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Finite Simple Graphs II

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DIAMETER AND CONNECTIVITY OF FINITE SIMPLE GRAPHS II

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ABSTRACT. Let G be a finite simple non-complete connected graph on $[n] = \{1, \dots, n\}$ and $\kappa(G) \geq 1$ its vertex connectivity. Let $f(G)$ denote the number of free vertices of G and $\text{diam}(G)$ the diameter of G . The final goal of this paper is to determine all sequences of integers (n, f, d, k) with $n \geq 8$, $f \geq 0$, $d \geq 2$ and $k \geq 1$ for which there exists a finite simple non-complete connected graph on $[n]$ with $f = f(G)$, $d = \text{diam}(G)$ and $k = \kappa(G)$.

INTRODUCTION

In this paper, we are interested in the diameter, connectivity and the number of free vertices of a finite simple non-complete connected graph and how they are related to each other. Even though the approach of this paper is purely combinatorial, our initial motivation has arisen from the combinatorial study of binomial edge ideals [4, 7], where these graphical invariants are closely related to a classical invariant in commutative algebra.

Let G be a finite simple connected graph on the vertex set $[n] = \{1, \dots, n\}$ with $n \geq 3$ and the edge set $E(G)$. The *distance* $\text{dist}_G(i, j)$ of i and j in $[n]$ with $i \neq j$ is the smallest length, (i.e. the number of edges) of paths connecting i and j in G . Especially, if $\{i, j\} \in E(G)$, then $\text{dist}_G(i, j) = 1$. The *diameter* of G is

$$\text{diam}(G) = \max\{\text{dist}_G(i, j) : i, j \in [n]\}.$$

The *induced subgraph* of G on $T \subset [n]$ is the subgraph G_T of G on the vertex set T whose edges are those $\{i, j\} \in E(G)$ with $i \in T$ and $j \in T$. Let $\kappa(G) \geq 1$ denote the *vertex connectivity* of G . In other words, $\kappa(G)$ is the smallest cardinality of $T \subset [n]$ for which $G_{[n] \setminus T}$ is disconnected.

A vertex $i \in [n]$ is *free* if $\{i, j\} \in E(G)$ and $\{i, j'\} \in E(G)$ with $j \neq j'$, yield $\{j, j'\} \in E(G)$. In particular, a free vertex belongs to exactly one maximal clique (i.e., complete subgraph) of G . If $\Delta(G)$ is the clique complex of G , i.e. the simplicial complex whose faces are the vertex sets of the cliques of G , then a vertex is a free vertex of G if and only if it belongs to exactly one facet of $\Delta(G)$. Such a vertex is sometimes called a *simplicial vertex*. If a free vertex has degree equal to 1, then it is just the same as a *leaf* in classical graph theory. Free vertices of graphs play an important role in the literature of monomial and binomial edge ideals, especially in the case of chordal graphs, see for example [3]. Let $f(G)$ denote the number of free vertices of G .

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In the study of binomial edge ideals [4, 7], it was shown in [1, 8] that, when G is non-complete, one has

$$(1) \quad f(G) + \text{diam}(G) \leq n + 2 - \varkappa(G).$$

Our final goal is to determine the possible sequences (n, f, d, k) of integers for which there is a finite simple non-complete connected graph G on $[n]$ with $f = f(G)$, $d = \text{diam}(G)$ and $k = \varkappa(G)$. Toward our goal, in the previous paper [5], the possible sequences (n, f, d, k) of integers for which there is a finite simple non-complete connected graph G on $[n]$ with $f = f(G)$, $d = \text{diam}(G)$ and $k = \varkappa(G)$ satisfying $f + d = n + 2 - k$ was determined. Furthermore, finite simple non-complete connected graphs G on $[n]$ satisfying $f(G) + \text{diam}(G) = n + 2 - \varkappa(G)$ were classified. To simplify the notation, we let

$$\varphi(G) = f(G) + \text{diam}(G) + \varkappa(G)$$

and

$$\delta(G) = (f(G), \text{diam}(G), \varkappa(G)).$$

The present paper is a continuation of [5]. First, in Section 1, a purely combinatorial proof of (1) is given and also a classification of graphs G with $\varphi(G) = 3$, i.e. the minimum possible value for $\varphi(G)$, is provided. Second, in Section 2, for each integer $4 \leq i \leq n + 1$, a finite simple non-complete connected graph G on $[n]$ with $\varphi(G) = i$ is constructed. Moreover, given an integer $n \geq 8$, all possible sequences $\delta = (f, d, k)$ of integers for which there is a finite simple non-complete connected graph G on $[n]$ with $\delta = \delta(G)$ and $\varphi(G) = 4, 5, 6$ are determined. In Section 3, based on the computations done in the previous sections, we provide the complete list of all possible triples $\delta(G)$, which is our main goal in this paper. Finally, we reprove the classifications of $\delta(G)$ with $\varphi(G) = n + 1, n + 2$, given in [6] and [5], respectively.

