from Kerov [2, §6]. Let $\mathcal{H}$ be a Hilbert space of dimension n with inner product $\langle\cdot,\cdot\rangle$. Let $A = A^* \in B(\mathcal{H})$ be given with eigenvalues $x_1 \le \cdots \le x_n$, repeated according to multiplicity. We let $\xi \in \mathcal{H}$ be such that $\|\xi\| = 1$ and create a state on $B(\mathcal{H})$ be setting $\varphi(T) = \langle T\xi, \xi\rangle$ for $T \in B(\mathcal{H})$. Let us find the spectral measure of A relative to φ; namely, we seek a probability measure, μ, on $\mathbb{R}$ such that $\varphi(f(A)) = \int f(t)\,d\mu(t)$ for all continuous functions, f, on $\mathbb{R}$.

To this end, we let $\eta_1, \dots, \eta_n$ be an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A: $A\eta_i = x_i\eta_i$ for $1 \le i \le n$. Write ξ in terms of $\{\eta_i\}_i$:

$$\xi = a_1\eta_1 + \cdots + a_n\eta_n.$$

Then $\mu = |a_1|^2\delta_{x_1} + |a_2|^2\delta_{x_2} + \cdots + |a_n|^2\delta_{x_n}$, and G_μ, the Cauchy transform of μ, is given by

$$(1)\quad G_\mu(z) = \langle (z-A)^{-1}\xi, \xi\rangle = \frac{|a_1|^2}{z-x_1} + \cdots + \frac{|a_n|^2}{z-x_n} = \frac{(z-y_1)\cdots(z-y_{n-1})}{(z-x_1)\cdots(z-x_n)}.$$

(That the coefficient of z is 1 in the numerator of the last term of (1) follows from $|a_1|^2 + \cdots + |a_n|^2 = \|\xi\| = 1$.) Also, as G_μ has a pole at each x_i, we must have that the y_is can be relabelled so that $x_1 \le y_1 \le x_2 \le \cdots \le x_{n-1} \le y_{n-1} \le x_n$ (see Figure 1 below). We say the sequences $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^{n-1}$ *interlace.* Cauchy's interlacing theorem asserts that the eigenvalues of a minor interlace the eigenvalues of the whole matrix. Indeed, if $P \in B(\mathcal{H})$ is the projection onto the subspace perpendicular to ξ and $B = PAP|_{P(\mathcal{H})}$, then the eigenvalues of B are exactly the numbers $y_1 \le \cdots \le y_{n-1}$ above.

Now, $\mathrm{Tr}(A^m) = \sum_{k=1}^n x_k^m$, and $\mathrm{Tr}(B^m) = \sum_{k=1}^{n-1} y_i^m$. If τ is the signed measure $\tau = \delta_{x_1} - \delta_{y_1} + \delta_{x_2} - \cdots + \delta_{x_{n-1}} - \delta_{y_{n-1}} + \delta_{x_n}$, then

$$G_\tau(z) = \frac{1}{z-x_1} + \cdots + \frac{1}{z-x_n} - \frac{1}{z-y_1} - \cdots - \frac{1}{z-y_{n-1}}.$$

For us, the main point will be the relation between G_μ and G_τ; in [2], Kerov calls μ the *Markov transform* of τ. Let

$$F_\mu(z) = \frac{1}{G_\mu(z)} = \frac{(z-x_1)\cdots(z-x_n)}{(z-y_1)\cdots(z-y_{n-1})}.$$

Then a simple calculation gives us that

$$G_\tau(z) = \frac{F'_\mu(z)}{F_\mu(z)}. \tag{2}$$

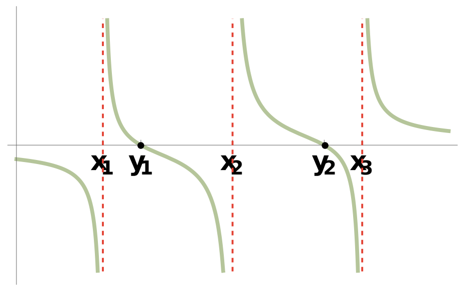

FIGURE 1. The graph of G_μ.

Random matrices. Let $\{X_N\}_N$ be the ensemble of Gaussian unitary matrices: $X_N = N^{-1/2}(x_{ij})_{i,j=1}^N$ with $X_N^* = X_N$, $\mathrm{E}(x_{ij}) = 0$, $\mathrm{E}(|x_{ij}|^2) = 1$, and the entries $\{x_{ii}\}_i \cup \{x_{ij}\}_{i<j}$ an independent complex (real when $i = j$) Gaussian family. The celebrated semi-circle law of Wigner says that when n is even $\mathrm{E}(\mathrm{tr}(X_N^n)) \to C_{n/2}$, where $C_{n/2}$ is the Catalan number $\binom{n}{n/2}/(n/2+1)$. (When n is odd, it is easy to see that $\mathrm{E}(\mathrm{tr}(X_N^n)) = 0$; $\mathrm{tr} = N^{-1}\mathrm{Tr}$ is the normalized trace.)

