Mathematisches
Forschungsinstitut
Oberwolfach

Member of Leibniz Association

Oberwolfach Preprints

Number 2025-06

Chiara Boccato
Joachim Kerner
Maximilian Pechmann
Wolfgang Spitzer

Generalized Bose-Einstein Condensation in the Kac-Luttinger Model

July 2025

Oberwolfach Preprints (OWP)

The MFO publishes the Oberwolfach Preprints (OWP) as a series which mainly contains research results related to a longer stay in Oberwolfach, as a documentation of the research work done at the MFO. In particular, this concerns the Oberwolfach Research Fellows program (and the former Research in Pairs program) and the Oberwolfach Leibniz Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

All information about the publication process can be found at
https://www.mfo.de/scientific-programme/publications/owp

All published Oberwolfach Preprints can be found at https://publications.mfo.de

ISSN 1864-7596

License Information

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany
https://www.mfo.de

DOI 10.14760/OWP-2025-06

GENERALIZED BOSE–EINSTEIN CONDENSATION IN THE KAC–LUTTINGER MODEL

CHIARA BOCCATO, JOACHIM KERNER, MAXIMILIAN PECHMANN, AND WOLFGANG SPITZER

Abstract. In this article, we prove generalized Bose–Einstein condensation (BEC) at zero temperature in the random Kac–Luttinger model for repulsive two-particle interactions that are scaled suitably in the limit of large volume. Compared to previous works, by proving generalized condensation rather than the macroscopic occupation of finitely many single-particle states (type-I BEC), we can allow for stronger two-particle interactions. We discuss implications of the result which include a possible transition in the type of the condensation.

1. Introduction

In this article we discuss the occurrence of generalized Bose–Einstein condensation (BEC) in a gas of interacting bosons in a random higher-dimensional continuum model known as the Kac–Luttinger model [KL73, KL74]. In mathematical physics, to prove BEC in a system of interacting particles is generally considered an interesting but intricate problem at the heart of many-body quantum mechanics [LSSY05]. Furthermore, random models are often invoked to describe more realistic systems since they typically exhibit certain defects. This being said, with this paper, we follow up on recent advances in the description of BEC in the random Kac–Luttinger model [KP23, Szn23, BKP24]. More explicitly, we shall focus on the notion of generalized BEC (g-BEC) introduced by Girardeau [Gir60, BLP86] and study its occurrence in the higher-dimensional Kac–Luttinger model. While conventional BEC refers to the macroscopic occupation of a single-particle state, generalized condensation instead refers to the macroscopic occupation of a certain band or family of states (note that it has been suggested that g-BEC is thermodynamically more stable [Gir60, JPZ10]). In this sense, g-BEC means a weaker form of condensation and one may therefore prove this for stronger interparticle interactions. Most importantly, however, one may expect a transition in the nature of the condensate going from type I to type III; type III means that one has g-BEC but – at the same time – no individual state of the family is itself macroscopically occupied, whereas type I is present if at least one but at most finitely many such states are macroscopically occupied. In [KPS19], such a transition has been proved for the one-dimensional analogue of the Kac–Luttinger model, the so-called Luttinger–Sy model, with contact interaction between the bosons; consequently, such a transition can be expected

Date: July 9, 2025.

to also occur in the Kac–Luttinger model. Here, one should take into account the fundamental differences in the behaviour of BEC comparing the random to the non-random setting. More explicitly, in the random setting certain spectral effects such as Lifshitz tails and large spectral gaps support the existence of BEC whereas localization effects suppress the existence of condensation as soon as repulsive interactions are present. In short, it is this competition between different effects what makes an investigation of BEC in the random setting particularly interesting.

