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Calculus of Variations

Organized by
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Abstract. The Calculus of Variations is at the same time a classical subject, with long-standing open questions which have generated exciting discoveries in recent decades, and a modern subject in which new types of questions arise, driven by mathematical developments and emergent applications. It is also a subject with a very wide scope, touching on interrelated areas that include geometric variational problems, optimal transportation, geometric inequalities and domain optimization problems, elliptic regularity, geometric measure theory, harmonic analysis, physics, free boundary problems, etc. The workshop will balance the traditional interests of past conferences with new emerging perspectives. The topics described in this proposal are linked to each other via the methods of Calculus of Variations that are employed, and it is our belief that a meeting with such a large group of experts will lead to substantial advances in these areas, as well as building bridges between them.

Mathematics Subject Classification (2020): 49-06.

Introduction by the Organizers

The workshop *Calculus of Variations* organized by Lia Bronsard (Hamilton), Maria Colombo (Lausanne), László Székelyhidi (Leipzig) and Yoshihiro Tonegawa (Tokyo) was very well attended with 41 in person participants, with broad geographic and gender representation. In this workshop we observed exactly what we hoped for: a mixture of theoretical and applied problems of interest to all participants. The large number of on-site participants allowed for a schedule where all talks were in-person. The carefully weighed and prepared hygiene measures by the MFO staff and a schedule allowing for ample free discussion time led to a productive, convivial atmosphere where participants interacted with new people, new directions and new techniques, and were also able to advance in current and new projects. Several talks included very recent, as yet unpublished results; a level of timeliness that is almost never observed in virtual talks. In addition, a large number of talks were blackboard talks and this was very much appreciated as it leads to slower pace and better understanding of the presentations.

Overall we had 15 "long talks" of 45 minutes+ 15 minutes for discussions, as well as 12 "short talks" of 25 minutes+5 minutes for discussions. In addition, there were two 30 minute talks in a special evening session after dinner on Tuesday. This mix of the format allowed us to offer many of the more junior participants to present their work, as well as plenty of space between talks for informal discussions. MFO is the perfect venue for this. In particular the short talks were very well received, as their format make their delivery very dynamic.

One of the topical focus points of the workshop was the study of problems in the calculus of variations which arise or are inspired by physical applications. Vincent Millot presented new results (joint with T. Gabard) on the singular limit to multiphase transitions for problems involving a fractional perimeter involving a nonlocal term for vector valued functions. This problem present new features related to nonlocal minimal surfaces and some partial regularity and open questions were presented. Radu Ignat presented the resolution of some conjecture related to the minimality of the vortex solution for Ginzburg-Landau systems. When considering N-dimensional maps, he presented the proof of the symmetry of the ground state of the Ginzburg-Landau system when defined on the unit ball with boundary data corresponding to a vortex of topological degree one and in the general case when the singular parameter $\varepsilon > 0$. For $N = 4, 5, 6$, the same result holds when considering curl-free maps. Hans Knüpfer presented fascinating results on the Gamma limit for zigzag walls. Using *Gamma* convergence techniques, he was able to recover observed zigzag walls in a supercritical regime for thin strip with appropriate boundary conditions. Filip Rindler presented a new framework, based on the notion of varifold, for the homogenization of elasto-plasticity driven by dislocation motion, thus allowing this classical problem to be attacked by the powerful tools of geometric measure theory. Angkana Rüland presented the state of the art on the various scaling regimes between rigidity and flexibility for variational problems in elasticity arising in the context of shape-memory allows - an exciting class of problems with close connections to the Nash-Kuiper theory of isometric immersions.

Elise Bonhomme presented recent results on variational methods applied to discrete models in brittle damage involving several parameters. Using very delicate analysis to study the convergence rates to effective limit models, she was able to characterize the appropriate effective models according to various limiting regime, generalizing previous results by Babadjian, Iurlano and Rindler. Michael Novack presented new impressive results (joint with F. Maggi, D. Restrepo and A. Skorobogatova) on soap films, Plateau's laws and a connection with an Allen–Cahn free boundary problem. This result is the first to capture features of real soap film and present a very important generalization of the homotopic spanning condition of Harrison-Pugh that promises to be very helpful in the understanding of soap films and other minimal surfaces problems. Dominik Stantejsky presented new results on minimizing Harmonic Maps with Planar boundary anchoring. Using a reflection method and new monotonicity formula derived for these new type of boundary conditions, he was able (jointly with L. Bronsard and A. Colinet), to study the symmetry and the regularity of minimizers and used it to describe the location of point defects. Dean Louizos presented new technical results (jointly with L. Bronsard and D. Stantejsky), related to a Γ convergence result for the Landau-de Gennes functional for nematic liquid crystal where a colloid with planar anchoring is imposed in the presence of a weak magnetic field. In particular, he was able to obtain the optimal orientation of general shapes, including multiply connected surfaces. Riccardo Cristoferi presented recent results (joint with R. Ferreira, I. Fonseca and J. Iglesias) on the monotonicity of the jump set for denoising models. His presentation (at 8pm) was very interactive albeit on a very technical subject and very much appreciated by the audience. Finally, Marc Pegon presented (joint with M. Goldman and B. Merlet) exciting results (at 8:30pm) on an isoperimetric problem involving competition between the perimeter and a nonlocal term related to the liquid drop model, in case of large masses, also very well received by a large audience.

The research talks on the aspects involving minimal surfaces and free boundaries were well represented and the audience could appreciate the connections to other relatively distant fields as well. R. Tione (joint with J. Hirsch and C. Mooney) presented a breakthrough result, namely the resolution of Lawson-Osserman conjecture for the two dimensional case which concerns the minimal graphs with Lipschitz regularity, employing a variety of techniques at the heart of the conference theme, such as techniques arising in the context of elliptic differential inclusions, the theory of quasiconformal mappings, a fine analysis of the codimension one case. While many talks in free boundaries regarded minimizers of variational problems, including the boundary regularity theory and global configurations, the workshop featured important discussions and results regarding the analysis of stable configurations, which are as well very important to study and observed in nature. One context where there are striking differences between our understanding of minimizers vs stable solutions is the one of the Allen-Cahn equation, an approximation of the minimal-surface equation which takes into account scale-dependent effects. This was the context of J. Serra talk (joint with Chan, Fernandez-Real, Figalli). He presented a work on the classification of stable solutions of the "free-boundary version" of the Allen-Cahn equation, namely an Alt-Caffarelli obstacle problem for non-local setting. On the same theme, F. Franceschini (joint with A. Figalli) described the kind of blow up of stable solutions to semilinear elliptic equations at singular points, obtaining consequences on the dimension of the singular set.

U. Menne (joint with C. Scharrer) presented a fundamental result relating geodesic diameter and curvature in the framework of geometric measure theory.

A. Pigati (joint with G. De Philippis and A. Halavati) presented a recent work showing a tilt-excess decay of abelian Higgs model motivated by Kelei Wang's work on the Allen-Cahn functional. Velichkov (joint with R. Ognibene) presented regularity results for free interface up to the boundary for minimizing problem of the sum of the principal eigenvalues.

Finally, there was a good selection of talks on recent progress on questions of regularity for minimizers or critical points, where a recurring theme was the point of view of differential inclusions. Tione's talk, mentioned above, is a good example of this. In a different direction, Xavier Lamy presented new results (jointly with T. Lacombe) on C^1 regularity for degenerate elliptic equations in the plane. He showed very delicate and technical sufficient conditions for these equations that ensures Lipschitzness implies C^1. The emphasis on degenerate elliptic problems was also the topic of Andre Guerra's talk, focussing on the functional given by $\det^2 X$. Zhuolin Li discussed in her talk extensions of the classical partial regularity theory for quasiconvex functionals to the A-quasiconvex case. Finally, Zemas presented his recent groundbreaking work with S. Luckhaus on the stability of the Möbius group.

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Workshop: Calculus of Variations

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Abstracts

On the Lawson-Osserman Conjecture

Riccardo Tione
(joint work with Jonas Hirsch, Connor Mooney)

The area of (the graph of) a Lipschitz map $v : B_1 \subset \mathbb{R}^m \to \mathbb{R}^n$ is measured by the energy

$$A(v) = \int_{B_1} \sqrt{\det(id_m + Dv^T Dv)}dx.$$

A stationary point u for this energy is called a *minimal graph.* In geometric terms, this is equivalent to requiring that the graph of u is a minimal surface. Analytically, stationarity of u corresponds to the following system of partial differential equations:

$$\begin{cases} \sum_{i,j} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}g^{ij}\frac{\partial u^l}{\partial x_j}\right) = 0, & l = 1,\dots, n, \quad (1) \\ \sum_i \frac{\partial}{\partial x_i}(\sqrt{\det g}g^{ij}) = 0, & j = 1, \dots, m, \quad (2) \end{cases}$$

where we have set $g = id_m + Du^T Du$. Equations (1) correspond to the requirement that u is a critical point for *outer* variations of this functional, while (2) are the equations associated to *inner* (or *domain*) variations. Minimal graphs are a classical subject of study. Regularity theory for weak solutions of the system (1)-(2) for Lipschitz regular u is well-understood. If $n = 1$, this follows from the De Giorgi-Nash-Moser theory (precisely because we are assuming u Lipschitz). If $n > 1$, then solutions are smooth *everywhere* for $m = 2, 3$ [1, 2, 3] and *partially* smooth for $m = 4$ [6]. From these works, we also deduce that the set of singular points is of Hausdorff dimension at most $m - 4$.

In the celebrated paper [6], H. B. Lawson and R. Osserman conjectured that equations (1) are enough to imply (2) for all m and n. In low dimensions $m \leq 3$, this is equivalent to conjecturing that (1) imply smoothness of u. The subject of my talk was the solution to this conjecture in the case $m = 2$ (and arbitrary n), recently obtained in the collaboration [5]. Our proof relies on a variety of tools and observations, such as the theory of quasiconformal mappings, a fine analysis of the codimension one case and techniques arising in the context of elliptic differential inclusions introduced by V. Šverák [9]. It is important to observe that smoothness of a solution to (1) cannot be deduced simply by convexity properties of the area functional. Indeed, the area functional is *polyconvex*, namely it is a convex function of the minors of the gradients. For such functionals, one cannot expect any *general* regularity theory of solutions to the outer variations equations, as shown via convex integration by L. Székelyhidi in [10] (see also [7]).

Our work leaves open the obvious natural question of what happens for higher dimensions. In this respect, it is interesting to draw a connection between the Lawson-Osserman conjecture and the same problem arising for harmonic maps. Indeed, in that theory, it has been shown by F. Hélein [4] that two-dimensional weakly harmonic maps are smooth, while a counterexample by T. Rivière [8] has shown that the same fails if $m \geq 3$. However, nothing is known concerning the Lawson-Osserman conjecture if $m \geq 3$.

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Regularity up to the boundary for optimal partition problems

Bozhidar Velichkov

(joint work with Roberto Ognibene)

Setting. Fixed an integer $N \geq 1$ and a bounded open set $D \subset \mathbb{R}^d$ with C^1 boundary, we consider the optimal partition problem

$$(1) \qquad \inf\left\{\sum_{i=1}^{N} \lambda_1(\Omega_i) \colon \Omega_i \subset D \, - \text{ open and such that } \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j\right\},$$

where $\lambda_1(\Omega_i)$ denotes the first Dirichlet eigenvalue on Ω_i. It is immediate to check that this problem can be re-written as

$$(2) \qquad \min\Big\{\sum_{j=1}^{N} \int_D |\nabla u_j|^2 \, dx \ : \ u_j \in H^1_0(D),\ u_j \geq 0,\ \int_D u_i u_j \, dx = \delta_{ij}\Big\},$$

the optimal sets Ω_i being exactly the positivity sets $\{u_i > 0\}$. Problems of this type has been studied in several frameworks such as dynamics of populations (see for instance [4, 5, 6]) and harmonic maps with values in singular spaces (see [3, 2, 7]).

Regularity in the interior of D. The regularity of the free interface

$$\mathcal{F} := \bigcup_{i=1}^{N} \partial\Omega_i \cap D, \quad \text{with } (\Omega_1, \ldots, \Omega_N) \text{ being a minimizer of (1)},$$

has been extensively studied in the interior of D. We refer for instance to [1, 3, 2, 4, 5, 7] (and the references therein), where it was proved that the interior interface $\mathcal{F}$ can be decomposed into the disjoint union of a *regular set* (a smooth $(d-1)$-dimensional manifold) and a *singular set* (a closed $(d-2)$-rectifiable set). The main steps in the interior regularity theory are the following:

- **(Almost-)monotonicity of the Almgren's frequency function**
$$N(u; x_0, r) := \frac{r \sum_{i=1}^{N} \int_{B_r(x_0)} |\nabla u_i|^2}{\sum_{i=1}^{N} \int_{\partial B_r(x_0)} u_i^2},$$
with respect to $r > 0$, for any $x_0 \in \mathcal{F}$ and $u = (u_1, \ldots, u_N)$ solution to (2). In particular, it implies that the limit
$$\gamma(x_0) := \lim_{r \to 0} N(u; x_0, r),$$
exists for every $x_0 \in \mathcal{F}$.
- **Admissible frequencies.** It is known that for every $x_0 \in \mathcal{F}$
$$\gamma(x_0) \in \{1\} \cup [1 + \varepsilon_d, +\infty),$$
for some dimensional constant $\varepsilon_d > 0$. Moreover, in [9], we prove that in every dimension the frequency gap is precisely $1/2$, that is:
$$\gamma(x_0) \in \{1\} \cup [3/2, +\infty).$$
The regular and the singular parts of the free interface are defined in terms of the frequency function as follows: $Reg(\mathcal{F})$ are the points of frequency 1, while $Sing(\mathcal{F})$ are the points of frequency $\geq 1 + \varepsilon_d$.
- **Clean-up and smoothness of $Reg(\mathcal{F})$.** It was proved in [2] that around any point $x_0 \in Reg(\mathcal{F})$ there are exactly two non-zero components u_i and u_j. Moreover, by the optimality of u, the difference $u_i - u_j$ solves a PDE in that neighborhood. Then, by the classical elliptic regularity theory, $u_i - u_j$ is $C^{1,\alpha}$ and by the implicit function theorem the free interface $\partial\Omega_i \cap \partial\Omega_j$ is also smooth around x_0.

Regularity up to the boundary ∂D. In [8], with Roberto Ognibene, we studied

- the behavior of the free interface $\mathcal{F} \subset D$ up to the fixed boundary ∂D;
- the regularity of $\mathcal{F}_{\partial D} := \overline{\mathcal{F}} \cap \partial D$ (as a subset of ∂D).

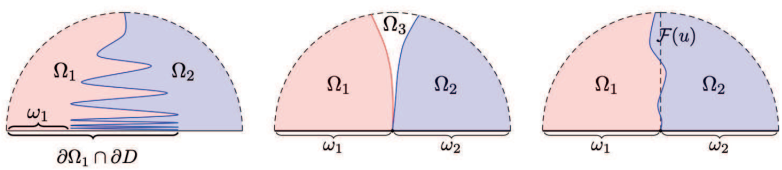
We introduce a family $(\omega_1, \dots, \omega_N)$ of relatively open subsets of ∂D such that:

$$\omega_i \cap \omega_j = \emptyset \text{ for } i \neq j\ , \qquad \bigcup_{i=1}^{N} \overline{\omega_i} = \partial D \qquad \text{and} \qquad \mathcal{F}_{\partial D} = \bigcup_{i=1}^{N} \partial_{\partial D}\omega_i\,,$$

which play the role of *traces* of the sets Ω_i. Furthermore, $\mathcal{F}_{\partial D}$ can be decomposed as disjoint union of a regular and singular parts, $\mathcal{R}_{\partial D}$ and $\mathcal{S}_{\partial D}$, where

- $\mathcal{R}_{\partial D}$ is locally a $(d-2)$-dimensional manifold of class C^1, the modulus of continuity of the normal derivative being given in terms of the one of ∂D;
- in a neighborhood of any point in $\mathcal{R}_{\partial D}$, the interior free boundary $\mathcal{F}$ is a $(d-1)$-dimensional smooth manifold that approaches ∂D orthogonally and is the interface between two components only (like in the picture on the right in the figure below).

In particular, around points in $\mathcal{R}_{\partial D}$, we are able to exclude pathological boundary behavior such as on the left and the central figures below.



The analysis of $\mathcal{F}$ up to ∂D faces issues, of both technical and topological nature, which are not present inside D: first, the monotonicity of the frequency function is a major technical obstacle even when ∂D is $C^{1,\alpha}$ smooth; second, even the definition of the optimal partition on the boundary is not straightforward and requires some fine analysis of the behavior of the eigenfunctions near ∂D; third, the smoothness of the *regular part* of the free boundary cannot be deduced from the implicit function theorem, but relies on decay rate of the blow-up sequences, which we obtain through epiperimetric inequalities.

Open problems.

- **Frequency gap at the boundary.** In [8] we prove that the limit

$$\gamma(x_0) = \lim_{t\to 0^+} \frac{r\sum_{i=1}^{N}\int_{B_r(x_0)\cap D} |\nabla u_i|^2}{\sum_{i=1}^{N}\int_{\partial B_r(x_0)\cap D} u_i^2},$$

exists for every $x_0 \in \mathcal{F}_{\partial D}$ and that

$$\gamma(x_0) \in \{2\} \cup [2+\delta_d, +\infty),$$

for some dimensional constant $\delta_d > 0$ (in dimension 2, $\delta_2 = 1$). The exact value of δ_d is not known for $d > 2$. We conjecture that $\delta_d = 5/2$ and is achieved by three domains forming a triple junction of 3/2-points disposed on a line that reaches ∂D orthogonally.

