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THE SUBGROUP STRUCTURE OF PSEUDO-REDUCTIVE GROUPS

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Abstract. Let k be a field. We investigate the relationship between subgroups of a pseudo-reductive k-group G and its maximal reductive quotient G', with applications to the subgroup structure of G. Let k'/k be the minimal field of definition for the geometric unipotent radical of G, and let $\pi' : G_{k'} \to G'$ be the quotient map. We first characterise those smooth subgroups H of G for which $\pi'(H_{k'}) = G'$. We next consider the following questions: given a subgroup H' of G', does there exist a subgroup H of G such that $\pi'(H_{k'}) = H'$, and if H' is smooth can we find such a H that is smooth? We find sufficient conditions for a positive answer to these questions. In general there are various obstructions to the existence of such a subgroup H, which we illustrate with several examples. Finally, we apply these results to relate the maximal smooth subgroups of G with those of G'.

1. Introduction

A fundamental problem in group theory is to describe the subgroup structure of a given group. In this paper we study the subgroup structure of pseudo-reductive groups over a field k. Pseudo-reductive groups have received considerable attention over the last decade and a half [BRSS], [CGP], [CP], [CP1].

Let G be a pseudo-reductive k-group. Let $\overline{k}$ be an algebraic closure of k, and let $k^{\rm s}$ be the separable closure of k in $\overline{k}$. Let k'/k be the minimal field of definition for the geometric unipotent radical $\mathscr{R}_u(G_{\overline{k}})$. Then $\mathscr{R}_u(G_{\overline{k}})$ descends to k', and $G' := G_{k'}/\mathscr{R}_u(G_{k'})$ is a reductive k'-group. We denote the canonical projection by $\pi' : G_{k'} \to G'$, and the associated map under the adjunction of extension of scalars and Weil restriction by $i_G : G \to R_{k'/k}(G')$. Given a (not necessarily smooth) subgroup H of G, we obtain a subgroup $H' := \pi'(H_{k'})$ of the reductive k'-group G'. This gives a mapping $\mathcal{S}$ from the set of subgroups of G to the set of subgroups of G' defined by $H \mapsto \pi'(H_{k'})$.

We investigate the relationship between subgroups H of G and subgroups H' of G'. Our motivation is that it is easier to understand subgroups of a reductive group than those of a pseudo-reductive group; for example, a great deal is known about the reductive subgroups of a simple exceptional group when $k = \overline{k}$ [LS2]. The mapping $\mathcal{S}$ allows us to pass from subgroups of G to subgroups of G'. To reverse this process, we need the following definition.

Definition 1.1. Let H' be a subgroup of G'. A subgroup H of G satisfying $\pi'(H_{k'}) = H'$ is called a *levitation* of H' in G; we say that H' *levitates* to H in G. If such a subgroup H exists then we say that H' *levitates* or *is levitating* in G. If, in addition, H (and hence H') is smooth then H is called a *smooth levitation* of H' in G, and we say that H' *smoothly levitates* or is *smoothly levitating* (to H) in G.

Usually $\mathcal{S}$ is not injective: that is, a given subgroup of G' can admit more than one levitation. We show, however, that if H' levitates then H' has a unique largest levitation, and if H' levitates smoothly then H' has a unique largest smooth levitation (Proposition 4.1). An obvious question is whether $\mathcal{S}$ is surjective: that is, does every subgroup of G' levitate? This is more delicate. We prove some results showing that certain subgroups of G' levitate under certain additional assumptions, and we give a collection of examples that exhibit various obstructions to levitation. Some of these subtleties can occur only when the root system of G is non-reduced, so they are restricted to characteristic 2.

We start with the case $H' = G'$. Here is our first main result. Recall that a subgroup L of G is called a *Levi subgroup* if it satisfies $G_{\overline{k}} = \mathscr{R}_u(G_{\overline{k}}) \rtimes L_{\overline{k}}$ (scheme-theoretically). We introduce a slightly weaker notion. A smooth subgroup L of G is an *almost Levi subgroup* if $G(\overline{k}) = \mathscr{R}_u(G(\overline{k})) \rtimes L(\overline{k})$ (as abstract groups). Given an almost Levi subgroup L of G, if L is not a Levi subgroup then k has characteristic 2 and the root system of $G_{k^{\mathrm{s}}}$ is non-reduced.

Theorem 1.2. *Let G be a pseudo-reductive k-group. Let H be a smooth subgroup of G, and denote $H' := \pi'(H_{k'})$.*

(i) *Suppose $k = k^{\mathrm{s}}$. Then $H' = G'$ if and only if H contains an almost Levi subgroup of G.*
(ii) *Suppose the root system of $G_{k^{\mathrm{s}}}$ is reduced. If $H' = G'$ then the restriction map $\pi'|_{H_{k'}} : H_{k'} \to G'$ is smooth.*

Next we turn our attention to tori.

Theorem 1.3. *Let G be a pseudo-reductive k-group. Suppose the root system of $G_{k^{\mathrm{s}}}$ is reduced. The following are equivalent.*

(a) *$i_G(G)$ contains $\mathscr{D}(R_{k'/k}(G'))$.*
(b) *$i_G(G)$ contains every torus of $R_{k'/k}(G')$.*
(c) *For every torus S' of G', there exists a subgroup S of G such that $\pi'(S_{k'}) = S'$.*
(d) *For every torus S' of G', there exists a torus S of G such that $\pi'(S_{k'}) = S'$.*
(e) *For every maximal torus T' of G', there exists a maximal torus T of G such that $\pi'(T_{k'}) = T'$.*
(f) *G is standard, and the minimal field of definition for the geometric unipotent radical of each pseudo-simple component of G equals k'/k.*

We find an example where a maximal torus of G' does not levitate (Example 4.6), and an example where a maximal torus of T levitates but does not levitate smoothly (Example 4.8).

Things are murkier when we move away from the case when H' is a torus, even when the root system of $G_{k^{\mathrm{s}}}$ is reduced and conditions (a)–(e) of Theorem 1.3 hold. We find an example where a smooth wound unipotent subgroup of G' does not levitate at all (Example 4.19), and an example where G is of the form $R_{k'/k}(G')$ for some reductive G' and there is a non-smooth subgroup of G' that does not levitate (Example 4.18).

On the other hand, the following result shows that the relationship between subgroups of G and G' is quite well-behaved if we restrict ourselves to regular smooth subgroups. (Recall that a subgroup of G is called *regular* if it is normalised by some maximal torus of G.)

Theorem 1.4. *Let G be a pseudo-reductive k-group. Let H' be a smooth subgroup of G'. Suppose there exists a maximal torus T of G such that $\pi'(T_{k'})$ normalises H'.*

(i) *There exists a largest smooth subgroup H of G such that $\pi'(H_{k'}) = H'$. Moreover H is normalised by T.*
(ii) *$(H')^\circ$ is a maximal torus (resp. root group, parabolic subgroup) of G' if and only if H° is a Cartan subgroup (resp. root group, pseudo-parabolic subgroup) of G.*

We finish the paper by considering maximal smooth subgroups of G and G'. It is not true in general that $\mathcal{S}$ takes maximal smooth subgroups of G to maximal smooth subgroups of G', or that a smooth levitation of a maximal smooth subgroup of G' must be a maximal smooth subgroup of G: for instance, in Example 4.8 we show that a maximal smooth subgroup of G' need not have a smooth levitation, while in Example 5.3 we find a maximal smooth subgroup H' of G' such that H' levitates smoothly but the largest smooth levitation of H' is not maximal in G. We do have the following result, which gives a step towards reducing the problem of describing maximal smooth subgroups from the case of an arbitrary pseudo-reductive group to the case of a reductive group.

Theorem 1.5. *Let G be a pseudo-reductive k-group.*

(i) *Let H be a maximal smooth subgroup of G. Suppose at least one of the following conditions hold:*
  (a) *$G = R_{k'/k}(G')$; or*
  (b) *H is a regular subgroup of G.*
  Then either $\pi'(H_{k'})$ is a maximal smooth subgroup of G', or $H_{k^{\mathrm{s}}}$ contains an almost Levi subgroup of $G_{k^{\mathrm{s}}}$.

(ii) *Let H' be a maximal smooth subgroup of G'. Suppose H' smoothly levitates in G, and let M be a smooth subgroup of G that properly contains the largest smooth levitation of H'. Then $M_{k^{\mathrm{s}}}$ contains an almost Levi subgroup of $G_{k^{\mathrm{s}}}$.*

The paper is set out as follows. We cover some preliminary material in Section 2, and in Section 3 we deal with the case $H' = G'$ and we prove Theorem 1.2. The proof of Theorem 1.3 and various counter-examples appear in Section 4. In Section 5 we consider maximal subgroups and we prove Theorem 1.5.

2. Preliminaries

Let k be a field. Let $\overline{k}$ be an algebraic closure of k, and let k^{s} be the separable closure of k in $\overline{k}$. By a *k-group* we mean a group scheme over k, and by an *algebraic k-group* we mean a group scheme of finite type over k. We denote by AlgGrp$/k$ the category of algebraic k-groups and homomorphisms between them. We do not assume that algebraic k-groups are smooth. By a *subgroup H* of an algebraic k-group G we mean a locally closed subgroup scheme; note that H is automatically closed by [Stacks, 047T] and is of finite type by [Stacks, 01T5, 01T3]. In this section we allow for not-necessarily-affine algebraic k-groups, whilst for the remainder of the paper we are only concerned with affine groups. We denote by α_p and μ_p the first Frobenius kernels of the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$, respectively.

2.1. **Some structure theory.** Let G be an algebraic k-group. We denote the centre of G by $Z(G)$. That is, for A a k-algebra, $Z(A)$ represents the functor

$$A \mapsto \{g \in G(A) \,|\, g^{-1}xg = x, \text{ for all } A\text{-algebras } A' \text{ and all } x \in G(A')\}.$$

If k'/k is a field extension then $G_{k'}$ denotes the algebraic k'-group obtained from G by extension of scalars. Let G_{red} denote the reduction of G (i.e. the unique reduced closed k-subscheme of G with the same underlying topological space). Let G^{sm} denote the subgroup of G that is generated by $G(k^{\mathrm{s}})$ (more precisely take the subgroup of $G_{k^{\mathrm{s}}}$ that is generated by its k^{s}-points, observe that it is stable under the action of $\mathrm{Gal}(k^{\mathrm{s}}/k)$, then take its k-descent). Then G^{sm} is the largest smooth subgroup of G. Note that G^{sm} is contained in G_{red} but it is not necessarily normal in G (see [Mi, Ex. 2.35], for example).

Now assume that G is a smooth connected affine k-group. The *unipotent radical* $\mathscr{R}_u(G)$ of G is the largest smooth connected unipotent normal subgroup of G. If $\mathscr{R}_u(G)$ is trivial then G is *pseudo-reductive*. For a comprehensive account of pseudo-reductive groups and their properties, see [CGP].

The notions of a root system and root groups are particularly important; see [CGP, §2.3], where the necessary constructions are carried out for an arbitrary smooth connected affine k-group. The root system of a reductive group is always reduced, but a pseudo-reductive group can have a non-reduced root system — see [CGP, 9.3] — and this leads to complications (see, e.g., the paragraph before Lemma 4.7). To clarify, we fix some maximal torus T of G. By a *root* (resp. *root group*) of G we mean a root (resp. root group) of $G_{k^{\mathrm{s}}}$ with respect to $T_{k^{\mathrm{s}}}$ which is stable under the action of $\mathrm{Gal}(k^{\mathrm{s}}/k)$. Note that we are referring to the "absolute" notions of root systems and root groups, which are not to be confused with the "relative" notions in the sense of [CGP, C.2.13].

The *split unipotent radical* $\mathscr{R}_{us}(G)$ of G is the largest smooth connected split unipotent normal subgroup of G. If $\mathscr{R}_{us}(G)$ is trivial then G is *quasi-reductive.* The group G is called *pseudo-split* if it contains a split maximal torus, and *pseudo-simple* if it is non-commutative and has no nontrivial smooth connected proper normal subgroup. We say that G is *absolutely pseudo-simple* if $G_{k^{\mathrm{s}}}$ is pseudo-simple. There are natural notions of pseudo-semisimple groups and pseudo-simple components.

The geometric unipotent radical of G is the group $\mathscr{R}_u(G_{\overline{k}})$. If this group is trivial then we say that G is *reductive.* Note our convention is that reductive and pseudo-reductive groups are connected.