Throughout the present paper, all graphs are finite and simple, and we denote the vertex set and the edge set of a finite simple graph G by $V(G)$ and $E(G)$, respectively, unless we mention something else.

1. BOUNDS ON $\varphi(G)$ AND ITS MINIMUM POSSIBLE VALUE

In this section, first we give an independent and totally combinatorial proof for the inequality (1). We need to recall some notation and terminologies from graph theory.

Recall that a connected graph G with n vertices is called k -connected for an integer k with $1 \leq k < n$, if for each $T \subseteq V(G)$ with $|T| < k$, the induced subgraph $G_{[n] \setminus T}$ of G is connected. In particular, any connected graph G is $\varkappa(G)$ -connected, but not $(\varkappa(G) + 1)$ -connected.

Let $P : u_1, u_2, \dots, u_\ell$ be a path of length $\ell - 1$ in a graph. Then we say that P is a (u_1, u_ℓ) -path and $u_2, \dots, u_{\ell-1}$ are *internal* vertices of P . Then, two paths $P : u_1, u_2, \dots, u_\ell$ and $Q : v_1, v_2, \dots, v_s$ are called *internally disjoint* if they do not have any common internal vertex. It is a well known fact in graph theory that a connected graph G is k -connected if and only if for any two distinct vertices u and v there exist at least k internally disjoint (u, v) -paths in G .

Theorem 1.1. *Let G be a finite simple non-complete connected graph on $[n]$. Then*

$$f(G) + \text{diam}(G) \leq n + 2 - \varkappa(G).$$

In particular,

$$3 \leq \varphi(G) \leq n + 2.$$

Proof. For simplicity, let $f = f(G)$, $d = \text{diam}(G)$ and $k = \varkappa(G)$. Since G is connected, we have $k \geq 1$, and hence $d - 2 \leq k(d - 2)$. Therefore, we have

$$(2) \quad d - 2 + k \leq k(d - 2) + k = k(d - 1).$$

Now, let u and v be two vertices of G such that $\text{dist}_G(u, v) = d$. Since G is k -connected, there exist at least k induced internally disjoint paths between u and v of length at least d . In particular, all internal vertices of these paths are not free. Thus, the number of non-free vertices of G is at least $k(d - 1)$. Therefore, we have

$$n \geq f + k(d - 1) \geq f + d - 2 + k$$

where the last inequality follows from (2). Hence we get the desired inequality. The lower bound 3 is also achieved by $f(G) = 0$, $\text{diam}(G) = 2$ and $\varkappa(G) = 1$, where the last two follow since G is non-complete and connected, respectively. \square

In [5] all graphs G on n vertices for which $\varphi(G) = n + 2$ were characterized. Namely, the case in which $\varphi(G)$ is maximized has been characterized. It is now interesting to characterize the case in which $\varphi(G)$ is minimized, i.e. $\varphi(G) = 3$.

First we recall the *join product* of two vertex disjoint graphs G_1 and G_2 which is denoted by $G_1 * G_2$. The vertex set of $G_1 * G_2$ is just $V(G_1) \cup V(G_2)$ and its edge set is

$$E(G_1 * G_2) = E(G_1) \cup E(G_2) \cup \{\{i, j\} : i \in V(G_1), j \in V(G_2)\}.$$

Also recall that a vertex v of a connected graph G is called a *cut vertex* if the graph $G - v$ is disconnected. In this paper, we denote the complete graph with n vertices by K_n .

Theorem 1.2. *Let G be a finite simple non-complete connected graph on $[n]$. Then the following hold:*

- (1) *Let $n < 9$. Then there is no graph G with $\varphi(G) = 3$.*
- (2) *Let $n \geq 9$. Then $\varphi(G) = 3$ if and only if $G = K_1 * H$ where H is a disconnected graph with no free vertex.*

Proof. Let G be a finite simple non-complete connected graph with $\varphi(G) = 3$, which implies that $f(G) = 0$, $\varkappa(G) = 1$ and $\text{diam}(G) = 2$. Since $\varkappa(G) = 1$, G has a cut vertex v . Then $G - v$ is a disconnected graph with the connected components H_1, \dots, H_s with $s \geq 2$. Since $\text{diam}(G) = 2$, it follows that any vertex in H_i for $i = 1, \dots, s$ is adjacent to the vertex v in G . This implies that $G = v * (G - v)$. If H_i for some $i = 1, \dots, s$ has a free vertex, then according to the construction of the join product, this vertex is a free vertex in G as well, which is a contradiction to the assumption $f(G) = 0$. Thus, none of H_i 's has a free vertex, and hence $|V(H_i)| \geq 4$ for each $i = 1, \dots, s$. This argument in particular implies that for $n < 9$, there is no desired graph and for $n \geq 9$ the statement (2) of the theorem holds. \square