- **Full boundary regularity.** In [8] we defined the set $\mathcal{R}_{\partial D}$ as the set of boundary points of frequency $\gamma = 2$; moreover, in dimension $d = 2$ we showed that the whole set $\mathcal{F}_{\partial D}$ is composed of a finite number of points and that at each of these points the frequency γ is an integer. In particular, it is not true that $\mathcal{F}_{\partial D} \setminus \mathcal{R}_{\partial D}$ has codimension 2. This means that the results above provide only partial description of the free boundary (the situation is similar to the one in the regularity theory for the thin-obstacle problem) and leads us to the following questions:
  Let Reg be the set of all points $x_0 \in \mathcal{F}_{\partial D}$ for which $\gamma(x_0)$ is an integer. Is it true that Reg is a C^1 submanifold of codimension 1 in ∂D? Is it true that the remaining set $\mathcal{F}_{\partial D} \setminus Reg$ has codimension 2 in ∂D?

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Rigidity of global solutions to the thin obstacle problem

Hui Yu

(joint work with Simon Eberle, Xavier Fernández-Real)

Let Ω denote a domain in $\mathbb{R}^{d+1} := \{(x,y) : \ x \in \mathbb{R}^d, y \in \mathbb{R}\}$ that is symmetric with respect to the hyperplane $\{y = 0\}$. The *thin obstacle problem* in Ω refers to the following system

$$(1) \qquad \begin{cases} \Delta u \leq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \Omega \cap \{y = 0\}, \\ \Delta u = 0 & \text{in } \Omega \cap (\{y \neq 0\} \cup \{u > 0\}). \end{cases}$$

Here the solution u denotes the height of an elastic membrane resting on an obstacle at height 0 in $\{y = 0\}$. In this context, the *contact set* denotes the region where the membrane is supported by the obstacle, namely,

$$\Lambda(u) := \{u = 0\} \cap \{y = 0\}.$$

This is one of the most well-studied elliptic free boundary problems. For classical results, the reader might consult the monograph Petrosyan-Shahgholian-Uraltseva [7]. For recent developments, the reader might refer to [1, 8, 9].

Most of previous results focus on local regularity properties of the solution as well as the contact set.

In [5, 6], we begin the study of *rigidity properties of global solutions* to this problem in the compact setting. To be precise, let u be a solution to (1) with $\Omega = \mathbb{R}^{d+1}$, we aim to classify possible compact contact sets $\Lambda(u)$.

For the classical obstacle problem, such a program has been completed in [2, 4]. Comparing with solutions to the classical obstacle problem, however, our solutions to (1) lack certain important properties.

Firstly, solutions to the thin obstacle problem can grow at many different homogeneities. As a result, we need to stratify all solutions according to their rate of growth. For $m \in \mathbb{N}$, define

$$\mathcal{S}_c^m := \{u : u \text{ solves (1) in } \mathbb{R}^{d+1} \text{ with compact } \Lambda(u), \text{ and } \sup \frac{|u|}{1 + |x, y|^m} < +\infty\}.$$

Secondly, the measure Δu is supported in sets of high co-dimensions and has variable density.

Based on these, Eberle, Ros-Oton and Weiss conjectured in [3] that the global solutions to (1) are not rigidit. To be precise, their conjecture reads

Conjecture. For each $m \in \mathbb{N}$, there is $u \in \mathcal{S}_c^m$ such that $\Lambda(u)$ is not an ellipse.

We confirm this conjecture for $m \geq 4$ in [6]. Our results show that the contact sets to global solutions of (1) are very flexible for large m.

Theorem 1 ([6])**.** *Given a compact set $K \subset \{y = 0\}$ and $\varepsilon > 0$, there is $u \in \cup_m \mathcal{S}_c^m$ such that*

$$\mathrm{dist}_H(K, \Lambda(u)) + \mathrm{dist}_H(\partial K, \partial\Lambda(u)) < \varepsilon.$$

Here dist_H denotes the Hausdorff distance between sets in $\{y = 0\}$.

Even when we restrict to specific order of growth $m \in \mathbb{N}$, the contact sets can still be quite flexible.

Theorem 2 ([6])**.** *Given $\varepsilon > 0$ and a polynomial $q \geq 0$ on $\{y = 0\}$ of degree $m \geq 4$, there is $u \in \mathcal{S}_c^m$ such that*

$$\{q = 0\} \cap B_1 \subset \Lambda(u) \subset \{q = 0\}^\varepsilon \cap B_{R_d}.$$

Here $\{q = 0\}^\varepsilon$ denotes the ε neighborhood of $\{q = 0\}$, and R_d is a dimensional constant.

The remaining question concerns the rigidity of $\Lambda(u)$ for $u \in \mathcal{S}_c^2$. It turns out here we have exactly the same rigidity as in the classical obstacle problem.

Theorem 3 ([5]).

$$\{\Lambda(u): \ u \in \mathcal{S}_c^2\} = \{\textit{Ellipsoids in } \{y=0\}\}.$$

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Minimizing constraint maps

Sunghan Kim

(joint work with Alessio Figalli, André Guerra, Henrik Shahgholian)

In this talk, we study energy minimizing constraint maps: these are the natural extension of the obstacle problem in higher codimension, where one minimizes the Dirichlet energy among maps that take value inside an open set M (so its complement acts as an obstacle). Because of their vectorial character, they share similarities with minimizing harmonic maps with target into manifolds, branch points in minimal surfaces, and free boundary problems of obstacle-type.

Interestingly, constraint maps reveal an extremely rare interplay between the free boundaries, the set of discontinuous points, and the set of branch points altogether. Despite the long history of constraint maps since their initial appearance in the literature, dating back to the 1970s by the work of Hilderbrandt [8, 9] and Tomi [10, 11] and the partial regularity theory established in the 1980s by the work of Duzaar and Fuchs [2, 3], the interaction between these apparently correlated objects was addressed only recently by us [4] in a different but related setting.

The goal of this paper is to address the following fundamental question:

Does the free boundary meet the mapping singularities?

The main result in our most recent work [5] asserts that minimizing constraint maps are uniformly continuous in a uniform neighborhood of the free boundary, provided the complement of the target manifold M^c is uniformly convex. To prove this, a key tool is a new quantitative unique continuation principle near singularities, which is new even in the setting of classical harmonic maps and which holds in great generality (also for nonconvex obstacles). Using degree theory, we support the assumption of uniform convexity by providing examples of manifolds with flat pieces where the map is discontinuous.

Yet another new feature in our problem is the so-called branch points, i.e., points where Du vanishes, producing degeneracy-like behavior for solutions to the obstacle problem. It is worth noting that the regularity of the free boundary away from the set of branch points is studied in [7, 6]. Nevertheless, the phenomenon of branching is intrinsic in this vectorial problem and does not appear in the scalar obstacle problem, unless dictated by external forces. By considering some class of maps with special symmetries, we show in [5] that branch points can exist on the free boundary and give rise to singularities to the free boundary itself.

Our work opens up uncharted territory regarding the interplay between free boundaries and mapping singularities. Here, we propose a few open problems:

- What can be said *beyond* energy-minimality? Can we address the same regularity issue, for example, in the case of *stationary* constraint maps?
- Related to the previous point, what can be said about the corresponding *heat flow* problem? Note that energy-minimality is neither applicable in the parabolic setting. The theory, aside from the partial regularity result by Chen-Musina [1], is largely unexplored.
- Returning to energy-minimality, what is the *sharp geometric condition* for obstacles that resolves the regularity issue near free boundaries? The presenter conjectures that it holds for any convex obstacle with all but one principal curvature being uniformly positive, and that this condition is sharp.
- What can be said about target constraints lying on (compact) *manifolds*? This problem already appears intriguing when the supporting manifold is a sphere.

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Fractional multiphase transitions & nonlocal minimal partitions

Vincent Millot
(joint work with Thomas Gabard)

In the classical van der Waals-Cahn-Hilliard theory of phase transitions, two-phase systems are driven by energy functionals of the form $\varepsilon\int_\Omega|\nabla u|^2+\varepsilon^{-1}\int_\Omega W(u)$ with $\varepsilon\in(0,1)$, where $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ is a normalized density distribution of the two phases in the container Ω (a smooth and bounded open set), and the smooth potential $W:\mathbb{R}\to[0,\infty)$ has exactly two global minima at ±1 with $W(\pm1)=0$. Critical points u_ε satisfy the scalar Allen-Cahn equation, and one is interested to describe phase separation in the singular limit $\varepsilon\to0$. When ε is small, a control on the potential implies that $u_\varepsilon=\pm1$ away from a region whose volume is of order ε. Formally, the transition layer from the phase -1 to the phase $+1$ has a characteristic width of order ε, and it should take place along an hypersurface which is expected to be a critical point of the area functional, i.e., a minimal surface. For energy minimizing solutions, this picture has been justified first in [9] through one of the first examples of Γ-convergence. The case of general critical points has been treated much more recently in [6].

Multiple phase systems appear to be more involved as they rely on a vectorial version of the Allen-Cahn equation, and their description in the singular limit is not yet fully understood. In this case, one consider vector valued maps $u:\Omega\to\mathbb{R}^m$ and the potential $W:\mathbb{R}^m\to[0,\infty)$ vanishes at exactly d distinct points $a_1,\dots,a_d$ of $\mathbb{R}^m$. Concerning minimizing solutions, their behavior as $\varepsilon\to0$ has again been obtained through a Γ-convergence result in [1, 10]. It shows that $u_\varepsilon\to a_i$ in A_i, where the sets $(A_1,\dots,A_d)$ form a partition of the domain Ω minimizing a geometric energy of the form

$$\sum_{i,j=1,i\neq j}^{d}\alpha_{ij}\mathcal{H}^{n-1}\big(\partial A_i\cap\partial A_j\cap\Omega\big)\,,$$

with coefficients $\alpha_{ij}=\alpha_{ji}>0$ determined through a generalized geodesic distance problem between a_i and a_j involving the potential W. In particular, these coefficients satisfy the triangular inequality. In case of a *strict* triangle inequality between the α_{ij}'s, the regularity theory for the boundaries of minimizing partitions has been obtained in [7]. Concerning arbitrary critical points of the vectorial Allen-Cahn equation, the result analogue to the scalar case is still unknown. Only very recently, it has been obtained in dimension $n = 2$ in [2].

We are here interested in a non local analogue of the multiple phase transitions theory where the elastic part of the energy is replaced by a fractional Dirichlet energy leading to the fractional Allen-Cahn equation

$$(-\Delta)^s u_\varepsilon + \frac{1}{\varepsilon^s}\nabla W(u_\varepsilon) = 0 \quad \text{in } \Omega$$

(complemented with a suitable Dirichlet condition outside Ω), where $(-\Delta)^s$ denotes the usual fractional Laplacian of order $s \in (0, 1/2)$ as defined in Fourier space. In this range of order exponents s ($< 1/2$), a different geometric problem appears in the limit $\varepsilon \to 0$ compared to the classical case. The scalar case has been handled in [8] where it is proven that $u_\varepsilon \to \pm 1$ away from a region asymptotic to a *stationary nonlocal minimal surface* (or minimizing nonlocal minimal surface in case u_ε is minimizing). The concept of nonlocal minimal surface has been introduced in [3] and studied in terms of regularity theory.

The present study is the vectorial analogue of [8], i.e., the asymptotic analysis of solutions (minimizing or not) of solutions of the vectorial fractional Allen-Cahn equation. In this setting, phase separation still occurs and the limiting geometric problem is again of the nature of nonlocal minimal surfaces. More precisely, assuming a uniform energy bound and a prescribed behavior outside Ω, $u_\varepsilon \to a_i$ in A_i, where the sets $(A_1, \ldots, A_d)$ form a partition of the whole $\mathbb{R}^n$, and $(A_1, \ldots, A_d)$ is stationary in Ω for the geometric energy

$$\sum_{i,j=1}^{d} \alpha_{ij} \Big(\mathcal{I}_s(A_i \cap \Omega, A_j \cap \Omega) + \mathcal{I}_s(A_i \cap \Omega, A_j \setminus \Omega) + \mathcal{I}_s(A_i \setminus \Omega, A_j \cap \Omega) \Big) ,$$

with

$$\mathcal{I}_s(A, B) := \iint_{A \times B} \frac{1}{|x - y|^{n+2s}} \, dxdy \, ,$$

and coefficients α_{ij} given by

$$\alpha_{ij} := |a_i - a_j|^2 \, .$$

In case u_ε is minimizing in Ω, then the limiting partition $(A_1, \ldots, A_d)$ is minimizing the nonlocal geometric energy in Ω.

We have also addressed the regularity issue of stationary or minimizing nonlocal partitions. In case $(A_1, \ldots, A_d)$ is stationary in Ω, then each $A_i \cap \Omega$ is (essentially) open and $\partial A_i \cap \Omega$ has a Minkowski dimension equal to $n - 1$. For minimizing partitions, much more can be said about the set of boundaries $\Sigma := \cup_i A_i \cap \Omega$. The case of constant coefficients $\alpha_{ij} = 1$ has been first treated in [4] showing that Σ is locally a smooth hypersurface away from a set of $(n - 2)$ Hausdorff dimension. The same regularity has been shown for $d = 3$ in [5] for coefficients satisfying the additivity condition $\alpha_{ij} = c_i + c_j$ with $c_i, c_j > 0$. On one hand we have generalized the regularity of [4] to nearly constant coefficients, and on the other hand (and more interestingly) proved the same regularity result in the case of three phases ($d = 3$) for coefficients satisfying one *reversed strict triangle inequality*, e.g., $\alpha_{23} > \alpha_{12}+\alpha_{13}$, for s close enough to $1/2$. This is in sharp contrast with the classical case [7] where the strict triangle inequality is required to derive (partial) regularity. The regularity theory for general coefficients, satisfying or not the triangle inequality, is still an open question.

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Variational methods applied to discrete models in brittle damage

Elise Bonhomme

In a numerical analysis [1], Allaire-Jouve-Van Goethem have conjectured that the mechanical model of brittle damage introduced by Francfort and Marigo [4] - within a specific scaling law and specified to the discrete setting where the total energies are restricted to piecewise affine continuous displacements - converges to a model of fracture. On the other hand, Babadjian-Iurlano-Rindler have recently proved in [3] that the conjecture fails when one considers the continuous setting (without restriction to piecewise affine displacements). In this talk, I will introduce Francfort-Marigo's model of brittle damage in the discrete setting, in different regimes where the damaged zone concentrates on vanishingly small sets, and identify the nature of the effective models obtained by means of an asymptotic analysis based on the Γ-convergence of the total energies (*work in preparation*).

More precisely, given ε and $\eta_\varepsilon > 0$, we consider a linearly elastic material, whose reference configuration is a bounded open set $\Omega \subset \mathbb{R}^2$, which is composed of only two phases: a damaged phase (where the elasticity of the medium is altered) and a sound one, whose elasticity properties are given by η_ε and 1 respectively. Introducing the characteristic function of the damaged region, $\chi \in L^\infty(\Omega; \{0, 1\})$, Francfort-Marigo's model consists in defining the total energy associated to a displacement $u \in H^1(\Omega;\mathbb{R}^2)$ as the sum of the elastic energy stored inside the material and a dissipative energy, taken as proportional to the volume of the damaged zone:

$$\mathcal{E}_\varepsilon(u,\chi) = \frac{1}{2}\int_\Omega (\eta_\varepsilon\chi + (1-\chi))\,|e(u)|^2\,dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi\,dx,$$

where $e(u) = \left(\nabla u + \nabla u^T\right)/2$ is the linearized elastic strain and $\kappa/\varepsilon > 0$ is the material's toughness in the damaged regions. Note that the elasticity coefficient η_ε of the weak material degenerates, while the diverging character of κ/ε forces the damaged zones to concentrate on vanishingly small sets as $\varepsilon \searrow 0$. Here, we consider the total energies restricted to couples $(u,\chi) \in \mathcal{C}^0(\Omega;\mathbb{R}^2) \times L^\infty(\Omega,\{0,1\})$ in the finite element set

$$(u,\chi) \in X_{h_\varepsilon}(\Omega),$$

for which there exists a triangulation $\mathbf{T}_{h_\varepsilon}$ of Ω, whose mesh-size is of order $h_\varepsilon > 0$, such that u is affine and χ is constant on each of its triangle. Noticing that the mesh-size imposes a minimal oscillation scale for the displacements, one can expect that the scaling of h_ε with respect to the other parameters will influence the nature of the limit models. Indeed, we obtain five effective models:

Regime	**Effective limit model**
$\varepsilon \ll \eta_\varepsilon$ or $\varepsilon \ll h_\varepsilon$	linear elasticity
$h_\varepsilon \ll \varepsilon$ and $\eta_\varepsilon \ll \varepsilon$	trivial model
$h_\varepsilon \ll \varepsilon$ and $\eta_\varepsilon \sim \varepsilon$	Hencky plasticity
$h_\varepsilon \sim \varepsilon$ and $\eta_\varepsilon \ll \varepsilon$	brittle fracture (see [2])
$h_\varepsilon \sim \varepsilon \sim \eta_\varepsilon$	in between plasticity and brittle fracture

In particular, when the mesh-size is negligible with respect to ε and/or η_ε, we recover the three asymptotic models obtained in [3]. Formally, the discrete approximation is so fine that the asymptotic behaviour is qualitatively the same as in the continuous setting. Whereas when h_ε is not negligible with respect to neither ε and η_ε, we recover the fracture model conjectured in [1] as well as an intermediate model in between plasticity and fracture (according to the scaling of η_ε with respect to ε). Heuristically, fracture type models can only be asymptotically obtained when we impose a sufficiently large oscillation scale on the displacements.

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Soap films, Plateau's Laws, and an Allen-Cahn free boundary problem

Michael Novack
(joint work with Francesco Maggi, Daniel Restrepo, and Anna Skorobogatova)

The Plateau problem of minimizing area among surfaces with a given boundary is the classical mathematical model for soap films and leads to theory of minimal surfaces. Among the many versions of the Plateau problem, the choice of a particular model often depends on the properties, such as the presence/type of singularities, that one wishes to see in minimizers. The goal of the works [5, 6, 7] is to analyze two related Plateau-type problems with the hope of capturing features of soap films and minimal surfaces outside the scope of previous models.