Now, let $J_N = J_N^* = J_N^2 \in M_N(\mathbb{C})$ be a non-random matrix. Let us assume that $\mathrm{tr}(J_N) \to 1$. Then by Voiculescu's theorem, X_N and J_N are *asymptotically free* and $\mathrm{E}(\mathrm{tr}((X_N J_N)^n)) - \mathrm{E}(\mathrm{tr}(X_N^n)) \to 0$, i.e., in the limit, the matrices X_N and the minors $J_N X_N J_N|_{J_N(\mathbb{C}^N)}$ are the same. However, when scaled by N, we get a limit law τ satisfying equation (2). Indeed, by a calculation using Wick's formula, we have for n even,

$$\mathrm{E}(\mathrm{Tr}((X_N J_N)^n)) - N C_{n/2} = C_{n/2} \frac{\mathrm{Tr}(J_N)^{\frac{n}{2}+1} - N^{\frac{n}{2}+1}}{N^{\frac{n}{2}}} + O(N^{-1}). \tag{3}$$

So if we take the case where $\mathrm{Tr}(J_N) = N - 1$, then

$$\mathrm{E}(\mathrm{Tr}((X_N J_N)^n)) - N C_{n/2} = -\Big(\frac{n}{2} + 1\Big) C_{n/2} + O(N^{-1}) = -\binom{n}{n/2} + O(N^{-1}).$$

So if we let τ be the signed measure with moments $-\binom{n}{n/2}$, then

$$G_\tau(z) = -\frac{1}{\sqrt{z^2 - 4}},$$

and we again have $G_\tau = F'_\mu / F_\mu$, where $F_\mu = 1/G_\mu$ and μ is the semi-circle law. We now have two occurrences of equation (2), and we shall use the results of [1, 3] to exhibit the simple connection between the two. Note that X_N and $X_N J_N$ have *the same limit distribution but a different infinitesimal distribution.*

Infinitesimal probability spaces. With a random matrix ensemble, $\{X_N\}_N$, we get a linear functional $\mu_N : \mathbb{C}[x] \to \mathbb{C}$ by mapping the polynomial $p \in \mathbb{C}[x]$ to $\mu_N(p) = \mathrm{E}(\mathrm{tr}(p(X_N)))$. If there is $\mu : \mathbb{C}[x] \to \mathbb{C}$ such that $\mu_N(p) \to \mu(p)$ we say that X_N has a *limit distribution.*[1] Now, we let $\mu'_N = N(\mu_N - \mu)$, and if this converges pointwise to μ' on $\mathbb{C}[x]$, we say that X_N has a *limit infinitesimal law.* In the example above, we saw from equation (3) that both X_N and $J_N X_N J_N$ have infinitesimal laws: 0 for the former and negative arcsine for the latter.

We make this abstract by letting $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and $\varphi, \varphi' : \mathcal{A} \to \mathbb{C}$ linear with $\varphi(1) = 1$ and $\varphi'(1) = 0$.[2] The triple $(\mathcal{A}, \varphi, \varphi')$ is an *infinitesimal probability space.* Now, we let $\widetilde{\mathbb{C}} = \{\left[\begin{smallmatrix} \alpha & \alpha' \\ 0 & \alpha \end{smallmatrix}\right] \mid \alpha, \alpha' \in \mathbb{C}\}$, $\widetilde{\mathcal{A}} = \{\left[\begin{smallmatrix} a & a' \\ 0 & a \end{smallmatrix}\right] \mid a, a' \in \mathcal{A}\}$, and $\widetilde{\varphi} = \left[\begin{smallmatrix} \varphi & \varphi' \\ 0 & \varphi \end{smallmatrix}\right] : \widetilde{\mathcal{A}} \to \widetilde{\mathbb{C}}$. Then the triple $(\widetilde{\mathcal{A}}, \widetilde{\varphi})$ is a non-commutative probability space over the commutative algebra $\widetilde{\mathbb{C}}$. Unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_s \subseteq \mathcal{A}$ are infinitesimally free if the algebras $\widetilde{\mathcal{A}}_1, \dots, \widetilde{\mathcal{A}}_s$ are $\widetilde{\varphi}$-free.[3]

