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Median Geometry and Applications

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Abstract. This was the first-ever workshop on median geometry and its applications, bringing together the leading experts from areas of pure and applied expertise in fields currently seen as distinct. Invited experts from mathematical biology, and computer science interacted with specialists from geometry, combinatorics and group theory, all fields in which median geometry plays a key role. The participants worked to understand the similarities and differences in the way in which they modelled and exploited median geometry in their work, developing a dictionary of terms that facilitated communication and fostering collaboration. As well as exploring recent activity in their fields, the participants developed a problem list to guide future development and promote cross disciplinary research.

Mathematics Subject Classification (2020): 05Cxx, 06Dxx, 20F65, 20F67, 51Kxx, 68R10, 91B14, 92D15.

Introduction by the Organizers

The workshop *Median Geometry and Applications*, held at the Mathematisches Forschungsinstitut Oberwolfach from February 15–20, 2026, brought together a diverse group of researchers working at the interface of pure and applied mathematics. It was the first event of its kind dedicated specifically to median geometry and its wide-ranging applications. Participants included experts in geometric group theory, combinatorics, and topology, alongside researchers from computer science, and mathematical biology. This broad participation created a uniquely collaborative atmosphere in which ideas could flow across traditional disciplinary boundaries.

A central aim of the workshop was to identify common structures underlying seemingly disparate problems. Median geometry, through the framework of median algebras, median graphs, and CAT(0) cube complexes, played a key role as a unifying language. Early expository talks set the stage by highlighting two key motivating applications: the median voter theorem from economics and the reconstruction of phylogenetic trees in biology. These examples illustrated how median structures can encode notions of choice, consensus and optimality, and how geometric interpretations can lead to both conceptual clarity and theoretical and computational advances.

One of the significant achievements of the workshop was the collaborative development of a *dictionary of terms* bridging different fields. Researchers observed that similar concepts such as "intervals", "consensus points", or "independent choices/events" appear, sometimes under different names, in economics, computer science, and biology. By systematically identifying these correspondences, participants created a shared vocabulary that will facilitate future interdisciplinary work. This effort proved particularly valuable during the problem session, where open questions from a range of areas were presented, with the dictionary allowing the participants to reframe and reformulate the questions, with the potential of addressing them using tools from another field.

The scientific program reflected a balance between foundational theory and concrete applications. On the theoretical side, several talks advanced the understanding of median spaces and their role in geometric group theory. Notably, new developments in the study of asymptotic cones of groups introduced higher-dimensional analogues of real trees, known as real cubings. These provide a powerful framework for analyzing hierarchically hyperbolic groups, a wide and important class including hyperbolic groups and mapping class groups. Other contributions examined actions of groups on median spaces, revealing deep connections between geometric properties such as non-positive curvature and analytic properties like Kazhdan's property (T) and a-T-menability.

Breakthrough results were also presented in topology and combinatorics. A highlight was the resolution of longstanding conjectures related to the modified Hauptvermutung and weighted strong factorization problems, demonstrating the continuing impact of combinatorial and geometric methods on classical questions.

From a combinatorial and algorithmic perspective, median structures were shown to offer powerful tools for efficient computation. New results on centrality and optimization problems in median graphs demonstrated how geometric properties can be exploited to design faster algorithms. In particular, recent progress on the weighted center problem highlighted how the intrinsic convexity-like features of median spaces can break classical complexity barriers. These developments are of direct relevance to network analysis and data science, where median-based models arise naturally.

The workshop also showcased important applications in computer science. Median graphs and CAT(0) cube complexes play a central role in concurrency theory, where they provide geometric models for event structures and Petri nets. Several talks reported significant progress on long-standing conjectures in this area, including new counterexamples and refined classifications of event structures. These results deepen the understanding of distributed computation and highlight the surprising interplay between discrete geometry and theoretical computer science.

In biology, median structures appeared prominently in phylogenomics. Researchers discussed the geometry of tree spaces and median networks as models for evolutionary relationships. Efficient algorithms for computing geodesics in these spaces enable statistical comparisons of phylogenetic trees, with applications to evolutionary biology and genomics. These contributions illustrated how abstract geometric ideas can inform practical data analysis in the life sciences.

Economics provided another classical, but important, domain of application. The study of preference aggregation on median spaces, including generalizations of the median voter theorem, demonstrated how geometric insights can inform social choice theory. In particular, results on Condorcet winners and single-peaked preferences highlighted the role of median structures in ensuring stability and strategy-proofness in voting systems.

A defining feature of the workshop was the active participation of researchers at all career stages. Early-career mathematicians presented innovative ideas alongside established experts, contributing to a dynamic and inclusive scientific environment. Many of the most exciting developments arose from informal discussions and collaborations initiated during the week, underscoring the importance of the Oberwolfach format in fostering deep engagement.

The program also included several survey lectures aimed at making advanced topics accessible to a broad audience. These talks were instrumental in bridging gaps between disciplines and enabling participants to engage meaningfully with unfamiliar areas. Evening sessions and informal discussions further enriched the scientific exchange, allowing for the exploration of speculative ideas and long-term research directions.

In summary, the workshop successfully demonstrated the unifying power of median geometry across a wide spectrum of mathematical and applied disciplines. By bringing together diverse perspectives, it not only advanced the state of knowledge in several areas but also laid the groundwork for future collaborations. The development of a shared dictionary of concepts, the identification of common problems, and the presentation of significant new results all point to a vibrant and growing field at the intersection of pure and applied mathematics. The overwhelmingly positive feedback from participants confirms that the workshop achieved its goal of fostering sustained interdisciplinary dialogue, and it is anticipated that the connections established here will lead to further breakthroughs in the years to come.

The organisers would like to thank the staff and leadership of the MFO for their hospitality throughout our visit and for all their efforts in preparing for the workshop. The institute is a paradise for mathematicians, and that reputation

is maintained by the professionalism and diligence of the entire team. Their efforts make a major contribution to the atmosphere of friendship, collegiality and successful collaboration, which is characteristic of the MFO experience.

Workshop: Median Geometry and Applications

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Abstracts

Hauptvermutung, Tietze's dream and Oda's strong factorization conjecture

Karim Adiprasito

Consider two triangulations of the same space. How can they be related? This is the subject of the infamous Hauptvermutung, stated by Steinitz and Tietze in 1908 in slightly different forms. In particular, Tietze's formulation modernized, is as follows:

Question 1 (Hauptvermutung (version Tietze), [3])**.** Given two homeomorphic simplicial complexes, is there a common refinement that is reached by elementary moves?

This Steinitz asked essentially the same question, but without asking for elementary moves. Now, of course, the story goes on that Milnor in 1961 proved that this is not true [2].

Now, the question remained: what happens if one is additionally interested in PL homeomorphisms only. So, the revised Tietze question would be:

Question 2 (Modified Hauptvermutung after Tietze)**.** Given two PL homeomorphic simplicial complexes, is there a common refinement that is reached by elementary moves?

That is perhaps more realistic, because PL homeomorphic complexes are related by elementary moves called stellar moves. But a common refinement is more difficult, and it remained an open conjecture, often attributed to Alexander (though it should have been probably attributed to Tietze).

A similar conjecture emerged in toric geometry, often called the strong factorization conjecture of Oda.

The author and Pak proved the following theorem in 2024, resolving Tietze's modified problem and a slight weakening of Oda's conjecture

Theorem 1 (Adiprasito-Pak)**.** *The weighted strong factorization conjecture is true, and the modified Hauptvermutung is true as well.*

I presented a proof, which relies on ideas from earlier work of the author [1].

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Median metric groups

Pénélope Azuelos

A *metric group* is a group equipped with a metric which is invariant under left-multiplication. Equivalently, it is a metric space equipped with a free transitive group action by isometries. Important examples of such groups include finitely generated groups equipped with the word metric induced by a finite generating set and Lie groups equipped with an invariant Riemannian metric. In this talk, I discussed some constructions of metric group structures on connected finite rank median spaces. With the exception of $\mathbb{R}^n$, these groups cannot be locally compact by a result of Messaci [3] and in general they are also not topological groups (while left-multiplication is isometric, right-multiplication often fails to be continuous).

Definition. An $\mathbb{R}$-*tree* is a geodesic metric space (X, d) such that any pair of points in X is connected by a unique arc. For any cardinal $\kappa \geq 2$, the *universal* κ*-tree* T_κ is the unique complete $\mathbb{R}$-tree such that every point has valence κ.

