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Brauer’s problems: 60 years of legacy

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Richard Brauer (1901–1977) was a German–American mathematician who is regarded as the founder of a highly active mathematical area known as modular representation theory. This area grew from group theory, which can be thought of as the mathematical study of symmetries. In this snapshot, we hope to impress on the reader the legacy left by Brauer and celebrate the 60th anniversary of “Brauer’s problems”, a list of 43 conjectures and objectives suggested by Brauer in 1963. These problems inspired an entire branch within character theory, studying “local-global conjectures”.

1 Groups and representations

Groups are mathematical objects that encode symmetries. They can also be thought of as abstract number systems: roughly speaking, a group is a set G on which there is an operation $*$ that allows you to combine two elements a and b

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in G to obtain another element $a * b$ in G. This operation has to satisfy certain rules. Before specifying these, let us consider an example of a group that we are all familiar with: the set of integers $\mathbb{Z}$ under addition. Two integers a and b can be added up to obtain another integer $a + b$. The number 0 plays a special role as adding it to any integer "does nothing". Furthermore, every integer n has the "inverse" $-n$, because adding $-n$ to n results in 0. Another example is given by the real numbers $\mathbb{R}$, this time under multiplication. Here, multiplying by 1 does nothing, and every real number r has the inverse $\frac{1}{r}$.

The definition of a group models the properties that we are used to from these two examples. A set G together with an operation $*$ is called a *group* if it satisfies the following properties:

- There exists an *identity element* $e \in G$ that "does nothing", that is, such that $e * g = g = g * e$ for all $g \in G$.
- Every $g \in G$ has an *inverse* $g^{-1} \in G$ that satisfies $g * g^{-1} = e = g^{-1} * g$.
- The operation is *associative*, meaning that $(g * h) * k = g * (h * k)$ for all $g, h, k \in G$.

In practice, the abstract group operation is often simply referred to as "multiplication", and the symbol $*$ is omitted.

Perhaps the reader has already caught us in a lie – since we cannot divide by 0, this number has no inverse. This means that the set of real numbers under multiplication does not truly form a group! Instead, we obtain a group from the set of non-zero real numbers under multiplication. (This time, we have not lied!) The reader may also have noticed that the two sets in the examples given so far have infinitely many elements. However, our protagonist Brauer was particularly interested in finite groups, where the underlying set only has a finite number of elements. For an example of a finite group, consider the set $\{0, 1\}$ with binary addition given by $0 + 0 = 0$, $1 + 0 = 1$, $0 + 1 = 1$, and $1 + 1 = 0$.

1.1 "That napkin example"

The reader may be asking themselves what this has to do with symmetry. Indeed, one of the most powerful uses of groups lies in their ability to describe the symmetry of objects.[3] As an example, consider a square napkin at your dinner table, with one flat side parallel to the edge of the table. Your task is to consider all of the ways you can move the napkin and place it back on the table so that one side is again parallel to the edge of the table (and without any folding). You might notice that you can rotate by 90°, 180°, 270°, or 360°. (The latter is equivalent to not moving the napkin at all!) You might also notice

[3] We refer the reader to another snapshot [15], which gives a beautiful overview of the idea of group theory from this perspective.

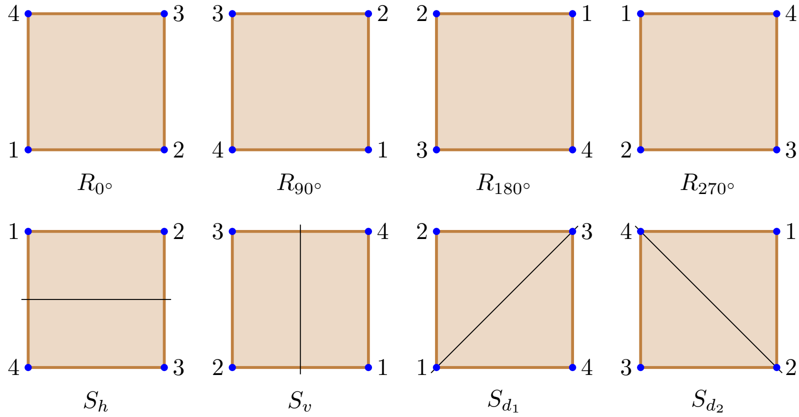

Figure 1: The dihedral group D_8.

that you can flip the napkin over an imaginary horizontal, vertical, or one of two diagonal lines through the center of the napkin. It turns out that these eight transformations are the only ones! You can do them in sequence, and always end up with your napkin in one of the same eight positions.