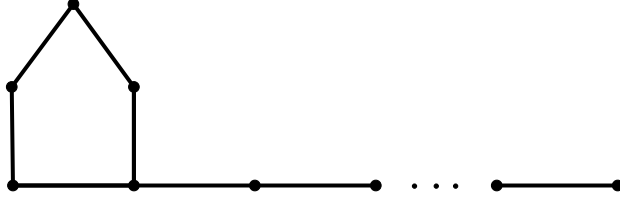


FIGURE 1. The graph G on $[n]$ with $\delta(G) = (1, n - 3, 1)$

2. POSSIBLE VALUES OF $\varphi(G)$

After considering the minimum and maximum possible values for $\varphi(G)$, it natural to ask if any possible number between 3 and $n + 2$ can be achieved by the invariant $\varphi(G)$ for some graph G . Motivated by a conjecture on Cohen-Macaulay property of binomial edge ideals of graphs posed by Bolognini et al. in [2], and the fact that any connected graph with Cohen-Macaulay binomial edge ideal has connectivity equal to 1, we provide the desired graphs G to answer the aforementioned question with $\varkappa(G) = 1$.

Let G and H be two graphs on disjoint sets of vertices. Then we denote the *disjoint union* of G and H by $G \cup H$ which is the graph on the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$.

Let G be a graph on $[n]$. Then we say that a graph G' is obtained by *duplicating* a vertex, say n , if $V(G') = V(G) \cup \{n'\}$ where $n' \notin V(G)$ and

$$E(G') = E(G) \cup \{\{j, n'\} : j \in N_G(n)\}$$

where $N_G(n)$ is the set of all neighbors, i.e. the adjacent vertices, of n in G .

Theorem 2.1. *Given any integer $n \geq 8$, for any integer i with $4 \leq i \leq n + 1$, there exists a finite simple non-complete connected graph G on $[n]$ such that $\varkappa(G) = 1$ and $\varphi(G) = i$.*

Proof. First assume that $i = 4$. Let C_{n-2} be the $(n - 2)$ -cycle. Then, let $G = K_1 * (C_{n-2} \cup K_1)$. Then $f(G) = 1$, $\text{diam}(G) = 2$, $\varkappa(G) = 1$ and $\varphi(G) = 4$.

Next, suppose that $i = 5$. Let C_{n-3} be the $(n - 3)$ -cycle. Then, let $G = K_1 * (C_{n-3} \cup K_2)$. Then $f(G) = 2$, $\text{diam}(G) = 2$, $\varkappa(G) = 1$ and $\varphi(G) = 5$.

Now, assume that $6 \leq i \leq n - 2$. Let $P_i : 1 \dots, i$ be the path graph. Let G be the graph on $[n]$ obtained from P_i by duplicating the vertex 2 and $n - i - 1$ times duplicating the vertex $i - 1$ in P_i . Then $f(G) = 0$, $\text{diam}(G) = i - 1$, $\varkappa(G) = 1$ and $\varphi(G) = i$.

Next assume that $i = n - 1$. Then, let G be the graph depicted in Figure 1. Then $f(G) = 1$, $\text{diam}(G) = n - 3$, $\varkappa(G) = 1$ and $\varphi(G) = n - 1$.

Now suppose that $i = n$. Then, let G be the graph depicted in Figure 2. Then $f(G) = 1$, $\text{diam}(G) = n - 2$, $\varkappa(G) = 1$ and $\varphi(G) = n$.

Finally, assume that $i = n + 1$. Then, let G be the graph depicted in Figure 3. Then $f(G) = 2$, $\text{diam}(G) = n - 2$, $\varkappa(G) = 1$ and $\varphi(G) = n + 1$. \square

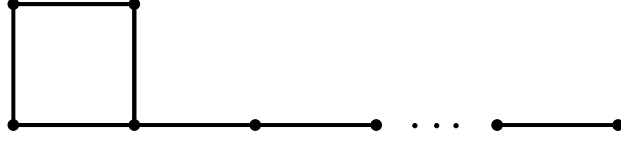


FIGURE 2. The graph G on $[n]$ with $\delta(G) = (1, n - 2, 1)$

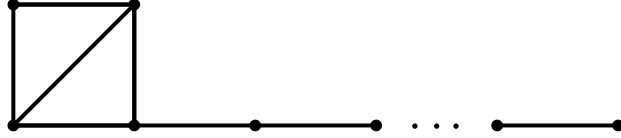


FIGURE 3. The graph G on $[n]$ with $\delta(G) = (2, n - 1, 1)$

In the proof of Theorem 2.1, in the case of $i = 4$, we provided a graph G with $n \geq 8$ vertices with $\delta(G) = (1, 2, 1)$. In the next example, we also provide graphs G with all other possible triples $\delta(G) = (f(G), \text{diam}(G), \varkappa(G))$ which yield $\varphi(G) = 4$.

