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Voronoi Cells: Or How to Find the Nearest Bakery

Sarah Hess[1] • Angélica Torres[2]
Mirte van der Eyden[3]

Deciding which mall, hospital or school is closest to us is a problem we face everyday. It even comes on holidays with us, when we optimize our plans to make sure that we have enough time to visit all the attractions we want to see. In this article, we show how concepts from metric algebraic geometry help us to rise to this task while planning a weekend trip to the Black Forest.

A Weekend trip to the Black Forest

Planning a vacation is never easy. You want to strike a balance between sightseeing, new food experiences, and relaxing. Striving for this equilibrium, however, can add even more stress to your holiday plans. Luckily, there are some mathematical tools that we can use to help with the planning process. Let us take a look at those and plan a weekend trip to the Black Forest together!

[1] Sarah Hess is supported by the *Studienstiftung des Deutschen Volkes* and the scholarship program of the University of Konstanz under the *Landesgraduiertenfördergesetz*.

[2] Angélica Torres is supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M).

[3] Mirte van der Eyden is supported by the Stand Alone Project P33122-N of the Austrian Science Fund (FWF)

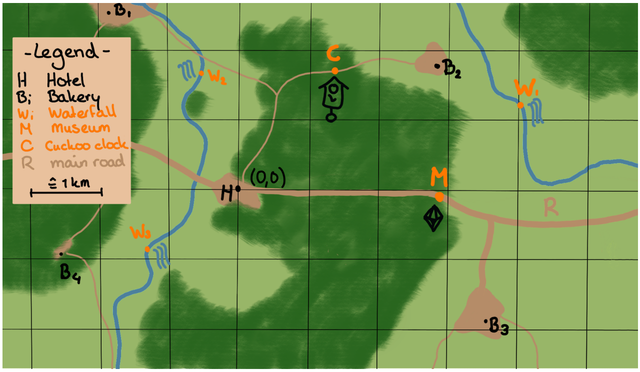


Figure 1: Map of the surroundings of our hotel. The origin is fixed at the hotel $H = (0, 0)$, and each square on the grid corresponds to a unit.

Our base camp is the hotel marked with an H on the map in Figure 1 and there are two things we absolutely cannot miss on our weekend trip: tasting the famous Black Forest cake (a classic!) and hiking in the beautiful mountains. We will plan our trip around these must-do's.

Day 1: Euclidean distance and Voronoi cells

On our first day, we plan to go for a nice walk, see some of the waterfalls marked on our map, and have a bite of Black Forest cake. Since we do not want to be too far away from the hotel at the end of the day, our goal is to finish at the bakery closest to the hotel.

If we bring a ruler on our vacation, then deciding which is the closest bakery is an easy task because we can simply measure the distances between the hotel and the bakeries on the map. In fact, we might even manage to pick the correct one by sight but, if we want to mathematically verify our guess for the closest bakery, then we need to explicitly calculate all distances.

Every point X on the map can be described by two coordinates (x_1, x_2). For example, our hotel H has coordinates $(0, 0)$, the waterfall W_1 has coordinates $(4, 1.2)$, and the bakery B_1 has coordinates $(-1.8, 2.7)$. The distance $d(X, Y)$ between two points $X = (x_1, x_2), Y = (y_1, y_2) \in \mathbb{R}^2$ can be computed by an application of the Pythagorean Theorem according to which

$$d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Let us use this for selecting the closest bakery to our hotel. We calculate:

$$\begin{aligned} d(H, B_1) &= \sqrt{(0+1.8)^2 + (0-2.7)^2} = \sqrt{10.53}, \\ d(H, B_2) &= \sqrt{(0-2.8)^2 + (0-1.8)^2} = \sqrt{11.08}, \\ d(H, B_3) &= \sqrt{(0-3.5)^2 + (0+1.9)^2} = \sqrt{15.86}, \\ d(H, B_4) &= \sqrt{(0+2.5)^2 + (0+0.9)^2} = \sqrt{7.06}. \end{aligned}$$

The minimal distance in the above list is $d^* = d(H, B_4) = \sqrt{7.06}$. So, we should plan our first day in such a way that our tour ends at bakery B_4.

To make this decision we solved two mathematical problems:

1. Computation of the distance between two points.
2. Minimization of the distance between a point and a finite set of points.