Abstractly, these transformations on your napkin correspond exactly to the symmetries of the square. They form what we call the *dihedral group* D_8, a finite group of size 8 (see Figure 1). The identity element is the rotation by 0°, and the reader is invited to verify that every symmetry has an inverse (for example, the inverse of the rotation by 90° is the rotation by 270°). Moreover, this is the first example we have given where the operation is not *commutative* – that is, it matters in which order we carry out the transformations. A group whose operation is commutative is called *abelian*.

In the same way, one can form the group of symmetries of an equilateral triangle, a regular pentagon, hexagon, and so on! On the surface, this may seem like nothing more than a mediocre dinner party trick, but the reader might imagine how this same concept can be applied to study the spin of particles in physics, the structure of chemical molecules, neural or social networks, and the list goes on!

1.2 Group representations

Given a group G, one wants to understand its *group structure*, that is, in what way the elements of the group interact with each other under the group operation. This structure can be quite complicated, even if G is finite. To get a better understanding of it, one studies its "representations". This means that

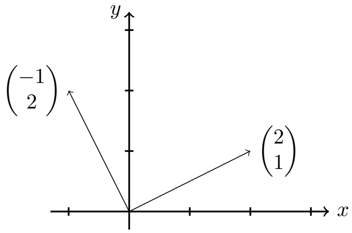

Figure 2: A vector of $\mathbb{R}^2$ rotated by 90°.

instead of looking at the original group, one "represents" the group by certain groups of "invertible matrices", since these are often simpler to deal with.

Matrices are arrays of numbers, which are often first introduced to students to solve systems of linear equations. Seen differently, matrices describe *linear transformations*, which are structure-preserving functions between vector spaces. For example, the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

describes the linear transformation of the real 2-dimensional vector space $\mathbb{R}^2$ which is given by (counterclockwise) rotation by 90°: If one multiplies an arbitrary vector $(x, y) \in \mathbb{R}^2$ with the above matrix, one obtains

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \cdot x + (-1) \cdot y \\ 1 \cdot x + 0 \cdot y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \in \mathbb{R}^2,$$

which is exactly the rotated vector (see Figure 2).

Here, we multiplied the matrix with the vector by the rules of *matrix multiplication*. To obtain a group of matrices under matrix multiplication, one has to consider square matrices (matrices that have the same number of rows and columns), because multiplying two of these yields again a square matrix of the same size. Furthermore, the matrices have to be *invertible*, meaning that they have an inverse with respect to matrix multiplication (where the identity element is the square matrix who has only 1's on the main diagonal and 0's everywhere else). The invertible square matrices of size n with entries in $\mathbb{R}$ form the *general linear group*, which is denoted $GL_n(\mathbb{R})$. Each matrix in this group represents an invertible linear transformation of $\mathbb{R}^n$, and the multiplication of two matrices corresponds to carrying out two transformations after each other.

Now a (real) *group representation* of a group G is a group homomorphism

$$\rho\colon G \to GL_n(\mathbb{R}).$$

This means that ρ assigns to every element $g \in G$ a matrix $\rho(g) \in GL_n(\mathbb{R})$ in a way that is compatible with the two group operations. That is, the multiplication of two elements in G is sent to the multiplication of the two corresponding matrices in $GL_n(\mathbb{R})$. Written formally, one has

$$\rho(g * h) = \rho(g) \cdot \rho(h),$$

where $*$ denotes the operation of G and $\cdot$ denotes matrix multiplication. Given such a representation, one can study the behaviour of the matrices instead of that of the elements of G. Although matrix multiplication can be onerous, it is well understood, unlike the abstract type of operation found in many groups! With this, we are able to use the full power of linear algebra to study our sometimes complicated abstract group.

Let us look at an example of a group representation of our napkin group D_8.[4] There is a group homomorphism $\rho\colon D_8 \to GL_2(\mathbb{R})$ that sends the elements of D_8 to the following matrices:

$$\rho(R_{0^\circ}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(S_h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho(R_{90^\circ}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho(S_v) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho(R_{180^\circ}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(S_{d_1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho(R_{270^\circ}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \rho(S_{d_2}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

These matrices are exactly the symmetries of the square but seen as linear transformations on the whole plane. The reader is invited to verify that this really defines a group homomorphism.

