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Mathematical Aspects of General Relativity

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Abstract. General relativity is an area at the interface of partial differential equations, differential geometry, global analysis, mathematical physics and dynamical systems. It interacts with astrophysics, cosmology, high energy physics, and numerical analysis. The field is rapidly expanding and has witnessed remarkable developments and interconnections with other fields in recent years.

The workshop Mathematical Aspects of General Relativity was organised by Carla Cederbaum (Tübingen), Mihalis Dafermos (Cambridge/Princeton), Jim Isenberg (Eugene) and Hans Ringström (KTH Stockholm). There were 48 on-site and 4 online participants. There were 16 one hour talks, nine 30 minute talks and four 10 minute talks.

Mathematics Subject Classification (2020): 83-XX, 35-XX, 53-XX.

Introduction by the Organizers

The talks during the workshop covered a wide range of remarkable new developments in the field. Below we briefly summarise some of them, grouped below in various thematic units.

1. A central focus of the workshop concerns the dynamics of asymptotically flat solutions to the Einstein equations (isolated self-gravitating systems). This continues to be an area of rapid mathematical developments, concerning problems of long-standing physical interest.

The workshop was opened by an overview talk by Igor Rodnianski on the state of the art on the *weak cosmic censorship* conjecture, a fundamental problem in general relativity, and on the related question of the construction, structure and stability properties of so-called "naked singularities".

Elena Giorgi presented results on linear stability for the very slowly rotating, small charge Kerr–Newman black holes, while Rita Teixeira da Costa presented upcoming results about the Maxwell equations on general Kerr backgrounds, including the difficult case of extremality. Stefan Hollands' talk also touched upon potential instabilities near extremality, seen from a more heuristic point of view.

Keeping to the theme of extremality, Ryan Unger presented results on *extremal critical collapse*, showing that extremal black holes can form dynamically in gravitational collapse for the Einstein–Maxwell–Vlasov system, and in fact occur at the threshold of black hole formation. Scattering results on the Maxwell–Vlasov system on a fixed Minkowski background, on the other hand, were presented in the talk of Leo Bigorgne.

Jonathan Luk presented a general method to understand the precise power-law asymptotic fall-off for a wide variety of both linear and nonlinear massless wave equations on very general asymptotically flat spacetimes, including black holes, unifying and generalising many previous results in this setting. Yakov Shlapentokh-Rothman, on the other hand, considered the problem of *massive* fields, presenting recent results on the asymptotics of the Klein–Gordon equation on the Schwarzschild spacetime, where the sharp statements turn out to depend on difficult conjectures in number theory concerning exponential sums.

Turning to the case of asymptotically AdS spacetimes, Gustav Holzegel described a linear stability result for Schwarzschild-AdS backgrounds, but with slow logarithmic decay, while Christoph Kehle described an upcoming result on weak turbulent instability for a nonlinear wave equation on Schwarzschild-AdS backgrounds, suggesting that these backgrounds themselves may in fact become *unstable* as solutions to the vacuum Einstein equations in the full nonlinear theory.

2. In addition to the setting of isolated self-gravitating systems, an important domain for general relativity is *cosmology*, and questions about the dynamics of cosmological solutions of the Einstein equations is an important field of study.

In recent years, several results concerning big bang stability have appeared. In his talk, Oliver Petersen formulated a general condition, in the absence of symmetries and without requiring proximity to a specific background solution, guaranteeing the formation of a big bang with curvature blow up. Warren Li described results on Belinski–Khalatnikov–Lifshitz bounces in a spatially *inhomogeneous* setting and Andrés Franco Grisales described the asymptotics of solutions to Maxwell's equations on a fixed Kasner background.

There is also interest regarding stability results for cosmological solutions in the *expanding* direction. Here several results concerning future global non-linear stability of cosmological solutions to the Einstein-Euler equations have appeared in the last ten years or so. There are also related results concerning the Euler equations on a fixed expanding background. David Fajman described results relating the stability of the fluid with the rate of expansion of the cosmological spacetime. On the basis of his results in his numerical studies, he also conjectured a relation between the speed of sound of the fluid and the rate of expansion necessary and sufficient in order to obtain stability.

Finally, cosmological solutions can also have local black hole regions, for instance the Kerr–de Sitter solution, and the stability of these regions is an important problem. Andras Vasy described recent and upcoming results on linear and nonlinear waves on general Kerr–de Sitter backgrounds in the regions between the event and cosmological horizons, with applications to the stability of these regions themselves as solutions to the Einstein equations.

3. The workshop also featured the presentation of two very general, completely local results about solutions of the Einstein equations, independent of asymptotic structure.

The first is a result of Cécile Huneau, who presented a proof of *Burnett's conjecture*. In particular, she considered the case that the relevant family of metrics satisfy a generalized wave coordinate condition and she described a proof of the fact that the limit metric g then satisfies the Einstein–massless Vlasov system. In contrast to previous results, her result does not require any symmetry.

The second is a result of Peter Hintz. In his lecture, he presented a new, very general theorem on gluing small Kerr black holes in an ambient vacuum spacetime, preserving the Einstein equations. As an important novel application, this gives a technique to construct solutions of the Einstein vacuum equations with two black holes in the extreme mass-ratio limit.

4. An additional important focus of the workshop was to discuss constructions and properties of initial data for the Einstein equations, an established field with vibrant new developments.

From Gerhard Huisken, we learned about some new geometric notions of quasi-local mass based on Inverse Mean Curvature Flow and how these can be used to prove a version of Kip Thorne's widely open hoop conjecture, improving upon seminal results by Rick Schoen and Shing-Tung Yau. Lan-Hsuan Huang dedicated her talk to the moduli space of Einstein manifolds with boundary, combining older aspects established by Robert Bartnik with new methods from the analysis of static and stationary extensions. This may open a new window on studying initial data from a different, very powerful perspective.

Both Albachiara Cogo and Sven Hirsch discussed results on the existence of initial data sets with special properties: Albachiara Cogo showed the existence of entire maximal *boosted* asymptotically Euclidean slices of the Schwarzschild spacetime, answering an open question posed by Robert Bartnik. Sven Hirsch presented results refining findings by Lan-Hsuan Huang and Dan Lee, namely showing the existence of geodesically complete, asymptotically Euclidean initial data sets in certain null dust pp-wave spacetimes in dimensions ≥ 6 and proving that these are the only non-Minkowskian initial data sets of zero ADM-mass.

In her talk, Olivia Vičánek Martínez described a new geometric construction of asymptotically Euclidean coordinates on asymptotically flat initial data sets via the spacetime constant mean curvature foliation by Carla Cederbaum and Anna Sakovich. These can be thought of as center-of-mass coordinates and might provide an avenue to a coordinate-free treatment of asymptotically Euclidean initial data sets. Relatedly, Rodrigo Avalos presented a sufficient criterion for the existence of asymptotically Schwarzschildean coordinates based on faster decay assumptions on the Cotton tensor and on techniques related to Rick Schoen's contribution to the resolution of the Yamabe problem.

Sung-Jin Oh explained a very flexible and promising newly-developed method of generating initial data sets with good localisation and fall off properties, generalising and complementing existing gluing results.

Considering asymptotically hyperbolic instead of asymptotically Euclidean initial data sets, Anna Sakovich described recent results on the definition and geometry of the mass aspect function. This opens the door for studying such initial data under very weak decay assumptions; this could for example be relevant for studying stability of geometric inequalities such as the positive mass theorem.

5. A final focus of the workshop was Lorentzian geometry and its connections to general relativity, a very diverse field with lots of activity.

Transferring ideas from the study of the initial data sets and combining them with techniques from Lorentzian and null geometry, Markus Wolff described a new geometric flow along null hypersurfaces which allows one to construct foliations by surfaces of constant spacetime mean curvature (in the sense of the foliations considered by Carla Cederbaum and Anna Sakovich in initial data sets) in null cones near null infinity. In her talk, Annegret Burtscher described a new characterization of global hyperbolicity via completeness of the null distance, providing a Lorentzian analogue of the Hopf–Rinow theorem.

Focusing on stationary spacetimes hosting a degenerate Killing horizon, James Lucietti presented recent breakthroughs of fully characterizing their near-horizon geometry and explained how this can be used to prove rigidity.

Finally, Jan Sbierski presented a result showing Lipschitz inextendibility of the weak null singularities which are expected to form in the interior of generic rotating black holes. Such a statement would be necessary to deduce versions of Penrose's *strong cosmic censorship* conjecture.

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Workshop: Mathematical Aspects of General Relativity

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Abstracts

Regularity of compactified 3-manifolds and the ADM Center of Mass

Rodrigo Avalos

In general relativity (GR), isolated gravitational systems are modelled by asymptotically Euclidean (AE) initial data sets, which are defined as sets of the form $\mathcal{I} \doteq (M^3, g, K, \mu, S)$, where (M^3, g) is a 3-dimensional Riemannian manifold; K is a symmetric $(0,2)$-tensor field, while μ and S are, respectively, a function and a 1-form which stand for the energy and momentum densities induced by physical sources. In this context, AE 3-manifolds are non-compact manifolds which outside a compact core $\mathcal{K}$ satisfy $M^3\backslash\mathcal{K} \cong \mathbb{R}^3\backslash\overline{B_1(0)}$ and, with respect to some fixed asymptotic chart $\{x^i\}_{i=1}^3$, satisfy decay conditions of the form:

$$g_{ij} = \delta_{ij} + O_k(|x|^{-\tau}), \;\; K_{ij} = O_{k-1}(|x|^{-\tau-1}) \text{ and } \mu, S_i = O_{k-2}(|x|^{-\rho}),$$

for some $\tau, \rho > 0$ and $k \geq 0$. For these types of initial data sets one can physically motivate a set a *asymptotic charges* (E, P, C, J), respectively denoting their energy, momentum, centre of mass (COM) and angular momentum. These so-called *ADM charges* have proven to be intimately related to the analysis of scalar curvature deformations, foliations of infinity and rigidity phenomena, as is notably exemplified by the famous positive mass/energy theorems and their role in the resolution of the Yamabe problem [13, 15]. In the case of an AE 3-manifold (M^3, g) with asymptotic coordinates $\{x^i\}_{i=1}^3$, the ADM energy is defined as

$$E \doteq \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii})\, \nu^j\, d\omega_r \;\; ; \tag{1}$$

whenever the limit exists, and where $S_r \hookrightarrow \mathbb{R}^3\backslash\overline{B_{R_0}(0)}$ denotes a topological sphere of radius $r > R_0$ contained within the end of M^3, while ν denotes the outward-pointing Euclidean unit normal to it and $d\omega_r$ the volume form on S_r induced by the Euclidean metric. Precise geometric conditions can be formulated to guarantee the convergence of (1) as well as the independence of this limit on the asymptotic coordinates [2]. In this context, the COM of such an AE manifold (M^3, g) with $E > 0$ is given by a vector $C = (C^1, C^2, C^3) \in \mathbb{R}^3$, whose components are defined by the limits

$$C^k \doteq \frac{1}{16\pi E} \lim_{r\to\infty} \left(\int_{S_r} x^k\, (\partial_i g_{ij} - \partial_j g_{ii})\, \nu^j\, d\omega_r - \int_{S_r} \left(g_{ik}\nu^i - g_{ii}\nu^k\right) d\omega_r \right), \tag{2}$$

whenever the limits exist. The COM has proven to be related to interesting problems in geometry, most notably the existence of geometric foliations in AE manifolds and the convergence of the centre of such foliations, which has been proposed as a geometric way of defining the COM. Notably, G. Huisken and S. T. Yau introduced such a definition via constant mean curvature (CMC) foliations near infinity for the first time in [10]. This triggered plenty of research related to other asymptotic geometric foliations, such as [6, 4, 5]. In all these works, a critical aspect of the analysis is the precise asymptotic behaviour of the AE manifold

(M^3, g), where one typically needs to impose (at least) certain asymptotic parity conditions known as the Regee-Teiltelboim (RT) conditions. Actually, in many cases a *Schwarzschildian* expansion near infinity is demanded [5, 6, 10], which implies that, asymptotically,

$$g_{ij} = \left(1 + \frac{A}{|x|}\right)\delta_{ij} + O_2(|x|^{-2}), \tag{3}$$

where A is a constant, related to the ADM energy of g. Clearly, the Schwarzschildian case implies a strong version of the RT parity conditions. A generally recognised problem in this context is that, except in very special cases, the existence of an a priori expansion for the AE metric g with Schwarzschildian asymptotics (or at least RT-type parity properties) is not known to be a consequence of some set of geometric hypotheses. This got translated into a conjecture by Cederbaum-Sakovich in [4] concerning the convergence of the COM under clearer geometric conditions which, in the case of totally geodesic initial data sets ($K = 0$), can be rephrased as

Conjecture 1 (Cederbaum-Sakovich). *Given an AE manifold (M^3, g) of order $\tau \in (-1, -\frac{1}{2})$ with respect to a structure of infinity Φ_z with coordinates $\{z^i\}_{i=1}^3$, if $z^i R_g \in L^1(M\backslash\mathcal{K}, \Phi_z)$, then there is a geometric condition on the coordinates $\{z^i\}_{i=1}^3$ ensuring the existence of a compatible asymptotic chart $\{\bar{z}^i\}_{i=1}^3$ such that (2) converges in the $\bar{z}$-coordinates.*

One of our main results is to provide an answer to this conjecture under slightly stronger conditions and identifying the right geometric objects which control the existence of such suitable asymptotic coordinates. This result is presented below:

Theorem 1. *(R. Avalos [1]) Let (M^3, g) be a smooth $W^{4,p}_{-\tau}$-AE manifold with respect to a structure of infinity with coordinates $\{z^i\}_{i=1}^3$, with $\tau \in (\frac{1}{2}, 1)$ and $p > 2$. Assume furthermore that:*

1. *$R_g \in L^r_{-3-\epsilon}(M, \Phi_z)$ for some $r > 3$ and $\epsilon > 0$;*
2. *$C_g \in L^{p_1}_{\sigma}(M, \Phi_z)$ for some $-6 < \sigma < -4$ and $p_1 = \frac{3}{6+\sigma}$.*

Then, there is a structure of infinity with coordinates $\{\bar{z}^i\}_{i=1}^3$, which is $C^{1,\alpha}$-compatible with the original one, such that

$$g(\partial_{\bar{z}^i}, \partial_{\bar{z}^j}) = \left(1 + \frac{4C}{|\bar{z}|}\right)\delta_{ij} + O_1(|\bar{z}|^{-1-\alpha}), \tag{4}$$

for some $\alpha > 0$ and a constant C. If moreover $R_g \in L^r_{-4-\epsilon}(M, \Phi_z)$, with $r > 3$ and $\epsilon > 0$, then the center of mass converges in the coordinates given by (4).

In Theorem 1 the structure of conformal classes is critical, and insights into this relation can be obtained thinking about the stereographic projections of (smooth) Yamabe positive closed manifolds $(\hat{M}^3, \hat{g})$. This procedure provides an interesting connection between special classes of closed Riemannian manifolds and AE manifolds obeying the RT conditions. In general, the analysis of the conformal class of an AE 3-manifold needed to prove Theorem 1 enforces the study of conformal classes of geometric objects with controlled singularities. This is because, in general, conformal compactifications of AE 3-manifolds will be of $W^{2,q}(\hat{M})$-regularity for some $q > \frac{3}{2}$, even if g is smooth [8]. Further regularity on $\hat{g}$ can be obtained by analysis of the decay properties of the conformally invariant Cotton tensor C_g, as was noted by M. Herzlich [9]. Another of our main results is to interpolate between the results of [8] and [9], providing weaker decay condition which are enough to provide $C^{1,\alpha}$-controls on $\hat{g}$, which is the threshold needed in the analysis of Theorem 1.

Theorem 2. *(R. Avalos [1]) Let (M^3, g) be a smooth $W^{k,p}_{\tau}$-AE manifold, relative to a structure of infinity with coordinates $\{z^i\}_{i=1}^3$ and $p > 2$, $\tau \in (-1, -\frac{1}{2})$, $k \geq 4$, and let $\hat{M}$ be the one point compactification of M. If $C_g \in L^{p_1}_{\sigma}(M, dV_g)$ with $-6 < \sigma < -4$ and $p_1 = \frac{3}{6+\sigma}$, then (M^3, g) can be conformally compactified into $(\hat{M}, \hat{g})$ where $\hat{M}$ stands for the 1-point compactification of M, such that $\hat{g} \in W^{2,q}(\hat{M})$ for some $q > 3$. In particular $\hat{g} \in C^{1,\alpha}(\hat{M})$, for some $\alpha \in (0, 1)$.*

The idea of the proof of Theorem 2 exploits that a weighted L^p-control of C_g will translate on an a priori L^p control of $C_{\hat{g}}$, in turn translating into $W^{-1,p}(\hat{M})$-control for $\Delta_{\hat{g}}\mathrm{Ric}_{\hat{g}}$. Once $W^{-1,p}$-control for $\Delta_{\hat{g}}\mathrm{Ric}_{\hat{g}}$ is obtained, the remaining improvements of regularity follow through some delicate new elliptic regularity theorems. It is interesting to note that the associated elliptic regularity theory is rather close to critical cases, being related to the conjecture posed by J. Serrin in [14] about the celebrated De Giorgi-Nash regularity theory, which has been addressed, for instance, in [3, 12, 16, 11]. The geometric problem analysed in Theorem 2 naturally poses the related problem of bootstrapping an a priori $L^{q'}$-solution to the tensor equation

$$\Delta_g u = f \tag{5}$$

for a $W^{2,q}(M^n)$ metric on a closed manifold M, with $q > \frac{n}{2}$, u a symmetric $(0, 2)$-tensor field, and $f \in L^p$, which is outside of the scope of the referred papers. Our associated regularity results find applications of their own within mathematical GR and the analysis of the rough initial data sets, something that has recently motivated related results analysed in [7].

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Modified scattering for the small data solutions to the Vlasov-Maxwell system

Léo Bigorgne

The Vlasov-Maxwell system is a classical model in collisionless plasma physics. We will focus here on the dynamics of its small data solutions. The system reads

$$v^0 \partial_t f + v \cdot \nabla_x f + v^\mu F_\mu{}^j \partial_{v^j} f = 0, \tag{V}$$

$$\nabla^\mu F_{\mu\nu} = J(f)_\nu, \qquad \nabla^\mu {}^*F_{\mu\nu} = 0, \tag{M}$$

where

- $f : \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+$ is the density distribution function of the particles.
- $J(f)_\nu := \int_{\mathbb{R}^3_v} \frac{v_\nu}{v^0} f \mathrm{d}v$ is the current density and $v^0 = \sqrt{1+|v|^2}$.
- The electromagnetic field is represented by a 2-form F, the Faraday tensor. The Hodge dual of F is given by ${}^*F_{\mu\nu} = \frac{1}{2} F^{\lambda\sigma} \varepsilon_{\lambda\sigma\mu\nu}$.

In order to lighten the notations, we consider, as it is usually done in the mathematical community, plasmas composed by one species of particles of charge $q = 1$ and mass $m = 1$.

The existence of global-in-time classical solutions to the Vlasov-Maxwell system is only known in the pertubative regime or under certain symmetry assumptions. Although the problem remains open besides these specific cases, various continuation criteria have been obtained during the past decades (see for instance [7, 9]).

Much more is known for the small data solutions to (V)–(M). In particular, Glassey-Strauss proved in [8] that these solutions are global and decay with the same rate as the solutions to the linearised Vlasov-Maxwell system around 0. More precisely, any small data solution (f, F) to (V)–(M) verifies

$$(1) \qquad \int_{\mathbb{R}^3_v} f(t,x,v)\mathrm{d}v \lesssim \frac{1}{\langle t+|x|\rangle^3}, \qquad |F|(t,x) \lesssim \frac{1}{\langle t+|x|\rangle \langle t-|x|\rangle}.$$

Many refinement of this result has been obtained recently. In particular, no compact support assumption is required anymore on the data, the derivatives of the solutions can be controlled [3, 10] and the smallness assumption on the electromagnetic field has been relaxed [11, 4].

Once the optimal decay estimates (1) are proved, the next question one may ask is whether or not (f, F) can be approached by a linear solution. For this, we would like to prove that *asymptotic completeness* holds, that is (f, F) converges in a suitable sense to an asymptotic state. Since the characteristics of the linearised Vlasov operator $v^\mu \partial_{x^\mu}$ are timelike straight lines, the trajectories of isolated massive bodies, we could expect

$$(2) \qquad f(tv^0, x+tv, v) \to f_\infty(x,v), \qquad \text{as } t \to +\infty \text{ and in } L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v).$$

Concerning the electromagnetic field, we could expect it to converge along null rays,

$$(3) \qquad rF(r+u, r\omega) \to F^\infty(u,\omega), \qquad \text{as } r \to +\infty \text{ and in } L^2(\mathbb{R}_u \times \mathbb{S}^2_\omega).$$

It turns out that, because of the long range effect of the Lorentz force $v^\mu {F_\mu}^j$ and unless f vanishes identically, the distribution function does not verify linear scattering (that is (2) does not hold). This has been proved in the context of the Vlasov-Poisson system by Choi-Ha [6]. One can already see it at the level of the characteristics of the Vlasov equation (V), which verify

$$\dot{X}^j = \frac{V^j}{V^0}, \qquad \dot{V}^j = \frac{V^\mu}{V^0}{F_\mu}^j.$$

Then, as $|F| \lesssim t^{-2}$ along timelike straight lines, we can expect $t \mapsto V(t)$ to converge to V_∞ as $t \to +\infty$ and

$$\big|V(t) - V_\infty\big| \lesssim \frac{1}{t}, \qquad \Big|\dot{X}(t) - \frac{V_\infty}{V^0_\infty}\Big| \lesssim \frac{1}{t},$$

suggesting that the spatial characteristics does not verify $X(tv^0_\infty) \approx x_\infty + tv_\infty$ for $t \gg 1$. In fact, f satisfies a modified scattering dynamics.

In contrast, the electromagnetic field does verify linear scattering. In order to state a refined version of (3), we introduce a null frame

$$\underline{L} := \partial_t - \partial_r, \qquad L := \partial_t + \partial_r, \qquad e_\theta := \frac{1}{r}\partial_\theta, \qquad e_\varphi := \frac{1}{r\sin(\theta)}\partial_\varphi$$

and the null components of the Faraday tensor

$$\underline{\alpha}_{e_A} := F_{e_A \underline{L}}, \qquad \alpha_{e_A} := F_{e_A L}, \qquad \rho := \frac{1}{2}F_{\underline{L}L}, \qquad \sigma := F_{e_\theta e_\varphi}.$$

It is well-known that in the vacuum setting and if $F|_{t=0}$ decays sufficiently fast, then ρ, $\sigma \sim r^{-2}$ and $\alpha \sim r^{-3}$ along null rays. The next result is obtained in [4, 5].