A subgroup L of G satisfying $G_{\overline{k}} = \mathscr{R}_u(G_{\overline{k}}) \rtimes L_{\overline{k}}$ is called a *Levi subgroup* of G; Levi subgroups are reductive. If G is pseudo-split and pseudo-reductive, or if k has characteristic 0, then G admits a Levi subgroup (see [CGP, 3.4.6] and [Mi, 25.49]). However in general G need not contain a Levi subgroup: this failure of existence may occur over any algebraically closed field of positive characteristic (see [CGP, A.6]), or if G is pseudo-reductive but not pseudo-split (refer to [CGP, 7.2.2]). We introduce a related notion.

Definition 2.1. A smooth subgroup L of G is an *almost Levi subgroup* if $G(\overline{k}) = \mathscr{R}_u(G(\overline{k})) \rtimes L(\overline{k})$ (as abstract groups).

Fix a cocharacter $\lambda : \mathbb{G}_m \to G$. For any k-algebra A and $g \in G(A)$, we say that $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ *exists* if the A-scheme map $\mathbb{G}_m \to G_A$ defined by $t \mapsto \lambda(t)g\lambda(t)^{-1}$ extends (necessarily uniquely) to an A-scheme map $\mathbb{A}^1 \to G_A$. The functor

$$A \mapsto \{g \in G(A) \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}$$

is representable as a subgroup (scheme) of G, we denote it by $P_G(\lambda)$. A subgroup of G is called *pseudo-parabolic* if it is of form $P_G(\lambda)\mathscr{R}_u(G)$ for some cocharacter $\lambda : \mathbb{G}_m \to G$; so if G is pseudo-reductive then each $P_G(\lambda)$ is pseudo-parabolic. Every pseudo-parabolic subgroup of G is smooth and connected. A pseudo-parabolic subgroup of G is called *maximal pseudo-parabolic* if it is maximal amongst all proper pseudo-parabolic subgroups of G.

Now specialise to the case where G is a pseudo-split pseudo-reductive k-group. Let T be a maximal torus of G and let $\Phi := \Phi(G,T)$ be the root system of G with respect to T. For any $\alpha \in \Phi$ there exists a unique T-stable smooth connected subgroup U_α of G such that $\mathrm{Lie}(U_\alpha)$ is the span of the root spaces in $\mathrm{Lie}(G)$ which correspond to multiples $n\alpha$ for $n \in \mathbb{N}$. We call U_α the *T-root group* of G associated to α. Without the pseudo-splitness assumption, a subgroup H of G is called a *root group* if $H_{k^{\mathrm{s}}}$ is a root group of $G_{k^{\mathrm{s}}}$.

2.2. **Extension of scalars and Weil restriction.** Let k'/k be a finite field extension. Let G' be an algebraic k'-group. The *Weil restriction* of G' by k'/k is the functor $(\mathrm{Alg}/k)^{\mathrm{op}} \to \mathrm{Grp}$ given on objects by $A \mapsto \mathrm{Hom}_{k'}(\mathrm{Spec}\, A \otimes_k k', G') =: G'(A \otimes_k k')$, where Alg/k refers to the category of finite-dimensional commutative k-algebras, and Grp the category of (abstract) groups. This functor is representable as an algebraic k-group (see [BLR, §7.6/4] or [CGP, A.5.1]); let us denote it by $R_{k'/k}(G')$. We have an induced functor

$$R_{k'/k}(-) : \mathrm{AlgGrp}/k' \to \mathrm{AlgGrp}/k \tag{1}$$

called *Weil restriction* by k'/k.

Many useful properties of algebraic groups over a field, in particular affineness and smoothness, are preserved by both the extension of scalars functor and the Weil restriction functor. Both functors also preserve the property of being a monomorphism.

It is well-known that the Weil restriction functor is right adjoint to the extension of scalars functor (see for instance [BLR, Lem. 7.6/1]). It follows that Weil restriction preserves pullbacks: in particular, it preserves preimages. We make explicit some other properties of this adjunction, given that we will use them repeatedly.

For an algebraic k-group G and an algebraic k'-group G' there exists a natural bijection

$$\mathrm{Hom}_k(G, R_{k'/k}(G')) \xrightarrow{\sim} \mathrm{Hom}_{k'}(G_{k'}, G'). \tag{2}$$

Let

$$j_G : G \to R_{k'/k}(G_{k'}) \tag{3}$$

be the component at G of the unit of this adjunction: that is, j_G is the map corresponding to $G_{k'} \xrightarrow{\mathrm{id}} G_{k'}$ via (2). Given any homomorphism of k-groups $f : H \to G$, the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\quad j_H \quad} & R_{k'/k}(H_{k'}) \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle R_{k'/k}(f_{k'})} \\ G & \xrightarrow{\quad j_G \quad} & R_{k'/k}(G_{k'}) \end{array} \tag{4}$$

Given a k'-homomorphism $\psi' : G_{k'} \to G'$, let $\psi : G \to R_{k'/k}(G')$ be the k-homomorphism associated to ψ' under the bijection in (2). The universal property of j_G says that the following triangle commutes:

$$\begin{array}{ccc} G & \xrightarrow{\ j_G\ } & R_{k'/k}(G_{k'}) \\ & {\scriptstyle \psi}\searrow & \Big\downarrow{\scriptstyle R_{k'/k}(\psi')} \\ & & R_{k'/k}(G') \end{array} \tag{5}$$

Now let

$$q_{G'} : R_{k'/k}(G')_{k'} \to G' \tag{6}$$

be the component at G' of the counit of this adjunction: that is, $q_{G'}$ is the map corresponding to $R_{k'/k}(G) \xrightarrow{\mathrm{id}} R_{k'/k}(G)$ via (2). If G' is smooth then [CGP, A.5.11(1)] tells us that $q_{G'} : G_{k'} \to G'$ is smooth and surjective. Given any homomorphism of algebraic k'-groups $f' : H' \to G'$, the following diagram commutes:

$$\begin{array}{ccc} R_{k'/k}(H')_{k'} & \xrightarrow{\quad q_{H'} \quad} & H' \\ {\scriptstyle R_{k'/k}(f')_{k'}}\Big\downarrow & & \Big\downarrow{\scriptstyle f'} \\ R_{k'/k}(G')_{k'} & \xrightarrow{\quad q_{G'} \quad} & G' \end{array} \tag{7}$$

Given a k-homomorphism $\phi : G \to R_{k'/k}(G')$, let $\phi' : G_{k'} \to G'$ be the k'-homomorphism associated to ϕ under the bijection in (2). The universal property of $q_{G'}$ says that the following triangle commutes:

$$\begin{array}{ccc} G_{k'} & \xrightarrow{\phi_{k'}} & R_{k'/k}(G')_{k'} \\ & \searrow^{\phi'} & \downarrow q_{G'} \\ & & G' \end{array} \tag{8}$$

The following two equations are respectively known as the first and second counit-unit equations of the adjunction; they follow from the universal properties of j_G and $q_{G'}$. If $G_{k'} = G'$ then

$$q_{G'} \circ (j_G)_{k'} = \mathrm{id}_{G'}\,. \tag{9}$$

If $G = R_{k'/k}(G')$ then

$$R_{k'/k}(q_{G'}) \circ j_G = \mathrm{id}_G\,. \tag{10}$$

In this paper we are mostly concerned with the following special situation. Let G be a pseudo-reductive k-group, and let k'/k be the minimal field of definition for its geometric unipotent radical $\mathscr{R}_u(G_{\overline{k}})$. Let $\pi' : G_{k'} \to G_{k'}/\mathscr{R}_u(G_{k'}) =: G'$ be the natural projection. We define a map

$$i_G : G \to R_{k'/k}(G') \tag{11}$$

to be the k-homomorphism associated to π' under the bijection in (2). Note that $\ker i_G$ is unipotent, so $(\ker i_G)^{\mathrm{sm}}$ is étale (see [CGP, 1.6 and Lem. 1.2.1]), and $\ker i_G$ is central if the root system of G_{k^s} is reduced [CP, 2.3.4]. Applying (8) with $\phi = i_G$ gives

$$q_{G'} \circ (i_G)_{k'} = \pi'. \tag{12}$$

In the special case when $G = R_{k'/k}(G')$, i_G gives an isomorphism from G onto itself [CGP, Thm. 1.6.2(2)]. More generally, if G is a subgroup of $R_{k'/k}(G')$ then we may regard i_G as the inclusion map.

We wish to study the relationship between subgroups of G' and subgroups of G, which motivates the following terminology.

Definition 2.2. Suppose G is pseudo-reductive and $\pi' : G_{k'} \to G'$ is the map just defined. Let H' be a subgroup of G'. A subgroup H of G satisfying $\pi'(H_{k'}) = H'$ is called a *levitation* of H' in G; we say that H' *levitates* to H in G. If such a subgroup H exists then we say that H' *levitates* or *is levitating* in G. If, in addition, H (and hence H') is smooth then H is called a *smooth levitation* of H' in G, and we say that H' *smoothly levitates* or is *smoothly levitating* (to H) in G.

2.3. **Subgroups of a Weil restriction.** In this subsection we prove two useful lemmas which relate the subgroup structure of a smooth algebraic k'-group G' to that of its Weil restriction $R_{k'/k}(G')$. Keep the assumption that k'/k is a finite field extension. Let G' be a smooth algebraic k'-group (not necessarily affine), and let $G := R_{k'/k}(G')$. If G' is reductive then G is pseudo-reductive [CGP, Prop. 1.1.10], if in addition k'/k is purely inseparable then k'/k is the minimal field of definition for $\mathscr{R}_u(G_{\overline{k}})$ and $i_G\colon G \to G$ is an isomorphism [CGP, Thm. 1.6.2(2)]. Note that most pseudo-reductive groups are obtained from a group of the form $R_{k'/k}(G')$ via the so-called standard construction [CGP, Thm. 1.5.1].

Lemma 2.3. *Let H' be a subgroup of G'. There is a canonical inclusion $q_{G'}(R_{k'/k}(H')_{k'}) \subseteq H'$, with equality if H' is smooth. Consequently, if H' is smooth and $R_{k'/k}(H') = R_{k'/k}(G')$ then $H' = G'$.*

Proof. Let $\phi' : H' \hookrightarrow G'$ be the inclusion. We may interpret $R_{k'/k}(\phi')_{k'}$ as an inclusion. Consider the k'-homomorphism

$$\nu' := q_{G'} \circ R_{k'/k}(\phi')_{k'} : R_{k'/k}(H')_{k'} \to G'.$$

That is, ν' is the restriction of $q_{G'}$ to $R_{k'/k}(H')_{k'}$. By the functoriality of $q_{G'}$ (i.e., Diagram (7)), we have that

$$\phi' \circ q_{H'} = q_{G'} \circ R_{k'/k}(\phi')_{k'} = \nu'.$$

Taking the image of this map ν' and leaving the inclusion maps as implicit gives us a canonical inclusion $q_{G'}(R_{k'/k}(H')_{k'}) \subseteq H'$. If H' is smooth then $q_{H'}$ is smooth and surjective, so indeed $q_{G'}(R_{k'/k}(H')_{k'}) = H'$. The second assertion then follows immediately. □

The inclusion $q_{G'}(R_{k'/k}(H')_{k'}) \subseteq H'$ in Lemma 2.3 may be strict if H' is not smooth; for an example with reductive G' see Example 4.18.

Lemma 2.4. *Let H be a subgroup of G. Then there are canonical inclusions $H \subseteq R_{k'/k}(q_{G'}(H_{k'})) \subseteq G$.*

Proof. Consider the subgroup $H' := q_{G'}(H_{k'})$ of G'. Let $\rho' : H_{k'} \to H'$ be the k'-homomorphism obtained by restricting the domain of $q_{G'}$ to $H_{k'}$, and then restricting the codomain to its image H'. Consider the k-homomorphism

$$\iota : H \to R_{k'/k}(H')$$

associated to ρ' under the bijection in (2).

Let $\phi : H \hookrightarrow G$ be the inclusion, and let $\phi' : H' \hookrightarrow G'$ be the inclusion induced by ϕ. By definition $\phi' \circ \rho' = q_{G'} \circ \phi_{k'}$. The universal property of j_H (i.e., Diagram (5)) says that $\iota = R_{k'/k}(\rho') \circ j_H$. Recall from Equation (10) that $R_{k'/k}(q_{G'}) \circ j_G$ is the identity map on G. By the functoriality of j_G (i.e., Diagram (4)), we have that $R_{k'/k}(\phi_{k'}) \circ j_H = j_G \circ \phi$. Combining all of the above gives us

$$R_{k'/k}(\phi') \circ \iota = \phi.$$

Since ϕ is a monomorphism, this is also true of ι. Leaving the inclusions as implicit, we have shown that $H \subseteq R_{k'/k}(H') \subseteq G$. □

3. Overgroups of almost Levi subgroups

In this section we prove Theorem 1.2, and then we illustrate it with some examples. We will need the following easy lemmas about reductive groups.