In our example above, X_N and J_N have a joint limit distribution, and we let $x, j \in \mathcal{A}$ represent these limits: $\varphi(x^n) = C_{n/2}$ for n even and 0 for n odd, $\varphi(j) = 1$. In addition, these random variables have limit infinitesimal laws: $\varphi'(x^n) = 0$ and $\varphi'(j^n) = -1$ for all n. By Shlyakhtenko's (2018) theorem, x and j are infinitesimally free. We saw that if we let $\tilde{m}_n = \varphi'(x^n) - \varphi'((xj)^n)$ and $G_\tau(z) = \sum_{n=0}^\infty \tilde{m}_n z^{-(n+1)}$, then $G_\tau = F'_\mu / F_\mu$, where F_μ is the reciprocal of the Cauchy transform of x.

Infinitesimal operators. An element $a \in (\mathcal{A}, \varphi, \varphi')$ is *infinitesimal* if $\varphi(x^n) = 0$ for all n. If j is our projection above, $j^\perp = 1 - j$ gives us an example of an infinitesimal operator with $\varphi'(j^\perp) = 1$. In [3, §5], we showed that if $j^\perp$ is an infinitesimal idempotent and is infinitesimally free from a, then

$$\tilde{m}_n =: \varphi'(a^n) - \varphi'((aj^\perp)) = (n-1)\varphi(a^n) - \sum_{\pi \in CI(n)} (-1)^{\#(\pi)} \varphi_\pi(a, \dots, a).$$

In this expression, $CI(n) \subseteq NC(n)$ is the subset of the non-crossing partitions where the blocks are intervals when drawn on a circle, and $\#(\pi)$ is the number of blocks of π. Using the moment-cumulant formula, one can show that

$$\tilde{m}_n = - \sum_{\pi \in NC(n)} \#(K(\pi)) \kappa_\pi,$$

where $K(\pi)$ is the Kreweras complement of π and $\{\kappa_n\}_n$ are the free cumulants of a. By the theorem of Fujie and Hasebe, we again have that when $G_\tau(z) = \sum_{n=0}^\infty \tilde{m}_n z^{-(n+1)}$, we have $G_\tau = F'_\mu / F_\mu$, where μ is the distribution of a. Thus, we can model the Markov transform by the pair (a, j).

[1] We saw above that both X_N and $J_N X_N J_N$ have the same semi-circle limit distribution, μ.

[2] These capture μ and μ' above.

[3] This was not the original definition of Belinschi and Shlyakhtenko but was shown by Tseng to be equivalent.

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Reporter: Evangelos A. Nikitopoulos

Participants

Dr. Meric Augat
Department of Mathematics and Statistics
Bucknell University
One Dent Drive
Lewisburg, PA 17837
UNITED STATES

Prof. Dr. Marwa Banna
New York University Abu Dhabi
Division of Science and Mathematics
P.O. Box 129188
11111 Abu Dhabi
UNITED ARAB EMIRATES

Prof. Dr. Hari Bercovici
Department of Mathematics
Indiana University
Bloomington, IN 47405-4301
UNITED STATES

Prof. Dr. Kelly Bickel
Department of Mathematics
Bucknell University
Lewisburg, PA 17837
UNITED STATES

Dr. Mireille Capitaine
Institut de Mathématiques de Toulouse, CNRS
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Dr. Guillaume Cébron
Laboratoire de Statistique et Probabilités
Université Paul Sabatier
118 route de Narbonne
31602 Toulouse Cedex 9
FRANCE

Dr. Nikolaos Chalmoukis
Dipartimento di Matematica
Universita Milano-Bicocca
Roberto Cozzi 55
20125 Milano
ITALY

Dr. Ian Lorne Charlesworth
School of Mathematics
Cardiff University
ATSACWS
Senghennydd Road
Cardiff CF24 4AG
UNITED KINGDOM

Dr. Alberto Dayan
Fachrichtung - Mathematik
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken
GERMANY

Dr. Charles-Philippe Diez
Université du Luxembourg
Unite de Recherche en Mathématiques
1359 Belvaux
LUXEMBOURG

Prof. Dr. Ken Dykema
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Nicolas Faroß
Fachrichtung Mathematik
Universität des Saarlandes
Campus
66123 Saarbrücken
GERMANY

Dr. Malte Gerhold
Fachrichtung Mathematik
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken
GERMANY

Prof. Dr. Michael Hartz
Fachrichtung Mathematik
Universität des Saarlandes
Campus
66123 Saarbrücken
GERMANY

Prof. Dr. J. William Helton
Department of Mathematics
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
UNITED STATES