In [5], joint with F. Maggi and D. Restrepo, our aim is to add a volume parameter $v > 0$, representing the amount of liquid in the soap film, to the Plateau problem. Such a model would describe for example *Plateau borders*, which are thickened tubes of liquid wetting a line of Y-point singularities and which play an important role in film drainage. In the physical literature, films with positive volume are known as "wet" films. They can be described mathematically as the union of a "wet" region E of volume $v > 0$ (one may imagine $E \approx$ the Plateau borders) and an interface K containing ∂E and the "dry" portions of the soap film. We formulate the corresponding Plateau problem using the spanning condition of Harrison-Pugh [3] since it is allows for Plateau type singularities (Y- and T-points) when $v = 0$. More precisely, for a fixed compact set $\mathbf{W} \subset \mathbb{R}^{n+1}$ and a homotopically closed family $\mathcal{C}$ of smooth embeddings of $\mathbb{S}^1$ into $\mathbf{W}^c$, we consider

$$\begin{aligned}(1)\qquad \inf\big\{&\mathcal{H}^n(\partial^* E \setminus \mathbf{W}) + 2\mathcal{H}^n(K \setminus \partial^* E) : (K \cup E) \cap \gamma \neq \emptyset\ \forall \gamma \in \mathcal{C},\ |E| = v,\\ &E \subset \mathbf{W}^c \text{ is open, } K \subset \mathbf{W}^c \text{ is rel. closed, } \partial E \setminus \mathbf{W} \subset K\big\};\end{aligned}$$

in short, we minimize the surface tension energy of a wet soap film spanning a wire frame. The main results of [5] are as follows:

- *Existence*: There exists a pair (K, E) such that (K, E) is minimal for (1).
- *Regularity*: There exists Σ of Hausdorff dimension at most $n - 7$ such that $(\partial^* E \setminus \mathbf{W}) \setminus \Sigma$ is smooth with constant mean curvature and $K \setminus (\partial E \cup \Sigma)$ is smooth with zero mean curvature. Also, $\Gamma = (\partial E \setminus \mathbf{W}) \setminus (\partial^* E \cup \Sigma)$ is locally $\mathcal{H}^{n-1}$-rectifiable, and, for any $x \in \Gamma$, there is $r > 0$ such that $K \cap B_r(x)$ is a union of two $C^{1,1}$ hypersurfaces touching tangentially at x.
- *Convergence to the Plateau problem*: Up to subsequences, a sequence energy measures associated to minimizers for (1) converges in the sense of Radon measures to $2\mathcal{H}^n \llcorner S$, where S is a minimizer for the Harrison-Pugh Plateau problem

$$(2)\qquad \inf\{\mathcal{H}^n(S) : S \subset \mathbf{W}^c \text{ is rel. closed, } S \cap \gamma = \emptyset\ \forall \gamma \in \mathcal{C}\}\,.$$

The existence of minimizers answers an open question from [4]. The difficulty is fundamentally about compactness, that is, proving that the limit of a minimizing sequence for (1) is an object in the same class. To address this, we generalize the spanning condition "$(K \cup E) \cap \gamma \neq \emptyset\ \forall \gamma \in \mathcal{C}$" in a fashion that allows us to obtain compactness of energy bounded sequences in a larger class. Then we prove that the minimizer in the larger class is in fact a pair (K, E) as in (1). The compactness also yields the asymptotic convergence of (1) to the Plateau problem (2) as $v \to 0$. The reformulated spanning condition relies on a measure theoretic notion of connectedness originally introduced in [1, 2] in the study of rigidity in symmetrization inequalities. Regarding the regularity of K, the delicate part is the analysis of Γ, which Allard's regularity theorem only guarantees to have empty interior. To prove that Γ is $(n-2)$-rectifiable and that K can be resolved near $x \in \Gamma$ as two tangential $C^{1,1}$ surfaces, a new comparison argument is used in conjunction with the regularity of solutions to the double membrane problem.

The motivation for the second Plateau-type problem is the following rigidity theorem [8]: the $\varepsilon \to 0$ limit of any sequence $\{u_\varepsilon\}$ of stable, bounded energy solutions to the Allen-Cahn equation $\varepsilon^2 \Delta u_\varepsilon = W'(u_\varepsilon)$ is a smooth minimal surface away from a singular set of co-dimension at least 8. In particular, Plateau-type singularities *cannot* be approximated by stable solutions to Allen-Cahn, despite the usefulness of Allen-Cahn in physical modelling. With the goal of modifying Allen-Cahn to address this, consider the following diffuse interface analogue of (1):

$$(3) \quad \inf\Big\{ \int_{\mathbf{W}^c} \varepsilon |\nabla u|^2 + W(u)/\varepsilon : \int_{\mathbf{W}^c} V(u) = v,\ \{u = 1\} \text{ spans } \mathbf{W},\ 0 \le u \le 1 \Big\}\,.$$

The function u can be thought of as the density distribution of soap particles, and V is a volume potential. With F. Maggi and D. Restrepo [6], we proved the existence of minimizers, which are formally critical for the free boundary problem

$$(4) \qquad \begin{cases} \varepsilon^2\, \Delta u = W'(u) - \varepsilon\, \lambda V'(u)\,, & \text{on } \{u \neq 1\}\,, \\ |\partial_\nu^+ u| = |\partial_\nu^- u|\,, & \text{on } \{u = 1\}\,. \end{cases}$$

Furthermore, as $\varepsilon \to 0$, minimizers converge to minimizers of (1). Together with the convergence of (1) to (2) and the appearance of Plateau-type singularities in minimizers of (2), this allows for the approximation of singular minimal surfaces by minimizers for a diffuse interface free boundary problem.

Lastly, in forthcoming work with A. Skorobogatova and D. Restrepo, we study the regularity of minimizers to (3) and their free boundaries. We prove:

if u minimizes (3)*, then u is locally Lipschitz and the free boundary $\{u = 1\}$ decomposes into* $\mathrm{Sing}\,(u) \sqcup \mathrm{Reg}\,(u)$*, where* $\mathrm{Reg}(u)$ *is a smooth hypersurface and* $\mathrm{Sing}(u)$ *has co-dimension 2.*

In light of the transmission condition, the Lipschitz regularity is optimal. However, since (4) is formal (the derivation of the usual Euler-Lagrange equations is complicated by the spanning constraint), our arguments are based instead on three other criticality conditions satisfied by minimizers: a modified equation that encodes the possible singularities of Δu along the free boundary $\{u = 1\}$, the inner variation version of the volume-constrained Allen-Cahn equation, and a differential inequality.

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Minimality of the vortex solution for Ginzburg-Landau systems

Radu Ignat

We consider the Ginzburg-Landau system for N-dimensional maps defined in the unit ball for some parameter $\varepsilon > 0$. For a boundary data corresponding to a vortex of topological degree one, the aim is to prove the symmetry of the ground state of the system. We show this conjecture for every $\varepsilon > 0$ in any dimension $N \geq 7$, and then, we also prove it in dimension $N = 4, 5, 6$ provided that the admissible maps are gradient fields. It comes from a series of articles [8, 9, 5, 10, 6] in collaboration with Luc Nguyen (Oxford), Mickael Nahon (Grenoble), Mircea Rus (Cluj), Valeriy Slastikov (Bristol) and Arghir Zarnescu (Bilbao).

The Ginzburg-Landau model. Let $B^N \subset \mathbb{R}^N$ be the unit ball, $N \geq 2$. For $u : B^N \to \mathbb{R}^N$, consider the Ginzburg-Landau functional for a parameter $\varepsilon > 0$:

$$G_\varepsilon(u) = \int_{B^N} \frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2)\, dx,$$

where $W : (-\infty, 1] \to \mathbb{R}_+$ is C^1 convex, $W(0) = 0$, $W(t) > 0$ for $t \neq 0$. Typically, $W(t) = \frac{t^2}{2}$. As $\varepsilon \to 0$, the limit maps take values into the unit sphere $\mathbb{S}^{N-1}$, so the limit model is the $\mathbb{S}^{N-1}$-harmonic map problem (HMP). Thus, our results are expected to be closely related with those obtained for HMP.

We focus on critical points u of G_ε for fixed $\varepsilon > 0$:

$$-\Delta u = \frac{1}{\varepsilon^2} W'(1 - |u|^2) u \quad \text{in } B^N \tag{1}$$

under the boundary condition

$$u(x) = x \quad \text{on } \partial B^N = \mathbb{S}^{N-1}. \tag{2}$$

Such critical points u (e.g., minimizers) exist. In particular, by the maximum principle, $|u| \leq 1$ in B^N and then, the standard elliptic theory yields $u \in W^{2,p} \cap C^{1,\alpha}$ for every $p < \infty$ and $\alpha \in (0,1)$. Moreover, the topological constraint in (2) implies that u has a zero point inside B^N that plays an important role in this theory. The main question concerns the uniqueness of solutions in (1) & (2).

The vortex solution. For every $\varepsilon > 0$, there exists a unique solution to (1) & (2) that is invariant under the special orthogonal group $SO(N)$, i.e., the group action $u \mapsto u^R(x) = R^{-1}u(Rx)$ for every $R \in SO(N)$ that keeps invariant the functional G_ε and the boundary condition (2). This is the so-called *vortex solution* (of topological degree 1) given by

$$u_\varepsilon(x) = f_\varepsilon(|x|)\frac{x}{|x|}, \quad x \in B^N \setminus \{0\}.$$

The radial profile $f_\varepsilon : [0,1] \to \mathbb{R}$ is the unique solution to the singular ODE:

$$\begin{cases} -f_\varepsilon'' - \frac{N-1}{r} f_\varepsilon' + \frac{N-1}{r^2} f_\varepsilon = \frac{1}{\varepsilon^2} W'(1-f_\varepsilon^2) f_\varepsilon & \text{in } (0,1), \\ f_\varepsilon(0) = 0, f_\varepsilon(1) = 1, \end{cases} \tag{3}$$

where $r = |x|$ (see [3, 4, 7]). In particular, $1 > f_\varepsilon > 0$ and $f_\varepsilon' > 0$ in $(0,1)$. The aim is to study the minimality of the vortex solution:

Question 1. Is $u_\varepsilon(x) = f_\varepsilon(|x|)\frac{x}{|x|}$ the (unique) minimiser of G_ε under the boundary condition (2) for every $\varepsilon > 0$?

For large ε, i.e., $\varepsilon \geq \varepsilon_{conv}$, the functional G_ε is strictly convex yielding uniqueness in (1) & (2) (in particular, the positive answer to Question 1), see [1, 9]. For $\varepsilon < \varepsilon_{conv}$, there are only some partial results. In dimension $N = 2$, Bethuel-Brezis-Hélein [1] proved in the regime $\varepsilon \to 0$ that a minimizer u of G_ε under (2) has a unique topological zero converging to the origin, while Pacard-Rivière [17] proved that u_ε is the unique solution to (1) & (2) for very small $\varepsilon > 0$; we also mention the work of Mironescu [16] for the corresponding blow-up problem in the domain $\mathbb{R}^2$. In dimension $N \geq 3$, we quote the works of Millot-Pisante [14] and Pisante [18] for the blow-up problem in the domain $\mathbb{R}^N$. Finally, for the $\mathbb{S}^{N-1}$-harmonic map problem, $u_*(x) = \frac{x}{|x|}$ is the unique minimizing harmonic map in B^N under (2) if $N \geq 3$ (see Jäger-Kaul [11], Brezis-Coron-Lieb [2], Lin [13]).

Main results. Our first result gives a positive answer to Question 1 in dimension $N \geq 7$ (see [8, 9]):

Theorem 2. If $N \geq 7$, then $u_\varepsilon(x) = f_\varepsilon(|x|)\frac{x}{|x|}$ is the unique minimiser of G_ε under (2) for every $\varepsilon > 0$.

Sketch of the proof. The idea is to linearize the potential energy in G_ε. More precisely, the convexity of W yields for every $v \in H_0^1(B^N, \mathbb{R}^N)$:

$$G_\varepsilon(u_\varepsilon + v) - G_\varepsilon(u_\varepsilon) \geq \frac{1}{2} F_\varepsilon(v) \tag{4}$$

where $F_\varepsilon(v) = \int_{B^N} |\nabla v|^2 - \frac{1}{\varepsilon^2} W'(1-|u_\varepsilon|^2)|v|^2\, dx$. To conclude, we need to prove that for every $\varepsilon > 0$, $F_\varepsilon(v) = \displaystyle\int_{B^N} L_\varepsilon v \cdot v \, dx \geq 0$, $\forall v \in H_0^1(B^N, \mathbb{R}^N)$, where $L_\varepsilon = -\Delta - \frac{1}{\varepsilon^2} W'(1-f_\varepsilon^2)$. Let $\ell(\varepsilon) = \lambda_1(L_\varepsilon, B^N)$ be the first eigenvalue of L_ε in B^N under zero Dirichlet condition. The conclusion follows by:

Lemma 3. If $N \geq 7$, then $\ell(\varepsilon) \geq c_N = \frac{(N-2)^2}{4} - (N-1) > 0$, $\forall \varepsilon > 0$.

Sketch of the proof. For $v \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R})$, we use the Hardy decomposition $v = f_\varepsilon s$. Integration by parts combined with (3) imply

$$F_\varepsilon(v) = \int_{B^N} L_\varepsilon v \cdot v = \int_{B^N} (f_\varepsilon^2 |\nabla s|^2 + s^2 L_\varepsilon f_\varepsilon \cdot f_\varepsilon) = \int_{B^N} f_\varepsilon^2 \big(|\nabla s|^2 - \frac{N-1}{r^2} s^2\big).$$

The limit case $\varepsilon \to 0$ follows from the fact that $f_\varepsilon \to 1$ in $(0,1]$ combined with Hardy's inequality:

$$\int_{B^N} L_\varepsilon v \cdot v \to \int_{B^N} |\nabla s|^2 - \frac{N-1}{r^2} s^2 \geq \int_{B^N} \big(\frac{(N-2)^2}{4} - (N-1)\big) \frac{s^2}{r^2} \geq c_N \int_{B^N} s^2.$$

For the general case $\varepsilon > 0$ (fixed), one decomposes $s = \phi \tilde{s}$ with $\phi = r^{-\frac{N-2}{2}}$ and obtains $F_\varepsilon(v) \geq c_N \int_{B^N} \frac{v^2}{r^2}$ yielding the conclusion of Lemma 3 together with the uniqueness of the minimizer u_ε in Theorem 2. □

In dimension $N \in [2,6]$, the above argument does not yield the answer to Question 1. Indeed, the first eigenvalue $\ell(\varepsilon)$ of L_ε in B^N becomes negative for small $\varepsilon > 0$ if $2 \leq N \leq 6$. However, the above argument improves the range of ε where u_ε is the unique minimizer of G_ε under (2) (with respect to ε_{conv} above which G_ε is strictly convex), see [5, 10]:

Lemma 4. If $2 \leq N \leq 6$, then there is $\varepsilon_N \in (0, \varepsilon_{conv})$ such that $\ell(\varepsilon_N) = 0$ and $\ell(\varepsilon) < 0$ if $\varepsilon < \varepsilon_N$ (resp. $\ell(\varepsilon) > 0$ if $\varepsilon > \varepsilon_N$). In particular, if $\varepsilon > \varepsilon_N$, then the vortex solution u_ε is the unique minimizer of G_ε under (2).

The minimality of u_ε is still an open question if $\varepsilon < \varepsilon_N$ and $N \in [2,6]$. A partial result is the *local* minimality of u_ε for every $\varepsilon > 0$. This is known in dimension $N = 2$ thanks to the works of Mironescu [15] and Lieb-Loss [12], while in dimension $N \in [3,6]$, this is proved by Ignat-Nguyen [5]:

Theorem 5. If $3 \leq N \leq 6$, then $u_\varepsilon = f_\varepsilon(|x|)\frac{x}{|x|}$ is a *local* minimizer of G_ε under (2) for every $\varepsilon > 0$.

Sketch of the proof. The aim is to prove that for every $\varepsilon > 0$, $G_\varepsilon(u_\varepsilon + v) - G_\varepsilon(u_\varepsilon) \geq C\|v\|_{H^1}^2$ if $\|v\|_{H^1} \leq \delta$ for some $\delta = \delta(\varepsilon) > 0$ and $C = C(\varepsilon) > 0$ small. For that, we analyse the second variation of G_ε at u_ε in direction $v \in H_0^1(B^N, \mathbb{R}^N)$:

$$Q_\varepsilon(v) = \frac{d^2}{dt^2}\bigg|_{t=0} G_\varepsilon(u_\varepsilon + tv) = F_\varepsilon(v) + \frac{2}{\varepsilon^2} \int_{B^N} W''(1-f_\varepsilon^2) f_\varepsilon^2 (v \cdot \frac{x}{|x|})^2 \, dx.$$

This is done by writing $v(x) = s(x)\frac{x}{|x|} + \tilde{v}(x)$ for some scalar function s and a tangent vector field $\tilde{v}(x) \cdot x = 0$ and then use the Hodge decomposition in the tangent space $T_x\mathbb{S}^{N-1}$ for every $x \in B^N \setminus \{0\}$: $\tilde{v}(r, \cdot) = v^\circ(r, \cdot) + \nabla\!\!\!\!/\, \psi(r, \cdot)$ on

$\mathbb{S}^{N-1}$ where $\nabla\!\!\!/ \cdot v^\circ(r,\cdot) = 0$ in $\mathbb{S}^{N-1}$ and ψ is a scalar function. The spectral decomposition of $s(r,\cdot)$ and $\psi(r,\cdot)$ in $L^2(\mathbb{S}^{N-1})$ yields a decomposition of $v - v^\circ$ in modes v_k and furthermore, the following decomposition of the second variation

$$Q_\varepsilon(v) = Q_\varepsilon(v^\circ) + \sum_{k\geq 0} Q_\varepsilon(v_k).$$

Using Hardy decompositions for each v_k, we obtain $Q_\varepsilon(v) \geq C(\varepsilon)\|v\|^2_{H^1}$ for every $v \in H^1_0(B^N, \mathbb{R}^N)$ and $\varepsilon > 0$. An extra argument yields local minimality of u_ε. □

The Aviles–Giga model. Note that the vortex solution is a gradient field, i.e., $u_\varepsilon = \nabla\phi_\varepsilon$ for some radial function $\phi_\varepsilon : B^N \to \mathbb{R}$ determined by $\phi'_\varepsilon = f_\varepsilon$ in $(0,1)$. Therefore, in dimension $N \in [2,6]$, it is natural to study the minimality of u_ε restricted to the class of gradient fields.