Theorem 1. *Let (f,E,B) be a solution to the Vlasov-Maxwell system arising from sufficiently small and regular initial data. Then,*

- *The distribution function f verifies modified scattering along logarithmical corrections of the linear characteristics, defined teleologically. There exists $f_\infty : \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}$, such that, for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v$,*

$$\Big| f\Big(v^0 t - \frac{\log(t)}{|v^0|^2} v^\mu \mathbb{F}_\mu{}^0(v), x^k + tv^k - \frac{\log(t)}{|v^0|^2} v^\mu \mathbb{F}_\mu{}^k(v), v \Big) - f_\infty(x,v) \Big| \lesssim \frac{\log^3(2+t)}{\sqrt{1+t}},$$

  where the constant in time 2-form $v \mapsto \mathbb{F}(v)$ is a functional of $\int_{\mathbb{R}^3_x} f_\infty(x,\cdot)\mathrm{d}x$ capturing the asymptotic behavior of the electromagnetic field along time-like geodesics,

$$F\big(v^0 t, x+tv\big) = t^{-2}\mathbb{F}(v) + o\big(t^{-2-\delta}\big), \qquad \delta > 0.$$

- *The Maxwell field F verifies linear scattering. There exists $\underline{\alpha}^\infty$, ρ^∞ and σ^∞ defined on $\mathbb{R}_u \times \mathbb{S}^2_\omega$ such that*

$$\big| r\underline{\alpha}(r+u, r\omega) - \underline{\alpha}^\infty(u,\omega) \big| \lesssim \langle r + |u| \rangle^{-1},$$
$$\big| r^2 \rho(r+u, r\omega) - \rho^\infty(u,\omega) \big| + \big| r^2 \sigma(r+u, r\omega) - \sigma^\infty(u,\omega) \big| \lesssim \langle u \rangle \langle r + |u| \rangle^{-1}.$$

  Moreover, ρ^∞ and σ^∞ are fully determined by $\underline{\alpha}^\infty$ and $\int_{\mathbb{R}^3_x} f_\infty(x,\cdot)\mathrm{d}x$ through constraint equations.

Conversally, given a sufficiently regular scattering state $(f_\infty, \underline{\alpha}^\infty)$ satisfying the same constraint equations, there exists a unique global classical solution (f,E,B) to the Vlasov-Maxwell system verifying the previous convergence estimates.

The constraint equations are related to the electromagnetic memory effect [2]. They differ significantly to the ones that one would obtain for the massless Vlasov-Maxwell system or in higher dimensions.

Finally, let us mention the independent work of Ben-Artzi-Pankavich [1] where modified scattering for the distribution function f is proved.

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Local and global properties of the null distance

Annegret Burtscher
(joint work with Leonardo García-Heveling, Brian Allen)

A spacetime (M, g) does not admit a canonical metric space structure. If one attempts a length metric space construction as in Riemannian geometry, now based on the maximation of length $L_g(\gamma) = \int_I \sqrt{-g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))}ds$ of future-directed timelike curves $\gamma\colon I \to M$, one arrives ats the *Lorentzian distance* (sometimes also called time separation function)

$$d_g(p,q) := \begin{cases} \sup\{L_g(\gamma); \gamma \text{ future-directed causal curve from } p \text{ and } q\} & p \le q, \\ 0 & \text{else,} \end{cases}$$

The Lorentzian distance has many important features, especially when (M, g) is globally hyperbolic and g is sufficiently regular. These properties have already been exploited successfully in Lorentzian geometry [3] and in recent synthetic approaches to Lorentzian geometry [4, 12, 17, 20]. Nonetheless, the Lorentzian distance is very far from being a metric: it is not symmetric, not positive definite, and only satisfies a reverse triangle inequality.

In 2016 Sormani and Vega [25] succeeded in constructing an honest metric on stably causal spacetimes, called *null distance*. It is the aim of this short note to discuss the construction and important basic properties of the null distance that can render it useful for analyzing low-regularity structures in general relativity.

Recall that stably causal spacetimes are precisely those spacetimes that admit a time function [16]. We fix a time function $\tau\colon M \to \mathbb{R}$ that is locally anti-Lipschitz, for example, a smooth temporal functions [5] or, if available, a regular cosmological time function [2]. To each continuous, piecewise causal curves $\beta\colon I \to M$ one can assign the null length $\hat L_\tau(\beta) = \sum_{i=1}^k |\tau(\beta(s_i)) - \tau(\beta(s_{i-1}))|$, where the s_i denote the parameter values where β changes time orientation. Then

$$\hat d_\tau(p,q) = \inf\{\hat L_\tau(\beta); \beta \text{ piecewise causal curve from } p \text{ to } q\}.$$

Already in [25] it was shown that $\hat d_\tau$ is a conformally invariant metric on M that induces the manifold topology. It is clear that the metric topology should depend on both g and τ. But how much? García-Heveling and I [10] have shown that if one restricts oneself to the class of (weak) temporal functions, then the null distances generated by any pair (g_i, τ_i), $i = 1, 2$, on M are *locally* bi-Lipschitz. By using a semi-global argument this result can also be extended to compact sets and is in line with the corresponding bi-Lipschitz property of Riemannian distances [8].

Globally, the metric structure crucially depends on both g and τ. As a general paradigm one should aim to work with a time foliation that closely reflects the causal properties and global structure of (M, g) one wishes to investigate. If (M, g) is globally hyperbolic, for instance, one is best served with a Cauchy temporal function. Then $\hat{d}_\tau$ encodes causality *globally* [10], i.e., for all $p, q \in M$

$$p \leq q \Longleftrightarrow \hat{d}_\tau(p, q) = \tau(q) - \tau(p). \tag{1}$$

Note that $\Longrightarrow$ is trivial and that the assumptions for $\Longleftarrow$ can be weakened to time functions with slices that are future/past Cauchy [10] or future/past complete [15]. This relaxation implies that the null distance of a spacetime with regular cosmological time function encodes causality. Before the global result was published Sakovich and Sormani [22] had already obtained that the null distance of any stably causal spacetime encodes causality *locally* (see [10] for a variation). Heuristically this makes sense also from the perspective of the global result as every point in a spacetime admits a globally hyperbolic neighborhood. In [10] we have provided several examples that show that the global result does not generalize to spacetimes that are not globally hyperbolic nor to globally hyperbolic spacetimes with more general time functions. The question when causality encoding property (1) holds was already mentioned as an open problem in [25] and by Sormani [24] in the Oberwolfach Workshop ID 1832. The above results fully answer the question for spacetimes without (timelike) boundary.

Another, rather surprising, global result concerns the completeness of the metric space $(M, \hat{d}_\tau)$. In Riemannian geometry, thanks to the Hopf–Rinow theorem, metric completeness is equivalent to geodesic completeness. In the Lorentzian setting such a result is impossible but in many other situations global hyperbolicity fills the role of complete Riemannian manifolds (for example, in the Avez–Seifert theorem; see [9] for a thorough discussion). Let us independently investigate the meaning of completeness for $(M, \hat{d}_\tau)$. It is clear that forcefully removing a point of a spacetime results in noncompleteness. The same is true for Riemannian completeness but there the Nomizu–Ozeki theorem at least guarantees the existence of a complete Riemannian metric (and distance). It is thus natural to expect that for a completeness result the time function τ has to be special in some way. Allen and I [1] have shown that a time function that is *globally* anti-Lipschitz with respect to a complete Riemannian metric results in a complete null distance. For a temporal function τ this simply means that there is a complete Riemannian metric h such that $d\tau(v) \geq \|v\|_h$ for all future-directed causal vectors v. Conveniently, it was shown by Bernard and Suhr [6, 7] almost simultaneously (but in a completely different context) that temporal functions with such a property exist precisely when (M, g) is globally hyperbolic. Hence, when (M, g) is globally hyperbolic there always exists a time function such that $(M, \hat{d}_\tau)$ is a complete metric space. That the converse also holds was observed in joint work with García-Heveling [10]. Following a question of Jim Isenberg, I have shown that both the null distance and this result generalize to proper cone structures (see [19] for an overview of this setting) and to a large class of semi-Riemannian manifolds [9].

To conclude, let me recall that one of the main motivations of Sormani to introduce the null distance was to investigate a question of Yau about spacetime convergence and stability [24]. In this direction, Allen and I have already analyzed the special case of convergence of warped product spacetimes [1] and Sakovich and Sormani [22] have obtained a crucial isometry theorem. A candidate for a limiting space could be the Lorentzian synthetic spaces [18] mentioned in the beginning of this extended abstract. For those spaces we have recently shown that all important classes of time functions can be constructed [11]. The general existence of locally anti-Lipschitz time functions in this setting, which is needed to guarantee the positive definiteness of the null distance, however, is still open.

Apart from the original motivation I expect, based on the work of Chruściel and Grant [13] and the fact that Lipschitz is the correct regularity for metric spaces, that the null distance can in the future also be used to investigate low regularity situations in general relativity where Lipschitz regularity is or could be at the threshold (such as matter–vacuum boundaries, spacetime extensions [23], strong cosmic censorship conjecture [14, 21]).

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Maximal surfaces and boosts in the Schwarzschild spacetime

Albachiara Cogo

In Minkowski spacetime, there is a one-to-one correspondence between inertial observers, boosts and complete spacelike maximal (vanishing mean curvature) surfaces by the renowned Cheng-Yau Bernstein type theorem. We address the question of whether there is a correspondent generalization of this idea in non-flat spacetimes, starting from the Schwarzschild spacetime. Towards this direction, we will discuss the existence of a maximal surface approaching a coordinate-dependent hypersurface related to a boost of Minkowski in the asymptotically flat end and describe some of its properties. This is work in progress.

A stability phase transition for cosmological fluids

David Fajman
(joint work with Maciej Maliborski, Maximilian Ofner, Todd Oliynyk, Zoe Wyatt)

Consider a cosmological spacetime of topology $\mathbb{R}_+ \times M$ with Lorentzian metric

$$g = -dt^2 + a(t)^2 g_0, \tag{1}$$

with covariant derivative ∇, where (M, g_0) is a 3-dimensional Riemannian manifold without boundary and $a : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotonically increasing function called the scale factor.

The corresponding relativistic Euler equations take the form

$$
\begin{aligned}
u^\mu \nabla_\mu \rho + (\rho + P) \nabla_\mu u^\mu &= 0, \\
(\rho + P) u^\mu \nabla_\mu u^\nu + (g^{\mu\nu} + u^\mu u^\nu) \partial_\mu P &= 0,
\end{aligned} \tag{2}
$$

where the functions ρ and P denote the energy density and pressure of the fluid, respectively. u^ν is the velocity vector field of the fluid. We consider the particular type of a barotropic fluid, which has the equation of state

$$
P = K\rho, \tag{3}
$$

where $K = c_S^2$ is the square of the speed of sound of the fluid and takes values $K \in [0, 1/3]$. The system (2) with g of the form (1) and equation of state (3) models a relativistic barotropic fluid in an expanding spacetime. It is a common matter model in cosmology.

Solutions (ρ, u), where $\rho = \rho(t)$ and u is a time-like unit normal vector field are referred to as quiet fluid states.

It is known by the work of Christodoulou that in the absence of expansion ($a(t) = 1$) fluids generically form shocks (gradient blow-up of the fluid variables) in finite time [2]. This also holds for solutions which arise from initial data arbitrarily close to a quiet fluid state. This scenario is referred to in the following as *unstable*.

In the regime of exponential expansion, i.e. $a(t) = e^t$, it was first observed by Brauer, Rendall and Reula, in a Newtonian setting with gravitational backreaction, that solutions arising from initial data sufficiently close to a quiet fluid state lead to future global solutions, which do not form shocks [1]. Following the corresponding pioneering work in the context of Einstein's equations by Rodnianski and Speck [12] a long series of works established similar results in the regime of exponential expansion [8, 9, 10, 11, 13]. We refer to this behaviour of the fluid as *stable*. Hence quiet fluid solutions are stable in spacetimes with fast expansion. The heuristic interpretation of these results is a dissipating effect in the fluid induced by the spacetime expansion, which regularizes the fluid.

A natural resulting question connecting the aforementioned results is whether there exists a critical expansion rate $a(t) = t^{\alpha_{\text{crit}}}$ where the stabilizing effect is too weak to regularize the fluid. Remarkably, for the case of linear expansion $a(t) = t$ it was shown by Speck that radiation fluids ($K = 1/3$) are unstable and dust ($K = 0$) is stable [14] (for dust with backreaction cf. [5]), while some of the present authors showed that fluids with $K \in (0, 1/3)$ are stable [4, 6] (even in the presence of backreaction). These results gave the first hint that the stabilization effect not only depends on the expansion rate but as well on the speed of sound of the fluid.

Indeed, in two recent works by the authors it was shown that a non-trivial phase transition occurs in the regime of decelerated expansion $a(t) = t^\alpha$ with $\alpha < 1$.

A high-precision numerical scaling analysis of (2) for initial data with toroidal symmetry and $M = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ provides strong evidence that the critical expansion rate, where the fluid behaviour changes from stable to unstable, is given by

$$\alpha_{\mathrm{crit}} = \frac{2}{3(1-K)}, \tag{4}$$

for $0 < K < 1/3$ [3]. This is the first evidence of a stability phase transition of this type for cosmological fluids. In a complementary rigorous result the authors show that (4) holds with an inequality ($\leq$ instead of $=$) by proving stability whenever $\alpha > \alpha_{\mathrm{crit}}$. The theorem below states the precise result. It requires a rescaling of the system (2). For that purpose we introduce the expansion-normalized variables

$$v^i = \frac{t^\alpha u^i}{\sqrt{1+t^{2\alpha}u^2}} \qquad \text{and} \qquad L = \log(t^{3\alpha(1+K)}\rho). \tag{5}$$

In those variables (2) reads

$$\begin{aligned} \partial_t v^i = &-\frac{\alpha(1-3K)}{t}v^i - \frac{K}{1+K}\frac{t^{1-\alpha}}{t}(1-v^2)\partial_i L \\ &- \frac{t^{1-\alpha}}{t}v^j\partial_j v^i + \frac{t^{1-\alpha}}{t}\left(1-\frac{1-K}{1-Kv^2}\right)v^i\partial_j v^j \\ &+ \frac{t^{1-\alpha}}{t}\frac{1-K}{1+K}\left(1-\frac{1-K}{1-Kv^2}\right)v^i v^j \partial_j L \\ &+ \frac{\alpha(1-3K)}{t}\frac{1-K}{1-Kv^2}v^2 v^i \end{aligned} \tag{6}$$

$$\begin{aligned} \partial_t L = &-\frac{t^{1-\alpha}}{t}\frac{(1+K)}{1-Kv^2}\partial_j v^j - \frac{t^{1-\alpha}}{t}\frac{(1-K)}{1-Kv^2}v^j\partial_j L \\ &+ \frac{\alpha(1+K)}{t}\frac{1-3K}{1-Kv^2}v^2 \end{aligned} \tag{7}$$

The main theorem then reads as follows, where $\overline{v}$ and $\overline{L}_0$ denote the respective spatial mean values.

Theorem. ([7]) *Let $0 < K < 1/3$ and $a(t) = t^\alpha$ with $\alpha > \frac{2}{3(1-K)}$. Let $\mu > 0$ be such that $\alpha(1-3K) - \mu > 2(1-\alpha)$. Let $(v_0, L_0) \in H^3(\mathbb{T}^3) \times H^3(\mathbb{T}^3)$ be a vector field and a function, respectively. Then there exists an $\varepsilon > 0$ such that for*

$$\overline{v}_0 + \|\nabla v_0\|_{H_2} + \|\nabla L_0\|_{H_2} < \varepsilon \tag{8}$$

the solution $(v(t), L(t))$ to the system (6), (7) with initial data (v_0, L_0) exists future-global in time and the following decay rates hold.

$$\begin{aligned} |\overline{v}(t)| &\leq C\varepsilon t^{(-\alpha(1-3K)+\mu)/2} \\ \|L(t) - \overline{L}_0\|_{L^\infty} &\leq C\varepsilon \\ \|\nabla v(t)\|_{H_2} + \|\nabla L(t)\|_{H_2} &\leq C\varepsilon t^{(-\alpha(1-3K)+\mu)/2} \end{aligned} \tag{9}$$

The theorem implies that the expansion-normalized variables remain close to the background in case of the energy density, while the normalized velocity even decays, which implies orbital stability of the corresponding homogeneous fluid solutions. It constitutes the first result on fluid stabilization in the regime of decelerated expansion.

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Asymptotics of solutions to silent wave equations

Andrés Franco-Grisales

We study the asymptotics of solutions to systems of linear wave equations on cosmological spacetimes. Here the asymptotics refer to the behavior of the solutions near a cosmological singularity, or near infinity in the expanding direction. The asymptotics correspond to the heuristics of the BKL conjecture. We present an improvement upon the results obtained by Ringström in [1], by obtaining asymptotic estimates of all orders for the solutions, and showing that solutions are uniquely determined by the asymptotic data contained in the estimates.

As an application, we then study solutions to the source free Maxwell's equations in Kasner spacetimes near the initial singularity. Our results allow us to show that the energy density of generic solutions blows up along generic timelike geodesics when approaching the singularity.

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Decay for the Teukolsky system in Kerr-Newman

Elena Giorgi

The stability problem for the Einstein equation is a well-studied topic of research in the field of General Relativity, whose goal is to study perturbations of known solutions to the Einstein equation

$$\mathrm{Ric}(g) - \frac{1}{2}\,\mathrm{R}(g)g + \Lambda g = \mathrm{T},$$

where $\mathrm{Ric}(g)$ is the Ricci curvature of a Lorentzian metric g of a 4-dimensional manifold, $\mathrm{R}(g)$ is the scalar curvature of g, Λ is the cosmological constant, and T is called the stress-energy tensor, and contains information about the matter fields present in the spacetime.

The asymptotically flat case, for $\Lambda = 0$, has been extensively developed as the problem of stability for Minkowski [5], Schwarzschild [6, 24, 8], Kerr [28, 7, 1, 20, 25] [26, 27, 30, 19, 31, 32], Reissner-Nordström [12, 11, 13, 14, 3] and Kerr-Newman [15, 16, 21] spacetimes as solutions to the Einstein equation has been studied in various levels of difficulties in the past few decades. Stability results in the case of expanding spacetimes, for $\Lambda > 0$, for de Sitter [10], Kerr-de Sitter [23, 9], and Kerr-Newman-de Sitter [22] have also been obtained. The case of negative cosmological constant, $\Lambda < 0$, presents instabilities properties [29].

Here, we will discuss the stability property of the case of the exterior region of the Kerr-Newman family of black holes as solutions to the *Einstein-Maxwell equation*:

$$\mathrm{Ric}(g) = 2\,\mathrm{F}\cdot\mathrm{F} - \frac{1}{2}|\,\mathrm{F}\,|^2 g, \qquad d\,\mathrm{F} = 0, \qquad \mathrm{div}\,\mathrm{F} = 0$$

where F is a 2-form, called the *electromagnetic tensor*, satisfying the Maxwell equations.

As it is well known, in the case of vacuum solutions the Einstein equation can be studied through the Teukolsky equations [33] for the components of the Weyl curvature W

$$\alpha_{ab} := W(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} := W(e_a, e_3, e_b, e_3),$$

where e_3, e_4 are respectively the incoming and outgoing null vectors and e_a, e_b denote vectors orthogonal vectors to e_3, e_4. The scalar version of the Teukolsky equations are wave equations with first order terms and a potential and energy-Morawetz estimates for these equations have been obtained in Schwarzschild [6], in Kerr for $|a| \ll M$ [28, 7] and for $|a| < M$ [31, 32]. In these works, the Teukolsky equations are transformed to more treatable equations, called (generalized) Regge-Wheeler equations, through a transformation now known as Chandrasekhar transformation. The Teukolsky equations then unlock all the other gauge-dependent quantities [6, 1]. Note that α and $\underline{\alpha}$ are (quadratically) gauge invariant.

The Maxwell equations $d\,\mathrm{F} = 0$, $\mathrm{div}\,\mathrm{F} = 0$ on the Kerr background also admit two decoupled Teukolsky equations for the components of the electromagnetic tensor

$$^{(F)}\beta_a := F(e_a, e_4), \qquad ^{(F)}\underline{\beta}_a := F(e_a, e_3).$$

On the other hand, if the background charge is non zero, then $^{(F)}\beta$ and $^{(F)}\underline{\beta}$ are not (quadratically) gauge invariant. Gauge invariance is related to the gauge freedom of the Einstein equation due to its tensorial nature, and a quantity is said (quadratically) gauge invariant if it only changes quadratically with a linear change of frame.

We are interested in gauge-invariant quantities as they are believed to represent physical quantities, such as gravitational and electromagnetic waves, which should not depend on the chosen coordinates or gauge. On the other hand, the coupled gravitational and electromagnetic perturbations of a charged and rotating black hole have long known to be problematic due to the coupling of spin and the failure to find separated equations [4], due to the behavior of spheroidal harmonics with respect to the adjoint operators appearing on the coupled system.

It is therefore natural to ask what the relevant gauge invariant quantities are in electrovacuum, what equations they satisfy and if they can be analyzed without separating in harmonics.

In [15], we define novel gravitational and electromagnetic radiation quantities $\mathfrak{f}$ and $\mathfrak{b}$, respectively a 2-tensor and a 1-tensor, which are gauge-invariant, and related to the Weyl curvature α. As a consequence of the Einstein-Maxwell equation, the quantities $\mathfrak{b}$ and $\mathfrak{f}$ satisfy a system of Teukolsky-type equations coupled between each other through angular operators.

We found [15] that their respective Chandrasekhar transforms $\mathfrak{p}$ and $\mathfrak{q}$ in this case satisfy a symmetric system of *generalized Regge-Wheeler equations* in Kerr-Newman spacetime, of the following form:

$$\Box_g \mathfrak{p} - V_1(r)\ \mathfrak{p} - ia\frac{\cos\theta}{r^2 + a^2\cos^2\theta}\nabla_{\partial_t}\mathfrak{p} = C_1[\mathfrak{q}] + L_1$$
$$\Box_g \mathfrak{q} - V_2(r)\ \mathfrak{q} - 2ia\frac{\cos\theta}{r^2 + a^2\cos^2\theta}\nabla_{\partial_t}\mathfrak{q} = C_2[\mathfrak{p}] + L_2,$$

where L_1, L_2 are lower order terms depending on the gauge invariant quantities $\mathfrak{f}, \mathfrak{b}, \alpha$ and C_1, C_2 are coupling operators involving angular derivatives which satisfy the following *spacetime adjointness* property:

$$\psi_1 \cdot \overline{C_1[\psi_2]} = -C_2[\psi_1] \cdot \overline{\psi_2} + D_\alpha\big(f(r,\theta)(\psi_1 \cdot \overline{\psi_2})^\alpha\big), \tag{1}$$

for a function $f(r,\theta)$, where D denotes the spacetime covariant derivative. Observe that since the principal null frame in Kerr-Newman spacetime is not integrable, the adjointness operator with respect to the *spacetime* divergence is crucial, as the integration upon the spheres as in the case of Minkowski, Schwarzschild or Reissner-Nordström is not allowed.

