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Uncertainty as an ingredient in financial modeling

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Uncertainty – as opposed to risk – is used to describe events to which we are not able to assign a probability due to lack of information. Instead of assigning a probability to an uncertain event, we only assume that such an event is possible or that its probability is within some range. We illustrate the effects of the inclusion of uncertainty in modeling by looking at simple cases of an optimal investment problem.

1 A little bit of randomness

As financial markets are not predictable, a model based on probabilities is a suitable one for their future evolution. Thus, before we start the modeling, we provide some basic concepts and facts of probability theory.

A real-valued *random variable* X assigns a real number to the outcome of a stochastic experiment (such as coin tossing or playing dice). It is fully described by its *(probability) distribution function*

$$F(x) = \mathbb{P}(X \leq x),\ x \in \mathbb{R},$$

which is the probability that the value of X does not exceed x. Examples are discrete distributions where the distribution function only increases in jumps, such as the uniform distribution on $\{1, 2, 3, 4, 5, 6\}$ in playing dice where each of these values has a probability of $1/6$ to occur while all the remaining values have a total probability of zero.

Another example are distributions with a *(probability) density function* $f(x)$. These are non-negative functions with an integral of 1[1]. The distribution function is then

$$F(x) = \int_{-\infty}^{x} f(z)\mathrm{d}z.$$

The most popular distribution with a density is the *normal distribution* with mean μ and variance σ^2 (for short, $N(\mu, \sigma^2)$) and density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{\sigma^2}}.$$

As it is central in financial modeling, we present the graph of $f(x)$ in Figure 1.

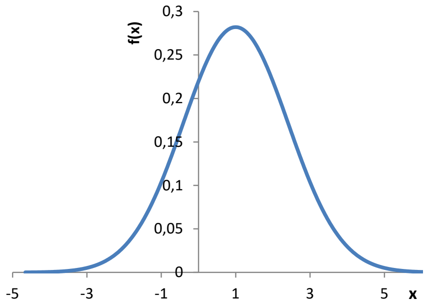

Figure 1: The density function $f(x)$ of the $N(1, 2)$-distribution.

A random experiment evolving in time is called a *stochastic process*, denoted $X = (X_t, t \geq 0)$, where each X_t is a (real-valued) random variable. Practical examples of a stochastic process can be the temperature curve over a day or the evolution of a stock price over time.

The most important stochastic process for us is *Brownian motion*, which we denote W_t for $t \geq 0$. It is a stochastic process starting at $W_0 = 0$ that generates continuous paths (such that one can draw them without lifting the pen). The changes follow a normal distribution $W_t - W_s \sim N(0, t-s)$. Further, the changes are assumed to be independent of the past, that is, of the part of the Brownian motion before time s. Note in particular, that a Brownian motion W_t is $N(0, t)$-distributed. Its typical behavior is illustrated in Figure 2.

[1] This is a continuous analogue of summing over all possibilities.

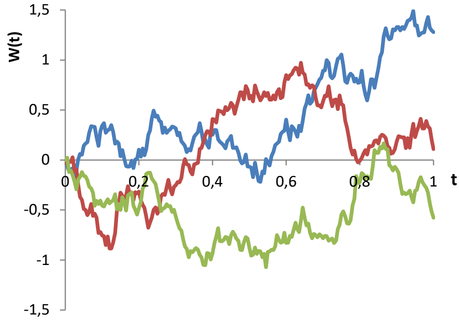

Figure 2: Three simulated paths of a Brownian motion $W_t(\omega), 0 \leq t \leq 1$.

Of course, we can only simulate a Brownian motion discretely, that is, simulate the steps independent of the past and then interpolate linearly between them. The three paths in Figure 2 are so irregular, that they do not have a time derivative.

While this non-differentiability sounds a bit obscure, it proves to be ideal for financial modeling. If the path of a stock price were differentiable, then we would know whether the price increases or decreases in the next moment, depending on the sign of the derivative. Thus, if we want continuous paths for a stock price, the use of a nowhere-differentiable stochastic process is a must.

The non-differentiability of Brownian motion requires the use of *Itô calculus* (see for example [3]) as a key tool in financial modeling. However, its detailed explanation is beyond the scope of this snapshot.

2 Risk, uncertainty, and financial modeling

Now we are well equipped to start financial modeling and to consider the terms risk and uncertainty. Both are often associated with events whose occurrence and scope are unknown. With regards to probabilistic modeling, however, there is a clear distinction between risk and uncertainty.

