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MFO-RIMS Tandem Workshop: Algebraic Geometry and Noncommutative Projective Varieties

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Abstract. The workshop focused on noncommutative projective varieties, and the role they play in algebraic geometry. The talks ranged from discussing noncommutative ring theory and the role derived categories play in the subject to novel constructions, classifications, and moduli theory for noncommutative varieties. The interactions between the Japanese and European sides of the workshop were particularly enriching.

Mathematics Subject Classification (2020): 14A22, 14A30, 14F08, 16S38.

Introduction by the Organizers

The MFO-RIMS TANDEM workshop *Algebraic Geometry and Noncommutative Projective Varieties*, was organized by Pieter Belmans (Luxembourg) and Sue Sierra (Edinburgh) on the MFO side, and Izuru Mori (Shizuoka) and Shinnosuke Okawa (Osaka) on the RIMS side.

In noncommutative algebraic geometry, one studies noncommutative algebras using geometric methods, and varieties using ring-theoretic methods. The focus of the workshop was on noncommutative projective varieties. These are abelian categories constructed from noncommutative homogeneous coordinate rings, following Yuri Manin's quote

> *"As Grothendieck has taught us, to do geometry you really don't need a space, all you need is a category of sheaves on this would-be space."*

Ever since the introduction of Artin–Schelter reguler algebras in 1987 as the right setting for noncommutative projective spaces, and the introduction of geometric tools to study these, the subject has been thriving through its many interactions with algebraic geometry, symplectic geometry, representation theory, and mathematical physics.

The workshop's goal was to take stock of the subject's state of the art and set up the stage for its future developments. The interactions between the strong Japanese and European schools (and beyond), each with their own focus points, made the workshop a timely event with a substantial impact on the subject's future development.

The workshop attracted a rich blend of experts in noncommutative algebraic geometry, and early-career researchers working both in the subject itself and some of its adjacent areas. The combination of mathematicians who have worked on the subject for the 35+ years since its inception, and a new generation of researchers was instrumental in creating a collaborative atmosphere that supported interactions across generations and subjects. The speakers took care to make their talks accessible to the participating non-experts, which is important for creating new collaborations, and crossing divides between subareas.

Because of the time difference between Germany and Japan, the workshop combined

- synchronous talks (live-streamed between institutes during common working hours);
- asynchronous talks (live in one institution and recorded for later viewing in the other institution).

There was on average 2 hours of synchronous talks, and 2 hours of asynchronous talks per day, leaving ample room for discussions locally and across continents. We had 17.5 hours of talks: 15 hour-long talks, and 5 half-hour long talks by more junior participants.

By virtue of being a TANDEM workshop, the approach was inherently hybrid. However, we are happy to see that participants were highly motivated to participate in-person on both sides. This made it possible to set up synergies between the two sides in the distraction-free setting that the MFO and RIMS provide. The participation of a small number of online participants who could not travel for various reasons was a valuable asset in the workshop's success. We are grateful to the support staff on both sides, for their excellent work in setting up the technology and making this TANDEM workshop possible.

The workshop was also an ideal opportunity to interact with collaborators, locally and across continents. as evidenced by the posting of arXiv preprints and updates during the workshop, and progress on preprints to be posted in the weeks following the workshop.

Some of the highlights were James Zhang's synchronous opening talk, who described the state of the art in the subject and highlight many interesting open problems. Other highlights were Dan Rogalski's talk on what closed subcategories are supposed to be, and the double-header talk across continents by Zheng Hua and Alexander Polishchuk on their joint works into modular Poisson varieties and Feigin–Odesskii brackets. Finally, we want to mention Travis Schedler's talk on deforming skew polynomial algebras, and the construction of a $\mathbb{P}^3_{\mathrm{nc}}$ using a 2-periodic $\mathbb{Z}$-algebra, which gave important new insights in the moduli theory of noncommutative projective spaces and answering several long-standing questions in the subject.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Daniel Rogalski in the "Simons Visiting Professors" program at the MFO.

Workshop: MFO-RIMS Tandem Workshop: Algebraic Geometry and Noncommutative Projective Varieties

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Abstracts

Affine Weyl group actions on moduli spaces of marked noncommutative del Pezzo surfaces

Kazushi Ueda
(joint work with Shinnosuke Okawa)

A *del Pezzo lattice* is either the odd unimodular lattice $\mathsf{I}_{1,9-d}$ for $1 \le d \le 9$ generated by $\{\mathbf{e}_i\}_{i=0}^{9-d}$ with the intersection form

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & i = j = 0, \\ -1 & 1 \le i = j \le 9-d, \\ 0 & \text{otherwise} \end{cases}$$

or the even unimodular hyperbolic lattice $\mathsf{II}_{1,1}$ generated by $\{\mathbf{e}_1, \mathbf{e}_2\}$ with the intersection form

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & i \ne j, \\ 0 & \text{otherwise.} \end{cases}$$

The *canonical element* is defined by

$$\omega := -3\mathbf{e}_0 + \sum_{i=1}^{9-d} \mathbf{e}_i$$

for $\mathsf{I}_{1,9-d}$ and

$$\omega := -2\mathbf{e}_1 - 2\mathbf{e}_2$$

for $\mathsf{II}_{1,1}$. The *affine Weyl group* is defined by

$$\mathsf{W}^{\mathrm{aff}}(\omega^\perp) := \omega^\perp \rtimes \mathsf{W}(\omega^\perp),$$

where $\omega^\perp$ is the orthogonal lattice to ω in the del Pezzo lattice and the *Weyl group* $\mathsf{W}(\omega^\perp)$ is the subgroup of the orthogonal group $\mathrm{O}(\omega^\perp)$ generated by reflections along (-2)-vectors.

Fix a del Pezzo lattice Λ. A *marking* of a del Pezzo surface X is an isometry $\varphi \colon \Lambda \xrightarrow{\sim} \operatorname{Pic} X$ sending ω to ω_X. A *marked del Pezzo surface* is a pair (X, φ) consisting of a del Pezzo surface X and a marking φ. An *isomorphism* $(X, \varphi) \xrightarrow{\sim} (X', \varphi')$ of marked del Pezzo surfaces is an isomorphism $\phi \colon X \xrightarrow{\sim} X'$ such that $\varphi = \phi^* \circ \varphi'$. Assume $d \le 5$. The moduli scheme of marked del Pezzo surfaces is an open subscheme $\mathcal{M}_{9-d}^{\mathrm{comm}}$ of the configuration space $(\mathbf{P}^2)^{9-d} / \operatorname{Aut} \mathbf{P}^2$ of $9-d$ points on $\mathbf{P}^2$. The moduli stack of del Pezzo surfaces is the quotient $\left[\mathcal{M}_{9-d}^{\mathrm{comm}} / \mathrm{W}(\mathsf{E}_{9-d})\right]$ of $\mathcal{M}_{9-d}^{\mathrm{comm}}$ by the action of the Weyl group $\mathrm{W}(\mathsf{E}_{9-d})$.

An object E of a dg category $\mathcal{D}$ is *exceptional* if $\hom(E, E) \simeq \mathbf{k} \operatorname{id}_E$. A sequence $(E_1, \ldots, E_\ell)$ of exceptional objects is an *exceptional collection* if $\hom(E_i, E_j) \simeq 0$ for $i > j$. An exceptional collection is *full* if it generates $\mathcal{D}$. A *helix* of dimension d and period ℓ is a sequence $(E_i)_{i \in \mathbf{Z}}$ of objects such that $(E_1, \ldots, E_\ell)$ is a full exceptional collection and $E_{i-\ell} = \mathbf{S}(E_i)[-d]$ for any $i \in \mathbf{Z}$, where $\mathbf{S}$ is the Serre functor of $\mathcal{D}$.

We say a helix $(E_i)_{i\in\mathbf{Z}}$ is *pure* if $\operatorname{Hom}^k(E_i,E_j)=0$ for $i<j$ and $k\neq 0$. A pure helix $(E_i)_{i\in\mathbf{Z}}$ produces a connected $\mathbf{Z}$-algebra. When we say *a pure helix* $(E_i)_{i\in\mathbf{Z}}$ *on a del Pezzo surface* X, we assume $E_i\in\operatorname{coh}X$ for all $i\in\mathbf{Z}$. Any del Pezzo surface has a pure helix consisting of vector bundles.

A pure helix on a del Pezzo surface defines a *type* of an AS-regular $\mathbf{Z}$-algebra specified by a quiver. A *noncommutative weak del Pezzo surface* is qgr of an AS-regular $\mathbf{Z}$-algebra of that type. It is a *noncommutative del Pezzo surface* if the pair $(\mathcal{O},(\mathbf{S}[-2])^{-k})$ of an appropriately defined 'structure sheaf' $\mathcal{O}$ and some power $k\geq 1$ of the shifted Serre functor is ample in the sense of Artin–Zhang.

A (noncommutative) *marking* of a noncommutative weak del Pezzo surface is a choice of a pure helix $(E_i)_{i\in\mathbf{Z}}$. The algebra $\bigoplus_{i,j=1}^{\ell}\operatorname{Hom}(E_i,E_j)$ is described by a quiver with relations. The *moduli stack of marked noncommutative del Pezzo surfaces*, defined as the rigidified moduli stack of relations, contains the moduli scheme of marked commutative del Pezzo surfaces as a locally closed substack. A particularly nice (3-block) helix, known to exist except for $\mathbf{P}^2$ blown up at one or two points by Karpov–Nogin, allows one to define a compact moduli scheme $\overline{M}_{\mathrm{rel}}$ of relations as a GIT quotient with respect to a reductive group [AOU1, AOU2].

Conjecture 1. *An exceptional object on a noncommutative del Pezzo surface is determined by its class in the Grothendieck group, so that a marking is equivalent to its image in the Grothendieck group. Any full exceptional collections on the derived category of a noncommutative del Pezzo surface are related by mutations.*

Conjecture 1 is known for commutative del Pezzo surfaces by Kuleshov–Orlov. Given a full exceptional collection on a commutative del Pezzo surface, one can produce the corresponding full exceptional collection on a noncommutative del Pezzo surface by describing the collection as the result of a sequence of mutations from the standard collection. The affine Weyl group $\mathsf{W}^{\mathrm{aff}}(\omega^\perp):=\omega^\perp\rtimes\mathsf{W}(\omega^\perp)$ acts on the set of noncommutative markings of a commutative del Pezzo surface in such a way that $\omega^\perp$ acts by tensoring with line bundles.

Conjecture 2. *A pure helix on a del Pezzo surface deforms to a pure helix on any noncommutative del Pezzo surface.*

Conjecture 2 implies the existence of an affine Weyl group action on the moduli stack of marked noncommutative del Pezzo surfaces.

Conjecture 3. *A pair of geometric points on the moduli stack of marked noncommutative del Pezzo surfaces give an isomorphic pair of noncommutative del Pezzo surfaces if and only if they are related by the action of* $\mathsf{W}^{\mathrm{aff}}(\omega^\perp)$.

References

[AOU1] T. Abdelgadir, S. Okawa, and K. Ueda, *Compact moduli of noncommutative projective planes*, arXiv:1411.7770.

[AOU2] T. Abdelgadir, S. Okawa, and K. Ueda, *Compact moduli of marked noncommutative cubic surfaces*, arXiv:2404.00175.

Some open questions in noncommutative algebra and noncommutative algebraic geometry

James Zhang

We survey several open questions in noncommutative algebra, motivated by developments in noncommutative algebraic geometry, noncommutative invariant theory, representation theory, Poisson geometry, and theory of quantum groups. One major project is the classification of Artin-Schelter regular algebras of global dimension four, or geometrically, the classification of quantum projective 3-spaces. Another major project is trying to resolve Artin's conjecture on division algebras of transcendence degree 2. In noncommutative invariant theory one important project is to understand a version of Shephard-Todd-Chevalley Theorem, namely, to workout the corresponding Hopf reflection algebras acting on Artin-Schelter regular algebras. Other open questions related to deformation quantization, Poisson fields, cancellation problem are briefly discussed.

Closed subschemes of noncommutative projective schemes

Daniel Rogalski

In commutative algebraic geometry, the closed subschemes $Z \subseteq S$ play an important role in the study of a scheme S. This work concerns one possible generalization of this idea to noncommutative geometry, following work of Kanda, Rosenberg, Smith, Van den Bergh, and others. We take as the basic object of noncommutative geometry a Grothendieck category X, that is, an abelian category with exact direct limits and a generator. Strictly speaking, this generalizes not a commutative scheme S but rather its category of quasi-coherent sheaves $\operatorname{Qch} S$.

Definition 1. Let X be a Grothendieck category. A full subcategory Z of X is *closed* if it is closed under subquotients, direct sums, and products.

The definition above is equivalent to requiring that the inclusion functor $i : Z \to X$ has both a left and right adjoint. For example, if Z is closed then the left adjoint $i^* : X \to Z$ is given by assigning to an object $M \in X$ the quotient M/M', where M' is the unique smallest subobject of M such that $M/M' \in Z$.

Closed subcategories are the most direct analogues of closed subschemes in the commutative case. Indeed, if S is a quasi-projective scheme over a field, then the closed subschemes of S are in bijection with the closed subcategories of $\operatorname{Qch} S$ [2, Theorem 4.1]. As further evidence that the definition of closed is reasonable, if R is any unital ring we can consider the category $\operatorname{Mod}-R$ of right R-modules as a noncommutative affine quasi-scheme. In this case, the closed subcategories of $\operatorname{Mod}-R$ are precisely those of the form $\operatorname{Mod}-(R/I)$ for ideals I of R, as proved by Rosenberg [1, Proposition 6.4.1].

This project grew out of a desire to understand the closed subcategories of noncommutative projective schemes. Let $A = \bigoplus_{n=0}^{\infty} A_n$ be an $\mathbb{N}$-graded k-algebra which is connected ($A_0 = k$) and finitely generated as an algebra by A_1. Let

$\mathrm{Gr}-A$ be the category of $\mathbb{Z}$-graded right A-modules, and let $\mathrm{Tors}-A$ be the subcategory of $\mathrm{Gr}-A$ consisting of direct limits of modules which are finite dimensional over k. Using a construction of Gabriel, one forms the quotient category $\mathrm{Qgr}-A = \mathrm{Gr}-A/\mathrm{Tors}-A$, the *noncommutative projective scheme* associated to A. In case A is commutative, $\mathrm{Qgr}-A \simeq \mathrm{Qch}\, S$ for the projective scheme $S = \mathrm{Proj}\, A$, and as we noted the closed subcategories of $\mathrm{Qch}\, S$ simply correspond to the closed subschemes of S, or equivalently ideal sheaves on S. Moreover, by standard results in algebraic geometry, these ideal sheaves are in bijection with non-irrelevant graded ideals I of A such that I is *saturated* (i.e. A/I has no finite-dimensional submodules). So closed subcategories of $\mathrm{Qgr}-A$ are classified by saturated graded ideals.

By contrast, closed subcategories of noncommutative projective schemes $X = \mathrm{Qgr}-A$ are not so easy to describe when A is a noncommutative $\mathbb{N}$-graded k-algebra. When A is noetherian and I is a graded ideal, Smith proved that $Z = \mathrm{Qgr}-(A/I)$ is indeed closed in $\mathrm{Qgr}-A$, but the proof is non-trivial [3, Theorem 1.2]. Moreover, in general X often has many closed subcategories that are not of this form. For example, if I is a graded right ideal of A such that $\dim_k(A/I)_n = 1$ for all $n \geq 1$, then A/I is called a *point module*. Under mild additional hypotheses, the image of the point module in $X = \mathrm{Qgr}-A$ is a simple object p and the semisimple subcategory consisting of all direct sums of copies of p forms a closed subcategory of X [2, Proposition 5.8], but I is generally not a 2-sided ideal.

The problem of describing the closed subcategories of a noncommutative projective scheme naturally breaks into two pieces: (1) understand the closed subcategories of the category $\mathrm{Gr}-A$ of $\mathbb{Z}$-graded modules over a graded ring A; and (2) understand how the closed subcategories of $\mathrm{Gr}-A$ and those of its quotient category $\mathrm{Qgr}-A = \mathrm{Gr}-A/\mathrm{Tors}-A$ are related. The following two general theorems solve these problems.

Recall that an object M of an abelian category X is *compact* if the functor $\mathrm{Hom}_X(M, -)$ commutes with arbitrary direct sums. Compactness is a natural categorical analog of the notion of finite generation of a module.

Theorem 2. *Let X be a Grothendieck category, with small set of compact projective generators $\{\mathcal{O}_\alpha\}_{\alpha\in S}$. A collection $\{J_\alpha\}$, where $J_\alpha \subseteq O_\alpha$ for all α, is called an* ideal *in the set of generators if whenever $f : \mathcal{O}_\alpha \to \mathcal{O}_\beta$ is a morphism in X, then $f(J_\alpha) \subseteq J_\beta$.*

Every closed subcategory Z of X is generated by the objects $\{O_\alpha/J_\alpha\}$ for a unique ideal $\{J_\alpha\}$ in the set of generators. Thus, there is a bijective correspondence between closed subcategories in X and ideals in the set of generators.

One can make the notion of ideal in a set of generators more natural by showing that the category X in Theorem 2 is equivalent to a category of modules over the non-unital "S-ring" $\bigoplus_{\alpha,\beta\in S} \mathrm{Hom}_X(\mathcal{O}_\beta, \mathcal{O}_\alpha)$. In this description an ideal in the set of generators becomes an actual ideal of the S-ring.

Given a Grothendieck category X, recall that a subcategory Y is *localizing* if it is closed under subquotients, direct sums, and extensions. In this case there is a quotient category X/Y together with an exact quotient functor $\pi : X \to X/Y$

which has a right adjoint $\omega : X/Y \to X$. The object M is called Y-torsionfree if it has no nonzero subobjects in Y.

Theorem 3. *Let X be a Grothendieck category such that products are exact (Grothendieck's (AB4*) condition). Let Y be a localizing subcategory of X and consider the quotient category X/Y. A closed subcategory Z of X is called* Y-closed *if (i) Z is* Y-torsionfree generated, *that is the full subcategory generated by the Y-torsionfree objects in Z is all of Z; and (ii) Z is* Y-essentially stable*: if $M \in Z$ then $\omega\pi(M) \in Z$.*

(1) *If Z is a closed subcategory of X which is Y-essentially stable, then $\pi(Z) = \{\pi(M) | M \in Z\}$ is a closed subcategory of X/Y.*
(2) *If $\mathcal{Z}$ is a closed subcategory of X/Y, then the subcategory Z of X generated by all Y-torsionfree $M \in X$ with $\pi(M) \in \mathcal{Z}$ is a Y-closed subcategory of X.*
(3) *There is a bijective correspondence between Y-closed subcategories Z of X and closed subcategories $\mathcal{Z}$ of X/Y given by the inverse operations in (1) and (2).*

We immediately get a characterization of closed subcategories of noncommutative projective schemes. For a graded right ideal I of a graded ring A, let $\widetilde{I}$ be its saturation inside A, i.e. the largest graded right ideal such that $\widetilde{I}/I$ is in $\mathrm{Tors}-A$.