In [2], Casals-Ruiz, Hagen and Kazachkov prove that, for any cardinal $0 \leq \kappa \leq 2^{\aleph_0}$, there is a metric group structure on $T_{2^{\aleph_0}}$ such that there are precisely κ orbits of lines in $T_{2^{\aleph_0}}$ which have transitive stabiliser. In particular, there are infinitely many non-isomorphic groups which act freely and transitively on this $\mathbb{R}$-tree. One feature of their construction is that any line stabiliser will be either trivial, cyclic or transitive, raising the question:

Question. Does there exist a group G which acts freely and transitively on $T_{2^{\aleph_0}}$ such that, for some line $L \subseteq T_{2^{\aleph_0}}$, the stabiliser $\mathrm{Stab_G(L)}$ is a dense proper subgroup of $\mathbb{R}$?

To answer this question, and some related questions regarding actions on higher rank median spaces, one approach is to abstract the construction of the free group from the set of words in some alphabet. This gives rise to the notion of an *ore*, which is a cancellative monoid with involution satisfying some additional assumptions, including that it is equipped with a median structure arising from a partial order. From an ore Y, one finds a canonical subset of *reduced* elements on which a group operation can be defined which preserves the median structure. I explained how this can be used to prove the following result from [1]:

Let $\mathrm{Sub}_{NC}(\mathbb{R})$ denote the set of non-cyclic subgroups of $\mathbb{R}$ and let $\mathcal{K}$ denote the set of cardinals $0 \leq \kappa \leq 2^{\aleph_0}$.

Theorem 1. *Let $\iota : \mathrm{Sub}_{NC}(\mathbb{R}) \to \mathcal{K}$ be any map which is supported on $\leq 2^{\aleph_0}$ elements. Then there exists a group G which acts freely and transitively on $T_{2^{\aleph_0}}$ such that the following holds. For all $H \in \mathrm{Sub}_{NC}(\mathbb{R})$, if A_H is the set of orbits of lines $G \cdot L$ such that the induced action of the stabiliser* $\mathrm{Stab}_G(L)$ *on L is isomorphic to the action of H on $\mathbb{R}$ by translations, then $|A_H| = \iota(H)$.*

In particular, taking ι to be the characteristic map of $\mathbb{Q}$ (or any other dense proper subgroup of $\mathbb{R}$), we get a positive answer to the question above. One can also use this to strengthen Casals-Ruiz, Hagen and Kazachkov's result as follows:

Corollary 2. *There are* $2^{2^{\aleph_0}}$ *isomorphism classes of groups which act freely and transitively on* $T_{2^{\aleph_0}}$*.*

An analogue of this theorem can also be proven in the setting of ℓ^1 products of $T_{2^{\aleph_0}}$, where one controls the stabilisers of maximal flats rather than lines. One consequence of this is the following (also from [1]):

Theorem 3. *There exists a group* G *acting freely and transitively on the* ℓ^1 *product* $X = T_{2^{\aleph_0}} \times T_{2^{\aleph_0}}$ *such that any subgroup* $H \leq G$ *which splits non-trivially as a direct product fails to act coboundedly on* X*.*

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Universal real cubings

Montserrat Casals-Ruiz
(joint work with Mark Hagen, Ilya Kazachkov)

Finitely generated groups are discrete objects, and therefore the traditional tools of local geometry cannot be applied directly to their study. In his influential work on asymptotic invariants, Gromov proposed recovering a geometric viewpoint by changing perspective and observing the group from infinitely far away. This idea leads to the notion of the *asymptotic cone of a group*, a more continuous object that can be studied using topological, dynamical, and infinitesimal methods.

Using this framework, Gromov proved that every asymptotic cone of a hyperbolic group is a (universal) real tree. Moreover, this property completely characterizes hyperbolic groups: a finitely generated group is hyperbolic if and only if each of its non-principal asymptotic cones is a real tree.

The fact that the asymptotic cones are real trees plays a central role in the study of hyperbolic groups. In particular, it provides one of the main tools used to describe JSJ decompositions encoding their splittings, to analyze the structure of their outer automorphism groups, and to classify them up to isomorphism. More surprisingly, Sela made deep use of the Rips machine in his study of the first-order theory of hyperbolic groups. Using this machinery, together with other tools, he showed that hyperbolic groups are equationally Noetherian, described the finitely generated models of their first-order theory, and proved that this theory is stable and admits elimination of imaginaries.

Motivated by the coarse geometry of mapping class groups and right-angled Artin groups, and more generally of hierarchically hyperbolic groups, we introduce the class of *real cubings* as a natural generalization of real trees. They form a subclass of complete geodesic median metric spaces of finite rank, in roughly the same way that hierarchically hyperbolic spaces form a subclass of coarse median spaces.

Using results of Bowditch and Zeidler on medians in asymptotic cones of coarse median spaces, together with Fioravanti's work on measured halfspaces, we show that every non-principal asymptotic cone of a hierarchically hyperbolic group is bilipschitz equivalent to a real cubing. This generalizes the fact that asymptotic cones of hyperbolic groups are real trees and also strengthens a result of Behrstock-Drutu-Sapir on asymptotic cones of mapping class groups.

The construction of an asymptotic cone involves several choices, namely a non-principal ultrafilter on $\mathbb{N}$ and a rescaling sequence, and these choices are crucial for the resulting coarse geometry. Indeed, Thomas–Velickovic and Drutu–Sapir constructed examples of finitely generated groups, and Ol'shanskii–Sapir and Osin–Ould Houcine examples of finitely presented groups, for which different choices of ultrafilters yield non-homeomorphic asymptotic cones.

Nevertheless, if G is word-hyperbolic, then every asymptotic cone of G is isometric to a (universal) real tree: a point if G is finite, a line if G is virtually $\mathbb{Z}$, or the unique complete homogeneous $2^{\aleph_0}$-valent real tree constructed by Dyubina and Polterovich.

In view of these results, it is natural to ask whether hierarchically hyperbolic groups have unique asymptotic cones. To address this question, we introduce the notion of a *local real cubing*, a higher-dimensional analogue of a sheaf of lines. Our first result establishes the existence of a canonical global object determined by such local data.

Theorem 1. *Given a local real cubing $\mathcal{L}$, there exists a unique complete homogeneous real cubing with local structure $\mathcal{L}$ at each point, which we call the* universal real cubing *with local structure $\mathcal{L}$.*

The proof is inspired by the classical construction of the universal real tree. We begin by reviewing this construction and show that, starting from an action of $\mathbb{R}$ on the line, one obtains the following.

Theorem 2. *There exists a group acting freely and transitively on a complete $2^{\aleph_0}$-valent real tree.*

We then generalize this construction to real cubings with prescribed local structure.

Theorem 3. *There exists a group acting freely and transitively on a complete real cubing with local structure $\mathcal{L}$.*

These constructions have direct applications to the asymptotic geometry of hierarchically hyperbolic groups. In particular, one can show that the non-principal asymptotic cones of a hierarchically hyperbolic group G are bilipschitz equivalent to universal real cubings for some local structure $\mathcal{L}$. Consequently, the problem of determining whether G has unique asymptotic cones reduces to the uniqueness of the associated local structure.

Under suitable algebraic conditions on G, satisfied, for example, by mapping class groups of surfaces, fundamental groups of compact special cube complexes, and hyperbolic groups, we prove that, up to bilipschitz equivalence, the local structure (and hence the asymptotic cones of G) is independent of the choice of ultrafilter and rescaling sequence.

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Automorphism groups of RAAGs: a geometric approach

Ruth Charney

(joint work with Corey Bregman, Nathaniel Stambaugh, Karen Vogtmann)

Given a finite, simplicial graph Γ, we define the associated Right-Angled Artin Group (RAAG) to be the group A_Γ whose generators correspond to the vertices of Γ and relations are commutators $[x, y]$ of generators joined by an edge in Γ. RAAGs span a range of groups from free groups (when Γ has no edges) to free abelian groups (when Γ is a complete graph). Associated to A_Γ is a cube complex, $\mathbb{S}_\Gamma$, known as the Salvetti complex, whose fundamental group is A_Γ and its universal covering space is CAT(0). RAAGs and their Salvetti complexes have played a major role in geometric group theory and geometric topology. In this talk I discuss some recent work on outer automorphism groups of RAAGs.