Question 2. Is u_ε the (unique) minimizer of G_ε over gradient fields

$$\mathcal{V} = \{u = \nabla\phi \,:\, \phi \in H^2(B^N, \mathbb{R}),\, \nabla\phi = Id \text{ on } \partial B^N\}?$$

This is the so-called Aviles-Giga model corresponding to the functional

$$G_\varepsilon(\nabla\phi) = \int_{B^N} \frac{1}{2}|\nabla^2\phi|^2 + \frac{1}{2\varepsilon^2}W(1 - |\nabla\phi|^2)\,dx.$$

We are able to improve Theorem 3 to the dimensions $N = 4, 5, 6$ in this restricted class $\mathcal{V}$, see Ignat-Nahon-Nguyen [6].

Theorem 6. If $N \geq 4$, then u_ε is the unique global minimizer of G_ε over $\mathcal{V}$ for every $\varepsilon > 0$.

Sketch of the first proof. As before, for every $\nabla\psi \in H^1_0(B^N, \mathbb{R}^N)$, we have $G_\varepsilon(u_\varepsilon + \nabla\psi) - G_\varepsilon(u_\varepsilon) \geq \frac{1}{2}F_\varepsilon(\nabla\psi)$. As $\nabla\psi = 0$ on ∂B^N, we have

$$F_\varepsilon(\nabla\psi) = \int_{B^N} (\Delta\psi)^2 - \frac{1}{\varepsilon^2}W'(1 - f_\varepsilon^2)|\nabla\psi|^2\,dx.$$

In the limit case $\varepsilon \to 0$, we expect that $F_\varepsilon(\nabla\psi) \to \int_{B^N}(\Delta\psi)^2 - \frac{N-1}{r^2}|\nabla\psi|^2$ and the conclusion would follow by the Hardy inequality in $\mathcal{V}$: $\int_{B^N}(\Delta\psi)^2 \geq K_N \int_{B^N} \frac{|\nabla\psi|^2}{r^2}$ with $K_N = \begin{cases} N^2/4 & \text{if } N \geq 5 \\ N-1 & \text{if } N = 4 \\ 25/36 & \text{if } N = 3 \end{cases}$. For the general case $\varepsilon > 0$, we use a spherical harmonic decomposition for ψ and based again on some Hardy decompositions, we get $F_\varepsilon(\nabla\psi) \geq 0$ provided that $N \geq 4$.

Sketch of the second proof if $N \geq 5$: This second proof is based on the following symmetrization of gradient fields. More precisely, for the stream function $\phi \in H^1(B^N, \mathbb{R})$, we associate the radial function $\phi_* = \phi_*(r)$ defined by

$$\phi'_*(r) = \Big(\fint_{\mathbb{S}^{N-1}} |\nabla\phi(r\theta)|^2 d\sigma(\theta)\Big)^{1/2} \geq 0, \quad r \in (0,1).$$

As W is convex, Jensen's inequality yields

$$\int_{B^N} W(1-|\nabla\phi|^2)\,dx \geq \int_{B^N} W(1-|\nabla\phi_*|^2)\,dx.$$

Moreover, if $\nabla\phi = Id$ on ∂B^N and $N \geq 5$ then

$$\int_{B^N} |\nabla^2\phi|^2\,dx \geq \int_{B^N} |\nabla^2\phi_*|^2\,dx$$

with equality if and only if ϕ is radial. Thus, for every $N \geq 5$ and any $\varepsilon > 0$, $G_\varepsilon(\nabla\phi) \geq G_\varepsilon(\nabla\phi_*) \geq G_\varepsilon(u_\varepsilon = \nabla\phi_\varepsilon)$. □

$\mathbb{R}^{N+1}$-valued vortex solutions. We can solve completely Question 1 when we add one target dimension, i.e., the admissible maps are $U = (u, U_{N+1}) : B^N \to \mathbb{R}^{N+1}$ satisfying the boundary condition

$$U(x) = (x, 0) \in \mathbb{S}^{N-1} \times \{0\} \text{ on } \partial B^N. \tag{5}$$

We prove that for every $\varepsilon > 0$, minimizers of G_ε under (5) are vortex type solutions that are either *non-escaping* (i.e., their $(N+1)$-component vanishes in B^N), or they are *escaping*, i.e., their $(N+1)$-component is positive (or negative) in B^N, see Ignat-Rus [10].

Theorem 7. Every minimizer of G_ε under (5) is symmetric of vortex type and the following dichotomy holds for $2 \leq N \leq 6$:

a) if $\varepsilon \geq \varepsilon_N$, then the *non-escaping* vortex solution $\bar{U}_\varepsilon = (f_\varepsilon(|x|)\frac{x}{|x|}, 0)$ is the unique minimizer of G_ε under (5).

b) if $\varepsilon < \varepsilon_N$, then the two *escaping* vortex solutions $(\tilde{f}_\varepsilon(|x|)\frac{x}{|x|}, \pm g_\varepsilon(|x|))$ with $g_\varepsilon > 0$ are the only minimizers of G_ε under (5). In this case, the non-escaping solution $\bar{U}_\varepsilon$ is unstable.

The idea is the following: point a) is implied by the proof of Theorem 2. For point b), if an escaping critical point $U = (u, U_{N+1})$ of G_ε exists under (5), then it is a minimizer and the set of minimizers is given by $\{(u, \pm U_{N+1})\}$ (this phenomenon is explained in [9]). Restricting to the class of symmetric vortex type maps, Lemma 4 implies that the non-escaping vortex solution $\bar{U}_\varepsilon$ is unstable if $\varepsilon < \varepsilon_N$ and therefore, an escaping symmetric vortex solution exists, which determines the set of minimizers. Of course, by the proof of Theorem 2, the non-escaping vortex solution $\bar{U}_\varepsilon$ is the unique minimizer of G_ε under (5) in dimension $N \geq 7$.

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On C^1 regularity for degenerate elliptic equations in the plane

Xavier Lamy
(joint work with Thibault Lacombe)

We show that Lipschitz solutions u of $\operatorname{div} G(\nabla u) = 0$ in $B_1 \subset \mathbb{R}^2$ are C^1, for strictly monotone vector fields $G \in C^0(\mathbb{R}^2; \mathbb{R}^2)$ satisfying a mild ellipticity condition. If $G = \nabla F$ for a strictly convex function F, and $0 \leq \lambda(\xi) \leq \Lambda(\xi)$ are the two eigenvalues of $\nabla^2 F(\xi)$, our assumption, stated loosely, is that the bad set $\mathcal{B} = \{\lambda = 0\} \cap \{\Lambda = \infty\} \subset \mathbb{R}^2$, where ellipticity degenerates *both* from below and from above, is finite. This extends results by De Silva and Savin [1] which assumed either that set empty, or the larger set $\{\lambda = 0\}$ finite. Our main new input is to transfer estimates in $\{\lambda > 0\}$ to estimates in $\{\Lambda < \infty\}$ by means of a conjugate equation. This also gives new results on the regularity of autonomous nonlinear

Beltrami equations. When G is not a gradient, the ellipticity assumption needs to be interpreted in a specific way and we provide an example highlighting the nontrivial effect of the antisymmetric part of ∇G. We conjecture that, for any general strictly monotone vector fields $G \in C^0(\mathbb{R}^2;\mathbb{R}^2)$ and any Lipschitz solution u of $\operatorname{div} G(\nabla u) = 0$, the function $x \mapsto \operatorname{dist}(\nabla u(x), \mathcal{B})$ is continuous.

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On Minimizing Harmonic Maps with Planar Boundary Anchoring

Dominik Stantejsky

(joint work with Lia Bronsard and Andrew Colinet)

Motivated by experiments with nematic liquid crystal droplets [3], we study harmonic maps that arise as minimizers of the one-constant approximation of the Oseen-Frank energy subject to strong anchoring tangential boundary condition. More precisely, we are interested in minimizers of the functional

$$E(\mathbf{n}) \;=\; \frac{1}{2}\int_{B_1(0)} |\nabla \mathbf{n}|^2 \,\mathrm{d}x\,,$$

subject to the constraint $|\mathbf{n}| = 1$ and the boundary condition $\mathbf{n}\cdot\nu = 0$, where $\nu = \mathbf{e}_r$ is the normal vector of the domain $B_1(0) \subset \mathbb{R}^3$.

Through a reflection method, we are able to study the regularity of minimizers close to the boundary. For the case of flat boundaries, see [2].

Furthermore, we study the symmetry of minimizers. Assuming the existence of an axis for cylindrical coordinates such that

$$\int_{B_1(0)} \frac{1}{\rho^2}\Big(n_\rho \partial_\theta n_\theta - n_\theta \partial_\theta n_\rho\Big)\,\mathrm{d}x \;\geq\; 0\,,$$

we are able to show that minimizers must be equivariant.

We believe that our assumption is not necessary, and a stronger statement holds:

Conjecture. Every global minimizer must satisfy $n_\theta \equiv 0$ with respect to some axis of symmetry.

The regularity result implies that all singularities must be point singularities. Under the assumption $n_\theta \equiv 0$, these points can only occur on the axis of symmetry.

Adapting methods from [1], we show that in this situation no interior singularities are possible and the only defects occur on the boundary at two antipodal points.

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Nematic liquid crystal colloid with planar anchoring and a weak magnetic field

Dean Louizos

(joint work with Lia Bronsard and Dominik Stantejsky)

We study minimizers of the non-dimensionalized Landau-de Gennes energy functional on an exterior domain $\Omega \subset \mathbb{R}^3$ given by

$$E_{\xi,\eta}(Q) = \int_\Omega \frac{1}{2}|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) + \frac{1}{\eta^2} g(Q)\, dx\,,$$

for two small parameters $\xi, \eta > 0$. In this setup we consider a particle immersed in nematic liquid crystal where the surface of the particle is described by a manifold $\mathcal{M}$ and Ω is the region in $\mathbb{R}^3$ exterior to $\mathcal{M}$. Similar results have been obtained for a homeotropic anchoring condition in [1] and [2], but in our setup we consider tangential anchoring of the liquid crystal molecules. This study began with [3] where $\mathcal{M}$ is taken to be the unit sphere and now we generalize to a larger class of manifolds.

The aim is to understand the limiting energy of minimizers in the large-particle limit, where we consider a regime corresponding to a weak magnetic field described by the asymptotics

$$\frac{\eta}{\xi} \to \infty \quad \text{as} \quad \xi, \eta \to 0\,.$$

We are able to obtain estimates on the energy of minimizers $Q_{\xi,\eta}$ in this regime which allow us to study the possible defects that may occur. The estimates also lead to the following limiting energy:

$$\lim_{\xi,\eta\to 0} \eta E_{\xi,\eta}(Q_{\xi,\eta}) = \int_{\mathcal{M}} \sqrt[4]{24}\Big(1 - \sqrt{1-\nu_3^2}\Big)\, d\mathcal{H}^2\,.$$

Using this energy we can examine a related optimization problem in which we allow the orientation of $\mathcal{M}$ with respect to the magnetic field direction to vary.

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The dimension and behaviour of singularities of stable solutions to semilinear elliptic equations

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(joint work with Alessio Figalli)

1. Setup

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex, positive, increasing nonlinearity and $\Omega \subset \mathbb{R}^n$ a bounded smooth domain.

Definition 1. We say that $u \in H^1_{\mathrm{loc}}(\Omega)$ is a *stable weak solution* of

$$-\Delta u = f(u), \tag{1}$$

if $f \circ u$ and $f' \circ u$ are in $L^1_{\mathrm{loc}}(\Omega)$ and, for all test functions $\phi \in C^1_c(\Omega)$, it holds

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f(u)\phi \quad \text{and} \quad \int_\Omega |\nabla \phi|^2 \geq \int_\Omega f'(u)\phi^2.$$

By the mean value inequality, u is locally bounded from below in Ω.

By standard elliptic regularity,

$$u \in L^\infty_{\mathrm{loc}}(\Omega) \implies u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega), \text{ for all } \alpha < 1.$$

As $\Longleftarrow$ holds as well, we define

$$\mathrm{reg}(u) = \text{"The largest open subset of } \Omega \text{ where } u \text{ is locally bounded above"},$$

and $\mathrm{sing}(u) := \Omega \setminus \mathrm{reg}(u)$. If $\mathrm{sing}(u) = \emptyset$, we say that $u^\star$ is a classical solution.

Concerning the behaviour of u around singular points, in this work we show that, if $0 \in \mathrm{sing}(u)$, then

$$\limsup_{x \to 0} f'(u(x))|x|^2 \geq c(n) > 0,$$

which is optimal thanks to the counterexamples of [13]. Concerning the size of $\mathrm{sing}(u)$, we show that, for a large subclass of nonlinearities f, $\dim \mathrm{sing}(u) \leq n-10$, which is optimal thanks to classical counterexamples.

In the rest of this abstract we explain more in detail these results and explain their relationship with two questions of Brezis.

2. Motivation

Weak stable solutions arise naturally as extremal solutions of the so called Gelfand problem (see the book [9]): given a constant $\lambda > 0$ consider

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

If we assume further that

$$f(0) > 0, \int_1^\infty \frac{dt}{f(t)} < \infty \text{ and } f(t)/t \to +\infty \text{ as } t \to +\infty,$$

the main result concerning (2) can be summarized as follows

Theorem 2 ([1, 3, 9, 11, 2])**.** There exists a constant $\lambda^\star \in (0, +\infty)$ such that:

(i) For every $\lambda \in (0, \lambda^\star)$, (2) has a unique weak stable solution u_λ which is bounded (and thus C^2). Furthermore, $u_\lambda < u_{\lambda'}$ for $\lambda < \lambda' < \lambda^\star$.

(ii) For every $\lambda > \lambda^\star$ there is no solution, not even in the following L^1-weak sense: $u \in L^1(\Omega)$, $f(u)\mathrm{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$, and

$$-\int_\Omega u\Delta\phi\, dx = \lambda \int_\Omega f(u)\phi\, dx \quad \text{for all } \phi \in C^2(\overline{\Omega}) \text{ with } \phi|_{\partial\Omega} = 0.$$

(iii) For $\lambda = \lambda^\star$ there exists a unique L^1-weak solution $u^\star$, which is also the unique weak stable solution in the sense of Definition 1. This solution is called the *extremal solution* of (2) and satisfies $u_\lambda \uparrow u^\star$ as $\lambda \uparrow \lambda^\star$.

We remark that, for $\lambda \in (0, \lambda^\star)$ there could be many classical solutions to (2), but only one is stable. For $\lambda = \lambda^\star$ instead, the weak solution is unique and it is necessarily stable, but may, or may not, be classical.

The model cases to keep in mind are:

Example 3. If $f(u) = e^u$, $n \geq 10$ and $\Omega = B_1$ then $u^\star = -2\log|x|$ and $\lambda^\star = 2(n-2)$. The restriction on $n \geq 10$ comes from the sharp constant in Hardy's inequality:

$$\int \frac{\phi(x)^2}{|x|^2} \leq \frac{4}{(n-2)^2} \int |\nabla\phi|^2.$$

Example 4. Assume $f(u) = (1+u)^p$, $n \geq 11$, $p \geq p_n$ and $\Omega = B_1$. Then $u^\star = |x|^{-2/(p-1)} - 1$ and $\lambda^\star = 2(np - n - 2p)(p-1)^{-2}$. Here $p_n > 1$ is the root of

$$\frac{n-2}{2} = \frac{p_n}{p_n - 1} + \sqrt{\frac{p_n}{p_n - 1}}.$$

2.1. **Shape of singularities.** In [1], inducting from these examples, Brezis asked: "**Open problem 4.** *Suppose $u^\star$ has an isolated singularity at $x_0 \in \Omega$. Is it true that*

$$f'(u^\star(x)) \simeq \frac{1}{|x - x_0|^2} \text{ as } x \to x_0 \text{ ?"} \tag{3}$$

Concerning the lower bound, Villegas constructed ([13]) f and a radial $u^\star$ such that

$$\liminf_{r\to 0} f'(u^\star(r))r^2 = 0.$$

Our first contribution goes in the positive direction. For any u as in Definition 1 and convex, increasing and positive f:

Theorem 5 ([10])**.** There is a dimensional constant $\varepsilon = \varepsilon(n) > 0$ such that:

$$\limsup_{r\to 0} r^{2-n} \int_{B_r \setminus B_{\varepsilon r}} f'(u) < \varepsilon \implies 0 \in \mathrm{reg}(u).$$

As an immediate corollary, if $0 \in \mathrm{sing}(u)$ then

$$\limsup_{x\to 0} f'(u(x))|x|^2 \geq \varepsilon^3 > 0.$$

3. Size of the singular set

Concerning the regularity of $u^\star$, in [1], Brezis also asked:

"**Open problem 1:** *Is there something "sacred" about dimension 10? More precisely, is it possible in "low" dimensions to construct some f (and some Ω) for which the extremal solution $u^\star$ is unbounded? Alternatively, can one prove in "low" dimension that $u^\star$ is smooth for every f and every Ω?"*

After many partial results ([12, 7, 4, 6, 14, 8, 5]), Cabré, Figalli, Serra and Ros-Oton proved this conjecture in [3], in particular showing that all weak stable solutions are bounded if $n \leq 9$ and that their gradient is always $\nabla u \in L^{2+\gamma}_{\mathrm{loc}}(\Omega)$, with $\gamma(n) > 0$.

It is natural to conjecture that, in all dimensions $\dim \mathrm{sing}(u) \leq n - 10$. This cannot follow from a Federer-type dimension reduction, for example because no monotonicity formula is available in this setting.

Instead, we prove that $f'(u) \in L^q_{\mathrm{loc}}$ for some $q > 1$. This, joined with Theorem 5 and a standard covering argument, shows that $\dim \mathrm{sing}(u) \leq n - 2q$.