- Example 2.2.** (1) Let $n \geq 8$ and let C_{n-4} be the $(n - 4)$ -cycle on $[n - 4]$ and $P_3 : n - 3, n - 2, n - 1$ be the path graph. Then, let $H = n * (C_{n-4} \cup P_3)$ and let G be the graph on $[n]$ obtained from H by removing the edge $\{n - 2, n\}$. Then $\delta(G) = (0, 3, 1)$ and $\varphi(G) = 4$.
- (2) Let $n \geq 4$ and let $G = K_{2, n-2}$ be the complete bipartite graph. Then $\delta(G) = (0, 2, 2)$ and $\varphi(G) = 4$.

In the proof of Theorem 2.1, in the case of $i = 5$, we provided a graph G on $n \geq 8$ vertices with $\delta(G) = (2, 2, 1)$. In the next example, we also provide graphs G with all other possible triples $\delta(G)$ which yield $\varphi(G) = 5$.

- Example 2.3.** (1) Let $n \geq 6$ and let $K_{2, n-4}$ be the complete bipartite graph on the bipartition of its vertex set $\{1, 2\} \cup \{3, 4, \dots, n - 2\}$. Then we set G to be the graph on $[n]$ obtained by adding the edges $\{1, n - 1\}$, $\{1, n\}$, $\{2, n\}$ and $\{n - 1, n\}$ to $K_{2, n-4}$. Then $\delta(G) = (1, 2, 2)$ and $\varphi(G) = 5$.
- (2) Let $n \geq 5$ and let $K_{2, n-4}$ be the complete bipartite graph on the bipartition of its vertex set $\{1, 2\} \cup \{3, 4, \dots, n - 2\}$. Then we set G to be the graph on $[n]$ obtained by adding the edges $\{1, n\}$, $\{2, n\}$ and $\{n - 1, n\}$ to $K_{2, n-4}$. Then $\delta(G) = (1, 3, 1)$ and $\varphi(G) = 5$.
- (3) Let $n \geq 6$ and let $G = K_{3, n-3}$ be the complete bipartite graph. Then $\delta(G) = (0, 2, 3)$ and $\varphi(G) = 5$.
- (4) Let $n \geq 7$ and let $K_{2, n-6}$ be the complete bipartite graph on the bipartition of its vertex set $\{1, 2\} \cup \{3, 4, \dots, n - 4\}$. Then we set G to be the graph on $[n]$ obtained by adding the edges $\{1, n - 2\}$, $\{2, n - 2\}$, $\{n - 3, n - 2\}$, $\{n - 2, n - 1\}$, $\{n - 1, n\}$ and $\{n - 3, n\}$ to $K_{2, n-6}$. Then $\delta(G) = (0, 4, 1)$ and $\varphi(G) = 5$.
- (5) Let $n \geq 6$ and let $K_{2, n-5}$ be the complete bipartite graph on the bipartition of its vertex set $\{1, 2\} \cup \{3, 4, \dots, n - 3\}$. Then we set G to be the graph on $[n]$

obtained by adding the edges $\{1, n-1\}$, $\{2, n-2\}$, $\{n-1, n\}$ and $\{n-2, n\}$ to $K_{2, n-3}$. Then $\delta(G) = (0, 3, 2)$ and $\varphi(G) = 5$.

In the proof of Theorem 2.1, in the case of $i = 6$, we provided a graph G with $n \geq 8$ vertices with $\delta(G) = (0, 5, 1)$. In the next example, we also provide graphs G with all other possible triples $\delta(G)$ which yield $\varphi(G) = 6$.

Example 2.4. (1) Let $n \geq 8$ and let $G = K_1 * (K_3 \cup C_{n-4})$. Then $\delta(G) = (3, 2, 1)$ and $\varphi(G) = 6$.

(2) Let $n \geq 8$ and let $H = 1 * C_{n-4}$ on the vertex set $\{1, 2, \dots, n-3\}$. Then we set G to be the graph on $[n]$ obtained by adding the edges $\{1, n-2\}$, $\{n-2, n\}$, $\{n-2, n-1\}$ and $\{n-1, n\}$ to H . Then $\delta(G) = (2, 3, 1)$ and $\varphi(G) = 6$.

(3) Let $n \geq 8$ and let $G = K_2 * (C_{n-4} \cup K_1 \cup K_1)$. Then $\delta(G) = (2, 2, 2)$ and $\varphi(G) = 6$.