Corollary 4. *Let A be a connected $\mathbb{N}$-graded k-algebra, finitely generated in degree 1. Every closed subcategory of $\mathrm{Qgr}-A$ is generated by $\{(A/I^{(i)})(i) | i \in \mathbb{Z}\}$, for a unique collection of graded right ideals $\{I^{(i)} | i \in \mathbb{Z}\}$ of A such that $\widetilde{A_1 I^{(i)}} = I^{(i+1)}$ for all i.*

It may still not be easy in practice to determine the sequences of right ideals occurring in the theorem for a given A, other than the obvious ones that come from taking $I^{(i)} = I$ for all i, for a graded ideal I.

Here is one case in the setting of noncommutative projective surfaces where we are able to classify the closed subschemes explicitly. Given closed subcategories Z_1, Z_2, the *Gabriel product* $Z_1 \cdot Z_2$ is defined to be the set of all extensions of an object in Z_2 by an object in Z_1; it is also closed.

Example 5. Let A be an *elliptic algebra*, so there is a central element $g \in A_1$ such that $A/gA \cong B(E, \mathcal{L}, \sigma)$ is a twisted homogeneous coordinate ring for some elliptic curve E, invertible sheaf $\mathcal{L}$ of degree ≥ 2, and infinite order automorphism $\sigma : E \to E$. Assume that A has a unique smallest ideal K such that $\mathrm{GK.dim}(A/K) = 1$ and A/K is g-torsionfree. Then every closed subscheme of $\mathrm{Qgr}-A$ is of the form

$$\mathcal{Z} = [\mathcal{V} \cdot (\mathrm{Qgr}-A/g^m A)] \cup \mathcal{W}.$$

Here, $\mathcal{V}$ is a closed subcategory supported at finitely many points of E— this is the only part that is not associated to a 2-sided ideal of A. The subcategory $\mathcal{W}$ is contained in $\mathrm{Qgr}-A/K$, which is a finite set of exceptional points disjoint from E. Roughly, the theorem says that every closed subscheme is a copy of E of multiplicity m (possibly $m = 0$) with some additional multiplicity at certain points of E, together with a discrete exceptional point set. The possibilities for $\mathcal{V}$ can be described explicitly using results of Van den Bergh and Van Gastel. The set $\mathcal{W}$ is empty when A is the Sklyanin $\mathbb{P}^2$, but may be non-trivial for non-commutative Del Pezzo surfaces (see [4, Chap. 11]).

References

[1] Alexander L. Rosenberg, *Noncommutative algebraic geometry and representations of quantized algebras*, Mathematics and its Applications, vol. 330, Kluwer Academic Publishers Group, Dordrecht, 1995. MR 1347919

[2] S. Paul Smith, *Subspaces of non-commutative spaces*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2131–2171 (electronic). MR 1885647

[3] ———, *Corrigendum to "Maps between non-commutative spaces"[MR2052602]*, Trans. Amer. Math. Soc. **368** (2016), no. 11, 8295–8302.

[4] Michel Van den Bergh, *Blowing up of non-commutative smooth surfaces*, Mem. Amer. Math. Soc. **154** (2001), no. 734, x+140. MR MR1846352 (2002k:16057)

Quantum-symmetric equivalence of algebras

Charlotte Ure

(joint work with Hongdi Huang, Van C. Nguyen, Kent B. Vashaw, Padmini Veerapen, Xingting Wang)

Let k be a field and let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded algebra. Twisting of the multiplicative structure of A can be done in various ways, such as twisting by a graded automorphism, by a twisting system, or, if A is a comodule of a Hopf algebra H, by a 2-cocycle of H. We explore these below.

The *(right) twist of A by a graded automorphism $\phi : A \to A$*, first discussed by Artin, Tate, and Van den Bergh in [1], is the graded algebra A^ϕ that coincides with A as a k-vector space and that has twisted multiplication defined by

$$a \cdot_\phi b = a\phi^i(b)$$

for any $a \in A_i, b \in A_j$. This concept was generalized by Zhang in [9] where the notion of *twisting system* was introduced as a set $\tau = \{\tau_i : i \in \mathbb{Z}\}$ of $\mathbb{Z}$-graded bijective k-linear maps $\tau_i : A \to A$ so that

$$\tau_i(a\tau_j(b)) = \tau_i(a)\tau_{i+j}(b)$$

for $a \in A_j, b \in A_l$. The *twist of A by τ* is the graded algebra A^τ that coincides with A as a k-vector space and that has twisted multiplication

$$a \cdot_\tau b = a\tau_i(b)$$

for any $a \in A_i, b \in A_j$. In the same article, Zhang proved that if A and B are two connected $\mathbb{Z}$-graded algebras that are generated in degree one, then A and B are Morita equivalent, i.e. their graded module categories are equivalent, if and only if A is isomorphic to a twist of B by some twisting system [9, Theorem 1.1].

There is a different twisting of the multiplcative structure of A that arises if A can be equipped with a comodule structure by a Hopf algebra H that was introduced by Doi and Takeuchi in [2, 3]. A bilinear map $\sigma : H \otimes H \to k$ is called a *(right) 2-cocycle* if it is convolution invertible and it satisfies the 2-cocycle condition

$$\sum \sigma(x_2, y_2)\sigma(x_1y_1, z) = \sum \sigma(y_2, z_2)\sigma(x, y_1z_1)$$

for any $x, y, z \in H$, where we use Sweedler's notation $\Delta(x) = \sum x_1 \otimes x_2$. We assume that all 2-cocycles are *normal*, that is

$$\sigma(x, 1) = \sigma(1, x) = \epsilon(x)$$

for any $x \in H$. The *2-cocycle twist of H by σ* is the Hopf algebra H^σ that agrees with H as a bialgebra, with original unit, and twisted multiplication

$$x \cdot_\sigma y = \sum \sigma^{-1}(x_1, y_1)x_2y_2\sigma(x_3, y_3)$$

for $x, y \in H$. The Hopf algebras H and H^σ are *Morita–Takeuchi equivalent*, i.e. their graded comodule categories are tensor equivalent. In particular, given an H-comodule algebra A with coaction $\rho(a) = a_0 \otimes a_1 \in A \otimes H$, there is an H^σ-comodule algebra A_σ that agrees with A as a k-vector space and that has twisted multiplication

$$a \cdot_\sigma b = \sum \sigma(a_1, b_1)a_0b_0$$

for any $a, b \in A$. In [5], building on [4], we realize twists of A by a graded automorphism via 2-cocycle twisting. This was generalized to twisting systems by Huang, Nguyen, Vashaw, Veerapen, and Wang in [6]. The 2-cocycle in these results is a 2-cocycle on the (right) universal quantum group $\underline{aut}^r(A)$ in the sense of Manin (see [7]), the universal Hopf algebra that universally (right) coacts on A preserving the grading.

Theorem 1 ([5, Theorem D])**.** *Let A be a connected graded algebra finitely generated in degree one subject to m-homogeneous relations and let ϕ be a graded automorphism of A. There is a 2-cocycle σ on $\underline{aut}^r(A)$ so that $A^\phi \cong A_\sigma$ as graded algebras.*

The previous theorem implies that for A and ϕ as above, the categories $\underline{aut}^r(A)$ and $\underline{aut}^r(A^\phi)$ are Morita–Takeuchi equivalent. This observation motivates the following definition.

Definition 2 ([5, Definition A])**.** Let A and B be two connected graded algebras that are finitely generated in degree one. A and B are *weakly quantum-symmetrically equivalent* if $\underline{aut}^r(A)$ and $\underline{aut}^r(B)$ are Morita–Takeuchi equivalent. If this equivalence takes A to B, we say that A and B are *quantum-symmetrically* equivalent.

It is natural to ask which properties A and B share if they are (weakly) quantum-symmetrically equivalent. We say that a connected graded algebra A is *Artin–Schelter regular of dimension d* if it is of finite global dimension d and it is Gorenstein.

Theorem 3 ([5, Theorem 5.1.1])**.** *Suppose k is algebraically closed. Let A be a Koszul Artin–Schelter regular algebra, and B be any connected graded algebra* *generated in degree one. Then A and B are quantum-symmetrically equivalent if and only if B is a Koszul Artin–Schelter regular algebra of the same global dimension as A. In this case, A and B are 2-cocycle twists of one another if and only if they have the same Hilbert series.*

The proof builds on a result by Raedschelders and Van den Bergh in [8, Theorem 7.2.3], which states that if A and B are two Artin–Schelter regular algebras of the same dimension, then they are weakly quantum-symmetrically equivalent. We remark that the previous theorem implies that the quantum-symmetric equivalence class of the polynomial ring $k[x_1, \ldots, k_d]$ in d variables is given by all Koszul Artin–Schelter regular algebras of the same dimension.

References

[1] M. Artin, J. T. Tate and M. Van den Bergh, Modules over regular algebras of dimension 3, Invent. Math. **106** (1991), no. 2, 335–388.
[2] Y. Doi, Braided bialgebras and quadratic bialgebras, Comm. Algebra **21** (1993), no. 5, 1731–1749.
[3] Y. Doi and M. Takeuchi, Multiplication alteration by two-cocycles–the quantum version, Comm. Algebra **22** (1994), no. 14, 5715–5732.
[4] H. Huang, V. C. Nguyen, C. Ure, K. B. Vashaw, P. Veerapen and X. Wang, Twisting of graded quantum groups and solutions to the quantum Yang–Baxter equation, Transform. Groups (2022)
[5] H. Huang, V. C. Nguyen, C. Ure, K. B. Vashaw, P. Veerapen and X. Wang, Twisting Manin's universal quantum groups and comodule algebras, Adv. Math. **445** (2024), no. 109651, 55 pp.
[6] H. Huang, V. C. Nguyen, K. B. Vashaw, P. Veerapen and X. Wang, Quantum-symmetric equivalence is a graded Morita invariant, Preprint, arXiv:2405.12201 (2024)
[7] Y. I. Manin, Quantum groups and noncommutative geometry, second ed., CRM Short Courses, Centre de Recherches Math´ematiques, [Montreal], QC; Springer, Cham, 2018, With a contribution by Theo Raedschelders and Michel Van den Bergh.
[8] T. Raedschelders and M. Van den Bergh, The Manin Hopf algebra of a Koszul Artin-Schelter regular algebra is quasihereditary, Adv. Math. **305** (2017), 601–660.
[9] J. J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. London Math. Soc. (3) **72** (1996), no. 2, 281–311.

Clifford quadratic complete intersections

Haigang Hu
(joint work with Izuru Mori)

This talk is based on the paper [5].

Let $F = (F_1, \ldots, F_n)$ be a sequence of linearly independent symmetric matrices of size n, and $\mathcal{F} = \sum_{i=1}^{n} F_i y_i$. A graded Clifford algebra S associated to $\mathcal{F}$ is the Clifford algebra of $\mathcal{F}$ over the polynomial algebra $k[y_1, \ldots, y_n]$ with variables $x_1, \ldots, x_n$, where $\deg x_i = 1, \deg y_j = 2$ for $i, j = 1, \ldots, n$ [3]. We say S is an *n-dimensional Clifford quantum polynomial algebra* if S is an n-dimensional Koszul AS-regular algebra [1].

Definition 1. We say an algebra A is a *Clifford quadratic complete intersection of length r in n variables* if $A = S/(f_1, \dots, f_r)$ for some n-dimensional Clifford quantum polynomial algebra S and some central regular sequence $f_1, \dots, f_r \in Z(S)_2$.

The notion of Calabi-Yau algebra was introduced by Ginzburg, which is an algebraic incarnation of the notion of Calabi-Yau manifold [4].

Theorem 2 (Hu-Mori). *An n-dimensional Clifford quantum polynomial algebra is Calabi-Yau if and only if n is odd.*

It is well known that a quadratic graded Clifford algebra S is Koszul AS-regular if and only if its quadratic dual is isomorphic to a commutative quadratic complete intersection of length n in n variables [3]. The following result generalizes the duality to the case of Clifford quadratic complete intersections.

Theorem 3 (Hu-Mori). *There is a one-to-one correspondence*

$$\mathscr{C}_{n,r} := \left\{ \begin{array}{c} \textit{Clifford quadratic} \\ \textit{complete intersections} \\ \textit{of length } r \\ \textit{in } n \textit{ variables} \end{array} \right\} / \cong \xleftrightarrow{1:1} \left\{ \begin{array}{c} \textit{commutative quadratic} \\ \textit{complete intersections} \\ \textit{of length } n-r \\ \textit{in } n \textit{ variables} \end{array} \right\} / \cong$$

for all n and $0 \le r \le n$ by taking quadratic dual.

By using the above theorem and the classification results of conics $k[x, y, z]/(f)$, pencils of conics $k[x, y, z]/(f_1, f_2)$, we get the following result.

Corollary 4. $\#(\mathscr{C}_{3,1}) = 6$*, and* $\#(\mathscr{C}_{3,2}) = 3$.

We say that a quadratic algebra A satisfies condition (G1) if it determines a geometric pair $\mathcal{P}(A) = (E, \sigma)$ in the sense of Artin-Tate-Van den Bergh [2]. We call E the *point variety* of A.

Theorem 5 (Hu-Mori). *Let $A = S/(f_1, \dots, f_r)$ be a Clifford quadratic complete intersection. Then*

(1) *S satisfies the condition (G1);*
(2) *There are $g_1, \dots, g_r \in S_1$ such that $g_i^2 = f_i$ for $1 \le i \le r$;*
(3) *A satisfies the condition (G1).*

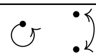
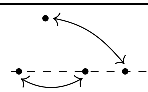
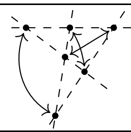
For a Clifford quadratic complete intersection $A = S/(g_1^2, \dots, g_r^2)$ where F is normalized and $g_1, \dots, g_r \in S_1$, there is an interesting result connecting the point variety E_A of A and the *characteristic variety* of A defined as follows: Define

$$X^{(s)}(F) := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{P}^{n-1} \mid \operatorname{rank}(\lambda_1 F_1 + \cdots + \lambda_n F_n) < s\}.$$

We call $X_A^{(s)} := X^{(s)}(F) \cap \mathcal{V}(\tilde{g}_1, \dots, \tilde{g}_r)$ the characteristic varieties of A. We show there is a double cover map of varieties $\Phi : E \to X^{(3)}(F)$, where E is the point variety of S, which restricts to a double cover map of varieties $\Phi|_{E_A} : E_A \to X_A^{(3)}$.

At last, we give the classification result of $\mathscr{C}_{3,r}$, where $r = 1, 2$, in the following table.

Table 1. $\mathscr{C}_{3,r}$ for $r = 1, 2$.

$A \in \mathscr{C}_{3,1}$	(E_A, σ_A)	$(\#(X_A^{(3)}), \#(X_A^{(2)}))$
$S^{(0,0,0)}/(x^2)$	E_A is a line, σ_A fixes exactly 2 points	$(\infty, 2)$
$S^{(1,0,0)}/(y^2)$		$(1, 1)$
$S^{(1,1,0)}/(3x^2 + 3y^2 + 4z^2)$		$(1, 0)$
$S^{(0,0,0)}/(x^2 + y^2)$		$(2, 1)$
$S^{(1,0,0)}/(x^2 + y^2 + z^2)$		$(2, 0)$
$S^{(0,0,0)}/(x^2 + y^2 + z^2)$		$(3, 0)$

$A \in \mathscr{C}_{3,2}$	(E_A, σ_A)	$(\#(X_A^{(3)}), \#(X_A^{(2)}))$
$S^{(0,0,0)}/(x^2, y^2)$		$(1, 1)$
$S^{(0,0,0)}/(x^2 + y^2, z^2)$		$(1, 0)$
$S^{(0,0,0)}/(x^2 + y^2, x^2 + z^2)$	$E_A = \emptyset$	$(0, 0)$

Where $S^{(a,b,c)} := k\langle x, y, z\rangle/(yz + zy + ax^2, zx + xz + by^2, xy + yx + cz^2)$.

References

[1] M. Artin and W. F. Schelter, *Graded algebras of global dimension* 3, Adv. Math. **66** (1987), 171–216.

[2] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, vol. 1, Progress in Mathematics **86** (Brikhäuser, Basel, 1990) 33–85.

[3] L. Le Bruyn, *Central singularities of quantum spaces*, J. Algebra **177** (1995), 142–153.

[4] V. Ginzburg, *Calabi-Yau algebras*, arXiv:math/0612139 (2007).

[5] H. Hu and I. Mori, *Clifford quadratic complete intersections*, Math. Z. **308** (2024), 1–33.

On three-dimensional noncommutative resolutions

Wahei Hara

(joint work with Michael Wemyss)

This talk presents my recent joint work with Michael Wemyss. The main theorem of this talk is as follows.

Theorem 1. *Let $f\colon X \to \operatorname{Spec} R$ be a projective morphism of noetherian schemes. Assume that there exists a (possibly noncommutative) algebra A such that* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$*. Then for any $\mathbb{C}$-valued point $p \in \operatorname{Spec} R$, $H^{\mathrm{odd}}(f^{-1}(p); \mathbb{C})$*

$= 0$ *and* $\dim H^{\mathrm{even}}(f^{-1}(p);\mathbb{C})$ *is equal to the number of the isomorphism classes of simple* A_p*-modules.*

If R is a local ring of 3-dimensional Gorenstein canonical singularity that admits a *noncommutative crepant resolution* in the sense of Van den Bergh [VdB], then it is known that there is a crepant resolution $f\colon X \to \operatorname{Spec} R$ such that $\mathrm{D^b}(\operatorname{coh} X) \simeq \mathrm{D^b}(\operatorname{mod} A)$. Thus, by the main theorem, it holds that any crepant resolution $g\colon Y \to \operatorname{Spec} R$ satisfies $H^{\mathrm{odd}}(g^{-1}(\mathfrak{m}_R);\mathbb{C}) = 0$. As a consequence, the following holds.

Corollary 2. *Let* $(R, \mathfrak{m}_R)$ *be a local ring of* 3*-dimensional Gorenstein canonical singularity that admits a crepant resolution* $f\colon X \to \operatorname{Spec} R$*. If the third Betti number* $b_3(f^{-1}(\mathfrak{m}_R))$ *is non-zero, then* R *does not admit a noncommutative crepant resolution.*

Note that the first Betti number b_1 is zero since R has a rational singularity, and the fifth Betti number b_5 is zero by the dimension reason. Thus with the assumptions in the corollary, the only odd Betti number that can be non-zero is the third Betti number b_3.

In the following, the dimension is always assumed to be three. For Gorenstein domains, Iyama and Wemyss [IW1] defined a notion of their *maximal modification algebra*, which is a generalisation of noncommutative crepant resolutions to singular situations. They conjecture that any local ring of 3-dimensional Gorenstein canonical singularity admits a maximal modifying algebra. Thus even when R admits a crepant resolution with non-vanishing b_3, R is expected to have a maximal modifying algebra, which is not a noncommutative crepant resolution (i.e. singular). The definition of maximal modification algebra comes from an analogy to the minimal model theory, and they are expected to play a role of "noncommutative minimal model". The existence conjecture was posed with this philosophy. (MMP tells us that $\mathbb{Q}$-factorial terminalization exists for Gorenstein canonical singularities.)