We need unfortunately some additional assumptions on f, the precise result is:

Theorem 6 ([10])**.** Let $f \in C^2(\mathbb{R})$ be convex, positive and increasing such that

$$\liminf_{t\to+\infty} \frac{f''(t)f(t)}{f'(t)^2} > 0. \tag{4}$$

Let u be a weak stable solution as in Definition 1. Then $f'(u) \in L^{q-}_{\mathrm{loc}}(\Omega)$ and $\dim \mathrm{sing}(u) \leq n - 2q$, where we set

$$q := 1 + 2\liminf_{t\to\infty} \frac{\log f(t) + \int^t \sqrt{\frac{f''(s)}{f(s)}}\, ds}{\log f'(t)}. \tag{5}$$

For a "typical" f with (super)exponential growth, one finds $q = 5$. For a "typical" f with p-growth one finds $q = \frac{p+\sqrt{p(p-1)}}{p-1}$, which is in accordance with the critical exponent p_n of Example 4.

This is of course heuristic, indeed the ratio $\int^t \sqrt{\frac{f''(s)}{f(s)}}\, ds / \log f'(t)$ is very unstable by affine approximation of f (formally because $\sqrt{\delta_0} = 0$).

3.1. **Open questions.** Concerning Brezis' Open Question 4, it is not known whether

$$\limsup_{x\to 0} f'(u^\star(x))|x|^2 < +\infty.$$

Nevertheless, by the stability inequality, the averaged version (as well as the "liminf" version) is true:

$$\int_{B_r} f'(u) \leq C r^{n-2}.$$

Concerning Theorem 6, we would like to relax the assumptions on f needed to show that $f'(u) \in L^{5-}_{\mathrm{loc}}$. Heuristically, one could hope to have only qualitative assumptions involving f and f', instead of (4) and (5).

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On the one-phase problem

Xavier Fernández-Real
(joint work with Max Engelstein, Hui Yu)

In this talk, we give an introduction to the one-phase problem, and relate the study of its properties to other problems in geometric analysis, such as minimal surfaces, the obstacle problem, or the Alt–Phillips problem.

The classical one-phase problem is the study of nonnegative solutions (critical points, stable points, or minimizers) of the *Alt–Caffarelli functional*

$$\mathcal{J}_\Omega(v) = \int_\Omega |\nabla v|^2 + |\{v > 0\}|, \qquad \text{for} \quad v \geq 0, \quad v \in H^1(\Omega), \tag{1}$$

where Ω is a domain in $\mathbb{R}^n$.

Motivated by models in flame propagation and jet flows, this energy was originally studied from a mathematical point of view by Alt and Caffarelli in [1]. Since then, regularity of the minimizer and its free boundary has been extensively studied; see, for instance, [1, 3, 7, 9, 10]. We refer to [5] for a thorough introduction to the classical theory, and refer to [12] for a modern treatment of the one-phase problem and related topics.

The study of (1) follows closely the steps originally done in the study of the regularity theory for minimal surfaces (which nowadays also apply to many other settings, such as the obstacle problem or the understanding of harmonic maps). As such, the regularity of the free boundary follows by a blow-up argument and classification of global solutions, once one observes that also a monotonicity formula is available for this setting. However, contrary to what happens for minimal surfaces, not all homogeneous blow-ups have been classified in low dimensions yet.

Even for minimizers, not all homogeneous global solutions to (1) (also known as *minimizing cones*) have been fully classified. By the works of Caffarelli–Jerison–Kenig [4] and Jerison–Savin [11], it is known that for $n \leq 4$, the only homogeneous minimizer[1] is, up to a rotation, the *half-plane solution*

$$u(x) = x_n^+. \tag{2}$$

While in dimension 7, De Silva-Jerison [6] provides a nonflat minimizing cone. What happens in the remaining dimensions is still, nowadays, a mystery.

Inspired by the parallels with minimal surfaces, in the second half of the talk we present recent results in collaboration with M. Engelstein and H. Yu.

The first result concerns the existence of Bernstein-type theorems for the one-phase problem.

Originally, the Bernstein theorem for minimal surfaces reads as:

Theorem 1 (Fleming, DeGiorgi, Almgren, Simons, 1960s)**.** *Let Γ be a C^2 minimal graph in $\mathbb{R}^n$ for $n \leq 8$. Then Γ is a hyperplane. Moreover, the statement does not hold for $n \geq 9$.*

In our setting, we prove the following analogous result, where k_0^* is the lowest dimensions for which non-trivial minimizing blow-ups appear:

Theorem 2 ([8])**.** *Let u be a viscosity solution to the classical one-phase problem in $\mathbb{R}^n$, whose contact set $\{u = 0\}$ is the subgraph of a continuous function (the free boundary is a continuous graph). Then, if $n < k_0^* + 1$, up to a rotation we have $u = (x_n)_+$, and in particular, the free boundary is a hyerplane.*

Our technique is so versatile that can also be applied to the case of the thin one-phase problem, for which we also need to develop the regularity theory up to the fixed boundary.

As a consequence, by the results from Audrito–Serra, [2], we also obtain a characterization of monotone global solutions to a wide class of semilinear equations:

Theorem 3 ([8])**.** *Let u satisfy $\Delta u = f(u)$ in $\mathbb{R}^n$ for some $f \geq 0$ compactly supported with $f(0) = 0$ and $f'(0) > 0$, and $\partial_n u > 0$. Suppose, moreover, that*

$$\lim_{x_n \to -\infty} u(x', x_n) = 0 \quad \text{and} \quad \lim_{x_n \to +\infty} u(x', x_n) = +\infty.$$

Then, if $n < k_0^* + 1$, u is one-dimensional (level sets are hyperplanes).

[1] The result applies to a larger class called *stable solutions* for which one has positive density of the contact set. They are critical points of the functional (1) with nonnegative second variations.

The second recent result we have presented in this talk, concerns the generic uniqueness and regularity of solutions to the one-phase problem:

In general, minimizers to the one-phase problem are not unique. In our recent result, we show that, however, such a phenomenon is rare:

Theorem 4 ([10]). *Minimizers of $\mathcal{J}_0$ are generically unique. That is,* almost every boundary datum *admits a unique minimizer to $\mathcal{J}_0$.*

We remark that genericity is understood in a measure-theoretic or prevalence sense. That is, given a monotone family of boundary datum $\{\varphi_t\}_{t\in(0,1)}$, then the set of t for which the corresponding family of minimizers with boundary datum φ_t is non-unique is countable.

We also show a generic-type result for the regularity of free boundaries:

Theorem 5 ([10]). *Free boundaries for minimizers of the one-phase or Alt–Caffarelli problem are smooth up to dimension k_0^* generically. That is, for* almost every boundary datum*, the corresponding minimizer has a singular set of dimension $n - k_0^* - 1$.*

In fact, our results are a bit more general, and also apply to the Alt–Phillips free boundary problem.

We finish the talk by stating an important conjecture for the one-phase problem, whose minimal surface analogue has gathered some attention in the past months:

Conjecture 6. *Let u be a global stable solution to the one-phase problem in $\mathbb{R}^n$. Then, if $n \leq 6$, u is one-dimensional.*

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Stable phase transitions: open questions and new results

Joaquim Serra
(joint work with Hardy Chan, Alessio Figalli, Xavier Fernández-Real)

Surface tension and similar forces give rise to area-minimizing interfaces in various physical phenomena, which are readily observable at macroscopic scales.

However, the principle of surface area minimization does not hold uniformly across all scales, as the underlying physical energies often vary with scale. For instance, describing a soap film as an area-minimizing surface becomes implausible at scales around 5 nanometers—the approximate size of a soap molecule.

This naturally raises the question: do all stable configurations in such scale-dependent, area-like models necessarily resemble minimal surfaces at macroscopic scales? Alternatively, can certain "microscopic effects" have observable consequences at the macroscopic level?

To address this general, somewhat philosophical question more concretely, we turn to a well-known example: the Allen-Cahn energy. This phenomenological model exhibits scale-dependent behavior, approximating area minimization only at larger scales. When restricted to a domain $\Omega \subset \mathbb{R}^3$, it is given by

$$E(u;\Omega) = \int_\Omega |\nabla u|^2 + W(u)\,dx,$$

where $u : \Omega \to (-1,1)$, and $W(u)$ is a double-well potential with minima at ± 1. That is, $W(u) = 0$ for $u = \pm 1$, and $W(u) > 0$ for $-1 < u < 1$. Typical examples of $W(u)$ include $(1-u^2)^2$ or $\cos(\pi u/2)$.

Over the past two decades, the regularity theory for absolute energy-minimizing minimal surfaces has been successfully extended to several scale-dependent models, including the Allen-Cahn model (as shown by Savin in [4]). However, extending these results to encompass all stable configurations —i.e. those 'observable' in nature— remains a significant and challenging open question.

A function $u : \mathbb{R}^3 \to [-1,1]$ is called *stable critical point* of E in $\mathbb{R}^3$ if

$$\left.\frac{d}{dt}\right|_{t=0} E(u\circ\phi;\Omega) = 0 \quad \text{and} \quad \left.\frac{d^2}{dt^2}\right|_{t=0} E(u\circ\phi;\Omega) \geq 0$$

for every domain Ω compactly supported in $\mathbb{R}^3$ and for every smooth variation $\phi = \phi(x,t)$, that is for any smooth function $\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ such that $\phi(x,0) = x$ and $\phi(x,t) - \phi(x)$ is compactly supported in Ω for every t.

Classifying stable critical points of E in $\mathbb{R}^3$ remains a long-standing open problem . Solving this problem would essentially close or complete the regularity theory for stable phase transitions in three dimensions, using the deep regularity results for stable critical points developed in [5].

In the case where $W(u) = \mathbf{1}_{(-1,1)}(u)$ (that is, $W(u) = 1$ if $-1 < u < 1$ and $W(\pm 1) = 0$), critical points of the energy functional E solve a free boundary problem. See [1, 2] and references therein for more details.

In an upcoming joint work with Chan, Fernández-Real, and Figalli, we establish the following result:

> Let $u : \mathbb{R}^3 \to [-1,1]$ be a stable critical point of E, in the free boundary case where $W(u) = \mathbf{1}_{(-1,1)}(u)$. Then $D^2 u \equiv 0$ in the open set $\{-1 < u < 1\}$. In particular, the free boundaries are parallel planes.

A key and challenging step in proving the above result is the classification of stable solutions to the Alt-Caffarelli free boundary problem in $\mathbb{R}^3$, which shows that such solutions have flat free boundaries. The Alt-Caffarelli problem naturally arises from the free boundary Allen-Cahn equation after a blow-up.

The classification of stable solutions to the Alt-Caffarelli problem in $\mathbb{R}^3$ is a subtle question. Indeed, there exist finite Morse index (and hence stable outside from a compact set) solutions to this problem in $\mathbb{R}^n$ with non-flat free boundaries for every $n \geq 2$ (see [3]).

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On the monotonicity of the jump set for a denoising problem

Riccardo Cristoferi

(joint work with Rita Ferreira, Irene Fonseca, Josè Iglesias)

We consider a standard variational model for denoising of signals, the so called ROF model. Let $f \in L^2(0,1)$ and $\alpha \in (0,+\infty)$, and consider the functional $\mathcal{I}_\alpha : BV(0,1) \to [0,+\infty)$ defined, for $u \in BV(0,1)$, by

$$\mathcal{I}_\alpha(u) := \frac{1}{2}\|u - f\|^2_{L^2(0,1)} + \alpha TV(u,(0,1)).$$

Here, $TV(u,(0,1))$ denotes the total variation norm of u in $[0,1]$. For each $\alpha > 0$, we know that the minimization problem

$$\min_{u \in BV(0,1)} \mathcal{I}_\alpha(u).$$

admits a unique solution $u_\alpha \in BV(0,1)$.

The goal of this investigation is to understand the behavior of J_{u_α}, the jump set of u_α, with respect to α. In particular, we aim at proving that $\alpha \mapsto J_{u_\alpha}$ is a decreasing function. Namely, that

$$J_{u_{\alpha_2}} \subset J_{u_{\alpha_1}}, \tag{1}$$

for all $\alpha_1 < \alpha_2 \in (0, \infty)$. This behavior was conjectured in [1], and the problem has been largely open for many years. The interest in this issue is that it sheds light on the features of the ROF model, that can help practitioners with choosing it over others based on what properties of the noiseless signal they care about.

The proof of the validity of (1) in the case where the initial data f is a piecewise constant function was treated in [3]. Moreover, combining several works on the monotonicity of the jump set for the TV flow, and the equivalence of this latter with the functional $\mathcal{I}_\alpha$ in the one dimensional scalar case, in [1] it is indicated how obtain, in a rather intricate way, the result in the case the initial data $f \in BV(0,1) \cap L^\infty(0,1)$.

In this talk, we present a more direct proof of this result that also allows to weaken the assumptions on f. The strategy of the proof relies on duality arguments (see [2]). The advantage of this, is that we work with a geometric problem that, roughly speaking, it allows to compare the energy of u_α for different parameters α's.

Further investigations will tackle the higher dimensional case.

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An isoperimetric problem involving the competition between the perimeter and a nonlocal perimeter

Marc Pegon

(joint work with Michael Goldman, Benoît Merlet)

In this talk, I will present an isoperimetric problem in which the perimeter is replaced by $P - \gamma P_\varepsilon$, where $\gamma \in (0,1)$, P stands for the classical perimeter and P_ε is a nonlocal energy which converges to the perimeter as ε vanishes. This problem is derived from Gamow's liquid drop model for the atomic nucleus in the case where the repulsive potential is sufficiently decaying at infinity and in the large mass regime. I will discuss existence of minimizers, uniform (w.r.t. ε) regularity of quasi-minimizers, and characterization of minimizers for small ε.

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Critical points of degenerate polyconvex energies

André Guerra
(joint work with Riccardo Tione)

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain, and consider the polyconvex energy

$$\mathbb{E}[u] \equiv \int_\Omega g(\det \mathrm{D}u)\,\mathrm{d}x, \qquad u\colon \Omega \to \mathbb{R}^2,$$

where $g \in C^1(\mathbb{R})$ is strictly convex. Energies of this type where studied extensively in the literature, for instance in connection with elastic fluids [3, 4, 5, 8, 9]. In [6], we consider critical points of $\mathbb{E}$, i.e. solutions of the Euler–Lagrange system

$$\operatorname{div}(g'(\det \mathrm{D}u)\operatorname{cof}(\mathrm{D}u)) = 0. \tag{1}$$

It is easy to see that any $u \in C^1(\Omega, \mathbb{R}^2)$ solving (1) satisfies $\det \mathrm{D}u = c$ in Ω, for some $c \in \mathbb{R}$. Our first result is that the same rigidity continues to hold for Lipschitz solutions:

Theorem 1. Let $g \in C^1(\mathbb{R})$ be strictly convex and let $u \in \mathrm{Lip}(\Omega, \mathbb{R}^2)$ solve (1). Then there is $c \in \mathbb{R}$ such that $\det \mathrm{D}u = c$ a.e. in Ω.

There is a simple proof of Theorem 1 when $\det \mathrm{D}u \geq \delta > 0$ a.e. in Ω [8], and so the main difficulty is reducing to this case. To do so, we combine tools from continuity equations and quasiconformal maps. To be more precise, using the results of [1, 2] one can show that any Lipschitz solution of (1) is *renormalized* (in the sense of DiPerna–Lions), i.e. that

$$\operatorname{div}(\beta[g'(\det \mathrm{D}u)]\operatorname{cof}(\mathrm{D}u)) = 0 \qquad \text{for all } \beta \in C_c^\infty(\mathbb{R}).$$

Equivalently, for a fixed β we can guarantee the existence of $v \in \mathrm{Lip}(\Omega, \mathbb{R}^2)$ such that

$$\mathrm{D}v = \beta[g'(\det \mathrm{D}u)]\,\mathrm{D}u,$$

provided that Ω is simply connected. It is then easy to verify that v is a quasiregular map. The unique continuation properties of such maps, together with a suitable choice of β, then yield the conclusion.

Theorem 1 gives a rigidity result for exact solutions of (1), but it is also interesting to consider sequences of *approximate solutions*; in this case, the natural question is whether such sequences converge to an exact solution, and if so in which sense. The following theorem provides the answer to this question:

Theorem 2. Consider a sequence such that $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{Lip}(\Omega, \mathbb{R}^2)$. If

$$\mathrm{div}(g'(\det \mathrm{D}u_j)\,\mathrm{cof}(\mathrm{D}u_j)) = \mathrm{div}(F_j), \qquad \text{where } F_j \to 0 \text{ in } L^1(\Omega),$$

then $\det \mathrm{D}u_j \to \det \mathrm{D}u$ in $L^1(\Omega)$ and, in particular, u solves (1).

This last result implies that the differential inclusion associated with (1) is *quasiconvex* (i.e. it is closed under weak convergence), answering positively [7, Question 10].

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On the homogenization problem for elasto-plasticity driven by dislocation motion

Filip Rindler

(joint work with Paolo Bonicatto)

The homogenization problem in elasto-plasticity concerns the passage from discrete to fields of dislocations. While much consensus exists on what the physical laws are for the individual dislocations, it is not clear - and in fact one of the most pressing open problems of solid mechanics - how to formulate laws for the movement of dislocation fields and the corresponding elasto-plastic effects.

The main contribution of this work is to show that in a prototypical model of small-strain, geometrically linear plasticity in single crystals and with rate-independent dynamics, such a homogenization procedure from discrete dislocation lines to dislocation fields can indeed be carried out.

We represent a (single) crystal specimen as occupying a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^3$ at the initial time. A map $u = u(t)\colon \Omega \to \mathbb{R}^3$ describes the total displacement of the body at time t. A fundamental relation in linearized elasto-plasticity is the geometrically linear splitting of the displacement gradient as

$$\nabla u = e + p,$$

where $e, p\colon \Omega \to \mathbb{R}^3$ are the elastic and plastic distortion fields, respectively.