Outer automorphisms of free groups have been much studied. An essential tool in this study is Culler-Vogtmann's "Outer Space", a contractible space with a proper action of $\mathrm{Out}(F_n)$. Our goal in the current project is to construct an analogous space for the outer automorphism group of any RAAG. There is a well-known generating set for $\mathrm{Out}(A_\Gamma)$ due to Laurence and Servatius, consisting of graph permutations, inversions, partial conjugations, and transvections. We divide the transvections $v \mapsto vw$ into two types. If v, w do not commute, we call this a fold. If v, w do commute, we call it a twist. The subgroup of $\mathrm{Out}(A_\Gamma)$ generated by all of the Laurence-Servatius generators *excluding twists*, is called the "untwisted subgroup" and denoted $U(A_\Gamma)$.

We begin by focusing on the untwisted subgroup. In joint work with Stambaugh and Vogtmann [1], we construct an Outer Space for $U(A_\Gamma)$, that is, a contractible space with a proper action of $U(A_\Gamma)$. Moreover, we show that this space equivariantly retracts onto a simplicial "spine" on which $U(A_\Gamma)$ acts properly and cocompactly. Points in this space are pairs (X, h) where X is a locally CAT(0) cube complex which can be reduced to $\mathbb{S}_\Gamma$ by a series of hyperplane collapses, and h is a homotopy equivalence from X to $\mathbb{S}_\Gamma$. Composing h with the inverse of the collapse map gives a homotopy equivalence from $\mathbb{S}_\Gamma$ to $\mathbb{S}_\Gamma$. We require that the resulting map be untwisted, that is, it represents an element of $U(A_\Gamma)$.

Next, in joint work with Bregman and Vogtmann [2], we extend this construction to an outer space for the full outer automorphism group. To do this, we allow the cubes in X to "skew", giving rise to a locally CAT(0) parallelotope complex. The marking h is again a homotopy equivalence with the Salvetti complex, but no longer required to be untwisted. However, a subtlety arises here as a twist may occur either by skewing cubes or by changing the marking. In order to get a well-defined action of $\mathrm{Out}(A_\Gamma)$ on this space, we must forget the explicit parallelotope structure on X, and just remember the induced metric. We then prove that the resulting space is contractible and the action of $\mathrm{Out}(A_\Gamma)$ is well-defined and has finite point stabilizers.

We expect that, as in the case of free groups, these outer spaces will serve as useful tools in studying outer automorphism groups of RAAGs. As a start, we consider finite subgroups $G < U^0(A_\Gamma)$ (where $U^0(A_\Gamma)$ is the subgroup of $U(A_\Gamma)$ excluding graph permutations). The Nielsen realization problem asks whether any such subgroup fixes a point in our untwisted outer space. In joint work with Bregman and Vogtmann [3], we prove that, in fact, it fixes a point in the *spine* of outer space. Since the action on the spine is cocompact, as a corollary, we conclude that $U^0(A_\Gamma)$ contains at most finitely many conjugacy classes of finite subgroups.

In the remainder of the talk, I briefly discuss some other applications and modifications of these spaces by various authors, as well as some open questions for future research.

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Condorcet winners on median spaces

INDIRA CHATTERJI

We discuss the following result by Berno Buechel in 2013 that appeared in [1].

Proposition 1. *Let $(>)$ be a single-peaked non-degenerate preference profile for N voters on a median space $(X, \mathcal{H})$. If there is a Condorcet winner then there is a median alternative (but the median alternative need not be a Condorcet winner).*

We start by explaining all the terms used in this proposition. A *property space* $(X, \mathcal{H})$ is here a finite set X of cardinality at least 3 (of policies, political positions or social states), endowed with a set of partitions $\mathcal{H}$ of X that are called *issues*, such that each two points in X are separated by at least one issue (these are the *walls* that are used in geometric group theory). The number of issues separating two points in X gives a metric on X:

$$d(x, y) = |\{\bar{H} = \{H, H^c\} \mid x \in H, y \in H^c\}|$$

that allows to define *intervals between a pair of points*

$$I(x, y) = \{t \in X \mid d(x, t) + d(t, y) = d(x, y)\}$$

and the property space $(X, \mathcal{H})$ is called *median* if given any 3 points, the triple intersection of intervals contains exactly one point, called the *median*.

A *preference profile* for N voters is a collection of total orders on X, denoted by $(>) = (>_n)$, one for each voter $n = 1, \dots, N$. An order $>_n$ is *single peaked* if there is $x^* \in X$ such that for each distinct elements $y, z \in X$, if $y \in I(x^*, z)$ then $y >_n z$ (it's a form Morse function, expressing compatibility with the median structure). The profile is *single peaked* if all the individual orders are. The profile is *non-degenerate* if all the issues have a winning side, and that allows to define the *median alternative* as the intersection of all the heavy sides.

A *Condorcet winner* is an element $x \in X$ such that for any other element y we have

$$|\{n \in \{1, \dots, N\} \mid x >_n y\} \geq |\{n \in \{1, \dots, N\} \mid y >_n x\}|$$

One can give examples of preference profiles with no Condorcet winner, but the median alternative always exists and is unique. Single-peaked preference profiles were studied by Nehring and Puppe in 2007, showing that median alternatives were strategy-proof and giving equivalent characterizations as dividing the issues in good and bad sides in a consistent way, meaning that two good sides cannot combine into a bad one. For instance, a department agreeing that hiring either candidate A or candidate B is better than no hire, but that hiring both candidates will result in a war that would be worse than no hire at all. While in theory voters should have single peaked preferences, we can all notice that in practice people are not that rational, showing the need to understand other types of preference profiles.

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On Thiagarajan's conjectures

Victor Chepoi

(joint work with Jérémie Chalopin)

Introduction. Median structures (median semilattices, median graphs, and CAT(0) cube complexes) found unexpected and decisive applications in completely different areas of research. One of such applications is the solution of three Thiagarajan's conjectures about event structures. Although my talk presented an overview of characterizations, properties, and applications of median structures, in this extended abstract I will focus on this precise application.

Event structures and their domains. Event structures, introduced by Nielsen, Plotkin, and Winskel (1981) are a widely recognized abstract model of concurrent computation. An event structure is a partially ordered set of the occurrences of actions, called events, together with a conflict relation. The partial order captures the causal dependency of events. The conflict relation models incompatibility of events so that two events that are in conflict cannot simultaneously occur in any state of the computation. Two events that are neither ordered nor in conflict may occur concurrently. Formally, an *event structure* is a triple $\mathcal{E} = (E, \leq, \#)$, consisting of a set E of events, and two binary relations $\leq$ and $\#$, the causal dependency $\leq$ and the conflict relation $\#$ with the requirement that the conflict is inherited by $\leq$. The pairs of events not in $\leq \cup \geq \cup \#$ define the concurrency relation $\|$. The *domain* of an event structure consists of all computation states, called configurations. Each computation is a subset of events such that no two conflicting events can occur together in the same computation and if an event occurred in a computation then all events on which it causally depends have occurred too. The domain of $\mathcal{E}$ is the set $\mathcal{D}(\mathcal{E})$ of all finite configurations ordered by inclusion. An event e is enabled by a configuration c if $e \notin c$ and $c \cup \{e\}$ is a configuration. The degree of $\mathcal{E}$ is the maximum number of events enabled by a configuration. The future (or the principal filter) of a configuration c is the set of all finite configurations c' containing c. Barthélemy and Constantin (1993) proved that the domains of the event structures are in bijection with pointed median graphs and CAT(0) cube complexes: the events correspond to hyperplanes, concurrency $e\|e'$ to intersecting hyperplanes e, e', causal dependency $e \leq e'$ to separation of e' from the origin by e, and the conflict $e\#e'$ to non-separation and non-crossing.

The nice labeling conjecture. The nice labeling conjecture was formulated by Rozoy and Thiagarajan (1991) and asserts that *every event structure with finite degree admits a nice labeling with a finite number of labels.* A nice labeling is a labeling of events with the letters from some finite alphabet such that any two co-initial events (i.e., any two events which are concurrent or in minimal conflict) have different labels. The nice labelings of event structures arise when studying the equivalence of three different models of distributed computation: labeled event structures, net systems, and distributed monoids. The nice labeling conjecture can be viewed as a question about a local-to-global finite behavior of such models.