In virtue of this property, we are able to prove the following.

Theorem 1 (G.[17], G.-Wan [18])**.** *For $|a| \ll M$, for $|Q| \ll M$ or for $|Q| < M$ in axial symmetry, the gauge-invariant quantities $\mathfrak{p}$ and $\mathfrak{q}$ satisfy Energy-Morawetz estimates.*

The system admits a combined energy-momentum tensor for the system $\mathcal{Q}_{\mu\nu}[\mathfrak{p}, \mathfrak{q}]$ as

$$\mathcal{Q}_{\mu\nu}[\mathfrak{p}, \mathfrak{q}] := \mathcal{Q}_{\mu\nu}[\mathfrak{p}] + 8Q^2\mathcal{Q}_{\mu\nu}[\mathfrak{q}]$$

where $\mathcal{Q}_{\mu\nu}[\psi] = \nabla_\mu\psi \cdot \nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\left(\nabla_\lambda\psi \cdot \nabla^\lambda\psi + V|\psi|^2\right)$ is the energy-momentum tensor associated to the Regge-Wheeler equation $\square_g\psi - V\psi = 0$. In virtue of the spacetime adjointness property (1), the combined energy-momentum tensor $\mathcal{Q}_{\mu\nu}[\mathfrak{p}, \mathfrak{q}]$ presents cancellation of the highest order terms in its divergence, effectively decoupling the system, therefore avoiding the issue of decomposition in spherical harmonics.

By adapting [16] the generalized vectorfield method of Andersson-Blue's [2] to the current $\mathcal{Q}_{\mu\nu}[\mathfrak{p}, \mathfrak{q}]X^\nu$ for a vectorfield X, we obtain energy estimates and Morawetz estimates of the system for $|a|, |Q| \ll M$ [17]. In axial symmetry, a simple choice of multiplier $X = \mathcal{F}(r)\partial_r$ allows to prove Morawetz estimates for the system for $|a| \ll M$ and $|Q| < M$ [18].

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Gluing small black holes along timelike geodesics

Peter Hintz

Black holes are fundamental predictions of Einstein's theory of General Relativity (GR): they are spacetimes (M, g) which solve the Einstein vacuum equations

$$\mathrm{Ric}(g) - \Lambda g = 0; \tag{1}$$

here $\Lambda \in \mathbb{R}$ is the cosmological constant. (For brevity, we only discuss the case $\Lambda \geq 0$ in this abstract.) Explicit solutions include the Schwarzschild spacetime

$$M = \mathbb{R}_t \times (2M, \infty) \times \mathbb{S}^2, \qquad g_{\mathfrak{m},0} := -\Big(1 - \frac{2\mathfrak{m}}{r}\Big)\mathrm{d}t^2 + \Big(1 - \frac{2\mathfrak{m}}{r}\Big)^{-1}\mathrm{d}r^2 + r^2 g_{\mathbb{S}^2},$$

where $\mathfrak{m} > 0$ is the mass of the black hole, and its generalizations—the Kerr and (for $\Lambda > 0$) Kerr–de Sitter (KdS) spacetimes. We write $g_{\mathfrak{m},\mathfrak{a}}$ for the Kerr metric with subextremal specific angular momentum $\mathfrak{a}$, i.e. $|\mathfrak{a}| < \mathfrak{m}$.

Recent years have seen remarkable progress in the understanding of the linear and nonlinear stability properties of these black hole solutions under small perturbations [HV18, ABBM19, HHV21, DHRT21, KS23, GKS22, She23]. A vague formulation of the Final State Conjecture asserts that the evolution of arbitrary (generic) initial data for the Einstein vacuum equations should, at late times, be described by a collection of Kerr(–de Sitter) black holes moving away from each other, plus gravitational radiation. The initial stages of the evolution are expected to be very complicated (e.g. formation of black holes, black hole mergers). In this context, black hole stability results can be thought of as describing the late stages of the evolution of a single black hole, far from the influence of any other black hole. In my talk, I describe very recent work [Hin23b, Hin24a, Hin24b] concerning the *rigorous construction and precise description of spacetimes describing the merger of two black holes with extreme (i.e. very large) mass ratios.* The main result allows for the insertion of a small black hole along a timelike geodesic in (almost) arbitrary spacetimes:

Theorem A (H.). *Let (M, g) denote a smooth, connected, globally hyperbolic spacetime solving $\mathrm{Ric}(g) - \Lambda g = 0$. Let $\mathcal{C} \subset M$ be an inextendible timelike geodesic, and let $X \subset M$ be a spacelike Cauchy hypersurface. Assume that*

- (I) *X is noncompact, or*
- (II) *(M, g) does not have nontrivial Killing vector fields in a neighborhood U of the point p in $X \cap \mathcal{C}$.*

Write (t, x) for Fermi normal coordinates around $\mathcal{C}$. (In particular, $g|_{(t,0)} = g^{\mathrm{Mink}} := -\mathrm{d}t^2 + \mathrm{d}x^2$.) Let $K^\circ \subset M$ be an open set with compact closure lying in the future of X. Then for all sufficiently small $\epsilon > 0$, there exists a smooth solution g_ϵ of (1) on $K \cap M_\epsilon$ where $M_\epsilon = M \setminus \{|x| < \epsilon\mathfrak{m}\}$ so that

- (i) *$g_\epsilon \to g$ in $\mathcal{C}^\infty(\bar{V}; S^2T^*M)$ for all open $V \subset K$ with $\bar{V} \cap \mathcal{C} = \emptyset$;*
- (ii) *near $\mathcal{C}$, we have*

$$(g_\epsilon)_{\mu\nu}(t, x) = g_{\mu\nu}(t, x) - g^{\mathrm{Mink}}_{\mu\nu} + (g_{\epsilon\mathfrak{m},\epsilon\mathfrak{a}})_{\mu\nu}(x) + (h_\epsilon)_{\mu\nu}(t, x), \tag{2}$$

where $(h_\epsilon)_{\mu\nu} \to 0$ *as* $\epsilon \to 0$ *together with all derivatives along* ∂_t *and* $(\epsilon^2 + |x|^2)^{\frac{1}{2}}\partial_x$*;*

(iii) *in the setting* (II)*, we can moreover ensure that* $g_\epsilon = g$ *outside of the domain of influence of* $\bar{U}$*. This also holds when* (M, g) *is a neighborhood of the domain of outer communications (DOC) of a subextremal Kerr(dS) spacetime, with* U *being a connected set containing both the point* p *and a point in the black hole interior.*

The meaning of (2) is that g_ϵ arises from g by inserting a Kerr black hole with small mass $\epsilon\mathfrak{m}$ (and subextremal angular momentum $\epsilon\mathfrak{a}$) along $\mathcal{C}$: the Minkowski metric $g^{\rm Mink}$ is replaced by the Kerr metric $g_{\epsilon\mathfrak{m},\epsilon\mathfrak{a}}$.

As a special case when $\Lambda > 0$, take (M, g) to be a neighborhood of the DOC of a very slowly rotating unit mass KdS black hole, and $\mathcal{C}$ to be a maximal timelike geodesic crossing the future event horizon of M in finite affine time. Split $M = \mathbb{R}_{t_*} \times X$ where the level sets of t_* are transversal to the future event and cosmological horizon, and $X = (r_- - 3\delta, r_+ + 3\delta) \times \mathbb{S}^2$ where r_-, resp. r_+ is the radius of the event, resp. cosmological horizon, and $\delta > 0$ is small. Apply then Theorem A to the subset $K = [0, T] \times r^{-1}([r_- - \delta, r_+ + \delta])$ where T is large enough so that $\mathcal{C}$ has crossed into $r < r_- - 2\delta$ by that time. Then on K, the metrics g_ϵ describe the merger of a mass $\sim \epsilon$ Kerr black hole with a unit mass KdS black hole; and the initial data induced by g_ϵ at $t_* = T$ are ϵ-close to those of the original KdS black hole. Therefore, the KdS stability result proved in [HV18] applies, and shows that, in an appropriate gauge, g_ϵ is equal to a final KdS metric, with parameters close to the original ones, plus an exponentially decaying remainder term. In short, we have

> *a complete description of the merger of a very slight Kerr with a unit mass KdS black hole, followed by the relaxation of the resulting single black hole to its stationary (KdS) state.*

A similar construction, now using [KS23, GKS22, She23] and relying on part (iii) of Theorem A, applies for $\Lambda = 0$ in the case that (M, g) is a neighborhood of the DOC of a very slowly rotating Kerr black hole.

Theorem A is optimal in the following two senses. First, metrics with the stated regularity (2) can only exist if $\mathcal{C}$ is a geodesic. This is a rigorous version of the statement that small bodies must move along geodesics *as a consequence of the Einstein vacuum equations*; previous arguments to this effect are due to a variety of authors including, in the recent literature, [EG04, GW08]. (The motion of small bodies in GR was studied for an Einstein–Klein–Gordon toy model in [Stu04, Yan14].) Second, if in (2) we tried to replace $g_{\epsilon\mathfrak{m},\epsilon\mathfrak{a}}(x) = g_{\mathfrak{m},\mathfrak{a}}(x/\epsilon)$ by $\hat{g}(t, x/\epsilon)$ for some another, possibly t-dependent, metric $\hat{g}$, then necessarily $\hat{g}$ would need to be asymptotically flat (so as to fit into M as a replacement for $g^{\rm Mink}$ near $\mathcal{C}$) and Ricci-flat (as a consequence of g_ϵ satisfying the field equations); and for the purpose of guaranteeing the well-posedness of the evolution problem for the Einstein equations, $\hat{g}$ would need to contain any event horizons (if present). According to the black hole uniqueness conjecture [CCH12, Conjecture 3.4], this *forces* $\hat{g}$ to be the Kerr metric for each time t. One can moreover show that the rescaled Kerr mass $\mathfrak{m}$ must be constant, and $\mathfrak{a}$ must be parallel, along $\mathcal{C}$.

The only prior work on gluing constructions for many-black-hole spacetimes is the author's work [Hin21a] in which KdS black holes are glued into neighborhoods of points on the conformal boundary of de Sitter space. Gluing methods have previously been successful for constructing special types of *initial data sets*; besides the proof that the spacetime evolving from the many-Kerr initial data of [CD03] has a disconnected black hole region as seen from a finite point along null infinity [CM03], the control of spacetimes evolving from initial data gluing constructions has, however, remained elusive. By contrast, Theorem A accomplishes a gluing construction *directly on the level of the spacetime metric*.

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Radiation in General Relativity

Sven Hirsch

(joint work with Yiyue Zhang)

In Special Relativity, massless objects are characterized as either vacuum states or as radiation propagating at the speed of light. We confirm in [1] that an analogous result also holds in General Relativity:

Theorem 1. *Let (M^n, g, k) be a $C^{2,a}$-asymptotically flat initial data set with decay rate $q \in (\frac{n-2}{2}, n-2]$ satisfying the dominant energy condition. Suppose that M^n is spin and that $E = |P|$. Then (M^n, g) isometrically embeds into Minkowski space of a pp-wave spacetime $(\overline{M}^{n+1}, \overline{g})$ with second fundamental form k.*

Here Minkowski space models vacuum and a pp-wave models radiation. More precisely, a Lorentzian manifold $(\overline{M}^{n+1}, \overline{g})$ is called a pp-wave spacetime if $\overline{M}^{n+1} = \mathbb{R}^{n+1}$ and

$$\overline{g} = -2dudt + Fdu^2 + g_{\mathbb{R}^{n-1}},$$

where F is a function on $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, independent of t, which is superharmonic on $\mathbb{R}^{n-1} \times \{u\}$ for all $u \in \mathbb{R}$.

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Extremal Black Hole Weather

Stefan Hollands

Highly spinning Kerr black holes are known to possess a tower of long-lived ("zero damped") quasi normal modes (QNMs). Conveniently denoting the extremality parameter of the black hole by

$$\epsilon = \frac{r_+ - r_-}{2r_+} > 0, \tag{1}$$

where r_+, r_- are the radii of the outer and inner horizons, the long-lived QNM frequencies scale as $\omega \sim [m - i\epsilon(N + h_{\ell m})]/(2r_+)$, where m, ℓ are standard labels of the solution to the angular Teukolsky equation, h is a parameter called the conformal weight, and $N = 0, 1, 2, \ldots$ is an additional label called the overtone number. The mode label m is an integer, which, as has been observed previously, is opening up the possibility of a (near) resonant interaction between long-lived QNMs once non-linear effects in perturbation theory are taken into account. It has therefore been speculated, see e.g., [1, 2], that some interesting energy transfer processes between modes of different energy might occur, possibly leading to a sort of cascade. So far, however, it has been impossible to analyze this idea concretely beyond the highly simplified analysis of [1] which does not provide a self-consistent framework of the non-linear effects, does not consider the Einstein equations but a linear scalar model equation on a perturbed background, and furthermore restricts attention to a finite number (triplet) of modes, which is hard to justify if all modes are resonant.

In this work [4], we develop this idea in a new framework in the context of the Einstein equations, keeping the leading non-linearities ("three wave interactions"). Our framework is partly based on prior work [2] showing that, up to a so-called "corrector tensor" which can be dealt with straightforwardly, non-linear metric perturbations of Kerr can be written in so-called "reconstructed form", i.e. in terms of a Hertz potential, Φ, solving a sourced Teukolsky equation. Starting from this formalism, we have derived [4] a dynamical system for the QNM mode amplitudes $c_q(t), q = (N, \ell, m)$, assuming that

$$\Phi = \sum_q c_q(t) \Upsilon_q \tag{2}$$

can be accurately described by a superposition of QNMs with time-dependent coefficients c_q. Our analysis of this dynamical system then proceeds by going to the so-called nNHEK approximation of the near zone of the near extremal ($\epsilon \ll 1$) Kerr geometry. We also consider a range of QNM parameters such that $\ell \gg m$, in which case the long-lived QNM spectrum becomes doubly resonant, $\omega \sim [m - i\epsilon(N + \ell + 1)]/(2r_+)$. Using also prior work on "QNM mode projection" [3], it turns out that in these limits, our dynamical system simplifies drastically, being of the schematic form

$$\dot{c}_1 = \sum_{2,3} \delta_{\ell_1,\ell_2+\ell_3} (\delta_{m_1,m_2+m_3} U_{123} c_2 c_3 + \delta_{m_1,-m_2+m_3} V_{123} c_2^* c_3). \tag{3}$$

Here, the overdot indicates a time derivative with respect to a "slow time" (given by ϵ times Boyer-Lindquist time), And U_{123}, V_{123} are certain homogenous functions of ℓ_1, ℓ_2, ℓ_3 that are of the order $O(\sigma)$ in the approximation considered when $m_1 = 0$. Furthermore, V_{123} vanishes unless one of the angular momenta is small. From this system, we pass to a phase randomized version of the system in the spirit of [5], assuming that, the QNM amplitude ensemble average behaves as

$$\langle c_{q_1}^* c_{q_2} \rangle = \delta_{q_1 q_2} n_{q_1}, \tag{4}$$

imposing conditions on the higher order cumulants identical to [5]. From our dynamical system, and a corresponding dynamical system for the "number densities" n_q, one can draw the conclusions that, in the approximation considered [4]:

(1) The QNM amplitudes associated with the axisymmetric mode are essentially constant on the time-scale $1/\epsilon$.
(2) The dynamical system possesses an equilibrium solution (on the time-scale $1/\epsilon$) $n^{\text{eq}}_{N\ell m}$ sharply peaked at $m = N = 0$, with a power-law tail in ℓ.

1) and 2) indicate that a kind of inverse cascade might be possible, from high to low values of m, hence a kind of "weather of QNMs" around the black hole.

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Linear Stability of Schwarzschild-Anti de Sitter spacetimes

Gustav Holzegel

(joint work with Olivier Graf)

The study of stability of black hole spacetimes is a classical subject in general relativity. While satisfactory non-linear stability results have recently been obtained for Schwarzschild and slowly rotating Kerr black holes both in the case of vanishing [2, 3] and positive cosmological constant [9], the case of negative cosmological constant (which is that of asymptotically anti-de Sitter (AdS) black holes) is much less understood. The PDE nature of the problem is quite different as the presence of the conformal boundary requires studying a boundary initial value problem with the boundary located at infinity [4].

About ten years ago, in joint work with Jacque Smulevici [7, 8] we established the following result:

Theorem 1. *Let $\alpha < \frac{9}{4}$. Solutions to the linear wave equation $\Box_g \psi + \alpha\psi = 0$ for (M, g) the exterior of a Kerr-AdS black hole with parameters satisfying the Hawking-Reall bound, and Dirichlet boundary conditions for ψ imposed at the conformal boundary, decay inverse logarithmically and not better in time.*

This slow decay of waves on asymptotically AdS black holes is rooted in a stable trapping phenomenon for these spacetimes and lead to the conjecture that these spacetimes should be non-linearly unstable. See the related talk of Christoph Kehle for recent progress on non-linear instability.

In joint work with Olivier Graf [5, 6] we recently established that the slow logarithmic decay remains valid for the full linearised Einstein equations:

Theorem 2. *Solutions to the linearisation of the Einstein equations $Ric(g) = \Lambda g$ with $\Lambda < 0$ around a Schwarzschild-AdS metric arising from regular initial data and with standard Dirichlet-type boundary conditions imposed at the conformal* *boundary (inherited from fixing the conformal class of the non-linear metric) remain globally uniformly bounded on the black hole exterior and in fact decay inverse logarithmically in time to a linearised Kerr-AdS metric.*

The proof employs a double null gauge and exploits a hierarchical structure of the equations of linearised gravity in this gauge, a strategy analogous to the corresponding result in the asymptotically flat case [1]. In particular, a key step is to first establish boundedness and inverse logarithmic decay results for the Teukolsky variables, which – contrary to the asymptotically flat case – now couple to one another at the boundary. (This is a consequence of fixing the conformal class of the non-linear metric on the boundary, which is the perhaps most natural boundary condition in this problem.) To prove decay for the Teukolsky quantities, we rely on (1) a physical space transformation theory between the Teukolsky equations and the Regge-Wheeler equations on Schwarzschild-AdS backgrounds (which requires the construction of new quantities and coercive energies) and (2) novel energy and Carleman estimates handling the coupling of the two Teukolsky equations through the boundary conditions thereby generalising earlier work of [7] for the covariant wave equation. Specifically, we also produce purely physical space Carleman estimates.

In a second step, the aforementioned hierarchical structure in the system of linearised Einstein equations in double null gauge is exploited by integrating (redshifted) transport equations, which inherit the decay established for the Teukolsky quantities. Contrary to the asymptotically flat case [1], addition of a residual pure gauge solution to the original solution is not required to prove decay of all linearised null curvature and Ricci coefficients. Roughly speaking this is because the non-decaying quantities in the asymptotically flat case, now inherit decay from the decaying quantities through the boundary conditions relating the two quantities. However, one may in addition normalise the solution at the conformal boundary to be in standard AdS-form by adding such a pure gauge solution, which is constructed dynamically from the trace of the original solution at the conformal boundary and quantitatively controlled by initial data.

Applications of these results to the rigidity problem for black holes and generalisations to the Kerr-AdS case will be discussed elsewhere.

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Local structure theory of Einstein manifolds with boundary

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(joint work with Zhongshan An)

The existence of Einstein metrics and the structure of their moduli space are central topics in geometry and theoretical physics. While significant progress has been made for closed or complete manifolds without boundary, understanding the moduli space of Einstein metrics, particularly in higher dimensions, remains challenging. A potential step toward this understanding is to study Einstein metrics on compact manifolds with boundary, focusing on the boundary value problem. A natural choice of boundary condition is the induced metric. However, this boundary condition is not elliptic in general [3, 10]. M. Anderson proposed an elliptic boundary condition by specifying the conformal geometry of the boundary and the mean curvature [3, 4]. For a compact Riemannian manifold (Ω, g) whose boundary is Σ with induced metric $g^{\intercal}$, we denote the boundary mean curvature by H_g and the (pointwise) conformal class of $(\Sigma, g^{\intercal})$ by $[g^{\intercal}]$. The pair $([g^{\intercal}], H_g)$ is referred to as the *Anderson boundary data* of (Ω, g). We investigate the moduli space of (Riemannian) Einstein metrics under this boundary condition in dimensions three and higher [2].

The Anderson boundary condition has been studied in various contexts. P. Gianniotis [8] established short-time existence and uniqueness of the Ricci-DeTurck flow with prescribed Anderson boundary data. Z. An and Anderson have examined the well-posedness of the initial boundary value problem for Lorentzian Einstein metrics with a generalized form of Anderson boundary data in the context of Cauchy problem [1]. There is also growing interest in Anderson boundary data due to its potential applications in Euclidean quantum gravity (see [10, 11, 9]). Another motivation to study Anderson boundary data is its possible connection to the existence of conformally compact Einstein metrics with prescribed conformal infinity boundary. There are some partial existence results (see, for example, the survey [7]). Our boundary value problem might be viewed as an analogy on a bounded region.

Let $n \geq 3$, and let Ω be a compact, connected n-dimensional manifold with a smooth nonempty boundary Σ. Assume the relative fundamental group $\pi_1(\Omega, \Sigma) = 0$. For $k \geq 2$ and $\alpha \in (0, 1)$, we denote the space of $\mathcal{C}^{k,\alpha}$ Riemannian metrics on Ω by $\mathcal{M}^{k,\alpha}(\Omega)$. We focus on the following subsets:

- The subset consisting of Einstein metrics with the fixed Einstein constant
$$\mathcal{M}_\Lambda^{k,\alpha}(\Omega) = \Big\{ g \in \mathcal{M}^{k,\alpha}(\Omega) : \mathrm{Ric}_g = (n-1)\Lambda g \Big\}.$$
- The subset consisting of negative Einstein metrics
$$\mathcal{M}_-^{k,\alpha}(\Omega) = \Big\{ g \in \mathcal{M}^{k,\alpha}(\Omega) : \mathrm{Ric}_g = \lambda g \text{ for some } \lambda < 0 \Big\}.$$

Let $\mathcal{D}^{k+1,\alpha}(\Omega)$ be the space of $\mathcal{C}^{k+1,\alpha}$ diffeomorphisms of Ω that restrict to the identity on Σ. Both the moduli spaces $\mathcal{M}_\Lambda^{k,\alpha}(\Omega)/\mathcal{D}^{k+1,\alpha}(\Omega)$ or $\mathcal{M}_-^{k,\alpha}(\Omega)/\mathcal{D}^{k+1,\alpha}(\Omega)$ are infinite-dimensional smooth Banach manifolds.