The elasticity of the specimen will lead to u being minimized over all candidate deformations. Computing the Euler–Lagrange equation, noting that $\operatorname{curl} p$ cannot be cancelled by any *gradient* ∇u, one obtains the following PDE system for the *geometrically-necessary distortion* β due to the dislocations, which are represented by $\operatorname{curl} p$:

$$\begin{cases} -\operatorname{div} \mathbb{E}\beta = 0 & \text{in } \Omega, \\ \quad \operatorname{curl} \beta = -\operatorname{curl} p & \text{in } \Omega, \\ \quad n^T \mathbb{E}\beta = 0 & \text{on } \partial\Omega. \end{cases}$$

Taking into account also further deformation that is not caused by dislocations (but by external or internal forces), we now introduce the notion of the *free (non-dislocation) deformation* as the remainder

$$\mathrm{fr}[\nabla u - p] := \nabla u - p - \beta,$$

where β is a solution to the above PDE system.

Following the approach of [2, 3, 4] we represent the movement of dislocations via their *slip trajectories* in space-time. So, let S^b be a 2-dimensional integral or normal current in $\mathbb{R}^{1+3}$, where $b \in \mathcal{B} = \{\pm b_1, \dots, \pm b_m\}$ is a Burgers vector. We then recover the dislocations for the Burgers vector b at a time t via slicing and forgetting the t-coordinate, that is,

$$S^b(t) := \mathbf{p}_*(S|_t),$$

where $\mathbf{p}(t, x) := x$. The space-time approach has many benefits, among them the straightforward definition of the *dislocation velocity* via

$$g^b := \star\left(\frac{\mathbf{p}_*(S^b|_t)}{|\nabla^{S^b}\mathbf{t}(t, \cdot)|}\right),$$

where "$\star$" denotes the Hodge star operation (which here transforms the 2-vector $\mathbf{p}_*(S^b|_t)$ to a normal vector that is orthogonal to the tangent space of the slice $S^b|_t$), and $\nabla^{S^b}\mathbf{t}$ is the projection of the gradient of $\mathbf{t}(t, x) := t$ onto the tangent space to S^b. Here we need of course assume that $\nabla^{S^b}\mathbf{t} \neq 0$ to make g^b well-defined. This condition is discussed at some length in [1], where it is shown to mean that there is no jump part in the slip trajectories, not even a jump "smeared-out" in space and time (the so-called Flat Mountain singularity).

If instead we were given a time-indexed family $(S^b(t))_t$ of slices, it would be very difficult to determine the above quantity g^b since $|\nabla^{S^b}\mathbf{t}|$ is not computable directly from the slices. Nevertheless, the (quite nontrivial) Rademacher-type differentiability result of [1] shows that such a formulation is essentially equivalent to the space-time approach when S^b is Lipschitz-continuous in time, up to dealing with a number of regularity issues. However, from an analytical point of view, the space-time formulation has many benefits due to the good compactness properties of integral (and normal) currents.

Based on the preceding discussion, we then consider the total energy functional to be given as

$$\mathcal{E}(t,u,p,(T^b)_b) := \frac{1}{2}\int_\Omega |\mathrm{fr}[Du-p]|^2_{\mathbb{E}}\,\mathrm{d}x + \frac{1}{2}\sum_{b\in\mathcal{B}} \mathbf{M}_{\psi^b}(T^b) - \int_\Omega f(t)u\,\mathrm{d}x,$$

where $\mathbb{E}$ is a symmetric fourth-order elasticity tensor, $|A|^2_{\mathbb{E}} := A : (\mathbb{E}A)$ is the associated quadratic form, and $\mathbf{M}_{\psi^b}$ is the (possibly) anisotropic mass (length) of the dislocation T^b (an integral or normal 1-current).

The coercivity of $\mathcal{E}$ implies that for all times t, the plastic distortion $p(t)$ is a measure with the property that also $\mathrm{curl}\, p$ is a measure. This implies strong restrictions on the singularities that can be present in $p(t)$. Roughly, $p(t)$ has the same dimensionality, rectifiability, and polar rank-one properties (i.e., the validity of Alberti's rank-one theorem) as BV-derivatives. We cannot expect any regularity beyond boundedness in mass for p since slips over surfaces are only representable by measures p. It is well-known that such effects can occur in real materials as *slip lines.*

The collection $\mathbf{S} = (S^b)_b$ of slip trajectories also allows us to express the corresponding evolution of plastic distortion via the *plastic flow formula*

$$p_{\mathbf{S}}(t) := p + \frac{1}{2}\sum_{b\in\mathcal{B}} b \otimes \star\mathbf{p}_*(S^b \llcorner (0,t)\times\mathbb{R}^3).$$

This plastic flow formula furthermore implies the *consistency relation*

$$\mathrm{curl}\, p(t) = \alpha(t) = \frac{1}{2}\sum_{b\in\mathcal{B}} b\otimes S^b(t),$$

meaning that our flow retains the correct relationship between the plastic distortion $p(t)$ and the dislocation system $(S^b(t))_t$ for all times t (assuming it at the initial time).

It is a crucial feature of the model introduced in the present work that the dislocation evolutions are modelled using the space-time approach. In fact, the more classical Kröner dislocation density tensor is not rich enough to account for all effects that are of relevance. On the other hand, the space-tine internal variables, from which the Kröner dislocation density can be recovered by integrating out the additional information, contain just enough additional information to make the discrete-to-field limit passage possible.

Technically, our main results shows the existence of *Mielke–Theil energetic solutions* to the limit evolutionary system (involving dislocation fields) that are obtained as limits of discrete evolutions (involving individual dislocation lines). Thus, the limit model of elasto-plasticity driven by dislocation fields may be considered to be well-justified from microscopic principles.

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A priori bounds for geodesic diameter

Ulrich Menne

(joint work with Christian Scharrer)

We presented selected results from our series [MS22, MS23, MS24] whose terminology we employ. In particular, the *geodesic diameter* of a closed subset A of $\mathbf{R}^n$ is the supremum of all numbers $\sigma(a,x)$ corresponding to $a, x \in A$, where $\sigma(a,x)$ is the infimum of the set of lengths of continuous paths in A connecting a and x.

Always, *suppose m and n are integers with $2 \le m \le n$.* The starting point of our research was the following result of P. Topping (for immersions) which is based on the monotonicity identity through the Sobolev inequality on M.

Theorem (see [Top08, 1.1])**.** *Suppose M is a connected m dimensional compact submanifold of class 2 of $\mathbf{R}^n$.*

Then, the geodesic diameter of M does not exceed

$$\Gamma \textstyle\int_M |\mathbf{h}(M,x)|^{m-1} \,\mathrm{d}\mathscr{H}^m x,$$

where Γ is a positive real number determined by m.

In generalising this result, we had three aims:

(1) the transfer to the non-smooth setting,
(2) the inclusion of boundary in the treatment, and
(3) the applicability to geometric variational problems.

These aims were accomplished by means of our following theorem phrased in the varifold setting which is the natural one due to the involvement of mean curvature.

Theorem (see [MS24, 7.4])**.** *Suppose V and W are varifolds in $\mathbf{R}^n$, $\dim V = m$, $\dim W = m-1$, $\|\delta V\|$ and $\|\delta W\|$ are Radon measures,*

$$\boldsymbol{\Theta}^m(\|V\|, x) \ge 1 \quad \text{for } \|V\| \text{ almost all } x,$$

$$\boldsymbol{\Theta}^{m-1}(\|W\|, x) \ge 1 \quad \text{for } \|W\| \text{ almost all } x,$$

$$\|\delta V\| \le \|V\| \llcorner |\mathbf{h}(V,\cdot)| + \|W\|, \quad \|\delta W\| \le \|W\| \llcorner |\mathbf{h}(W,\cdot)|,$$

$(\|V\| + \|W\|)(\mathbf{R}^n) < \infty$, V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, and d is the geodesic diameter of $\operatorname{spt} \|V\|$*.*

Then, for some positive finite number Γ *determined by* m*, there holds*

$$d \le \Gamma\big(\textstyle\int |\mathbf{h}(V,\cdot)|^{m-1} \,\mathrm{d}\|V\| + \int |\mathbf{h}(W,\cdot)|^{m-2} \,\mathrm{d}\|W\| \big);$$

here, by convention, we stipulate $\int |\mathbf{h}(W,\cdot)|^0 \,\mathrm{d}\|W\| = \|W\|(\mathbf{R}^n)$ *regarding* $m = 2$*.*

This includes the smooth case of a submanifold-with-boundary and entails the immersed case by means of differential-topological density results. The implication that the finiteness of the sum of the mean curvature integrals implies the finiteness of d is new even when $W = 0$. Aside of including boundary, the two most significant challenges involved in establishing our preceding theorem were

(i) how to rephrase the connectedness hypothesis for varifolds; and,
(ii) how to handle the low summability of the mean curvature.

The following simple examples of varifolds comprised of countable sums of spheres already capture the corresponding key phenomena.

Example. There exists an m dimensional varifold V in $\mathbf{R}^{m+1}$ corresponding to a countable sum of spheres such that $\operatorname{spt} \|V\|$ is compact, $\|\delta V\|$ is a Radon measure absolutely continuous with respect to $\|V\|$, hence $\|\delta V\| = \|V\| \llcorner |\mathbf{h}(V,\cdot)|$, and

$$\textstyle\int |\mathbf{h}(V,\cdot)|^{m-1} \,\mathrm{d}\|V\| < \infty = d.$$

In fact, $\operatorname{spt}\|V\|$ can be prescribed to equal the closure of any bounded open subset of $\mathbf{R}^n$ if $m < n$, see [Men16, 14.1].

The indecomposability hypothesis in the theorem rules out such behaviour; in that setting, the critical scaling is given by cylinders rather than spheres.

Example (see [MS23, 6.4, 6.5, 10.20])**.** Suppose V corresponds to the union of two touching spheres. Then, V is decomposable but indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$.

Indecomposability of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$ can be exploited by co-area type considerations, see [MS23, 7.11, 7.12], and is entailed by connectedness of $\operatorname{spt}\|V\|$ in case δV is well-behaved, see [MS23, 10.20, 10.22]. Here, well-behaved means that δV satisfies a dimensionally critical summability condition with respect to $\|V\|$ possibly in conjunction with a smooth Dirichlet or Neumann boundary condition, see [MS23, 9.1, 9.16].

Our theorem admits a wide range of applications to geometric variational problems including several formulations of Plateau's problem. For the Plateau problem in the sense of Reifenberg, thoroughly studied by the group of G. David, we arrive at the following corollary which combines our previous theorem with results of F. Almgren, H. Pugh, and C. Labourie in [Alm76, Pug19, Lab22].

Corollary (see [MS24, Theorem B])**.** *Suppose* B *is a nonempty compact connected* m *dimensional submanifold of class* 2 *of* $\mathbf{R}^n$*,* G *is a commutative group,* L *is a subgroup of the* $(m-1)$*-th Čech homology group of* B *with coefficients in* G*,* $\check{\mathscr{C}}(B, L, G)$ *denotes the family of compact subsets of* $\mathbf{R}^n$ *spanning* L*,*

$$E \in \check{\mathscr{C}}(B, L, G), \quad \mathscr{H}^m(E) = \inf\big\{ \mathscr{H}^m(F) : F \in \check{\mathscr{C}}(B, L, G) \big\},$$

$A = \operatorname{spt}(\mathscr{H}^m \llcorner E)$*, and* d *is the geodesic diameter of* A*.*

Then, for some positive finite number Γ *determined by* n, *there holds*

$$d \le \Gamma \operatorname{reach}(B)^{-m} \mathscr{H}^{m-1}(B)^{m/(m-1)} \int_B |\mathbf{h}(B,b)|^{m-2} \, \mathrm{d}\mathscr{H}^{m-1}\, b;$$

here, by convention, we stipulate $\int_B |\mathbf{h}(B,b)|^0 \, \mathrm{d}\mathscr{H}^1\, b = \mathscr{H}^1(B)$ *regarding* $m = 2$.

Due to the inevitable singularities of A, even finiteness of d is new. F. Almgren's study in [Alm76] ensures that E is $(\mathscr{H}^m, m)$ rectifiable; H. Pugh's isoperimetric inequality in [Pug19] yields the natural estimate of $\mathscr{H}^m(E)$ in terms of $\mathscr{H}^{m-1}(B)$; C. Labourie's homological considerations in [Lab22] crucially entail

$$B \subset A;$$

and the m dimensional varifold V associated with E clearly satisfies $\operatorname{spt} \delta V \subset B$. With these results at hand, we merely require two observations to make our preceding theorem applicable to V: Firstly, refining W. Allard's estimates regarding boundary behaviour in [All75], we show that

$$\|\delta V\| \le \Gamma \lambda \mathscr{H}^{m-1} \llcorner B, \quad \text{where } \lambda = \operatorname{reach}(B)^{-m} \|V\|(E)$$

and Γ is a positive finite number determined by m, see [MS24, 8.12]. Secondly, the study of connected components of $A \sim B$ in [Men16, 6.14] in conjunction with the isoperimetric inequality allows to readily deduce connectedness of A from that of B, see [MS24, 8.13].

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On Scaling Laws for Shape-Memory Alloys – Between Rigidity and Flexibility

Angkana Rüland
(joint work with Janusz Ginster, Antonio Tribuzio, Barbara Zwicknagl)

In the modelling of many shape-memory alloys a striking dichotomy between rigidity and flexibility emerges. On the one hand, if the surface energy is assumed to be bounded (e.g., by imposing BV regularity on the deformation gradient), exactly stress-free solutions are rather rigid [6, 8]. They obey the kinematic compatibility conditions and often form microstructures such as simple laminates or crossing twins which are also experimentally well documented [1]. If, on the other hand, no additional surface energy constraints are imposed, in many settings a plethora of highly non-unique solutions exists which can deviate substantially from the rigid configurations [10, 5, 2, 14].

In this talk, based on the observations from [12], I adopt a scaling perspective to infer finer information on this transition and the potential complexity of microstructures in situations between rigidity and flexibility. To this end, I consider energies of the form

$$E_\epsilon(u) := E_{el}(u) + \epsilon E_{surf}(u),\ \epsilon > 0.$$

Here $E_{el}(u)$ encodes the elastic energy and is typically of the following form

$$E_{el}(u) = \int\limits_{\Omega} \mathrm{dist}^2(\nabla u, K)dx.$$

In this context $\Omega \subset \mathbb{R}^n$ is the reference configuration, $u : \Omega \to \mathbb{R}^n$ models the deformation and $K \subset \mathbb{R}^{n\times n}$ denotes the stress-free strains which are typically of the form $K = \bigcup\limits_{j=1}^{m} SO(n)U_j$ for $U_j = U_j^t \in \mathbb{R}^{n\times n}_{>0}$. The energy $E_{surf}(u)$ is a higher order penalization, e.g., $E_{surf}(u) = \|\nabla^2 u\|_{TV(\Omega)}$, penalizing the formation of extremely fine structures. The constant $\epsilon > 0$ is a material specific parameter, which in many situations should be considered as being very small.

This talk particularly focuses on two settings: Firstly, I consider the situation of the square-to-rectangular phase transformation with identity boundary conditions which physically corresponds to austenite boundary conditions. In this setting, it is possible to form quite flexible microstructures at extremely low energetic cost. For instance, this includes the formation of self-similar star-type constructions [3, 4] with an only linear scaling in $\epsilon > 0$ and which are thus also of substantial interest in nucleation phenomena. In addition to this qualitative information on particular microstructures, in this setting, I discuss a scaling law quantifying the Hadamard jump condition. In terms of scaling, this provides rather sharp information on isoperimetric geometries. More precisely, I show that in generic domains logarithmic losses to the linear scaling law occur, while in specific geometries which are tailored to the well geometry it is possible to deduce linear scaling [7].

Secondly, I discuss a very rigid setting displaying a first (weak) transition between rigidity and flexibility. More precisely, I focus on the Tartar square and present an associated scaling law for it [9]. In this setting, dropping frame-indifference, the set K consists of four matrices only. These are of a particular structure ruling out the presence of rank-one connections and are such that the associated 4-well problem in the exact setting is rigid, while in the approximate setting it becomes flexible. The transition to flexibility is accompanied with the formation of infinite-order laminates, a rather complex class of microstructures. In this talk, I show how this is captured in a scaling law which is of subalgebraic but superlogarithmic behaviour [13].

I further present various generalizations of these ideas and scaling laws.

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Stability aspects of the Möbius group of $\mathbb{S}^{n-1}$

Konstantinos Zemas

(joint work with Stephan Luckhaus, Jonas Hirsch, André Guerra and Xavier Lamy)

We discuss quantitative stability aspects of the class of Möbius transformations of the sphere among maps in the critical Sobolev space $W^{1,n-1}(\mathbb{S}^{n-1})$. The special case of $\mathbb{S}^{n-1}$-valued and the more general case of $\mathbb{R}^n$-valued maps will be addressed. In the latter, more flexible setting, unlike similar in flavour results for maps defined on domains, not only a conformal deficit is necessary, but also a deficit measuring the distortion of $\mathbb{S}^{n-1}$ under the maps in consideration. The latter is introduced as an associated isoperimetric deficit. In all cases, the corresponding stability estimates are optimal in terms of the conformally invariant deficits and the distance in $\dot{W}^{1,n-1}(\mathbb{S}^{n-1})$ to the Möbius group.

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Γ-limit for zigzag domain walls

Hans Knüpfer

(joint work with W. Shi)

Transition layers in the classical Aviles–Giga problem are typically one-dimensional. In micromagnetism one also observes two-dimensional transition layers. One such example is the so called zigzag wall observed in thin films. We derive the Γ-limit to an anisotropic perimeter problem.

Generic Uniqueness and Multiplicity One Property for area-minimizing Currents

Simone Steinbrüchel

(joint work with G. Caldini, A. Marchese, A. Merlo)

Finding the surface with least area among those having the same boundary is called the Plateau Problem and has been an active topic of research for more than hundred years now. Several mathematical models have been proposed to study such surfaces. The one we focus on, are the area-minimizing integral currents introduced by Federer and Fleming in the 60's. Similarly to Sobolev functions, currents are functionals acting on differential forms and in case of smooth oriented submanifolds, the action is given by integrating the differential form ω over the manifold M

$$T(\omega) := \int_M \omega \,. \tag{1}$$

For such T, its operator norm (which we call *mass*) is then exactly the area of M. Integral currents are the compactification of such "smooth" currents as in (1). To be more precise, they integrate over a rectifiable set against an integer-valued multiplicity function which counts how often a piece of the rectifiable set is counted. Together with a bound on their mass and on the mass of their boundary, they form a compact set of currents. This implies that when we fix the boundary, we can minimize the mass and find an optimal integral current which we call area-minimizing.