A counterexample to the nice labeling conjecture was constructed in [3]. It is based on the bijection between domains of event structures and CAT(0) cube complexes and on Burling's construction (1965) of 3-dimensional box hypergraphs with clique number 2 and arbitrarily large chromatic numbers k. Namely, we consider the chain of all Burling box hypergraphs and transform it into a 4-dimensional CAT(0) cube complex, which is further transformed into the domain of an event structure with degree at most 10 not admitting a finite nice labeling.

The conjecture on regular event structures. Finite 1-safe Petri nets, also called net systems, are natural models of asynchronous concurrency. Nielsen, Plotkin, and Winskel (1981) proved that every net system N unfolds into an event structure $\mathcal{E}_N$ describing all possible executions of N: the events of $\mathcal{E}_N$ are all prime Mazurkiewicz traces on the transitions of N, equipped with the causal dependency and conflict relations. To deal with net systems, Thiagarajan (1996, 2002) introduced the notions of regular event structure and trace regular event structure. The main difference is that the regularity is defined for unlabeled event structures while trace regularity is defined under the stronger assumption of a given trace regular labeling. These definitions were motivated by the fact that the event structures $\mathcal{E}_N$ arising from *finite* 1-safe Petri nets N are regular: Thiagarajan proved that event structures of *finite* 1-safe Petri nets correspond to regular trace event structures. This led him to conjecture that *regular event structures and regular trace event structures are the same.* Equivalently, *an event structure $\mathcal{E}$ is isomorphic to the event structure unfolding of a net system iff $\mathcal{E}$ is regular.*

In [1], we presented a counterexample to this conjecture based on a geometric and combinatorial view on event structures. To deal with regular event structures, we showed how to construct regular event domains from CAT(0) cube complexes. Of particular importance for us are the CAT(0) cube complexes arising as universal covers of *finite* NPC (Nonpositively Curved) cube complexes. We adapted the universal cover construction to directed NPC complexes (Y, o) and showed that every principal filter of the directed universal cover $(\widetilde{Y}, \widetilde{o})$ is the domain of an event structure. Furthermore, if the NPC complex Y is finite, then this event structure is regular. Motivated by this result, we called an event structure *strongly regular* if its domain is the principal filter of the directed universal cover $\widetilde{\mathbf{Y}} = (\widetilde{Y}, \widetilde{o})$ of a finite directed NPC complex $\mathbf{Y} = (Y, o)$. Our counterexample to Thiagarajan's conjecture is a strongly regular event structure not admitting a finite regular nice labeling. It is derived from Wise's (2007) nonpositively curved square complex $\mathbf{X}$ obtained from a tile set with six tiles.

In view of this counterexample, one can ask if *the event structures arising as unfoldings of finite 1-safe Petri nets are strongly regular* and *under which conditions is a regular event structure trace regular*? Haglund and Wise (2008,2012) detected five types of pathologies which may occur in NPC complexes. They called the NPC complexes without such pathologies *special*. The main motivation for introducing special cube complexes was the idea of Wise that the famous virtual Haken conjecture for hyperbolic 3-manifolds can be reduced to solving problems about special cube complexes. Agol (2013) completed this program and solved the virtual Haken conjecture using the deep theory of special cube complexes developed by Haglund and Wise. The main ingredient in this proof is Agol's theorem that finite NPC complexes whose universal covers are hyperbolic are virtually special (i.e., they admit finite covers which are special cube complexes).

In [1] we proved that *Thiagarajan's conjecture is true for event structures whose domains arise as principal filters of universal covers of finite special cube complexes.* On the other hand, in [2], we established the converse result: we prove that to any 1-safe Petri net N, one can associate a finite directed labeled special cube complex $\mathbf{X}_N$ such that the domain of the event structure $\mathcal{E}_N$ (obtained as the unfolding of N) is a principal filter of the universal cover $\widetilde{\mathbf{X}}_N$ of $\mathbf{X}_N$. This proves that *the trace regular event structures are exactly the special strongly regular event structures* and that *the trace labeling is obtained via the covering map.* This shows that all event structures arising as unfoldings of finite 1-safe Petri nets are strongly regular, answering in the positive previous question. This also shows that specialness must be added to strong regularity to ensure a positive answer to Thiagarajan's second conjecture. Therefore, the trace regular event structures can be characterized as the event structures whose domains arise from *finite special* cube complexes. This establishes a surprising bijection between 1-safe Petri nets (fundamental objects in concurrency) and special cube complexes (fundamental objects in geometric group theory).

The conjecture on decidability of MSO logic. Thiagarajan and Yang (2014) defined the monadic second order (MSO) theory of an event structure unfolding $\mathcal{E}_N$ of a net system N. This immediately leads to the following fundamental question: *When MSO of $\mathcal{E}_N$ is decidable?* It turns out that the MSO theory of trace event structures is not always decidable: Thiagarajan and Yang defined grid event structures and showed that the MSO theory of event structures containing grids is undecidable. This led Thiagarajan to conjecture that *the MSO theory of a trace regular event structure $\mathcal{E}_N$ is decidable iff $\mathcal{E}_N$ is grid-free.* In [2] we constructed a counterexample to this conjecture. Namely, we constructed an NPC square complex $\mathbf{Z}$ with one vertex, four edges, and three squares. We showed that $\mathbf{Z}$ is virtually special and thus any principal filter of the universal cover of $\mathbf{Z}$ is the domain of a trace regular event structure $\mathcal{E}_Z$, whose graph has infinite treewidth and bounded hyperbolicity. The first result implies that MSO of $\mathcal{E}_Z$ is undecidable while the second result shows that $\mathcal{E}_Z$ is grid-free.

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Connections between median graphs, spaces of non-positive curvature and actions on Hilbert spaces

Cornelia Druţu
(joint work with Indira Chatterji, Frederic Haglund, Davide Spriano, Stefanie Zbinden)

The fist part of this talk explored how key properties of infinite groups relate to key classes of graphs. Two combinatorial structures are, in a sense, at the two extremes of the spectrum: expander graphs (robust, highly connected networks, non-embeddable in Hilbert spaces, difficult to construct) and median graphs (economic networks, easy to embed in Hilbert spaces, easy to construct). These two classes turn out to be closely connected to two important properties of infinite groups: Kazhdan's property (T), respectively a-T-menability.

Thus, on one hand the first hands-on example of expander graphs has been constructed by Margulis using infinite groups with property (T) that are residually finite: the Cayley graphs of all the finite quotients endowed with the image of a fixed generating set of the initial group compose a family of expanders.

Conversely, using expanders one can construct new infinite groups with property (T). Building on work of Garland, Ballmann and Świątkowski proved the following [1]. Given a 2-dimensional simplicial complex X such that the link of every vertex is a strong expander (i.e. the first non-zero eigenvalue of the Laplacian is $> \frac{1}{2}$) a group G that acts simplicially, properly discontinuously and cocompactly on X has property (T).

The other combinatorial geometry, the median one, can be defined in the more general setting of metric spaces. A metric space (X, d) is median if every triple of points $x_1, x_2, x_3 \in X$ admits a unique median point $m \in X$ satisfying

$$d(x_i, m) + d(m, x_j) = d(x_i, x_j)$$

for all $i, j \in \{1, 2, 3\}, i \neq j$. A graph is then called median if its set of vertices with the combinatorial structure satisfies the above property.

In joint work with I. Chatterji and F. Haglund [2], we proved that a group has property (T) $\Leftrightarrow$ any action by isometries on a median space has bounded orbits.

Likewise, a group is a-T-menable $\Leftrightarrow$ it admits a proper action by isometries on a median space.

This comes to illustrate that indeed a-T-menability can be reformulated in terms of median geometry, while property (T) is incompatible with such a geometry.

There are various degrees of compatibility with the median structure among a-T-menable groups: some are cubulable (i.e act cocompactly on CAT(0) cube complexes) while others cannot act properly on either CAT(0) cube complexes or on median spaces of finite rank, even if they act cocompactly on median spaces of infinite rank (as is the case for instance for irreducible uniform lattices of isometries of products of real hyperbolic spaces) [3].

Median spaces share various properties of non-positive curvature. To begin with, spaces of finite rank can have their metric deformed in a bi-Lipschitz and equivariant way so that it becomes CAT(0).