For each fixed k, α, define the boundary map $\Pi : \mathcal{M}^{k,\alpha}(\Omega)/\mathcal{D}^{k+1,\alpha}(\Omega) \to \mathcal{S}_1^{k,\alpha}(\Sigma) \times \mathcal{C}^{k-1,\alpha}(\Sigma)$ by
$$\Pi(g) = ([g^\intercal], H_g)$$
where $\mathcal{S}_1^{k,\alpha}(\Sigma)$ denotes the space of conformal classes $[\gamma]$ of $\mathcal{C}^{k,\alpha}$ Riemannian metrics on Σ. We consider the boundary map restricted to either the moduli space $\mathcal{M}_\Lambda^{k,\alpha}(\Omega)/\mathcal{D}^{k+1,\alpha}(\Omega)$ or $\mathcal{M}_-^{k,\alpha}(\Omega)/\mathcal{D}^{k+1,\alpha}(\Omega)$, and the boundary map Π is smooth.

Since an Einstein metric is of constant sectional curvature in dimension 3 and thus is more rigid, Anderson made the following conjecture.

Conjecture 1 (Anderson [5, p. 2]). *Let Ω be a 3-dimensional manifold with smooth boundary Σ and $\pi_1(\Omega, \Sigma) = 0$. For each fixed Λ, the boundary map*
$$\Pi : \mathcal{M}_\Lambda^{k,\alpha}(\Omega)/\mathcal{D}^{k+1,\alpha}(\Omega) \to \mathcal{S}_1^{k,\alpha}(\Sigma) \times \mathcal{C}^{k-1,\alpha}(\Sigma)$$
is regular for generic metrics.

Anderson noted that the genericity condition cannot be removed [6]. Specifically, a round ball is a critical point of Π. We confirm Conjecture 1 in a stronger form, proving that Π is generically a local *diffeomorphism*.

Theorem 1. *Let Ω be a 3-dimensional manifold with smooth boundary Σ and $\pi_1(\Omega, \Sigma) = 0$. The boundary map Π is a local diffeomorphism on an open dense subset.*

For higher-dimensional manifolds, we introduce non-degenerate boundary conditions and prove that the boundary map is regular on an open dense subset under these conditions. We refer to [2, Definition 1.1] for the definition of the non-degenerate boundary conditions and just note that in special cases that the induced scalar and mean curvatures R_Σ, H_g on Σ are non-zero constants, they become equivalent to the induced metric $g^\intercal$ being a strictly stable critical point of the normalized total scalar curvature functional among non-homothetic conformal transformations on the boundary.

We first consider the space of Ricci flat metrics $\mathcal{M}_0^{k,\alpha}(\Omega)$. Define $\widehat{\mathcal{M}}_0^{k,\alpha}(\Omega)$ as the open subspace of $\mathcal{M}_0^{k,\alpha}(\Omega)$ where the boundary is non-degenerate.

Theorem 2. *Let $n \geq 4$ and Ω be an n-dimensional manifold with smooth boundary Σ satisfying $\pi_1(\Omega, \Sigma) = 0$. The boundary map $\Pi : \widehat{\mathcal{M}}_0^{k,\alpha}(\Omega)/\mathcal{D}^{k+1,\alpha}(\Omega) \to \mathcal{S}_1^{k,\alpha}(\Sigma) \times \mathcal{C}^{k-1,\alpha}(\Sigma)$ is a local diffeomorphism on an open dense subset.*

Consider the space of negative Einstein metrics $\mathcal{M}_-^{k,\alpha}(\Omega)$. Define $\widehat{\mathcal{M}}_-^{k,\alpha}(\Omega)$ as the open subspace of $\mathcal{M}_-^{k,\alpha}(\Omega)$ where the boundary is non-degenerate.

Theorem 3. *Let $n \geq 4$ and Ω be an n-dimensional manifold with smooth boundary Σ satisfying $\pi_1(\Omega, \Sigma) = 0$. The boundary map $\Pi : \widehat{\mathcal{M}}_-^{k,\alpha}(\Omega)/\mathcal{D}^{k+1,\alpha}(\Omega) \to \mathcal{S}_1^{k,\alpha}(\Sigma) \times \mathcal{C}^{k-1,\alpha}(\Sigma)$ is regular on an open dense subset.*

Furthermore, at a regular point g, for (γ, ϕ) sufficiently close to its Anderson boundary data $([g^{\intercal}], H_g)$, there exists a family of Einstein metrics g_s, for $|s|$ small, such that g_s realizes the same Anderson boundary data (γ, ϕ).

The Einstein constant of g_s depends on the volume of g_s, which we do not have control over. As a result, g_s might have the same Einstein constant for different values of s, preventing us from improving the boundary map Π to a local diffeomorphism when restricted to the smaller space with a fixed Einstein constant. This subtlety differentiates negative Einstein metrics from the Ricci flat case. It remains an open question whether a similar result holds for positive Einstein constants.

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Turbulent instabilities for quasilinear waves on Schwarzschild–AdS black holes

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(joint work with Georgios Moschidis)

In the presence of a negative cosmological constant $\Lambda < 0$, the Einstein vacuum equations take the form

$$\mathrm{Ric}[g] = \Lambda g \tag{1}$$

and the *Schwarzschild–Anti-de Sitter (AdS)* metrics

$$g_M = -\Big(1-\frac{2M}{r}-\frac{1}{3}\Lambda r^2\Big)dt^2+\Big(1-\frac{2M}{r}-\frac{1}{3}\Lambda r^2\Big)^{-1}dr^2+r^2\big(d\theta^2+\sin^2\theta\, d\varphi^2\big) \tag{2}$$

give rise to a 1-parameter family, parametrized by $M \geq 0$, of spherically symmetric solutions to (1). For $M = 0$, the metric (2) reduces to the Anti-de Sitter spacetime, the maximally symmetric solution to (1). For $M > 0$, the metric describes a black hole spacetime with the event horizon located at $r = r_+$, the positive root of $1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2$. The cosmological constant Λ introduces a length scale which we assume without loss of generality to be $\Lambda = -3$. A conformal boundary $\mathcal{I}$ can be naturally attached to Schwarzschild–AdS at $r = \infty$, with $\mathcal{I}$ having the conformal structure of a timelike hypersurface diffeomorphic to $\mathbb{R}\times S^2$. Since the boundary $\mathcal{I}$ is timelike, boundary conditions must be imposed to ensure well-posedness. In the following, we will only consider the case of reflecting Dirichlet boundary conditions imposed at $\mathcal{I}$.

In the remarkable works [7, 8] Moschidis showed that the Anti-de Sitter spacetime is unstable for the Einstein-null dust system and the Einstein–Vlasov system. He showed that there exist arbitrarily small spherically symmetric perturbations of the exact AdS initial data giving rise to solutions which eventually form Schwarzschild–AdS black holes. While within spherical symmetry the exterior region of Schwarzschild–AdS is stable [3], the stability problem is much more delicate for non-spherically symmetric perturbations. Indeed, even solutions to the linear Klein–Gordon equation

$$\Box_{g_M}\phi + 2\phi = 0 \tag{3}$$

merely decay at an inverse logarithmic rate [4]. Moreover, in [5] this decay rate has been proven to be sharp using high-frequency quasimode solutions supported near $\mathcal{I}$. The existence of such quasimodes is intimately tied to the presence of stably trapped null geodesics near $\mathcal{I}$. This weak stability and the sharp decay were generalized to the full system of linearized gravity [1, 2], and we refer to the talk of Gustav Holzegel for a more detailed discussion.

Motivated by the question of the *nonlinear* stability of the Schwarzschild–AdS exterior as a solution to (1), we study solutions $\phi : \mathcal{M}^{(M)}_{ext} \to \mathbb{C}$ to the initial-boundary value problem for the cubic quasilinear defocusing wave equation on the exterior region $\mathcal{M}^{(M)}_{ext} = \mathbb{R}_t \times (r_+, \infty)_r \times S^2_{\theta,\varphi}$ of Schwarzschild–AdS:

$$\begin{cases} \Box_{g_M}\phi + 2\phi - \mathcal{N}[\phi] = 0, \\ (\phi, \partial_t\phi)|_{t=0} = (f_0, f_1), \\ r\phi|_{r=\infty} = 0, \end{cases} \tag{4}$$

where

$$\mathcal{N}[\phi] = r^{-6}\left(|\partial_t\phi|^2\phi + |\phi|^2\partial_t^2\phi\right).$$

The nonlinearity $\mathcal{N}[\phi]$ can be seen as a defocusing quasilinear cubic interaction and the weight r^{-6} arises as the natural r-weight for cubic terms by a formal analysis of (1) in generalized harmonic coordinates. Our main result below establishes a proof of principle that Schwarzschild–AdS exhibits the phenomenon of weak turbulence, namely the growth of higher-order Sobolev norms as a consequence of nonlinear mode interactions.

Theorem 1 (K.–Moschidis [6], forthcoming)**.** *For any $s > 0$ sufficiently large and any $\varepsilon > 0$, there exists an open and dense set $\mathcal{J}_{\varepsilon,s} \subseteq (0,\infty)$ such that the following statement holds: For any black hole mass parameter $M \in \mathcal{J}_{\varepsilon,s}$, there exists a smooth initial data set (f_0, f_1) for the initial–boundary value problem* (4) *with*

$$\|f_0\|_{H^s} + \|f_1\|_{H^{s-1}} \le \varepsilon \quad \textit{and} \quad \mathrm{supp} f_0, \mathrm{supp} f_1 \subset \{r > 3M\}$$

and a time $T_1 > 0$ such that the smooth solution for (4) *exists for $t \le T_1$ and satisfies*

$$\|\phi|_{t=T_1}\|_{H^s} > \frac{1}{\varepsilon}.$$

The Sobolev norm growth instability of Theorem 1 is a consequence of nonlinear quasimode interactions and energy transfer from low to high frequencies. More precisely, by fine-tuning the Schwarzschild–AdS mass parameter M, we can ensure that three *dominant* high-frequency quasimodes ϕ_0, ϕ_+, ϕ_- have *resonant* frequency space (and physical space) support while at the same time being quantitatively *non-resonant* to all other quasimodes. As a result of this, we show that for initial data (f_0, f_1) supported on these dominant quasimodes, the corresponding solution to (4) remains suitably close up to time $t = T_1$ to the sum of dominant modes

$$\phi_k(t, r, \theta, \varphi) = \sum_{k\in\{0,+,-\}} \frac{a_k(t)}{r} e^{im_k\varphi} R_{n_k\ell_k}(r) Y_{m_k\ell_k}(\theta) e^{\pm_k i\omega_k t}, \tag{5}$$

where the weights $a_k(t)$ satisfy a nonlinear ODE system. In addition to ensuring the resonant conditions, the frequencies $\omega_0, \omega_+, \omega_-$ are chosen such that $1 \ll \frac{|\omega_\pm|}{|\omega_0|} \ll L$ holds, where $|\omega_0|, |\omega_+|, |\omega_-| \sim L \gg 1$. We show that the initial data can be chosen such that the solution of the ODE system for $a_k(t)$ has the property that mass is transferred from $a_0(t)$ to $a_+(t)$ and $a_-(t)$ and thereby causing the Sobolev norm to grow. To close our argument, we have to estimate the error term, i.e. the difference between the approximate solution consisting of weighted quasimodes and the true solution. For this, we use energy estimates based on the redshift and ∂_t multiplier. We also introduce the *helical* vector field

$$V = \big(1 + L^{-1}\big)\partial_t + \partial_\varphi$$

which has good commutation properties when acting on functions with physical space and frequency support as in (5).

Our proof also extends to other types of nonlinearities satisfying the null condition but does not apply to power nonlinearities such as $\mathcal{N}(\phi) = |\phi|^2\phi$. It would be interesting to identify a suitable structure in the nonlinearity which ensures that the proof method applies. Another interesting problem is reducing the value of s in Theorem 1 because a value of $s = 3/2$ could be interpreted as an indication of trapped surface formation for (1).

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BKL bounces outside homogeneity

Warren Li

Due to the Hawking–Penrose singularity theorems, see for instance [1], it is known that "singularities" are a robust prediction of Einstein's equations on a spacetime $(\mathcal{M}, \mathbf{g})$:

$$\text{Ric}_{\mu\nu} - \frac{1}{2}R\,\mathbf{g}_{\mu\nu} = 2\,\mathbf{T}_{\mu\nu} \tag{1}$$

However, it remains a wide open problem to understand the dynamical description of spacetimes near their singularities.

One approach towards this goal in the physics literature is the heuristics of Belinski, Khalatnikov and Lifshitz (BKL), who proposed the following leading order ansatz for near-singularity solutions of the Einstein equations (in vacuum), based on the Kasner metric discovered in [2]:

$$\mathbf{g} = -dt^2 + \sum_{I=1}^{3} t^{2p_I(x)} \boldsymbol{\omega}^I(x) \wedge \boldsymbol{\omega}^I(x) + \cdots, \tag{2}$$

where the spacetime manifold is $\mathcal{M} = \{(t,x) : t \in (0,T), x \in \Sigma^3\}$, and the p_I and ω^I are functions and one-forms on the spatial slices Σ. In [3], via a formal power series expansion, it is suggested that (2) is a valid ansatz provided: (1) the Kasner relations $\sum p_I(x) = \sum p_I^2(x) = 1$ hold, as well as asymptotic momentum constraints and (2) whenever $p_I < 0$, the associated one-form $\boldsymbol{\omega}^I$ is integrable in the sense of Frobenius: $\boldsymbol{\omega}^I \wedge d\boldsymbol{\omega}^I = 0$. Furthermore, in this case the metric (2) is singular in the sense that the Kretschmann scalar blows up as $O(t^{-4})$.

Since the Kasner relations force $p_I \leq 0$ for some $I \in \{1,2,3\}$, it is exactly this condition (2), along with a function counting argument, that leads the authors of [3] to suggest that singularities of the form (2) are non-generic i.e. are not in one-to-one correspondence with Cauchy initial data for the Einstein equations. This was revisited together with Belinski in [4], where they propose that generic spacelike singularities are such that (this is often called the BKL conjecture):

(1) (*AVTD behaviour*) For two different points on the singularity, say $(0, x_1)$, $(0, x_2) \in \partial\mathcal{M}$ with $x_1, x_2 \in \Sigma$, the dynamics in the future of $(0, x)$ and the future of $(0, y)$ quantitatively decouple in the sense that close to the singularity: $|\partial_x\text{-derivatives}| \ll |\partial_t\text{-derivatives}|$.

(2) (*Mixmaster*) As a result, the dynamics in the future of any $(0, x)$ are well-approximated by that of a spatially homogeneous spacetime, and thus by a finite dimensional, autonomous system of ODEs. All fixed points in this ODE system are unstable, and general orbits of this ODE system are approximately a cascade of heteroclinic orbits (called *BKL bounces*) joining pairs of these fixed points.

Treating the exponents p_I as part of the ODE variables (so that now they also depend on time), these heteroclinic orbits change the p_I via the *Kasner map*: when $p_1 < 0$:

$$p_1 \mapsto \acute{p}_1 = -\frac{p_1}{1+2p_1}, \quad p_2 \mapsto \acute{p}_2 = -\frac{p_2+2p_1}{1+2p_1}, \quad p_3 \mapsto \acute{p}_3 = -\frac{p_3+2p_1}{1+2p_1}. \tag{3}$$

The question remains to understand the extent to which BKL's heuristics apply in the rigorous study of solutions to Einstein's equation (1) arising from (regular) initial data away from the singularity. There are some results in the spatially homogeneous class, where the ODE system is exact, see [5, 6, 7]. The difficulty in the non-homogeneous case is that one must *prove* the AVTD behaviour in the process of showing singularity formation.

One setting in which this is better understood is the Einstein–scalar field system where $\phi : \mathcal{M} \to \mathbb{R}$ solves $\Box_{\mathbf{g}}\phi = 0$ and $\mathbf{T}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\mathbf{g}(\nabla\phi,\nabla\phi)\mathbf{g}_{\mu\nu}$. According to [8], here the ansatz (2) is supplemented with the leading order ansatz $\phi = A\log t + \cdots$, and the Kasner relations become $\sum p_I = \sum p_I^2 + 2A^2 = 1$. The benefit is that for $A \neq 0$ one can have $\min p_I > 0$, and thus one does not require the additional integrability assumption for $\boldsymbol{\omega}^I$. In the ODE picture, this corresponds to the existence of *stable* fixed points in the dynamical system; in fact generic orbits will converge to one of these points. Mathematically, there are recent results such as:

Theorem 1 (Fournodavlos–Rodnianski–Speck [9]). *Consider initial data for the Einstein–scalar field system which are perturbations of that of a subcritical generalized Kasner spacetime (these spacetimes have $\Sigma = \mathbb{T}^3$, $\mathbf{g} = -dt^2 + \sum_{i=1}^3 t^{2p_i}(dx^i)^2$ and $\phi = A \log t$ with p_i, A constants satisfying $\sum p_i = \sum p_i^2 + 2A^2 = 1$ and $p_i > 0$).*

Then the maximal past development arising from such data terminates in a Big Bang singularity in the sense that near the singularity one can foliate by spacelike hypersurfaces Σ_t, $t > 0$ such that the Kretschmann scalar is $O(t^{-4})$ on Σ_t, and moreover the spacetime exhibits AVTD behaviour and $\mathbf{g}$ roughly takes the form (2) *near $t = 0$.*

See the report of O. Petersen for another recent extension of this theorem. The major difficulty of these results is proving the AVTD behaviour; in particular in the proof of AVTD it is essential that the subcritical generalized Kasner spacetimes correspond to *stable* fixed points of the associated ODE system. To get closer to the full BKL conjecture the next step is to prove AVTD behaviour even in the presence of nonlinear BKL bounces.

We now introduce our new results, which apply to solutions of the Einstein equations in various symmetry classes. The first concerns the Einstein vacuum equations in Gowdy symmetry, where the metric takes the form:

$$\mathbf{g} = -t^{-1/2} e^{\lambda/2}(-dt^2 + d\theta^2) + t\left[e^P (d\theta + Q d\delta)^2 + e^{-P} d\delta^2\right], \tag{4}$$

where λ, P, Q are functions of $(t, \theta) \in (0, +\infty) \times \mathbb{S}^1$ and the Einstein vacuum equations take the form of a (semilinear) wave-transport system for P, Q, λ.

According to the monumental Strong Cosmic Censorship result of Ringström in Gowdy symmetry [10], for an open and dense set of initial data, solutions to these equations exist up to $t = 0$ and moreover are such that for all but finitely many $\theta \in \mathbb{S}^1$, one has the convergence $-t\partial_t P(t, \theta) \to V(\theta) \in (0, 1)$ as $t \to 0$. Further, this roughly corresponds to Kasner exponents:

$$p_1(\theta) = \frac{V^2 - 1}{V^2 + 3}, \quad p_2(\theta) = \frac{2(1 - V)}{V^2 + 3}, \quad p_3(\theta) = \frac{2(1 + V)}{V^2 + 3}. \tag{5}$$

Note that the condition $V < 1$ arises since the 1-form $\boldsymbol{\omega}^2 = d\theta + Q d\delta$ is not (generically) integrable in the sense of Frobenius and one therefore requires $p_2(\theta) > 0$.

However, this result concerns only the eventual asymptotics, and leaves open the question of the intermediate dynamics that could occur between initial data at $t = t_0 > 0$ and the eventual singularity at $t = 0$. Our new result is the following, exhibiting the existence of Gowdy spacetimes with (a) up to one BKL bounce along causal curves but (b) AVTD behaviour in spite of such bounces.

Theorem 2 (L.[11])**.** *There exists an open set of smooth initial data for the Gowdy symmetric Einstein vacuum equations at $t = t_0$ satisfying:*

$$V_0(\theta) \doteq -t\partial_t P(t_0, \theta) \in (0, 2),$$

such that t_0 is sufficiently small, then: The solution exists in $t \in (0, t_0)$ and that for "most" $\theta \in \mathbb{S}^1$, one has the convergence:

$$-t\partial_t P(t, \theta) \to \acute{V}(\theta) \approx \min\{V_0(\theta), 2 - V_0(\theta)\} \text{ as } t \to 0,$$

noting that via (5)*, the transition $V(\theta) \mapsto 2 - V(\theta)$ is exactly the Kasner map* (3)*, The solution remains AVTD in $t \in (0, t_0)$, in the sense that for instance:*

$$|\partial_\theta P| + |t\partial_t \partial_\theta P| + \cdots \lesssim t^{-\sigma} \text{ for some } 0 < \sigma < 1.$$

We remark that we call this last point AVTD because while a ∂_t-derivatives costs an entire power of t^{-1}, a spatial ∂_θ-derivative costs only $t^{-\sigma}$. As well as this result in Gowdy symmetry, we prove a similar result for the spherically symmetric (or more generally surface symmetric) Einstein–Maxwell–scalar field system in the gauge $\mathbf{g} = -e^{2\mu}dr^2 + e^{2\lambda}dx^2 + r^2 d\sigma_{\mathbb{S}^2}$. Here, the role of $-t\partial_t P$ is replaced by the derivative $r\partial_r\phi$ of the scalar field.

Theorem 3 (L.[12])**.** *There exists an open set of smooth initial data for the spherically symmetric Einstein–Maxwell–scalar field equations at $r = r_0$ satisfying:*

$$\Psi_0(\theta) \doteq r\partial_r\phi(r_0, x) \in (0, +\infty),$$

such that r_0 is sufficiently small, then: The solution exists in $r \in (0, r_0)$ and that for $x \in \mathbb{S}^1$, one has the convergence:

$$r\partial_r\phi(r, x) \to \acute{\Psi}(x) \approx \max\{\Psi_0(x), \Psi_0^{-1}(x)\} \text{ as } r \to 0,$$

and the transition $\Psi(x) \mapsto \Psi^{-1}(x)$ is exactly the Kasner map (3)*, The solution remains AVTD in $t \in (0, t_0)$, in the sense that for instance:*

$$|\partial_x\mu| + |\partial_x\phi| + |r\partial_r\partial_x\phi| + \cdots \lesssim t^{-\sigma} \text{ for some } 0 < \sigma < 1.$$

While Theorem 1 and Theorem 2 apply to symmetric systems that reduce to $1+1$-dimensional PDEs, the method of proof is insensitive to the dimension and one would hope that in future similar results can apply to full $1+3$-dimensional problems.

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Intrinsic rigidity of extremal horizons and black hole uniqueness

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(joint work with Alex Colling, Maciej Dunajski, David Katona)

The no-hair theorem states that, under certain global assumptions, any stationary (analytic) vacuum spacetime that contains a black hole must be a Kerr solution [1]. This celebrated result rests on several remarkable theorems that constrain the topology, symmetry and geometry of such black hole spacetimes. Notably, Hawking's rigidity theorem establishes that the event horizon of a stationary analytic spacetime must be a Killing horizon, and if the black hole is rotating, the spacetime is axially symmetric [2]; this has subsequently been generalised to extremal black holes and to higher dimensional spacetimes [3, 4, 5].