Lemma 3.1. *Let G be a reductive k-group. Let N be a unipotent normal subgroup of G. Then N is infinitesimal.*

Proof. Since G is reductive, the subgroup of $G_{\overline{k}}$ generated by $N(\overline{k})$ is finite, and so it is central in G. But $Z(G)$ is of multiplicative type as G is reductive, hence $N(\overline{k})$ is trivial. That is, N is infinitesimal. □

Lemma 3.2. *Let $f : G_1 \to G_2$ be an isogeny between reductive k-groups. Suppose f restricts to an isomorphism between each corresponding pair of simple components and between maximal tori. Then f is an isomorphism.*

Proof. Since G_1 is reductive, its center $Z(G_1)$ is of multiplicative type. Hence $Z(G_1) \cap \ker f$ is trivial, as it is contained in every maximal torus of G_1. Assume $\ker f$ is non-trivial. Then by assumption $\ker f$ must be a non-central diagonally embedded subgroup of at least two of the simple factors of G_1. However these simple factors are all pairwise commuting, so $\ker f$ centralises all of them, so it must be central. We have a contradiction. □

Henceforth let G be a pseudo-reductive k-group, let k'/k be the minimal field of definition for $\mathscr{R}_u(G_{\overline{k}})$, and let $\pi' : G_{k'} \to G_{k'}/\mathscr{R}_u(G_{k'}) =: G'$ be the natural projection. Recall that a smooth subgroup L of G is an almost Levi subgroup if $G(\overline{k}) = \mathscr{R}_u(G(\overline{k})) \rtimes L(\overline{k})$ (Definition 2.1). The following result shows that this notion is only slightly more general than that of a Levi subgroup.

Proposition 3.3. *Let L be an almost Levi subgroup of G. Then L is reductive, and π' restricts to an isomorphism $Z(L_{k'})^\circ \to Z(G')^\circ$. Let S be a pseudo-simple component of $G_{k^{\mathrm{s}}}$. Then either*

(i) *$L_{k^{\mathrm{s}}} \cap S$ is a Levi subgroup of S, or*
(ii) $\mathrm{char}(k) = 2$ *and there exists $n := n(S) \geq 1$ such that S is of type BC_n, $L_{k^{\mathrm{s}}} \cap S \cong$ SO_{2n+1} and $L_{\overline{k}} \cap S_{\overline{k}} \cap \ker \pi'_{\overline{k}} \cong (\alpha_2)^{2n}$.*

In particular, if the root system of $G_{k^{\mathrm{s}}}$ is reduced – for instance if $\mathrm{char}(k) \neq 2$ *– then L is a Levi subgroup of G.*

Proof. Without loss of generality we can assume that $k = k^{\mathrm{s}}$, since being a Levi subgroup or an almost Levi subgroup is invariant under base change by k^{s}/k. Thus we may assume that G is pseudo-split.

We first show that L is reductive. By definition of an almost Levi subgroup and since G' is smooth, the restriction map $\pi'|_{L_{k'}} : L_{k'} \to G'$ is bijective on $\overline{k}$-points and its kernel is infinitesimal and unipotent. In other words, $\pi'|_{L_{k'}}$ is a unipotent isogeny. Then L is connected, since $\ker \pi'|_{L_{k'}}$ is connected and as connectedness is preserved by group extensions. Since $\mathscr{R}_u(G_{\overline{k}}) \cap L_{\overline{k}}$ has trivial $\overline{k}$-points and as $\pi'_{\overline{k}}(\mathscr{R}_u(L_{\overline{k}})) \subseteq \mathscr{R}_u(G'_{\overline{k}}) = 1$, it follows that L is reductive.

Next consider the central torus $Z(L_{k'})^\circ$ of $L_{k'}$. Since $\pi'|_{L_{k'}}$ is an isogeny between reductive k'-groups, it restricts to an isogeny $Z(L_{k'})^\circ \to \pi'(Z(L_{k'})^\circ) = Z(G')^\circ$. But $\ker \pi'$ is unipotent, so $Z(L_{k'})^\circ \to Z(G')^\circ$ is an isomorphism.

Let S be a pseudo-simple component of G. Since $\pi'|_{L_{k'}}$ is an isogeny between split reductive k'-groups, it induces a bijection between the respective sets of simple components of $L_{k'}$ and G'. Consider the simple component $S' := \pi'(S_{k'})$ of G', and let S_0 be the simple component of L satisfying $\pi'((S_0)_{k'}) = S'$. Certainly $S_0 \subseteq S$. Let us define $\upsilon : (S_0)_{k'} \to S'$ to be the composition of the inclusion $(S_0)_{k'} \hookrightarrow S_{k'}$ with $\pi'|_{S_{k'}}$. If υ is an isomorphism then S_0 is a Levi subgroup of S.

Now assume that υ is not an isomorphism; then it is a unipotent isogeny between split simple k-groups. Such isogenies were classified in [PY, Lem. 2.2] and [Va, Thm. 2.2]. These results tell us that $\mathrm{char}(k) = 2$ and there exists $n := n(S) \geq 1$ such that $S_0 \cong \mathrm{SO}_{2n+1}$, $S' \cong \mathrm{Sp}_{2n}$ and $\ker \upsilon = (S_0)_{k'} \cap \ker \pi' \cong (\alpha_2)^{2n}$. Let U be the (direct) product of the first Frobenius kernels of the short root subgroups of S_0. Then $\ker \upsilon = U_{k'}$ by [BRSS, Lem. 2.2]. Consider the map $i_S : S \to R_{k'/k}(S')$ defined in (11) (certainly $\mathscr{R}_u(S_{\overline{k}})$ descends to k', even if it is not the minimal such field). Observe that $U \subseteq \ker i_S$, but U is non-central in S, and hence by [CP1, 6.2.15] the root system of S is non-reduced. The only irreducible non-reduced root system of rank n is BC_n.

It remains to show that $S_0 = L \cap S$. Let T be a maximal torus of L. Then $T \cap S_0$ is a maximal torus of S_0 by [CGP, A.2.7]. Observe that L has maximal rank in G. Hence, by a similar argument, $T \cap S = T \cap S_0$ is a maximal torus of S. Since L is reductive, we have $Z_{L \cap ST}(T) = ST \cap Z_L(T) = T$. Then

$$\mathrm{Lie}(L \cap ST) = \mathrm{Lie}(T) \oplus \bigoplus \{\mathfrak{g}_\alpha \mid \alpha \in \Phi(L,T) \cap \Phi(ST,T)\} = \mathrm{Lie}(S_0 T).$$

That is, the inclusion $S_0 T \subseteq L \cap ST$ induces an equality on Lie algebras. But $S_0 T$ is smooth and connected, and so $S_0 T = (L \cap ST)^\circ$. Taking the derived subgroup gives us $S_0 = (L \cap S)^\circ$. Since $L_{k'} \cap S_{k'} \cap \ker \pi'$ has trivial $\overline{k}$-points, it is infinitesimal and in particular connected. But $L_{k'} \cap S_{k'}$ is an extension of the connected k'-group S' by $L_{k'} \cap S_{k'} \cap \ker \pi'$, and so $L \cap S$ is also connected. This shows that the third property in (ii) holds.

It remains to prove the second assertion of the proposition. Suppose the root system of $G_{k^{\mathrm{s}}}$ is reduced. By the (already proved) first assertion, the isogeny $\pi'_{\overline{k}}|_{L_{\overline{k}}} : L_{\overline{k}} \to G'_{\overline{k}}$ restricts to an isomorphism between each corresponding pair of simple components. By [Bo, 11.14] and since $\ker \pi'$ is unipotent, $\pi'_{\overline{k}}|_{L_{\overline{k}}}$ sends any maximal torus of $L_{\overline{k}}$ isomorphically onto a maximal torus of $G'_{\overline{k}}$. Hence $\pi'_{\overline{k}}|_{L_{\overline{k}}}$ is an isomorphism by Lemma 3.2. That is, L is a Levi subgroup of G. □

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Recall that G is a pseudo-reductive k-group, and H is a smooth subgroup of G.

We first prove (i). We assume that $k = k^{\mathrm{s}}$. In particular, G and H are pseudo-split.

If H contains an almost Levi subgroup L of G then the restriction of $\pi'(\overline{k})$ to $L(\overline{k})$ is an isomorphism, so indeed $\pi'(H_{k'}) = G'$. For the converse, suppose that H is a levitation of G' in G. We will show that H is pseudo-reductive so that we can then apply [CGP, 3.4.6].

Let U be a smooth connected unipotent normal subgroup of H. Then $\pi'(U_{k'})$ is trivial as G' is reductive, so $U_{k'} \subseteq \ker \pi' = \mathscr{R}_u(G_{k'})$. Consider the subgroup of $G_{k^{\mathrm{s}}}$ generated by all $G(k^{\mathrm{s}})$-conjugates of $U_{k^{\mathrm{s}}}$; it admits a k-descent which we call N. Clearly N is smooth, connected and unipotent, and $N_{k'}$ is contained in $\mathscr{R}_u(G_{k'})$. Moreover, N is normal in G since $G(k^{\mathrm{s}})$ is dense in G. Hence N is contained in $\mathscr{R}_u(G)$, which is trivial. So indeed H is pseudo-reductive.

Since H is pseudo-split and pseudo-reductive, applying [CGP, 3.4.6] tells us there exists a Levi subgroup L of H. Consider the maximal reductive quotient map $\kappa' : H_{\overline{k}} \to H_{\overline{k}}/\mathscr{R}_u(H_{\overline{k}})$. Note that κ' sends $L_{\overline{k}}$ isomorphically onto $H_{\overline{k}}/\mathscr{R}_u(H_{\overline{k}})$. Since H is a levitation of G' in G, the restriction $\pi'_{\overline{k}}|_{H_{\overline{k}}} : H_{\overline{k}} \to G'_{\overline{k}}$ is a quotient map. Hence $\pi'_{\overline{k}}|_{H_{\overline{k}}}$ factors through κ', as G' is reductive. It follows that $\pi'(\overline{k})$ sends $L(\overline{k})$ onto $G'(\overline{k})$. Since L is reductive, the normal subgroup $L(\overline{k}) \cap \mathscr{R}_u(G(\overline{k}))$ of $L(\overline{k})$ must be trivial by Lemma 3.1. Hence L is an almost Levi subgroup of G. This completes the proof of (i).

We next prove (ii). Since smoothness is a geometric property, we can assume that $k = k^{\mathrm{s}}$. Suppose the root system of G is reduced. Assume that H is a levitation of G' in G. Then, by part (i) along with Proposition 3.3, H contains a Levi subgroup L of G. The restriction $\pi'|_{L_{k'}} : L_{k'} \to G'$ is an isomorphism, and so it induces an isomorphism on Lie algebras. Since H contains L, the restriction $\pi'|_{H_{k'}} : H_{k'} \to G'$ induces a surjection on Lie algebras. Hence $\pi'|_{H_{k'}}$ is smooth by [Mi, 1.63]. □

The following example shows that both parts of Theorem 1.2 can fail if we allow G to be quasi-reductive rather than pseudo-reductive.

Example 3.4. Let k be an imperfect field of characteristic 2, and let $N = \alpha_2 \times \alpha_2$. Choose any smooth connected wound unipotent k-group U in which N embeds as a central subgroup. Let G be the central product $(\mathrm{PGL}_2 \times U)/N$, where N embeds into PGL_2 as the intersection of its Frobenius kernel with its root groups. The unipotent radical $\mathscr{R}_u(G)$ of G is isomorphic to U — in particular, G is quasi-reductive — and $\mathscr{R}_u(G)$ is a k-descent of $\mathscr{R}_u(G_{\overline{k}})$. So our map π' is simply the natural projection $G \to G/\mathscr{R}_u(G) =: G'$. Let H be the canonical subgroup of G that is isomorphic to PGL_2, and let $H' := \pi'(H)$. Then $H' = G' \cong \mathrm{SL}_2$, yet G does not admit a Levi subgroup. Moreover, the restriction of π' to H has non-smooth kernel.