2. The Kac–Luttinger model

Consider a system of N bosons, $N \in \mathbb{N}$, confined within the box

$$\Lambda_N := (-L_N/2, +L_N/2)^d \subset \mathbb{R}^d \quad \text{where} \quad L_N := \rho^{-1/d} N^{1/d}$$

where $d \geq 2$ and where $\rho > 0$ represents the particle density. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the one-particle Hamiltonian in the Kac–Luttinger model is then informally given by the random Schrödinger operator $-\Delta + V$ restricted to Λ_N with V being the random potential

$$V : \Omega \times \mathbb{R}^d \to \mathbb{R}_+ \cup \infty, \quad (\omega, x) \mapsto V^\omega(x) := \sum_m \infty \cdot B_r(x - x_m^\omega) \ , \tag{1}$$

where $\{x_m^\omega\}_m$ represents the points of a Poisson point process on $\mathbb{R}^d$ with a constant intensity $\nu > 0$ and $B_r(x)$ denotes a closed ball with fixed radius $r > 0$ centered at $x \in \mathbb{R}^d$. On a more rigorous level, the random potential (1) yields the random domain

$$\Lambda_N^\omega := \Lambda_N \setminus \bigcup_m B_r(x_m^\omega), \quad \omega \in \Omega, \ N \in \mathbb{N} \ ,$$

which we refer to as the *vacancy set*. It can be shown that the vacancy set consists, $\mathbb{P}$-almost surely and for all $N \in \mathbb{N}$, of finitely many components only [MR96, Proposition 4.1]. In this paper, we shall always refer to such a *typical* $\omega \in \Omega$. For each $N \in \mathbb{N}$, we denote the number of components of the vacancy set Λ_N^ω by $K_N^\omega \in \mathbb{N}_0$ (with the understanding that $K_N^\omega = 0$ if $\Lambda_N^\omega = \emptyset$). Defining the index set $\mathcal{K}_N^\omega := \{1, \ldots, K_N^\omega\}$ if $K_N^\omega \geq 1$ (and $\mathcal{K}_N^\omega := \emptyset$ if $K_N^\omega = 0$), we label the components by $k \in \mathcal{K}_N^\omega$. Therefore, if $K_N^\omega \geq 1$, a component of Λ_N^ω is denoted by $\Lambda_N^{k,\omega}$ where $k \in \mathcal{K}_N^\omega$, and the collection $\{\Lambda_N^{k,\omega}\}_{k \in \mathcal{K}_N^\omega}$ forms a partition of Λ_N^ω. We remark that $\mathbb{P}$-almost surely and for all but finitely many N, $\mathcal{K}_N^\omega \neq \emptyset$, see for example [Szn98, Chapter 4, Theorem 4.6].

To define the N-particle Hamiltonian that describes our random, interacting quantum system, we introduce the potential

$$v_N : \mathbb{R}^d \to \mathbb{R}_+, \quad x \mapsto v_N(x), \quad N \in \mathbb{N} \ ,$$

which represents the interaction between two bosons. We assume that $v_N \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ is a non-negative function. For any $\omega \in \Omega$ and $N \in \mathbb{N}$ such that $K_N^\omega \geq 1$, our system is now governed by the random, self-adjoint N-particle Hamiltonian

$$H_N^\omega := -\sum_{j=1}^{N} \Delta_j + \sum_{1 \leq i < j \leq N} v_N(x_i - x_j) \tag{2}$$

which acts on the space $L^2_{\mathrm{s}}((\Lambda_N^\omega)^N)$, subject to Dirichlet boundary conditions on the boundary of $(\Lambda_N^\omega)^N$. Here, the index s indicates the symmetric subspace of $L^2((\Lambda_N^\omega)^N)$. The form domain of H_N^ω is given by $\mathcal{D}[H_N^\omega] = H_0^1((\Lambda_N^\omega)^N) \cap L^2_s((\Lambda_N^\omega)^N)$. If $K_N^\omega = \emptyset$, we set $H_N^\omega := 0$.

Remark 1. *In this paper, we abbreviate L^p-norms by writing, for instance, $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(\Lambda_N^\omega)}$. In addition, we adopt the following notation for the asymptotic behavior of sequences: Whenever $(a_N)_{N\in\mathbb{N}}$ and $(b_N)_{N\in\mathbb{N}}$ are two positive sequences, we write $a_N \ll b_N$ if and only if $\lim_{N\to\infty} a_N/b_N = 0$, and $a_N \sim b_N$ if and only if there exist constants $c, C > 0$ such that $ca_N \leq b_N \leq Ca_N$ for all but finitely many $N \in \mathbb{N}$. We use the notation $a_N \lesssim b_N$ if and only if $a_N \ll b_N$ or $a_N \sim b_N$.*