On the other hand, the Einstein equation restricted to a Killing horizon implies that the intrinsic geometry decouples from the extrinsic geometry precisely if the horizon is extremal (i.e. the surface gravity vanishes). The intrinsic geometry of a cross-section of an extremal horizon in an $(n+2)$-dimensional Einstein spacetime corresponds to a *quasi-Einstein structure*: an n-dimensional manifold M (a cross-section of the horizon), a Riemannian metric g on M and a vector field $X \in \mathfrak{X}(M)$, which satisfy a quasi-Einstein equation,

$$\mathrm{Ric}(g) = \frac{1}{2} X^\flat \otimes X^\flat - \mathcal{L}_X g + \Lambda g \ , \tag{1}$$

where $\mathrm{Ric}(g)$ is the Ricci tensor of g, $\mathcal{L}_X$ is the Lie-derivative, the one-form $X^\flat$ is the g-dual of X and $\Lambda \in \mathbb{R}$ is the cosmological constant [6]. This quasi-Einstein structure also describes the intrinsic geometry of extremal isolated horizons [7]. It is also equivalent to the Einstein equation for the *near-horizon geometry*, an associated $(n+2)$-dimensional spacetime that can be obtained by a near-horizon scaling limit of the original spacetime [6]. Motivated by applications to black hole spacetimes, we will assume M is a compact manifold (without a boundary). We refer to a quasi-Einstein structure as trivial if X vanishes identically.

Numerous classification results have been obtained for the quasi-Einstein equation (1) under a variety of symmetry assumptions [6]. For instance, solutions with $X^\flat$ closed, which is equivalent to the near-horizon geometry being static, are trivial if $n = 2$ (four spacetime dimensions), or either trivial or the product of a circle with an Einstein metric if $\Lambda \leq 0$ [8, 9, 10]. Interestingly, if $\Lambda \geq 0$ and $n \geq 3$, the classification of static near-horizon geometries remains an open problem.

Perhaps the most notable result is that the general solution to (1) on $M = S^2$ that admits an axial Killing field (which preserves X) is isometric to the horizon geometry of the extremal Kerr black hole [11, 7, 12]. It has been an open problem to determine whether spherical topology of M together with the equations (1) imply the existence of a Killing vector field. If so, this would be an intrinsic version of Hawking's rigidity theorem; furthermore, it would imply that all solutions to (1) on $M = S^2$ arise from an extremal Kerr horizon. Recently, we have solved this problem and, in fact, proven much more: the existence of a Killing field holds in any dimension and also with a cosmological constant, as follows.

Theorem 1 ([13])**.** *Any n-dimensional compact Riemannian manifold (M, g) with a non-gradient vector field X that satisfies* (1)*, must admit a Killing vector field K such that $[K, X] = 0$.*

The proof makes use of the existence of a positive function $\Gamma > 0$ that ensures that $K^\flat := \Gamma X^\flat + d\Gamma$ is divergence-free, together with a remarkable tensor identity which reduces the g–norm $|\mathcal{L}_K g|^2$ to a total divergence. Thus its integral over a closed manifold M vanishes which implies that K is a Killing vector field. It is worth emphasising that the validity of the identity is based on several cancellations which depend crucially on the precise numerical coefficient of the term quadratic in X of the quasi-Einstein equation (1). The identity then implies that the vector field X is also invariant under the Killing field provided $\Lambda \leq 0$; for $\Lambda > 0$ a further argument employing the Fredholm alternative establishes this [14] (recently a more general identity which unifies these proofs has been derived [15]). The symmetry inheritance of X is significant because it implies that K extends to a Killing field of the associated spacetime near-horizon geometry.

Theorem 1 is complementary to the aforementioned classification of static horizons which correspond to $X^\flat$ closed. Furthermore, taken together with triviality of solutions to (1) on higher genus surfaces M (which by the Gauss-Bonnet theorem can only occur for $\Lambda < 0$) [16], this establishes the following.

Corollary 1. *The extremal Kerr horizon, possibly with cosmological constant, is the unique non-trivial solution to* (1) *on a compact surface M.*

This completes the classification of extremal horizons with a compact cross-section, and their near-horizon geometry, in four-dimensional vacuum spacetimes. Recently, an analogous result has been proven for Einstein-Maxwell theory [17].

An important application of the classification of extremal horizons is to the classification of extremal black hole spacetimes [6]. In the presence of a cosmological constant, analogues of the no-hair theorem are not known (even for non-extremal black holes), apart from a few limited cases. For instance, a uniqueness theorem for non-extremal Schwarzschild de Sitter black holes has been established under an assumption on the level sets [18]; remarkably, numerical evidence has been presented for the existence of static binary black holes in de Sitter that evades this assumption [19], so even the classification of static black holes in de Sitter is incomplete. On the other hand, for negative cosmological constant, the uniqueness of spherical Schwarzschild-AdS remains a notable open problem. In the extremal case such classification problems should be more tractable due to the above uniqueness theorems for extremal horizons, which are also valid with a cosmological constant. To this end, we have established the following theorem.

Theorem 2 ([20]). *Any $d \geq 4$-dimensional analytic Einstein spacetime with $\Lambda > 0$, that contains a static extremal Killing horizon with a maximally symmetric compact cross-section, must be locally isometric to the extremal Schwarzschild de Sitter spacetime or its near-horizon geometry (the Nariai solution).*

The proof is elementary and starts by noting that the first transverse derivative of the metric at the horizon is unique (up to scale) and if non-vanishing corresponds to the extremal Schwarzschild-de Sitter solution [21]. We showed this persists for the second transverse derivative and via an inductive argument for all transverse derivatives. The key fact is that the nth transverse derivative of the Einstein equation reduces to an eigenvalue equation of the laplacian, on a cross-section of the horizon $M = S^{d-2}$, for the traceless part of the nth transverse derivative of the metric, which is sourced by the lower order transverse derivatives of the metric. If $\Lambda > 0$ these eigenvalues are strictly negative and hence the solution is unique at each order. Therefore, for analytic spacetimes, the full metric is determined by the transverse derivatives of the metric at the horizon. This proof was inspired by a similar analysis due to Isenberg and Moncrief which showed that four-dimensional vacuum spacetimes with an extremal toroidal horizon must be a plane wave spacetime [22]. If $\Lambda < 0$ an analogous result can be established for the extremal hyperbolic Schwarzschild-AdS horizon where M is a hyperbolic surface, however, the corresponding eigenvalues are now positive so there are other solutions at non-generic points in the moduli space of hyperbolic surfaces; it would be interesting to determine the spacetime interpretation of these solutions.

Theorem 2 is the first uniqueness theorem for extremal black holes with a cosmological constant. In contrast to typical black hole uniqueness theorems, this result does not make any global assumptions on the spacetime such as asymptotics or topology. Indeed, somewhat surprisingly, the full spacetime is determined by the intrinsic geometry of the extremal horizon! Furthermore, since $d = 4$ static near-horizon geometries with compact cross-sections are unique [8], Theorem **??** implies that any analytic spacetime containing a static extremal horizon with a compact cross-section, must be (locally) isometric to an extremal Schwarzschild-de Sitter spacetime or its near-horizon geometry. This solves the classification problem for $d = 4$ static extremal vacuum black holes in de Sitter (assuming analyticity); in particular, it rules out the possibility of extremal multi-black holes in de Sitter. An analogous result has been proven for four-dimensional Einstein-Maxwell theory [23]. It would be interesting to determine if such transverse uniqueness results persist for the extremal Kerr-(A)dS horizons.

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Late time tails

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(joint work with Sung-Jin Oh)

1. Motivation

Our main motivation concerns the stability of the smooth Cauchy horizon in rotating Kerr black holes. It has been proven by Dafermos–Luk that the Cauchy horizon is stable in C^0. However, consistent with the strong cosmic censorship conjecture, it is expected that the Cauchy horizon is unstable in the $C^{0,1}$ topology, and that generic perturbations lead to black hole interiors where the Cauchy horizon is a so-called weak null singularity.

With the progress in the past decade, it is now well-understood that the key to the aforementioned conjecture is a precise rate that a generic dynamical black hole settles down to Kerr. In order to study this problem, we first consider a slightly simpler question of the precise asymptotic behavior *nonlinear* waves on possibly *dynamical* backgrounds. We stress that in view of the intended application, it is important to allow for both nonlinearity and nonstationarity.

2. Examples

Example 1 (Price's law)**.** *The best-known example of late-time tail is the Price law rate predicted by Price's heuristics. Though originally only proposed only for Schwarzschild, it applies also to Kerr, and the expected rates for solutions* ϕ *to the linear wave equation* $\Box_g \phi = 0$ *is* $\simeq t^{-3}$*. In the Schwarzschild case, one can further decompose into spherical harmonics and the* ℓ*-spherical modes decay as* $\simeq t^{-3-2\ell}$*.*

Price's predictions are now established theorems. As upper bounds, this was proven by [6, 7, 10, 11]*. More recently, it is proven that they hold as generic precise asymptotics* [1, 2, 9]*. The proofs in* [1, 2] *and* [9] *used independent methods but both rely on stationary in a crucial way.*

Example 2 (Higher dimensional Schwarzschild black holes)**.** *For general spatial dimension d, the Schwarzschild metric takes the form*

$$g = -(1 - \frac{2m}{r^{d-2}})\mathrm{d}t^2 + (1 - \frac{2m}{r^{d-2}})^{-1}\mathrm{d}r^2 + r^2 g_{\mathbb{S}^{d-1}}.$$

For generic stationary metric in $(d+1)$ *dimensions with* $d \geq 3$ *odd and decaying as* r^{-k} *as* $r \to \infty$*, considerations as Price suggest a decay rate of solutions to wave equation* $\phi \simeq t^{-d-(k-1)}$*. In the specific case of Schwarzschild, however, the decay rate is faster, consistent with previous heuristic and numerical works* [3, 5]*:*

Theorem. *(L.–O. (2024)) When* $d = 5$*,* $\phi \simeq t^{-10}$*.*

Example 3 (Dynamical black holes)**.** *Consider generic dynamical black holes settling down to d-dimensional Schwarzschild. The generic decay rate of solutions to the wave equation is* <u>*slower*</u> *than the stationary case. (Compare the rates to those in Examples 1, 2.) See also the numerical and heuristic works* [8, 4]*.*

Theorem. *(L.–O. (2024)) When $d \geq 5$, $\phi \simeq t^{-6}$. When $d = 3$, if we assume the background is spherically symmetric and decompose the solution into spherical harmonics, then* $\phi_{(\ell)} \simeq \begin{cases} t^{-3} & \ell = 0 \\ t^{-2-2\ell} & \ell \geq 1 \end{cases}$.

3. Main result

The vast array of possibilities can in fact be understood in a unified manner.

Theorem 1 (L.–O. (2024)). *Let the spatial dimension $d \geq 3$ be odd. Consider a nonlinear wave equation*

$$P\phi \doteq \Box_g \phi + B^\mu \partial_\mu \phi + V\phi = \mathcal{N}(\phi, \partial\phi, \partial^2\phi)$$

such that

(1) *ϕ and its vector field derivatives satisfy some weak decay bounds,*
(2) *g, B, V are asymptotically flat,*
(3) *P_0 (the "elliptic" part of P) is invertible, and*
(4) *the nonlinearity $\mathcal{N}$ obeys the null condition when $d = 3$.*

Then the following holds:

(1) *The solution admits an expansion in Bondi-type coordinates for $r \gtrsim u$:*
$$r^{\frac{d-1}{2}}\phi(u, r, \vartheta) = \Phi_0(u, \vartheta) + r^{-1}\Phi_1(u, \vartheta) + \cdots + r^{-J}\Phi_J(u, \vartheta) + \rho_J,$$
where $J \geq \frac{d-1}{2}$, $\lim_{u\to\infty} \Phi_j(u, \vartheta) = 0$ for $0 \leq j \leq J - 1$, and ρ_J is error.
(2) *The precise asymptotics is given by*
$$\phi(u, r, \vartheta) = \eta(r, \vartheta)u^{-J-\frac{d-1}{2}} + O_r(u^{-J-\frac{d-1}{2}-\delta}), \quad \delta > 0,$$
where the profile $\eta(r, \vartheta)$ is given by solving an "elliptic" equation associated to P_0 with boundary condition determined by $\mathfrak{L}(\vartheta) = \lim_{u\to\infty} \Phi_J(u, \vartheta)$.

4. Application to $(5 + 1)$-dimensional Schwarzschild

As an illustration of how Theorem 1 gives the precise rates, consider $(5 + 1)$-dimensional Schwarzschild and reduce to spherical symmetry for simplicity. The expansion of $r^2\phi$ when $r \gtrsim u$ implies that the following recurrence equations hold:

$$\begin{aligned} &\partial_u \Phi_1 = c_1 \Phi_0, && \partial_u \Phi_2 = 0, \\ &\partial_u \Phi_3 = 0, && \partial_u \Phi_4 = c_2 \Phi_3 + c_3 \Phi_0, \\ &\partial_u \Phi_5 = c_4 \Phi_4 + c_5 \Phi_1, && \partial_u \Phi_6 = c_6 \Phi_5 + c_7 \Phi_2, \\ &\partial_u \Phi_7 = c_8 \Phi_6 + c_9 \Phi_3, && \partial_u \Phi_8 = c_{10} \Phi_7 + c_{11} \Phi_4. \end{aligned}$$

Here, all constants $c_j \neq 0$ are explicitly computable and all $\Phi_j(-\infty) = 0$. By Theorem 1 (since $J \geq \frac{d-1}{2} = 2$) we have Φ_0, $\Phi_1 \to 0$ as $u \to +\infty$, and we need to find the first J for which $\Phi_J \not\to 0$. In the process, we have the freedom to perturb Φ_1 on any compact u-interval to make statements about generic solutions.

Clearly, the equations imply $\Phi_2, \Phi_3 \equiv 0$. Combining $\partial_u \Phi_4$ with $\partial_u \Phi_1$ we get $\partial_u(\Phi_4 - \frac{c_3}{c_1}\Phi_1) = 0$ so that $\Phi_4 = \frac{c_3}{c_1}\Phi_1 \to 0$. The $\partial_u \Phi_5$ equation and information so far give $\partial_u(\Phi_5 - (\frac{c_3 c_4}{c_1} + c_5)\Phi_1) = 0$. Specific (and highly non-generic) Schwarzschild calculations give $\frac{c_3 c_4}{c_1} + c_5 = 0$ so that $\Phi_5 \equiv 0$. Then $\Phi_6 \equiv \Phi_7 \equiv 0$. Finally, $\partial_u \Phi_8 = \frac{c_3 c_{11}}{c_1} \Phi_1$, which gives $\Phi_8 \not\to 0$ generically so that decay rate is $t^{-(8+2)} = t^{-10}$.

Notice, however, that in the nonstationary setting, the conclusion is completely different. The difference in the equations is only that the coefficients c_j may now depend on u. Thus, $\Phi_4 = \int_{-\infty}^{u} \frac{c_3(u')}{c_1(u')} \partial_u \Phi_1(u')\, du'$. As long as $\frac{c_3(u)}{c_1(u)}$ is non-constant, we can now prescribe Φ_1 in a way that $\Phi_4 \not\to 0$. Hence, one obtains the much slower decay rate of t^{-6} in the dynamical setting.

5. Comments on the proof

The proof is based on an iteration argument in physical space: starting with weak decay bounds, we slowly improve the rate until we reach an identifiable obstruction that gives the leading order term.

An important ingredient of the proof is the use of the *strong Huygen's principle* of $\Box_{\mathbf{m}}$ by rewriting the equation as $\Box_{\mathbf{m}} \phi = \text{error}$ ($\Box_{\mathbf{m}} =$ Minkowski wave operator):

(1) This allows one to say that the main contribution comes from the wave zone, and thus the tails is completely determined by the higher radiation fields Φ_j.
(2) This also captures explicit Minkowski cancellations: for instance Φ_j never contributes to the late-time tail when $j \leq \frac{d-3}{2}$.
(3) The error terms have extra r decay due to asymptotic flatness. This allows for improving the estimates in terms of r-decay. There are then standard techniques to upgrade r-decay to t-decay when $r \lesssim t$ [1, 10].

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Flexibility of initial data sets via solution operators with prescribed support properties

Sung-Jin Oh
(joint work with Philip Isett, Yuchen Mao and Zhongkai Tao)

The subject of this talk is initial data sets in general relativity. For simplicity, we focus on *vacuum initial data sets* on $\mathbb{R}^3$, i.e., symmetric two-tensors (g_{ij}, k_{ij}) on $\mathbb{R}^3$ that solve the *Einstein vacuum constraint equations*

$$\begin{cases} R + (\mathrm{tr}_g k)^2 - |k|_g^2 = 0, \\ \nabla^i k_{ij} - \nabla_i \mathrm{tr}_g k = 0. \end{cases} \tag{1}$$

When $\mathrm{tr}_g k = 0$ also holds, we say (g, k) is in *maximal gauge*.

I will discuss some new results from [6, 7, 8] concerning *flexibility* of the nonlinear set of objects that solve (1) (resp. (1) and $\mathrm{tr}_g k = 0$), such as extension, gluing, etc. The main focus of the talk, however, will be to explain our new approach to this problem based on *construction of solution operators with prescribed support properties*. This method was already used in [6, 7, 8]; moreover, in the upcoming work [5], we will provide a systematic development of our approach.

Linearized constraint equations. Like prior works in the subject, our approach is based on the understanding of the linearization of (1). Specifically, consider a perturbation of the flat data $g_{ij} = \delta_{ij} + \dot{g}_{ij}$ and $k_{ij} = \dot{k}_{ij}$, where δ_{ij} is the Euclidean metric on $\mathbb{R}^3$, and also introduce the change of variables $(h^{ij}, \pi^{ij}) = (\dot{g}^{ij} - \delta^{ij}\mathrm{tr}\dot{g}, \dot{k}^{ij} - \delta^{ij}\mathrm{tr}\dot{k})$. Then (1) becomes

$$\mathcal{P}\begin{pmatrix} h \\ \pi \end{pmatrix} := \begin{pmatrix} \partial_i \partial_j h^{ij} \\ \partial_i \pi^{ij} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(h, \partial^2 h) + \mathcal{O}(\partial h, \partial h) + \mathcal{O}(\pi, \pi) \\ \mathcal{O}(h, \partial \pi) + \mathcal{O}(\partial h, \pi) \end{pmatrix}. \tag{2}$$

Moreover, (1) in maximal gauge also takes the form (2), but with π replaced by its traceless part $\hat{\pi} = \pi - \frac{1}{3}\delta \mathrm{tr}\pi$.

Here, we are working with the following conventions: (i) repeated upper and lower indices are summed, (ii) indices are lowered (and raised) using the flat metric δ_{ij} (and δ^{ij}); and (iii) tr is defined using δ^{ij}. On the right-hand side, $\mathcal{O}(h_1, h_2)$ refers to an expression that is bilinear in h_1 and h_2, whose coefficients are smooth functions of $\dot{g}$ (at least for $\dot{g}$ small, where such an expansion is relevant).

Solution operator with prescribed support properties, take 1. Our approach is based on the construction of solution operators for the underdetermined differential operator $\mathcal{P}$ with prescribed support properties. To demonstrate the ideas in a simpler setting, let us first consider the divergence operator on $\mathbb{R}^3$, i.e., $Pu := \partial_i u^i$. We break down our approach into two steps:

Step 1. To solve the equation $Pu = f$, we look for a Green's function for this problem, i.e., for $y \in \mathbb{R}^3$, a solution $K(\cdot, y)$ to $PK(\cdot, y) = \delta_y$. This is an underdetermined problem; the point is not finding one solution (as we will see, there are plenty), but rather finding a useful one. To this end, we also consider a ray $\mathbf{x}(y, \cdot) : [0, \infty) \to \mathbb{R}^3$ emanating from y (i.e., $\mathbf{x}(y, 0) = y$) that is non-trapped (i.e., exits every compact set), and require the support property $\operatorname{supp} K(\cdot, y) \subseteq \mathbf{x}([0,\infty))$.

Our simple but key first observation is that finding such a K is straightforward if we consider the dual problem. Indeed, observe that the equation for K is equivalent to $\varphi(y) = \langle K, P^*\varphi\rangle$ for every $\varphi \in C_c^\infty(\mathbb{R}^3)$. Hence, the existence of a distribution solution K with the prescribed support properties amounts to:

(RC) There exists a way to recover $\varphi(y)$ from $P^*\varphi$ on $\mathbf{x}(y,[0,\infty))$ in a linear continuous manner.

We will refer to this as the *Recovery on Curves* condition. Phrased in this way, and noting that P^* is simply the differential (i.e., $P^*\varphi = -\partial_i\varphi$), we see that the desired $K \in \mathcal{D}'(\mathbb{R}^3)$ is given by

$$\langle K(\cdot, y), \varphi\rangle = \int_0^\infty \dot{\mathbf{x}}(y,t) \cdot P^*\varphi(\mathbf{x}(y,t))\,\mathrm{d}t.$$

Step 2. The Green's function obtained in Step 1 has the advantage of having precisely prescribed support properties on curves; however, it is too singular to be useful in solving nonlinear problems. Our next key idea is that this can be remedied by taking a (suitable) smooth average.

To demonstrate, let us consider straight rays $\mathbf{x}(y,\omega,t) = y + t\omega$, where $\omega \in \mathbb{S}^2$, and denote by $K_\omega(\cdot, y)$ the corresponding Green's function from Step 1. We fix $\eta \in C^\infty(\mathbb{S}^2)$ with $\int \eta\,\mathrm{d}A = 1$ and define

$$\langle K_\eta(\cdot, y), \psi\rangle := \int \langle K_\omega(\cdot, y), \psi\rangle \eta(\omega)\,\mathrm{d}A(\omega).$$

The following properties of the *smoothly averaged Green's function* K_η can be easily verified: (1) $\mathcal{S}_\eta f(x) := \int K_\eta(x,y) f(y)\,\mathrm{d}y$ defines a right-inverse for P, which is furthermore a singular integral operator that is regularizing of maximal order (i.e., of order -1 for the order 1 operator P); and (2) if f is supported in a cone C whose angles contain $\operatorname{supp}\eta$, then so is $\mathcal{S}_\eta f$. Such an operator was first obtained in Oh–Tataru [9]; we will refer to it as a *conic operator*.

Application: sharp Carlotto–Schoen gluing. Our presentation should make it clear that the same ideas can be applied to the operator $\mathcal{P}$, as long as we can verify (RC)! For $P_1 h := \partial_i\partial_j h^{ij}$, this is obvious since $P_1^*\varphi = \partial_i\partial_j\varphi$. For $P_2\pi := \partial_i \pi^{ij}$, $P_2^* u = -\frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is (essentially) the Killing operator, and (RC) follows by the standard proof of the rigidity of Killing vector fields. Interestingly, (RC) may also be verified for $P_3\hat{\pi} := \partial_i\hat{\pi}^{ij}$ with $\operatorname{tr}\hat{\pi} = 0$; in this case, P_3^* is the conformal Killing operator, and one may follow the rigidity argument of Reshetnyak [10]. Note that P_1, P_2 and P_3 are the constituents of $\mathcal{P}$.