It is known by Bridgeland and Chen [B, C] that all minimal models (= $\mathbb{Q}$-factorial terminalizations) of $\operatorname{Spec} R$ are derived equivalent. In particular, all crepant resolutions (= smooth minimal models) are derived equivalent. Similarly, it is known by Iyama and Wemyss that all maximal modifying algebras of R are derived equivalent. This is a generalisation of the result by Van den Bergh that all noncommutative resolutions are derived equivalent.

Another conjecture by Iyama and Wemyss [IW2] expected that a minimal model of $\operatorname{Spec} R$ and a maximal modification algebra of R are derived equivalent. However, the main theorem and the corollary above give counterexamples of this derived equivalence conjecture.

Example 3. For $k \in \{1, 2, 3, \cdots\}$, put

$$R_k := \mathbb{C}[\![x, y, z, w]\!]/(x^3 + y^3 + z^3 + w^k).$$

(a) If $k = 1$, R_1 is regular. If $k = 2$, R_2 has a factorial terminal singularity. In both cases, $\operatorname{Spec} R_k$ is a minimal model of itself, and a maximal modification algebra is also the ring R_k itself. Of course, the derived equivalence $\mathrm{D^b}(\operatorname{coh} \operatorname{Spec} R_k) \simeq \mathrm{D^b}(\operatorname{mod} R_k)$ holds.

(b) If $k = 3$, R_3 is the affine cone over the Fermat cubic surface $Z \subset \mathbb{P}^3$. The total space X of the canonical bundle ω_Z (completed along the zero-section) gives a crepant resolution of $\operatorname{Spec} R_3$. By Van den Bergh [VdB, arXiv version], a noncommutative crepant resolution A of R_3 also exists, and they are derived equivalent $\mathrm{D^b}(\operatorname{coh} X) \simeq \mathrm{D^b}(\operatorname{mod} A)$.

(c) If $k = 3m + 1$ with $m \geq 1$, an iterated blowing up of $\operatorname{Spec} R_k$ at singular points gives a crepant resolution $f \colon X \to \operatorname{Spec} R_k$. However, by Dao [D], R_k is a maximal modification algebra of itself, and has no noncommutative crepant resolution.

Since X is regular and R_k is not, two categories $\mathrm{D^b}(\operatorname{coh} X)$ and $\mathrm{D^b}(\operatorname{mod} R_k)$ are not equivalent. Alternatively, it is not difficult to see that $b_3(f^{-1}(\mathfrak{m}_{R_k})) = 2m \neq 0$, which implies that X cannot be derived equivalent to any algebra by the main theorem. This gives a counterexample of the derived equivalence conjecture by Iyama and Wemyss.

(d) If $k = 3m + 2$ with $m \geq 1$, then a minimal model $f \colon X \to \operatorname{Spec} R_k$ is singular, and has $b_3(f^{-1}(\mathfrak{m}_{R_k})) = 2m \neq 0$. Similarly, by Dao [D], R_k is a maximal modification algebra of itself. Since X cannot be derived equivalent to any algebra by the main theorem, two categories $\mathrm{D^b}(\operatorname{coh} X)$ and $\mathrm{D^b}(\operatorname{mod} R_k)$ are not equivalent.

(e) If $k = 3m$ with $m > 1$, then an iterated blowing up of $\operatorname{Spec} R_k$ at singular points gives a crepant resolution $f \colon X \to \operatorname{Spec} R_k$, which satisfies $b_3(f^{-1}(\mathfrak{m}_{R_k})) = 2m - 2 \neq 0$. Thus the corollary shows that R_k has no noncommutative crepant resolution. We don't know the existence of a maximal modification algebra of R_k at the point of writing. However, even if it exists, it cannot be derived equivalent to a minimal model, which is a crepant resolution in this case.

References

[B] T. Bridgeland, *Flops and derived categories*, Invent. Math. **147** (2002), no. 3, 613–632.

[C] J.-C. Chen, *Flops and equivalences of derived categories for threefolds with only terminal Gorenstein singularities*, J. Differential Geom. **61** (2002), no. 2, 227–261.

[D] H. Dao, *Remarks on non-commutative crepant resolutions of complete intersections*, Adv. Math. **224** (2010), no. 3, 1021–1030.

[IW1] O. Iyama and M. Wemyss, *Maximal modifications and Auslander-Reiten duality for non-isolated singularities*. Invent. Math. **197** (2014), no. 3, 521–586.

[IW2] O. Iyama and M. Wemyss, *Singular derived categories of $\mathbb{Q}$-factorial terminalizations and maximal modification algebras*, Adv. Math. **261** (2014), 85–121.

[VdB] M. Van den Bergh, *Non-commutative crepant resolutions*, The legacy of Niels Henrik Abel, 749–770, Springer, Berlin, 2004.

Modular Poisson structure and its applications

Zheng Hua and Alexander Polishchuk
(joint work with Nikita Markarian, Eric Rains)

Let C be a Calabi-Yau (CY) curve, i.e. a connected projective curve C such that $\omega_C \cong \mathcal{O}_C$. The moduli stack of perfect complexes $\mathfrak{Perf}(C)$ admits a 1-shifted symplectic structure by the seminal work of Pantev, Toën, Vaquié and Vezzosi [8]. For a subset $I \subset \mathbb{Z}$, denote by $\mathfrak{Vect}^I(C)$ the stack of I-graded vector bundles. Denote by $\mathfrak{Cplx}(C)$ the moduli stack of bounded chain complex of vector bundles modulo chain isomorphisms. We have a diagram

$$\mathfrak{Vect}^I(C) \xleftarrow{\;p_I\;} \mathfrak{Cplx}(C) \xrightarrow{\;(p_{\mathbb{Z}\setminus I},q)\;} \mathfrak{Vect}^{\mathbb{Z}\setminus I}(C) \times \mathfrak{Perf}(C)$$

where p_I sends a complex to the underlying I-graded vector bundle and q sends it to the corresponding perfect complex. It is proved in [9] that the above diagram is a Lagrangian correspondence with respect to the 1-shifted symplectic structure on the target. It follows that $\mathfrak{Cplx}(C)$ admits a canonical Poisson structure, which we refer as the *modular Poisson structure*. The fibers of p_I and of $(p_{\mathbb{Z}\setminus I}, q))$ are Poisson substacks and their intersection is symplectic.

Many classical Poisson varieties can be realized as the coarse moduli schemes of the Poisson substacks of $\mathfrak{Cplx}(C)$. For example, it includes Poisson structure on matrix algebra over the field of meromorphic functions over complex elliptic curves [10], Feigin-Odesskii elliptic Poisson structure [1, 11] and Drinfeld-Jimbo standard Poisson structure on grassmannians [13].

1. Symplectic foliation on Positroid varieties

For $0 < k < n$, let $G(k, n)$ be the complex grassmannian of k-planes in $\mathbb{C}^n$. Let $\chi : \mathfrak{gl}_n \to TG(k, n)$ be the infinitesimal action of GL_n on $G(k, n)$. The *standard Poisson structure* is defined by the bivector field

$$\pi_{st} = -\sum_{i<j} \chi(E_{ij}) \wedge \chi(E_{ji})$$

where E_{ij} is the matrix with one at the (ij)-th entry and zero elsewhere. Let T be the maximal torus of SL_n. The standard Poisson structure π_{st} is T-invariant. One may ask **how to describe moduli stack of symplectic leaves of π_{st}?**

We represent a point in $G(k, n)$ by a $k \times n$ matrix $A = [v_1, \dots, v_n]$ of rank k modulo the row transformations. Define a bijection $f_A : \mathbb{Z} \to \mathbb{Z}$ by

$$f_A(i) = \min\{j | v_i \in \mathrm{span}(v_{i+1}, \dots, v_j)\}$$

where the index i of v_i is considered modulo n. Such a f_A belongs to $B(k, n)$: the set of bounded affine permutation of period n and average k. For $f \in B(k, n)$, denote $\Pi_f := \{A | f_A = f\}\Big/ GL_k$ where GL_k acts by row transformations. We call Π_f the positroid variety of type f and we have a stratification $G(k,n) = \bigsqcup_{f\in B(k,n)} \Pi_f$.

Let $C = \bigcup_{i=1}^n C_i$ be the Kodaira cycle with n irreducible components and L be a line bundle such that $L|_{C_i} \cong \mathcal{O}(1)$. We consider the stack $N(k,L)$ of complexes $\mathcal{O}^k \to V$ such that $V/\mathcal{O}^k \cong L$ and the connecting morphism $H^0(L) \to H^1(\mathcal{O}^k)$ is surjective. It coarse moduli scheme is $G(k,n)$. In [12], we prove that the modular Poisson structure on $N(k,L) \subset \mathfrak{Cplx}(C)$ descends to π_{st} on $G(k,n)$. The $(n-1)$-dimensional torus $T = \{t_1 \cdot \ldots \cdot t_n = 1\} \subset \mathrm{Aut}(C)$ preserves L, therefore acts on the stack $\mathfrak{Vect}^L(C)$ of vector bundles with determinant L and on $N(k,L)$. The map $p : N(k,L) \to \mathfrak{Vect}^L(C)$ sends $\mathcal{O}^k \to V$ to V. We establish an one to one correspondence between $B(k,n)$ and the set of T-orbits in the image of p. Let $V_f \to \mathfrak{Vect}^L(C)$ be the T-orbit for $f \in B(k,n)$ and let $p_f : N_f \to V_f$ be the base change of p. We prove that N_f is isomorphic to Π_f and p_f is a smooth surjective morphism with connected symplectic fibers. This leads to an answer to the question about classification of symplectic leaves of π_{st}. In particular it gives a new proof for the result of Goodearl and Yakimov about T-leaves of π_{st} [13].

2. Feigin-Odesskii Poisson brackets and del Pezzo surfaces

Recall that given an elliptic curve E with a stable vector bundle V of rank r and degree d, there is a natural construction of a Poisson bracket (up to proportionality) on the projective space $\mathbb{P}H^0(E,V)^*$ (see [1], [6]). We refer to these brackets as FO-brackets of type $q_{d,r}$.

Two Poisson brackets are called *compatible* if the corresponding bivectors satisfy $[\Pi_1, \Pi_2] = 0$ (equivalently, any linear combination of these brackets is again Poisson). It was shown in [2] that if $(\mathcal{O}_S, V)$ an exceptional pair of vector bundles on a smooth projective surface S, then for any smooth anticanonical divisor $E \subset S$ the restriction map $H^0(S,V) \to H^0(E, V|_E)$ is an isomorphism, so we can view the FO-bracket associated with $(E, V|_E)$ as a Poisson bracket on the projective space $\mathbb{P}H^0(S,V)^*$, and furthermore, this gives a linear map

$$\kappa : H^0(S, \omega_S^{-1}) \to H^0(\mathbb{P}H^0(S,V)^*, \bigwedge^2 T),$$

whose image consists of (compatible) Poisson brackets.

For example, taking V to be an appropriate line bundle either on $\mathbb{P}^1 \times \mathbb{P}^1$ or on the Hirzebruch surface F_1, one recovers in this way 9-dimensional families of compatible Poisson FO brackets of type $q_{n,1}$ on $\mathbb{P}^{n-1}$, originally discovered by Odesskii-Wolf in [5].

One can ask conversely, when two FO-brackets of the same type on the same projective space are compatible. For example, FO-brackets of type $q_{n,1}$ are associated with normal elliptic curves of degree n in $\mathbb{P}^{n-1}$. It was shown in [3] that given two such curves E_1 and E_2, the corresponding brackets are compatible if and only if E_1 and E_2 are anticanonical divisors on a certain surface in $\mathbb{P}^{n-1}$. For example, for $n = 4$, this happens if and only if E_1 and E_2 lie on the same quadric surface in $\mathbb{P}^3$. There is also a similar criterion for FO-brackets of type $q_{5,2}$ proved in [4]. Namely, such brackets are associated with elliptic curves obtained as linear sections of the Grassmannian $G(2,5)$ in its Plücker embedding. We show that two such FO-brackets are compatible if and only if the corresponding elliptic curves lie on a del Pezzo surface obtained as a linear section of $G(2,5)$.

A natural question is what is the maximal dimension of a space of compatible Poisson brackets on $\mathbb{P}^{d-1}$ containing a given FO-bracket of type (d,r). To begin with, one can ask what is the maximal dimension one gets from the construction of [2]. In [7] we studied this question for exceptional pairs on del Pezzo surfaces. We proved that for every relatively prime (r,d), with $r>0$, there exists an exceptional pair $(\mathcal{O}_S, V)$ on a del Pezzo surface of degree 5 such that V has rank r and degree $c_1(V)\cdot c_1(\omega_S^{-1}) = d$. Furthermore, we proved that for $d>r+1$, the corresponding map κ is injective, so this gives a 5-dimensional subspace of compatible Poisson brackets of type (d,r).

References

[1] B. L. Feigin, A. V. Odesskii, *Vector bundles on an elliptic curve and Sklyanin algebras*, Topics in quantum groups and finite-type invariants, 65–84, Amer. Math. Soc., Providence, RI, 1998.

[2] Z. Hua, A. Polishchuk, *Elliptic bihamiltonian structures from relative shifted Poisson structures*, J. Topology *16* (2023), no. 4, 1389–1422.

[3] N. Markarian, A. Polishchuk, *Compatible Feigin-Odesskii Poisson brackets*, Manuscripta Math. *173* (2024), no. 3-4, 907–923.

[4] N. Markarian, A. Polishchuk, *Compatible Poisson brackets associated with elliptic curves in* $G(2,5)$, SIGMA *20* (2024), Paper No. 037.

[5] A. Odesskii, T. Wolf, *Compatible quadratic Poisson brackets related to a family of elliptic curves*, J. Geom. Phys. *63* (2013), 107–117.

[6] A. Polishchuk, *Poisson structures and birational morphisms associated with bundles on elliptic curves*, IMRN *13* (1998), 683–703.

[7] A. Polishchuk, E. Rains, *Exceptional pairs on del Pezzo surfaces and spaces of compatible Feigin-Odesskii brackets*, arXiv:2407.19307.

[8] Pantev, Tony, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi, *Shifted symplectic structures*, Publications mathématiques de l'IHÉS 117 (2013): 271-328.

[9] Hua, Zheng, and Alexander Polishchuk, *Shifted Poisson structures and moduli spaces of complexes*, Advances in Mathematics 338 (2018): 991-1037.

[10] Hua, Zheng, and Alexander Polishchuk, *Shifted Poisson geometry and meromorphic matrix algebras over an elliptic curve*, Selecta Mathematica 25 (2019): 1-45.

[11] Hua, Zheng, and Alexander Polishchuk. *Bosonization of Feigin-Odesskii Poisson varieties*, arXiv preprint arXiv:2306.14719 (2023).

[12] Hua, Zheng, and Alexander Polishchuk. *Feigin-Odesskii brackets associated with Kodaira cycles and positroid varieties*, arXiv preprint arXiv:2404.03935 (2024).

[13] Goodearl, K., and Milen Yakimov, *Poisson structures on affine spaces and flag varieties. II*, Transactions of the American Mathematical Society 361, no. 11 (2009): 5753-5780.

Phantoms and exceptional collections on rational surfaces

Johannes Krah

Let $\mathcal{T}$ be a triangulated category with a full exceptional collection $\mathcal{T} = \langle E_1, \dots, E_n\rangle$. In 1993 Bondal–Polishchuk conjectured that the action by mutations and shifts is transitive on the set of full exceptional collections of $\mathcal{T}$ [2, Conj. 2.2].

Although a counterexample was recently constructed by Chang–Haiden–Schroll [3], the conjecture is known to be true if $\mathcal{T} = \mathrm{D}^b(\mathrm{Coh}(X))$ for X a del Pezzo surface [7]. In [6] we study a numerical variant of this question, namely the action by mutations and shifts on numerically exceptional collection of maximal length in the numerical Grothendieck group of a smooth projective surface. Furthermore, we extend the result of Kuleshov–Orlov in [7] to the blow-up of the projective plane in 9 very general points:

Theorem 1 ([6, Thm. 1.3]). *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 9 very general points. Then*

(1) *any numerically exceptional collection of maximal length consisting of line bundles is a full exceptional collection, and*
(2) *any two such collections are related by mutations and shifts.*

Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 10 points. Then the Picard lattice $\mathrm{Pic}(X)$ (with bilinear form given by the intersection pairing) admits a certain involution $\iota\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ which fixes the canonical class. Explicitly, if $E_i \in \mathrm{Pic}(X)$ denotes the (-1)-curve over the i-th blown up point and H denotes the pullback of a hyperplane class in $\mathbb{P}^2_{\mathbb{C}}$, then

$$\mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_{10}$$

is an orthogonal basis satisfying $H^2 = 1$ and $E_i^2 = -1$ for $1 \leq i \leq n$. The canonical class is $K_X = -3H + \sum_i E_i$ and the involution ι is given by

$$D_i := \iota(E_i) = -6H + 2\sum_{j=1}^{10} E_j - E_i,$$

$$F := \iota(H) = -19H + 6\sum_{i=1}^{10} E_i.$$

Equivalently ι can be defined by using the orthogonal decomposition $\mathrm{Pic}(X) = \mathbb{Z}K_X \oplus K_X^{\perp}$ and setting $\iota(K_X) = K_X$ and $\iota|_{K_X^{\perp}} = -\mathrm{id}_{K_X^{\perp}}$.

Theorem 2 ([5, Thm. 1.1]). *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 10 general points. Then the collection*

$$\langle \mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F) \rangle \subseteq \mathrm{D}^b(\mathrm{Coh}(X)) \tag{1}$$

is an exceptional collection which is not full.

Since $\mathrm{D}^b(\mathrm{Coh}(X))$ contains the full exceptional collection

$$\mathrm{D}^b(\mathrm{Coh}(X)) = \langle \mathcal{O}_X, \mathcal{O}_X(E_1), \ldots, \mathcal{O}_X(E_{10}), \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$$

of the same length, Theorem 2 disproves the following conjecture of Kuznetsov:

Conjecture 3 ([8, Conj. 1.10]). *Let $\mathcal{T} = \langle E_1, \ldots, E_n \rangle$ be a triangulated category generated by an exceptional collection. Then any exceptional collection of length n in $\mathcal{T}$ is full.*

As a consequence of Theorem 2, the right- or left-orthogonal complement of the collection (1) is a phantom category. In general, if $\mathcal{A}$ is an admissible subcategory of $\mathrm{D}^b(\mathrm{Coh}(X))$ and $\mathrm{D}^b(\mathrm{Coh}(X))$ admits a full exceptional collection, then, by [9, Cor. 3.4], $\mathcal{A}$ has a dg-enhancement quasi-equivalent to $\mathrm{Perf} - \mathcal{R}$, where $\mathcal{R}$ is a smooth finite-dimensional dg-algebra. Hence, Theorem 2 disproves the following conjecture of Orlov:

Conjecture 4 ([9, Conj. 3.7]). *There are no phantoms of the form* $\mathrm{Perf} - \mathcal{R}$*, where* $\mathcal{R}$ *is a smooth finite-dimensional dg-algebra and* $\mathrm{Perf} - \mathcal{R}$ *is the dg-category of perfect dg-modules over* $\mathcal{R}$*.*

Previous examples of phantoms were obtained by Gorchinskiy–Orlov [4] and Böhning–Graf von Bothmer–Katzarkov–Sosna [1].