3. Preliminaries

Let an $\omega \in \Omega$ and $N \in \mathbb{N}$ be given. Suppose $K_N^\omega \geq 1$. The lowest eigenvalue $E^{1,\omega}_{\mathrm{QM},N}$ of the Hamiltonian H_N^ω, hence the N-particle ground state energy, is given by

$$E^{1,\omega}_{\mathrm{QM},N} := \inf\left\{\langle \Psi, H_N^\omega \Psi\rangle : \Psi \in \mathcal{D}[H_N^\omega], \|\Psi\|_2 = 1\right\} \ ,$$

where we understand $\langle \Psi, H_N^\omega \Psi\rangle$ in the form sense (if $\mathcal{K}_N^\omega = \emptyset$, we set $E^{1,\omega}_{\mathrm{QM},N} := 0$). There exists a minimizer $\Psi_N^{1,\omega}$, which we also refer to as a ground state of H_N^ω, satisfying

$$\langle \Psi_N^{1,\omega}, H_N^\omega \Psi_N^{1,\omega}\rangle = E^{1,\omega}_{\mathrm{QM},N} \ .$$

Given a ground state $\Psi_N^{1,\omega}$, the one-particle density matrix $\varrho^{(1)}_{\Psi_N^{1,\omega}}$ corresponding to $\Psi_N^{1,\omega}$ is defined as a non-negative, trace-class operator on $L^2(\Lambda_N^\omega)$ with integral kernel, $x, y \in \Lambda_N^\omega$,

$$\varrho^{(1)}_{\Psi_N^{1,\omega}}(x,y) = \int\limits_{\Lambda_N^\omega} \dots \int\limits_{\Lambda_N^\omega} \Psi_N^{1,\omega}(x, x_2, \dots, x_N) \Psi_N^{1,\omega}(y, x_2, \dots, x_N)\, \mathrm{d}x_2 \dots \mathrm{d}x_N \ .$$

This one-particle density matrix is normalized, meaning that its trace satisfies

$$\mathrm{tr}^{(1)}(\varrho^{(1)}_{\Psi_N^{1,\omega}}) = \sum_{j\in\mathbb{N}} \langle \varphi_j, \varrho^{(1)}_{\Psi_N^{1,\omega}} \varphi_j\rangle = 1 \ ,$$

where $(\varphi_j)_{j\in\mathbb{N}}$ is an arbitrary orthonormal basis of $L^2(\Lambda_N^\omega)$. The number of particles occupying a one-particle state $\varphi \in L^2(\Lambda_N^\omega)$, given the N-particle system is in the N-particle ground state $\Psi_N^{1,\omega}$, is then given by

$$n^{\varphi}_{\Psi_N^{1,\omega}} := N\, \mathrm{tr}^{(1)}\left(|\varphi\rangle\langle\varphi| \varrho^{(1)}_{\Psi_N^{1,\omega}}\right) \ ,$$

where $|\varphi\rangle\langle\varphi|$ is the projection operator onto the linear subspace generated by φ (we set $n^{\varphi}_{\Psi^{1,\omega}_N} := 0$ if $K^{\omega}_N = 0$). Informally, a normalized state $\varphi_N \in L^2(\Lambda^{\omega}_N)$ is called macroscopically occupied if the quotient $n^{\varphi_N}_{\Psi^{1,\omega}_N}/N$ does not converge to zero.

4. Main results

We are now ready to formulate and prove our main result, namely, the occurrence of generalized BEC in the random Kac–Luttinger model.

Theorem 2 (Generalized BEC)**.** *Let an $\omega \in \Omega$ be given. Assume that*

$$\|v_N\|_\infty \ll \frac{1}{N} \quad \text{or} \quad \|v_N\|_1 \ll \frac{\ln N}{N} \ .$$