As a result, we obtain a conic solution operator $\mathcal{S}_\eta$ for $\mathcal{P}$. Using this solution operator and setting up a simple Picard iteration for (2), the gluing theorem of Carlotto–Schoen [2] on conic regions immediately follows. Thanks to the simplicity of the method, this proof is sharp in a number of ways, including giving the $|x|^{-1}$ asymptotics as $|x| \to +\infty$ that was conjectured in [2]. By tweaking the solution operator, a gluing theorem in a degenerate region of the form $\{(x^1)^2 + (x^2)^2 <$

$(x^3)^{2\alpha}\}$ for some $\alpha < 1$ may also be established. For more details, see [6] (non-maximal case) and [7] (maximal case)

Solution operator with prescribed support properties, take 2. We now return to $Pu = \partial_i u^i$ and discuss the construction of solution operators that preserve compact support. We modify the above construction as follows.

Step 1. We consider a segment $\mathbf{x}(y, y_1, \cdot) : [0,1] \to \mathbb{R}^3$ such that $\mathbf{x}(y, y_1, 0) = y$ and $\mathbf{x}(y, y_1, 0) = y_1$, and look for $K_{y_1}(\cdot, y)$ that is supported in $\mathbf{x}(y, y_1, [0,1])$ and solves $PK_{y_1}(\cdot, y) = \delta_y - b_{y_1}$, where b_{y_1} is some distribution supported in $\{y_1\}$. Instead of (RC), we need P to satisfy:

(RC') There exists a way to recover $\varphi(y)$ from $P^*\varphi$ on $\mathbf{x}(y, y_1, [0,1])$ and (the jet of) φ at y_1 in a linear continuous manner.

Step 2. We now consider $\eta \in C_c^\infty(\mathbb{R}^3)$ with $\int \eta(y_1)\, \mathrm{d}y_1 = 1$ and define (formally) $K_\eta(\cdot, y) = \int K_{y_1}(\cdot, y)\eta(y_1)\, \mathrm{d}y_1$. As before, it can be checked that K_η defines a singular integral operator $\mathcal{S}_\eta$ that this regularizing of maximal order. Furthermore, $\mathcal{S}_\eta$ preserves any set U that is star-shaped with respect to $\operatorname{supp} \eta$. Finally, $\mathcal{S}_\eta$ solves as long as $\int f\, \mathrm{d}y = 0$, which is in fact a necessary condition for the existence of a compactly supported solution since $1 \in \ker P^*$.

In fact, this operator coincides with the classical Bogovskii operator [1]. Nevertheless, from our presentation, it should be clear that this construction can be generalized to any P satisfying (RC'). See [6] (non-maximal) and [7] (maximal) for the case of $\mathcal{P}$, and see [5] for more general cases (such as linearized constraints on a non-flat background).

Application: sharp Corvino–Schoen and obstruction-free gluing. Using Bogovskii-type operators for $\mathcal{P}$ and setting up a simple Picard iteration for (2), we may now immediately deduce a gluing theorem similar to Corvino–Schoen [3]. Combined with further ideas for locally manipulating the linear obstructions in Corvino–Schoen gluing, we also obtain the following obstruction-free gluing result à la Czimek–Rodnianski [4]:

Theorem 1. *Let $s > \frac{3}{2}$ and $\alpha > \frac{1}{2}$. Let $(g_{in} - \delta, k_{in})$, $(g_{out} - \delta, k_{out}) \in H^s_\alpha \times H^{s-1}_{\alpha+1}(B^c_{R_0})$ be pairs solving (1) (resp. and $\mathrm{tr}_g k = 0$). Assume the positivity condition $\Delta\mathbf{E} > |\Delta\mathbf{P}|$, where $\Delta\mathbf{E}$ is the difference between the ADM energies of (g_{out}, k_{out}) and (g_{in}, k_{in}), and $\Delta\mathbf{P}$ is the difference between the ADM linear momenta. Then there exists $(g - \delta, k) \in H^s_\alpha \times H^{s-1}_{\alpha+1}(B^c_{R_0})$ solving (1) (resp. and $\mathrm{tr}_g k = 0$) and $r \geq R_0$ such that $(g, k) = (g_{in}, k_{in})$ on B_{2r} and $(g, k) = (g_{out}, k_{out})$ on B^c_{32r}.*

Here, H^s_α is the weighted Sobolev space consistent with decay $r^{-\alpha}$. This result improves upon [4] in a number of ways: first, we attain the sharp constant in the positivity condition (namely, 1 in front of $|\Delta P|$), second, we extend the result to the maximal gauge, and third, the decay and regularity assumptions are sharpened. Our proof is also purely spacelike, as opposed to [4] that employed characteristic gluing. For details, see [6] (non-maximal case) and [7] (maximal case).

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Formation of quiescent big bang singularities

Oliver Petersen

(joint work with Hans Oude Groeniger and Hans Ringström)

The purpose of this talk is to present *a condition on initial data ensuring the formation of a big bang singularity* (in the sense of curvature blow-up at the singularity). The condition does not refer to any particular background solution. In particular, the main goal is not to prove a stability result of some particular solution. In order to describe our condition, it is however natural to start with a simple explicit example:

Example 1. *For an integer $n \geq 2$, consider the spacetime*

$$M := (0,\infty) \times (S^1)^n, \quad g := -\mathrm{d}t^2 + \sum_{i=1}^n t^{2p_i} \left(\mathrm{d}x^i\right)^2,$$

$$\phi := a\ln(t) + b, \quad \sum_{i=1}^n p_i = \sum_{i=1}^n p_i^2 + a^2 = 1.$$

For fixed $(a, b, p_1, \ldots, p_n)$, the triple (M, g, ϕ) will in this talk be called the generalized Kasner spacetime *and can easily be checked to satisfy the Einstein-scalar field equations*

$$\begin{aligned} \mathrm{Ric}(g) &= \mathrm{d}\phi \otimes \mathrm{d}\phi, \\ \Box\phi &= 0. \end{aligned} \tag{1}$$

It is straightforward to check that the Kretschmann scalar

$$\mathrm{R}_{\alpha\beta\gamma\delta}\mathrm{R}^{\alpha\beta\gamma\delta} = \frac{4}{t^4}\left(\sum_{j=1}^{n} p_i^2(1-p_i)^2 + \sum_{i<j} p_i^2 p_j^2\right).$$

Consequently, (M, g, ϕ) *has a big bang singularity in the sense of curvature blow-up as* $t \to 0$ *unless all* p_i *vanish or one* p_i *equals one and the rest vanish. The case that all* p_i *vanish is the* Minkowski metric *and the case that one* p_i *equals one and the rest vanish is called the* flat Kasner metric*.*

The following remarkable result was recently proven by Fournodavlos-Rodnianski-Speck in [3], building on previous work by Rodnianki-Speck in [6], [7], [8] and Speck in [9].

Theorem 1 (Fournodavlos-Rodnianski-Speck)**.** *Assume that*

$$\max_{i,j,k,i\neq j}(p_i + p_j - p_k) < 1.$$

Then the generalized Kasner spacetime (M, g, ϕ) *presented in Example 1 is CMC-stable under the evolution of the Einstein-scalar field equations* (1) *toward the big bang singularity, i.e. as* $t \to 0$*.*

Fournodavlos-Rodnianski-Speck do assume that the initial data has to be CMC, but it expected that the CMC assumption should not be too hard to remove. We also mention the important related results by Beyer-Oliynyk in [1], where a *local gauge* is used, and the result by Fajman-Urban in [2], where simultaneous non-linear stability in *both the future and the past direction* is proven.

Remark 1. *One can interpret Theorem 1 as a formation of big bang singularity result, since it in particular says that any CMC initial data close to the induced initial data on any constant t-Cauchy hypersurface in* (M, g, ϕ) *produces a maximal global hyperbolic spacetime with a big bang singularity (in the sense of curvature blow-up).*

Our condition, presented below, has (the big bang formation interpretation of) Theorem 1 as a special case. However, the main point is that our condition *does not refer to any background solution to Einstein's equations.* In order to formulate our main theorem, we need to introduce four **expansion-normalized** quantities, closely related to the objects used by Ringström to formulate initial data at the singularity in [5]:

Definition 1. *Let* M *be a spacetime of dimension* $n + 1 \geq 3$*, let* $\Sigma \subset M$ *be a spacelike hypersurface and let* $\phi : M \to \mathbb{R}$ *be a smooth function. Let* (h, k) *be the first and second fundamental forms and let* $\phi_0 := \phi|_\Sigma$ *and* $\phi_1 := \nu(\phi)$*, where* ν *is the future pointing unit normal on* Σ*. We assume that the mean curvature is positive, i.e.* $\mathrm{tr}_h k > 0$*. The* expansion-normalized Weingarten map *is given by*

$$\mathcal{K}(X) := \frac{k(X, \cdot)^\sharp}{\mathrm{tr}_h k},$$

for any vector $X \in T\Sigma$. The expansion-normalized first fundamental form *is given by*

$$\mathcal{H}(X,Y) := h\left((\mathrm{tr}_h k)^{\mathcal{K}}(X), (\mathrm{tr}_h k)^{\mathcal{K}}(Y)\right),$$

for any vectors $X, Y \in T_p\Sigma$ and any $p \in \Sigma$. The expansion-normalized normal derivative of the scalar field *is given by*

$$\Phi_1 := \frac{\phi_1}{\mathrm{tr}_h k},$$

and the expansion-normalized induced scalar field *is given by*

$$\Phi_0 := \phi_0 + \Phi_1 \log(\mathrm{tr}_h k).$$

Example 2. *If (M, g, ϕ) is the generalized Kasner spacetime, introduced in Example 1, then the expansion-normalized quantities are easily verified to be*

$$\mathcal{K} = \sum_{i=1}^n p_i \partial_{x^i} \otimes \mathrm{d}x^i, \quad \mathcal{H} = \sum_{i=1}^n \left(\mathrm{d}x^i\right)^2,$$
$$\Phi_1 = a, \quad \Phi_0 = b$$

and the mean curvature is given by $\mathrm{tr}_h k = \frac{1}{t}$.

Note that the expansion-normalized quantities in the generalized Kasner spacetimes are in fact independent of time. Loosely speaking, we think in this talk of *quiescent* big bang singularities as those where we find a CMC foliation of Cauchy hypersurfaces up to the singularity for which the mean curvature goes to inifinity while the eigenvalues $p_1, \dots, p_n$ of $\mathcal{K}$ stay bounded (as is obviously the case in the generalized Kasner spacetimes). Our main theorem in [4] is the following:

Theorem 2 (Oude Groeniger - P. - Ringström '23). *Fix first a margin $\sigma > 0$ and integers $n \geq 2$,*

$$k_0 \geq \frac{n+1}{2},$$
$$k_1 \geq \frac{c_1 k_0 + c_2}{\sigma},$$

where c_1 and c_2 are some (explicitly computable) combinatorial constants independent of n and σ. Let Σ be a parallelizable closed manifold. For every $\zeta_0 > 0$, there is a $\zeta_1 > 0$, such that if $(\bar{h}, \bar{k}, \bar{\phi}_0, \bar{\phi}_1)$ are CMC-initial data on Σ to the Einstein-scalar field equations (1) *with expansion-normalized quantities satisfying*

- $\max_{i,j,k,i\neq j} (\bar{p}_i + \bar{p}_j - \bar{p}_k) < 1 - \sigma$,
- $||\bar{\mathcal{H}}^{-1}||_{C^0} + ||\bar{\mathcal{H}}||_{H^{k_1}} + ||\bar{\mathcal{K}}||_{H^{k_1}} + ||\bar{\Phi}_0||_{H^{k_1}} + ||\bar{\Phi}_1||_{H^{k_1}} < \zeta_0$,
- $|\bar{p}_i - \bar{p}_j| > \zeta_0^{-1}$ *for all $i \neq j$,*
- $\mathrm{tr}_{\bar{h}} \bar{k} > \zeta_1$,

then the corresponding maximal globally hyperbolic development with respect to the Einstein-scalar field equations (1) *has a big bang singularity towards the past (including curvature blow-up) and there is a foliation by CMC Cauchy hypersurfaces* *such that the mean curvature goes to infinity and the expansion-normalized quantities* $p_1, \ldots, p_n$ *(i.e. the eigenvalues of the expansion-normalized Weingarten map* $\mathcal{K}$ *on the leaf) and* Φ_0, Φ_1 *converge to limits*

$$p_i \to \mathring{p}_i, \quad \Phi_0 \to \mathring{\Phi}_0, \quad \Phi_1 \to \mathring{\Phi}_1$$

in C^{k_0} *towards the singularity.*

We refer to [4, Thm. 12] for a more precise version, including the possibility of a potential (in particular allowing for a cosmological constant). This result can then be used to prove stability of any solution to Einstein-scalar field equations with non-degenerate robust Ringström initial data on the singularity, see [4, Sec. 1.4]. In particular, we recover Theorem 1 as a special case. However, the main point of Theorem 2 is not to prove a stability result of a background solution, but to provide a general condition ensuring big bang formation without reference to a background solution.

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A definition of the mass aspect function for weakly regular asymptotically hyperbolic manifolds

Anna Sakovich
(joint work with Romain Gicquaud)

Let $(\mathbb{H}^n, b)$ denote the hyperbolic space of dimension $n \geq 3$, that is

$$\mathbb{H}^n = B_1(0), \quad b = \rho^{-2}\delta, \quad \rho = \frac{1-|x|^2}{2},$$

where $B_1(0)$ denotes the open unit ball in $\mathbb{R}^n$, δ is the Euclidean metric, and $|x| = \sqrt{(x^1)^2 + \ldots + (x^n)^2}$ where $x^1, \ldots, x^n$ are Cartesian coordinates on $\mathbb{R}^n$.

A complete Riemannian manifold (M, g) is said to be *asymptotically hyperbolic of order* $\tau > 0$ if there exist compact subsets $K \subset M$ and $K' \subset \mathbb{H}^n$, a diffeomorphism $\Phi : M \setminus K \to \mathbb{H}^n \setminus K'$ and a constant $C > 1$ such that $\frac{1}{C} b \le \Phi_* g \le C b$ and $e = \Phi_* g - b$ satisfies

$$\int_{\mathbb{H}^n \setminus K'} \rho^{-2\tau} \left(|e|_b^2 + |De|_b^2 \right) d\mu^b < \infty.$$

Φ is then called a *chart at infinity* for (M, g).

Assuming additionally that $g \in C^1$, and $\rho^{-1}\mathrm{Scal} \in L^1$, Chruściel and Herzlich [1] defined for asymptotically hyperbolic manifolds of order $\tau \ge 1/2$ a *mass functional*

$$p(e, V) = \lim_{r \to 1} \int_{S_r(0)} \left[V(\mathrm{div}(e) - d\,\mathrm{tr}(e)) + \mathrm{tr}(e) dV - e(DV, \cdot) \right] (\nu) d\mu^b \tag{1}$$

where ν is the outward pointing unit normal to $S_r(0) = \{|x| = r\} \hookrightarrow (\mathbb{H}^n, b)$ and

$$V \in \mathcal{N} := \mathrm{span} \left\{ \frac{1 - \rho}{\rho}, \frac{x^1}{\rho}, \dots, \frac{x^n}{\rho} \right\}. \tag{2}$$

This notion can be seen as the closest analogue of the notion of ADM mass for asymptotically Euclidean manifolds in the asymptotically hyperbolic setting, and Chruściel and Herzlich [1] showed that the limit (1) is well-defined under the above conditions. They also showed that the mass functional depends covariantly on the chart at infinity in the following sense: if Φ_i, $i = 1, 2$, are two asymptotically hyperbolic charts at infinity such that $e_i := (\Phi_i)_* g - b$ satisfy $|e_i| + |De_i| = o(\rho^{n/2})$ then there exists an isometry A of the hyperbolic space such that

$$p(e_2, V) = p(e_1, V \circ A) \text{ for any } V \in \mathcal{N}. \tag{3}$$

The mass functional of Chruściel and Herzlich [1] generalizes an earlier notion of mass for asymptotically hyperbolic manifolds given by Wang in [2]. This notion requires more stringent assumptions on geometry near infinity, namely that $e = \rho^n \overline{e} + o(\rho^n)$, where $\overline{e}$ is a smooth tensor on $B_1(0)$ which is transverse, meaning that its components satisfy $\sum_{j=1}^n e_{ij} x^j = 0$ for all $i, j = 1, \dots, n$. The components of Wang's *mass vector* p are then given by

$$p^0 = \int_{S_1(0)} m(x) d\mu^\sigma(x), \quad p^i = \int_{S_1(0)} x^i m(x) d\mu^\sigma(x), \quad i = 1, \dots, n,$$

where σ denotes the round metric on $S_1(0)$ and $m := \mathrm{tr}^\sigma \overline{e}$ is the so-called *mass aspect function*.

Note that both $\overline{e}$ and m have covariance properties under a change of chart at infinity, see Cortier, Dahl and Gicquaud [3], so the idea that the mass of an asymptotically hyperbolic manifold should be encoded by the mass aspect function is geometrically very appealing. However, when passing from $\overline{e}$ to m and then to p we drastically loose information about the asymptotic geometry of (M, g). In our upcoming work [4] we remedy this by showing that the mass aspect function admits an ADM-style definition similar to (1) that applies in very low regularity. To achieve this, we first show that the definition of mass functional (1) can be extended to encompass metrics with local regularity $L^\infty \cap W^{1,2}$ and with distributional scalar curvature as follows:

$$(4)\qquad p(e,V) = \lim_{k\to\infty} \int_{\mathbb{H}^n} \left[V(\mathrm{div}(e) - d\mathrm{tr}(e)) + \mathrm{tr}(e)dV - e(DV,\cdot)\right](-D\chi_k)d\mu^b.$$

Here $(\chi_k)_k$ is a sequence of compactly supported functions over $\mathbb{H}^n$ with bounded C^1-norm and such that the sets $\Omega_k = \chi_k^{-1}(1)$ is an increasing sequence of compact sets with $\mathbb{H}^n = \bigcup_k \Omega_k$. We observe that the limit (4) is well-defined not only for $V \in \mathcal{N}$ as in (1) but more generally for functions $V : \mathbb{H}^n \to \mathbb{R}$ satisfying

$$(5)\qquad \Delta V = nV, \qquad |\mathrm{Hess}V - Vb| = O(\rho)$$

provided that (M,g) is asymptotically hyperbolic of order $\tau > \frac{n-3}{2}$ which imposes a mild additional restriction in dimensions $n > 3$. Based on this observation, we prove that given $v \in C^2(S_1(0))$ there is a unique $V \in \mathbb{H}^n$ such that $\rho V \equiv v$ on $S_1(0)$ satisfying (5), and that the map $v \mapsto p(e,V)$ is continuous. This defines mass aspect function as a distribution on $S_1(0)$ and one can show that this notion agrees with the aforementioned definition of Wang whenever his more restrictive asymptotic assumptions are satisfied. Furthermore, by suitably adapting the argument used by Bartnik in [5] to show that the ADM mass is a coordinate invariant, we confirm that our definition (4) transforms covariantly under changes of the chart at infinity in the sense that (3) holds. For this, we additionally requre that $e \in W^{1,p}_\tau$ with $p > n$ and $\tau > 1$ such that $2 \le \tau + \frac{n-1}{p} < n$.

We also show that the integral at infinity (4) can be expressed in terms of the Ricci tensor of the metric, similar to Herzlich [6]. This approach has the merit of being more geometric, but nevertheless requires additional regularity assumptions.

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Lipschitz inextendibility of weak null singularities from curvature blow-up

Jan Sbierski

This talk presented forthcoming work by the author [14] which gives a criterion for the Lipschitz inextendibility of weak null singularities in terms of curvature blow-up. Since weak null singularities are expected to form in the interior of generic (vacuum) black holes, this result provides one of the ingredients needed for an eventual resolution of the $C^{0,1}_{\text{loc}}$-formulation of the strong cosmic censorship conjecture, which asserts that the maximal globally hyperbolic development (MGHD) of generic asymptotically flat initial data is $C^{0,1}_{\text{loc}}$-inextendible.[1]

Let us illustrate this by considering the future MGHD of generic one-ended asymptotically flat initial data which is sufficiently close to sub-extremal Kerr exterior initial data outside some compact set. Then by Kerr stability (cf. [4]) we expect to obtain region I in the Penrose diagram below and by [2] we expect to obtain a Cauchy horizon in the black hole interior II to which the metric extends continuously. Moreover, it is anticipated that this Cauchy horizon is weakly singular. This goes back to heuristics by Penrose [8], see also the recent work [12]. We call this piece of the boundary of the MGHD a weak null singularity. Nothing definite is known yet about the remaining structure of the boundary in the black hole interior in Figure 1.

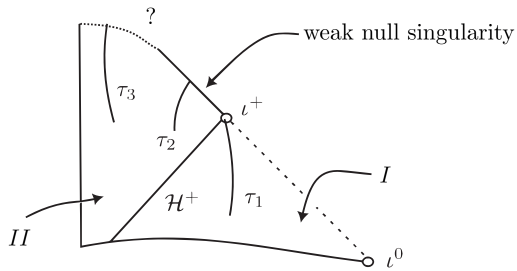

Figure 1. An example of a future MGHD

Now, in order to show that the future MGHD, let's call it (M, g), cannot be extended as a $C^{0,1}_{\text{loc}}$-regular Lorentzian manifold towards the future, it suffices to show that no future inextendible timelike geodesic τ in M can have a future limit point in a $C^{0,1}_{\text{loc}}$-extension of M, see [3], [10], [7]. This reduces the global problem of inextendibility to finding local obstructions in the vicinity of timelike geodesics. For example the timelike geodesic τ_1 will not be able to enter a $C^{0,1}_{\text{loc}}$-extension of M since it is already future complete ([3], [11]). The result presented here shows that if we assume that curvature blows up suitably at the weak null singularity then the timelike geodesic τ_2 will not be able to enter a $C^{0,1}_{\text{loc}}$-extension. In this way, by piecing together these local results, a global inextendibility result is obtained.

Before we state the main result, we summarise in some more detail what is already known about the structure of the weak null singularity. Going back to the interior of exact sub-extremal Kerr and using Boyer-Lindquist coordinates

[1]Recall that a $C^{0,1}_{\text{loc}}$-*extension* of a Lorentzian manifold (M, g) consists of a smooth isometric embedding $\iota : M \hookrightarrow \tilde{M}$ of M into another Lorentzian manifold $(\tilde{M}, \tilde{g})$ of the same dimension as $\tilde{M}$ with $\tilde{g}$ being $C^{0,1}_{\text{loc}}$-regular and such that $\partial\iota(M) \neq \emptyset$. If no such extension exists, then we say that (M, g) is $C^{0,1}_{\text{loc}}$-*inextendible*.