4. Levitating subgroups

Let G be a pseudo-reductive k-group with minimal field of definition k'/k for its geometric unipotent radical, and quotient map $\pi' : G_{k'} \to G_{k'}/\mathscr{R}_u(G_{k'}) := G'$. In this section we investigate the following general question: *which subgroups of G' levitate in G, and which smooth subgroups of G' levitate smoothly in G?* Along the way we prove Theorem 1.3. Before proceeding, we give a brief overview of the results below to help with navigation. Our basic idea is to consider three progressively more general cases: first, where $G = R_{k'/k}(G')$ is a Weil restriction; second, where $i_G(G)$ contains $\mathscr{D}(R_{k'/k}(G'))$; finally, we consider the general case, for arbitrary pseudo-reductive G.

In the first case, if H' is a smooth subgroup of G' then Lemma 2.3 tells us that H' levitates in G, its largest levitation being $R_{k'/k}(H')$, which moreover is smooth. However Example 4.18 shows that a non-smooth subgroup H' of G' need not levitate in G.

In the second case, it turns out that the condition that $i_G(G)$ contains $\mathscr{D}(R_{k'/k}(G'))$ is equivalent to requiring that all tori of G' levitate in G. If, in addition, the root system of G_{k^s} is reduced, this condition is equivalent to requiring that all tori of G' *smoothly* levitate in G. This is a consequence of Theorem 1.3 and its proof. A similar statement holds if we require that H' is generated by tori; see Corollary 4.11. However, unlike when $G = R_{k'/k}(G')$, a smooth subgroup H' of G' need not levitate in G; refer to Example 4.19.

Finally consider the general case, for arbitrary pseudo-reductive G. If H' is a smooth connected normal subgroup of G' then it smoothly levitates in G by [CGP, 3.1.6]. However in the absence of normality a torus of G' need not levitate in G, and even if it does, it need not smoothly levitate in G. Refer to Examples 4.6 and 4.8, respectively.

4.1. **Basic results and examples.** Our first basic result shows that in case a subgroup does levitate, there is a largest levitation.

Proposition 4.1. *Let H' be a subgroup of G'. If there exists at least one levitation (resp. smooth levitation) of H' in G, then there exists a largest such levitation (resp. smooth levitation) of H' in G: namely $i_G^{-1}(R_{k'/k}(H'))$ (resp. $(i_G^{-1}(R_{k'/k}(H')))^{\mathrm{sm}}$).*

Proof. Assume that H is a levitation of H' in G. Combining Lemmas 2.3 and 2.4 tells us that $i_G(H)$ is contained in $R_{k'/k}(H')$, and that $q_{G'}(R_{k'/k}(H')_{k'}) = H'$. That is, $R_{k'/k}(H')$ is the largest subgroup Z of $R_{k'/k}(G')$ that satisfies $q_{G'}(Z_{k'}) = H'$. Since $\pi' = q_{G'} \circ (i_G)_{k'}$ by (12), it follows that $i_G(H) \subseteq R_{k'/k}(H')$. So H is contained in $i_G^{-1}(R_{k'/k}(H')) =: \widetilde{H}$, which satisfies $\pi'(\widetilde{H}_{k'}) = H'$. In other words $\widetilde{H}$ is the largest levitation of H' in G. If H is smooth (which implies that H' is smooth), then H is contained in $\widetilde{H}$, so $H \subseteq (\widetilde{H})^{\mathrm{sm}}$. Hence $(\widetilde{H})^{\mathrm{sm}}$ is the largest smooth levitation of H' in G. □

An obvious example of a smooth subgroup of G' that smoothly levitates in G is the trivial subgroup; its largest levitation in G is $\ker i_G$, and its largest smooth levitation is étale.

Now consider the (abstract group) homomorphism given by composing the canonical inclusion $G(k) \hookrightarrow G(k')$ with $\pi'(k') : G(k') \to G'(k')$; we abuse notation and call this map

$$\pi'(k) : G(k) \to G'(k'). \tag{13}$$

We give a basic criterion for subgroups of G' to levitate in G.

Proposition 4.2. *Let H' be a subgroup of G' that levitates (resp. smoothly levitates) in G. Let H'_1 be another subgroup of G'. If there exists g' in the image of $\pi'(k)$ such that $H'_1 = g'H'(g')^{-1}$ then H'_1 levitates (resp. smoothly levitates) in G.*

Proof. Let $g \in G(k)$ such that $\pi'(k)(g) =: g'$ satisfies $H'_1 = g'H'(g')^{-1}$. Let H be a levitation of H' in G. Then

$$\pi'((gHg^{-1})_{k'}) = g'H'(g')^{-1} = H'_1,$$

so gHg^{-1} is a levitation of H'_1 in G. If H is smooth then of course so is gHg^{-1}. □

It is remarkable that a partial converse to Proposition 4.2 holds, when H' is a maximal rank smooth subgroup of G'. We prove this in Theorem 4.12, and apply it to maximal tori in Corollary 4.13. In Theorem 1.4 we find a sufficient condition for a smooth subgroup H' of G' to smoothly levitate in G, namely, there exists a maximal torus T' of G' which normalises H' and smoothly levitates in G. If in addition the root system of $G_{k^{\mathrm{s}}}$ is reduced we can relax this condition slightly, requiring only that T' levitates in G rather than smoothly levitates. We prove this in Corollary 4.16.

We continue with examples which exhibit various possible behaviours.

Remark 4.3 (**k-structures on G'**)**.** Suppose $k = k^{\mathrm{s}}$. Fix a maximal torus T of G. Choose a Levi subgroup L of G containing T; this exists by [CGP, 3.4.6] as G is pseudo-split. Then G' inherits a canonical k-structure from L, via base changing by k'/k and applying the isomorphism $\pi'|_{L_{k'}} : L_{k'} \to G'$. Denote $T' := \pi'(T_{k'})$. Observe that T' is a k-defined maximal torus of G' with respect to this k-structure. A k-structure on a split reductive k'-group is completely determined by a choice of maximal torus and a pinning (see for instance [CGP, A.4.13]). So, up to a choice of simple roots and a scaling of the root groups, this k-structure on G' does not depend on the choice of L for fixed T.

By [Mi, 23.39], any two pinnings of the split reductive pair (G', T') are conjugate by some element of $N_{G'}(T')(k')$. Now let T_1 be another maximal torus of G. Since $k = k^{\mathrm{s}}$, there exists $g \in G(k)$ such that $gTg^{-1} = T_1$. Denote $g' := \pi'(k)(g) \in G'(k')$. Hence any k-structure on G' induced by T_1 is conjugate by g' to one that is induced by T. However, if $\pi'(k)$ does not give rise to a surjection onto $G'(k')/T'(k')$ then not every k-structure on G' is induced by a choice of maximal torus of G. So there is a set of privileged k-structures on G', each of which arises from a choice of maximal torus of G.

In the following (very general) example we present a family of subgroups H' of G' that levitate in G; moreover, they smoothly levitate if H' is smooth.

Example 4.4. Suppose $k = k^{\mathrm{s}}$. Let H' be a subgroup of G'. Fix a maximal torus T of G. Assume H' is k-defined with respect to the k-structure on G' induced by T (as in Remark 4.3). Choose any Levi subgroup L of G containing T. Then, under the identification $G' = L_{k'}$, there exists an algebraic k-group J and an inclusion $J \hookrightarrow L$ whose base change by k'/k is precisely $H' \hookrightarrow G'$. Since the composition

$$G' = L_{k'} \hookrightarrow G_{k'} \xrightarrow{\pi'} G'$$

is the identity, J is indeed a levitation of H' in G. If H' is smooth then J is also smooth, as smoothness is invariant under base change.

Not all subgroups H' of G' that levitate in G can be constructed as in Example 4.4, even if G is pseudo-split. We illustrate this as follows.

Example 4.5. Let k be a local function field of characteristic 2. Let k' be a degree 2 extension of k, and let $q' : V' \to k'$ be an anisotropic non-degenerate quadratic form over k' in 3 variables. It is shown in [CGP, Ex. 7.2.2] that $H' := \mathrm{SO}(q')$ does not descend to k. Consider the natural inclusion of H' in $\mathrm{SL}(V') =: G'$. Denote $H := R_{k'/k}(H')$ and $G := R_{k'/k}(G')$. Since H' is smooth, Lemma 2.3 tells us that H is a levitation of H' in G. Let L be a Levi subgroup of G (which exists as G is pseudo-split). If we had $\pi'((H \cap L)_{k'}) = H'$ then $H \cap L$ would be a Levi subgroup of H, which violates [CGP, 7.2.1].

4.2. **Levitating tori.** Let T' be a torus of G'. It is natural to ask whether T' levitates to a torus of G. Obstructions to this come in two flavours. First, T' need not levitate at all in G. Consider the following example.

Example 4.6. Let G_1 be a reductive k-group which is not a torus. Let k'/k be a non-trivial purely inseparable finite field extension. Set $G_1' := (G_1)_{k'}$ and $G := G_1 \times R_{k'/k}(\mathbb{G}_m)$. Note that $\ker i_G$ is trivial. The minimal field of definition for $R_u(G_{\overline{k}})$ is k'/k, the maximal reductive k'-quotient of G is $G_1' \times \mathbb{G}_m =: G'$, and $L := G_1 \times \mathbb{G}_m$ is the unique Levi subgroup of G. As explained in Remark 4.3, G' inherits a k-structure from L. Choose a maximal torus T' of G' which is not k-defined. Suppose T' levitates. Then T' levitates to a torus T of G by Lemma 4.10 below, since the root system of G is reduced. But T is contained in L, so T' must be k-defined by Example 4.4, a contradiction.

Even if G is absolutely pseudo-simple, it need not be the case that T' levitates in G. It turns out that one can find such a non-levitating torus if and only if G is non-standard; this will be proved in Theorem 1.3.

The second obstruction is as follows: even if T' levitates in G, it need not smoothly levitate in G. We demonstrate this in Example 4.8 (via Lemma 4.7), for G pseudo-simple of type BC_1. In fact, this obstruction can only occur if the root system of $G_{k^{\mathrm{s}}}$ is not reduced: if the root system is reduced then we prove in Lemma 4.10 that any torus of G' that levitates in G must smoothly levitate in G.

Lemma 4.7. *Assume that $k = k^{\mathrm{s}}$. Let T be a maximal torus of G. Suppose there exists some root $\alpha \in \Phi(G,T)$ such that the restriction $U_\alpha \to i_G(U_\alpha)$ is not surjective on k-points. Then there exists a maximal torus (resp., Borel subgroup) of G' that does not levitate to any maximal torus (resp., minimal pseudo-parabolic subgroup) of G.*

Proof. Choose $g \in (i_G(U_\alpha))(k)$ such that $g \notin i_G(U_\alpha(k))$. Set $T_0 := i_G(T)$, $S_0 := gT_0g^{-1}$ and $S' := q_{G'}((S_0)_{k'})$. Observe that S_0 is a smooth levitation of S' in $i_G(G)$. Recall that $\pi' = q_{G'} \circ (i_G)_{k'}$ by (12), so $i_G^{-1}(S_0)$ is a levitation of S' in G.

Assume (for a contradiction) that S' levitates to some maximal torus S of G. Since $R_{k'/k}(S')$ is commutative it contains a unique maximal torus, which must equal S_0. Again using the universal property of $q_{G'}$, we deduce that $i_G(S) = S_0$.

Since $i_G(S) = S_0 = gT_0g^{-1} \subseteq T_0 i_G(U_\alpha)$ we have that $S \subseteq TU_\alpha \ker i_G$, and hence $S \subseteq TU_\alpha$ since $(\ker i_G)(k)$ is finite. Applying [CGP, 3.2.1] to the map $TU_\alpha \to T_0 i_G(U_\alpha)$ got by restricting i_G tells us that the induced map

$$N_{TU_\alpha}(T)/Z_{TU_\alpha}(T) \to N_{T_0 i_G(U_\alpha)}(T_0)/Z_{T_0 i_G(U_\alpha)}(T_0)$$

is an isomorphism. Consequently there exists $x \in U_\alpha(k)$ such that $S = xTx^{-1}$ and such that $g^{-1} i_G(x)$ centralises T_0. But $i_G(U_\alpha) \cap Z_{i_G(G)}(T_0)$ is trivial, and so $g = i_G(x)$. This contradicts our assumption that $g \notin i_G(U_\alpha(k))$. That is, S' does not levitate to any maximal torus of G.