For solving the first problem, we used the *Euclidean distance*. This notion of distance is familiar because it represents the length of the line segment connecting the two end-points. In our particular example of a 2-dimensional map, the points only have two coordinates. However, it is possible to generalize the Euclidean distance to n dimensions by setting distance $d(X, Y)$ between two points $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ to be

$$d(X, Y) := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

For solving the second problem, we minimized the distance between the hotel H and the finite set $\mathcal{S} = \{B_1, B_2, B_3, B_4\}$ of all the bakeries on our map. It is again possible to generalize this concept to n dimensions. Given a point $X \in \mathbb{R}^n$ and a finite set of points $\mathcal{S} \subseteq \mathbb{R}^n$, the *minimal distance* $d(X, \mathcal{S})$ from X to $\mathcal{S}$ is given by the minimum of the individual distances $d(X, Y)$ for all the points $Y \in \mathcal{S}$. Any $Y \in \mathcal{S}$ with $d(X, Y) = d(X, \mathcal{S})$ is said to *minimize* the distance between X and $\mathcal{S}$.

Let us now get back to planning our tour for the first day. Having determined that we would end our excursion at bakery B_4, we must now choose a sight to explore along the way. From looking at our map, it is clear that waterfall W_3 is a good choice since all other attractions are much further away from bakery B_4. If we want to verify our intuition, then we have to show that B_4 minimizes the distance between the waterfall W_3 and the set $\mathcal{S} = \{B_1, B_2, B_3, B_4\}$ of all bakeries on the map. However, it can be exhausting to carry out these computations each time we want to decide on something.

We propose a different strategy. The *Voronoi cell* $\mathrm{Vor}_{\mathcal{S}}(B_i)$ of the bakery B_i is the region around B_i on our map for which B_i minimizes the distance between the set of all bakeries $\mathcal{S} = \{B_1, B_2, B_3, B_4\}$ and any point X of that region. In

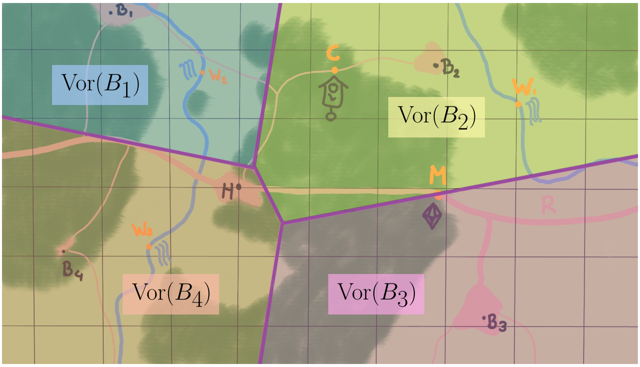


Figure 2: Surroundings of our hotel $H = (0, 0)$ with Voronoi Cells of the bakeries delimited by the purple lines.

other words, $\text{Vor}(B_i)$ consists of the points X for which all bakeries are at least as far away from X as B_i. Mathematically,

$$\text{Vor}_{\mathcal{S}}(B_i) := \{X \in \mathbb{R}^2 \mid d(X, \mathcal{S}) = d(X, B_i)\}.$$

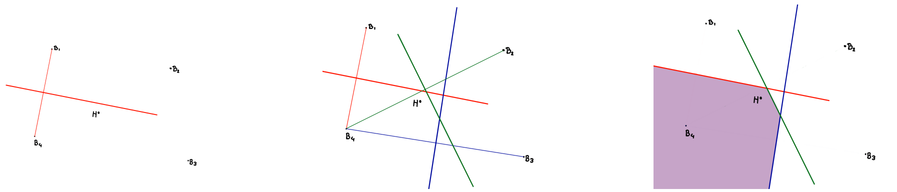
Figure 2 shows the Voronoi cells of the bakeries delimited by purple line segments. From this, we see that the hotel H, the waterfall W_3, and the bakery B_4 all belong to $\text{Vor}_{\mathcal{S}}(B_4)$, while all other sights lie in different Voronoi cells. Our intuition was indeed correct: the best plan is to explore waterfall W_3 during the first day in order to stay in $\text{Vor}_{\mathcal{S}}(B_4)$.