References

[1] C. Böhning, H.-C. Graf von Bothmer, L. Katzarkov, and P. Sosna, *Determinantal Barlow surfaces and phantom categories*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 7, 1569–1592.

[2] A. I. Bondal and A. Polishchuk, *Homological properties of associative algebras: the method of helices*, Izv. Ross. Akad. Nauk Ser. Mat. **57** (1993), no. 2, 3–50.

[3] W. Chang, F. Haiden, and S. Schroll, *Braid group actions on branched coverings and full exceptional sequences*, (2023), arXiv:2301.04398v2.

[4] S. O. Gorchinskiĭ and D. O. Orlov, *Geometric phantom categories*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 329–349.

[5] J. Krah, *A phantom on a rational surface*, Invent. Math. **235** (2024), no. 3, 1009–1018.

[6] J. Krah, *Mutations of numerically exceptional collections on surfaces*, Math. Z. **307** (2024), no. 4, Paper No. 78, 28 pp.

[7] S. A. Kuleshov and D. O. Orlov, *Exceptional sheaves on Del Pezzo surfaces*, Izv. Ross. Akad. Nauk Ser. Mat. **58** (1994), no. 3, 53–87.

[8] A. G. Kuznetsov, *Semiorthogonal decompositions in algebraic geometry*, in Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, 635–660, Kyung Moon Sa, Seoul.

[9] D. O. Orlov, *Finite-dimensional differential graded algebras and their geometric realizations*, Adv. Math. **366** (2020), 107096, 33 pp.

Central Curves in Noncommutative Planes

Okke van Garderen

(joint work with Thilo Baumann, Pieter Belmans)

1. Noncommutative hypersurfaces

Noncommutative projective spaces are prime examples of noncommutative varieties. They are defined by an AS regular algebra A with Hilbert series of the form $H_A(t) = (1-t)^{-(n+1)}$ as a noncommutative graded coordinate ring, and they have a “category of coherent sheaves” defined via Serre's construction

$$\mathsf{qgr}\, A = \mathsf{gr}\, A/\mathsf{fd}\, A.$$

An important problem is to understand *noncommutative hypersurfaces*: subcategories of the form

$$\mathsf{qgr}(A/(f)) \subset \mathsf{qgr}\,A,$$

where $f \in Z(A)$ is a homogeneous central element of some degree d. The case of quadrics ($d = 2$) has been studied extensively, starting with the introduction in [1] of a Koszul duality technique that gives a criterion for understanding when such a noncommutative hypersurface is smooth. Subsequently [2] gave a description of the derived category of such noncommutative quadrics in some examples, and [3] gave a full classification for quadrics in noncommutative surfaces ($n = 2$).

2. Relation to orders

In my talk I explained a new approach to noncommutative hypersurfaces from our paper [5] for the case when A is finite over its center. Under this last assumption one can consider the *central Proj*: a pair $(Y, \mathcal{A})$ of a variety Y with a finite map $Y \to \operatorname{Proj} Z(A)$ equipped with a maximal order $\mathcal{A}$, and a generalisation of Serre's construction yields an equivalence

$$\mathsf{coh}(Y, \mathcal{A}) \simeq \mathsf{qgr}\,A.$$

Now given a central element $f \in Z(A)$ it defines a function on Y whose vanishing locus $V \subset Y$ is an ordinary hypersurface, enhanced with the restricted sheaf of algebras $\mathcal{A}|_V$, and there is (as we show) a restricted equivalence

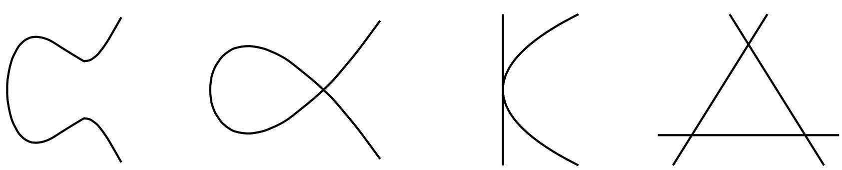
$$\mathsf{coh}(V, \mathcal{A}|_V) \simeq \mathsf{qgr}(A/(f)).$$

Hence, the study of these noncommutative hypersurfaces fits into a general program of understanding how well-behaved orders restrict to ordinary hypersurfaces.

Now $\mathcal{A}$ is an Azumaya algebra over a dense locus, and in particular a coherent $\mathcal{A}$-module is generically just equivalent to a twisted sheaf. The interesting behaviour appears on the complement, the *ramification locus* $\Delta \subset Y$. The same holds for any restriction $(V, \mathcal{A}|_V)$, which acts like a (possibly twisted) ordinary hypersurface outside the ramification locus. If V itself is chosen smooth then the possible singular behaviour is therefore localised on $V \cap \Delta$, and it suffices to consider the (complete) local structure of orders, studied e.g. for surfaces in [6].

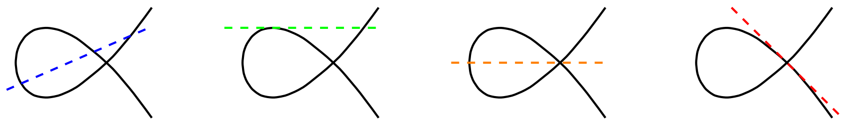
3. Curves in noncommutative planes

Motivated by the above, one would hope to control the singularities of noncommutative hypersurfaces via the geometry of the central Proj. We show that this indeed works for $n = 2$, that is: curves in noncommutative planes. Here the possible pairs $(Y, \mathcal{A})$ are classified in [7] and consist of $Y = \mathbb{P}^2$ and $\mathcal{A}$ an order with discriminant locus given by a cubic curve $\Delta \subset \mathbb{P}^2$ of one of four types:



Given a central element $f \in Z(A)$ defining a smooth curve $V \subset \mathbb{P}^2$, we characterise the singularities by studying the intersection of V and Δ. If the intersections are transverse then we show that $\mathcal{A}|_V$ is always hereditary, while for non-transverse intersections on the smooth locus of Δ we show that the order is never hereditary.

We recover many of the results for the degree $d = 2$ case alluded to at the start, using the explicit model of Clifford algebras. In this setting the hypersurface V is always a line and there are 6 possible isomorphism classes for $\mathcal{A}|_V$, which correspond precisely to the types of intersections it might have with the cubic. In this way we recover a geometric reinterpretation of the classification of [3]. The main benefit of our approach is that we can use explicit order-theoretic properties to characterise the singularities of such noncommutative hypersurfaces. As an example, we can consider the case where Δ is a nodal cubic:



The first one case yields a hereditary order, the others a singular one. The second and third case are nodal orders, while the second is also a tiled order. All examples above are moreover Bass orders, a certain strengthening of the Gorenstein property, which one would expect for a hypersurface.

4. Relation to stacks

There is another dimension to the story, which is the relation between orders and stacks. It was shown in [8] that hereditary orders on curves are naturally equivalent to smooth stacky curves with trivial generic stabilisers. More recently [9] showed a similar result for surfaces: for any sufficiently well-behaved order $\mathcal{A}$ on a normal surface Y they construct a sequence of stacky birational modifications

$$(\mathcal{Y}_{\text{can}}, \mathcal{A}_{\text{can}}) \to (\mathcal{Y}_{\text{root}}, \mathcal{A}_{\text{root}}) \to (Y, \mathcal{A}),$$

which subsequently resolve the ramification of $\mathcal{A}$ in codimension 1 and 2, yielding an Azumaya algebra on a stack. The categories of coherent modules are moreover equivalent, so this procedure exchanges the noncommutative structure for an equivalent stacky one.

Taking $V \subset Y$ to be a hypersurface we can restrict the construction of [9], to obtain a sequence of stacky curves equipped with a sheaf of algebras

$$(\mathcal{V}_{\text{can}}, \mathcal{A}_{\text{can}}|_{\mathcal{V}_{\text{can}}}) \to (\mathcal{V}_{\text{root}}, \mathcal{A}_{\text{root}}|_{\mathcal{V}_{\text{root}}}) \to (V, \mathcal{A}|_V),$$

In our paper we show that the associated categories of coherent modules are again equivalent, so every order obtained by restricting to a curve has an equivalent description as a stacky curve with an Azumaya algebra. This gives a natural extension of [8] to non-hereditary orders, in which case the corresponding stacky curve has singularities.

The above applies to noncommutative plane curves: taking $(Y, \mathcal{A})$ to be the central Proj and V a curve defined by a central element f we obtain equivalences

$$\mathsf{qgr}(A/(f)) \simeq \mathsf{coh}(V, \mathcal{A}|_V) \simeq \mathsf{coh}(\mathcal{V}_{\mathrm{can}}, \mathcal{A}_{\mathrm{can}}|_{\mathcal{V}_{\mathrm{can}}}).$$

Hence, every noncommutative plane curve also has a description as a (possibly twisted) stacky curve, and with some luck an analogous result may hold in higher dimensions.

References

[1] S. Smith, M. Van den Bergh, *Noncommutative quadric surfaces*, Journal of Noncommutative Geometry **7** (2013), 817-856.

[2] K. Ueyama, *Derived categories of skew quadric hypersurfaces*, Israel Journal of Mathematics **253** (2023).

[3] H. Hu, M. Matsuno, I. Mori, *Noncommutative conics in Calabi-Yau quantum projective planes*, Journal of Algebra **620** (2023), 194–224.

[4] A. Kanazawa, *Non-commutative projective Calabi-Yau schemes*, Journal of Pure and Applied Algebra **219** (2015), 2771–2780.

[5] T. Baumann, P. Belmans, O. van Garderen, *Central curves on noncommutative surfaces*, arXiv:2410.07620 [math.AG].

[6] D. Chan, C. Ingalls. *The minimal model program for orders over surfaces*, Inventiones Mathematicae **161.2** (2005), 427–452.

[7] M. Van Gastel, *On the center of the Proj of a three dimensional regular algebra*, Communications in Algebra **30** (2002), 1–25.

[8] D. Chan, C. Ingalls. *Non-commutative coordinate rings and stacks*, In: Proceedings of the London Mathematical Society **88.1** (2004), 63–88.

[9] E. Faber, C. Ingalls, S. Okawa, M. Satriano, *On stacky surfaces and noncommutative surfaces*, arXiv:2206.13359v1 [math.AG].

Categorical absorptions for hereditary orders

Thilo Baumann

Let (B, o) be a smooth pointed curve over an algebraically closed field $\mathbf{k}$ and $\mathscr{A}$ a hereditary $\mathcal{O}_B$-order which is only ramified over the point $o \in B$ with ramification index $e \in \mathbb{Z}_{>0}$. It is a classical fact that, up to Morita equivalence, $\mathscr{A}$ is given by the $\mathcal{O}_B$-subalgebra

$$\mathscr{A} = \begin{pmatrix} \mathcal{O}_B & \dots & \mathcal{O}_B & \mathcal{O}_B \\ \mathcal{O}_B(-o) & \ddots & \mathcal{O}_B & \mathcal{O}_B \\ \vdots & \ddots & & \vdots \\ \mathcal{O}_B(-o) & \dots & \mathcal{O}_B(-o) & \mathcal{O}_B \end{pmatrix} \subset \mathrm{Mat}_e(\mathbf{k}(B)). \tag{1}$$

Consider $\mathrm{coh}(B, \mathscr{A})$ the category of coherent $\mathcal{O}_B$-modules with a right $\mathscr{A}$-module structure.

There is a two-step process of constructing a semiorthogonal decomposition of the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathrm{coh}(B, \mathscr{A}))$, which goes as follows:

(1) Use the dictionary [2] between hereditary orders on smooth separated curves and smooth separated Deligne–Mumford stacks with trivial generic stabilizer in order to obtain a root stack $\sqrt[e]{B;o}$ associated to $\mathscr{A}$.
(2) Apply [1, 3] to obtain a semiorthogonal decomposition for the root stack.

Apart from the contribution of $\mathsf{D}^{\mathsf{b}}(\operatorname{coh} B)$ to the semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\operatorname{coh} \sqrt[e]{B;o})$, the remaining components lack a direct interpretation in terms of the order $\mathscr{A}$.

In our work in progress, we present a direct way of constructing a B-linear semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\operatorname{coh}(B,\mathscr{A}))$ using the recently introduced notion of *categorical absorption* [5].

Categorical absorption. Let $\mathcal{T}$ be the bounded derived category of a projective variety X (resp. of a finite-dimensional $\mathbf{k}$-algebra A) and denote by $\mathcal{T}^{\mathsf{hf}} \subset \mathcal{T}$ the triangulated subcategory of perfect complexes, see [6, Definition 4.10] for the notation. If $\mathcal{T}^{\mathsf{hf}}$ is properly contained in $\mathcal{T}$, the variety X (resp. A) is not smooth. Kuznetsov–Shinder define so-called $\mathbb{P}^{\infty,2}$-objects, which can only exist if $\mathcal{T}$ is not smooth.

Definition 1. An object $S \in \mathcal{T}$ is called a $\mathbb{P}^{\infty,2}$-*object* if

$$\operatorname{Ext}^{\bullet}_{\mathcal{T}}(S,S) := \bigoplus_{k\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(S,S[k]) \cong \mathbf{k}[\theta], \tag{2}$$

where $\deg\theta = 2$.

A semiorthogonal collection $(S_1,\dots,S_r)$ of $\mathbb{P}^{\infty,2}$-objects *absorbs singularities of* $\mathcal{T}$ if the triangulated category $\mathcal{S} = \langle S_1,\dots,S_r\rangle$ generated by these objects is admissible in $\mathcal{T}$ and the orthogonal complement ${}^{\perp}\mathcal{S}$ is smooth and proper.

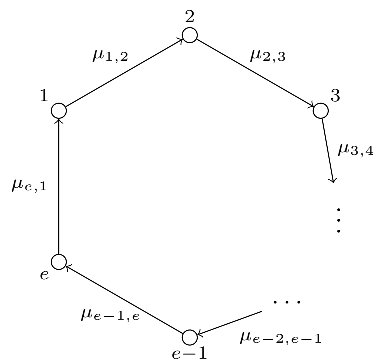
We focus on the following special case. Assume that $\mathcal{T} = \mathsf{D}^{\mathsf{b}}(X)$, where $X \cong \mathcal{X}_o$ is the fiber of a smoothing $\mathcal{X} \to B$ of X. If there is an absorption of singularities of X by $\mathbb{P}^{\infty,2}$-objects, [5, Theorem 1.5, Theorem 1.8] show that:

(1) the semiorthogonal collection of $\mathbb{P}^{\infty,2}$-objects on X pushes forward to an exceptional collection on $\mathcal{X}$ (via the closed immersion $\iota\colon X \to \mathcal{X}$).
(2) there is a B-linear semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\mathcal{X}) = \langle \iota_*\mathcal{S}, \mathcal{D}\rangle$ such that
    - the admissible subcategory $\mathcal{D}$ is smooth and proper,
    - over $o \in B$: $\mathcal{D}_o = {}^{\perp}\mathcal{S}$,
    - over every other point $b \in B \setminus \{o\}$: $\mathcal{D}_b = \mathsf{D}^{\mathsf{b}}(\mathcal{X}_b)$.

We explain how this result can be applied to hereditary orders.

Hereditary orders as smoothing of finite-dimensional algebras. Given the hereditary $\mathcal{O}_B$-order $\mathscr{A}$ of the form (1), we can view it as a smoothing of the finite-dimensional $\mathbf{k}$-algebra $A = \mathbf{k}Q/I$, where Q is the cyclic quiver

(3)



and $I \trianglelefteq \mathbf{k}Q$ is the admissible ideal generated by all cycles. Then $\mathsf{D}^{\mathsf{b}}(\operatorname{mod} A)$ is not smooth and admits an absorption of singularities by $\mathbb{P}^{\infty,2}$-objects.

Lemma 2. *Denote by S_i the simple A-module associated with the vertex i.*

1. *The sequence $(S_1, \dots, S_{e-1})$ forms a semiorthogonal collection of $\mathbb{P}^{\infty,2}$-objects in $\mathsf{D}^b(\operatorname{mod} A)$.*
2. *The triangulated subcategory $\mathcal{S} = \langle S_1, \dots, S_{e-1}\rangle$ is admissible and ${}^{\perp}\mathcal{S}$ is smooth and proper, and equivalent to $\mathsf{D}^b(\mathbf{k})$.*

Note that $A \cong \mathscr{A} \otimes_B \mathbf{k}(o)$, whereas $\mathscr{A} \otimes_B \mathbf{k}(b) \cong \operatorname{Mat}_e(\mathbf{k}(b))$ for all $b \in B \setminus \{o\}$. The notion of morphisms of noncommutative schemes as in [4, 7] allows us to view

$$\mathbf{f} = (\mathrm{id}_B, \mathrm{incl})\colon (B, \mathscr{A}) \to (B, \mathcal{O}_B) \tag{4}$$

as a smoothing of A, where A can be interpreted as the fiber of $\mathbf{f}$ over o via the morphism of noncommutative schemes

$$\mathbf{i}_o = (i_o, \mathrm{id}_{i_o^* \mathscr{A}})\colon (\operatorname{Spec} \mathbf{k}(o), A) \to (B, \mathscr{A}), \tag{5}$$

obtained from the closed immersion $i_o\colon \operatorname{Spec} \mathbf{k}(o) \to B$. Our main result combines the observation of Lemma 2 with an adapted version of [5, Theorem 1.5, Theorem 1.8] to noncommutative schemes.

Theorem 3. *There is a B-linear semiorthogonal decomposition*

$$\mathsf{D}^b(\operatorname{coh}(B, \mathscr{A})) = \langle \mathbf{i}_{o,*}S_1, \dots, \mathbf{i}_{o,*}S_{e-1}, \mathcal{D}\rangle \tag{6}$$

such that

- *the sequence $\mathbf{i}_{o,*}S_1, \dots, \mathbf{i}_{o,*}S_{e-1}$ is exceptional,*
- *the admissible subcategory $\mathcal{D}$ is smooth and proper,*
- *the fibers of $\mathcal{D}$ over $b \in B$ are equivalent to $\mathsf{D}^b(\operatorname{mod} \mathbf{k}(b))$.*

The result gives a better understanding of the semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\operatorname{coh}(B, \mathscr{A}))$ as we obtain the B-linearity of the decomppsition as well as a direct interpretation of the components in terms of the order $\mathscr{A}$.

References

[1] D. Bergh, V. Lunts, O. Schnürer, *Geometricity for derived categories of algebraic stacks*, Selecta Math. (N.S.) **22** (2016), 2535–2568.

[2] D. Chan, C. Ingalls, *Non-commutative coordinate rings and stacks*, Proc. London Math. Soc. (3) **88** (2004), 63–88.

[3] A. Ishii, K. Ueda, *The special McKay correspondence and exceptional collections*, Tohoku Math. J. (2) **67** (2015), 585–609.

[4] A. Kuznetsov, *Hyperplane sections and derived categories*, Izv. Ross. Akad. Nauk Ser. Mat. **70** (2006), 23–128.

[5] A. Kuznetsov, E. Shinder, *Categorical absorptions of singularities and degenerations*, Épijournal Géom. Algébrique (2023), 12–42.

[6] A. Kuznetsov, E. Shinder, *Homologically finite-dimensional objects in triangulated categories*, preprint, arXiv:2211.09418 (2023).