Now let P be a minimal pseudo-parabolic subgroup of G that contains both T and U_α. Then $P_0 := i_G(P)$ is a minimal pseudo-parabolic subgroup of $i_G(G)$ that contains the maximal torus S_0. Since $i_G(G)$ is pseudo-reductive, we can choose a cocharacter $\lambda : \mathbb{G}_m \to i_G(G)$ such that $P_0 = P_{i_G(G)}(\lambda)$ and $Z_{i_G(G)}(S_0) = Z_{i_G(G)}(\lambda)$. Consider the minimal pseudo-parabolic subgroup $Q_0 := P_{i_G(G)}(-\lambda)$ of $i_G(G)$; by construction $P_0 \cap Q_0 = Z_{i_G(G)}(S_0)$. Define another cocharacter $\lambda' := q_{G'} \circ \lambda_{k'} : \mathbb{G}_m \to G'$, and consider the associated Borel subgroups $P' := P_{G'}(\lambda')$ and $Q' := P_{G'}(-\lambda')$ of G'. By construction $P' \cap Q' = S'$. Observe that $Q' = q_{G'}((Q_0)_{k'})$ by [CGP, 2.1.4, 2.1.9]. In other words Q' smoothly levitates to Q_0 in $i_G(G)$. So Q' levitates to $i_G^{-1}(Q_0)$ in G.

Assume (for a contradiction) that Q' levitates to some minimal pseudo-parabolic subgroup Q of G. Observe that $P \cap Q$ has maximal rank in G by [CGP, 3.5.12(1)], hence

$\pi'((P \cap Q)_{k'}) = S'$. This implies that any maximal torus of $P \cap Q$ is a levitation of S', which is a contradiction. □

It follows that neither the maximal torus S' nor the Borel subgroup Q' of G' constructed in the proof of Lemma 4.7 levitates to any smooth subgroup of G. One may either show this directly, or appeal to Theorem 1.4(ii).

We now use Lemma 4.7 to give a concrete example of a type BC_1 k-group G and a torus and Borel subgroup of G' both of which levitate, but do not smoothly levitate, in G.

Example 4.8. Let k be a separably closed imperfect field of characteristic 2. Let $K = k(a^{1/2})$, where $a \in k$ but $a^{1/2} \notin k$. Then $K = \{a^{1/2}x + y \mid x, y \in k\}$. Let V' be the k-subspace $ka^{1/2}$ of K. Now take $V = K$ and define $q\colon V \to K$ by $q(z) = z^2$. Set $V^{(2)} = q(V) = K^2 \subseteq k$. Then $V' \cap V^{(2)} = 0$. Note also that $V^{(2)} = K^2$ has dimension 1 as a vector space over K^2, and the subfield of K generated over k by $V' \oplus V^{(2)}$ is K itself.

Using this data, we can form a pseudo-simple BC_1 group G of minimal type as in [CGP, §9.8] or [BRSS, Def. 3.3]. Let α be a very short root (multipliable) for G with respect to a maximal torus T of G. We may identify $U_\alpha(G)(k)$ with $V \times V'$ (regarded as a k-vector space) and $U_{2\alpha}(i_G(G))(k)$ with K, and the map f from $U_\alpha(G)$ to $U_\alpha(i_G(G))$ induced by i_G is given on k-points by $(v, v') \mapsto q(v) + v'$ [CGP, 9.6.8, 9.6.9].

Now assume $[k : k^2] > 2$. We claim that f is not surjective on k-points, which follows as long as we can show that $V^{(2)} \oplus V' \neq K$; in fact, we show that k is not contained in the image of f. Given $c \in k$, if we want $c = f(v, v') = q(v) + v'$, then we must have $v' = 0$. Since $[k : k^2] > 2$, there exist $c \in k$ such that $c \neq ax^2 + y^2$ for any $x, y \in k$, and hence c does not lie in the image of f. This proves the claim.

Given the claim, by Lemma 4.7 and the subsequent remark, there exists a maximal torus and a Borel subgroup of $G' \cong \mathrm{SL}_2$ both of which levitate, but do not smoothly levitate, in G.

We now move on to the proof of Theorem 1.3. We will need the following lemmas.

Lemma 4.9. *Let Z be a commutative affine algebraic k-group, and let Z_t denote its unique maximal torus. If f is a surjective homomorphism from Z to a k-torus T then $f(Z_t) = T$.*

Proof. By [Mi, Thm. 16.13(a)] there is a subgroup Z_s of Z such that Z_s is of multiplicative type and Z/Z_s is unipotent. Now f induces a surjective homomorphism from Z/Z_s to $T/f(Z_s)$. Since Z/Z_s is unipotent and $T/f(Z_s)$ is a torus, $T/f(Z_s)$ is trivial. Hence we may assume without loss of generality that $Z = Z_s$ is of multiplicative type. Then by [Mi, Cor. 12.24] there is a short exact sequence

$$1 \to Z_t \to Z \to F \to 1, \tag{14}$$

where F is finite. Now f induces a surjective homomorphism from Z/Z_t to $T/f(Z_t)$. Since Z/Z_t is finite and $T/f(Z_t)$ is a torus, $T/f(Z_t)$ is trivial. Hence $f(Z_t) = T$. □

Lemma 4.10. *Let G be a pseudo-reductive k-group. Suppose the root system of G_{k^s} is reduced. Let T' be a torus of G' and assume that there exists some subgroup Z of $i_G(G)$ such that $q_{G'}(Z_{k'}) = T'$. Then there exists a unique torus T of G such that $\pi'(T_{k'}) = T'$. If T' is maximal in G' then T is maximal in G.*

Proof. Let T' be a torus of G'. By assumption there exists a subgroup Z of $i_G(G)$ such that $q_{G'}(Z_{k'}) = T'$. Then Z is contained in $R_{k'/k}(T')$ by Lemma 2.4; in particular Z is commutative, so Z contains a unique maximal torus T_0. By [CGP, C.4.4] this torus T_0 remains maximal in Z after base change by k'/k. Then applying Lemma 4.9 says that $q_{G'}$ sends $(T_0)_{k'}$ onto T'. That is, T_0 is a smooth levitation of T' in $i_G(G)$.

By assumption the root system of G_{k^s} is reduced, so $\ker i_G$ is central in G. Then, by the proof of [CGP, 2.2.12(1)], $i_G^{-1}(T_0)$ is commutative and its unique maximal torus T satisfies $i_G(T) = T_0$. Since $\pi' = q_{G'} \circ (i_G)_{k'}$ we deduce that $\pi'(T_{k'}) = T'$. Moreover, T is the unique torus that satisfies this property, as any other such torus must also map onto T_0 via i_G.

Finally, observe that $\mathrm{rank}_k(G) = \mathrm{rank}_{k'}(G')$ as k'/k is purely inseparable. The final assertion follows immediately. □

We can now prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that the root system of G_{k^s} is reduced. We start by observing that each of (a)–(f) is invariant under replacing k (resp., k') with k^s (resp., $(k')^s$). For (a) and (b), note that the formation of the i_G map commutes with separable algebraic field extensions (since this is true of the unipotent radical and minimal fields of definition, by [CGP, 1.1.9]). Moreover, the formation of the derived group commutes with arbitrary field extensions by [Mi, Cor. 6.19(a)]. For (c)–(e), if H' is a subgroup of G' and $(H')_{k^s}$ levitates to a subgroup M_1 of G_{k^s} then the subgroup M of G_{k^s} generated by the Galois conjugates of M_1 is a levitation of $(H')_{k^s}$ and M descends to a subgroup H of G which is a levitation of H'; it then follows from Lemma 4.10 that if H' is a (maximal) torus of G' then H' levitates to a (maximal) torus of G. For (f), it is clear from the standard construction (see [CGP, 1.4]) that if G is standard then G_{k^s} is standard, and the converse is [CGP, 5.2.3]; moreover, if k' is the minimal field of definition for $\mathscr{R}_u(G_{\overline{k}})$ then $(k')^s$ is the minimal field of definition for $R_u((G_{k^s})_{\overline{k}})$ by [CGP, 1.1.8], and one can show using a Galois descent argument that if G is pseudo-simple and S is a pseudo-simple factor of G_{k^s} then the minimal fields of definition for $\mathscr{R}_u((G_{k^s})_{\overline{k}})$ and $\mathscr{R}_u(S_{\overline{k}})$ are equal.

Hence we can assume without loss that $k = k^s$; in particular, G is pseudo-split.

(a) $\iff$ (b). Assume $i_G(G)$ contains $\mathscr{D}(R_{k'/k}(G'))$. Then $q_{G'}$ is smooth and surjective since G' is smooth, and $\ker q_{G'} = \mathscr{R}_u(R_{k'/k}(G')_{k'})$ since G' is reductive. Let T_0 be a maximal torus of $R_{k'/k}(G')$. Denote $T' := q_{G'}((T_0)_{k'})$. Let Z be the unique maximal torus of $Z(R_{k'/k}(G'))$. We claim that $Z \subseteq i_G(G)$. To see this, observe that $i_G(G)$ is a pseudo-split pseudo-reductive k-group, hence it contains a Levi subgroup L by [CGP, 3.4.6]. Now L is also a Levi subgroup of $R_{k'/k}(G')$ by [CGP, 9.2.1(2)], so it contains some maximal torus of $R_{k'/k}(G')$. But all such maximal tori are $R_{k'/k}(G')(k)$-conjugate since $k = k^s$, so L contains the central torus Z, and the claim is proved.

We have $Z(R_{k'/k}(G')) = R_{k'/k}(Z(G')$ by [CGP, A.5.15(1)], so $q_{G'}(Z(R_{k'/k}(G'))) = Z(G')$ by Lemma 2.4. Hence $q_{G'}(Z_{k'}) = Z(G')^0$ by Lemma 4.9. Choose a subtorus M of T_0 such that $MZ = T_0$ and $M \cap Z$ is finite. Then $q_{G'}(M_{k'})Z(G')^0 = q_{G'}(M_{k'})q_{G'}(Z_{k'}) = q_{G'}((T_0)_{k'}) = T'$ and $q_{G'}(M_{k'}) \cap Z(G')^0$ is finite. It follows that $q_{G'}(M_{k'}) \subseteq \mathscr{D}(G')$. To prove (b), it is enough by (a) to show that $M \subseteq \mathscr{D}(R_{k'/k}(G'))$.

Since $\ker q_{G'}$ is unipotent, combining $q_{G'}$ with base change by k'/k induces an inclusion of root systems $\iota : \Phi(G', T') \hookrightarrow \Phi(R_{k'/k}(G'), T_0)$. Let $\alpha' \in \Phi(G', T')$ and consider the associated T'-root group $U_{\alpha'}$ of G'. Let $\alpha := \iota(\alpha')$ and consider the associated T_0-root group U_α of $R_{k'/k}(G')$. Note that $U_{\alpha'}$ is 1-dimensional as G' is reductive, and hence $q_{G'}((U_\alpha)_{k'}) = U_{\alpha'}$. Recall that the derived subgroup of a pseudo-reductive k-group is perfect [CGP, 3.1], and that it is generated by all of the root groups [CGP, 3.1.5]. Consequently $q_{G'}(\mathscr{D}(R_{k'/k}(G'))_{k'}) = \mathscr{D}(G')$, and so

$$q_{G'}^{-1}(\mathscr{D}(G')) = \mathscr{D}(R_{k'/k}(G'))_{k'} \ker q_{G'}. \tag{15}$$

It follows that $M_{k'} \subseteq \mathscr{D}(R_{k'/k}(G'))_{k'} \ker q_{G'}$. Since $\ker q_{G'}$ is smooth and unipotent, there is a maximal torus T_1' of $\mathscr{D}(R_{k'/k}(G'))_{k'} \ker q_{G'}$ such that $T_1' \subseteq \mathscr{D}(R_{k'/k}(G'))_{k'}$. But $k = k^s$ and $\mathscr{D}(R_{k'/k}(G'))_{k'}$ is normal in $R_{k'/k}(G')_{k'}$, so every maximal torus of

$\mathscr{D}(R_{k'/k}(G'))_{k'} \ker q_{G'}$ is contained in $\mathscr{D}(R_{k'/k}(G'))_{k'}$. Hence $M_{k'} \subseteq \mathscr{D}(R_{k'/k}(G'))_{k'}$. It follows that $M \subseteq \mathscr{D}(R_{k'/k}(G'))$, so (b) holds.

On the other hand, since the derived subgroup $\mathscr{D}(R_{k'/k}(G'))$ of $R_{k'/k}(G')$ is perfect, it is generated by tori. So if (b) holds then so does (a).