(4) Let $n \geq 8$ and let $H = 1 * C_{n-4}$ with the vertex set $\{1, 2, \dots, n-3\}$. Then we set G to be the graph on $[n]$ obtained by adding the edges $\{1, n-2\}$, $\{n-2, n-1\}$ and $\{n-1, n\}$ to H . Then $\delta(G) = (1, 4, 1)$ and $\varphi(G) = 6$.

(5) Let $n \geq 8$ and let $G = K_3 * (C_{n-4} \cup K_1)$. Then $\delta(G) = (1, 2, 3)$ and $\varphi(G) = 6$.

(6) Let $n \geq 6$ and let $K_{2, n-5}$ be the complete bipartite graph with the bipartition of its vertex set $\{1, 2\} \cup \{3, 4, \dots, n-3\}$. Then we set G to be the graph on $[n]$ obtained by adding the edges $\{1, n-1\}$, $\{2, n-2\}$, $\{n-2, n-1\}$, $\{n-2, n\}$ and $\{n-1, n\}$ to H . Then $\delta(G) = (1, 3, 2)$ and $\varphi(G) = 6$.

(7) Let $n \geq 8$ and let $G = K_{4, n-4}$. Then $\delta(G) = (0, 2, 4)$ and $\varphi(G) = 6$.

(8) Let $n \geq 8$ and let C_8 be the 8-cycle on $[8]$. Then we set G to be the graph on $[n]$ obtained by $n-8$ times duplicating the vertex 1. Then $\delta(G) = (0, 4, 2)$ and $\varphi(G) = 6$.

(9) Let $n \geq 8$ and let $K_{3, n-7}$ be the complete bipartite graph with the bipartition of its vertex set $\{1, 2, 3\} \cup \{4, \dots, n-4\}$. Then we set G to be the graph on $[n]$ obtained by adding the edges $\{n-3, n-2\}$, $\{n-2, n-1\}$, $\{n-j, n\}$ and $\{j, n-j\}$ for each $j = 1, 2, 3$ to H . Then $\delta(G) = (0, 3, 3)$ and $\varphi(G) = 6$.

3. THE COMPLETE LIST OF POSSIBLE $\delta(G)$

In this section we make the complete list of all possible $\delta(G)$. First we start with the case $\varkappa(G) = 1$.

Theorem 3.1. Fix integers $n \geq 8$, $f \geq 0$ and $d \geq 2$. Then there exists a finite simple non-complete connected graph G on $[n]$ with $\delta(G) = (f, d, 1)$ if and only if one of the following conditions is satisfied:

- (1) $f = 0$ and $2 \leq d \leq n-3$;
- (2) $f = 1$ and $2 \leq d \leq n-2$;
- (3) $f \geq 2$, $d \geq 2$ and $f+d \leq n+1$.

Proof. Note that if $f(G) = 0$, then $\text{diam}(G)$ is never equal to $n-1$ or $n-2$. Also, if $f(G) = 1$, then $\text{diam}(G) \neq n-1$. Thus, if $\delta(G) = (f, d, 1)$ for some non-complete connected graph G , then one of the conditions (1), (2), (3) holds. For the converse, in the following, in each case (1), (2), (3) we construct a desired graph.

(1) Assume that $f = 0$ and $2 \leq d \leq n - 3$. If $d = 2, 3$, then $\varphi(G) = 3, 4$ which was discussed in Theorem 1.2 and Example 2.2. Now, let $4 \leq d \leq n - 3$. Let $P_{d+1} : 1, 2, \dots, d + 1$ be the path graph. Then, let G be the graph obtained by duplicating the vertex 2 and $n - d - 2$ times duplicating the vertex d . Then, $\delta(G) = (0, d, 1)$.

(2) Assume that $f = 1$ and $2 \leq d \leq n - 2$. If $d = 2$, then $\varphi(G) = 4$ which was discussed in Theorem 2.1. So, let $3 \leq d \leq n - 2$. Let $P_{d+1} : 1, 2, \dots, d + 1$ be the path graph. Then, let G be the graph obtained by $n - d - 1$ times duplicating the vertex 2. Then, $\delta(G) = (1, d, 1)$.

(3) Assume that $f \geq 2$, $d \geq 2$ and $f + d \leq n + 1$. First assume that $f + d = n + 1$. Then, let $P_{d+1} : 1, 2, \dots, d + 1$ be the path graph and let G be the graph obtained from P_{d+1} by $f - 2$ times duplicating the vertex $d + 1$. Then $\delta(G) = (f, d, 1)$. This case was also considered in [5, Theorem 1.4]. So, we assume that $f + d \leq n$.