An important class of group representations are the irreducible ones. A representation $\rho\colon G \to GL_n(\mathbb{R})$ is called *irreducible* if there is no linear subspace of $\mathbb{R}^n$ that remains fixed under the transformations $\rho(g)$ for all $g \in G$. For example, the linear subspaces of $\mathbb{R}^2$ are all the lines through the origin. Therefore, the example above is an irreducible representation, as there is no line that remains fixed under all the symmetries of the square. Maschke's Theorem states that every (real or complex) representation of a finite group can be decomposed into a sum of irreducible representations. This means that irreducible representations are in a sense the "building blocks" of all representations (analogous to viewing the prime numbers as the building blocks of all integers, or the chemical elements in the periodic table as the building blocks of matter).

[4] More examples of group representations can be found in the snapshot [14].

1.3 Character tables

In *ordinary* representation theory, the matrices take entries in the complex numbers, meaning that one studies group homomorphisms

$$\rho\colon G \to GL_n(\mathbb{C}).$$

In this case, a remarkable amount of information about the representations of a finite group is encoded in its *characters*. The character of an element $g \in G$ is obtained by simply taking the sum of the entries along the main diagonal of the matrix $\rho(g)$. As the reader might imagine, this condenses the information significantly! Instead of studying sets of large matrices, one only needs to study one number for each matrix! This allows us to condense the ordinary representation theory of a finite group G into what is known as a *character table*, which consists of the characters $\chi(g)$ for every group element $g \in G$.

In our example of the napkin group D_8, the matrices have real entries, but as real numbers are also complex numbers, one can interpret the representation as a group homomorphism $G \to GL_2(\mathbb{C})$. The character table for this representation is the following:

g	R_{0°	R_{90°	R_{180°	R_{270°	S_h	S_v	S_{d_1}	S_{d_2}
$\chi(g)$	2	0	-2	0	0	0	0	0

A complete character table would include the characters of every single irreducible complex representation of G (if G is finite, there are only finitely many of these).

This leads us to two overarching questions in character theory:

1. Can we calculate the character table of a given finite group G?
2. Given a character table, what can we say about the group that it came from?

Much work has been done on the first of these questions, and there is even a book called the *Atlas of Finite Groups* [10] that includes many of the known character tables. But unfortunately, this only goes so far – there are many even relatively small groups for which we still do not know the full character table!

For our purposes, we focus on the second question. The reader might initially hope that, given a character table, we can determine exactly what group it came from. But it turns out that the second question is more subtle than this – in fact, the groups D_8 and another group Q_8 of size 8 (the "quaternion group") have identical character tables! This leads us to Richard Brauer, who asked many questions related to this one.

2 Richard Brauer

Richard Brauer was born in 1901 in Berlin-Charlottenburg, Germany, and studied under Issai Schur (1875–1941) to obtain his doctorate in 1926. He held positions at the University of Königsberg, the University of Kentucky, the Institute for Advanced Study in Princeton, the University of Toronto, the University of Michigan, and finally Harvard University, from which he retired in 1971. He passed away in 1977. During his career, he won several famous awards (including the Cole Prize and the National Medal of Science), served as an editor on a number of top-tier journals, and left a lasting impression on the areas of group theory and representation theory that we can only begin to discuss in this snapshot.[5]

2.1 Brauer's problems

In 1963, Brauer published a paper [4] which enumerated 43 problems that he saw as the main questions to be answered in group theory and character theory. These problems had perplexed him for quite some time, and he gave some insight into his own thoughts on them. Many of Brauer's problems are in the realm of "local-global" theory (and, really, they began this sort of study in character theory). This means that they posit relationships between the character theory of a finite group and that of certain nice "local subgroups".

A *subgroup* is a smaller group sitting inside the original group under the same operation. For example, the non-zero rational numbers are a subgroup of the non-zero real numbers, because multiplication of rational numbers leaves you inside the rational numbers. Similarly, the four rotations of your napkin form a subgroup of the dihedral group D_8. Each group has two "trivial subgroups", namely the group itself and the subgroup $\{e\}$ that only consists of the identity element. Studying the properties of the group's subgroups and their relationship to each other is essential for understanding the group structure.