Even without the finite rank condition, the median geometry still has features in common with the CAT(0) one. In particular, a dichotomy similar to the Ballmann rigidity theorem occurs for groups acting cocompactly on spaces median or quasi-isometric to median.

Thus, in joint work with Davide Spriano and Stefanie Zbinden [4], we proved that a non-virtually cyclic group that acts properly discontinuously and with compact quotient on a metric space X quasi-isometric to a median space satisfies the following dichotomy: either the divergence of G is linear, equivalently all asymptotic cones are without cut-points; or G has a Morse element, is acylindrically hyperbolic, has free non-abelian subgroups. All asymptotic cones have cut-points.

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Radius functions in median graphs

Guillaume Ducoffe

(joint work with Pierre Bergé, Jérémie Chalopin, Victor Chepoi, Feodor F. Dragan, Michel Habib, Yann Vaxès)

The purpose of this talk was to report on the state of the art for the following problem on median graphs:

Definition. Let $\Gamma = (V, E)$ be a graph (possibly infinite), and let $\pi : V \mapsto \mathbb{R}_{\geq 0}$, with finite support, be called a *profile*. The *radius function* r_π maps every vertex v to $\max\{\pi(u) \cdot d(u, v) : u \in \mathrm{supp}(\pi)\}$. The Weighted Center problem asks to find a vertex c such that $r_\pi(c)$ is minimized, hereafter called a *center*.

The Weighted Center problem can also be defined for metric spaces. The special case where π can only take $0, 1$ values is sometimes called the Unweighted Center problem, or the Subset Center problem. Furthermore, for finite graphs or metric spaces the classical CENTER problem corresponds to the particular case of the all-one profile π, which has attracted considerable attention in algorithmic graph theory, and in computational geometry.

Motivations. The fine-grained complexity of the WEIGHTED CENTER problem on median graphs can be relevant in the study of *median networks* [5]: that are roughly the median closure of phylogenetic data (*e.g.*, mitochondrial DNA of human populations). Indeed, in network analysis, the inverse of the radius function is used as a centrality index to determine the relative importance of the nodes in a network. Since median networks are notoriously large, it is important to design algorithms for them that are as fast as possible. On general graphs, the CENTER problem requires $n^{2-o(1)}$ time, under plausible complexity hypotheses, even on n-vertex graphs with at most $n^{1+o(1)}$ edges [2]. However, it is possible to break this quadratic barrier for median graphs, due to their metric and geometric properties.

1. State of the art

For n-node trees, an $O(n)$-time algorithm for the WEIGHTED CENTER has long been known [12]. A crucial ingredient to this algorithm is that *radius functions in trees are convex.* However, there exist median graphs with non convex radius functions: for instance, consider a cycle on four vertices.

The following results were presented during the talk.

Parameterized complexity. Progress on much larger subclasses of median graphs, beyond trees, has only been achieved quite recently. A first result in this direction was obtained for median graphs of bounded *tree-dimension* [11]. (For a median graph Γ, or more generally for a partial cube, its tree-dimension is the least k such that Γ can be isometrically embedded in the Cartesian product of k trees). More specifically, given an isometric embedding of a graph Γ in the product of k N-node trees, the WEIGHTED CENTER problem can be solved in $2^{O(k \log k)} N^{1+o(1)}$ time. Note that median graphs of tree-dimension two can be recognized in linear time [4]. *Squaregraphs* are another well-studied subclass of median graphs with tree-dimension at most five: they are the plane graphs such that all the inner faces are quadrangles and every vertex of degree at most three lies on the outer-face [3]. However, the problem of computing the tree-dimension of median graphs is NP-hard, and no efficient approximation algorithm is likely to exist [6]. This inherently limits the use of this parameter in algorithmic design. By comparison, the *cube-dimension* is a more tractable parameter: the latter is just the largest dimension of a hypercube in a median graph. In particular, the cube-dimension of any median graph is at most its tree-dimension, and it is much smaller for some median graphs (*e.g.*, the tree-dimension is unbounded for the median graphs of cube-dimension two). For n-vertex median graphs of cube-dimension d, there exists a $2^{O(d)} n^{1+o(1)}$-time algorithm for the UNWEIGHTED CENTER problem [9].

Open problem 1. Is there an almost linear-time algorithm for the WEIGHTED CENTER problem on median graphs of bounded cube-dimension?

At present, such an algorithm is only known for median graphs of cube-dimension two [10]. This algorithm is based on the following G^2*-unimodality property* of radius functions in these graphs, namely: every vertex v that minimizes $r_\pi(v)$ within its ball of radius 2 is a center.

Open problem 2. Characterize the median graphs with G^p-unimodal radius functions. In particular, does there exist some constant $c > 0$ such that every median graph of cube-dimension d has all its radius functions that are $G^{c \cdot d}$-unimodal?

Arbitrary median graphs. An important notion for the design of efficient algorithms on general median graphs is that of Θ-*classes*. Roughly, a Θ-class of a median graph Γ is made of all its edges that are mapped to the same dimension of the smallest hypercube in which Γ isometrically embeds. Every Θ-class of a median graph is a matching cutset whose removal leaves two gated components, called *halfspaces*. Therefore, Θ-classes are used in the design of divide-and-conquer algorithms on median graphs. In particular, for $0, 1$ profiles, the radius function of a median graph Γ can be computed in linear time from its restriction to two complementary halfspaces, with respect to any Θ-class. However, this does not immediately lead to subquadratic-time algorithms for the UNWEIGHTED CENTER problem because there may exist a linear number of Θ-classes. The generic framework presented in [7] consists of disconnecting the largest Θ-classes so as to reduce the cube-dimension of the resulting subgraphs to $\alpha \cdot \log n$, for some arbitrarily small α. Doing so, an $O(n^{\frac{5}{3}+o(1)})$-time algorithm for the UNWEIGHTED CENTER problem was claimed. More recently, the concept of "balanced" Θ-class was introduced, which has led to the proof of the following stronger result:

Theorem 1 ([8])**.** *The* UNWEIGHTED CENTER *problem can be solved in* $O(n \log^4 n)$ *time on* n*-vertex median graphs.*

2. PERSPECTIVES

It was asked during the talk whether the results above can be extended to edge-weighted median graphs. This is likely to require additional assumptions on the edge-weight distribution. For example, a natural choice is to assign the same positive weight to every two edges in the same Θ-class. Generalizations of the results above for median graphs to (unweighted) partial cubes also look challenging. For that, it could be interesting to identify families of partial cubes with a median closure of sub-quadratic size. However, for ℓ_1-spaces, a conditional quadratic lower bound was proved recently [1].

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An invitation to quasi-median graphs

Anthony Genevois

Loosely speaking, quasi-median graphs provide the smallest reasonnable family of median-like graphs that include both median graphs and (products of) complete graphs. Formally, mimicking the characterisation of median graphs as retracts of hypercubes, quasi-median graphs can be defined as retracts of Hamming graphs (i.e. products of complete graphs). Other characterisations of median graphs can be turned into characterisations of quasi-median graphs, usually just by replacing edges and cubes with cliques and Hamming graphs. For instance, in the same way that a graph is median if and only if it is the one-skeleton of a CAT(0) cube complex, a graph is quasi-median if and only if it is the one-skeleton of a CAT(0) prism complex. (Here, a prism refers to a product of simplices.)

More generally, it seems that most of the properties satisfied by median graphs have natural analogues for quasi-median graphs, always obtained by applying the same dictionnary. This motivates the idea that median and quasi-median graphs are tightly connected. This motivates the following question: are quasi-median graphs actually useful or is it sufficient to work with median graphs instead?

In this talk, our goal is to motivate the idea that, despite the fact that every quasi-median graph can be turned into a canonical median graph, quasi-median graphs pop up in various places so naturally that they help significantly to think and to prove. In other words, quasi-median graphs should not be underestimated.

Data visualisation. Let S be a set and $\mathfrak{C}$ a set of partitions of S. We think of $\mathfrak{C}$ as a set of *characters* that differentiate the objects in S. Define a *selector* as a map $\sigma : \mathfrak{C} \to 2^S$ satisfying $\sigma(\mathscr{C}) \in \mathscr{C}$ for every character $\mathscr{C} \in \mathfrak{C}$. In other words, σ chooses an element of the partition $\mathscr{C}$ for every $\mathscr{C} \in \mathfrak{C}$. Through the embedding

$$s \mapsto (\mathscr{C} \mapsto \text{ element of } \mathscr{C} \text{ containing } s),$$

one can think of S as living inside the graph $H(S, \mathscr{C})$

- whose vertices are selectors;
- and whose edges connect any two selectors that differ on a single character.