We refer to *risk* when we know all possible values that a random variable can attain and are able to assign a probability distribution for their occurrence. Here, we assume that all parameters of the distribution (for example the mean μ and the variance σ^2 of a normal distribution $N(\mu, \sigma^2)$) are either known or can be estimated from data.

On the contrary, the term *uncertainty* dates back to the economist Frank Knight [5]. He used it to describe events for which we have *no quantifiable knowledge*. In other words, we know that an uncertain event might happen, but we have no model for the probability of its outcomes as there is no data from the past that would allow us to specify a probability distribution neither for the likelihood of its occurrence nor for its possible realizations.

A popular example of a (seemingly) random event is the evolution of stock prices in financial markets. A central modeling task in financial mathematics is the evolution of a stock price in time. The standard stock price model is given by

$$S_t = S_0 e^{(b-\frac{1}{2}\sigma^2)t+\sigma W_t}. \tag{1}$$

Here, S_0 is the initial price, b the average increase (or decrease) of the stock price per unit time, and σ quantifies how much we expect the price increase (or decrease) to deviate from the average. Hence, the Brownian motion term W_t describes the randomness of a process with a known probability distribution, that is, risk. We write out the explicit forms of the mean $\mathbb{E}(S_t)$ and the variance $\mathrm{Var}(S_t)$ of S_t for later comparison.

$$\mathbb{E}(S_t) = S_0 e^{bt}, \quad \mathrm{Var}(S_t) = \mathbb{E}(S_t^2) - \mathbb{E}(S_t)^2 = S_0^2 e^{2bt}\left(e^{\sigma^2 t} - 1\right). \tag{2}$$

We compare the profits of stock investment to the profits of a *money market account*, which is the value of money following the market rates. The evolution of a money market account with continuous accumulation of interest at rate r is given by

$$B_t = B_0 e^{rt}. \tag{3}$$

Notice in particular, that this quantity is not random. There are two ways in which uncertainty typically affects our model.

- If we do not know r, b, and/or σ exactly, but have some boundaries $r_0 \leq r \leq r_1$, $b_0 \leq b \leq b_1$, and/or $\sigma_0 \leq \sigma \leq \sigma_1$, then we speak of *parameter ambiguity* of the market. We further explore ambiguity in Section 4.
- If we are afraid of a stock price crash with a maximum possible height of, say, 20% until some final time T (but know all model parameters), then we call this *(crash) uncertainty* when we neither have a probability distribution for the crash height, for the crash time, nor for the crash occurrence. We dive deeper into crash uncertainty in Section 3.

In both cases, we do not have a full probabilistic description of the evolution of the market. We demonstrate the effect of these forms of uncertainty on the solution of the so-called *continuous-time portfolio problem*. For this, we describe the investment strategy of an investor via a *portfolio process* $p_t, t \in [0, T]$. We denote with p_t the fraction of wealth we invest into the stock at time t, while the remaining fraction $1-p_t$ is invested in the money market account. The evolution of the resulting wealth process X_t^p with initial wealth $x>0$ is given by

$$X_t^p = x\exp\left(\int_0^t\left(r+p_s(b-r)-\frac{1}{2}p_s^2\sigma^2\right)\mathrm{d}t+\int_0^t p_s\sigma\mathrm{d}W_s\right). \tag{4}$$

We give a heuristic derivation of the wealth process in the appendix, but one can see that the wealth process follows the stock price in case of $p_t=1$ and the money market account in case of $p_t=0$. The aim of the investor is to find the portfolio process p_t, which maximizes the *expected utility from final wealth*

$$\max_{p\in A(x)}\mathbb{E}\left(U\left(X_T^p\right)\right), \tag{5}$$

where we denoted by $A(x)$ the set of possible portfolio processes with initial wealth x. In case of the log-utility function $U(x)=\ln(x)$, the optimal portfolio process equals a constant investment of $p_t^*=\frac{b-r}{\sigma^2}$. The optimal expected utility is (see for example [6])

$$\mathbb{E}\left(\ln\left(X_T^{p^*}\right)\right)=\ln(x)+rT+\frac{1}{2}\frac{(b-r)^2}{\sigma^2}T. \tag{6}$$

Note that we have not taken uncertainty into account. In the following sections, we explore how uncertainty affects the optimal portfolio process.