If $f = 2$ and $d = 2$, then $\varphi(G) = 5$ which was discussed in Theorem 2.1. Now, let $f \geq 3$ and $d = 2$. Let $K_{1, n-1}$ be the complete bipartite graph with the bipartition of its vertex set $\{1\} \cup \{2, 3, \dots, n\}$. Let G be the graph on $[n]$ which is obtained from $K_{1, n-1}$ by adding the edges $\{j, j + 1\}$ for each $j = 2, \dots, n - f + 1$. Then, $\delta(G) = (f, 2, 1)$.

Now, let $f \geq 2$, $d \geq 3$ and $f + d \leq n - 2$. Let $P_{d+1} : 1, 2, \dots, d + 1$ be the path graph. Then, let H be the graph on $[d + 2]$ which is obtained by adding the edges $\{d, d + 2\}$ and $\{d + 1, d + 2\}$ to P_{d+1} . Now, let G be the graph on $[n]$ obtained from H by $f - 1$ times duplicating the vertex $d + 2$ and by $n - d - f - 1$ times duplicating the vertex 2 in H . Then, $\delta(G) = (f, d, 1)$.

Next, let $f \geq 2$, $d \geq 3$ and $f + d = n - 1$. Let $P_{d+1} : 1, 2, \dots, d + 1$ be the path graph. Then, let G be the graph on $[n]$ obtained from P_{d+1} by $f - 1$ times duplicating the vertex $d + 1$ and by duplicating the vertex 2. Then, $\delta(G) = (f, d, 1)$.

Finally, let $f \geq 2$, $d \geq 3$ and $f + d = n$. Let $P_{d+1} : 1, 2, \dots, d + 1$ be the path graph. Then, let H be the graph on $[d + 2]$ obtained from P_{d+1} by duplicating 2 and adding the edge $\{2, d + 2\}$. Then, let G be the graph on $[n]$ obtained from H by $f - 2$ times duplicating the vertex $d + 1$. Then, $\delta(G) = (f, d, 1)$. \square

Next, we consider the case $\text{diam}(G) = 2$.

Theorem 3.2. *Fix integers $n \geq 8$, $f \geq 0$ and $k \geq 2$. Then there exists a finite simple non-complete connected graph G on $[n]$ with $\delta(G) = (f, 2, k)$ if and only if one of the following conditions is satisfied:*

- (1) $f = 0$ and $2 \leq k \leq n - 2$;
- (2) $f = 1$ and $2 \leq k \leq n - 3$;
- (3) $f \geq 2$, $k \geq 2$ and $f + k \leq n$.

Proof. Note that for any non-complete connected graph G , we have $\varkappa(G) \leq n - 2$. If $f(G) = 1$, then $\varkappa(G) \leq n - 3$. In fact, if $\varkappa(G) = n - 2$, then each vertex of G is adjacent to at least $n - 2$ other vertices in G . Suppose that the vertex 1 is free. If 1 is adjacent to all other vertices of G , then G must be complete graph, since 1 is a free vertex. Thus, we may assume that there is no edge connecting the vertices 1 and 2 in G . This implies that 2 is adjacent to all vertices other than 1 which means

that 2 is also a free vertex of G . This contradicts the assumption that $f(G) = 1$. Thus, if $\delta(G) = (f, 2, k)$ for some non-complete connected graph G , then one of the conditions (1), (2), (3) holds. For the converse, in the following, in each case (1), (2), (3) we construct a desired graph.

(1) Assume that $f = 0$ and $2 \leq k \leq n - 5$. If $k = 2$, then $\varphi(G) = 4$ and the result follows from Example 2.2. So, let $3 \leq k \leq n - 5$. Let K_k be the complete graph on $[k]$, and let $P_{n-k-2} : k + 1, k + 2, \dots, n - 2$ and $P_2 : n - 1, n$ be path graphs. Then, let $H = K_k * (P_{n-k-2} \cup P_2)$. Then, let G be the graph on $[n]$ obtained by removing the edges $\{1, k + 2\}$, $\{2, n - 3\}$, $\{1, n - 1\}$ and $\{2, n\}$ from H . Then, $\delta(G) = (0, 2, k)$. Note that if $n - k = 5$, then the vertices $k + 2$ and $n - 3$ coincide.

Now, assume that $f = 0$ and $k = n - 4$. Let K_{n-4} be the complete graph on $[n - 4]$, let $P_2 : n - 3, n - 2$ and $P'_2 : n - 1, n$ be path graphs. Then, let $H = K_k * (P_2 \cup P'_2)$. Then, let G be the graph on $[n]$ obtained by removing the edges $\{1, n - 3\}$, $\{2, n - 2\}$, $\{1, n - 1\}$ and $\{2, n\}$ from H . Then, $\delta(G) = (0, 2, n - 4)$.