Notice that $H(S, \mathscr{C})$ is a Hamming graph. This is a huge graph, and the challenge in data visualisation is to find a smaller subgraph of $H(S, \mathscr{C})$ that still contains S and that encodes in a meaningfull way how the characters in $\mathscr{C}$ differentiate the points of S.

When all the characters in $\mathscr{C}$ are bipartitions, a relevant choice is to take the median closure of S in $H(S, \mathscr{C})$. In phylogenomics, this is known as the *Buneman graph* of a *split system*; and, in geometric group theory, this is known as the *cubulation* of a *wallspace*. In the general case, we explain how to construct a quasi-median graph from $(S, \mathfrak{C})$. See [8, 9] for more information on the construction.

Median sets. Most commonly, median graphs are defined by the property that any three vertices admit a unique median vertex. But, more generally, it is possible to define the *median set* $\mathrm{Med}(S)$ of an arbitrary finite collection S of vertices as the set of the vertices x minimising the sum $\sum_{s \in S} d(x, s)$. Determining median sets is a classical problem in *location theory*. In median graphs, median sets are well-understood, and similar results exist for quasi-median graphs. In fact, it is possible to characterise quasi-median graphs as the only graphs for which a certain dynamical location problem is solvable [13]. There are also applications in geometric group theory, since median sets can be used in order to prove fixed-point theorems [14, 12].

Independency. Consider a physical system, which can be in a state $s \in S$ and on which can applied actions $a \in \mathcal{A}$. An action may not be applicable if the system is in a given state; but, if it is, the action changes the state of the system. Two actions are *independent* if, when applicable, applying a and then b, or applying b and then a, moves the system to same state. In this part of the talk, we explain how median graphs can be characterised in terms of such systems of actions with independency. This is related to *state complexes* in robotics [2] and *event structures* in informatics [1]. We also explain how adding *intensities* to the actions naturally leads to a characterisation of quasi-median graphs. Formally, this allows us to describe (quasi-)median graphs as Cayley graphs of some specific groupoids, leading to interesting constructions in geometric group theory (extending, for instance, diagram groups [15] and quandle products [11]). Details will be available in a forthcoming article.

Relatively median graphs. In geometric group theory, it is often desirable to be able to say that a given space X, equipped with a particular collection $\mathcal{P}$ of subspaces, has some geometry relative to $\mathcal{P}$ despite the fact that nothing is known about the geometry of the subspaces in $\mathcal{P}$. A remarkable illustration of this idea is provided by relatively hyperbolic spaces. In this part of the talk, we propose a definition of *relatively median* graphs and motivate its relevancy. For instance, a product of graphs is median relative to its factors and a pointed sum of graphs is median relative to its factors. Other examples include Cayley graphs of graph products of groups and some diadem products of graphs [4]. A notable characterisation of relatively median graphs, which highlights quasi-median graphs as sorts of universal median-like graphs, is the following: a graph is relatively median if and only if it is isometric to a quasi-median graph equipped with a parallel-friendly weight function on its edges. (Here, a weight function is referred to as *parallel-friendly* if two opposite edges in an induced 4-cycle always have the same weight.) More details and applications in geometric group theory will be available in a forthcoming article.

Other applications in geometric group theory. Now, quasi-median graphs have plenty of applications in geometric group theory, including: a generalisation of Haglund and Wise's theory of special cube complexes [5] (replacing right-angled Artin groups with graph products of groups); the introduction of (right-angled) rotation groups [10], which provides a new perspective and an exntesion of Kim and Koberda's extension graphs [12, 7]; the construction of embeddings between Thompson-like groups [6]; a geometric description of approximations of lamplighter groups [3]; a characterisation of the numbers of relative ends and of coends [9]; a reduction of the theory of group actions on right-angled buildlings to the theory of group actions on quasi-median graphs (in preparation).

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Introduction to Real Cubings

Mark Hagen
(joint work with Montserrat Casals-Ruiz, Ilya Kazachkov)

The purpose of this talk is to introduce the class of *real cubings*, which are complete, connected, finite-rank median spaces with an additional property allowing them to be "coordinatised" as a median subalgebra of a product of real trees. The class includes several familiar examples, like finite-dimensional CAT(0) cube complexes and real trees, but the motivating examples are asymptotic cones of *hierarchically hyperbolic spaces*. To explain the motivation, we begin with the main theorem of the work being presented [1]:

Theorem 1. *Let G be a finitely generated group that is (quasi-isometric to) one of the following:*

- *the mapping class group of a connected orientable surface of finite genus with finitely many punctures/boundary components;*
- *the fundamental group of a compact special cube complex;*
- *a hyperbolic group.*

Then any two non-principal asymptotic cones of G are bilipschitz equivalent.

In fact, the theorem holds more generally for a suitable class of hierarchically hyperbolic groups that also includes hyperbolic groups, most 3–manifold groups, certain Artin groups, etc. This generalises the well-known characterisation of nonelementary hyperbolic groups as those groups whose asymptotic cones are all isometric to the universal $2^{\aleph_0}$–tree. Theorem 1 has important predecessors, notably the Osin-Sapir and Sisto results on uniqueness of asymptotic cones of relatively hyperbolic groups [2, 3]. For context, note that uniqueness up to bilipschitz equivalence, or even just homeomorphism, is a strong property of a group, since it fails even in relatively straightforward examples, and there are numerous more elaborate examples where the set of homeomorphism types of asymptotic cones has various wild properties (see e.g. [5, 4, 6]).

Theorem 1 is also the subject of another talk in the workshop, by Montse Casals-Ruiz; the present talk therefore focuses on just one of the ingredients in the proof, namely the following theorem:

Theorem 2. *Let $(Z, \mathfrak{S})$ be a* hierarchically hyperbolic space *in the sense of* [8, Defn. 1.1]. *Let ω be a non-principal ultrafilter on $\mathbb{N}$, let $(o_n)_n$ be a sequence of observation points in Z, let $(r_n)_n$ be an unbounded increasing sequence of positive reals, and let $Cone(Z)$ be the asymptotic cone of Z with observation point $(o_n)_n$, rescaling $(1/r_n)_n$, and ultrafilter ω. Then $Cone(Z)$ is bilipschitz equivalent to a* real cubing.

In the case where Z is a mapping class group with a finitely generated word metric, most of the content of Theorem 2 is contained in a result of Behrstock-Druţu-Sapir that predates the language of real cubings [7].

The goal of the talk is to define *real cubings*, explain the "real tree coordinate system" on a real cubing, and indicate how the proof of Theorem 2 goes.

Let X be a median metric space, with metric d and median $\mu : X^3 \to X$. We require that the *rank* of X as a median algebra is finite, and that the metric space (X, d) is complete and connected, or, alternatively, discrete, i.e. X is the 0–skeleton of a CAT(0) cube complex. A *wall* is a pair $\hat{w} = \{w, w^*\}$ where $w \subset X$ is median-convex and $w^* = X - w$ is also. The set of walls is $W(X)$. More generally, given a subspace $A \subset X$, let $W(A)$ be the set of walls $\hat{w}$ such that $A \cap w, A \cap w^* \neq \emptyset$. If $\hat{u} \in W(X)$, we say that $\hat{w}$ and $\hat{u}$ *cross* if $\hat{w} \in W(u) \cap W(u^*)$ and let $W(\hat{u})$ be the set of walls that cross $\hat{u}$.

Given any set $I \subseteq W(X)$, let $I^\perp = \bigcap_{\hat{w} \in I} W(\hat{w})$. If $I = \emptyset$, we say $I^\perp = W(X)$, and note that $W(X)^\perp = \emptyset$. Let $\mathfrak{F} = \{I^{\perp\perp} : I \subseteq W(X)\} - \{\emptyset\}$. Then $\mathfrak{F}$ is partially ordered by inclusion and $W(X)$ is the unique maximal element. If $U \in \mathfrak{F}$ and $U \subset W(X)$, then $U^\perp \in \mathfrak{F} - \{W(X)\}$, so $^\perp$ is an involution on the non-maximal elements, and $U \subset V$ if and only if $V^\perp \subset U^\perp$.