As was the case with the definition of the Euclidean distance, the definition of a Voronoi cell can also be generalized to n dimensions and any finite set. For a fixed finite set $\mathcal{S} = \{S_1, \ldots, S_m\}$ of points in $\mathbb{R}^n$, the Voronoi cells are defined as

$$\text{Vor}_{\mathcal{S}}(S_i) := \{X \in \mathbb{R}^n \mid d(X, \mathcal{S}) = d(X, S_i)\}.$$

In general, the number of Voronoi cells is the same as the number of elements in $\mathcal{S}$. In our map, we have 4 Voronoi cells, one for each bakery. The Voronoi cells also have a number of useful mathematical properties. For instance, Voronoi cells are *convex* sets, which means that the line segment between any two points from the set lies entirely in that set. Hence, in our example, this means that if we walk in a straight line between any two points in $\text{Vor}_{\mathcal{S}}(B_4)$, then we are guaranteed to stay in $\text{Vor}_{\mathcal{S}}(B_4)$. That is, B_4 remains the bakery closest to us during our hike.

Moreover, Voronoi cells are delimited by linear spaces (In our example, the linear spaces are straight lines that contain the purple line segments). This allows us to geometrically construct Voronoi cells in the way indicated in



(a) Drawing the bisector perpendicular to the connection line of B_4 and B_1.

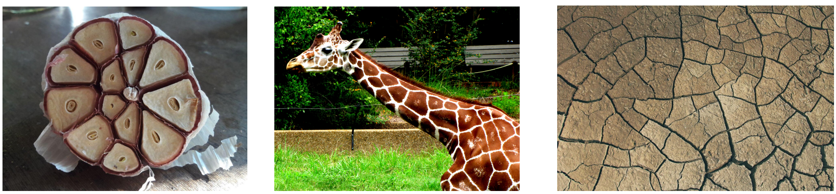
(b) Repetition for pairings with B_2 and B_3.

(c) Determination of the region $\mathrm{Vor}_{\mathcal{S}}(B_4)$ around B_4 cut out by all bisectors.

Figure 3: Construction of $\mathrm{Vor}_{\mathcal{S}}(B_4)$.

Figure 3. There $\mathrm{Vor}_{\mathcal{S}}(B_4)$ is delimited by the three lines highlighted in red, blue and green. The other Voronoi cells can be similarly constructed and together they split our map in four regions (see Figure 2). The partition of the maps into its Voronoi cells is known as the *Voronoi diagram* of $\mathcal{S}$.

Voronoi cells are extensively studied in algebraic geometry and, as mathematical as their definition is, they appear often in nature, as can be seen in Figure 4.



(a) Garlic loafs.

(b) Giraffe.

(c) Desert.

Figure 4: Voronoi diagrams in nature.

Turning our attention back to the planning of our weekend trip, we now understand that the Voronoi diagram of $\mathcal{S} = \{B_1, B_2, B_3, B_4\}$ allows us to list all the attractions contained in each Voronoi cell:

We see that the museum belongs to $\mathrm{Vor}_{\mathcal{S}}(B_2)$ and $\mathrm{Vor}_{\mathcal{S}}(B_3)$, due to the fact that M lies in the boundary of the two Voronoi cells. In other words, M is as close to B_2 as it is to B_3. Indeed, all the points equidistant to at least two bakeries together form the purple line segments in Figure 2. These segments are called the *medial axis* and their properties are studied in the area of metric algebraic geometry [2].

Voronoi Cell	Contained Attractions
$\text{Vor}_{\mathcal{S}}(B_1)$	B_1, W_2
$\text{Vor}_{\mathcal{S}}(B_2)$	B_2, C, W_1, M
$\text{Vor}_{\mathcal{S}}(B_3)$	B_3, M
$\text{Vor}_{\mathcal{S}}(B_4)$	B_4, H, W_3

Table 1: Attractions contained in the four Voronoi cells.

Day 2: Voronoi cells for algebraic plane curves

On our second day, we plan to go for a more challenging hike through the beautiful hills and woods. We wish to visit the cuckoo clock C, some waterfalls W_i and, maybe, even the museum M. Although we can pack some snacks, bringing a full lunch simply adds too much weight to our backpacks. So, we have to find a way to organize lunch to meet us while we are out. Fortunately, our hotel offers a service called the *Express Lunch Lane* where they deliver lunch to any point along the main road $\mathcal{R}$, which is the thick brown curve running from east to west through the hotel H on the map. This is a great option for us. For example, if we expect to be at the cuckoo clock around lunch time, then this allows us to go to the closest point on the main road to pick up our lunch there. This saves us lots of time and energy compared to walking all the way back to our hotel.