[7] F. Xie, *Residual categories of quadric surface bundles*, J. Reine Angew. Math., **796** (2023), 161–199.

Mirror symmetry for generalized K3 surfaces

Atsushi Kanazawa

Mirror symmetry for a K3 surface S is a very subtle problem, as the complex and Kähler structures are somewhat mixed in $H^2(S,\mathbb{C})$. In [2], Dolgachev formulated mirror symmetry for lattice polarized K3 surfaces as a duality between the algebraic and transcendental cycles. Although his formulation works well in many cases, it cannot be definitive due to an assumption that does not hold in general. For example, it does not work for a singular K3 surface, which admits no deformation of complex structure while retaining the maximal Picard number. A satisfactory formulation of mirror symmetry, particularly one that solves the puzzle of mirror symmetry for singular K3 surfaces, has been anticipated. In this talk, I will provide a solution to this problem by using the generalized Calabi-Yau (CY) structures developed in [3, 4], which can be viewed as a realization of non-commutative geometry.

Let M be a differentiable manifold underlying a K3 surface and $A^{2*}_{\mathbb{C}}(M) = \oplus_{i=0}^{2} A^{2i}_{\mathbb{C}}(M)$ the space of even differential forms with $\mathbb{C}$-coefficients. We define a pairing on $A^{2*}_{\mathbb{C}}(M)$ by

$$\langle \varphi, \varphi' \rangle := \varphi_2 \wedge \varphi'_2 - \varphi_0 \wedge \varphi'_4 - \varphi_4 \wedge \varphi'_0 \in A^4_{\mathbb{C}}(M),$$

where φ_i denotes the degree i part of φ. A *generalized CY structure* on M is a closed form $\varphi \in A^{2*}_{\mathbb{C}}(M)$ such that $\langle \varphi, \varphi \rangle = 0$ and $\langle \varphi, \overline{\varphi} \rangle > 0$.

Let φ be a generalized CY structure, then there are 2 types:

(A) If $\varphi_0 \neq 0$, then $\varphi = e^B(\varphi_0 e^{\sqrt{-1}\omega}) = \varphi_0 e^{B+\sqrt{-1}\omega}$ for a symplectic ω and a B-field B.

(B) If $\varphi_0 = 0$, then $\varphi = e^B \sigma = \sigma + B^{0,2} \wedge \sigma$ with a holomorphic 2-form σ (for a complex structure) and a B-field B.

For a generalied CY structure φ, we denote by P_φ the $\mathbb{R}$-vector spaces spanned by $\mathrm{Re}(\varphi)$ and $\mathrm{Im}(\varphi)$. We say that generalized CY structures φ, φ' are orthogonal if P_φ and $P_{\varphi'}$ are pointwise orthogonal. A *hyperKähler structure* of a generalized

CY structure φ is a generalized CY structure φ' which is orthogonal to φ and $\langle\varphi,\overline{\varphi}\rangle = \langle\varphi',\overline{\varphi'}\rangle$. A *generalized K3 surface* is a pair (φ,φ') of generalized CY structures such that φ is a hyperKähler structure for φ'. For example, a K3 surface M_σ (one with a holomorphic 2-form σ) with a chosen hyperKähler structure ω (which means $2\omega^2 = \sigma\wedge\overline{\sigma}$) can be identified with a generalized K3 surface $(e^{\sqrt{-1}\omega},\sigma)$.

We next introduce the *Néron–Severi lattice* and the *transcendental lattice* of a generalized K3 surface $X = (\varphi,\varphi')$ as follows:

$$\widetilde{NS}(X) := \{\delta\in H^*(M,\mathbb{Z}) \mid \langle\delta,\varphi'\rangle = 0\},\ \widetilde{T}(X) := \{\delta\in H^*(M,\mathbb{Z}) \mid \langle\delta,\varphi\rangle = 0\}.$$

It is important that we define them on a completely equal footing, and hence the intersection $\widetilde{NS}(X)\cap\widetilde{T}(X)$ is in general non-trivial. Moreover, it is easy to see that

$$\varphi\in\widetilde{NS}(X)_{\mathbb{C}},\quad \varphi'\in\widetilde{T}(X)_{\mathbb{C}},$$

and

$$2\le \mathrm{rank}(\widetilde{NS}(X))\le 22,\quad 2\le\mathrm{rank}(\widetilde{T}(X))\le 22.$$

For integers $\kappa,\lambda\ge 2$ such that $\kappa+\lambda = 24$, let K and L be even lattices of signature $(2,\kappa-2)$ and $(2,\lambda-2)$ respectively. A (K,L)*-polarized generalized K3 surface* is a pair (X,i) of a generalized K3 surface $X = (\varphi,\varphi')$ and a lattice embedding $i : K\oplus L\hookrightarrow H^*(M,\mathbb{Z})$ such that

(1) the restrictions $i|_K$ and $i|_L$ are primitive embeddings,
(2) $i(K)\subset\widetilde{NS}(X)$ and $i(L)\subset\widetilde{T}(X)$.

Then a family $\mathcal{X}$ of (K,L)-polarized generalized K3 surfaces and a family $\mathcal{Y}$ of (L,K)-polarized generalized K3 surfaces are defined to be *mirror symmetric*. Indeed, for generic (K,L)-polarized generalized K3 surface X and (L,K)-polarized generalized K3 surface Y, we have

$$\widetilde{NS}(X)\cong K\cong\widetilde{T}(Y),\quad \widetilde{T}(X)\cong L\cong\widetilde{NS}(Y).$$

Namely, a duality between algebraic and transcendental cycles holds. A notable examples is given for $K = \langle -2n\rangle^{\oplus 2}\oplus U^{\oplus 2}\oplus E_8^{\oplus 2}$, $L = \langle 2n\rangle^{\oplus 2}$:

- (K,L)-polarized generalized K3 surfaces = singular K3 surfaces.
- (L,K)-polarized generalized K3 surfaces $\supset$ polarized K3 surfaces (S,H) with $H^2 = 2n$.

In this way, a mirror of a K3 surface may not be a K3 surface, but a generalized K3 surface (non-commutative one).

Our formulation is compatible with Aspinwall–Morrison's description of mirror symmetry from the viewpoint of SCFTs [1]. They identified the moduli space of $N = (2,2)$ SCFTs on a K3 surface with the Grassmannian

$$\mathrm{Gr}^{po}_{2,2}(H^*(M,\mathbb{R}))\cong \mathrm{O}(4,20)/(\mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{O}(20))$$

parametrizing the orthogonal pairs of positive oriented 2-planes in $H^*(M,\mathbb{R})$. To a K3 surface M_σ with a Kähler class ω, we may associate

$$(P_{[\sigma]},P_{[e^{\sqrt{-1}\omega}]})\in\mathrm{Gr}^{po}_{2,2}(H^*(M,\mathbb{R})).$$

There are "non-geometric points" that cannot be obtained in this way. However, these points are given by generalized K3 surfaces by a Torelli type theorem proven in [4]. Mirror symmetry is an involution of $\mathrm{Gr}^{po}_{2,2}(H^*(M,\mathbb{R}))$ given by swapping the two 2-planes, which does not preserve the geometric points. Therefore, a mirror partner of a K3 surface need not be a K3 surface, but rather a generalized K3 surface.

References

[1] P. Aspinwall and D. Morrison, *String theory on K3 surfaces, Mirror symmetry*, II, AMS/IP Stud. Adv. Math. **1** (1997), 703–716.

[2] I. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*, J. Math. Sci. **81** (1996), 2599–2630.

[3] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford Ser. **54** (2003) 281–308.

[4] D. Huybrechts, *Generalized Calabi-Yau structures, K3 surfaces, and B-fields*, Int. J. Math. **16** (2005) 13–36.

[5] A. Kanazawa, *Mirror symmetry and rigid structures of generalized K3 surfaces*, arXiv:2108.05197.

Tilting for 1-dimensional AS-Gorenstein algebras and noncommutative quadric hypersurfaces

Kenta Ueyama

(joint work with Osamu Iyama and Yuta Kimura)

In the first part of this talk, we explain our results on tilting theory for the triangulated category $\underline{\mathsf{CM}}_0^{\mathbb{Z}}\,A$ associated with an Artin-Schelter Gorenstein (AS-Gorenstein) algebra A of dimension 1.

Throughout this talk, let k be a field and let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be a noetherian locally finite $\mathbb{N}$-graded k-algebra. In addition, we always assume that A is a basic algebra with a complete set of primitive orthogonal idempotents $\{e_1, \dots, e_n\}$. We say that A is *AS-Gorenstein* of dimension d if

- $\operatorname{injdim} A_A = \operatorname{injdim} {}_AA = d < \infty$,
- there exist integers $p_1, \dots, p_n \in \mathbb{Z}$ and a permutation $\nu \in \mathfrak{S}_n$ such that
$$\mathrm{Ext}^i_A(S_i, A) \cong \begin{cases} D(S_{\nu(i)})(p_i) & \text{if } i = d \\ 0 & \text{if } i \neq d, \end{cases}$$
where $S_i := \operatorname{top} e_iA$ is the i-th simple right A-module and D is the k-dual.

We call p_i the *Gorenstein parameter* of S_i and ν the *Nakayama permutation* of A.

Let A be an AS-Gorenstein algebra. The category of graded *maximal Cohen-Macaulay* modules is defined to be

$$\mathsf{CM}^{\mathbb{Z}}\,A = \{M \in \mathsf{mod}^{\mathbb{Z}}\,A \mid \mathrm{Ext}^i_A(M, A) = 0 \text{ for all } i > 0\}.$$

It is a Frobenius category, so the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}\,A$ is a triangulated category. Moreover, we consider the Serre quotient category

$$\mathsf{qgr}\,A = \mathsf{mod}^{\mathbb{Z}}\,A/\{M \in \mathsf{mod}^{\mathbb{Z}}\,A \mid \dim_k M < \infty\}.$$

The category $\mathsf{qgr}\, A$ is called the *noncommutative projective scheme* of A.

Let A be an AS-Gorenstein algebra of dimension 1. We define the $\mathbb{Z}$-graded algebra Q by $Q = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{qgr}\, A}(A, A(i))$. Then the full subcategory

$$\mathsf{CM}_0^{\mathbb{Z}} A = \{M \in \mathsf{CM}^{\mathbb{Z}} A \mid M \otimes_A Q \text{ is a projective } Q\text{-module}\}$$

is also a Frobenius category, so the stable category $\underline{\mathsf{CM}}_0^{\mathbb{Z}} A$ is a triangulated category. The category $\underline{\mathsf{CM}}_0^{\mathbb{Z}} A$ behaves much nicer than $\underline{\mathsf{CM}}^{\mathbb{Z}} A$ from a point of view of Auslander-Reiten theory. The main theorem of the first part is as follows.

Theorem 1 ([2])**.** *Let A be an AS-Gorenstein algebra of dimension 1. Assume that A is ring-indecomposable and* $\operatorname{gldim} A_0 < \infty$*. Then the following hold.*

(1) *There exists a positive integer q such that $\bigoplus_{i=1}^{q} A(i)$ is a progenerator in* $\mathsf{qgr}\, A$.

(2) *If $p_i \le 0$ for all $1 \le i \le n$, then $\bigoplus_{j=1}^{n} \bigoplus_{i=1}^{-p_j+q} e_{\nu(j)} A(i)_{\ge 0}$ is a tilting object in $\underline{\mathsf{CM}}_0^{\mathbb{Z}} A$.*

(3) *Let p_A^{av} be the average of $p_1, \dots, p_n$, i.e., $p_A^{\mathrm{av}} = n^{-1} \sum_{i=1}^{n} p_i$. Then $\underline{\mathsf{CM}}_0^{\mathbb{Z}} A$ admits a tilting object if and only if either $p_A^{\mathrm{av}} \le 0$ or A is AS-regular.*

Note that Theorem 1 is a generalization of Buchweitz-Iyama-Yamaura's theorem [1].

In the second part of this talk, we apply Theorem 1 to study noncommutative quadric hypersurfaces.

Let S be a connected graded Koszul AS-regular algebra of dimension $d+1$, and $f \in S$ a homogeneous regular normal element of degree 2. Then $B := S/(f)$ is a connected graded Koszul AS-Gorenstein algebra of dimension d and Gorenstein parameter $d-1$ and $A := (\bigoplus_{i\in\mathbb{N}} \operatorname{Ext}_B^i(k,k))^{\mathrm{op}}$ is a connected graded Koszul AS-Gorenstein algebra of dimension 1 and Gorenstein parameter $1-d$. The main theorem of the second part is as follows.

Theorem 2 ([2])**.** *Let S be a connected graded Koszul AS-regular algebra of dimension $d+1$, and $f \in S$ a homogeneous regular normal element of degree 2. Put $B = S/(f)$ and $A = (\bigoplus_{i\in\mathbb{N}} \operatorname{Ext}_B^i(k,k))^{\mathrm{op}}$. Assume that $d \ge 1$ and* $\operatorname{gldim}(\mathsf{qgr}\, B) < \infty$*. Then the following hold.*

(1) $\underline{\mathsf{CM}}^{\mathbb{Z}} A = \underline{\mathsf{CM}}_0^{\mathbb{Z}} A$ *has a tilting object* $\bigoplus_{i=1}^{d} A(i)_{\ge 0}$.

(2) *The derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{qgr}\, B)$ has a tilting object $\bigoplus_{i=1}^{d} \Omega^i k(i)$.*

(3) *Consider a direct sum decomposition $\Omega^d k(d) = \bigoplus_{j=1}^{\ell} X_j^{\oplus m_j}$ in $\mathsf{qgr}\, B$, where X_j's are pairwise non-isomorphic indecomposable. Then $\mathsf{D}^{\mathrm{b}}(\mathsf{qgr}\, B)$ has a full strong exceptional collection*

$$(X_\ell, X_{\ell-1}, \dots, X_1, \Omega^{d-1} k(d-1), \dots, \Omega^1 k(1)).$$

References

[1] R.-O. Buchweitz, O. Iyama, and K. Yamaura, *Tilting theory for Gorenstein rings in dimension one*, Forum Math. Sigma **8** (2020), Paper No. e36, 37 pp.

[2] O. Iyama, Y. Kimura, and K. Ueyama, *Cohen-Macaulay representations of Artin-Schelter Gorenstein algebras of dimension one*, preprint, `arXiv:2404.05925`.

On non-commutative cubic surfaces and their moduli

Tarig Abdelgadir
(joint work with Shinnosuke Okawa and Kazushi Ueda)

1. Moduli of quiver relations

Given a quiver (with relations) one may construct moduli spaces of quiver representations. These usually display similar features to moduli of vector bundles and admit similar tools.

In [1], quivers with relations were used to develop a different type of moduli spaces called *moduli spaces of quiver relations*. They are defined in a similar fashion to moduli of hypersurfaces and capture the deformations of the quiver algebra; roughly speaking these are the moduli of varieties analogue. These are especially useful when the quivers studied come from strong exceptional collections on varieties when they allow us to capture non-commutative deformations of commutative objects.

2. Compact moduli of marked noncommutative cubics

The methods outlined in [1] work well for del Pezzo surfaces; they come with full strong exceptional collections. This abstract deals with cubic surfaces in particular.

The strategy is to take a full strong exceptional collection on a cubic surface and deform the relations on the corresponding quiver. The corresponding moduli space of relations is then a candidate for a 'moduli space of noncommutative cubic surfaces'. One could then go on to study how/when the points of this space define noncommutative spaces in the sense of Artin-Zhang.

A cubic surface X has a full strong exceptional collection consisting of nine line bundles

$$\begin{aligned} &E_{0,0} := \mathcal{O}_X, && E_{0,1} := \mathcal{O}_X(l_1), && E_{0,2} := \mathcal{O}_X(l-l_4), \\ &E_{1,0} := \mathcal{O}_X\left(-2l+\textstyle\sum_{i=1}^6 l_i\right), && E_{1,1} := \mathcal{O}_X(l_2), && E_{1,2} := \mathcal{O}_X(l-l_5), \\ &E_{2,0} := \mathcal{O}_X(-l+l_1+l_2+l_3), && E_{2,1} := \mathcal{O}_X(l_3), && E_{2,2} := \mathcal{O}_X(l-l_6), \end{aligned} \tag{1}$$

where l is the strict transform of the hyperplane in $\mathbf{P}^2$ and l_i for $1 \le i \le 6$ are exceptional divisors.

The total morphism algebra of the collection (1) is described by the quiver $Q = (Q_0, Q_1, s, t)$ shown in Figure 1. The sets Q_0 and Q_1 of vertices and arrows of the quiver Q consist of 9 and 18 elements respectively. It has 9 relations, each of which can be regarded as an element of $\mathbf{P}^2$. We define the *compact moduli scheme of marked noncommutative cubic surfaces* as the geometric invariant theory quotient $\overline{M}_\chi := ((\mathbf{P}^2)^9)^{\chi\text{-ss}} /\!/ (\mathbb{G}_\mathrm{m})^{Q_1}$ of $(\mathbf{P}^2)^9$ by the action of the group $(\mathbb{G}_\mathrm{m})^{Q_1} \cong (\mathbb{G}_\mathrm{m})^{18}$ rescaling the arrows, where $\chi \in \mathrm{Pic}^{(\mathbb{G}_\mathrm{m})^{Q_1}}(\mathbf{P}^2)^9 \cong \mathbf{Z}^{27}$ is the stability parameter. The same idea has been used in [1] to construct a compact moduli of noncommutative $\mathbf{P}^2$. The moduli scheme $\overline{M}_\chi$ is a toric variety depending on

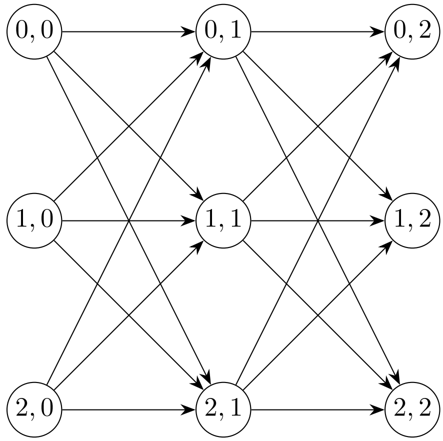


Figure 1. The quiver Q

the stability condition χ, and the subscheme $M^\circ \subset \overline{M}_\chi$ where all components of the homogeneous coordinate are non-zero is a dense torus $M^\circ \cong (\mathbb{G}_\mathrm{m})^8$ for any generic χ in the secondary fan.

The main result of [2] shows that one can construct an AS-regular $\{0,1,2\} \times \mathbf{Z}$-algebra of type $\widetilde{Q}$ from a decuple $\left(Y, (L_v)_{v \in Q_0}\right)$ consisting of a smooth projective curve Y of genus one and nine line bundles $(L_v)_{v \in Q_0}$ satisfying an admissibility condition. This construction gives a rational map $\mathscr{A} \colon M_\mathrm{ell} \dashrightarrow \overline{M}_\chi$ from the fine moduli scheme M_ell of admissible elliptic decuples.