We say that X is a *real cubing* provided there exists $n \in \mathbb{N}$, called the *depth* of $\mathfrak{F}$, such that any $\subset$–chain in $\mathfrak{F}$ contains at most n distinct elements.

This is an abstraction of the notion of a *cube complex with a factor system* as in [9, Sec. 8]; the latter notion is used to create hierarchically hyperbolic structures (i.e. "coarse coordinate systems" where the coordinates take values in hyperbolic spaces). If the depth of $\mathfrak{F}$ is finite, we can similarly produce maps $\pi_U : X \to TU$, $U \in \mathfrak{F}$, where TU is a real tree, such that $\prod_{U \in \mathfrak{F}} : X \to \prod_{U \in \mathfrak{F}} TU$ yields an isometric embedding into the base component of the codomain, equipped with the ℓ_1 metric and the product median.

The construction is roughly as follows. For each $U \in \mathfrak{F}$, work of Fioravanti [10] on *measured halfspaces* allows us to produce a closed convex (hence gated) subspace $F_U \subseteq X$ such that $U = W(F_U)$. Let $\pi_U : X \to F_U$ be the gate map. We will construct a pseudometric s_U on F_U such that s_U is median of rank 1 and π_U is a 1–lipschitz median-preserving map. We then take TU to be the metric quotient of the pseudometric space (F_U, s_U). Roughly speaking, s_U is constructed as follows. For each $V \in \mathfrak{F}$ properly contained in U, the gate map $\pi_V : F_V \to F_U$ is an isometric embedding. Given $x, y \in F_U$, there are countably many such V where $\pi_V(x) \neq \pi_V(y)$, and we subtract $d(\pi_V(x), \pi_V(y))$ from $d(x, y)$. An inclusion-exclusion argument lets one define $s_U(x, y)$ in such a way that $\sum_{V \subseteq U} s_U(x, y) = d(x, y)$, which is what is required to make the product map an isometric embedding. The reason why TU is a real tree is that nontrivial "median rectangles" in F_U provide pairs $V, V^\perp \in \mathfrak{F}$, both contained in U, and $U \neq U^\perp$, so the containment is proper, and hence the rectangle collapses to a point in TU. This ensures that the median rank of TU is at most 1.

Now we turn to Theorem 2. The proof proceeds roughly as follows, eliding some technical points. If Z is a hierarchically hyperbolic space, then it supports a coarse median μ_Z, and also comes equipped with a collection of μ_Z–quasiconvex *standard product regions* P_i, each of which is quasi-isometric to a product $F_i \times E_i$ of simpler hierarchically hyperbolic spaces. Simplicity is measured by a nonnegative integer, called the *complexity*. Now, by a result of Bowditch, up to modifying Z in its bilipschitz equivalence class, the map $\mu := \omega - \lim_n \mu_Z$ defines a median operator on Z making it a complete, connected, finite rank median metric space. There are many closed convex subspaces of the form $\mathbf{F} = \omega - \lim_n F_{i_n}$, where $(F_{i_n})_n$ is a sequence of factors of standard product regions P_{i_n}. One constructs $\mathfrak{F}$ as above, and then shows that for $U \in \mathfrak{F}$, there are various $\mathbf{F}$ as above such that walls belonging to U cross $\mathbf{F}$. This correspondence is close enough that $\subset$–chains in $\mathfrak{F}$ yield chains of comparable length in the *nesting order* on the hierarchically hyperbolic index set of Z, which has bounded chains by definition.

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Median Graphs from a Phylogenetics Perspective

Katharina Huber

Ever since Darwin, trees have been used to represent evolutionary histories of species and organisms. Over the years, it has however been found that trees are not always the best way to represent certain evolutionary processes. Championed by Bandelt and Dress in [1], one way to address this is to view a tree as a split system, that is, a set of bipartitions of its leaf set, and to relax the requirement that the splits should be compatible, or in other words, that they all “fit” onto the same tree.

Intriguingly, any split system can be represented by a split network and labelled median graphs are a special type of such a structure. For this reason, labelled median graphs have found use in, for example, phylogeography studies where they are employed to help shed light into the genetic make-up of populations [2, 8]. From a more combinatorial point of view, labelled median graphs have also been found to have interesting and far-reaching connections with discrete structures of interest in the area of Phylogenetic Combinatorics [6] such as tight spans of metrics [5] and geometries of split systems.

In this talk, we will first consider labelled median graphs within a phylogenetics context [7], and then review some of the mathematical results alluded to above, including some recent results that link labelled median graphs with diversities [3], a novel concept that generalises metrics introduced in [4].

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On a particular class of infinite rank median spaces

Lamine Messaci

(joint work with Indira Chatterji)

Median spaces contain an abundance of halfspaces: any two disjoint convex subsets are separated by a halfspace. The median geometry is entirely encoded in the set of halfspaces, which naturally carries the structure of a measured space satisfying a Crofton-type formula: the measure of the set of halfspaces separating two points equals the distance between them (see [2] and [6]).

In finite-rank median spaces, halfspaces are either open, closed, or both ([6]). This is no longer true in the infinite-rank case, where halfspaces may be dense with empty interior; this occurs, for instance, in L^1 spaces over a non-atomic measure space.

We aim to investigate a subclass of infinite-rank median spaces in which halfspaces still retain a topological meaning.

Definition. A median space X has the *topological separation property* if for every pair of points $x, y \in X$, there exists a halfspace $\mathfrak{h} \subseteq X$ such that $x \in \mathring{\mathfrak{h}}$ and $y \in \mathring{\mathfrak{h}^c}$.

We will show that many important properties follow from this topological separation. Before doing so, we first present a list of median spaces satisfying this property.

Examples.

- Any locally convex median space has the topological separation property, as any pair of disjoint convex subsets is separated by a halfspace.
- The 0-skeleton of a CAT(0) cube complex of arbitrary dimension, endowed with the combinatorial metric, is a median space that satisfies the topological separation property.
- The median space associated to the n-dimensional hyperbolic space $\mathbb{H}^n$ introduced in [1]. This follows from the fact that the set of halfspaces separating a point from a halfspace that does not contain it in its closure has positive measure.
- The class is closed under direct sums. If $(X_i, a_i)_{i\in\mathbb{N}}$ is a sequence of pointed complete median spaces with the topological separation property, then their direct sum
$$\bigoplus_{i\in\mathbb{N}}(X_i, a_i) := \{(x_i)_{i\in\mathbb{N}} \mid \sum_{i\in\mathbb{N}} d(a_i, x_i) < +\infty\}$$
also has the topological separation property.
- The class is closed under direct limits. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of complete median spaces with the topological separation property and
$$\phi_i : X_i \to X_{i+1}$$
be isometric embeddings with convex images. Then the direct limit $\varinjlim X_i$ is a median space with the topological separation property.

When dealing with group actions, it is important to have a compactification of the space in order to encode the global behaviour of the action. In the case of locally convex median spaces, this was achieved in [6] under the additional assumption that intervals are compact.

One approach is to embed the space into a product of intervals:
$$\begin{aligned} i : X &\to \prod_{a,b\in X} [a, b] \\ x &\mapsto (m(x, a, b))_{a,b}. \end{aligned}$$
The Roller compactification of X is defined as the closure of $i(X)$, where the target space is endowed with the product topology. This construction is called the *zero-completion* (see [6, Section 4]).

The assumption that intervals are compact, which is not true for every median spaces, ensures that the product topology is compact. For complete median spaces with the topological separation property, we prove that intervals are compact.

Theorem 1. *Let X be a complete median space with the topological separation property. Then every interval is compact.*

Another compactification of the space is in the spirit of Stone duality. One considers the set of ultrafilters $\mathcal{U}(X)$ on the set of halfspaces of X. This set is naturally endowed with a convex topology $\mathcal{C}$ induced by open halfspaces: for any thick open halfspace, consider the set of ultrafilters containing it. This family forms a basis for $\mathcal{C}$.