Let, for each $N \in \mathbb{N}$, $-\Delta$ be the Dirichlet Laplacian on Λ^{ω}_N with eigenfunctions $\varphi^{j,\omega}_N \in L^2(\Lambda^{\omega}_N)$ and corresponding eigenvalues $e^{j,\omega}_N$, $j \in \mathbb{N}$. Moreover, let

$$n^{j,\omega}_N := n^{\varphi^{j,\omega}_N}_{\Psi^{1,\omega}_N} = N\,\mathrm{tr}^{(1)}(|\varphi^{j,\omega}_N\rangle\langle\varphi^{j,\omega}_N|\varrho^{(1)}_{\Psi^{1,\omega}_N})$$

be the occupation number of the one-particle state $\varphi^{j,\omega}_N$ given the N-particle system is in the N-particle ground state $\Psi^{1,\omega}_N$ of H^{ω}_N (meaning the N-particle state is given by $\varrho^{(N)}_{\Psi^{1,\omega}_N} = |\Psi^{1,\omega}_N\rangle\langle\Psi^{1,\omega}_N|$). Then $\mathbb{P}$-almost surely,

$$\lim_{\epsilon\searrow 0}\liminf_{N\to\infty}\frac{1}{N}\sum_{j\in\mathbb{N}:e^{j,\omega}_N\le\epsilon} n^{j,\omega}_N = 1 \ ,$$

that is, complete g-BEC (into eigenstates of the free Dirichlet Laplacian on Λ^{ω}_N) occurs.

Proof. Let an $\omega \in \Omega$ and an arbitrary $N \in \mathbb{N}$ such that $K^{\omega}_N \geq 1$ be given. By inserting the trial state $\varphi^{1,\omega}_N \otimes ... \otimes \varphi^{1,\omega}_N$, we obtain the upper bound

$$\begin{aligned} E^{1,\omega}_{QM,N} &\le Ne^{1,\omega}_N + \frac{N(N-1)}{2}\int\limits_{\Lambda^{\omega}_N}\int\limits_{\Lambda^{\omega}_N} v_N(x-y)|\varphi^{1,\omega}_N|^2(x)|\varphi^{1,\omega}_N|^2(y)\,\mathrm{d}x\mathrm{d}y \\ &\le N\left(e^{1,\omega}_N + N\min\left\{\|v_N\|_\infty, C_1^2\|v_N\|_1(e^{1,\omega}_N)^{d/2}\right\}\right) \ , \end{aligned}$$

where we used a classical L^∞-estimate (see, e.g., [Szn23, Lemma 1.1]) and Young's convolution inequality to bound the second component in the minimum expression; in particular, $C_1 > 0$ is some constant independent on N. Regarding a lower bound, we obtain

$$\begin{aligned} E^{1,\omega}_{QM,N} &= \mathrm{tr}^{(N)}\left(H^{\omega}_N\,\varrho^{(N)}_{\Psi^{1,\omega}_N}\right) \\ &\ge \mathrm{tr}^{(N)}\left(\sum_{j=1}^{N}(-\Delta_j)\,\varrho^{(N)}_{\Psi^{1,\omega}_N}\right) \\ &= N\,\mathrm{tr}^{(1)}\left(-\Delta\,\varrho^{(1)}_{\Psi^{1,\omega}_N}\right) \end{aligned}$$

$$
\begin{aligned}
&= N \operatorname{tr}^{(1)}\left(\sum_{j=1}^{\infty} e_N^{j,\omega} |\varphi_N^{j,\omega}\rangle\langle\varphi_N^{j,\omega}|\, \varrho^{(1)}_{\Psi_N^{1,\omega}}\right)\\
&= N \sum_{j=1}^{\infty} e_N^{j,\omega} \operatorname{tr}^{(1)}\left(|\varphi_N^{j,\omega}\rangle\langle\varphi_N^{j,\omega}|\, \varrho^{(1)}_{\Psi_N^{1,\omega}}\right)\\
&\geq N\left(e_N^{1,\omega}\sum_{j=1}^{J_N^\omega-1}\frac{n_N^{j,\omega}}{N} + e_N^{J_N^\omega,\omega}\,\frac{N-\sum_{j=1}^{J_N^\omega-1} n_N^{j,\omega}}{N}\right)\ ,
\end{aligned}
$$

where

$$
J_N^\omega := \min\left\{ j\in\mathbb{N} : e_N^{j,\omega} \geq 2\max\left\{N\gamma_N \min\{\|v_N\|_\infty, C_1^2\|v_N\|_1 (e_N^{1,\omega})^{d/2}\}, e_N^{1,\omega}\right\}\right\}
$$

and $(\gamma_N)_{N\in\mathbb{N}}$ is a positive sequence such that $\gamma_N \gg 1$ and

$$
N\gamma_N \min\left\{\|v_N\|_\infty, C_1^2\|v_N\|_1(\ln N)^{-1}\right\} \ll 1\ . \tag{3}
$$

Here, note that $e_N^{1,\omega} \sim (\ln N)^{-2/d}$ $\mathbb{P}$-almost surely, see for example [Szn98, Chapter 4, Theorem 4.6].