(t, r, θ, φ), it was shown in [9] how to construct a function $r^*(r, \theta)$ such that $u := \frac{1}{2}(r^* - t)$ and $\underline{\tilde{u}} := \frac{1}{2}(r^* + t)$ are null coordinates. We also introduce the Kruskal-like coordinate $\underline{u} := \frac{1}{2\kappa_-} e^{2\kappa_- \underline{\tilde{u}}}$ which is regular at the right Cauchy horizon. Here, $0 > \kappa_-$ is the surface gravity of the Cauchy horizon. Moreover, one can introduce new angular coordinates θ^A, $A = 1, 2$, such that the metric in $(u, \underline{u}, \theta^1, \theta^2)$ coordinates takes the form

$$g = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \gamma_{AB}(d\theta^A - b^A d\underline{u}) \otimes (d\theta^B - b^B d\underline{u}) \tag{1}$$

and extends smoothly to the right Cauchy horizon at $\{\underline{u} = 0\}$. The result of Dafermos and Luk in [2] can now be roughly stated as follows:

Theorem 1 (Dafermos-Luk '17 (rough version))**.** *Consider the hypersurface* $\Sigma = \{u + \underline{\tilde{u}} = \text{const}\}$ *with the induced exact sub-extremal Kerr initial data* $(\overline{g}_K, k_K)$*. Let* $(\overline{g}, k)$ *be initial data on* Σ *for the vacuum Einstein equations such that* $|\overline{g} - \overline{g}_K|$, $|k - k_K|$ *decay sufficiently fast for* $\underline{\tilde{u}} \to \infty$*. Then there exists a* $u_f \in \mathbb{R}$ *such that the future evolution exists in the gauge* (1) *in the shaded region in Figure 2 and extends continuously to* $\{\underline{u} = 0\}$*. Moreover, the metric remains* C^0*-close to that of exact sub-extremal Kerr.*

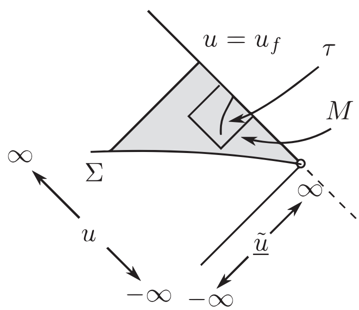

Figure 2. The spacetime constructed in [2].

The result in [2] only gives a stability result; no instability result for the vacuum Einstein equations is available yet that would ensure that $\{\underline{u} = 0\}$ is actually singular. For the linearised Einstein equations, in the form of the Teukolsky equation, instability results are given in [6] and [12]. The later result, in combination with [5], gives that for solutions of the linearised Einstein equations, arising from generic compactly supported initial data on a global Cauchy hypersurface for sub-extremal Kerr, we have

$$\int_{\{u=\text{const}\}\cap\{-\varepsilon_0 \le \underline{u} < 0\}} |\underline{u}|^3 (\log|\underline{u}|)^{14} |\overset{(1)}{R}(X_1, X_2, X_3, X_4)|^2 \,\mathrm{vol}_{\mathbb{S}^2}\, d\underline{u} = \infty \;, \tag{2}$$

where $\varepsilon_0 > 0$, $\overset{(1)}{R}$ denotes linearised curvature, and X_i, $i = 1, \ldots, 4$ are particular vector fields which extend continuously to the Cauchy horizon $\{\underline{u} = 0\}$.

In order to state the main theorem, given a metric of the form (1), we introduce the null vector fields $e_3 = -2\Omega^2 (d\underline{u})^\sharp = \frac{\partial}{\partial u}$ and $e_4 = -2(du)^\sharp = \frac{1}{\Omega^2}\big(\frac{\partial}{\partial \underline{u}} + b^A \frac{\partial}{\partial \theta^A}\big)$ which are normalised by $g(e_3, e_4) = -2$ and the angular vector fields $e_A = \frac{\partial}{\partial \theta^A}$. We define the connection coefficients $\underline{\chi}_{AB} = g(\nabla_{e_A} e_3, e_B)$, $\underline{\eta}_A = -\frac{1}{2} g(\nabla_{e_4} e_A, e_3)$, $\underline{\omega} = -\frac{1}{4} g(\nabla_{e_3} e_4, e_3)$ and χ, η, ω are defined analogously with e_3 and e_4 interchanged. Given now a timelike geodesic τ approaching the weak null singularity as in Figure 2, we phrase the theorem only in terms of a local piece (M, g) of the global spacetime as depicted in the above figure.

Theorem 2 (S. (forthcoming))**.** *Let $M = (-1,1) \times (-1,0) \times \mathbb{S}^2$ with $(u, \underline{u}, \theta^A)$ coordinates, g as in* (1) *which extends continuously to $\overline{M} = (-1,1) \times (-1,0], \times \mathbb{S}^2$ and assume*

$$
\begin{aligned}
&\sup_M \big(|\underline{\omega}| + |\eta|_\gamma + |\underline{\eta}|_\gamma + |\underline{\chi}|_\gamma\big) \leq C \\
&\int_{-1}^0 \Big(\sup_{(u,\theta)\in(-1,1)\times\mathbb{S}^2} |\chi|_\gamma(u, \underline{u}, \theta)\Big)\, d\underline{u} \leq C \\
&\int_{-1}^0 \Big(\sup_{(u,\theta)\in(-1,1)\times\mathbb{S}^2} |\partial_{\theta^B} b^A|(u, \underline{u}, \theta)\Big)\, d\underline{u} \leq C\ .
\end{aligned}
\tag{3}
$$

Suppose for any $\overline{p} \in \partial\overline{M}$ and any neighbourhood $\overline{W}$ of $\overline{p}$, there exists $\overline{q} \in \overline{W} \cap \partial\overline{M}$ and a compact neighbourhood $\overline{V} \subseteq \overline{W}$ of $\overline{q}$ and continuous vector fields $\overline{X}_i$ on $\overline{V}$, $i = 1,2,3,4$ and $\varepsilon > 0$ such that for any continuous vector fields X_i on $\overline{V}$, $i = 1,2,3,4$, with $||\overline{X}_i^\mu - X_i^\mu||_{\mathbb{R}^4} < \varepsilon$ we have

$$
\Big| \int_{\overline{V}\cap\{\underline{u}\leq\underline{u}_k\}} R(X_1, X_2, X_3, X_4)\mathrm{vol}_g \Big| \to \infty
\tag{4}
$$

along a sequence $\underline{u}_k \nearrow 0$.[2] *Then there is no $C^{0,1}_{\mathrm{loc}}$-extension $\iota : M \hookrightarrow \tilde{M}$ and a future directed timelike geodesic $\tau : [-1,0) \to M$ with $\lim_{s\to 0} \tau^{\underline{u}}(s) = 0$, $\lim_{s\to 0} \tau^u(s) < 1$, such that $\lim_{s\to 0} (\iota \circ \tau)(s)$ exists in $\tilde{M}$.*

Let us make a few remarks: i) the first two conditions on the connection coefficients in (3) are proven in [2]. ii) While the analogue of (4) for linearised curvature is not implied by (2), it is not difficult to slightly strengthen the result in [12] such that it is. This is unpublished work by the author and may appear as part of forthcoming work. iii) The advantage of the curvature-based approach presented here compared to the holonomy-based approach to Lipschitz inextendibility from [11] is that here we only need lower bounds on curvature and not on the connection

[2] To be quite precise, we assume that the third condition in (3) as well as the closeness condition of the coordinate components of $\overline{X}_i^\mu - X_i^\mu$ hold restricted to two coordinate charts which cover $\mathbb{S}^2$ and, moreover, that $\overline{V} \cap \{\underline{u} \leq \underline{u}'\}$ is also a smooth manifold with corners for $\underline{u}' \leq 0$ close enough to 0.

coefficients and also that we only need integrated lower bounds and not pointwise bounds. As a consequence, the assumptions presented here are easier to establish analytically.

The proof is by contradiction and proceeds in three steps: assume that there is a $C^{0,1}_{\mathrm{loc}}$-extension $\iota : M \hookrightarrow \tilde{M}$ together with a timelike geodesic τ as in the statement of the theorem. In the first step one shows that one can go over from the timelike geodesic τ that leaves M to a null geodesic σ, an integral curve of e_4, that also approaches $\{\underline{u} = 0\}$ and leaves M for $\tilde{M}$. This is done by verifying the assumptions of Proposition 5.1 in [13]. The second step is crucial: one shows that the C^1-structures of the two extensions $\tilde{M}$ and $\overline{M}$ are the same at the boundary. This is achieved by verifying the assumptions of Proposition 5.11 in [13]. Now, in the final step, one uses the equivalence of the C^1-structures of the extensions to push forward the vector fields $\overline{X}_i$ to $\tilde{M}$ and to smooth them out with respect to the smooth structure of $\tilde{M}$ to obtain vector fields X_i. Since the C^1-structures are equivalent, this can be done in such a way that with respect to the coordinates on $\overline{M}$, the components of X_i are arbitrarily close to those of $\overline{X}_i$. Then starting with the assumption (4) and pushing it forward via ι to $\tilde{M}$ one can do an integration by parts and use Stokes' theorem to obtain an expression which only contains first covariant derivatives of the smoothed out vector fields which are all uniformly bounded and which then concludes the proof.

As a final remark we point out that one can also write down a blow-up condition on curvature, analogous to (4), which ensures that if the C^1-structure of an extension $\iota : M \hookrightarrow \tilde{M}$ is the same as that of $\overline{M}$ at the boundary, that then $\tilde{M}$ cannot have locally square integrable Christoffel symbols. However, as is shown in [1], if $\iota : M \hookrightarrow \tilde{M}$ is only a continuous extension (i.e. not locally Lipschitz), then there is no rigidity of the C^1-structure. See [14] for more details.

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Polynomial decay for the Klein-Gordon equation on the Schwarzschild spacetime

Yakov Shlapentokh-Rothman
(joint work with Maxime Van de Moortel, Fedrico Pasqualotto)

In this talk we reported on the two works [1] and [2] which concern the long time behavior of solutions to the Klein–Gordon equation on the Reissner–Nordström spacetime. For the sake of exposition, we shall restrict attention in this abstract to the Schwarzschild spacetime.

In the work [1] we studied solutions ψ to the Klein–Gordon equation on the Schwarzschild spacetime which arise from compactly supported initial data and which are also supported on a single spherical harmonic. Letting (t^*, r, θ, ϕ) denote ingoing Eddington–Finklestein coordinates on Schwarzschild and $\tilde{\psi}(t^*, r)$ the projection of ψ onto the spherical harmonic, the main result then states that there exists a function $u(r)$ (depending only on the spherical harmonic) so that when r ranges over a compact set,

$$\tilde{\psi}\,(t^*, r) \sim u(r)(t^*)^{-5/6} f(t^*). \tag{1}$$

Here $f(t^*)$ satisfies $|f| \leq C$ and is a certain explicit oscillating function

In the work [2] we studied the long time dynamics of solutions ψ to the Klein–Gordon equation on the Schwarzschild spacetime without any restriction on the spherical harmonic support of ψ. Assuming that the initial data for ψ is compactly supported, we show that there exists $\delta \in (0, 1/23)$ so that when r ranges over a compact set,

$$|\psi| \leq C(t^*)^{-5/6+\delta}, \tag{2}$$

for a constant C which depends only on the initial data. The proof of (2) requires us to estimate a certain exponential sum; assuming that the famous "exponent pair conjecture" holds, we would obtain a sharp bound on this exponential sum and would be able to show that for every $\epsilon > 0$,

$$|\psi| \leq C_\epsilon (t^*)^{-5/6+\epsilon}. \tag{3}$$

In view of (1), we see that this estimate is essentially sharp.

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The Maxwell equations on the full $|a| \leq M$ Kerr family of black hole spacetimes

Rita Teixeira da Costa
(joint work with Gabriele Benomio)

The Kerr family of black hole spacetimes is a family of $(1+3)$ dimensional Lorentzian manifolds $(\mathcal{M}, g)$ satisfying the vacuum Einstein equations

$$\mathrm{Ric}(g) = 0, \tag{1}$$

which is described by two parameters, $M > 0$ and $|a| \leq M$. For instance, in Boyer–Lindquist coordinates $(t, r, \theta, \phi) \in \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$, we have

$$g = -\frac{\Delta}{\rho^2}(\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi)^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)^2,$$
$$\Delta \doteq (r-r_+)(r-r_-), \quad r_\pm \doteq M \pm \sqrt{M^2-a^2}; \qquad \rho^2 \doteq r^2 + a^2\cos^2\theta.$$

The Kerr family has a fundamental role in the theory of General Relativity, as it is thought to describe the endstate of generic gravitational collapse under (1).

The goal of the talk is to describe the propagation of electromagnetic waves on Kerr black holes, both in the subextremal $|a| < M$ range and in the extremal case $|a| = M$. Concretely we investigate the boundedness and decay properties of the Maxwell equations

$$\mathrm{d}\,\boldsymbol{F} = 0, \quad \mathrm{d} \star \boldsymbol{F} = 0, \tag{2}$$

where $\boldsymbol{F}$ is an antisymmetric 2-tensor on $(\mathcal{M}, g)$, given initial data on a suitable spacelike hypersurface in $(\mathcal{M}, g)$. To do so, it is convenient to choose a null frame (e_1, e_2, e_3, e_4) in which to decompose $\boldsymbol{F}$ and, therefore, (2). Following [5, 6], we choose (e_3, e_4) to be the principal null vectors of Kerr and take (e_1, e_2) to be any frame on the horizontal distribution $\mathfrak{D} \doteq \mathrm{span}\{e_3, e_4\}^\perp$. We set[1]

$$\boldsymbol{\alpha}(e_A) \doteq \boldsymbol{F}(e_A, e_4), \qquad \underline{\boldsymbol{\alpha}}(e_A) \doteq \boldsymbol{F}(e_A, e_3),$$
$$\boldsymbol{\rho} \doteq \frac{1}{2}\boldsymbol{F}(e_3, e_4), \qquad \boldsymbol{\sigma} \doteq \frac{1}{2}\not{\boldsymbol{\in}}^{\boldsymbol{AB}}\boldsymbol{F}(e_A, e_B).$$

With respect to $(\boldsymbol{\alpha}, \underline{\boldsymbol{\alpha}}, \boldsymbol{\rho}, \boldsymbol{\sigma})$, the Maxwell equations (2) can be rewritten as a system of first order partial differential equations. One can also derive, from this

[1] In this geometric language, the well-known Newman–Penrose formalism [19] for Maxwell is obtained by making a particular choice of (e_1, e_2) and then introducing three complex scalars from particular combinations of the six real functions $\boldsymbol{\alpha}(e_1)$, $\boldsymbol{\alpha}(e_2)$, $\underline{\boldsymbol{\alpha}}(e_1)$, $\underline{\boldsymbol{\alpha}}(e_1)$, $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$.

(first order) system, wave-type (second order) equations for the so-called extremal Maxwell components $\boldsymbol{\alpha}$ and $\underline{\boldsymbol{\alpha}}$, cf. [23], which are completely decoupled from each other and from the rest of the system. These independent equations, called Teukolsky equations, were studied recently in [21, 22], where it was shown:

Theorem 1 (Extremal components, $|a| < M$). *Fix $M > 0$, and let $|a| < M$. Solutions $\boldsymbol{\alpha}$ and $\underline{\boldsymbol{\alpha}}$ to the Teukolsky equations on $(\mathcal{M}, g)$ arising from suitably regular initial data satisfy energy boundedness (without derivative loss) and energy decay estimates, with explicit constants depending only on the initial data and the black hole parameters (a, M). These estimates imply the pointwise decay of $\boldsymbol{\alpha}$ and $\underline{\boldsymbol{\alpha}}$ (at a sufficiently fast inverse polynomial rate) on a suitable folliation of the Kerr black hole exterior, including the future event horizon $\mathcal{H}^+$.*

We direct the reader to [17, 18] for sharp pointwise decay results.

In the case $|a| = M$, based on the heuristic works [8, 12] and the recent proof of scalar azimuthal instabilities by Gajic [10], one may conjecture:

Conjecture 1 (Extremal components, $|a| \leq M$). *Fix $M > 0$ and let $|a| \leq M$. Solutions $\boldsymbol{\alpha}$ and $\underline{\boldsymbol{\alpha}}$ to the Teukolsky equations on $(\mathcal{M}, g)$ arising from suitably regular initial data supported on a fixed azimuthal mode $m \in \mathbb{Z}$ satisfy energy boundedness and energy decay estimates, with explicit constants depending only on m, the initial data, and the black hole parameter M. These estimates are consistent with the pointwise decay of $\boldsymbol{\alpha}$ and $\underline{\boldsymbol{\alpha}}$ on a suitable foliation of the Kerr black hole exterior away from future event horizon $\mathcal{H}^+$, decay of $\boldsymbol{\alpha}$ along $\mathcal{H}^+$, and generic (in m) **growth** of $\underline{\boldsymbol{\alpha}}$ along $\mathcal{H}^+$.*

We note that, in the non-generic case $m = 0$, $\underline{\boldsymbol{\alpha}}$ is expected to decay along $\mathcal{H}^+$, though its transverse derivatives to $\mathcal{H}^+$ are not, see [3, 4, 15].

The main result presented in the talk states that, starting from the above theorem and conjecture, one can control the remaining Maxwell components $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$):

Theorem 2 (Remaining components). *Fix $M > 0$. Consider solutions $(\boldsymbol{\alpha}, \underline{\boldsymbol{\alpha}}, \boldsymbol{\rho}, \boldsymbol{\sigma})$ to the Maxwell equations (2) on $(\mathcal{M}, g)$ arising from suitably regular initial data. Assume that either*

- (i) *$|a| < M$, or*
- (ii) *$|a| \leq M$, the initial data is supported on a fixed azimuthal mode $m \in \mathbb{Z}$, and Conjecture 1 holds.*

Then, $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ satisfy energy boundedness (without derivative loss) and, after subtracting a teleologically-determined stationary solution $(\boldsymbol{\alpha}_{\text{stat}} = 0, \underline{\boldsymbol{\alpha}}_{\text{stat}} = 0, \boldsymbol{\rho}_{\text{stat}}, \boldsymbol{\sigma}_{\text{stat}})$, to (2), energy decay estimates, with explicit constants depending only the initial data and the black hole parameters (a, M) in case (i), and m, the initial data and the black hole parameter M in case (ii). These estimates imply the pointwise decay of $\boldsymbol{\rho} - \boldsymbol{\rho}_{\text{stat}}$ and $\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\text{stat}}$ (at a sufficiently fast inverse polynomial rate) on a suitable folliation of the Kerr black hole exterior, including the future event horizon $\mathcal{H}^+$.

We emphasize that decay along $\mathcal{H}^+$ holds in spite of the azimuthal instabilities allowed by Conjecture 1 and conjectured in the aforementioned works [8, 12].

In the subextremal $|a| < M$ setting, where our Theorem 2 is unconditional, combining it with Theorem 1 yields a full energy boundedness (without derivative loss) and decay result for the Maxwell equations (2), cf. [1, 7, 16, 20]. Given the similarities between the Maxwell equations and the linearized gravity equations on Kerr spacetimes, we view this result as a toy problem to understand the orbital and asymptotic stability of the subextremal $|a| < M$ Kerr subfamily to (linearized) gravitational perturbations, cf. other works on this topic [2, 9, 11, 13, 14].

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Extremal critical collapse

Ryan Unger
(joint work with Christoph Kehle)

One of the most spectacular predictions of general relativity is the existence and formation of black holes. Solutions of the Einstein field equations,

$$\mathrm{Ric}(g) - \tfrac{1}{2}R(g)g = 2\mathbf{T},$$

can undergo *gravitational collapse* to form a black hole dynamically, starting from regular, one-ended Cauchy data. In contrast, for reasonable matter models, solutions with "small" initial data *disperse* without a black hole forming (i.e., Minkowski space is stable). It is a fundamental problem in classical general relativity to understand how these different classes of spacetimes—collapsing and dispersing—fit together in the moduli space of solutions. The interface between collapse and dispersion is known as the *black hole formation threshold* and families of solutions crossing this threshold are said to exhibit *critical collapse*.

Critical collapse has been extensively studied numerically (see the survey [3]), starting with the influential work of Choptuik [2] on the spherically symmetric Einstein-scalar field model, in a regime where the critical solutions are believed to be naked singularities. The regimes of the black hole formation threshold that have been numerically studied so far remain out of reach of mathematical techniques.

At first glance, the *Reissner–Nordström* family of metrics (indexed by the mass $M > 0$ and charge e) appears to exhibit a type of critical behavior: the solution contains a black hole when $|e| < M$ (*subextremal*) or $|e| = M$ (*extremal*) and does not contain a black hole when $|e| > M$ (*superextremal*). However, the Reissner–Nordström black holes are eternal and arise from two-ended Cauchy data, while the superextremal variants contain an eternal "naked singularity" that has historically caused much confusion. Moreover, it was long thought that extremal black holes could not form dynamically, a consideration closely related to the recently disproved *third law of black hole thermodynamics* [1, 4, 5, 9].

In my talk, I discussed recent work with C. Kehle in which we show that this formal critical behavior gives rise to genuine examples of critical behavior in gravitational collapse, which is a new phenomenon that we call *extremal critical collapse*.

Theorem. *There exist smooth one-parameter families of spherically symmetric Cauchy data for the Einstein–Maxwell–charged Vlasov system on* $\mathbb{R}^3$ *such that the resulting maximal developments* $\{\mathcal{D}_\lambda\}_{\lambda\in[0,1]}$ *have the following properties:*

(1) $\mathcal{D}_0$ *is Minkowski space and there exists* $\lambda_* \in (0,1)$ *such that for* $\lambda < \lambda_*$, $\mathcal{D}_\lambda$ *is future causally geodesically complete and disperses towards Minkowski space. No black hole or naked singularity forms.*

(2) *If* $\lambda = \lambda_*$, *an* <u>*extremal*</u> *Reissner–Nordström black hole forms. The spacetime contains no trapped surfaces.*

(3) *If* $\lambda > \lambda_*$, *a* <u>*subextremal*</u> *Reissner–Nordström black hole forms. The spacetime contains an open set of trapped surfaces.*

In addition, for every $\lambda \in [0,1]$, $\mathcal{D}_\lambda$ *is past causally geodesically complete and is isometric to Minkowski space near the center* $r = 0$ *for all time.*

As a direct consequence, we obtain:

Corollary. *The very "black hole-ness" of an extremal black hole arising in gravitational collapse can be unstable: There exist one-ended asymptotically flat Cauchy data for the Einstein–Maxwell–Vlasov system, leading to the formation of an extremal black hole, such that an arbitrarily small smooth perturbation of the data leads to a future causally geodesically complete, dispersive spacetime.*

This is in stark contrast to the subextremal case, where formation of trapped surfaces behind the event horizon—and hence *stable* geodesic incompleteness [8]—is expected. Despite this inherent instability of the critical solution, we expect extremal critical collapse itself to be a stable phenomenon: We conjecture that there exists a teleologically determined "hypersurface" in moduli space which consists of asymptotically extremal black holes, contains $\mathcal{D}_{\lambda_*}$, and locally delimits the boundary in moduli space between future complete and collapsing spacetimes.