Remark (Why do we need a utility function?)**.** *If we maximize the expected final wealth $\mathbb{E}\left(X_T^p\right)$ then – in the case of $b>r$ – we should invest all our money in the stock. Even more, if unlimited borrowing is allowed, we would take an infinite credit (an unbounded negative position in the money market account) to buy as many shares of the stock as possible. Of course, this is an extremely risky strategy. Utility functions are strictly concave, differentiable functions on $(0,\infty)$ with an infinite slope at 0 and a vanishing slope at infinity. They automatically rule out the above strategy of infinite borrowing. The main reason for this is the asymptotically vanishing derivative of the utility function, as it implies having one million euros in the bank is better than a one-in-a-thousand chance to win a billion euros.*

3 Portfolio problems with crashes – the worst-case approach

Worst-case portfolio problems under the threat of crashes were introduced in [9]. In the simplest case, the authors consider a portfolio problem with a money market account with constant interest rate r and a stock with price dynamics *in normal times* given by Equation (1). At a possible *crash time* τ, the stock price suddenly drops by a fraction κ. We face uncertainty because we cannot assign a probability to the crash height or time. We only assume $\kappa \leq k < 1$ and that a crash can happen. Thus, we cannot set up a portfolio problem of the form (5), but have to look at a max-min problem of the form

$$\max_{p \in A(x)} \min_{\tau \in [0,T+1], \kappa \in [0,k]} \mathbb{E}\left(\ln\left(X_T^p\right)\right), \tag{7}$$

to take crash uncertainty into account. The added minimization only means that we look at the worst-case scenario. Note that the case $\tau > T$ corresponds to the crash not happening.

The indifference principle. To solve problem (7), note that we know both the optimal portfolio process $p_t^* = \frac{b-r}{\sigma^2}$ and the optimal expected utility *after the crash* (see Equation (6)). To be independent of the crash, we choose a portfolio process $\hat{p}_t$ such that we are indifferent between the worst possible crash happening at an arbitrary random time $\tau \in [0,T]$ and no crash happening at all. This condition can be expressed in equations as

$$\mathbb{E}\left(\ln\left(X_\tau^{\hat{p}}\left(1 - k\hat{p}_{\tau-}\right)\right)\right) + \left(r + \frac{1}{2}\frac{(b-r)^2}{\sigma^2}\right)(T-\tau) = \mathbb{E}\left(\ln\left(X_T^{\hat{p}}\right)\right). \tag{8}$$

This *indifference principle* is sufficient to characterize the worst-case optimal portfolio process $0 \leq \hat{p}_t < 1/k$ *before the crash*. This time, the optimal portfolio process is not constant, but changes in time. It is determined as the solution of the following differential equation (see [9] for a derivation)

$$p_t' = -\frac{\sigma^2}{2k}\left(1 - p_t k\right)\left(p_t - p_t^*\right)^2, \; p_T = 0. \tag{9}$$

Note that we start following $\hat{p}_t$ until a possible crash time τ and then switch to p_t^*. We illustrate the behaviour of this strategy in Figure 3. There, we present the optimal portfolio process $\hat{p}_t$ before the crash along with the optimal (constant) portfolio process $p_t^* = 0.8$.

Generalizations and modifications. There exist many generalizations of the above setting. Frank Seifried has developed a general framework including abstract optimality principles in a more general model setting for the stock prices in [12]. Optimal strategies under stress scenarios in the form of different possible crashes in a multi-asset framework are derived in [8] (which also contains a good list of further references). An application to dynamic reinsurance of large claims can be found in [7].

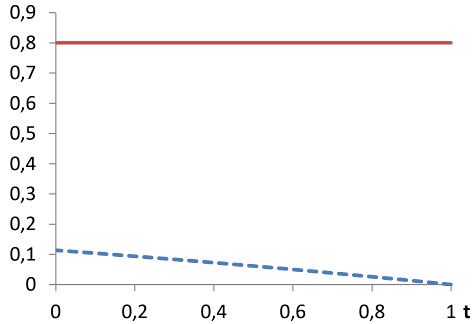

Figure 3: Optimal strategies before (dashed line) and after a possible crash (solid line) for parameter values $k = 0.15$, $b = 0.06$, $r = 0.01$, $\sigma = 0.25$, $T = 1$.

4 Portfolio problems with ambiguity

In the fundamental contribution by Gilboa and Schmeidler [1], the idea of ambiguity is related to a stochastic experiment that is not fully described. There is uncertainty about prior knowledge of ingredients, in our case about the distribution parameters as described in Section 2.