Together with G. Caldini, A. Marchese, and A. Merlo, we asked in [1] the question of uniqueness, e.g. whether for a fixed boundary, there exists only *one* area-minimizing integral current having this boundary. We were not the first people working on this question. Namely, Morgan proved in [3] that there are very symmetric examples where uniqueness failes. However, under some assumptions (for instance connected boundaries and codimension one), such boundaries with several minimizers are rare. Both Morgan's and our result rely on regularity theorems on the supporting set of the minimizer. We are using the recent result [2] to prove that in the space of $C^{3,\alpha}$-boundaries, the set of boundaries with a unique area-minimizing current, is residual. The main new aspect of our theory compared to the one of Morgan relies on the fact that we can exclude the currents with two-sided boundaries. In turn, this boils down to a taylor reminder estimate implying that most $C^{3,\alpha}$-curves of dimension $(m-1)$ do not lie in a (relatively) open set of a $C^{3,\beta}$-manifold of dimension m. By construction, these remaining currents then must have multiplicity one everywhere.

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Regularity in $\mathscr{A}$-quasiconvex variational problems

Zhuolin Li
(joint work with Bogdan Raiță)

The notion of *quasiconvexity* was introduced by Morrey to investigate the scope of the direct method of the calculus of variations, and the variational problems he considered are of the following form:

$$\mathcal{F}(u) = \int_\Omega f(Du(x))\,\mathrm{d}x,$$

where $u\colon \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^N$ and $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$. Under suitable assumptions, the quasiconvexity of f is equivalent to the lower semicontinuity of $\mathcal{F}$ [13], which, together with its connection with coercivity [5], makes it the natural framework for vectorial variational problems.

On the other hand, non-convex variational problems have been studied widely, for instance in connection with continuum mechanics or with gradient flows (see, for example, [1, 7, 2, 9, 4, 15]). Apart from the gradient operator D in $\mathcal{F}$, more general partial differential operators are involved in these problems, examples including div, curl, $\mathcal{E}(\mathcal{E}u = \frac{1}{2}(Du+(Du)^T))$, and d (the exterior derivative operator).

In this talk, we consider constant rank operators. Given a homogeneous partial differential operator $\mathscr{A} := \sum_{|\alpha|=\ell} A_\alpha \partial^\alpha$ with constant coefficients $A_\alpha \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^m)$, it is said to have *constant rank* $r \in \mathbb{N}$ if $\mathrm{rank}\mathscr{A}[\xi] = r$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$, where $\mathscr{A}[\xi] = \sum_{|\alpha|=\ell} A_\alpha \xi^\alpha$ is the symbol of $\mathscr{A}$.

Under the constant rank condition, Fonseca and Müller proved the equivalence of $\mathscr{A}$-quasiconvexity and the lower semicontinuity of the corresponding functional with respect to $\mathscr{A}$-free sequences [10]. An integrand $f\colon \mathbb{R}^d \to \mathbb{R}$ is *$\mathscr{A}$-quasiconvex* if there holds

$$\int_{(0,1)^n} f(z + w(x))\,\mathrm{d}x \geq f(z)$$

for any $z \in \mathbb{R}^d$ and any $w \in C_c^\infty((0,1)^n, \mathbb{R}^d)$ with $\mathscr{A}w = 0$. The result in [10] is done for non-negative integrands, and was generalised to signed integrand in [12].

With the lower semicontinuity result, it is then possible to consider the following variational problem

$$I_{v_0}(w) = \int_\Omega f(v_0 + w(x))\,\mathrm{d}x, \quad w \in C_c^\infty(\Omega, \mathbb{R}^d),\ \mathscr{A}w = 0,$$

where $f\colon \mathbb{R}^d \to \mathbb{R}$ is strongly $\mathscr{A}$-quasiconvex and of p-growth ($|f(z)| \leq L(1+|z|^p)$) for some $1 < p < \infty$, and $v_0 \in L^p(\Omega, \mathbb{R}^d)$ satisfies $\mathscr{A}v_0 = 0$. The direct method can be applied, with the lower semicontinuity and coercivity of I_{v_0}, to obtain the existence of (at least) one minimizer of I_{v_0} in $L^p_{0,\mathscr{A}}(\Omega, \mathbb{R}^d)$ (the closure of the space $\{w \in C_c^\infty(\Omega, \mathbb{R}^d) \colon \mathscr{A}w = 0\}$ in $L^p(\Omega, \mathbb{R}^d)$).

In this project, we investigate the regularity of the minimizers, and obtain the following partial regularity result:

Theorem 1. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, and the C^2 integrand $f\colon \mathbb{R}^d \to \mathbb{R}$ is strongly $\mathscr{A}$-quasiconvex and of p-growth, where $\mathscr{A}$ is a constant rank operator. Fix $v_0 \in L^p(\Omega, \mathbb{R}^d)$ with $\mathscr{A} v_0 = 0$, and suppose that $w \in L^p_{0,\mathscr{A}}(\Omega, \mathbb{R}^d)$ is a minimizer of I_{v_0}. Then there exists a closed subset $S_w \subset \Omega$ with $\mathcal{H}^n(S_w) = 0$ such that $v_0 + w \in C^{0,\alpha}_{loc}(\Omega \setminus S_w, \mathbb{R}^d)$ for any $\alpha \in (0,1)$.

For any minimizer $w \in L^p_{0,\mathscr{A}}(\Omega, \mathbb{R}^d)$ of I_{v_0}, there exists $\tilde{w}_0 \in L^p(\Omega, \mathbb{R}^d)$ with $\mathscr{A} \tilde{w}_0 = 0$ such that

$$I_{v_0}(w) = \inf\{\mathcal{F}_{w_0}(\varphi) \colon \varphi \in C^\infty_c(\Omega, \mathbb{R}^e)\},$$

where $w_0 = v_0 + \tilde{w}_0$,

$$\mathcal{F}_{w_0}(\varphi) = \int_\Omega f(w_0 + \mathscr{B}\varphi(x))\,\mathrm{d}x,$$

and $\mathscr{B}$ is a potential operator (of constant rank and of k-th order) of $\mathscr{A}$ ([14]). Moreover, $v_0 + w$ can be locally expressed as $\mathscr{B}u$ with some $W^{k,p}$ map u. Therefore, we can consider the following variational problem

$$\mathcal{F}(u) = \int_\Omega f(\mathscr{B}u(x))\,\mathrm{d}x$$

instead, one illustrative prototype example of which is $\int f(d\beta)$. Partial regularity results in this setting can also be found in [11] (where $\mathscr{B}$ is elliptic and $p = 1$), and [6] (where $\mathscr{B}$ is $\mathbb{C}$-elliptic and $1 < p < \infty$).

To show the partial regularity result, the excess decay estimate strategy is used here. Compared to the classical situation (see, for example, [8]), there are extra difficulties due to the degeneracy of constant rank operators. Given a function $u \in C^\infty_c(\mathbb{R}^n)$, it is possible to recover $\widehat{D^k u}$ from $\widehat{\mathscr{B}u}$ in the frequency space, and thus conclude the following estimate

$$\|u\|_{L^p} \leq C\|D^k u\|_{L^p} \leq C\|\mathscr{B}u\|_{L^p}$$

if $\mathscr{B}$ is an elliptic operator (see [3]). A local version of the above estimate can also be shown by doing cut-off. This is, however, not possible for constant operators since $\mathscr{B}u$ only carries partial information of $D^k u$. Consequently, the usual Poincaré inequality is not expected here, and the linearised Euler-Lagrange system is not elliptic, which make the excess decay estimate more challenging. We discuss in the talk how to overcome the difficulties caused by the degeneracy mentioned above.

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A Savin-type theorem in codimension two

Alessandro Pigati

(joint work with Guido De Philippis, Aria Halavati)

Starting from the pioneering ideas of De Giorgi, Modica, Ilmanen, and Hutchinson–Tonegawa, it was understood that smooth critical points $u : M \to \mathbb{R}$ of the *Allen–Cahn energy*

$$E_\epsilon(u) := \int_M \left[\epsilon|du|^2 + \frac{(1-u^2)^2}{4\epsilon}\right]$$

are effective diffuse approximations of minimal hypersurfaces in a given Riemannian ambient (M^n, g). The Allen–Cahn functional is a well studied model for phase transitions; a typical critical point u takes values in $[-1, 1]$, with $u \approx \pm 1$ (the pure phases) except in a transition region of thickness $\simeq \epsilon$, where most of the energy concentrates. Roughly speaking, this region is an ϵ-neighborhood of a minimal hypersurface, which acts as an interface between the two phases.

In codimension two, similar attempts have been made by looking at the same energy for maps $u : M \to \mathbb{C}$, replacing u with $|u|$ in the second term. This corresponds to a simplified version of the Ginzburg–Landau model of superconductivity, where one neglects the magnetic field. However, the asymptotic analysis of this energy is substantially more involved, due to a slow energy density decay off the zero set, and brought mixed results.

On the other hand, including the magnetic field and looking at the so-called *self-dual regime* (also called *critical coupling*), we can consider the energy

$$E_\epsilon(u, \alpha) := \int_M \left[|du - i\alpha u|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2} + \epsilon^2|d\alpha|^2\right].$$

It differs from the previous energies by an additional variable, the one-form $\alpha \in \Omega^1(M;\mathbb{R})$, which twists the Dirichlet term and appears in the Yang–Mills term $|d\alpha|^2$ (the latter equals $|F_\nabla|^2$, where F_∇ is the curvature of the unitary connection $\nabla := d - i\alpha$ on the trivial complex line bundle $\mathbb{C} \times M$).

This energy is well known in gauge theory, where it is often called $U(1)$-*Yang–Mills–Higgs*, or simply *abelian Higgs model.* It received a thorough treatment in dimension two, with a complete classification of critical planar pairs (u, ∇) of finite energy by Taubes. Recently, in [3], Stern and the speaker developed the asymptotic analysis in arbitrary Riemannian manifolds, obtaining the precise codimension-two analogue of the result by Hutchinson–Tonegawa.

Based on some new functional inequalities, Halavati recently obtained a quantitative refinement of the work of Taubes, who showed (among other facts) that critical pairs on the plane minimize the energy among pairs with the same degree at infinity: namely, in [2] a quantitative *stability* is proved. Together with the main result from [3], this result is instrumental for the proof of the results discussed below.

Since the work of De Giorgi and Allard [1], it is known that almost-flat minimal submanifolds enjoy an *improvement of flatness*, i.e., they become even closer to a plane at smaller scales, in a quantitative way. Iteration of this improvement of flatness is the key mechanism in proving regularity of minimal submanifolds.

A related question, in the spirit of the classical Liouville theorem, is *Bernstein's problem*, predicting that minimal graphs $\mathbb{R}^{n-1} \to \mathbb{R}$ are necessarily planar. More generally, one can ask the same for area-minimizing hypersurfaces in $\mathbb{R}^n$. In dimension $n \leq 7$, by dimension reduction and classification of stable smooth cones, several combined contributions proved that such hypersurfaces are flat at infinity. In view of improvement of flatness, this gives an affirmative answer for $n \leq 7$, while for $n \geq 8$ counterexamples were found (the dimension ranges are shifted up by 1 for the special case of minimal graphs).

By analogy, De Giorgi conjectured that critical points $u : \mathbb{R}^n \to \mathbb{R}$ of the Allen–Cahn energy with $\frac{\partial u}{\partial x_n} > 0$ are just rotations of a one-dimensional solution $u = u(x_n)$, when $n \leq 8$; usually, one also adds the assumption that $u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$, so that level sets $u^{-1}(\lambda)$ for $\lambda \in (-1, 1)$ are graphs on $\mathbb{R}^{n-1}$. After some important partial results by others, in [4] Savin settled the conjecture. In fact, his main contribution could be phrased as follows.

Theorem (Savin's theorem)**.** *A local minimizer u for Allen–Cahn enjoys improvement of flatness. In particular, if any blow-down is a hyperplane, then the blow-down is unique.*

Here the blow-downs can be understood in terms of energy concentration, or by looking at the blow-downs of the zero set $\{u = 0\}$ with respect to the (local) Hausdorff convergence of sets.

The previous statement implies the resolution of De Giorgi's conjecture for $n \leq 8$: the classification of blow-downs can be directly exported from the setting of minimal hypersurfaces; finally, using maximum principle techniques, from the uniqueness of the blow-down one can deduce that u is one-dimensional.

Savin's approach uses viscosity techniques, resembling the Krylov–Safanov theory in spirit. In particular, it is not always clear how one can extend these techniques to the vectorial setting, where the maximum principle does not apply. Recently, Wang [5] obtained a variational proof of Savin's theorem, following the strategy of Allard's proof of excess decay for stationary varifolds. Inspired by Wang's approach, we obtained the following theorem.

Theorem. *Savin's result, as stated above, holds for critical pairs (u, ∇) for E_ϵ, in any dimension $n \geq 2$.*

The following is the precise statement of the excess decay for critical points.

Theorem (Tilt-excess decay). *For any $n \geq 3$ and small enough $0 < \rho \leq \rho_0(n)$, there exist constants $\epsilon_0(n,\rho), \tau_0(n,\rho)$ such that the following holds. Let (u, ∇) be a critical point for E_ϵ on the unit ball $B_1^n \subset \mathbb{R}^n$, with $\epsilon \leq \epsilon_0$, $u(0) = 0$. If*

$$\int_{B_1^n} e_\epsilon(u, \nabla) \leq 2\pi\omega_{n-2} + \tau_0,$$

then at least one of the following statements is true: either

$$\mathrm{E}_1(u, \nabla, B_\rho^n, \bar{S}) \leq C\rho^2 \mathrm{E}_1(u, \nabla, B_1^n, S),$$

for some new $(n-2)$-plane $\bar{S}$ (where S minimizes excess on B_1^n), or

$$\mathrm{E}_1(u, \nabla, B_1^n, S) \leq \max\{C\epsilon^2 |\log \mathrm{E}|^2 \sqrt{\mathrm{E}}, e^{-K/\epsilon}\},$$

where $\mathrm{E} = \mathrm{E}(u, \nabla, B_1^n, S)$ and $C = C(n)$, $K = K(n)$ are independent of ρ.

Here E is the *excess*, which naturally splits into two parts, E_1 and E_2, measuring how far a solution is from being two-dimensional and from solving the first order *vortex equations*, respectively; in fact, E_1 parallels the notion of excess in the theory of varifolds and does not depend on the orientation of S, while E mimics the stronger notion of excess in the setting of currents. While the previous result establishes a quantitative decay only for E_1, it is enough to obtain the main result.

Differently from the codimension one setting, where uniqueness of the blow-down (with multiplicity one) implies via the maximum principle that u is one-dimensional, at the present time we are not able to conclude that the solution u is two-dimensional. We conjecture that it is always true in this low-energy regime, up to change of gauge. On the other hand, our excess decay is strong enough to give an affirmative answer in some cases, giving in particular a full analogue of Savin's theorem.

Theorem. *The previous conjecture holds for critical points in dimension $2 \leq n \leq 4$, as well as for local minimizers in all dimensions $n \geq 2$.*

Compared to [5], there are several key differences which require substantially new ideas. For instance, in order to construct the Lipschitz approximation, Wang uses a generic level set of u. For the abelian Higgs model, level sets of u can be arbitrarily irregular, due to gauge invariance. Rather, we rely on [2] in order to control in a fine way the behavior of u on many two-dimensional slices perpendicular to the reference plane, e.g., to bound the distance of the zero set from a certain "center of mass" of each slice.

In the case of minimizers, this refined control, used in a very involved gauge fixing argument, allows us to deform a nearly flat minimizing pair (u, ∇) in the interior to gain a *stronger* decay of the excess. This improved decay proves our conjecture in arbitrary dimension n, in the case of local minimizers.

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Boundary unique continuation of harmonic functions

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(joint work with Carlos Kenig)

The classical unique continuation theorem says: if a harmonic function u vanishes at a point at infinite order (that is, near that point u decays to zero faster than polynomials of any degree), then u must vanish everywhere in a connected set containing that point. This is a fundamental property of harmonic functions, as well as solutions to a large class of elliptic and parabolic PDEs. In the same spirit, mathematicians are interested in *quantitative* unique continuation results, which are to use the local information about the growth rate of a harmonic function to study its global behaviors. In particular, we are interested in studying, for a non-trivial harmonic function u, how big its singular set $\mathcal{S}(u) := \{u = 0 = |\nabla u|\}$ and critical set $\mathcal{C}(u) := \{|\nabla u| = 0\}$ can be.

As is well known, the classical unique continuation property is in general false at the boundary (even if the boundary is flat), so there are many new problems and obstacles to study boundary unique continuation properties. More precisely, the study of the size of the singular/critical set *at the boundary* is related to a classical question asked by L. Bers: considering a domain Ω in $\mathbb{R}^n$ (with $n \geq 3$) and a harmonic function u in Ω, if both u and ∇u vanish on a boundary set with positive surface measure (i.e. the $(n-1)$-dimensional Hausdorff measure restricted to the boundary), does it necessarily follow that u must vanish everywhere in the domain? In general, the answer is no even for the upper half-space (by a counter-example of Bourgain and Wolff in [2]), unless one assumes a priori that $u \equiv 0$ on a relative open set of the boundary. There have been many attempts to answer this question for different classes of domains, and so far the best known result is the following theorem by Tolsa (see also related work in [4]):

Theorem 1 (Theorem 1.1 in [15]). Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, B be a ball centered on $\partial\Omega$, and suppose that $\Sigma := B \cap \partial\Omega$ is a Lipschitz graph with slope at most τ_0 (τ_0 is a small positive constant depending only on the dimension n). Let $u \in C(\overline{\Omega})$ be a harmonic function in Ω. Suppose that u vanishes on Σ and that

$$\mathcal{H}^{n-1}(\{x \in \Sigma : \nabla u(x) = 0\}) > 0.$$

Then $u \equiv 0$ in Ω.