Next, assume that $f = 0$ and $k = n - 3$. Let K_{n-3} be the complete graph on $[n - 3]$ and let $P_3 : n - 2, n - 1, n$ be the path graph. Let $H = K_{n-3} * P_3$. Then let G be the graph on $[n]$ obtained by removing the edges $\{1, n - 2\}$, $\{2, n - 1\}$, and $\{3, n\}$ from H . Then, $\delta(G) = (0, 2, n - 3)$.

Next, assume that $f = 0$ and $k = n - 2$. Let K_{n-2} be the complete graph on $[n - 2]$ and let $P_2 : n - 1, n$ be the path graph. Let $H = K_{n-2} * P_2$. Then let G be the graph on $[n]$ obtained by removing the edges $\{1, n - 1\}$ and $\{2, n\}$ from H . Then, $\delta(G) = (0, 2, n - 2)$.

(2) Assume that $f = 1$ and $2 \leq k \leq n - 4$. If $k = 2$, then $\varphi(G) = 5$, and hence the result follows from Example 2.3. Now suppose that $3 \leq k \leq n - 4$. Let K_k be the complete graph on $[k]$ and let $P_{n-k-1} : k + 1, k + 2, \dots, n - 1$ be the path graph. Then, let $H = K_k * (P_{n-k-1} \cup n)$. Then, let G be the graph on $[n]$ obtained by removing the edges $\{1, k + 2\}$ and $\{2, n - 2\}$ from H . Then, $\delta(G) = (1, 2, k)$. Note that if $n - k = 4$, then the vertices $k + 2$ and $n - 2$ coincide.

Now, assume that $f = 1$ and $k = n - 3$. Let K_{n-3} be the complete graph on $[n - 3]$, let $P_2 : n - 2, n - 1$ be the path graph. Then, let $H = K_{n-3} * (P_2 \cup n)$. Then, let G be the graph on $[n]$ obtained by removing the edges $\{1, n - 2\}$ and $\{2, n - 1\}$ from H . Then, $\delta(G) = (1, 2, n - 3)$.

(3) First assume that $f = 2$ and $2 \leq k \leq n - 4$. Then let K_k be the complete graph on $[k]$, let $P_{n-k-2} : k + 1, k + 2, \dots, n - 2$ and $P_2 : n - 1, n$ be the path graphs. Then, let $H = K_k * (P_{n-k-2} \cup P_2)$. Then, let G be the graph on $[n]$ obtained by removing the edges $\{1, n - 1\}$ and $\{2, n\}$ from H . Then, $\delta(G) = (2, 2, k)$.

Now, assume that $f = 2$ and $k = n - 3$. Then let K_{n-3} be the complete graph on $[n - 3]$ and let $P_3 : n - 2, n - 1, n$ be the path graph. Then, let $H = K_{n-3} * P_3$. Then, let G be the graph on $[n]$ obtained by removing the edges $\{n - 1, n - 2\}$ and $\{1, n\}$ from H . Then, $\delta(G) = (2, 2, n - 3)$.

Next, assume that $f = 2$ and $k = n - 2$. Then, let $G = K_{n-2} * (K_1 \cup K_1)$. Then, $\delta(G) = (2, 2, n - 2)$.

Finally, assume that $f \geq 3$ and $2 \leq k \leq n - f$. Then let K_k be the complete graph on $[n] \setminus [n - k]$ and let C_{n-k} be the $(n - k)$ -cycle on the vertex set $[n - k]$. Let

$H = K_k * C_{n-k}$. Then let G be the graph on $[n]$ obtained by removing the edges $\{j, j+1\}$ for each $j = 1, \dots, f-1$ from H . Then, $\delta(G) = (f, 2, k)$. \square

Finally, we consider $\varkappa(G) \geq 2$ and $\text{diam}(G) \geq 3$.

Theorem 3.3. *Fix integers $f \geq 0$, $d \geq 3$ and $k \geq 2$. Then there exists a finite simple non-complete connected graph G on $[n]$ with $\delta(G) = (f, d, k)$ if and only if*

$$n \geq k(d-1) + \max\{2, f\}.$$

Proof. First let G be a non-complete connected graph on $[n]$ with $\delta(G) = (f, d, k)$ where $f \geq 0$, $d \geq 3$ and $k \geq 2$. Let $Q_1 : 1, 2, \dots, d+1$ be a longest path in G . Since G is k -connected, there exist k internally disjoint induced paths between the vertices 1 and $d+1$, say Q_1, \dots, Q_k . Since $d = \text{diam}(G)$, each Q_i for $i = 2, \dots, k$ has length at least d . Since none of the internal vertices of each Q_i for $i = 1, \dots, k$ is a free vertex, it follows that the number of non-free vertices of G is at least $k(d-1)$. Note that the vertices 1 and $d+1$ can be free or not. Then, we deduce that

$$n \geq k(d-1) + \max\{2, f\}.$$

Conversely, suppose that $n \geq k(d-1) + \max\{2, f\}$. Let Q_1, Q_2, \dots, Q_k be path graphs of length $d-2$ on disjoint sets of vertices. Let i and $i+k$ be the leaves of Q_i for each $i = 1, \dots, k$. Let $H = Q_1 \cup Q_2 \cup \dots \cup Q_k$ with $V(H) = [k(d-1)]$. Then, let H' be the graph on $[k(d-1) + 2]$ obtained by adding the edges $\{j, k(d-1) + 1\}$ and $\{k+j, k(d-1) + 2\}$ for each $j = 1, \dots, k$.