The problem now is that we don't know in advance if we will really be at the cuckoo clock around noon. The tour might unexpectedly take longer. For this reason, we will use a strategy that always allows us to determine the closet point on the main read from our current position. Although we know how to calculate distance from planning the first day of our trip, we face an even greater challenge: we are no longer minimizing the distance between a point and a finite set (of bakeries), but rather the distance between a point and the infinite set of points (forming the main road). So, our simple computations from before are no longer applicable. Fortunately, we are not out of options. Metric algebraic geometry also provides us with a way to compute the Voronoi diagram of the curve that represents the main road.

Formally, in algebraic geometry, an *algebraic plane curve* is a set of points in the plane that solve the equation $f(x_1, x_2) = 0$ for a given polynomial f. For example, the main road $\mathcal{R}$ in our particular case is the algebraic plane curve defined by the polynomial

$$f(x_1, x_2) = 155x_1^5 - 1145x_1^4 - 207x_1^3 + 12464x_1^2 - 28840x_1 - 100000x_2. \quad (1)$$

(see the right plot in Figure 5). So, mathematically, $\mathcal{R}$ corresponds to the set

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \mid 155x_1^5 - 1145x_1^4 - 207x_1^3 + 12464x_1^2 - 28840x_1 - 100000x_2 = 0\right\}.$$

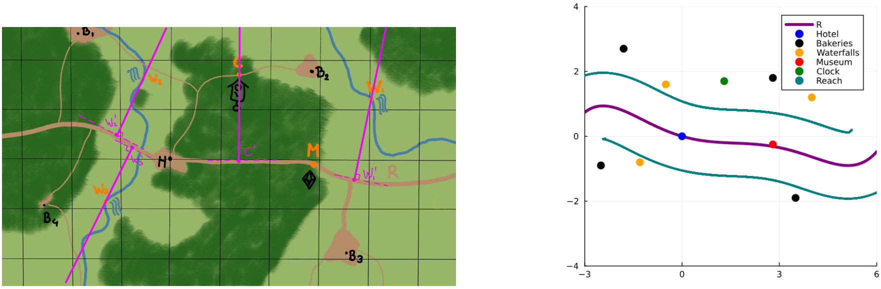


Figure 5: (Left) The map with subsets of the Voronoi cells of W_1', C', W_3' and W_2'. (right) A plot of the curve $f(x_1, x_2) = 0$ for f as in (1), and the boundary of the region of the plane with distance to the curve smaller than the reach.

The definition of a Voronoi cell of a point S_i in a finite set $\mathcal{S} = \{S_1, \ldots, S_m\}$ can now be generalized to a point K over an algebraic plane curve $\mathcal{K}$. The Voronoi cell of the point K consist of all the points in the plane that are closer to K than to all the other points on the curve. That is,

$$\mathrm{Vor}_{\mathcal{K}}(K) := \left\{ X \in \mathbb{R}^2 \mid d(X, \mathcal{K}) := \min_{Y \in \mathcal{K}} d(X, Y) = d(X, K) \right\}. \tag{2}$$

Metric algebraic geometry now tells us that $\mathrm{Vor}_{\mathcal{K}}(K)$ is generally a 1-dimensional set contained in the *normal space* of $\mathcal{K}$ through K. That is, a line through K perpendicular to the tangent of $\mathcal{K}$ in K.

Keeping this in mind, we interpret the magenta lines in Figure 5. The point W_1' belongs to the main road $\mathcal{R}$ and its tangent line is visualized by the magenta dashed segment. The corresponding normal space is thus the depicted perpendicular solid magenta line which can be shown to be contained in $\mathrm{Vor}_{\mathcal{R}}(W_1')$. So, the point on the road closest to any point on that solid magenta line is W_1'. We conclude that W_1 belongs to $\mathrm{Vor}_{\mathcal{R}}(W_1')$. This means that if we are at the waterfall W_1 around noon, then it is best for us to let our lunch be delivered to W_1'. Now, what about the cuckoo clock C? Can you determine where our lunch should be delivered to, if our hike goes according to plan and we are in C at noon?