It is still an open question whether the same statement holds if we only assume Ω is a Lipschitz domain (and remove the smallness assumption on the Lipschitz constant).

On the other hand, it has been observed that the singular set of a harmonic function (in the interior) is $(n-2)$-dimensional, see for example [6, 5, 10]; moreover, the work of Cheeger, Naber and Valtorta in [3, 12] give quantitative estimates of the $(n-2)$-dimensional size of the singular/critical set. Inspired by this, Carlos Kenig and I set out to give a fine estimate of the size of the singular/critical set *at the boundary*, as in the setting of Bers and the above theorem by Tolsa. Roughly speaking we prove the following theorem:

Theorem 2 (Theorem 1.1 in [7]). Let $\Omega \subset \mathbb{R}^n$ be a C^1-Dini domain, and B be a ball centered in $\partial\Omega$. Let $u \in C(\overline{\Omega})$ be a non-trivial harmonic function in $\Omega \cap 5B$ such that $u \equiv 0$ on $\partial\Omega \cap 5B$. Then the singular set $\mathcal{S}(u) := \{x \in \overline{\Omega} : u(x) = 0 = |\nabla u(x)|\}$ satisfies that $\mathcal{S}(u) \cap B$ is $(n-2)$-rectifiable, and

$$\mathcal{H}^{n-2}(\mathcal{S}(u) \cap B) \leq C,$$

where C depends only on the dimension n and the upper bound of the growth rate of u in $5B$ (or more precisely, the modified frequency function of u in the ball $5B$).

The assumption that Ω is a C^1-Dini domain means that locally, Ω is the region above the graph of a C^1-function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, such that the modulus of continuity of $\nabla\varphi$ satisfies a Dini condition. In particular, every $C^{1,\alpha}$ domain with $\alpha \in (0, 1)$ satisfies this assumption. It is worth remarking that in [9], we also give countrerexamples (using the tools of harmonic measures) to illustrate that C^1-Dini domains is the optimal class of domains for which the singular set has finite $\mathcal{H}^{n-2}$-measure, even when the harmonic function is additionally assumed to be non-negative.

For the full critical set $\mathcal{C}(u)$, the statement analogous to Theorem 2 does hold for $C^{1,\alpha}$ domains, which we prove in [11]. Our estimates of the size and structure of the singular/critical set are inspired by the work of Naber and Valtorta (see [13, 14]), which has since found many applications in geometric variational problems, as well as the work of Adolfsson and Escauriaza [1] to tackle the difficulties arising from the boundary.

In a follow-up work, we also use PDE methods to show that centered at every boundary point, the harmonic function has an asymptotic expansion as follows:

Theorem 3 (Theorem 1.1 in [8])**.** Under the same assumption as in Theorem 2, we have that for every $x \in \partial\Omega \cap B$, there exists $r > 0$ such that

$$u(y) = P_N(y - x) + \psi(y - x) \quad \text{ in } B_r(x) \cap \Omega,$$

where P_N is a non-trivial homogeneous harmonic polynomial of degree $N \in \mathbb{N}$, the error term ψ satisfies

$$|\psi(z)| \leq C|z|^N \theta(|z|), \quad |\nabla\psi(z)| \leq C|z|^{N-1}\theta(|z|),$$

and $\theta(r) \to 0$ as $r \to 0$, with a decay rate determined by the Dini parameter of the domain Ω.

Lastly but not least, it is worth mentioning that our studies of the singular/critical set achieve the sharp dimensions and regularities, but the volume bound depends exponentially on the growth rate of the function. The latter is far from being optimal. It is a challenging open question, even in the interior (for an elliptic operator with Lipschitz coefficient matrix), to achieve the optimal dependence on the growth rate. This has only been done in the two-dimensional case, see [16] (interior case) and [11] (boundary case).

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A variational approach to generalized Newtonian Navier-Stokes

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(joint work with Christina Lienstromberg, Stefan Schiffer)

We study the non-Newtonian Navier–Stokes system

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla \pi + \operatorname{div}\big(2\mu(|\epsilon(u)|)\epsilon(u)\big), & t > 0,\, x \in \mathbb{T}_d \\ \operatorname{div} u = 0, & t > 0,\, x \in \mathbb{T}_d \\ u(0,x) = u^0(x), & x \in \mathbb{T}_d, \end{cases} \tag{1}$$

describing the flow of an incompressible viscous non-newtonian fluid on the d-dimensional torus $\mathbb{T}_d$, $d \geq 2$. Here, $\epsilon = \epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ denotes the *rate-of-strain*, and the function $\mu \colon [0, \infty) \to \mathbb{R}_+$ is the strain-dependent *viscosity* of the fluid.

For $\eta > 0$ we define the functional

$$I_\eta(u) = \int_0^\infty \int_{\mathbb{T}_d} e^{-t/\eta} \left(\tfrac{1}{2} |\partial_t u + (u \cdot \nabla) u|^2 + \tfrac{1}{\eta} W(\epsilon(u)) + \tfrac{C_4}{4} |\nabla u|^4 \right) \mathrm{d}x \mathrm{d}t,$$

where W is such that $DW(\epsilon) = 2\mu(|\epsilon|)\epsilon$ and with p-growth and p-coercivity. Minimizing this functional corresponds to an elliptic regularization of (1) as can be seen from the Euler-Lagrange equation. In [3] we prove the following

Theorem 1 (Existence of Leray–Hopf solutions)**.** Let $p > \frac{2d}{d+2}$. For each $\eta > 0$ the functional I_η possesses a minimiser u_η in the energy class of I_η. Moreover, there exists a subsequence u_η (not relabeled) that converges weakly to a Leray–Hopf solution of the non-Newtonian Navier–Stokes system (1).

The linear case $W(\epsilon) = \frac{1}{2}\mu_0 |\epsilon|^2$ was already considered in [1] and the case $p > \frac{3d+2}{d+2}$ in [2]. For such $p > \frac{3d+2}{d+2}$, we actually obtain, refining [2], that the convergence is *strong* and that the limiting solution satisfies an energy *equality.*

The main challenge is the passage to the limit in the nonlinear viscosity term which is simplified by both linearity and by strong convergence ($p > \frac{3d+2}{d+2}$). In the case of genuine weak convergence ($p < \frac{3d+2}{d+2}$), in order to separate effects of concentration (which are not harmful) and oscillation, and based on the concept of (solenoidal) Lipschitz-truncation (see for example [4]) for parabolic problems, we introduce a novel elliptic-parabolic truncation that might be of independent interest for other elliptic regularizations of parabolic problems.

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The stability of Sobolev's inequality: best constants and optimizers

Tobias König

The *Sobolev inequality*

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq S_n \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}}, \tag{1}$$

with optimal constant $S_n > 0$, is optimized precisely by constant multiples of the bubble functions

$$B_{z,\lambda}(x) = \lambda^{\frac{n-2}{2}} (1 + \lambda^2 |x-z|^2)^{-\frac{n-2}{2}}, \quad x \in \mathbb{R}^n, \lambda > 0, \tag{2}$$

by classical results of Aubin [1] and Talenti [17].

Answering a question by Brezis and Lieb [6], Bianchi and Egnell have shown the *quantitative stability* of (1) in the ground-breaking work [3]. Namely, there is $c_{BE} > 0$ such that

$$\mathcal{E}_{BE}(u) := \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx - S_n \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}}}{d(u, \mathcal{M})^2} \geq c_{BE} \quad \text{for all } u \in \dot{H}^1(\mathbb{R}^n), \tag{3}$$

where $\mathcal{M} = \{ cB_{z,\lambda} : c \in \mathbb{R}, z \in \mathbb{R}^n, \lambda > 0 \}$ is the manifold of Sobolev optimizers and $d(u, \mathcal{M})^2 = \inf_{h \in \mathcal{M}} \int_{\mathbb{R}^n} |\nabla (u-h)|^2 \, dx$.

The $\dot{H}^1$ norm used to define the distance $d(u, \mathcal{M})$ is the strongest possible for this purpose. Likewise, the quadratic exponent of the distance $d(u, \mathcal{M})$ is optimal in (3) and cannot be replaced by a smaller power. On the other hand, it is a major open question to determine the value of the best constant c_{BE}. The proof from [3] proceeds by compactness and thus gives no information on c_{BE}. It has long been informally conjectured that the sharp constant c_{BE} in (3) is attained (uniquely) in the limit of sequences of functions converging to some $B_{z,\lambda}$, leading to the explicit value $c_{BE} = \frac{4}{n+4} =: c_{BE}^{\text{spec}}$ determined by the spectral gap of an associated linearized operator.

Very recently, advances concerning the value of c_{BE} have been made in two different directions. On the one hand, Dolbeault, Esteban, Figalli, Frank and Loss [11] have obtained the first-ever explicit lower bound on c_{BE} through a refined local analysis near $\mathcal{M}$ together with a delicate symmetrization procedure involving competing symmetries [9] and continuous Steiner symmetrization [8]. On the other hand, in [14] we show the *strict* inequality $c_{BE} < \frac{4}{n+4}$. This inequality comes as a surprise because it falsifies the above-mentioned conjecture that c_{BE} is attained near $\mathcal{M}$. In the follow-up paper [15], we prove the existence of optimizers for (3). The proof from [15] applies the classical strategy by Brezis and Lieb [5] by exploiting a subtle convexity property of the functional $\mathcal{E}_{BE}$. Besides the inequality $c_{BE} < \frac{4}{n+4}$, the strict inequality $c_{BE} < 2 - 2^{\frac{n-2}{n}}$ (this value corresponds to the value of $\mathcal{E}_{BE}$ for two non-interacting bubbles) is needed to conclude. Differently from structurally simpler inequalities like Sobolev's inequality, the Yamabe problem [2] or the Brezis–Nirenberg problem [7], here two different 'compactness thresholds' need to be beaten in order to ensure strong convergence of an optimizing sequence. This reflects the more complicated structure of the functional $\mathcal{E}_{BE}$, which allows two non-interacting bubbles to be reasonable competitors for the infimum that can only be excluded through the refined asymptotic analysis leading to $c_{BE} < 2 - 2^{\frac{n-2}{n}}$.

It remains an open problem to characterize the optimizers of c_{BE} and even to determine their basic qualitative properties. A major obstacle is the lack of efficient symmetrization techniques for $\mathcal{E}_{BE}$: since $\mathcal{E}_{BE}$ does not decrease in general under symmetric-decreasing rearrangement, it is unclear whether optimizers of c_{BE} are radial. Since $\mathcal{E}_{BE}(u) < \mathcal{E}_{BE}(|u|)$ for every $u \in \dot{H}^1(\mathbb{R}^n)$, it might even occur that optimizers change sign. (This is somewhat reminiscent of the stability inequality for the isoperimetric inequality [13, 12], whose optimizer in the planar case $n = 2$ is conjectured to be a certain non-convex 'mask-shaped' set [4].)

Under the condition that $n \geq 2$, all of the above remains true if one replaces (1) by the *fractional Sobolev inequality* $\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^n)} \geq S_{n,s}\|u\|^2_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}$ for $s \in (0, n/2)$, and (3) by its analogous fractional stability inequality proved in [10]. If $n = 1$ and $s \in (0, 1/2)$, however, the perturbative argument from [14] can no longer be used to prove that the best stability constant satisfies $c_{BE}(s) < c^{\text{spec}}_{BE}(s)$. Indeed, this argument relies on choosing a degree-two spherical harmonic ρ on $\mathbb{S}^n$ with $\int_{\mathbb{S}^n} \rho^3 \neq 0$; such ρ exists for $n \geq 2$, but not for $n = 1$. In [16] I showed that this problem is not technical: unlike for $n \geq 2$, one has in fact $\mathcal{E}_{BE}(u) \geq c^{\text{spec}}_{BE}(s)$ in a neighborhood of $\mathcal{M}_s$ for every $n = 1$ and $s \in (0, 1/2)$. This dimensional dichotomy is rather surprising and leads to the conjecture that in the latter case one must have $c_{BE}(s) = c^{\text{spec}}_{BE}(s)$ and optimizers do not exist.

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Dimension of the singular set of 2-valued stationary graphs

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(joint work with Jonas Hirsch)

In his groundbreaking work [1], Allard proved that the singular set of stationary integral varifolds is meager. Since then little to no progress has been made on the question of the optimal dimension of the singular set for integral stationary varifolds. In joint work with J. Hirsch we answer this question under two assumptions: multiplicity 2 and Lipschitz graphicality. Moreover we do this by applying Almgren's strategy for the first time to the stationary setting, that is without any minimizing (nor stability) assumption.

Given a domain $\Omega \subset \mathbb{R}^m$, we consider Lipschitz multiple valued functions f from a domain $\Omega \subset \mathbb{R}^m$ to the space of Q-points in $\mathbb{R}^n$ (see [7, 4] for the relevant definitions).

Definition 1. Given a function a multivalued function f from a set Ω, we say that a point $x \in \Omega$ *is regular* if there exists a neighborhood $B \subset \Omega$ of x and Q analytic functions $f_i \colon B \to \mathbb{R}^n$ such that

$$f(y) = \sum_{i=1}^{Q} \delta_{f_i(y)} \qquad \text{for almost every } y \in B\,,$$

and either $f_i(x) \neq f_j(x)$ for every $x \in B$ or $f_i \equiv f_j$. The *singular set of f* is the complement in Ω of the set of regular points.

Our main result is the following optimal dimensional bound on the singular set of stationary 2-valued Lipschitz maps.

Theorem 2 (Dimension of the singular set)**.** Let f be a 2-valued map from $\Omega \subset \mathbb{R}^m$ open, and assume it is Lipschitz map and its graph is a stationary varifold. Then the dimension of the singular set of f in Ω is at most $m - 1$ and all the points in the regular part of f have either multiplicity 1 or 2. Furthermore, in the second case the dimension of the singular set is in fact at most $m - 4$. Moreover both dimensional bounds are optimal.

In codimension 1 and under the additional assumption of stability of the regular part, works of Schoen-Simon, Wicramasekera, Minter and Minter-Wickramasekera, provide beautiful partial results. When the varifold is associated to an area minimizing current, then a celebrated result of Almgren [2], later revisited by De Lellis-Spadaro [7, 4, 3, 5, 6], shows that the optimal dimension of the singular set is $(m-2)$. Recently De Lellis-Minter-Skorobogatova and Krummel-Wickramasekera, proved that in fact such singular set is $(m - 2)$ rectifiable. When the varifold is associated to an area minimizing current mod p, then work of De Lellis-Hirsch-Marchese-Stuvard shows that the optimal dimension of the singular set is $(m-1)$, with a finer description achieved in codimension 1 in work of De Lellis-Hirsch-Marchese-Spolaor-Stuvard, combined with a result of Minter-Wickramasekera. Our situation is somewhat more similar to this case, at least in the fact that for stationary varifolds the singular set can be of dimension $(m - 1)$ and branch points can occur, however the minimizing assumption is used crucially in these works, while it's missing in our setting.

In the stationary case, for $C^{1,\alpha}$ multivalued maps, works of Simon and Wickramasekera, and Krummel and Wickramasekera investigate the size and the structure of the branching set. Our situation is more general as the singular set can be of codimension 1 due to the Lipschitz regularity assumption as opposed to $C^{1,\alpha}$.

Let us mention quickly the main new ideas of the proof. By standard arguments using monotonicity formula and dimension reduction, it is enough to understand the size of the branching set, that is the collection of points where at least one blow-up is a plane with multiplicity. To understand such set we use Almgren's approach [2] in the revisited form of De Lellis and Spadaro [3, 5, 6]. In order to do that, except for minor technicalities, the main difficulties are: the construction of a small Lipschitz approximation to the graph of f with errors that are superlinear in the excess, the development of a suitable linearization theory for stationary graphs and of unique continuation and regularity theories for multivalued maps that arise through such linearization (in particular which are stationary, but not necessarily minimizing for the Dirichlet energy), and a suitable capacity argument to reach a contradiction at the linearized level.

To overcome these difficulties, one of the main new ingredients of our proof with respect to Almgren's approach is a higher integrability estimate for the Dirichlet energy of f. Such an estimate was crucial in the new proof of Almgren's theorem by De Lellis and Spadaro. In particular it allows us to prove the existence of a superlinear in the excess small Lipschitz approximation. Our proof of the higher integrability is completely different than De Lellis and Spadaro's, since our current is not minimizing. In particular this is where we use crucially the $Q = 2$ assumption. Extending a similar estimate to higher Q is the only missing ingredient to removing the $Q = 2$ constraint in Theorem 2. A tempting conjecture is that the higher integrability holds true as well for any $Q \geq 2$.

Moreover, in order to have good compactness properties for stationary sequences, we build the theory of multivalued generalized gradient Young measures and we study their regularity and unique continuation type properties under various assumptions of stationariety: this seems to be the correct linear problem in the stationary setting and it allows us to prove for example a strong Dir-stationary approximation to the graph of f. We believe that this new notion will be useful in investigating further regularity properties of stationary varifolds, as it provides the right tools to linearize the problem in the absence of strong convergence in energy.

Finally we revisit the capacity argument of [6], replacing it with a weaker, but more general, argument that doesn't require any stronger regularity than Sobolev. This is needed, since we cannot guarantee that our final blow-up sequence converges to a strong solution, but only to a measure solution, as the higher integrability statement is not preserved when subtracting averages from multivalued functions.

Finally we conclude with three open questions, in what we believe to be their order of difficulty, that seem like natural next steps after our theorem.

- Can Theorem 2 be extended to any multiplicity $Q \geq 2$?
- For $Q = 2$, can the Lipschitz assumption be removed from Theorem 2, to handle for instance the case of stationary currents in multiplicity 2?
- For $Q = 2$, can the graphicality assumption be removed altogether from Theorem 2, to handle the general case of stationary varifolds in multiplicity 2?

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