Given a relation $I \subset \mathbf{k}Q$ of the quiver Q and a generic stability parameter $\theta \in \mathbf{Z}^{Q_0}$, the moduli scheme N_θ of θ-stable representations of $\Gamma = (Q, I)$ of dimension vector $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^{Q_0}$ is a projective variety carrying the universal line bundles $(\mathcal{U}_v)_{v \in Q_0}$. This construction gives a rational map $\mathscr{N}_\theta \colon \overline{M}_\chi \dashrightarrow M_\mathrm{ell}$.

Theorem 1. *There exists θ such that $\mathscr{A}$ and $\mathscr{N}_\theta$ are birational inverses of each other.*

We also prove the following:

Theorem 2. *By sending a marked cubic surface to the relation of the quiver, one obtains a locally closed immersion $\iota \colon M_\mathrm{cubic} \to M^\circ$.*

References

[1] Tarig Abdelgadir, Shinnosuke Okawa and Kazushi Ueda, *Compact moduli of noncommutative projective planes*, arXiv:1411.7770.

[2] Shinnosuke Okawa and Kazushi Ueda, *AS-regular algebras from acyclic spherical helices*, arXiv:2007.07620.

A relative version of Smith's theorem, Calabi-Yau completion and a generalization of derived quiver Heisenberg algebras

Hiroyuki Minamoto

Let K be a field. We recall Paul Smith's pioneering result.

Theorem 1 ([4]). *Let $A = K \oplus A_1 \oplus A_2 \oplus \cdots$ be a Koszul algebra (connected over K). Then A is AS-regular if and only if its Koszul dual $A^!$ is Frobenius. Moreover, A is Calabi-Yau if and only if $A^!$ is symmetric (in a graded sense).*

This beautiful and important correspondence has been generalized in several directions by many researchers. Among other things, we point out that Lu-Palmieri-Wu-Zhang [3] dropped the Koszul assumption. For this purpose, they employ the notion of Adams graded A_∞-algebras that are connected over the base field K.

The aim of this talk is to give a generalization of Smith's theorem for an Adams graded A_∞-algebra that is connected over a DG-algebra R, to show that the simplest case of this generalization provides Keller's Calabi-Yau completion [2] and to show that the second simplest case includes the derived quiver Heisenberg algebras which were introduced by M. Herschend and the speaker [1].

Definition 2. Let R be a DG-algebra over K.

(1) An Adams graded (not necessarily counital) A_∞-coalgebra

$$C = (\bigoplus_{p\in\mathbb{Z}} C_p, \{\Delta_i\}_{i\geq 1})$$

over R is a pair of an Adams graded cofibrant DG-R-bimodules $C = \bigoplus_{p\in\mathbb{Z}} C_p$ and a collection of morphisms $\Delta_i : C \to C^{\otimes_R i}$ of Adams graded DG-R-bimodules of cohomologically degree $2-i$ that satisfies the following conditions:

(a) For each $p \in \mathbb{Z}$, there exists i such that $\Delta_j|_{C_p} = 0$ for all $j \geq i$.

(b) The co-Stasheff identities:

$$\sum (-1)^{i+jk}(\mathrm{id}_A^{\otimes i} \otimes \Delta_j \otimes \mathrm{id}_A^{\otimes k})\Delta_{i+1+k} = 0$$

where $i, k \geq 0, \quad j \geq 1$. We note that by the condition (a) the above sum is well-defined.

- The morphisms Δ_i are collectively called higher multiplications.

For $i \geq 1$ and $p_1, p_2, \ldots, p_i \in \mathbb{Z}$, we denote by

$$\Delta_{i,(p_1,p_2,\cdots,p_i)} : C_{p_1+p_2+\cdots+p_i} \to C_{p_1} \otimes_R C_{p_2} \otimes_R \cdots \otimes_R C_{p_i}$$

the homogeneous component with respect to the Adams grading.

(2) An Adams graded A_∞-coalgebra $\overline{C}$ over R is said to be *positively graded* if $\overline{C}_i = 0$ for $i \leq 0$.

We note that from the condition on the Adams grading, the condition (1)-(a) for the morphisms Δ_i above is automatically satisfied.

(3) An Adams graded A_∞-coalgebra C over R is said to be *co-connected over R* if the following conditions are satisfied:

(a) $C_i = 0$ for $i \leq -1$ and $C_0 = R$ as a DG-R-module.

(b) For $p \geq 0$, the morphisms $\Delta_{2,(0,p)} : C_p \to R \otimes_R C_p$, $\Delta_{2,(p,0)} : C_p \to C_p \otimes_R R$ coincide with the inverse of the canonical isomorphism induced from the R-bimodule structure of C_p.

(c) For $i \geq 3$ and $p_1, p_2, \ldots, p_i \geq 0$, we have $\Delta_{i,(p_1,p_2,\ldots,p_i)} = 0$ if one of p_s is 0.

We note that from the conditions (3)-(a)(b)(c), the condition (1)-(a) for the morphisms Δ_i above is automatically satisfied.

(4) An Adams graded A_∞-coalgebra C over R is said to be *locally perfect* if for each $p \in \mathbb{Z}$, the p-the Adams graded component C_p is perfect as a DG-R-bimodule.

(5) Let $C = R \oplus C_1 \oplus C_2 \oplus \cdots$ be an Adams graded A_∞-coalgebra connected over R. The Koszul dual (or the cobar construction) $C^!$ of C is an Adams graded DG-algebra defined as follows: as an algebra equipped with Adams grading and cohomological grading $C^!$ is defined to be the tensor algebra $T_R(\Sigma^{-1}\overline{C})$ where Σ^{-1} denotes the shift by -1 in the cohomological degree. The differential $d_{C^!}$ is given by the derivation induced from the higher comultiplications $\{\Delta_i\}_{i\geq 1}$.

Let $(-)^\vee := \mathrm{Hom}_{R^e}(-, R^e)$ be the R^e-duality functor regarded as an endofunctor of the DG-category of DG-R-bimodules via the canonical isomorphism $R^e \cong (R^e)^{op}$.

Definition 3. Let $d \in \mathbb{Z}$ and $C = R \oplus C_1 \oplus C_2 \oplus \cdots \oplus C_l$ be a finitely Adams graded A_∞-coalgebra, locally perfect and connected over R. Then C is said to be *d-symmetric* if the dual A_∞-C-bicomodule $C^\vee$ is quasi-isomorphic to $\Sigma^{-d}C(l)$ as an Adams graded A_∞-C-bicomodule where Σ^{-d} denotes the shift by $-d$ in the cohomology degree, and (l) denotes the shift by l in the Adams grading.

The main result of this talk is the following theorem.

Theorem 4. *Assume that R is a homologically smooth DG-algebra. Let $d \in \mathbb{Z}$ and $C = R \oplus C_1 \oplus C_2 \oplus \cdots \oplus C_l$ be a finitely Adams graded A_∞-coalgebra, locally perfect and connected over R. Then C is d-symmetric if and only if its Koszul dual $C^!$ is d-Calabi-Yau.*

We observe that the simplest case of this theorem provides Keller's Calabi-Yau completion [2].

Example 5. Let $d \in \mathbb{Z}$ and R a homologically smooth DG-algebra. Thanks to the strict co-unital condition, the Adams graded DG-R-bimodule $C := R \oplus \Sigma^d R^\vee$ has a unique structure of an A_∞-coalgebra connected over R, which is d-symmetric. The Koszul dual $C^!$ of C is nothing but the d-Calabi-Yau completion:

$$C^! = T_R(\Sigma^{-1}\overline{C}) = T_R(\Sigma^{d-1}R^\vee).$$

Thus the main theorem recovers Keller's result that d-Calabi-Yau completion is actually d-Calabi-Yau.

Next we look at the second simplest case that the length l of the Adams grading is 2. Namely we deal with the case $C = R \oplus C_1 \oplus \Sigma^d R^\vee$. For the simplicity we set $X := C_1$. To ensure that C is d-symmetric, X must possess an appropriate symmetry. More precisely, a higher comultiplication $\{\Delta_i\}_{i\geq 1}$ on C which makes C d-symmetric can be reinterpreted to a quasi-isomorphism $\alpha : \Sigma^d X^\vee \to X$ of DG-R-bimodules which complete the following homotopy commutative diagram

$$\begin{array}{ccc} \Sigma^d R^\vee & \longrightarrow & \Sigma^d X^\vee \otimes X \\ \big\downarrow & & \big\downarrow {\scriptstyle \alpha\otimes X} \\ X \otimes \Sigma^d X^\vee & \xrightarrow[X\otimes\alpha]{} & X \otimes X \end{array}$$

where the top horizontal morphism and the left vertical morphism are the shifts of the canonical co-trace morphisms.

Applying this criterion, we can prove the following result.

Theorem 6. *Assume that the characteristic of the base field K is not 2. Let $d \in \mathbb{Z}$ and R a homologically smooth DG-algebra. Let X be the cone of a morphism $\theta : \Sigma^{d-1} R^\vee \to R$ of DG-R-bimodules. Then the Adams graded DG-R-bimodule $C := R \oplus X \oplus \Sigma^d R^\vee$ has a structure of an Adams graded A_∞-coalgebra connected over R, which is d-symmetric.*

In the case where $R = KQ$ is the path algebra of a finite quiver Q, a little modification of this construction provides the derived quiver Heisenberg algebra introduced in [1].

References

[1] Herschend, Martin; Minamoto, Hiroyuki, Quiver Heisenberg algebras: a cubic analogue of preprojective algebras, arXiv:2402.08162.

[2] Keller, Bernhard, Deformed Calabi-Yau completions. With an appendix by Michel Van den Bergh. J. Reine Angew. Math. 654 (2011), 125–180.

[3] Lu, Di Ming; Palmieri, John H.; Wu, Quan Shui; Zhang, James J. Koszul equivalences in A_∞-algebras. New York J. Math. 14 (2008), 325–378.

[4] Smith, S. Paul Some finite-dimensional algebras related to elliptic curves. Representation theory of algebras and related topics (Mexico City, 1994), 315-348, CMS Conf. Proc., 19, Amer. Math. Soc., Providence, RI, 1996.

Ozone groups of PI Artin–Schelter regular algebras

Jason Gaddis

(joint work with Kenneth Chan, Robert Won, James Zhang)

The goal of this talk is to define an invariant which is useful in the study of Artin–Schelter (AS) regular algebras. A good overview of the history and relevance of AS regular algebras may be found in the survey of Rogalski [7].

Throughout, let $\Bbbk$ be a characteristic zero field. Given $\mathbf{p} \in M_n(\Bbbk^\times)$ satisfying $p_{ij}p_{ji} = p_{ii} = 1$ for all i, j, the corresponding *skew polynomial ring* is the $\Bbbk$-algebra

$$S_{\mathbf{p}} = \Bbbk\langle x_1, \ldots, x_n \mid x_j x_i - p_{ij} x_i x_j \rangle.$$

It is well-known that $S_{\mathbf{p}}$ is an AS regular noetherian domain with global and Gelfand-Kirillov dimension n. It is module-finite over its center if and only if each p_{ij} is a root of unity. In this setting that is equivalent to satisfying a polynomial identity (that is, being a PI ring).

Work in [4] was motivated by the question of when the center $Z(S_{\mathbf{p}})$ is Gorenstein or regular. The strategy therein is to define the subgroup O of $\mathrm{Aut}_{\mathrm{gr}}(S_{\mathbf{p}})$ generated by the automorphisms ϕ_i in which $\phi_i(f) = x_i^{-1} f x_i$ for all $f \in S_{\mathbf{p}}$. This map is well-defined because each x_i is a normal element. It is not difficult to see that O is abelian and that $|O| = \mathrm{rank}_Z(A)$, but the key point is to observe that $O = \mathrm{Aut}_{Z-\mathrm{alg}}(S_{\mathbf{p}})$ and $Z = S_{\mathbf{p}}^O$. Hence, one can employ tools from *(noncommutative) invariant theory* to study $Z(S_{\mathbf{p}})$. For a thorough survey of work in this area, including means for generalizing invariant theoretic concepts like reflections and determinant to the noncommutative setting, see the survey of Kirkman [6].

If $S_{\mathbf{p}}$ is PI, then there is some ℓth root of unity ξ such that $p_{ij} = \xi^{b_{ij}}$ for each i, j. Let $B = (b_{ij})$. Let $\overline{B}$ be the matrix obtained from B by reduction mod ℓ.

Theorem 1 ([4])**.** *Suppose $n = 3$ and $S_{\mathbf{p}}$ is PI. Then the center $Z(S_{\mathbf{p}})$ is*

- *regular if and only if the orders of p_{12}, p_{13}, and p_{23} are pairwise coprime;*
- *Gorenstein if and only if $\overline{B}(b'_{23}, b'_{13}, b'_{12})^T = 0$ where $b'_{ij} = \gcd(b_{ij}, \ell)$.*

In [4], conditions are also given for $n = 4$. While some more general results are presented, the problem becomes significantly harder in higher dimension.

It is then reasonable to apply the concepts above to other families of AS regular algebras.

Definition 2. Let A be an algebra and C be a subalgebra of $Z(A)$. The *Galois group* of A over C is defined as

$$\mathrm{Gal}(A/C) := \{\sigma \in \mathrm{Aut}(A) \mid \sigma(c) = c \text{ for all } c \in C\}.$$

The corresponding *ozone group*[1] is then $\mathrm{Oz}(A) = \mathrm{Gal}(A/Z(A))$.

One can similarly define the *graded Galos group* $\mathrm{Gal}_{\mathrm{gr}}(A/C)$ of A over C and the *graded ozone group* $\mathrm{Oz}_{\mathrm{gr}}(A)$ by restricting to graded automorphisms. Moreover, if A is a $\mathbb{Z}^n$-graded domain which is prime and a finite module over its center, then an application of the Skolem-Noether Theorem gives that every ozone automorphism is given by conjugation by a normal homogeneous element. Consequently, $\mathrm{Oz}_{\mathrm{gr}}(A) = \mathrm{Oz}(A)$.

Now suppose A is a noetherian PI AS regular algebra with center Z. Then the order of $\mathrm{Oz}(A)$ is finite and divides $\mathrm{rank}_Z(A)$. Hence,

$$1 \leq |\mathrm{Oz}(A)| \leq \mathrm{rank}_Z(A).$$

At one extreme one has trivial ozone groups. This occurs if and only if every normal element is central. Furthermore, if A has trivial ozone group, then A is necessarily Calabi–Yau since the Nakyama automorphism of A belongs to the ozone group [2, Proposition 4.4]. The converse, however, is not true.

[1] *Over Z it is ONE!*

Example 3. Let q be a primitive ℓth root of unity q, $\ell \geq 2$ and $3 \nmid \ell$. Define
$$H_q = \Bbbk\langle x, y, z \mid zx - qxz, yz - qzy, xy - qyx - z^2 \rangle.$$
Then H_q is Calabi–Yau and $\mathrm{Oz}(H_q) \cong \mathbb{Z}_\ell$.

This raises the question of which Calabi–Yau algebras have trivial ozone groups. Using the classification of Itaba and Mori [5] of quantum projective planes finite over their centers, we obtain the following classification result.

Theorem 4 ([3])**.** *Let A be a PI quadratic AS regular algebra of global dimension three. Then A has trivial ozone group if and only if it is isomorphic to one of the following:*

- *(type E) an elliptic Skylanin algebra*
$$S(a, b, c) = \Bbbk\langle x, y, z \mid axy + byx + cz^2, ayz + bzy + cx^2, azx + bxz + cy^2 \rangle$$
with defining automorphism of order $n > 1$ and $3 \nmid n$, or
- *(type NC) an algebra*
$$B_q = \Bbbk\langle x, y, z \mid xy - qyx, zx - qxz - y^2, zy - q^{-1}yz - x^2 \rangle$$
where $q \neq 1$ is a root of unity and 3 does not divide the order of q.

Most cyclic groups are realized as $\mathrm{Oz}(H_q)$ for some q. The remaining ones are realized using other quadratic AS regular algebras. We show that if A and B are noetherian PI AS regular algebras, then
$$\mathrm{Oz}(A \otimes B) = \mathrm{Oz}(A) \otimes \mathrm{Oz}(B).$$
Hence, every finite abelian group is realizable as the ozone group of a noetherian PI AS regular algebra. As a sort of converse, we make the following conjecture which is supported by a host of examples.

Conjecture 5. *If A is a PI AS regular algebra, then* $\mathrm{Oz}(A)$ *is abelian.*

The hypothesis that A is connected graded is necessary. If Q is the extended Dynkin quiver $\widetilde{A_2}$, then the ozone group of preprojective algebra Π_Q is both infinite and nonabelian. However, it is not clear if one needs the full power of AS regularity.

This leaves the question of which PI AS regular algebras have ozone groups of maximal rank. It turns out that this property characterizes skew polynomial rings amongst PI AS regular algebras.

Theorem 6 ([3])**.** *Suppose $\Bbbk$ is algebraically closed. Let A be a noetherian PI AS regular algebra that is generated in degree one. The following are equivalent:*

(1) $\mathrm{Oz}(A)$ *is abelian and* $|\mathrm{Oz}(A)| = \mathrm{rank}_Z(A)$.
(2) $\mathrm{Oz}(A)$ *is abelian and* $A^{\mathrm{Oz}(A)} = Z(A)$.
(3) *A is generated by normal elements.*
(4) *A is* isomorphic *to $S_{\mathbf{p}}$.*

We hope that future research on ozone groups will present additional applications to the study of PI AS regular algebras. The above theorem shows that we cannot reasonably expect the invariant ring of the ozone group to be the center in general. However, if we iterate correctly one may obtain critical information on the center.

Example 7. Let $A = \Bbbk\langle x, y \mid x^2y + yx^2, xy^2 + y^2x\rangle$. Then A is PI AS regular. In particular, A is a graded noetherian down-up algebra.

A computation shows that $\mathrm{Oz}(A) \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Then $B = A^{\mathrm{Oz}(A)}$ is commutative and $Z(A) \subsetneq B$. Moreover, B is Gorenstein since $\mathrm{Oz}(A)$ acts by trivial homological determinant on A.

Let $G = \mathrm{Gal}(B/Z(A)) = \{1, \tau\}$. Since B is commutative and $\det \tau|_{B_1} = 1$, then $B^G = Z(A)$ is Gorenstein.

To finish, we show that the ozone group my be applied to the study of the Zariski Cancellation Problem.

We say an algebra A is *cancellative* (in a class of algebras $\mathcal{C}$) if for all algebras B (in $\mathcal{C}$) such that $A[t] \cong B[t]$, we have $A \cong B$. Using the tensor product result above, $\mathrm{Oz}(A[t]) = \mathrm{Oz}(A)$ for a PI AS regular algebra A. Hence, we are able to apply ozone groups to help prove the following theorem, which improves on an earlier result of Bell and Zhang [1, Theorem 9].

Theorem 8 ([3])**.** *If S is a PI skew polynomial ring in three variables, then S is cancellative in the class of connected graded algebras.*

References

[1] J. Bell and J. J. Zhang. An isomorphism lemma for graded rings. *Proc. Amer. Math. Soc.*, 145(3):989–994, 2017.

[2] K. A. Brown and J. J. Zhang. Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras. *J. Algebra*, 320(5):1814–1850, 2008.

[3] K. Chan, J. Gaddis, R. Won, and J. J. Zhang. Ozone groups of PI Artin-Schelter algebras. *arxiv preprint arxiv:2312.13014*, 2023.