Dr. David Jekel
Fields Institute for Research in
Mathematical Sciences
222 College St
Toronto M5T 3J1
CANADA

Prof. Dr. Michael T. Jury
Department of Mathematics
University of Florida
358 Little Hall
P.O. Box 118105
Gainesville, FL 32611-8105
UNITED STATES

Prof. Dr. Matthew Kennedy
Department of Pure Mathematics
University of Waterloo
200 University Avenue West
Waterloo ON N2L 3G1
CANADA

Prof. Dr. Torben Krüger
Lehrstuhl Mathematische Stochastik
Department Mathematik
Universität Erlangen-Nürnberg
Cauerstr. 11
91058 Erlangen
GERMANY

Prof. Dr. Wing Suet Li
School of Mathematics
Georgia Institute of Technology
686 Cherry Street
Atlanta, GA 30332-0160
UNITED STATES

Prof. Dr. Constanze Liaw
Department of Mathematical Sciences
University of Delaware
501 Ewing Hall
Newark, DE 19716-2553
UNITED STATES

Dr. Tobias Mai
Fachrichtung Mathematik
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken
GERMANY

Dr. Robert T.W. Martin
Dept. of Mathematics
University of Manitoba
Winnipeg R3T 2N2
CANADA

Prof. Dr. John E. McCarthy
Department of Mathematics
Washington University in St. Louis
Campus Box 1146
One Brookings Drive
St. Louis, MO 63130-4899
UNITED STATES

Prof. Dr. James A. Mingo
Department of Mathematics and Statistics
Queen's University
Jeffery Hall
48 University Avenue
Kingston ON K7L 3N6
CANADA

Dr. Brent Nelson
Department of Mathematics
Michigan State University
C 212 Wells Hall
East Lansing, MI 48824
UNITED STATES

Evangelos A. Nikitopoulos
Department of Mathematics
University of California San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
UNITED STATES

Assistant Prof. Dr. James Eldred Pascoe
Department of Mathematics
Drexel University
Korman Center 269
3141 Chestnut St.
Philadelphia, PA 19104
UNITED STATES

Marcel Scherer
Fachrichtung Mathematik
Universität des Saarlandes
66123 Saarbrücken
GERMANY

Prof. Dr. Felix Schwenninger
Department of Applied Mathematics
University of Twente
P.O. Box 217
7500 AE Enschede
NETHERLANDS

Rikhav Shah
Department of Mathematics
University of California, Berkeley
1514 Oxford Street, Unit 304
Berkeley, CA 94720-3840
UNITED STATES

Prof. Dr. Orr Shalit
Department of Mathematics
TECHNION
Israel Institute of Technology
Haifa 32000
ISRAEL

Dr. Eli Shamovich
Dept. of Mathematics
Ben-Gurion University of the Negev
P.O. Box 653
8410501 Beer-Sheva
ISRAEL

Prof. Dr. Dimitri Shlyakhtenko
Department of Mathematics
University of California, Los Angeles
405 Hilgard Avenue
Los Angeles, CA 90095-1555
UNITED STATES

Dr. Paul Skoufranis
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto ON M3J 1P3
CANADA

Prof. Dr. Anna Skripka
Dept. of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131-1141
UNITED STATES

Dr. Alan Sola
Department of Mathematics
Stockholm University
106 91 Stockholm
SWEDEN

Prof. Dr. Roland Speicher
Fachrichtung Mathematik
Universität des Saarlandes
Gebäude E2 4
Postfach 151150
66041 Saarbrücken
GERMANY

Zachary Stier
Department of Mathematics
University of California, Berkeley
1087 Evans Hall
Berkeley, CA 94720-3860
UNITED STATES

Tea Strekelj
Institute of Mathematics, Physics
and Mechanics
University of Ljubljana
Jadranska 19
1000 Ljubljana
SLOVENIA

Maximilian Tornes
Fachrichtung Mathematik
Universität des Saarlandes
Campus
66123 Saarbrücken
GERMANY

Dr. Ryan Tully-Doyle
Department of Mathematics
California Polytechnic State
University
San Luis Obispo, CA 93407
UNITED STATES

Prof. Dr. Victor Vinnikov
Department of Mathematics
Ben-Gurion University of the Negev
P.O. Box 653
Beer-Sheva 8410501
ISRAEL

Dr. Jurij Volčič
Department of Mathematics
Drexel University
Korman Center 269
3141 Chestnut St.
Philadelphia, PA 19104
UNITED STATES

Dr. Yi Wang
School of Mathematics and Statistics
Chongqing University of Technology
Xichenyuanzhu Community
East No. 1 Rd., Daxuecheng, Gaoxin
District,
Chongqing 401331
CHINA