The "local" subgroups appearing in Brauer's problems are defined by a given prime number p – one can think of a local-global problem as zooming in on the group through the point of view of that prime. For example, one of the most quintessential local subgroups is known as a *Sylow p-subgroup* (named after the mathematician Peter Ludvig Sylow (1832–1918)). This is a subgroup whose size is p^a, where p^a is the largest power of p that divides the size of the group. More generally, a *p-group* is one whose size is a power of p. Just this one example of a local subgroup already seems to control an immense amount of information about the character theory of the group. For example, one of the easier-to-state problems of Brauer is the following, number 12 on his list:

[5] See https://mathshistory.st-andrews.ac.uk/Biographies/Brauer/ for a biography of Brauer.

Brauer's Problem 12. *Given the character table of a finite group G, what can be said about a Sylow p-subgroup P? In particular, can it be determined whether or not P is abelian?*

This question allows different interpretations. For instance, it was proved in [19] that if two finite groups have the same character table and one of them has abelian Sylow p-subgroups, then the same holds for the other one. In [28], a different solution is given, providing an explicit criterion for the existence of abelian Sylow p-subgroups. The reader may notice that Problem 12 is quite open-ended. We can study this question forever and continue to find more relationships between the character table and properties of a Sylow p-subgroup! This is the case with many of the problems in Brauer's list. For example:

Brauer's Problem 7. *Study the irreducible characters of p-groups.*

3 The classification of finite simple groups

Brauer's problems inspired a number of other questions that are deep-rooted in group theory. At the same time that Brauer's problems were first being studied, group theorists were in the thick of another transformative program: the classification of finite simple groups (or simply the CFSG).

3.1 The classification

We saw in subsection 1.2 that we are interested in studying irreducible representations, because every group representation is in fact a sum of these. This principle occurs in many areas of mathematics: once a mathematical object is introduced, we become interested in studying the "building blocks" of those objects. For example, the area of number theory began as the study of prime numbers, which can be thought of as the building blocks of the integers. For group theory, the corresponding notion is that of "finite simple groups".

To introduce simple groups, we need to talk about an important relation between group elements: We say that two elements g and h of a group G are *conjugate* if there exists an element $x \in G$ such that $x^{-1}gx = h$. This means that g can be transformed into h by multiplying it by x (from the right) and by the inverse of x (from the left). Intuitively, g is almost the same as h, just shifted by x, and the two group elements behave in a similar way. For instance, in our napkin group D_8, the elements S_h and S_v are conjugate as $S_h = R_{90^\circ}^{-1} \circ S_v \circ R_{90^\circ}$ (where $\circ$ denotes the operation of D_8). This matches our intuition that reflection on the horizontal line is the same as reflection on the vertical line rotated by 90°.

We say that a subgroup N of G is *normal* in G if it is closed under conjugation, that is, if for every $n \in N$ and for every $x \in G$ we have that $x^{-1}nx \in N$. Normal subgroups are important as they allow us to "divide" our group into two smaller pieces, namely into the group N and a new group, the *factor group* G/N. In this new group, elements of G are identified if they only differ by an element of N. For example, the subgroup of D_8 that consists of the four rotations is a normal subgroup (the reader is invited to verify this). Let us denote it by R. The factor group D_8/R only has two elements: the element that represents all reflections (as these are all "the same" up to some rotation) and the one that represents all rotations (which is the identity element).

We say that a group is *simple* if it has no normal subgroups (except of the trivial subgroups $\{e\}$ and G, which are always normal). If a group is simple, we can not divide it into smaller groups using the above trick! For any finite group, we can do the "factoring trick" repeatedly until we get to the situation where all the factor groups appearing in the process are simple. In this way, we can decompose any finite group into finite simple groups.

Let us carry out this process for our napkin group D_8. We already found the normal subgroup R consisting of the four rotations and explained that D_8/R consists of only two elements, which implies that it is simple. Now we repeat this process with R: Inside R, there is another normal subgroup $R' = \{R_{0^\circ}, R_{180^\circ}\}$. In the factor group R/R', the rotations R_{0° and R_{180° as well as R_{90° and R_{270° are identified, as they only differ by R_{180°. Consequently, R/R' also has only two elements and is therefore simple. Since R' itself is simple, we can not continue, thus we are done. This means that for D_8, we have found the three simple groups D_8/R, R/R' and R'.