[4] K. Chan, J. Gaddis, R. Won, and J. J. Zhang. Ozone groups and centers of skew polynomial rings. *Int. Math. Res. Not. IMRN*, (7):5689–5727, 2024.

[5] A. Itaba and I. Mori. Quantum projective planes finite over their centers. *Canad. Math. Bull.*, 66(1):53–67, 2023.

[6] E. E. Kirkman. Invariant theory of Artin-Schelter regular algebras: a survey. In *Recent developments in representation theory*, volume 673 of *Contemp. Math.*, pages 25–50. Amer. Math. Soc., Providence, RI, 2016.

[7] D. Rogalski. Artin-Schelter regular algebras. In *Recent advances in noncommutative algebra and geometry*, volume 801 of *Contemp. Math.*, pages 195–241. Amer. Math. Soc., [Providence], RI, [2024] ©2024.

Calabi–Yau deformations of q-symmetric algebras

Travis Schedler
(joint work with Mykola Matviichuk, Brent Pym)

Let $\mathsf{A}_q := \mathbb{C}\langle x_0, \ldots, x_{n-1}\rangle/(x_ix_j - q_{ij}x_jx_i)$ be the q-symmetric algebra, defined in terms of an $n \times n$ matrix $q = (q_{ij})$ with $q_{ij} \in \mathbb{C}^\times$, $q_{ij}q_{ji} = 1$ for all $i \neq j$, and $q_{ii} = 1$ for all i. In this talk we explained how to construct an analytic family of graded deformations of A_q, which under some mild genericity properties on q

define all possible formal deformations of A_q. These give many new examples of quadratic, Koszul, and twisted Calabi–Yau (and hence Artin–Schelter regular) algebras, which have the same Hilbert series as projective space and define quantisations of projective space. These results are motivated by the work [3] of the authors on Poisson deformations of toric log symplectic structures (and more generally, log symplectic structures whose polar divisor is normal crossings).

The basic technique we use is the Hochschild cohomology of A_q together with its $\mathbb{Z}^n$ grading coming from dilations on each of the variables $x_0, \ldots, x_{n-1}$. We describe this structure (which is not new, and is a special case of results appearing in the literature, such as [1, Theorem 3.3]): $HH^*(A_q)$ is concentrated in the weights (indexing $\mathbb{Z}^n$ from 0 to $n-1$):

$$\Phi := \{w \in \mathbb{Z}^n \mid w_j \geq -1, \forall j, \prod_k q_{jk}^{w_k} = 1 \text{ whenever } w_j \geq 0\} \subseteq \mathbb{Z}^n.$$

Setting $\Phi_i := \{w \in \Phi \mid \#\{j \mid w_j = -1\} = i\}$ and $\Phi_{\leq i} := \bigcup_{j \leq i} \Phi_j$, we see that $HH^i(A_q)$ is concentrated in weights $\Phi_{\leq i}$. Let $\overline{\mathbb{Z}^n} := \{w \in \mathbb{Z}^n \mid \sum_i w_i = 0\}$ be the weights summing to zero, i.e., the ones appearing in the dilation invariant subspace $HH^*(A_q)^{\mathbb{C}^\times}$. Let $\overline{\Phi_{\leq i}} := \Phi_{\leq i} \cap \overline{\mathbb{Z}^n}$. It follows that infinitesimal graded deformations are in weights $\overline{\Phi_{\leq 2}}$ and the possible obstructions are in weights $\overline{\Phi_{\leq 3}}$.

Therefore, given a subset $\Psi \subseteq \overline{\Phi_{\leq 2}}$ of weights such that no sum of ≥ 2 elements of Ψ lies in $\overline{\Phi_{\leq 3}}$, the deformations in the direction Ψ are unobstructed. We explained that these infinitesimal deformations extend to filtered quadratic deformations $\widetilde{A}_q$ over $\mathbb{C}$, with the property that $\operatorname{gr} \widetilde{A}_q \cong A_q$, and if no elements of Ψ are sums of other elements of Ψ, then the filtered degree -1 part matches the original first-order deformation. The filtration is defined on A_q, with $F^m A_q$ the sum of all weight spaces with weights $w \in \mathbb{N}^n$ such that $w \geq w'$ for w' a sum of some m elements of Ψ (with $w \geq w'$ meaning $w_i \geq w'_i$ for all i). As filtered deformations preserve the property of being both Koszul and twisted n-Calabi–Yau, these deformations enjoy these properties (in particular, are Artin–Schelter regular).

We give several new examples of such filtered deformations. These examples include filtered deformations of A_q for $n = 3$ and $n = 4$ which give irreducible components of the moduli space of quadratic algebras with the same graded dimension as A_1 but which do not pass through A_1. One example had, for ζ a primitive seventh root of unity, $q = \begin{pmatrix} 1 & 1 & \zeta & \zeta^{-2} & \zeta \\ 1 & 1 & \zeta^{-1} & \zeta^{-2} & \zeta^3 \\ \zeta^{-1} & \zeta & 1 & \zeta & \zeta^{-1} \\ \zeta^2 & \zeta^2 & \zeta^{-1} & 1 & \zeta^{-3} \\ \zeta^{-1} & \zeta^{-3} & \zeta & \zeta^3 & 1 \end{pmatrix}$. This A_q is untwisted Calabi–Yau and we found a family of filtered Calabi–Yau deformations given by replacing three of the q-commutation relations by the following:

$$x_0 x_1 - x_1 x_0 = a x_3 x_4, \quad x_0 x_2 - \zeta x_2 x_0 = b x_4^2, \quad x_0 x_3 - \zeta^5 x_3 x_0 = c x_2^2.$$

Note that, for a, b, c nonzero, these algebras are all isomorphic to the one for $a = b = c = 1$ by dilating the variables. We show moreover that the centres of these algebras form a flat family, generated by x_i^7 for all variables x_i, together with a deformation of the element $x_0x_1x_2x_3x_4$. For $a = b = c = 1$ this deformation is

$$x_0x_1x_2x_3x_4 + \frac{1}{7}(-5\zeta^5 - 3\zeta^4 - \zeta^3 + \zeta^2 - 4\zeta - 2)x_4x_2^3x_1 +$$
$$\frac{1}{7}(-\zeta^5 + 5\zeta^4 + 4\zeta^3 + 3\zeta^2 + 2\zeta + 1)x_4^2x_3^2x_2 + \frac{1}{7}(\zeta^5 - 5\zeta^4 - 4\zeta^3 - 3\zeta^2 - 2\zeta - 1)x_4^3x_3x_1.$$

For another example, with $n = 3$, we set $q = \begin{pmatrix} 1 & -1 & i & i \\ -1 & 1 & i & i \\ -i & -i & 1 & -1 \\ -i & -i & -1 & 1 \end{pmatrix}$, with weights of $HH(A_q)^{\mathbb{C}^\times}$ given by

$$0, (-1,-1,1,1), (1,1,-1,-1), (3,-1,-1,-1),$$
$$(-1,3,-1,-1), (-1,-1,3,-1), \text{ and } (-1,-1,-1,3).$$

This A_q is untwisted Calabi–Yau, and we produce from this the filtered Calabi–Yau deformation of A_q given by replacing the relation $x_0x_1 + x_1x_0$ by $x_0x_1 + x_1x_0 + tx_2x_3$ for a parameter $t \in \mathbb{C}$. This example is particularly interesting because q is conjugate by a diagonal matrix to the one $q' = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$. Here, $A_{q'}$ is a graded Clifford algebra (now twisted Calabi–Yau), and its deformations include a twelve-parameter family of graded Clifford algebras (thanks to Colin Ingalls for pointing this out). The weights for these deformations are given by $(-1,-1,2,0)$ and permutations of this, so they are different from the previous weights; generic deformations can be given by replacing relations $x_ix_j + x_jx_i$ by $x_ix_j + x_jx_i + ax_k^2 + bx_\ell^2$ for i, j, k, ℓ all distinct, and $a, b \in \mathbb{C}$. Since q, q' are conjugate, A_q and $A_{q'}$ define equivalent categories $\mathrm{qgr} A_q \simeq \mathrm{qgr} A_{q'}$ of $\mathbb{Z}$-graded modules modulo modules of elements annihilated by powers of the augmentation ideal (x_0, x_1, x_2, x_3). Thus one can construct from the preceding an abelian category deformation $\mathrm{qgr} A_q$ which is not obtainable by deforming A_q itself (as an algebra, although it can be obtained from a $\mathbb{Z}$-algebra).

We then proceed to consider the case where A_q is related to toric log symplectic structures on $\mathbb{P}^{n-1}$ by deformation quantisation. According to our paper [3], the weights occurring in the second Poisson cohomology $HP^*(\mathbb{P}^{n-1}, \pi)$ for the latter type of Poisson structures have a rigid structure: other than the zero weight, they are given by weights $w \in \mathbb{Z}^n$ with $w_i = -1$ for exactly two values of i. For each such weight w with $w_i = w_j = -1$, colour the corresponding edge i—j of the complete graph on n vertices. Then in [3] we proved that the resulting coloured set has connected components which are cycles and segments. We additionally colour in the angles $\angle ikj$ opposite to coloured edges i—j when the weight w with $w_i = w_j = -1$ has $w_k > 0$; we colour it darkly if $w_k = 2$, and lightly if $w_k = 1$, so that there are either two lightly coloured angles or one darkly coloured angle opposite to each coloured edge. We call the resulting graph the *smoothing diagram* due to its geometric interpretation: it specifies which codimension-one singularities of the degeneracy locus of π (i.e., the union of coordinate hyperplanes of $\mathbb{P}^{n-1}$) can be removed under deformation.

If $\pi = \sum_{i,j} \pi_{ij} x_i \partial_i \wedge x_j \partial_j$ is a toric Poisson structure on $\mathbb{P}^{n-1}$, then for generic $\hbar$, setting $q_{ij} := \exp(\hbar \pi_{ij})$, the weights of $HH^*(A_q)$ are the same as the weights of the Poisson cohomology of $(\mathbb{P}^{n-1}, \pi)$. In fact, by work of Lindberg and Pym [2], for $\hbar$ a deformation parameter, A_q is the Kontsevich canonical deformation quantisation of $(\mathbb{P}^{n-1}, \pi)$.

Our main theorem shows that, when $HH^*(A_q)$ has the same weights as those occurring in the Poisson cohomology of a toric log symplectic structure (for example, $q_{ij} = \exp(\hbar \pi_{ij})$ for π log symplectic and $\hbar$ generic), then we can construct a formally versal family of actual quadratic deformations of A_q by a combination of modifying q, applying filtered deformations over the coloured components of the smoothing diagram which are segments, and replacing couloured cycles by Feigin–Odesskii elliptic algebras [4], and finally tensoring the elliptic algebras together with the aforementioned filtered deformation on the complement of the couloured cycles. Note that, unlike the examples considered at the beginning of this report, all of these families are obtained by analytic continuation of deformation quantisations of A_1, and in particular have A_1 on the closure.

Finally, as a consequence of these results and [2], we show that the filtered deformations we produce are, up to gauge equivalence, isomorphic to the canonical quantisations of the filtered deformations of the toric log symplectic structures on $\mathbb{P}^{n-1}$. Moreover we show that, under mild conditions on the weights, we do not require a gauge equivalence. This confirms Kontsevich's conjecture on convergence of quadratic Poisson structures in these cases, the first large class of examples to do so.

References

[1] L. Grimley, *Hochschild cohomology of group extensions of quantum complete intersections.* arXiv:1606.01727.

[2] A. Lindberg and B. Pym, *Semiclassical Hodge theory for log Poisson manifolds*, 2408.16685.

[3] M. Matviichuk, B. Pym, and T. Schedler, *A local Torelli theorem for log symplectic manifolds*, 2010.08692.

[4] A. V. Odesskiĭ and B. L. Feĭgin, *Sklyanin's elliptic algebras*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 3, 45–54, 96.

Derived categories of derived Grassmannians

Qingyuan Jiang

Although the title contains twice the word "derived," making it less user-friendly, the content of the talk will be concrete and relevant to noncommutative algebraic geometers. Throughout this talk, k is a field of characteristic zero, e.g., $k = \mathbb{Q}$.

1. (Non-)Commutative Resolutions of Determinantal Varieties

1.1. Determinantal Varieties and Resolutions.

- Let $W = k^m$, $V = k^n$ with $m \leq n$, and let $X = \mathrm{Hom}_k(W, V) \simeq \mathbb{A}^{mn}$. There exists a *tautological $\mathcal{O}_X$-module morphism* $\tau\colon W \otimes_k \mathcal{O}_X \to V \otimes \mathcal{O}_X$ whose valuation at a point $x = [A] \in X$ is A, i.e., $\tau([A]) = A$.
- For any integer $0 \leq \ell \leq m$, the *ℓ-th degeneracy locus* is

$$\mathbb{D}_\ell = \mathrm{Zero}(\wedge^{\ell+1}(\tau)) = \{[A] \in X \mid \mathrm{rank}\, A \leq \ell\} \subseteq X.$$

  It is well-known that $\mathbb{D}_\ell$ is a Cohen–Macaulay subscheme of codimension $(m-\ell)(n-\ell)$ and is singular if $0 < \ell < m$, with $\mathrm{Sing}(\mathbb{D}_\ell) = \mathbb{D}_{\ell-1} \subseteq \mathbb{D}_\ell$.

The degeneracy loci $\mathbb{D}_\ell$, $0 < \ell < m$, admit two natural types of resolutions.

- For a given pair of integers (d_+, d_-), consider the Grassmannian varieties

$$\mathbb{G}_+ = \mathbb{G}_{d_+}(V^\vee) \quad \text{and} \quad \mathbb{G}_- = \mathbb{G}_{d_-}(W),$$

  Let $\mathcal{U}_\pm$ denote the universal subbundle of rank $d_\pm$, and $\mathcal{Q}_\pm$ the corresponding tautological quotient bundle of rank $\ell_\pm$, where

$$\ell_+ = n - d_+ \quad \text{and} \quad \ell_- = m - d_-.$$

- We now consider the following varieties:

$$Z_+ = Z_+^{(d_+)} = \{([A], T_+ \subseteq V^\vee) \mid T_+ \subseteq \mathrm{Ker}(A^\vee)\} \subseteq X \times \mathbb{G}_+$$
$$Z_- = Z_-^{(d_-)} = \{([A], T_- \subseteq W) \mid T_- \subseteq \mathrm{Ker}(A)\} \subseteq X \times \mathbb{G}_-.$$

  If $0 < \ell_\pm < m$, then the natural projections $Z_\pm^{(d_\pm)} \to \mathbb{D}_{\ell_\pm}$ are *resolutions of singularities*. The map $Z_- \to D_{\ell_-}$ is in fact an IH-*small resolution*; see [4, Thm. 5.2]. Generally, the fibers of $Z_\pm \to X$ are Grassmannian varieties but with different dimensions over different degeneracy loci.

1.2. Buchweitz–Leuschke–Van den Bergh's Results on NCR.

Let $p_\pm\colon Z_\pm \to \mathbb{G}_\pm$ be the natural projections. Consider the following vector bundles over $Z_\pm$:

$$\mathcal{T}_+ = \bigoplus_{\alpha \in B_{\ell_+, d_+}} p_+^*\big(\mathbb{S}^\alpha(\mathcal{Q}_+)\big) \quad \text{and} \quad \mathcal{T}_- = \bigoplus_{\alpha \in B_{\ell_-, d_-}} p_-^*\big(\mathbb{S}^\alpha(\mathcal{Q}_-)\big).$$

Theorem 1 (Buchweitz–Leuschke–Van den Bergh [3]). *In the above situation:*

1. *$\mathcal{T}_\pm$ are classical tilting bundles over $Z_\pm$.*
   ($\implies$ $D(Z_\pm) = \langle p_\pm^*\big(\mathbb{S}^\alpha(\mathcal{Q}_\pm)\big)\rangle_{\alpha \in B_{\ell_\pm, d_\pm}}$, and $D(Z_\pm) \simeq D(E_\pm\text{-mod})$, where $E_\pm = \mathrm{End}_{\mathcal{O}_{Z_\pm}}(\mathcal{T}_\pm)$).
2. *E_- is a noncommutative resolution of $\mathbb{D}_{\ell_-}$.*
3. *If $\ell_+ = \ell_-$, then $Z_+ \to D_{\ell_+} = D_{\ell_-} \leftarrow Z_-$ is a flip if $m < n$ and a flop if $m = n$. If $m < n$, $D(Z_-) \hookrightarrow D(Z_+)$, if $m = n$, $D(Z_-) \simeq D(Z_+)$.*

Question 2. What are the relations between $D(Z_+^{(d_+)})$ and $D(Z_-^{(d_-)})$ in general?

1.3. **Beilinson–Bernstein–Deligne–Gabber Decomposition Theorem.** Fix $d_+ = d$. The BBDG decomposition theorem implies (for $k = \mathbb{C}$):

$$H^*(Z_+;\mathbb{Q}) = \bigoplus_{j=0}^{d} H^{*-2j(n-m-d+j)}\Big(Z_-^{(j)}; \underline{H^*}\big(\mathrm{Gr}_{d-j}(n-m)\big)\Big).$$

While this formula may seem intricate initially, its intuition is easier to grasp. It can be deduced from the following Chow-theoretical formula ([4]):

$$\mathfrak{h}(Z_+) = \bigoplus_{j=0}^{d} \mathfrak{h}(Z_-^{(j)}) \otimes \mathfrak{h}(\mathrm{Gr}_{d-j}(n-m)) \otimes \mathbb{L}^{j(n-m-d+j)}.$$

Essentially, these formulas say that the cohomology/Chow-theory of Z_+ can be decomposed into "$|\mathrm{Gr}_{d-j}(n-m)|$"-contributions from each small resolution $Z_-^{(j)}$ of degeneracy locus $X_j = \mathbb{D}_{m-j}$, $j = 0, \ldots, d$. We can upgrade the previous question:

Question 3. Can we categorify the above decomposition formulas?

2. Categorified Decompositions

Indeed, the answer is yes. The formula was conjectured by the author in [5], initially confirmed by Yukinobu Toda [9], and then by the author [8].