(b) $\Longrightarrow$ (c). Let S' be a torus of G', and let S_0 denote the unique maximal torus of the smooth commutative group $R_{k'/k}(S')$. Then $(S_0)_{k'}$ is the unique maximal torus of $R_{k'/k}(S')_{k'}$. Combining this with Lemmas 2.3 and 4.9 tells us that $q_{G'}((S_0)_{k'}) = S'$. By assumption (b) holds, so $S_0 \subseteq i_G(G)$. Since $\pi' = q_{G'} \circ (i_G)_{k'}$, we deduce that $\pi'(i_G^{-1}(S_0)_{k'}) = S'$. In other words, $i_G^{-1}(S_0)$ is a levitation of S' in G.

(c) $\Longrightarrow$ (d). This follows from Lemma 4.10, since $\pi' = q_{G'} \circ (i_G)_{k'}$.

(d) $\Longrightarrow$ (e) is clear, as $\operatorname{rank}_k G = \operatorname{rank}_{k'} G'$.

(e) $\Longrightarrow$ (b). Assume (e) holds. Let T_0 be a maximal torus of $R_{k'/k}(G')$. Consider the maximal torus $q_{G'}((T_0)_{k'}) =: T'$ of G'. By assumption there exists a maximal torus T of G such that $\pi'(T_{k'}) = T'$. Observe that $q_{G'}(i_G(T)_{k'}) = \pi'(T_{k'}) = T'$, and hence $i_G(T) \subseteq R_{k'/k}(T')$ by Lemma 2.4. Similarly, $T_0 \subseteq R_{k'/k}(T')$. But $\operatorname{rank}_{k'} G' = \operatorname{rank}_k R_{k'/k}(G') = \operatorname{rank}_k G$, so $T_0 = i_G(T)$ must be the unique maximal torus of $R_{k'/k}(T')$. As T_0 was chosen arbitrarily, we have shown that (b) holds.

(a) $\Longleftrightarrow$ (f). If G is an absolutely pseudo-simple k-group with reduced root system then G is standard if and only if $i_G(G) = \mathscr{D}(R_{k'/k}(G'))$, by [CGP, 5.3.8]. Now consider the general case, for arbitrary pseudo-reductive G. We have a decomposition $G = \mathscr{D}(G) \cdot C$, where C is any Cartan subgroup of G, and $\mathscr{D}(G)$ is a commuting product of (absolutely) pseudo-simple k-groups S_i for $i = 1, ..., r$. Observe that G is standard if and only if S_i is standard for each i (this follows from [CGP, 5.2.3, 5.2.6, 5.3.1]).

Henceforth fix some $i \in \{1, ..., r\}$. The aforementioned decomposition of G is preserved by the i_G map. So we have a (co)restriction map $i_G|_{S_i} : S_i \to R_{k'/k}(S_i')$, where $S_i' := \pi'((S_i)_{k'})$. By functoriality $i_G|_{S_i}$ is the map associated to $\pi'|_{(S_i)_{k'}} : (S_i)_{k'} \to S_i'$ under adjunction.

Assume (a) holds. Since the formation of the derived subgroup commutes with (commuting) products, and as S_i is perfect, we see that $i_G|_{S_i}$ maps onto $\mathscr{D}(R_{k'/k}(S_i'))$. It follows that k'/k is the minimal field of definition for $\mathscr{R}_u((S_i)_{\overline{k}})$, as otherwise surjectivity would fail by dimension considerations. So $i_G|_{S_i} = i_{S_i}$, i.e., it is the analogue of the i_G map for S_i. Since $k = k^{\mathrm{s}}$ we can apply [CGP, 5.3.8], so S_i is standard. Hence $\mathscr{D}(G)$ is a commuting product of standard pseudo-simple k-groups, and so G is itself standard.

Conversely, assume (f) holds. Since G is standard, S_i is also standard. By assumption k'/k is the minimal field of definition for $\mathscr{R}_u((S_i)_{\overline{k}})$, so we can again apply [CGP, 5.3.8], which tells us that $i_G|_{S_i} = i_{S_i}$ maps onto $\mathscr{D}(R_{k'/k}(S_i'))$. Again using the fact that the formation of the derived subgroup commutes with (commuting) products, it follows that $i_G(G)$ contains $\mathscr{D}(R_{k'/k}(G'))$. □

We can extend the results of Theorem 1.3 to subgroups of G' that are generated by tori.

Corollary 4.11. *Let G be a pseudo-reductive k-group. Suppose the root system of $G_{k^{\mathrm{s}}}$ is reduced, and that G satisfies the equivalent conditions of Theorem 1.3. Let H' be a subgroup of G' that is generated by tori (for example, this holds if H' is perfect). Then there exists a smooth subgroup H of G such that $\pi'(H_{k'}) = H'$.*

Proof. This is an immediate consequence of Theorem 1.3(d); simply take a generating set of tori $\{T_i' \mid i = 1, ..., l\}$ of H', levitate each T_i' to a torus T_i of G, and let H be the subgroup of G generated by $\{T_i \mid i = 1, ..., l\}$. The formation of H commutes with base change by

k'/k by [Mi, 2.47], and H is smooth by [Mi, 2.48], so indeed H is a smooth levitation of H' in G. □

4.3. **Levitating maximal rank subgroups.** Next we study levitations of maximal rank subgroups. Recall the map $\pi'(k) : G(k) \to G'(k')$ from (13).

Theorem 4.12. *Suppose $k = k^{\mathrm{s}}$. Let G be a pseudo-reductive k-group. Let H be a smooth subgroup of G such that $\pi'(H_{k'}) =: H'$ has maximal rank in G'. Let H_1' be another subgroup of G' that is $G'(k')$-conjugate to H'. If there exists a smooth subgroup H_1 of G satisfying $\pi'((H_1)_{k'}) = H_1'$ then there exists g' in the image of $\pi'(k)$ such that $H_1' = g'H'(g')^{-1}$.*

Proof. Let H be a smooth levitation of H' in G. Combining [CGP, C.4.4] with [Bo, 11.14] tells us that H is a maximal rank subgroup of G. So let T be a maximal torus of H; then $\pi'(T_{k'}) =: T'$ is a maximal torus of H'. Similarly let H_1 be a smooth levitation of H_1' in G, let T_1 be a maximal torus of H_1; then $\pi'((T_1)_{k'}) =: T_1'$ is a maximal torus of H_1'.

Since $k = k^{\mathrm{s}}$ there exists $x \in G(k)$ such that $T_1 = xTx^{-1}$. Denote $x' := \pi'(k)(x)$. Consider the subgroups H' and $(x')^{-1}H_1'x'$ of G'; by assumption they are $G'(k')$-conjugate, but they share a maximal torus T', so there exists $n' \in N_{G'}(T')(k')$ such that $(x')^{-1}H_1'x' = n'H'(n')^{-1}$.

We claim that $\pi'(k)$ induces a surjection $N_G(T)(k) \to N_{G'}(T')(k')/T'(k')$. Given the claim, there exists $n \in N_G(T)(k)$ such that $\pi'(k)(n) = n't'$ for some $t' \in T'(k')$. Let $g := xn$ and let $g' := \pi'(k)(g) = x'n't'$; then indeed $H_1' = g'H'(g')^{-1}$.

It remains to prove the claim. We need the following ingredients. First observe that $N_G(T)/Z_G(T)$ is a constant k-group, so the canonical inclusion

$$(N_G(T)/Z_G(T))(k) \to (N_G(T)/Z_G(T))(k') \tag{16}$$

is an isomorphism. Note that $Z_{G'}(T') = T'$, as G' is reductive. Then by [CGP, 3.2.1], as $\ker \pi'$ is unipotent, π' induces an isomorphism

$$(N_G(T)/Z_G(T))(k') \to (N_{G'}(T')/T')(k'). \tag{17}$$

Next observe that $(N_G(T)/Z_G(T))(k) = N_G(T)(k)/Z_G(T)(k)$: this holds as k is separably closed and $Z_G(T) = N_G(T)^{\circ}$ is smooth. Similarly $(N_{G'}(T')/T')(k') = N_{G'}(T')(k')/T'(k')$. Then composing the natural projection $N_G(T)(k) \to N_G(T)(k)/Z_G(T)(k)$ with (16) and (17) gives us the desired surjection $N_G(T)(k) \to N_{G'}(T')(k')/T'(k')$. This completes the proof. □

Corollary 4.13. *Suppose $k = k^{\mathrm{s}}$. Let G be a pseudo-reductive k-group with reduced root system. Let Z be a subgroup of G such that $\pi'(Z_{k'}) =: T'$ is a maximal torus of G'. Let T_1' be another maximal torus of G'. Then there exists a subgroup Z_1 of G satisfying $\pi'((Z_1)_{k'}) = T_1'$ if and only if there exists g' in the image of $\pi'(k)$ such that $T_1' = g'T'(g')^{-1}$.*

Proof of Corollary 4.13. Since k' is separably closed, T' and T_1' are $G'(k')$-conjugate. By Lemma 4.10, T' smoothly levitates in G. Similarly, T_1' levitates in G if and only if it smoothly levitates in G. The result then follows from combining Proposition 4.2 with Theorem 4.12. □

4.4. **Levitating regular subgroups.** Recall that a subgroup of a pseudo-reductive group is *regular* if it is normalised by some maximal torus. In this subsection we prove Theorem 1.4, which concerns smooth levitations of regular smooth subgroups. We first need the following lemmas.

Lemma 4.14. *Let G be a smooth connected affine k-group. Let S be a torus of G. Then $Z_G(S) \cap \mathscr{R}_u(G) = \mathscr{R}_u(Z_G(S))$.*

Proof. Let $\pi : G \to G/\mathscr{R}_u(G) =: \overline{G}$ be the natural projection, and denote $\overline{S} := \pi(S)$. Observe that $\pi(Z_G(S)) = Z_{\overline{G}}(\overline{S})$ by [Bo, 11.14, Cor. 2]. Now $Z_{\overline{G}}(\overline{S})$ is pseudo-reductive by [CGP, 1.2.4], hence $\pi(\mathscr{R}_u(Z_G(S)))$ is trivial. So $\mathscr{R}_u(Z_G(S)) \subseteq \mathscr{R}_u(G)$.

For the opposite inclusion, certainly $Z_G(S) \cap \mathscr{R}_u(G)$ is a unipotent normal subgroup of $Z_G(S)$. The same argument as in [CGP, C.2.23, proof, par. 4] tells us that $Z_G(S) \cap \mathscr{R}_u(G) = \mathscr{R}_u(G)^S$ is smooth and connected. Hence $Z_G(S) \cap \mathscr{R}_u(G) \subseteq \mathscr{R}_u(Z_G(S))$. □

Henceforth we use the notation of the previous subsection. That is, G is a pseudo-reductive k-group, k'/k is the minimal field of definition for $\mathscr{R}_u(G_{\overline{k}})$, and $\pi' : G_{k'} \to G_{k'}/\mathscr{R}_u(G_{k'}) := G'$ is the natural projection.

Lemma 4.15. *Let S' be a split torus of G'. Suppose that S' levitates to a torus S in G. Then any subgroup of S' also levitates in G.*

Proof. Every split multiplicative type k-group of M uniquely descends to $\mathbb{Z}$ (since $M \mapsto X(M)$ is an equivalence from the category of split multiplicative type k-groups of to the opposite category of finitely generated $\mathbb{Z}$-modules; see [Mi, 12.23]). Similarly, any embedding of split multiplicative type k-groups uniquely descends to $\mathbb{Z}$.

Let M' be a subgroup of S'. By the above remark the embedding $S_{k'} \cap (\pi')^{-1}(M') \hookrightarrow S_{k'}$ uniquely descends to $\mathbb{Z}$ and hence to k. Let us call this k-descent $M \hookrightarrow S$. Then π' sends $M_{k'}$ isomorphically onto M', since $\ker \pi'$ is unipotent. □

Proof of Theorem 1.4. Let H' be a smooth subgroup of G', and let T' be a maximal torus of G' that normalises H'. Assume that T' levitates to a (maximal) torus T of G. We first prove (i). Using a standard argument of Galois descent, we may assume without loss of generality that $k = k^{\mathrm{s}}$ (cf. the remarks about (c)–(e) in the proof of Theorem 1.3); note that the formation of normalisers commutes with base change.

Recall that maximal tori in reductive k-groups are self-centralising. Hence

$$T' \subseteq \pi'(Z_G(T)_{k'}) \subseteq Z_{G'}(T') = T'.$$

That is, the Cartan subgroup $Z_G(T)$ is a levitation of T' in G. Indeed $Z_G(T)$ is smooth by [Mi, 17.44].