Now, combining our upper and lower bound, we get

$$
\begin{aligned}
&N\min\left\{\|v_N\|_\infty, C_1^2\|v_N\|_1(e_N^{1,\omega})^{d/2}\right\}\\
&\quad\geq e_N^{1,\omega}\left(\sum_{j=1}^{J_N^\omega-1}\frac{n_N^{j,\omega}}{N}-1\right) + e_N^{J_N^\omega,\omega}\left(1-\sum_{j=1}^{J_N^\omega-1}\frac{n_N^{j,\omega}}{N}\right)\\
&\quad= \left(1-\sum_{j=1}^{J_N^\omega-1}\frac{n_N^{j,\omega}}{N}\right)\left(e_N^{J_N^\omega,\omega}-e_N^{1,\omega}\right)\ ,
\end{aligned}
$$

and therefore

$$
\left(1-\frac{1}{N}\sum_{j=1}^{J_N^\omega-1} n_N^{j,\omega}\right) \leq \frac{N\min\left\{\|v_N\|_\infty, C_1^2\|v_N\|_1(e_N^{1,\omega})^{d/2}\right\}}{e_N^{J_N^\omega,\omega}-e_N^{1,\omega}} \leq \frac{1}{\gamma_N}\ . \tag{4}
$$

Now, the right-hand side of (4) converges to zero, and $e_N^{J_N^\omega-1,\omega}$ converges $\mathbb{P}$-almost surely to zero, due to the definition of J_N^ω. Hence, $\mathbb{P}$-almost surely,

$$
\lim_{\varepsilon\searrow 0}\liminf_{N\to\infty}\frac{1}{N}\sum_{j\in\mathbb{N}:e_N^{j,\omega}\leq\varepsilon} n_N^{j,\omega} \geq \liminf_{N\to\infty}\frac{1}{N}\sum_{j=1}^{J_N^\omega-1} n_N^{j,\omega} = 1\ .
$$

□

Remark 3. *For weak interactions (as specified below), one may actually show that $\varphi_N^{1,\omega}$, the ground state eigenfunction of the Dirichlet-Laplacian $-\Delta$ on Λ_N^ω, is macroscopically occupied in probability; this then implies that the condensation* *is not of type III. To see this, we proceed similarly as in the proof of Theorem 2: One starts with the estimates*

$$E^{1,\omega}_{QM,N} \leq N e^{1,\omega}_N + \frac{N(N-1)}{2} \min\{\|v_N\|_\infty, C_1^2 \|v_N\|_1 (e^{1,\omega}_N)^{d/2}\} \ ,$$

with a constant $C_1 > 0$, *and*

$$E^{1,\omega}_{QM,N} \geq N \left(\frac{n^{1,\omega}_N}{N} e^{1,\omega}_N + \left(1 - \frac{n^{1,\omega}_N}{N}\right) e^{2,\omega}_N \right) \ .$$

Combining the upper and lower bound, we obtain

$$1 - \frac{n^{1,\omega}_N}{N} \leq \frac{N \min\{\|v_N\|_\infty, C_1^2 \|v_N\|_1 (e^{1,\omega}_N)^{d/2}\}}{e^{2,\omega}_N - e^{1,\omega}_N} \ .$$