First assume that $f = 0$. Then, let G be the graph on $[n]$ obtained by $n - (k(d-1) + 2)$ times duplicating the vertex 1 in H' . Then, $\delta(G) = (0, d, k)$.

Next assume that $f = 1$. Then, let G be the graph on $[n]$ obtained by $n - (k(d-1) + 2)$ times duplicating the vertex 1 in H' and by adding the edges $\{k+i, k+j\}$ for all $1 \leq i < j \leq k$. Then, the vertex $k(d-1) + 2$ is the only free vertex of G , and hence $\delta(G) = (1, d, k)$.

Finally, assume that $f \geq 2$. Let $e = \{s, t\}$ be an edge of Q_k . Then, let H'' be the graph on $[k(d-1) + 3]$ by adding the edge $\{s, k(d-1) + 3\}$ and $\{t, k(d-1) + 3\}$ to H' . Then let G be the graph on $[n]$ obtained from H'' by $f-3$ times duplicating the vertex $k(d-1) + 3$ and $n - (k(d-1) + f)$ times duplicating the vertex 1 and then by connecting all the neighbors of the vertex $k(d-1) + 1$ in the new graph to each other by an edge as well as connecting all the neighbors of the vertex $k(d-1) + 2$ to each other. Then, $\delta(G) = (f, d, k)$. \square

In the next two corollaries we consider, in particular, the characterizations due to $\varphi(G) = n + 2$ and $\varphi(G) = n + 1$ to recover [5, Theorem 1.4] and [6, Theorem 3.3], respectively.

Corollary 3.4. *Fix integers $n \geq 8$, $f \geq 0$, $d \geq 2$ and $k \geq 1$. Then there exists a finite simple non-complete connected graph G on $[n]$ with $\varphi(G) = n + 2$ and $\delta(G) = (f, d, k)$ if and only if $f + d + k = n + 2$ and one of the following conditions is satisfied:*

- (1) $f \geq 2$, $k \geq 1$ and $d = 2$;
- (2) $f \geq 2$, $k = 1$ and $d \geq 2$.

Proof. First suppose that G is a graph on $[n]$ with $\varphi(G) = n + 2$, $f = f(G) \geq 0$, $k = \varkappa(G) \geq 2$ and $d = \text{diam}(G) \geq 3$. Then by Theorem 3.3, we have

$$f + d + k - 2 \geq k(d - 1) + \max\{2, f\}.$$

Thus, we have

$$(k - 1)(d - 2) \leq f - \max\{2, f\},$$

which is impossible. Therefore, if $\varphi(G) = n + 2$, then $k = 1$ or $d = 2$, and hence the result follows from Theorem 3.1 and Theorem 3.2. \square

Corollary 3.5. *Fix integers $n \geq 8$, $f \geq 0$, $d \geq 2$ and $k \geq 1$. Then there exists a finite simple non-complete connected graph G on $[n]$ with $\varphi(G) = n + 1$ and $\delta(G) = (f, d, k)$ if and only if $f + d + k = n + 1$ and one of the following conditions is satisfied:*

- (1) $f \geq 2$, $k \geq 1$ and $d = 2$;
- (2) $f \geq 2$, $k = 1$ and $d \geq 2$;
- (3) $f \geq 2$, $k = 2$ and $d = 3$.

Proof. First suppose that G is a graph on $[n]$ with $\varphi(G) = n + 1$, $f = f(G) \geq 0$, $k = \varkappa(G) \geq 2$ and $d = \text{diam}(G) \geq 3$. Then by Theorem 3.3, we have

$$f + d + k - 1 \geq k(d - 1) + \max\{2, f\}.$$

Thus, we have

$$(k - 1)(d - 2) \leq f - \max\{2, f\} + 1,$$

which is possible if and only if $f \geq 2$, $d = 3$ and $k = 2$. Therefore, if $\varphi(G) = n + 1$, then $k = 1$ or $d = 2$ or $f \geq 2$, $d = 3$ and $k = 2$. Then the result follows from Theorem 3.1, Theorem 3.2 and Theorem 3.3. \square

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