Since T is split and π' restricts to an isomorphism $T_{k'} \to T'$, extending scalars by k'/k induces a natural bijection $\iota : X(T') \to X(T)$. One can check that ι restricts to an injection between the respective root systems $\Phi(G', T') \hookrightarrow \Phi(G, T)$, mapping onto the set of non-multipliable roots of $\Phi(G, T)$ (see for instance [CGP, 2.3.10]). Since H' is normalised by T', ι further restricts to an injection $\Phi(H', T') \hookrightarrow \Phi(G, T)$. (Note that the set of roots $\Phi(H', T')$ is still defined even when H' is not pseudo-reductive; it just may no longer be symmetric about the origin.)

Since H' is smooth and T'-stable, [Bo, 13.20] tells us that $(H')^\circ$ is generated by the T'-root groups $U_{\alpha'}$ for each $\alpha' \in \Phi(H', T')$ along with the torus $(H' \cap T')^\circ_{\mathrm{red}} =: S'$. By Lemma 4.15 (and its proof), there exists a subtorus S of T such that S is a levitation of S'.

Let $\alpha' \in \Phi(H', T')$ and consider the corresponding T'-root group $U_{\alpha'}$ of H'. Define a root $\alpha \in \Phi(G, T)$ as follows: if $\iota(\alpha')$ is non-divisible in $\Phi(G, T)$ then set $\alpha = \iota(\alpha')$, otherwise set $\alpha = \iota(\alpha')/2$. Let U_α be the T-root group of G associated to α. Observe that $\pi'((U_\alpha)_{k'})$ is a T'-stable smooth connected subgroup of G' whose Lie algebra is the α'-root space in $\mathrm{Lie}(G')$. But this condition uniquely defines a T'-root group of G' by [CGP, 2.3.11], and hence $\pi'((U_\alpha)_{k'}) = U_{\alpha'}$. That is, the root group U_α is a smooth levitation of $U_{\alpha'}$ in G.

Let Ψ be the subset of $\Phi(G, T)$ consisting of all roots α defined as above, that is

$$\Psi := \big\{\alpha \in \Phi(G, T) \,\big|\, \alpha \text{ is non-divisible, } \iota^{-1}(\alpha) \in \Phi(H', T') \sqcup \tfrac{1}{2}\Phi(H', T')\big\}.$$

Let us define H to be the subgroup of G generated by the T-root groups U_α for each $\alpha \in \Psi$ along with the torus S. Note that H is smooth and connected. Since the formation of H commutes with base change [Mi, 2.47], we deduce that $\pi'(H_{k'}) = (H')^\circ$. That is, H is a levitation of $(H')^\circ$ in G.

Now let Λ be a set of representatives of the cosets of $(H')^\circ(k')$ in $H'(k')$; we can choose them so that Λ normalises T' (since all maximal tori in $H'T'$ are conjugate by an element of $(H')^\circ(k')$). For each $h \in \Lambda$, we define an element $g := g(h) \in G(k)$ as follows. Since $\ker \pi'$ is unipotent, π' restricts to a surjection $\pi'|_{N_G(T)_{k'}} : N_G(T)_{k'} \to N_{G'}(T')$ by [CGP, 3.2.1]. Observe that

$$(\ker \pi'|_{N_G(T)_{k'}})^\circ = (N_G(T)_{k'} \cap \ker \pi')^\circ = Z_G(T)_{k'} \cap \mathscr{R}_u(G_{k'}) = \mathscr{R}_u(Z_G(T)_{k'}),$$

where the final equality is due to Lemma 4.14 (along with the fact that centralisers commute with base change). In particular, $\ker \pi'|_{N_G(T)_{k'}}$ is smooth. Hence, since k' is separably closed, π' induces a surjection $N_G(T)(k') \to N_{G'}(T')(k')$. So choose $g' \in N_G(T)(k')$ such that $\pi'(k')(g') = h$.

We claim that g' normalises H. To see this, we first observe that the formation of root groups of G commutes with base change by k'/k. Since g' normalises T and h normalises H', it follows that g' stabilises the set of T-root groups $\{U_\alpha \,|\, \alpha \in \Psi\}$. Note that h normalises S', hence g' normalises S. So indeed $g' \in N_G(H)(k')$.

The Cartan subgroup $Z_G(T)$ normalises each T-root group U_α of G for $\alpha \in \Psi$, and of course it centralises S, so $Z_G(T)$ is contained in $N_G(H)$. Let $M = N_G(H)^{\mathrm{sm}}$. Clearly T normalises M, so M is generated by H, $Z_G(T)$ and the root groups U_α such that α is perpendicular to all the roots of H; hence $N_G(H)$ normalises M. Applying [Mi, 17.47] to M tells us that $N_G(M)^0 = M^0$, and we deduce that $N_G(H)$ is itself smooth. Moreover $N_G(H)/N_G(H)^\circ$ is constant as $k = k^{\mathrm{s}}$. It follows that the inclusion $N_G(H)(k) \hookrightarrow N_G(H)(k')$ induces a surjection $N_G(H)(k) \to N_G(H)(k')/N_G(H)^\circ(k')$. So there exists $x' \in N_G(H)^\circ(k')$ such that $g'x' \in N_G(H)(k)$; denote $g = g(h) := g'x'$. Then the subgroup $\langle H, g(h) \,|\, h \in \Lambda\rangle$ of G is a smooth levitation of H' in G.

We have shown that H' smoothly levitates in G. Then Proposition 4.1 says that H' admits a largest smooth levitation in G, which must be normalised by T since all of its $T(k)$-conjugates are also smooth levitations of H' in G. This completes the proof of (i).

We next prove (ii). Observe that maximal tori, Cartan subgroups, root groups and pseudo-parabolic subgroups of G are all examples of regular subgroups of G. Since the formation of all of the aforementioned subgroups of G commutes with base change by separable field extensions, we can assume without loss of generality that $k = k^{\mathrm{s}}$.

In the course of proving (i), we showed that if H is a Cartan subgroup (resp. root group) of G then $\pi(H_{k'})$ is a maximal torus (resp. root group) of G'. If H is a pseudo-parabolic subgroup of G then $\pi(H_{k'})$ is a parabolic subgroup of G' by [CGP, 3.5.4]. This proves the "if" direction.

It remains to prove the converse. Without loss of generality we can assume that H' is connected. Henceforth take H to be the largest smooth levitation of H' in G; we know it exists by (i).

We first assume that H' is a maximal torus of G', i.e., $H' = T'$. By Proposition 4.1, $i_G^{-1}(R_{k'/k}(T'))$ is the largest levitation of T' in G, so

$$i_G(T) \subseteq i_G(H) \subseteq R_{k'/k}(T') \subseteq R_{k'/k}(G').$$

Observe that $i_G(T)$ is a maximal torus of $R_{k'/k}(G')$, since $i_G(G)$ contains a Levi subgroup of $R_{k'/k}(G')$ by [CGP, 9.2.1(2)]. Moreover, $R_{k'/k}(T')$ is a Cartan subgroup of $R_{k'/k}(G')$ by [CGP, A.5.15(3)]. Consequently $i_G(H)$ centralises $i_G(T)$. Now [Bo, 11.14, Cor. 2] says that i_G sends Cartan subgroups of G onto Cartan subgroups of $i_G(G)$, hence

$$H \subseteq i_G^{-1}\big(Z_{i_G(G)}(i_G(T))\big) = i_G^{-1} i_G(Z_G(T)) = Z_G(T) \ker i_G.$$

Observe that $(\ker i_G)(k)$ is finite, since G is pseudo-reductive. Hence $H^\circ \subseteq Z_G(T)$. But we showed earlier that $Z_G(T)$ is a smooth levitation of T' in G, so $H^\circ = Z_G(T)$.

We next assume that H' is a T'-root group of G', say $H' = U_{\alpha'}$ for some $\alpha' \in \Phi(G', T')$. The aforementioned injection $\iota : \Phi(G', T') \hookrightarrow \Phi(G, T)$ factors through a bijection $\iota_0 : \Phi(G', T') \to \Phi(i_G(G), i_G(T))$. Let $\alpha_0 := \iota_0(\alpha')$ and consider the associated $i_G(T)$-root group U_{α_0} of $i_G(G)$. Once again define $\alpha \in \Phi(G, T)$ as follows: if $\iota(\alpha')$ is non-divisible in $\Phi(G, T)$ then set $\alpha = \iota(\alpha')$, otherwise set $\alpha = \iota(\alpha')/2$. Consider the associated T-root group U_α of G. We showed earlier that U_α is a smooth levitation of H' in G, so $U_\alpha \subseteq H$.

Recall from Lemmas 2.3 and 2.4 that $R_{k'/k}(H')$ is the largest levitation of H' in $R_{k'/k}(G')$. Observe that $R_{k'/k}(H')$ is $i_G(T)$-stable, since it is smooth and all of its $i_G(T)(k)$-conjugates are also smooth levitations of H' in $R_{k'/k}(G')$. Combining [CGP, 2.3.6, 2.3.16] tells us that $i_G(G) \cap R_{k'/k}(H') = U_{\alpha_0}$; in particular $i_G(G) \cap R_{k'/k}(H')$ is smooth. So U_{α_0} is the largest levitation of H' in $i_G(G)$. Clearly $i_G(U_\alpha) \subseteq U_{\alpha_0}$; in fact equality holds by Lie algebra considerations. Then

$$H \subseteq i_G^{-1}(U_{\alpha_0}) = i_G^{-1} i_G(U_\alpha) = U_\alpha \ker i_G.$$

Once again since $(\ker i_G)(k)$ is finite, we deduce that $H^\circ = U_\alpha$.

Finally, assume that H' is a parabolic subgroup of G'. Observe that H' is generated by the T'-root groups $U_{\alpha'}$ for each $\alpha' \in \Phi(H', T')$ along with T'. Define $\alpha \in \Phi(G, T)$ as previously. Consider the subgroup P of G generated by each of these T-root groups U_α along with $Z_G(T)$. Certainly P is a pseudo-parabolic subgroup of G, and it is a levitation of H' in G. Hence P is contained in H. Then H is also a pseudo-parabolic subgroup of G by [CGP, 3.5.8]. In particular, H is connected. If H is strictly larger than P then the set of non-multipliable roots of $\Phi(H, T)$ is strictly larger than that of $\Phi(P, T)$, which is in natural bijection with $\Phi(H', T')$, contradicting the fact that H is a levitation of H' in G. So $H = P$. This completes the proof of (ii). □

Corollary 4.16. *Let G be a pseudo-reductive k-group, where the root system of $G_{k^{\mathrm{s}}}$ is reduced. Let H' be a smooth subgroup of G'. Suppose there exists a maximal torus T' of G' that normalises H', and suppose T' levitates. Then H' has a smooth levitation H. If H' is a torus then there exists a unique such H that is a torus.*

Proof. Combine Theorem 1.4(i) with Lemma 4.10. □

In the setting of Theorem 1.4 and Corollary 4.16, it is possible that H' is connected whilst the largest smooth levitation of H' in G is not connected. Consider the following example.

Example 4.17. Following [CGP, 9.1.10], over a suitably chosen imperfect field k of characteristic 2 one can construct a pseudo-simple k-group G of type A_1 with $\ker i_G \cong \mathbb{Z}/2\mathbb{Z}$. Let T be a maximal torus of G, and let U be a T-root group of G. Denote $T' := \pi'(T_{k'})$, and consider the T'-root group $\pi'(U_{k'}) =: H'$ of G'. Then $U \ker i_G$ is the largest levitation of H' in G; it is clearly smooth. Observe that $\ker i_G \subsetneq U$ since $\ker i_G$ is central in G, so $U \ker i_G$ is not connected.

4.5. **Non-levitating subgroups.** In this subsection we present two examples of non-levitating subgroups, both of which are variants of the same underlying construction. In each case the root system is reduced and the hypotheses of Theorem 1.3 are satisfied. Both examples are in characteristic 2; however one could construct analogues for arbitrary characteristic $p > 0$.

In the first example we construct a non-smooth subgroup H' of a reductive k'-group G' such that H' does not levitate in $G = R_{k'/k}(G')$.