Now, we use the facts that $\mathbb{P}$*-almost surely* $e^{1,\omega}_N \sim (\ln N)^{-2/d}$ *and that*

$$\lim_{\sigma \to 0} \liminf_{N \to \infty} \mathbb{P}\left(e^{2,\omega}_N - e^{1,\omega}_N \geq \sigma (\ln N)^{-(1+2/d)}\right) = 1 \ ,$$

see [Szn23, Theorem 6.1]. *Therefore, if the two-particle interactions is sufficiently weak in the sense that*

$$\|v_N\|_\infty \ll \frac{1}{(\ln N)^{1+(2/d)} N} \quad \text{or} \quad \|v_N\|_1 \ll \frac{1}{(\ln N)^{2/d} N} \ , \tag{5}$$

we have

$$\lim_{N \to \infty} \mathbb{P}\left(\left| \frac{n^{1,\omega}_N}{N} - 1 \right| < \zeta \right) = 1$$

for all $\zeta > 0$, *that is,* $\varphi^{1,\omega}_N$ *is macroscopically occupied in probability. This should be compared with the results obtained in* [BKP24].

Let us further remark on Theorem 2: In [BKP24] the authors proved macroscopic occupation (with probability almost one) of a certain one-particle state (that is, the minimizer of a Hartree-type functional) in the Kac–Luttinger model for, among others, an interaction potential $v_N : \mathbb{R}^d \to \mathbb{R}_+$ of the form

$$v_N(x) = \kappa V(x) \frac{1}{(\ln N)^{2/d} N} \tag{6}$$

where $V \in (L^1 \cap L^\infty)(\mathbb{R}^d, \mathbb{R}_+)$ is a suitable potential (satisfying some extra assumptions that we can neglect here) and $\kappa > 0$ is sufficiently small; note that $\|v_N\|_1 \sim 1/[(\ln N)^{2/d} N]$. Theorem 2 now shows that generalized BEC is present, for example, for an interaction potential of the form

$$v_N(x) = V(x) \frac{(\ln N)^{1-\epsilon}}{N} \tag{7}$$

with non-zero $V \in (L^1 \cap L^\infty)(\mathbb{R}^d, \mathbb{R}_+)$ and any $\epsilon > 0$ (here one refers to the L^1-norm condition in Theorem 2). In other words, g-BEC can be proved for interactions with a higher power of $\ln N$!

In order to better understand the meaning of this, let us compare Theorem 2 also with results from [KP23] where the authors investigate absence of macroscopic occupation of *sufficiently localized* states for interaction potentials of sufficient strength. While the setting discussed in [KP23] is fairly similar to the one discussed here, it is important to note that [KP23] assumes a positive temperature, whereas we assume zero temperature, and they work solely in the non-percolation regime (meaning one chooses the density $\nu > 0$ of the underlying Poisson process large enough; note that the components of the vacancy set are relatively small in this regime). To compare, consider

$$v_N(x) = V(\|x\|)\, c_N$$

with $c_N > 0$ and a bounded continuous and positive $V \in L^1(\mathbb{R}, x^{d-1}\mathrm{d}x)$; note that $\|v_N\|_1 \sim c_N$. For such a potential, it has been proved in [KP23, Theorem 4.2] that no state localized on only one component of the vacancy set (as an example, one could think of the ground state of the one-particle Dirichlet-Laplacian, which is supported on only one component) can be macroscopically occupied whenever

$$\frac{(\ln N)^3}{N} \ll c_N \ll \frac{1}{(\ln N)^2}\ .$$

Consequently, again comparing the potentials (6) and (7), we see that the extra powers of $\ln N$ are not negligible since a highly localized condensate can be excluded, at least when the temperature is non-zero, already for $c_N = (\ln N)^{3+\epsilon}/N$ with any $\epsilon > 0$.

In addition to that, it is insightful to compare Theorem 2 with results obtained in [KPS19]. In that paper, the authors discuss interacting Bose gases in the one-dimensional analogue of the Kac–Luttinger model, the so-called Luttinger–Sy model. More explicitly, the authors consider contact interactions between the particles of the form

$$v_N(x) = g_N \delta(x),$$

where $g_N > 0$ describes the interaction strength. Note that g_N can, informally, be understood as the L^1-norm of v_N. In [KPS19, Theorem 3.1] the authors then prove generalized condensation for interactions strengths that satisfy

$$g_N \ll \frac{1}{(\ln N)^2 N^\eta}$$

for an $0 < \eta \leq 1/3$. Most importantly, however, in [KPS19, Theorem 3.3] a transition in the type of condensation is established: whereas for interaction strengths