The Jordan–Hölder Theorem assures us that it does not matter how you do this process, you always end up with the same simple groups. This is why the finite simple groups can be thought of as the "building blocks" of all finite groups (and why it was so important to classify them!).

For the majority of the 20th century, especially in its second half, the primary goal in group theory was to completely classify the finite simple groups. After some hiccups,[6] it was finished in the early 2000's. The remarkable result of the CFSG is that every finite simple group belongs to one of four specific classes of groups.[7] Brauer himself made huge contributions to this program, especially notably his article with Fowler *On Groups of Even Order* [5], which set the stage for the strategy that was eventually successful in completing the classification.

3.2 Using the classification

As the classification project was coming to a close (or, so it was thought) in the 1980s, group theorists started to imagine the world post-classification. What

[6] The story of the CFSG is a tale in itself! For a more complete history, see [34].

[7] These four classes of simple groups are described in more detail in the snapshot [12].

would group theory be like once the CFSG was completed? Would it be the "death" of the area?[8] The authors of this article would, of course, argue that the answer to the latter is a resounding "no!", but that it certainly changed the landscape of group theory. In fact, the classification opened the doors to new possibilities. In 1984, Gerhard Michler published a paper [25] posing the question: "Which of the open problems [from Brauer's list] can now be proved by means of the classification theorem?" He goes on to specifically suggest that perhaps the CFSG could be used to make progress on Brauer's Problems 12, 23 (Brauer's Height Zero Conjecture), and 20 (Brauer's $k(B)$-Conjecture). The strategy would be (and, indeed, has become) to approach large problems in group theory using two steps:

1. Prove that *condition A* holds for every finite simple group by proving that it holds for every class of finite simple groups individually.
2. Prove that if *condition A* holds for every finite simple group, then *property B* holds for every finite group.

The idea behind the second step is to show that once *condition A* is verified for all simple building blocks of a finite group G, then G as a whole must inherit *property B* when assembling the simple pieces back together. In this way, many problems on finite groups can be reduced to problems on finite simple groups.

Among the first complete examples of the use of this strategy is the Itô–Michler Theorem, proved as [25, Theorem 2.3], in a sense as a proof of concept and as a step toward Brauer's Height Zero Conjecture and Brauer's Problem 12.

The Itô–Michler Theorem. *A finite group G has a normal abelian Sylow p-subgroup if and only if every irreducible character of G has degree not divisible by p.*

This theorem illustrates a common theme among local-global problems: the idea of the *degree* (the size of the matrices in a representation) not being divisible by your chosen prime p. This idea also appears in the *McKay Conjecture*, which was introduced in 1972 by John McKay in a short note [24] dedicated to Brauer's 70th birthday.

The McKay Conjecture. *For a finite group, the number of irreducible representations whose degree is not divisible by p is the same as that for the normalizer[9] of any of its Sylow p-subgroups.*

The McKay Conjecture was reduced in 2007 to a problem on simple groups by Isaacs, Malle and Navarro in [17]. The authors gave a list of conditions and

[8] There was even a conference in 1979 that was nicknamed "The Funeral"!

[9] The *normalizer* of a subgroup $H \subset G$ is the largest subgroup of G in which H is normal.

showed that the conjecture would hold if these were satisfied by every simple group. (This is step 2 of the strategy!) Using this, the conjecture was proved for the case $p = 2$ by Malle and Späth in 2016 [23]. Over the years, the conditions were also established for several cases for odd primes [35, 36, 37, 8, 9, 39]. Finally, in 2023, Cabanes and Späth announced that they had proven the conjecture for all finite groups and all primes. The proof will appear in [7].

4 Modular representation theory

When we introduced group representations in subsection 1.2, we assumed that our matrices had entries in the complex numbers. But there are many more types of representations that one can consider! The case of *modular representations* deals with matrices that take entries in a *field of characteristic* p, where p is a prime number. The reader may think of a field as a number system with *two* operations (for example, addition and multiplication) satisfying some nice conditions similar to the ones of a group. For example, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ form a field. A field of characteristic p is one where $p \cdot 1 = 0$ (and p is the smallest number with this property). For example, the set $\{0, 1\}$ with binary addition and multiplication is a field of characteristic 2.