Example 4.18. Let k be an imperfect field of characteristic 2 and let $k' = k(a)$, where $a^2 \in k$ and $a \notin k$. Let $K = k'(\sqrt{a})$. Let A' be a k'-algebra. Consider the free A'-module $A' \otimes_{k'} K$, and its A'-basis $\{1 \otimes 1, 1 \otimes \sqrt{a}\}$. The left regular representation of $(A' \otimes_{k'} K)^\times$ on $A' \otimes_{k'} K$ induces an embedding

$$R_{K/k'}(\mathbb{G}_m)(A') := (A' \otimes_{k'} K)^\times = \left\{ \begin{pmatrix} x & ya \\ y & x \end{pmatrix} \middle| x, y \in A';\ x^2 + y^2 a \neq 0 \right\} \subset \mathrm{GL}_2(A').$$

Below we write (x, y) as shorthand for $\begin{pmatrix} x & ya \\ y & x \end{pmatrix}$. This subfunctor $R_{K/k'}(\mathbb{G}_m)$ of the k'-group GL_2 is representable, and contains the center $Z(\mathrm{GL}_2) \cong \mathbb{G}_m$. Set $H' := R_{K/k'}(\mathbb{G}_m) \cap \mathrm{SL}_2$. A simple calculation shows that $H' = R_{K/k'}(\mu_2)$. In particular, $\dim H' = 1$ and H' is not smooth.

Now consider the map $q_{H'} : R_{k'/k}(H')_{k'} \to H'$ defined in (6). We claim that its image on $\overline{k}$-points is trivial, where $\overline{k}$ is an algebraic closure of k.

We identify $R_{k'/k}(H')(\overline{k})$ with $H'(\overline{k} \otimes_k k')$, and by [CGP, A.5.7] we can identify $q_{H'}(\overline{k})$ with the map $H'(m)\colon H'(\overline{k} \otimes_k k') \to H'(\overline{k})$ induced by the multiplication map $m\colon \overline{k} \otimes_k k' \to \overline{k}$, $c' \otimes d \mapsto c'd$. So let $(x, y) \in H'(\overline{k} \otimes_k k')$; then $x, y \in \overline{k} \otimes_k k'$ and $x^2 + (1 \otimes a)y^2 = 1$. We can write $(x, y) = (s \otimes 1 + t \otimes a, u \otimes 1 + v \otimes a)$ for some $s, t, u, v \in \overline{k}$. Then we have

$$\begin{aligned} 1 \otimes 1 &= (s \otimes 1 + t \otimes a)^2 + (1 \otimes a)(u \otimes 1 + v \otimes a)^2 \\ &= s^2 \otimes 1 + t^2 \otimes a^2 + u^2 \otimes a + v^2 \otimes a^3 \\ &= (s^2 + t^2 a^2) \otimes 1 + (u^2 + v^2 a^2) \otimes a. \end{aligned}$$

Since 1 and a are linearly independent over k, we deduce that $s^2 + t^2 a^2 = 1$ and $u^2 + v^2 a^2 = 0$, so $s + ta = 1$ and $u + va = 0$. Hence

$$H'(m)(x, y) = H'(m)(s \otimes 1 + t \otimes a, u \otimes 1 + v \otimes a) = (s + ta, u + va) = (1, 0).$$

The claim follows.

Now let G' be any reductive k'-group that contains H', and let $G := R_{k'/k}(G')$. Let H be any – not necessarily smooth – subgroup of G such that $q_{G'}(H_{k'}) \subseteq H'$. Then $H \subseteq R_{k'/k}(H')$ by Lemma 2.4, so

$$q_{G'}(H_{k'}) \subseteq q_{G'}(R_{k'/k}(H')_{k'}) = q_{H'}(R_{k'/k}(H')_{k'}).$$

But $q_{H'}\colon R_{k'/k}(H')_{k'} \to H'$ is not surjective by the claim, as H' is positive-dimensional. This shows that H' does not levitate in G.

We next give an example of a smooth wound unipotent subgroup U' of G' which does not levitate in G.

Example 4.19. Let k, k', K, $\overline{k}$ be as in Example 4.18, regard $R_{K/k'}(\mathbb{G}_m)$ as a k'-subgroup of GL_2, and set $H' := R_{K/k'}(\mathbb{G}_m) \cap \mathrm{SL}_2$. As we showed in Example 4.18, $q_{H'}$ is not surjective.

Let U' be the image of $R_{K/k'}(\mathbb{G}_m)$ under the canonical projection $\mathrm{GL}_2 \to \mathrm{PGL}_2$. Observe that U' is smooth and wound unipotent. Let $\phi'\colon \mathrm{SL}_2 \to \mathrm{PGL}_2$ be the canonical projection, and note that $(\phi')^{-1}(U') = H'$. Set $G' := \mathrm{PGL}_2$ as a k'-group, and $G := \mathscr{D}(R_{k'/k}(G'))$.

Suppose U is a levitation of U' in G. Since $\ker i_G$ is trivial, we have a canonical inclusion $U \subseteq R_{k'/k}(U')$ by Lemma 2.4. The Weil restriction functor is continuous so it preserves preimages, and hence

$$R_{k'/k}(\phi')^{-1}(U) \subseteq R_{k'/k}(\phi')^{-1}(R_{k'/k}(U')) = R_{k'/k}(H').$$

By functoriality of the counit (Diagram (7)), we have that $\phi' \circ q_{H'} = q_{U'} \circ R_{k'/k}(\phi')_{k'}$. Now $\phi'(\overline{k})$ gives a bijection from $H'(\overline{k})$ to $U'(\overline{k})$, and $q_{U'}(\overline{k})$ gives a surjection from $U(\overline{k})$ to $U'(\overline{k})$ by hypothesis. Moreover, the image of $R_{k'/k}(\phi')$ is G by [CGP, 1.3.4]. It follows that $q_{H'}$ is surjective, which is a contradiction. We conclude that U' does not levitate in G, smoothly or otherwise. On the other hand, every torus of G' levitates in G by Theorem 1.3.

5. Maximal subgroups

We turn to a study of maximal smooth subgroups of pseudo-reductive groups, using the results from the previous sections as a tool. There is a long history of studying maximal subgroups of a simple linear algebraic group over an algebraically closed field; in this context, "subgroup" is taken to mean "smooth subgroup". It is known that G has only finitely many conjugacy classes of maximal smooth subgroups of positive dimension and there are explicit lists of these due to various authors including Borel-de Siebenthal [BD], Dynkin [Dy, Dy1], Seitz [Se, Se1], Liebeck-Seitz [LS, LS1, LS2] and Testerman [Te]. This gives us a relatively good understanding of the maximal smooth subgroups of an arbitrary reductive group over an algebraically closed field. Over an imperfect field the situation is less clear – in Example 5.2 we demonstrate that a simple k-group can admit infinitely many isomorphism classes of positive-dimensional maximal smooth subgroups.

The smoothness requirement is indeed essential, as the following remark shows.

Remark 5.1. The notion of maximality is not very well-behaved if we allow non-smooth subgroups. For instance, let G be a split semisimple group defined over the finite field $\mathbb{F}_p$ and let H be any proper subgroup of G. For $r \geq 1$, let G_r be the rth Frobenius kernel of G, a normal subgroup of G. Since each G_r is infinitesimal, $\dim HG_r = \dim H < \dim G$. Let U be a root group of G that is not contained in H (one must exist, since G is generated by its root groups). Let n be the smallest integer such that the finite group $U \cap H$ does not contain a copy of α_{p^n}. Since $U \cap G_r \cong \alpha_{p^r}$ for all $r \geq 1$, we have an ascending chain of proper subgroups of G

$$HG_n \subseteq HG_{n+1} \subseteq \cdots$$

which never becomes stationary. On the other hand, if G is a pseudo-simple group then any positive-dimensional smooth subgroup H of G lies in a maximal smooth subgroup. To see this, let M be a smooth proper subgroup of G such that $M \supseteq H$ and M has maximal dimension. If K is a proper smooth subgroup of G and $M \subseteq K$ then $K^0 = M^0$, so $K \subseteq N_G(M^0)$, so $K \subseteq N_G(M^0)^{\mathrm{sm}}$. Note that $N_G(M^0)^{\mathrm{sm}}$ is proper as $\dim(M) > 0$ and G is simple. It follows that $N_G(M^0)^{\mathrm{sm}}$ is a maximal smooth subgroup.

Hence we restrict ourselves to smooth subgroups in the study of maximality.

Example 5.2. Let k be an imperfect field of characteristic 2 and let $k' = k(\sqrt{a})$ for some $a \in k \setminus k^2$. As in Example 4.18, consider the canonical embedding of $R_{k'/k}(\mathbb{G}_m)$ in GL_2. Taking the quotient of $R_{k'/k}(\mathbb{G}_m)$ by the center $Z(\mathrm{GL}_2) \cong \mathbb{G}_m$ gives us an embedding of $U := R_{k'/k}(\mathbb{G}_m)/\mathbb{G}_m$ into PGL_2. By choosing infinitely many elements $a \in k \setminus k^2$ that are pairwise distinct modulo k^2, we can construct infinitely many such subgroups U; they are pairwise non-isomorphic because the minimal fields of definition of their geometric unipotent radicals are all different. All such subgroups U constructed in this manner are smooth, connected, wound unipotent, 1-dimensional, and are maximal smooth in G.

Example 5.3. Let k'/k be a non-trivial purely inseparable finite field extension. Assume that k has characteristic 2. Let $G' = \mathrm{SO}_7$ and let H' be an irreducibly embedded copy of G_2 in G'; note that H' is maximal smooth in G'. Once again let $G := R_{k'/k}(G')$ and $H := R_{k'/k}(H')$. Observe that H is perfect since H' is simply connected [CGP, A.7.11]. Hence H is properly contained in the (smooth) derived subgroup $\mathscr{D}(G)$ of G. We claim that G is not perfect. To see this, consider the simply connected cover Spin_7 of G' and its centre $Z \cong \mu_2$. The central quotient $R_{k'/k}(\mathrm{Spin}_7)/R_{k'/k}(Z) \to \mathscr{D}(G)$ is an isomorphism by [CGP, 1.3.4]. But $\dim R_{k'/k}(Z) > 0$ since Z is infinitesimal, and so counting dimensions implies that G is not perfect. We conclude that H is not maximal.

Proof of Theorem 1.5. We first prove (i). Let H be a maximal smooth subgroup of G. Denote $H' := \pi'(H_{k'})$. If $H' = G'$ then Theorem 1.2(i) tells us that H_{k^s} contains an almost Levi subgroup of G_{k^s}. So henceforth we can assume that $H' \neq G'$. Then, by maximality, H is the largest smooth levitation of H' in G (which exists by Proposition 4.1).

Now let M' be a smooth proper subgroup of G' that contains H'. We claim that either of the conditions in the statement of the theorem is enough to ensure that M' smoothly levitates in G. Given the claim, again using Proposition 4.1, there exists a largest smooth levitation M of M' in G. Of course $H \subseteq M \subsetneq G$. Hence $H = M$ by maximality, so $H' = M'$. So indeed H' is a maximal smooth subgroup of G'. It remains to prove the claim.

Assume first that $G = R_{k'/k}(G')$. Then Lemma 2.3 says that there exists a smooth levitation of M' in G, namely $R_{k'/k}(M')$.

Next assume that H is a regular subgroup of G. Let T be a maximal torus of G that normalises H. By maximality, either H contains T or $HT = G$. Let $T' = \pi'(T_{k'})$, a maximal torus of G'. If H contains T then $T' \subseteq H' \subseteq M'$. If $HT = G$ then $H'T' = G'$, so H' must contain $\mathscr{D}(G')$, so every overgroup of H' is normal in G'. In both cases M' is normalised by T', so we can apply Theorem 1.4(i), which tells us that M' smoothly levitates in G.

We next prove (ii). For simplicity, we redefine/reuse some of the notation used in the proof of (i). Let H' be a maximal smooth subgroup of G'. By assumption there exists at least one smooth levitation of H' in G. Then, by Proposition 4.1, there exists a largest smooth levitation H of H' in G.

Let M be a smooth subgroup of G that properly contains H. By maximality of H' and H and since $\pi'(M_{k'})$ is smooth, we have that

$$H' = \pi'(H_{k'}) \subsetneq \pi'(M_{k'}) = G'.$$

That is, M is a smooth levitation of G' in G. But then Theorem 1.2(i) tells us that M_{k^s} contains an almost Levi subgroup of G_{k^s}. This completes the proof. □

Question 5.4. If H is a smooth maximal subgroup of G and $\pi'(H_{k'})$ is properly contained in G', must $\pi'(H_{k'})$ be a maximal smooth subgroup of G'? The answer is yes under the hypotheses of Theorem 1.5, but we do not know whether this is true in general.

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