Theorem 4 (Toda [9], J. [8])**.** *There exists a semiorthogonal decomposition:*

$$D(Z_+) = \Big\langle \binom{n-m}{d-j} \text{ copies of } (D(Z_-^{(j)})\Big\rangle_{j=0,1,\ldots,d}.$$

The methods in these two papers are quite different. We illustrate the author's approach:

- (Fourier–Mukai kernels). For each pair $(d_+ = d, d_- = j)$, we consider the following incidence correspondence:

$$\widehat{Z} = \{([A], T_+ \subseteq V^\vee, T_- \subseteq W) \mid T_+ \subseteq \mathrm{Ker}(A^\vee), T_- \subseteq \mathrm{Ker}(A)\} = Z_+ \times_X^{\mathrm{cl}} Z_-$$

  There exists a tautological two-term complex $[\widehat{\tau}\colon \mathcal{Q}_- \to \mathcal{Q}_+^\vee]$. Consider the Fourier–Mukai functors $\Phi^{(j,\lambda)}$ induced by the *Schur complexes* ([1, 7])

$$\mathbb{S}^\lambda([\mathcal{Q}_- \to \mathcal{Q}_+^\vee]) = [\mathbb{W}^{\lambda^t}(\mathcal{Q}_-) \to \cdots (\mathcal{Q}_-) \otimes \mathbb{S}^{\lambda/1}(\mathcal{Q}_+^\vee) \to \mathbb{S}^\lambda(\mathcal{Q}_+^\vee)] \in D(\widehat{Z})$$

  parametrized by Young diagrams $\lambda \in B_{n-m-d+j,d-j}$ (and twisted by the line bundle $\det(\mathcal{Q}_+)^{\otimes j}$). The main result is then [8, Thm. 3.2]:

$$D(Z_+) = \Big\langle \Phi^{(j,\lambda^{(j)})}\big(D(Z_-^{(j)}\big) \mid 0 \le j \le d, \lambda^{(j)} \in B_{n-m-d+j,d-j}\Big\rangle.$$

- The key is an inductive approach that considers $\{Z_+^{(d)}\}_{d=0,1,\ldots}$ altogether, along with the flag correspondences among them. These flag and incidence correspondences are shown to be compatible [8] (easier shown using DAG).

Example 5. Let $m = 3$, $n = 6$, $d = 3$. $X = \mathrm{Hom}(k^3, k^6)$ has four strata: $X_0 = X$, $X_1 = \{[A] \mid \mathrm{rank}\, A \leq 2\} \subseteq X$ (codimension 4), $X_2 = \mathrm{cone}(\mathbb{P}^2 \times \mathbb{P}^5)$, and $X_3 = \{0\}$. The small resolutions of these strata are given by $Z_-^{(0)} = X$, $Z_-^{(1)} = \mathbb{P}(\mathrm{Coker}(\tau^\vee))$, $Z_-^{(2)} = \mathrm{Grass}_2(\mathrm{Coker}(\tau^\vee))$, and $Z_-^{(3)} = X_3 = \{0\}$. Moreover, $Z_+^{(3)} = \mathrm{Bl}_{X_1}(X)$.

$$D(\mathrm{Bl}_{X_1}(X)) = \langle D(X_0), 3 \text{ copies } D(Z_-^{(1)}), 3 \text{ copies } D(Z_-^{(2)}), D(\{0\})\rangle.$$

3. The General Setup and Applications

The result extends to any derived stack X/k and any complex $\mathcal{E} \in \mathrm{Perf}_{[0,1]}(X)$ with rank $r \geq 0$. $Z_+^{(d)}$ is replaced by the derived Grassmannian $\mathrm{Grass}_d(\mathcal{E})$, which maps each $\eta\colon T \to X$ in dSch to the ∞-groupoid of maps $\eta^*\mathcal{E} \to Q_d$ that are surjective on H^0, where $Q_d \in \mathrm{Vect}_d(T)$. This extends Grothendieck's construction in EGA. For $d = 1$, this corresponds to the derived projectivization $\mathbb{P}(\mathcal{E})$ in [6]. In [7], it was proven that $\mathrm{Grass}_d(\mathcal{E}) \to X$ and $\mathrm{Grass}_j(\mathcal{E}^\vee[1]) \to X$ are all quasi-smooth (Kontsevich's hidden smoothness). Hence $\mathrm{Grass}_j(\mathcal{E}^\vee[1])$ are *quasi-smooth resolutions of degeneracy loci* X_j. It was shown by the author [8, Thm. 3.2] that for any X and $\mathcal{E}$ as above, and any $D = D^{\mathrm{perf}}, D^b_{\mathrm{coh}}, D^-_{\mathrm{pcoh}}, D_{\mathrm{qcoh}}$,

$$D(\mathrm{Grass}_d(\mathcal{E})) = \left\langle \binom{r}{d-j} \text{ copies of } (D(\mathrm{Grass}_j(\mathcal{E}^\vee[1]))\right\rangle_{j=0,1,\ldots,d}.$$

This general framework provides numerous examples and applications:

(1) The formula unifies various results (for $\mathrm{char}(k) = 0$), including the projective/Grassmannian bundle formula, blowup formula along lci or determinantal centers, and formulas for standard flips and Grassmannian flips.
(2) Fulton–Macpherson's degeneration spaces are examples of $\mathrm{Grass}_d(\mathcal{E})$.
(3) Abel–Jacobi maps for singular integral curves (Altman–Kleiman).
(4) Hecke modifications of $[0,1]$-perfect complexes on surfaces (Negut). In the case of $\mathrm{Hilb}_n(S)$, it is related to the Brill–Noether conjecture [2].
(5) Wall-crossing scenarios (Nakajima–Yoshioka, Markman, etc.).

References

[1] K. Akin, D. Buchsbaum, J. Weyman. *Schur functors and Schur complexes.* Adv. Math. no. 3 (1982): 207–278.
[2] A. Bayer, H. Chen, Q. Jiang, *Brill–Noether theory for Hilbert schemes of points on surfaces.* Int. Math. Res. Not. no. 10 (2024), 8403–8416.
[3] R. Buchweitz, G. Leuschke, M. Van den Bergh, *Non-commutative desingularization of determinantal varieties, II: arbitrary minors.* Int. Math. Res. Not., no. 9 (2016): 2748-2812.
[4] Q. Jiang, *On the Chow theory of Quot schemes of locally free quotients.* J. Pure Appl. Algebra. (To appear).
[5] Q. Jiang, *Derived categories of quot schemes of locally free quotients.* arXiv:2107.09193.
[6] Q. Jiang, *Derived projectivizations of complexes.* Mem. Amer. Math. Soc. (To appear).
[7] Q. Jiang, *Derived Grassmannians and derived Schur functors.* arXiv:2212.10488.
[8] Q. Jiang, *Derived categories of derived Grassmannians.* arXiv: 2307.02456.
[9] Y. Toda, *Derived categories of Quot schemes of locally free quotients via categorified Hall products.* Math. Res. Lett. no. 30 (1), 239–265 (2023).

Zeta functions of orders on surfaces

Daniel Chan
(joint work with Sean Lynch, Colin Ingalls)

The Riemann zeta function and its Euler product expansion

$$\zeta(s) = \sum_{n\in\mathbb{N}} n^{-s} = \prod_{p,\text{prime}} (1-p^{-s})^{-1}$$

are fundamental to number theory. There are two natural ways to generalise this zeta function. Serre's arithmetic zeta function considers a scheme X of finite type over $\mathbb{Z}$ and defines the zeta function analogously to the Euler product expansion as

$$\zeta_X^{\text{Serre}}(s) = \prod_{x\in X} (1-|\kappa(x)|^{-s})^{-1}.$$

This recovers Riemann's zeta function when $X = \operatorname{Spec}\mathbb{Z}$ and is relatively accessible in part thanks to the Weil Conjectures.

We are interested in studying zeta functions for maximal orders A on X. Here it is more natural to use the following definition which generalises easily to the case of finitely generated A-modules M.

$$\zeta_A(M;s) := \sum_{\text{cofinite } N\leq M} |M/N|^{-s} = \sum_n a_n n^{-s} = \prod_{\text{closed } x\in X} \zeta_{\hat{A}_x}(\hat{M}_x; s)$$

where $a_n = \#\{N \leq {}_A M \mid |M/N| = n\}$ and the product expansion comes formally from the Chinese Remainder Theorem. The difference between the two definitions (when $M = \mathcal{O}_X$ in the latter case) is best seen by observing that computing the a_n amounts to counting ideals or equivalently, points on the Hilbert scheme, whilst Serre's zeta function counts 0-cycles, or points on the symmetric product.

The zeta function for maximal orders on one-dimensional X was computed by Hey in his 1927 PhD thesis. Suppose $\dim X = 1$. If A is Azumaya degree d then

$$\zeta_A(s) := \zeta_A(A;s) = \prod_{j=1}^{d} \zeta_X^{\text{Serre}}(d(s-1)+j).$$

If A is a degree d maximal order ramified at $x_1,\dots,x_\ell$ with ramification indices $e_1,\dots,e_\ell$ then

$$\frac{\zeta_A(s)}{\zeta_{M_d(\mathcal{O}_X)}} = \prod_{i=1}^{\ell} \prod_{e_i \nmid j,\ j=1}^{d} \left(1-|\kappa(x_i)|^{-d(s-1)-j}\right).$$

Importantly, the zeta function encodes the ramification data of the order. Andre Weil exploited this feature in his treatment of class field theory [6].

We were interested in zeta functions of maximal orders on surfaces X. In the commutative case where the order is $\mathcal{O}_X$ and X is smooth, these were studied by Göttsche [3]. He was interested in the topology of Hilbert schemes of points on surfaces and the zeta functions allowed the computation of their Poincaré polynomials via the Weil conjectures. The analysis here is greatly aided here by the fact that the Hilbert schemes are themselves smooth and toric methods can be exploited in the complete local case. There has also been some work by Gyenge-Nemethi-Szendroi [4] in the case of the skew group ring $k[[x,y]]*G$ where $G < SL_2$ is a finite subgroup, or the invariant ring $k[[x,y]]^G$. The formulas there involve large sums which make it difficult to globalise.

Lynch, in his PhD thesis [5], developed an approach for the complete local case $X = \operatorname{Spec} R$. The hypotheses are that there is a regular normal element $z \in \operatorname{rad} A$ such that $\bar{A} := A/(z)$ is hereditary and M is projective. We may thus consider the z-adic filtration on M as well as the induced filtration on any submodule $N \leq M$. Let us fix a sequence $\bar{\mathbf{M}} = (\bar{M}_n)_{n\in\mathbb{N}}$ of projective $\bar{A}$-modules $\bar{M}_n$ generically isomorphic to $\bar{M}$. Lynch showed that, if instead of looking at all cofinite $N \leq M$, one restricts to those whose associated graded module satisfy $(\operatorname{gr} N)_n \simeq z^n\bar{M}$, then one obtains an infinite product formula for the zeta function, so the full zeta function is just the sum of these over the (usually infinitely many) possible $\bar{\mathbf{M}}$.

Suppose now that A comes from the complete localisation of a maximal order at some closed point x and contains some primitive d-th root of unity where $d = \deg A$. Suppose further, that the maximal order is either terminal of prime index > 5 or that the ramification locus has at worst nodes as singularities, and that, if x is a node, one of the ramification covers ramifies totally there. Then work with Ingalls [1] shows that with these assumptions, one can always pick the regular normal element z so that there is only one choice for the sequence $\bar{\mathbf{M}}$ above and furthermore, the zeta function is now given by

$$\zeta_A(s) = \prod_{n=1}^{\infty}\prod_{i=1}^{r}(1 - q^{-nr(s-1)-j})^{-m}$$

where $A/\operatorname{rad} A \simeq \prod_{i=1}^m M_r(\mathbb{F}_q)$.

Returning to the global situation, suppose that locally, the ramification conditions above hold. Then [2] there is a formula for the zeta function, analogous to Hey's formula above, which shows precisely how ramification data is encoded. Let $d = \deg A$. In the Azumaya case we have

$$\zeta_A(s) = \prod_{n=1}^{\infty}\prod_{j=1}^{d}\zeta_X^{\mathrm{Serre}}(dn(s-1)+j)$$

If A is ramified on a single regular curve Y with degree e ram cover $\tilde{Y} \to Y$ then

$$\frac{\zeta_A(s)}{\zeta_{M_d(\mathcal{O}_X)}(s)} = \prod_{n=1}^{\infty}\frac{\prod_{j=1}^{d/e}\zeta_{\tilde{Y}}^{\mathrm{Serre}}(\frac{d}{e}n(s-1)+j)}{\prod_{j=1}^{d}\zeta_{Y}^{\mathrm{Serre}}(dn(s-1)+j)}$$

The case of more ramification curves can also be handled by adding more factors.

This is a report on Sean Lynch's PhD thesis and joint work with Sean Lynch and Colin Ingalls.

References

[1] D. Chan, C. Ingalls, *Terminal orders on arithmetic surfaces*, to appear in Algebra and Number Theory, (2024)
[2] D. Chan, S. Lynch, *Zeta functions of orders on surfaces* in preparation
[3] L. Göttsche, *Hilbert schemes of zero-dimensional subschemes of smooth varieties*, Lecture Notes in Mathematics, **1572**, Springer-Verlag, Berlin, (1994)
[4] A. Gyenge, A. Nemethi, B. Szendroi, *Euler characteristics of Hilbert schemes of points on simple surface singularities*, Eur. J. MRH **4** (2018) 439–524
[5] S. Lynch, *The slivered into parts method for semilocal zeta functions*, PhD thesis UNSW, Sydney, (2023)
[6] A. Weil, *Basic Number Theory*, Springer Verlag, New York **32** (1967).

An explicit derived McKay correspondence for some complex reflection groups of rank 2

Eleonore Faber
(joint work with Anirban Bhaduri, Yael Davidov, Katrina Honigs, Peter McDonald, C. Eric Overton-Walker, Dylan Spence)

This talk is about an extension of the derived McKay correspondence for finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ to complex reflection groups of rank 2 generated by reflections of order 2. If G is such a reflection group, then it contains a subgroup $H := G \cap \mathrm{SL}(2,\mathbb{C})$ of index 2, indeed, there is a bijection between these reflection groups and finite subgroups H in $\mathrm{SL}(2,\mathbb{C})$.

The classical McKay correspondence relates the representation theory of a finite group H in $\mathrm{SL}(2,\mathbb{C})$ and the geometry of the exceptional divisor of the minimal resolution Y of the corresponding quotient singularity $\mathbb{C}^2/H$, in particular, the nontrivial irreducible representations of H are in bijection with the components of the exceptional divisor in Y, see [Buc12] for more on history and algebraic versions of this result. Kapranov and Vasserot [KV00] showed that the correspondence may be realized as a derived equivalence between the derived category of coherent sheaves on the minimal resolution Y, and the derived category of equivariant coherent sheaves on the two-dimensional vector space the group is acting on:

$$D^b(Y) \simeq D^H(\mathbb{C}^2) .$$

This result has been extended to the case of small finite subgroups G in $\mathrm{GL}(2,\mathbb{C})$, that is, groups not containing any complex reflections, see [Ish02]. Furthermore, in the seminal paper [BKR01] a derived McKay correspondence for finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ was established, using equivariant Hilbert schemes.

On the other hand, for a complex reflection group $G \subseteq \mathrm{GL}(2,\mathbb{C})$ acting on $\mathbb{C}^2$ the quotient $\mathbb{C}^2/G$ is smooth by the theorem of Chevalley, Shephard, and Todd. This makes the geometric picture quite different from the classical case, as there are no singularities to resolve. However, a recent algebraic version of the McKay correspondence for reflection groups [BFI20] shows a bijection of the nontrivial irreducible representations of G with certain Cohen–Macaulay modules over the coordinate ring of the *discriminant* of the reflection group, a singular curve in $\mathbb{C}^2$.

Furthermore, the following conjecture predicts a semi-orthogonal decomposition of the equivariant category $D^G(\mathbb{C}^2)$:

Conjecture 1 (Polishchuk–Van den Bergh [PVdB19])**.** *Suppose that G is a finite group acting effectively on a smooth variety X and that for all $g \in G$ the geometric quotient $\bar{X}^g = X^g/C(g)$ is smooth. Then there is a semi-orthogonal decomposition of $D^G(X)$ whose components $C_{[g]}$ are in bijection with conjugacy classes and $C_{[g]} \simeq D^b(\bar{X}^g)$.*

Coming back to our setting, we can show:

Theorem 2. *Let G be a finite group contained in $\mathrm{GL}(2,\mathbb{C})$ generated by order 2 reflections and acting on $\mathbb{C}^2$. There is a semi-orthogonal decomposition of $D^G(\mathbb{C}^2)$ of the following form, where $B_1,\dots,B_r$ are the normalizations of the irreducible components of the branch divisor $\mathbb{C}^2 \to \mathbb{C}^2/G$, $E_1,\dots,E_n$ are exceptional objects and $r+n+1$ is the number of distinct irreducible representations of G:*

$$D^G(\mathbb{C}^2) \simeq \langle D^b(\mathbb{C}^2/G), D(B_1),\dots,D(B_r),E_1,\dots,E_n\rangle \ .$$

The proof strategy is inspired by Potter's thesis [Pot18], who proved the analogous result for the dihedral groups $G(m,m,2)$. An essential step in our argument is to compute, for each group G appearing in Theorem 2, the action of $G/H \cong \mathbb{Z}/2\mathbb{Z}$ on the minimal resolution Y of $\mathbb{C}^2/H$. For this, we use the explicit description of Y as the H-equivariant Hilbert scheme $H-\mathrm{Hilb}(\mathbb{C}^2)$ due to Ito and Nakamura [IN99]. A crucial observation is further that the action of G/H extends to Y and the quotient $Y/(G/H)$ is smooth. The decomposition of $D^G(Y)$ is then obtained using the equivalence $D^G(\mathbb{C}^2) \simeq D^{G/H}(Y)$ and facts about root stacks, see [IU15], and Orlov's blowup formula [Orl92].

Analyzing the quotients $\mathbb{C}^2/C(g)$ for $g \in G$, one may also deduce that Conjecture 1 holds in our case.

It would be interesting to further analyze the structure of $D^G(\mathbb{C}^n)$ for reflection groups $G \subseteq \mathrm{GL}(n,\mathbb{C})$ of higher rank and also for those reflection groups of rank 2 that are generated by complex reflections of higher order than 2.

References

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14(3):535–554, 2001.

[BFI20] Ragnar-Olaf Buchweitz, Eleonore Faber, and Colin Ingalls. A McKay correspondence for reflection groups. *Duke Math. J.*, 169(4):599–669, 2020.

[Buc12] R.-O. Buchweitz. From platonic solids to preprojective algebras via the McKay correspondence. *Oberwolfach Jahresbericht*, pages 18–28, 2012.

[IN99] Y. Ito and I. Nakamura. Hilbert schemes and simple singularities. In *New trends in algebraic geometry (Warwick, 1996)*, volume 264 of *London Math. Soc. Lecture Note Ser.*, pages 151–233. Cambridge Univ. Press, Cambridge, 1999.

[Ish02] Akira Ishii. On the McKay correspondence for a finite small subgroup of $\mathrm{GL}(2,\mathbb{C})$. *J. Reine Angew. Math.*, 549:221–233, 2002.

[IU15] Akira Ishii and Kazushi Ueda. The special McKay correspondence and exceptional collections. *Tohoku Math. J. (2)*, 67(4):585–609, 2015.

[KV00] M. Kapranov and E. Vasserot. Kleinian singularities, derived categories and Hall algebras. *Math. Ann.*, 316(3):565–576, 2000.

[Pot18] Rory Potter. *Derived Categories of Surfaces and Group Actions.* Phd thesis, University of Sheffield, February 2018. Available at `https://etheses.whiterose.ac.uk/19643/`.

[Orl92] D. O. Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izv. Ross. Akad. Nauk Ser. Mat.*, 56(4):852–862, 1992.

[PVdB19] Alexander Polishchuk and Michel Van den Bergh. Semiorthogonal decompositions of the categories of equivariant coherent sheaves for some reflection groups. *J. Eur. Math. Soc. (JEMS)*, 21(9):2653–2749, 2019.

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