Much of Brauer's work was dedicated to relating this more complicated modular representation theory back to the ordinary representation theory. One of the essential tools he developed for studying this relationship is called a "p-block". When studying ordinary characters, Brauer's p-blocks can be thought of as a way to break a given character table into smaller pieces inherent to the prime p. By studying the characters of a finite group block by block, one obtains analogues to many of the pieces of data that we have discussed above.

For example, there is a type of local subgroup known as the "defect group" that plays the role that Sylow p-subgroups played before. The definition of this object is a bit technical and we don't really need it for the purpose of this exposition. It is enough to say that to each p-block, we can associate a uniquely determined class of p-subgroups of G, which are the defect groups of the p-block. We say that a p-block of a finite group G has *defect* d if its associated defect group has p^d elements.

In 1946, Brauer proved [2, Theorem 8] that the number of irreducible characters of a p-block B with defect d, which we denote by $k(B)$, is less than or equal to $p^{d(d+1)/2}$. In that same work, he says that "it is probable that the bound $p^{d(d+1)/2}$ here can be replaced by p^d, but I have been able to prove this stronger result only for $d = 0, 1, 2$." He later added this problem to his list:

Brauer's Problem 20. *Is it true that a* p*-block of defect* d *consists of at most* p^d *irreducible characters?*

This problem is known today as *Brauer's $k(B)$-Conjecture*, and although the bound $p^{d(d+1)/2}$ has been improved since 1946, it is still not known if the more ambitious bound should hold. In fact, this problem is still wide open, in the sense that there is not even a reduction to simple groups. This means that step 2 of the strategy is not solved yet – we don't know which conditions we need the finite simple groups to satisfy in order to obtain the result. However, some important cases have been solved (see [11] for an overview of the progress).

In his Problem 21, Brauer asks what kind of defect groups of a p-block can occur, assuming its number of irreducible characters is known. One way to state this problem is as follows:

Brauer's Problem 21. *Let n be a natural number. Are there finitely many classes of defect groups that can occur as defect groups of a p-block B with $k(B) = n$?*

In [20], Külshammer and Robinson show that the "blockwise" version of the McKay Conjecture (known as the *Alperin–McKay Conjecture*) implies Brauer's Problem 21. Since the Alperin–McKay Conjecture was reduced to a question on simple groups in [38], so is Brauer's Problem 21. However, step 1 of the strategy remains open, although the case of $p = 2$ is now known by [32] and the case of principal blocks was recently resolved in [26].

Another concept that arises when studying p-blocks of modular representations are characters of "height zero". Roughly speaking, a character in a p-block has height zero if its degree is "least divisible by p", meaning that it contains the smallest possible power of p. These characters appear for instance in Brauer's Problem 23, which is a generalization of Problem 12 involving p-blocks:

Brauer's Problem 23. *Is it true that the defect group of a p-block is abelian if and only if all its irreducible characters have height 0?*

This is known as *Brauer's Height Zero Conjecture*, first stated in [3]. Unlike the previous conjectures, here we have a positive solution. After the work of many authors, the final solution to Brauer's Height Zero Conjecture was obtained in [22] and announced in the Oberwolfach workshop "Representations of Finite Groups", held in April 2023. How did it happen after so many years? The first big push was in 1988, when Berger and Knörr reduced the "only if" direction to p-blocks of "quasisimple groups" [1], but it was only verified in 2013 by Kessar and Malle [18]. For the "if" direction, the process was a bit more intricate. After a reduction to simple groups by Navarro and Späth [29], the case $p = 2$ was solved in 2022 [32]. Finally, in [22], the authors gave a complete solution for the "if" direction for odd prime numbers, by means of a different reduction theorem.

We end this snapshot by listing some of Brauer's problems that have been completely solved, and their solutions.

Problem 2*: Dade (1971) [13].
Problem 3: Pyber (1992) [30].
Problem 10: Saksonov (1967) [33].
Problem 12: Kimmerle, Sandling (1995) [19].
Problem 14: Murray, Sambale (2023) [27].
Problem 19: Robinson (1984) [31].
Problem 22: Landrock (1973) [21].
Problem 23: Malle, Navarro, Schaeffer Fry, Tiep (2023) [22].
Problem 34: Herzog (1968) [16].
Problem 35: Bugeaud, Cao, Mignotte (2001) [6].

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