In my talk, I outlined the proof of our theorem, which has two main steps:

(1) Construct examples of extremal critical collapse in Ori's "charged null dust" model [7] which consists of an ingoing radial charged null dust "beam" glued to an outgoing radial charged null dust beam along a spacelike "bounce" hypersurface. This model is quite singular but the solutions are more or less explicit.

(2) Show that Ori's model arises as a certain hydrodynamic radial limit of Einstein–Maxwell–Vlasov and therefore the examples in Ori's model can be desingularized to give examples for Einstein–Maxwell–Vlasov. These spacetimes contain Vlasov matter which is smoothly "turning around" near the old dust bounce hypersurface.

To solve step (1), we found a new way to parametrize solutions of Ori's model via the geometry of the bounce hypersurface itself, which lets us control which Reissner–Nordström solution is formed and exactly where the beam bounces. The proof of step (2) involves a careful analysis of the electromagnetic geodesic flow near a bounce in order to control the energy-momentum tensor and particle current. As this hydrodynamic limit is very singular (the fluid density ρ in Ori's model is not bounded at the bounce), the analysis of the phase space volume necessitates introducing several scales and an "auxiliary beam" which bounces due to its angular momentum. This analysis uses the structure and monotonicities of the spherically symmetric Einstein equations in several crucial ways.

The charged Vlasov model seems to be the simplest setting of an impeccable matter model where extremal critical collapse can be mathematically observed, and so our result gives the first rigorous example of a critical solution in general relativity. It would be very interesting to understand if extremal critical collapse can occur for extremal Kerr black holes, possibly already in vacuum.

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Waves on Kerr–de Sitter space

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(joint work with Peter Hintz, Oliver Petersen)

In Einstein's theory of General Relativity, a vacuum spacetime with cosmological constant $\Lambda \in \mathbb{R}$ is a $(1+3)$-dimensional manifold M equipped with a Lorentzian metric g satisfying the Einstein vacuum equation

$$\text{(EVE)} \qquad \mathrm{Ric}(g) + \Lambda g = 0.$$

There are some particular 'simple' solutions such as de Sitter or Kerr-de Sitter (KdS) spacetimes, in these cases with $\Lambda > 0$. A fundamental question is whether (EVE) can be solved globally if we perturb the initial conditions of this spacetime and whether the global solution can be described asymptotically. In order to even make sense of this question, recall that after suitable modifications, namely gauge fixing, EVE is a tensorial wave equation.

This talk focuses on $\Lambda > 0$. Much recent progress has been made on $\Lambda = 0$, see especially the work of Klainerman–Szeftel, Giorgi, Shen, [KS23] and references therein, and Dafermos–Holzegel–Rodnianski–Taylor [DHRT21], but we cannot detail this due to the limitations of this abstract. In $\Lambda > 0$ the classical result is de Sitter stability, due to Friedrich [Fri86] in $3+1$ dimensions, in the mid-80's; later Anderson [And05] extended this to even dimensions, Ringström [Rin08] gave an even more general and furthermore localized treatment.

The gauge issue arises as EVE is diffeomorphism invariant: if g solves it, so does $\Psi^* g$, Ψ a diffeomorphism, hence one has infinite dimensional non-uniqueness. While this may not sound worrisome for existence, since the general existence theory for linearizations is via duality, and the adjoint carries the infinitesimal version of this invariance, there are indeed problems. (Cf. linear algebra: surjectivity implies that the adjoint is injective.) The *gauge conditions* used here are the harmonic/wave/DeTurck's gauge: one fixes a background metric $\mathfrak{g}_0$, and requires that the identity map $(M, g) \to (M, \mathfrak{g}_0)$ be a wave map (solve a wave equation).

The *implementation of gauge fixing* for EVE is to solve

$$\text{(GE)} \qquad \mathrm{Ric}(g) + \Lambda g - \Phi(g, \mathfrak{g}_0) = 0, \text{ where}$$

$$\Phi(g, \mathfrak{g}_0) = \delta_g^* \Upsilon(g, \mathfrak{g}_0), \ \Upsilon(g, \mathfrak{g}_0) = g \mathfrak{g}_0^{-1} \delta_g G_g \mathfrak{g}_0;$$

here δ_g is the (negative) divergence (adjoint of the symmetric gradient δ_g^*), and $G_g r = r - \frac{1}{2}(\mathrm{tr}_g\, r) g$. Note that $(\mathfrak{g}_0^{-1} \delta_g G_g \mathfrak{g}_0)^k = g^{ij}(\Gamma_{ij}^k - {}^{\mathfrak{g}_0}\Gamma_{ij}^k)$. To relate to EVE, apply $\delta_g G_g$ to (GE), use the 2nd Bianchi identity: $\delta_g G_g \mathrm{Ric}(g) = 0$ for all g, obtain an equation for $\Box^{\mathrm{CP}} = 2\delta_g G_g \delta_g^*$, a one-form wave operator, and use vanishing Cauchy data (needs arrangement from the geometric data using the constraint equations) to conclude $\Upsilon = 0$. Using this Choquet-Bruhat [CB52] proved local well-posedness. There are global issues though which often require first a modification of the implementation, namely δ_g^*, which is done via constraint damping, a zeroth order modification of δ_g^* to $\widetilde{\delta}^*$, and secondly of the gauge, which for Kerr-de Sitter space is done via fixing a suitable finite dimensional space Θ of smooth compactly supported one forms and requiring $\Upsilon(g, \mathfrak{g}_0) = \theta \in \Theta$. This is achieved by a global nonlinear iteration that finds both g and θ.

We now describe global results for KdS, which are black holes in de Sitter space. For this $\Sigma_0 = \{t_* = 0\}$ is the initial Cauchy surface, $g_{m,a}$ a KdS metric on

$$\Omega = [0, \infty)_{t_*} \times [r_e - \delta, r_c + \delta]_r \times \mathbb{S}^2, \ r = r_e, r_c \text{ horizons};$$

$m > 0$ is the mass, $a \in \mathbb{R}$ the angular momentum of the metric. The 'black hole' nature corresponds to the presence of certain null-hypersurfaces (horizons), lying at certain values of r. These are given by roots of a quartic polynomial

$$\mu(r) := \left(r^2 + a^2\right)\left(1 - \frac{\Lambda r^2}{3}\right) - 2mr.$$

The metric $g_{m,a}$ is *subextremal* if μ has four distinct real roots $r_- < r_C < r_e < r_c$, with the latter two giving the event and cosmological horizons respectively; this is equivalent to a discriminant condition that can be expressed in terms the dimensionless quantities Λm^2 and $\frac{a}{m}$. Here in terms of the well-known Boyer-Lindquist (B-L) coordinates (t, r, ϕ, θ), valid in $(r_e, r_c)_r$ (away from the poles of the sphere), $t_* = t - \Phi(r)$, $\phi_* = \phi - \Psi(r)$, where Φ and Ψ remove the apparent coordinate singularity of the B-L form of the metric: $\Phi' = b\frac{r^2+a^2}{\mu(r)}f(r)$, $\Psi' = b\frac{a}{\mu(r)}f(r)$, f smooth, $f(r_e) = -1$, $f(r_c) = 1$, and f is such that dt_* is timelike, so the constant t_* hypersurfaces are spacelike.

In the slowly rotating case, a few years ago, Hintz and the author [HV18] proved stability of the KdS family. Namely for initial data on Σ_0, satisfying the constraint equations (which are easy to satisfy in black hole spacetimes) close, in a high regularity norm, to the data of g_{m_0,a_0}, with $\frac{|a_0|}{m_0}$ small, [HV18] showed that there is a global solution g decaying exponentially, in t_*, to a nearby member of the KdS family. During the workshop Fournodavlos-Schlue [FS24] posted a paper showing that this implies stability of the cosmological region, $r > r_c$, extending the stability picture.

The purpose of this talk was to explain recent advances towards extending this to the full subextremal range of KdS parameters. For orientation, recall the analytic framework of [HV18], itself based on an extended version of the smooth linear one introduced in [Vas13]. The framework of [HV18] combines non-elliptic linear global analysis with coefficients of finite Sobolev regularity and a simple global Nash-Moser iteration to deal with the loss of derivatives corresponding to non-ellipticity and trapping (there are alternatives: Fang [Fan21]), to yield global solvability for quasilinear wave equations like (GE) provided two conditions hold. First, certain dynamical assumptions are satisfied, namely the only trapping is normally hyperbolic trapping, with an appropriate subprincipal symbol condition. Second, there are no exponentially growing modes (with a precise condition on non-decaying ones), i.e. non-trivial solutions of the linearized equation at g_{m_0,a_0} of the form $e^{-i\sigma t_*}$ times a function of the "spatial" (radial and spherical) variables r, ω only, with $\operatorname{Im}\sigma > 0$. While the nonlinear and finite Sobolev regularity coefficients require a careful treatment, this is relatively straightforward at the current stage of analytic developments, so we focus on the linear analysis with smooth coefficients.

First, for the *dynamical assumptions* of [Vas13], recently Petersen and the author proved in [PV24b] that these are indeed *satisfied in the full subextremal range. Second,* we would like to make sense of "there are no exponentially growing modes (with a precise condition on non-decaying ones)". This depends on the choice of a(n asymptotic) Killing vector field T and the quasinormal modes (QNM) are (distributional) eigenfunctions of T, i.e. $Tu = -i\sigma u$, as well as elements of $\operatorname{Ker} P$. We take $T = \partial_{t_*} + \frac{a}{r_0^2+a^2}\partial_{\phi_*}$, where $r_0 \in [r_e, r_c]$ is arbitrarily chosen, which is different from $T = \partial_{t_*}$ chosen in [HV18], though for a small (the case there) they are close. The main result of [PV24a] is that for T as above, the set of quasinormal frequencies is discrete and the space of quasinormal modes for each is finite dimensional. Moreover, *for sufficiently small $\epsilon > 0$, and for forcing decaying (in a Sobolev sense) as $e^{-\epsilon t_*}$, the solutions of the forced wave equations have a finite asymptotic expansion in quasinormal modes.* This shows that the only obstacle to exponential decay of solutions of wave equations is QNMs. For the non-linear problem the latter is an issue; one would want solvability in decaying spaces hence the need for gauge modifications, which we briefly discuss. This *completes the analytic discussion in the full subextremal range, modulo the mode analysis,* which is an important *open problem.* (Cf. recent work of Casals and Texeira da Costa [CTdC22], as well as of Hintz [Hin24]!) In particular, Hintz has shown mode stability in the scalar setting (with a tensorial version expected) in the subextremal 'physical regime', i.e. when Λm^2 is small.

Unfortunately, in the harmonic/wave/DeTurck gauge, while the dynamical assumptions are satisfied, there *are* growing modes, although only a finite dimensional space of these. The key to proving stability (given the analytic background) is to overcome this issue. One might then expect that the other non-decaying (including growing!) modes come from the diffeomorphism invariance, i.e. gauge issues, (plus KdS parameters) but this is not true at this stage. Constraint damping (Gundlach et al [GCHMG05], Pretorius [Pre05]... see also Ringström [Rin08]) modifies δ_g^* by a 0th order term in Φ, as already mentioned, so that $\Box^{\mathrm{CP}}$ *only has decaying mode solutions. Thanks to this* the a priori non-geometric QNMs of gauge-fixed Einstein become geometric modes of Einstein, and thus there is a chance to show that they are pure gauge or infinitesimal change of KdS family parameters, which is the form of *mode stability in this (ungauged Einstein) case.* Constraint damping depends on the choice of a certain timelike one-form $\mathfrak{c}$ as well as a large parameter γ. Small a of [HV18] uses $\mathfrak{c} = dt_*$ with a particular choice of t_* based on the $a = 0$ case. The last, in progress, development reported here is on joint work with Hintz and Petersen: *in the full subextremal range constraint damping can be arranged* using $\mathfrak{c}$ that is precisely tuned to the geometry of subextremal KdS spacetimes. The proof of the theorem still uses semiclassical analysis in γ^{-1}, just as [HV18]. As a corollary:

Theorem 1 (Hintz-Petersen-V., in progress, hopefully '24)**.** *If ungauged Einstein mode stability holds for any subextremal KdS parameter, then nonlinear stability also holds nearby.*

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A geometric choice of asymptotically Euclidean coordinates via STCMC-foliations

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(joint work with Annachiara Piubello)

Asymptotically Euclidean 3-dimensional initial data sets were shown to carry asymptotic foliations of closed hypersurfaces with constant spacetime mean curvature (Cederbaum-Sakovich, 2021). In order to prove the inverse implication of this result and hence the geometric characterization of being asymptotically Euclidean, we start from the purely geometric foliation and construct asymptotic coordinates from it, exploiting the properties of the induced Laplacian of the foliation leaves via a delicate analysis. We show that these coordinates are asymptotically Euclidean, and moreover seem well-adapted to the center of mass. This is joint work with A. Piubello.

Foliations of asymptotically Schwarzschildean lightcones by surfaces of constant spacetime mean curvature

Markus Wolff
(joint work with Klaus Kröncke)

Consider a codimension-2 surface Σ in an ambient 4-dimensional spacetime $(\overline{M}, \overline{g})$ with codimension-2 mean curvature vector $\vec{\mathcal{H}}$. The *spacetime mean curvature* $\mathcal{H}^2$ of Σ is defined as the Lorentzian length of the mean curvature vector $\vec{\mathcal{H}}$, i.e.,

$$\mathcal{H}^2 := \overline{g}\left(\vec{\mathcal{H}}, \vec{\mathcal{H}}\right),$$

and we say Σ is a *surface of constant spacetime mean curvature* (STCMC) if $\mathcal{H}^2$ is constant along Σ.

In the case that Σ is contained in an initial data set (M, g, K), i.e., a spacelike hypersurface (M, g) with second fundamental form K with respect to a future timelike unit normal $\vec{n}$, we have

$$\mathcal{H}^2 = H^2 - (\operatorname{tr}_\Sigma K)^2,$$

where H denotes the mean curvature of Σ in (M, g).

Theorem 1 (Cederbaum–Sakovich '21 [1]). *If (M, g, K) is an asymptotically flat initial data set with $E_{ADM} \neq 0$, there exists an asymptotic foliation of STCMC surfaces.*

The above foliation is unique within a suitable a-priori class of surfaces and therefore allows to define a notion of center of mass in general relativity. Theorem 1 reflects the state of the art and is preceeded by decades of various contributions, in particular in time symmetry ($K \equiv 0$), i.e., in the CMC case, see e.g. Ye [13], Metzger [8], Huang [5] and many more. In particular, Huisken–Yau [6] construct an asymptotic foliation of CMC surfaces using volume preserving mean curvature flow.

Motivated by both the work of Cederbaum–Sakovich and Huisken–Yau, we employ a geometric flow to prove a corresponding statement in the null case:

Theorem 2 (Kröncke–W. (in preparation)).
Let $\mathcal{N}$ be an asymptotically Schwarzschildean lightcone ($m > 0$). Given suitable initial data, the solution to area preserving null mean curvature flow (APNMCF) exists for all times and converges to an STCMC surface. Moreover, the limiting STCMC surfaces form an asymptotic foliation of $\mathcal{N}$.

Similar to the work of Huiken–Yau [6], we assume that $\mathcal{N}$ is close to the Schwarzschild lightcone up to 4-th order. Hence, we assume much stronger decay assumptions compared to an asymptotically flat null hypersurface, cf. Mars–Soria [7]. Recall that a null hypersurface $\mathcal{N}$ is ruled by affine null geodesics that are the integral curves of a choice of null generator $\underline{L}$, which is a tangent null vector field of $\mathcal{N}$ that is normal to all tangent directions. In particular, $\underline{L}$ is perpendicular to any spacelike cross section Σ of $\mathcal{N}$ ($\Sigma \subseteq \mathcal{N}$ is spacelike and intersects any integral curve of $\underline{L}$ exactly once), and there exists a unique null vector field L normal to Σ

such that $\overline{g}(\underline{L}, L) = 2$. Then, the *null second fundamental forms* of Σ are defined as

$$\underline{\chi}(V,W) := -g\left(\overline{\nabla}_V W, \underline{L}\right), \qquad \chi(V,W) := -g\left(\overline{\nabla}_V W, L\right),$$

for $X, Y \in \Gamma(T\Sigma)$. Note that the mean curvature vector $\vec{\mathcal{H}}$ satisfies

$$\vec{\mathcal{H}} = -\frac{1}{2}\theta\underline{L} - \frac{1}{2}\underline{\theta}L,$$

where $\underline{\theta} := \operatorname{tr}_\Sigma \underline{\chi}$, $\theta := \operatorname{tr}_\Sigma \chi$ are the *null expansions* of Σ. Note that $\underline{\chi}$, χ, $\underline{\theta}$, θ depend on the choice of null frame, i.e., the choice of null generator $\underline{L}$. We find

$$\mathcal{H}^2 = \underline{\theta}\theta,$$

and we further define the *scalar second fundamental form* A as

$$A := \underline{\theta}\chi.$$

Observe that $\operatorname{tr}_\Sigma A = \mathcal{H}^2$ and both A and $\mathcal{H}^2$ are independent of the choice of $\underline{L}$. Additionally, as every spacelike cross section Σ intersects the integral curves of $\underline{L}$ precisely once, Σ can be written as a graph $\Sigma = \operatorname{graph}_S \omega$ of a function ω with respect to a given (fixed) spacelike cross section S.

We say a family $x \colon [0,T) \times \Sigma \to \mathcal{N}$ is a solution of *area preserving null mean curvature flow* (APNMCF) if

$$\frac{\mathrm{d}}{\mathrm{d}\,t}x = -\frac{1}{2\underline{\theta}}\left(\mathcal{H}^2 - \frac{1}{|\Sigma|}\int_\Sigma \mathcal{H}^2\right)\underline{L}.$$

Note that the flow is gauge invariant, i.e., independent of the choice of $\underline{L}$. It is easy to see that STCMC surfaces are the stationary points of the flow. Moreover, as

$$\frac{\mathrm{d}}{\mathrm{d}\,t}|\Sigma| = \int_\Sigma \underline{\theta}\varphi \,\mathrm{d}\,\mu$$

for a general variation $\frac{\mathrm{d}}{\mathrm{d}\,t}x = \varphi\underline{L}$, the flow is indeed area preserving. Lastly, observe that the flow is equivalent to the following scalar parabolic equation

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\omega = -\frac{1}{2\underline{\theta}}\left(\mathcal{H}^2 - \frac{1}{|\Sigma|}\int_\Sigma \mathcal{H}^2\right).$$

In the Minkowski lightcone, this flow is equivalent to Hamilton's Ricci flow for topological spheres in 2-dimensions, cf. [11], where it arises as a rescaling of $2d$-Ricci flow in this special setting. Moreover, $2d$-Ricci flow in this special case is equivalent to the corresponding null mean curvature flow, which was first studied by Roesch–Scheuer [9] in a more general setting.

In the context of Theorem 2 initial data for APNMCF is contained in a suitably defined a-priori class: For $\sigma > 0$, and constants $B_1, B_2, B_3 \geq 0$, we define

$$B_\sigma(B_1,B_2,B_3) := \{\Sigma \subseteq \mathcal{N} \colon |\omega - \sigma| \leq B_1, |\mathring{A}| \leq \frac{B_2}{\sigma^4}, |\nabla\mathring{A}| \leq \frac{B_3}{\sigma^5}\}.$$

Note that any spacelike cross section in the Schwarzschild lightcone can be isometrically embedded in a canonical way (via the graph function ω) into the standard

Minkowski lightcone such that $\mathring{A}_{Schw} = \mathring{A}_{Mink}$. Furthermore, $\mathring{A}_{Mink} \equiv 0$ if and only if Σ has constant scalar curvature. In particular,

$$\omega = b_{\rho,\vec{a}} := \frac{\rho}{\sqrt{1+|\vec{a}|} - \vec{a}\cdot\vec{x}}$$

for some positive constant $\rho > 0$ and 3-vector $\vec{a}$, cf. [7, Proposition 6]. For a spacelike cross section Σ in the Minkowski lightcone, we can define a associate future timelike 4-vector $\vec{Z}$ via

$$\vec{Z} := \frac{1}{|\Sigma|}\begin{pmatrix} \int_\Sigma t \,\mathrm{d}\mu \\ \int_\Sigma \vec{x}\,\mathrm{d}\mu \end{pmatrix} = |\vec{Z}|\begin{pmatrix} \sqrt{1+|\vec{a}|^2} \\ \vec{a} \end{pmatrix}, \tag{1}$$

cf. [12]. See also [2]. It was shown in [12] that if $\mathring{A}_{Mink}$ is sufficiently small in an L^2-sense then ω is close to $b_{\rho,\vec{a}}$ in $W^{2,2}$, where ρ is chosen as the area radius of Σ and $\vec{a}$ is determined by (1). Adapting a corresponding C^1 estimate by Shi–Wang–Wu [10] and embedding $\Sigma \subseteq \mathcal{N}$ into the Minkowski lightcone via ω (not necessarily isometrically), we obtain the following a-priori estimates (omitting some mild additional assumptions for simplicity):

Proposition 1. *Let $\mathcal{N}$ be asymptotically Schwarzschildean, Σ be in $B_\sigma(B_1, B_2, B_3)$. Then*

$$|\vec{a}| \le \frac{C(B_1)}{\sigma} \tag{2}$$

for σ sufficiently large. Moreover

$$||\omega - b_{\rho,\vec{a}}||_{C^{2,\alpha}(\mathbb{S}^2)} \le \frac{C(B_1,B_2,B_3)}{\sigma^2}, \quad \left|\mathcal{H}^2 - \frac{4}{\rho^2} + \frac{8m}{b^3_{\rho,\vec{a}}}\right| \le \frac{C(B_1,B_2,B_3)}{\sigma^4}.$$

Using Proposition 1, one can show that that spacelike cross sections Σ in $B_\sigma(B_1, B_2, B_3)$ are strictly stable, which allows to conclude the convergence in Theorem 2 once the long-time existence is shown.

We expect that the foliation in Theorem 2 is unique at least within the a-priori class $B_\sigma(B_1, B_2, B_3)$. In fact, we conjecture that (under suitable asymptotic assumptions) there is a unique asymptotic foliation of stable STCMC surface (that agrees with the foliation constructed in Theorem 2). In particular, this would yield a unique asmyptotic background foliation at infinity (suitable to discuss energy, mass and momentum) that is formulated purely with respect to the geometry of the given lightcone. Note that in spherical symmetry, uniqueness of STCMC surfaces (without a stability assumption) already follows by work of Chen–Wang [3] assuming the null energy condition.

By Equation (2), we note that the constructed foliation has vanishing Bondi-momentum and is thus centered in the sense that the Bondi energy of this foliation agrees with the Bondi mass of the lightcone (which is m in the asymptotically Schwarzschildean setting). At this point, it is unclear if such an STCMC foliation is suitable to also discuss center of mass in the null setting. This is subject of future research.

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