$$g_N \ll \frac{1}{(\ln N)^2 N}$$

one almost surely has BEC of type I or type II (implying a macroscopic occupation of at least one one-particle state), BEC is almost surely of type III whenever

$$\frac{1}{(\ln N)N} \ll g_N \ll \frac{1}{(\ln N)^2 N^\eta}$$

with an $0 < \eta \leq 1/3$. Translating this to the higher-dimensional Kac–Luttinger model, and, in particular, recalling (5), one might ask whether there exists a $\gamma \leq 2/d$ such that type-III condensation occurs for interaction potentials v_N considered in Theorem 2 that additionally satisfy

$$\|v_N\|_1 \gg \frac{1}{(\ln N)^\gamma N} \ .$$

In other words, is it possible to prove for such potentials that for all $j \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{n_N^{j,\omega}}{N} = 0$$

in some probabilistic sense? This would imply that the family of low-energy eigenstates of the Dirichlet Laplacian is macroscopically occupied as a whole, since one has generalized BEC according to Theorem 2, but that each individual state of that family is itself not macroscopically occupied, manifesting a type-III condensate.

Acknowledgement. This research was supported through the program "Oberwolfach Research Fellows" by the Mathematisches Forschungsinstitut Oberwolfach in 2024.

References

[BKP24] C. Boccato, J. Kerner, and M. Pechmann, *Interacting many-particle systems in the random Kac–Luttinger model and proof of Bose–Einstein condensation*, J. Math. Pures Appl. **189** (2024), 103594.

[BLP86] M. van den Berg, J. T. Lewis, and J. V. Pulé, *A general theory of Bose-Einstein condensation*, Helv. Phys. Acta **59** (1986), no. 8, 1271–1288.

[Gir60] M. Girardeau, *Relationship between systems of impenetrable bosons and fermions in one dimension*, J. Math. Phys. **1** (1960), no. 6, 516–523.

[JPZ10] T. Jaeck, J. V. Pulé, and V. Zagrebnov, *On the nature of Bose–Einstein condensation enhanced by localization*, J. Math. Phys. **51** (2010), no. 10, 103302.

[KL73] M. Kac and J. M. Luttinger, *Bose–Einstein condensation in the presence of impurities*, J. Math. Phys. **14** (1973), no. 11, 1626–1628.

[KL74] ———, *Bose–Einstein condensation in the presence of impurities. II*, J. Math. Phys. **15** (1974), no. 2, 183–186.

[KP23] J. Kerner and M. Pechmann, *Bose–Einstein condensation for particles with repulsive short-range pair interactions in a Poisson random external potential in $\mathbb{R}^d$*, J. Appl. Probab. **60** (2023), no. 2, 382–393.

[KPS19] J. Kerner, M. Pechmann, and W. Spitzer, *Bose–Einstein condensation in the Luttinger–Sy model with contact interaction*, Ann. Henri Poincaré, vol. 20, Springer, 2019, pp. 2101–2134.

[LSSY05] E. H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason, *The mathematics of the Bose gas and its condensation*, vol. 34, Springer Science & Business Media, 2005.

[MR96] R. Meester and R. Roy, *Continuum percolation*, vol. 119, Cambridge University Press, 1996.

[Szn98] A.-S. Sznitman, *Brownian motion, obstacles and random media*, Springer Science & Business Media, 1998.

[Szn23] ______, *On the spectral gap in the Kac–Luttinger model and Bose–Einstein condensation*, Stoch. Process. Their Appl. (2023), 104197.

Department of Mathematics, Università di Pisa, 56127 Pisa, Italy
Email address: chiara.boccato@unipi.it

Department of Mathematics and Computer Science, FernUniversität in Hagen, 58084 Hagen, Germany
Email address: joachim.kerner@fernuni-hagen.de

Department of Mathematics, Tennessee Technological University, 110 University Drive, Cookeville, TN 38505, USA
Email address: mpechmann@tntech.edu

Department of Mathematics and Computer Science, FernUniversität in Hagen, 58084 Hagen, Germany
Email address: wolfgang.spitzer@fernuni-hagen.de