We now generalize the notion of the medial axis over a finite set of points to the case of an algebraic plane curve. Just as before, there might be points in the plane that are equally close to more than one point on the curve, so they belong to more then one Voronoi cell. We call the collection of all points that belong to at least two distinct Voronoi cells of $\mathcal{K}$ the *medial axis* of $\mathcal{K}$. Thus, if we find ourselves on the medial axis of $\mathcal{R}$, then we need to decide which point on $\mathcal{R}$ the food should be delivered to. Else we might run into quite a pickle with the hotel and end up in the wrong place without any lunch.

Of course, we can avoid all of this trouble, if we simply stay clear off the medial axis. Indeed, in the right hand side of Figure 5, we see the main road in purple, and

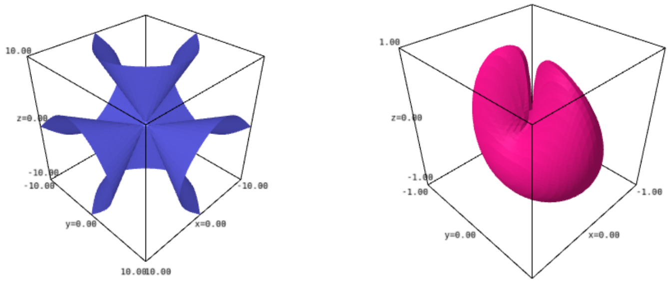


Figure 6: On the left, a Clebsch cubic. On the right, the variety defined by the polynomial $4x_1^2(x_1^2 + x_2^2 + x_3^2 + x_3) + x_2^2(x_2^2 + x_3^2 - 1)$.

two curves in teal that define a region surrounding the main road. This is a region that can be defined for algebraic plane curves $\mathcal{K}$ in general and it has the property that all the points inside it have a unique closest point. The distance between any point in the green curves and the main road is the minimal distance between $\mathcal{K}$ and its medial axis called the *reach* of $\mathcal{K}$. Thus, if we want to avoid confusion at all cost, we should order lunch only when we are closer to the main road $\mathcal{R}$ than its reach ε. This admissible area is the region between the teal curves. In particular, we are guaranteed to always find an unique closest point on the main road $\mathcal{R}$ as long as we stay in the gray area. Here, we do not run into any trouble whatsoever and always get our well-deserved lunch. With this in mind we can get our lunch in the main road and then head to the Museum to finish our second day of the trip.

A Final note

What we have done mathematically during our weekend trip planning, was to investigate the properties of the Voronoi diagram of a zero dimensional variety (the finite set $\mathcal{S}$ of bakeries from the first day) and of a 1-dimensional variety (the algebraic plane curve of the main road $\mathcal{R}$ from the second day). Mathematicians do not stop here. Quite the contrary, they study the Voronoi diagrams of varieties of any dimension, for example varieties as in Figure 6 in a 3-dimensional space. The above introduced concepts of Voronoi cells, medial axis, reaches and offset hypersurface can be generalized in all these instances. For the interested reader, we recommend the book *Voronoi Diagrams and Delauney Triangulations* [1] for the theoretical framework of Voronoi diagrams, and the book *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams* [4] for an introduction on the use of Voronoi diagrams in other areas. For a lighter reading, we recommend the Quanta article *How Geometry, Data and Neighbors Predict Your Favorite Movies* [3]. The latter is a science communication article which gives a friendly introduction to the topic.

To finish this journey, we highlight that our plan was done using the Euclidean distance. This was an adequate choice in our setting because it was possible to walk directly between any two places on the map. However, what would happen if we plan a trip to a city where we could only walk along the streets? There is plenty more to discover!

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Sarah Hess *is a doctoral student at the University of Konstanz.*

Angélica Torres *is a postdoctoral researcher at Centre de Recerca Matemàtica.*

Mirte van der Eyden *is a doctoral student at the University of Innsbruck.*

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Junior Editors
Matthew K. Tam, Carina Geldhauser
junior-editors@mfo.de

Senior Editor
Anja Randecker
senior-editor@mfo.de

Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9–11
77709 Oberwolfach
Germany

Director
Gerhard Huisken

Mathematisches
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