To illustrate how to deal with this kind of uncertainty, we consider the portfolio problem (7) where instead of a possible crash, we now face interest rate ambiguity, that is, we only know that $r_0 \le r \le r_1$ and look at

$$\max_{p \in A(x)} \min_{r \in [r_0, r_1]} \mathbb{E}\left(\ln\left(X_T^p\right)\right). \tag{10}$$

We know that for every fixed value of r, the optimal portfolio process is given by $p_t^* = (b-r)/\sigma^2$ (see Section 2) with optimal utility of

$$F(r,x) := \ln(x) + \left(r + \frac{1}{2}\frac{(b-r)^2}{\sigma^2}\right)T. \tag{11}$$

To solve the problem heuristically, we differentiate F with respect to r

$$\frac{\partial}{\partial r}F(r,x) := \left(1 - \frac{b-r}{\sigma^2}\right)T. \tag{12}$$

As a consequence, we see that if

$$r_0 \le b - \sigma^2 \le r_1,$$

the possible values of the derivative contain zero where indeed the minimum of $F(r, x)$ is attained. Hence, our only chance to be *ambiguity indifferent* with respect to the interest rate r in this case is to choose $p^* = 1$, that is, to *invest all our money in the stock*. This then also leads to the worst optimal performance in this setting. That this strategy is indeed the optimal one is a special case of the results in [10]. There, the authors consider uncertainty on all r, b, and σ. They derive explicit results for the ambiguity optimal portfolio process for all possible cases of the relations between the intervals for r, b, and σ.

Generalizations and modifications. In practice, solving the portfolio optimization problem under ambiguity quickly leads to non-linear partial differential equations, which are generally hard to solve. To weaken the influence of extreme parameters, the concept of *smooth ambiguity* has been introduced in [4]. This approach effectively recognises that extreme parameter values are extremely unlikely and lessens their influence. Considering the inner part of the portfolio problem has led to the concept of *non-linear expectation* and a corresponding *stochastic calculus under uncertainty* pioneered by Shige Peng (summarized in [11]). An interesting recent branch (with possible applications to insure climate risks) is presented in [2] which considers the optimal allocation in an exchange economy under ambiguity.

5 Conclusion

In this snapshot, we have highlighted different aspects and applications of uncertainty in financial modeling. Rather than being exhaustive, we focused on a case study of portfolio problems. Interested readers should, in particular, consult the references in the *Generalizations and modifications* sections.

We emphasize that recent research on uncertainty in financial models has led to new principles such as the indifference principle and the novel theoretical branch of non-linear expectations. On the applied side, many contributions have shown that including uncertainty does not result in moving completely away from investments with an uncertain component but treating them with caution.

As the future climate and political risks are hard to model conventionally, we believe that the role of uncertainty as a modeling tool will grow.

6 Appendix: Motivation of the form of the wealth process

We give a heuristic derivation of the form of the wealth process. We show that it behaves as the stock price process but with suitably adjusted parameters. We start by using the approximation

$$e^{\delta} \approx 1 + \delta + \frac{1}{2}\delta^2, \tag{13}$$

which works for small values of δ. Evaluate the relative change of the stock price over a short time δ:

$$\frac{S_{t+\delta} - S_t}{S_t} = e^{(b - \frac{1}{2}\sigma^2)\delta + \sigma(W_{t+\delta} - W_t)} - 1 \approx b\delta + \sigma(W_{t+\delta} - W_t), \tag{14}$$

where we ignored terms of order higher than δ, in particular all mixed terms with Brownian motion and δ. Since the variance of the Brownian step is δ, it has an order of $\delta^{1/2}$. Indeed, the second-order approximation of the Brownian step cancels the first-order approximation $-\frac{1}{2}\sigma^2\delta$ on average.

The relative change of the money market account is simpler since it does not involve a stochastic term. We have

$$\frac{B_{t+\delta} - B_t}{B_t} \approx r\delta. \tag{15}$$

Let us denote by p the fraction of our wealth invested into the stock, while the remaining fraction $1 - p$ is invested in the money market account. Then the wealth process is

$$X_t = (1 - p)B_t + pS_t.$$

The relative change of the wealth process X_t is estimated via equations (14) and (15) as

$$\frac{X_{t+\delta} - X_t}{X_t} \approx ((1 - p)r + pb)\delta + p\sigma(W_{t+\delta} - W_t). \tag{16}$$

The relative increase of the wealth process has the same behaviour as the stock price (Equation (14)) if we identify b with $(1 - p)r + pb$ and σ with $p\sigma$. Thus the wealth process looks exactly like a stock price process (Equation (1)). In case of a non-constant portfolio process, suitable integrals occur as in (4).

This derivation can be made rigorous with the Itô calculus at hand.

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