In [8], it is shown that the embedding of X into $\mathcal{U}(X)$ is convex. This allows the extension of the retractions onto intervals $[a, b]$ to the whole space $\mathcal{U}(X)$. In this way one obtains a canonical map

$$\begin{aligned} \phi : \mathcal{U}(X) &\to \bar{X} \\ \mathfrak{u} &\mapsto (\tilde{m}(\mathfrak{u}, a, b))_{a,b}. \end{aligned}$$

Theorem 2. *The map ϕ is a homeomorphism between $(\mathcal{U}(X), \mathcal{C})$ and $\bar{X}$.*

In finite-rank median spaces, the irreducibility assumption is mild: any finite-rank median space splits uniquely as an ℓ^1-product of irreducible median spaces, and its isometry group is, up to finite index, the product of the isometry groups of the factors.

In the infinite-rank case, even the existence of irreducible factors is not guaranteed. For instance, consider L^1 spaces over a non-atomic measure space (X, μ). Any measurable subset of X gives rise to a splitting of $L^1(X, \mu)$, but no such splitting yields an irreducible factor.

However, the topological separation property imposes the existence of irreducible factors. More precisely, we have the following decomposition.

Theorem 3. *Let X be a complete separable median space with the topological separation property. Then there exists a sequence of pointed median spaces $((X_i, a_i))_{i\in\mathbb{N}}$ with the topological separation property such that*

$$X = \bigoplus_{i\in\mathbb{N}} (X_i, a_i) := \{(x_i)_{i\in\mathbb{N}} \mid \sum_{i\in\mathbb{N}} d(a_i, x_i) < +\infty\}.$$

In his foundational work, Margulis established the phenomenon of superrigidity for higher-rank lattices, showing that any "non-elementary" morphism from such lattices into a semisimple Lie group extends to the ambient group. Since then, many related questions have been studied, extending these ideas to situations where the source group is a product of locally compact, second-countable groups and the target group is a more general topological group preserving a certain structure, namely a median structure ([3], [5], [7]).

Two main assumptions in Margulis' superrigidity theorem are the higher-rank condition and semisimplicity. In the general setting of lcsc groups, the higher-rank assumption is encoded by requiring the group to split as a product. The semisimplicity assumption is instead reflected in a "non-elementary" condition on the action, ensuring the presence of sufficiently rich dynamics (for instance the existence of free subgroups). In the median setting, this condition ensures the non-emptiness of the regular boundary, denoted $\partial_r X$, inside the Roller boundary.

The first step toward establishing superrigidity is the construction of a boundary map.

Theorem 4. *Let* Γ *be a finitely generated group acting non-elementarily by isometries on a complete irreducible median space* X *with the topological separation property. Then there exists a unique* Γ*-equivariant map from the Furstenberg–Poisson boundary of* Γ *to the regular boundary* $\partial_r X$*.*

When Γ is an irreducible lattice in a lcsc group $G := G_1 \times G_2$, we obtain the following superrigidity theorem.

Theorem 5. *Let* Γ *be an irreducible lattice in* $G = G_1 \times G_2$*. For any non-elementary isometric action of* Γ *on a complete irreducible median space* X *with the topological separation property, there exists a* Γ*-invariant median subspace* $Y \subseteq X$ *such that the* Γ*-action on* Y *extends to* G *and factors through one of the projections* $\pi_i : G \to G_i$*.*

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CAT(0) cube complexes in low dimensional topology

Michah Sageev

We discuss CAT(0) cube complexes as viewed through the lens of 3-manifold topology. We review the classical theorem of Kneser, reducing the study of closed 3-manifolds to the study of irreducible ones. This showed the connection with group action on trees, which are 1-dimensional CAT(0) cube complexes. We then discuss the notion of a Haken 3-manifold and a fibered 3-manifold, as well as the famous virtual Haken and virtual Fibering conjectures. We then turn to CAT(0) cubulations proper, starting with the pioneering work of Aichison and Rubinstein on CAT(0) cubulations of 3-manifolds. This led to the more general notion of CAT(0) cubulations via spaces with walls and later to the connection with median spaces and median graphs. We then turn to a discussion of Haglund and Wise's development of special complexes, a setting in which Stallings approach for building of covers of graphs can be generalized to building covers of non-positively curved cube complexes. Finally, we discuss how this led to Agol's solution of the virtual Haken and virtual Fibering conjecture. We end with a few questions stemming from this line of research.

From Kazhdan's property to higher-dimensional expansion

Roman Sauer
(joint work with Uri Bader)

A waist inequality for a space X is a statement of the following kind: for every map from X to a lower-dimensional Euclidean space, at least one fiber must be large. In a Riemannian setting, the size of a fiber is measured by its $(m-d)$-volume, where m is the dimension of X and d is the dimension of the target. In a discrete setting, one measures the size of a fiber by counting the number of d-simplices that intersect that fiber.

A prototypical waist inequality is Gromov's sphere waist theorem [3]. It says that if $1 \leq d < m$ and $f\colon S^m \to \mathbb{R}^d$ is continuous, then there exists $p \in \mathbb{R}^d$ with

$$\mathrm{vol}_{m-d}\big(f^{-1}(p)\big) \geq \mathrm{vol}_{m-d}\big(S^{m-d}\big).$$

In discrete geometry, related overlap statements appear in Boros–Furedi-type theorems [1] and lead naturally to the topological overlap property for simplicial complexes.

For a d-dimensional simplicial complex X, the *topological ε-overlap property* requires that for every continuous $f\colon X \to \mathbb{R}^d$, some point $p \in \mathbb{R}^d$ is covered by at least an ε-fraction of the f-images of the d-simplices.

Definition 1. A family $\{X_i\}_{i\in\mathbb{N}}$ of simplicial complexes with $|X_i| \to \infty$ is a *d-dimensional topological expander* if there is a uniform $\varepsilon > 0$ such that every X_i has the ε-overlap property.

There is an analogous Riemannian formulation.

Definition 2. A family $\{M_i\}_{i\in\mathbb{N}}$ of closed Riemannian m-manifolds with $\mathrm{vol}_m(M_i) \to \infty$ is a *d-dimensional topological expander* if there is a uniform $\varepsilon > 0$ such that

$$\mathrm{vol}_{m-d}\big(f^{-1}(p)\big) \geq \varepsilon\ \mathrm{vol}_m(M_i)$$

for every smooth map $f\colon M_i \to \mathbb{R}^d$.

For $d = 1$, the behaviour of the largest fiber is controlled by the Cheeger constant and by spectral gaps of the graph Laplacian. Margulis' construction of expanders from finite quotients of a Kazhdan group is a fundamental example [7]: for a finitely generated Kazhdan group Γ, finite quotients Γ/Γ' produce Cayley graphs that form an expander family. On the cohomological level, the Kazhdan property introduced by Kazhdan [4] is equivalent to the vanishing of $H^1(\Gamma, V)$ for unitary representations V.

The higher-dimensional analogue starts from Garland's vanishing theorem for lattices in $\mathrm{SL}_n(\mathbb{Q}_p)$ [2]: for unitary representations V,

$$H^i(\Gamma, V) = 0 \qquad (1 \leq i \leq n-2).$$

This can be interpreted as a higher Kazhdan property. The corresponding geometric model is the Bruhat–Tits building. One asks whether its finite quotients form high-dimensional topological expanders, as suggested by the connection between the vanishing of the first cohomology (that is, the Kazhdan property) and one-dimensional expanders. This program has seen major progress: first for $n = 3$ (Ramanujan complexes) [5], and more recently in higher rank [6].

On the Lie group side, an analogous vanishing theorem for lattices in $\mathrm{SL}_n(\mathbb{R})$ was obtained in joint work with Uri Bader [8]. This supports the following conjectural Riemannian picture.

Conjecture. *For a simple noncompact Lie group G of real rank $d \geq 2$ with symmetric space $X = G/K$, the family of compact locally symmetric spaces $\{\Gamma\backslash X\}_\Gamma$ associated with G is a d-dimensional topological expander.*

In the talk I discussed the following result, which partially proves the rank 2 case of the conjecture above [9].

Theorem. *If M is a closed Riemannian manifold with Kazhdan fundamental group, then the family of finite-sheeted covers of M is a 2-dimensional topological expander.*

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Nebeský's step systems and "smoothness" in graphs

Peter F. Stadler
(joint work with Boštjan Brešar, Manoj Changat, and Bruno Schmidt)

The basic idea behind step systems of graphs introduced by Ladislav Nebeský is to formalize a *local* view on shortest path, see e.g. [3]. Set $S(u,v) := I[u,v] \cap N(u)$, where $I[u,v]$ as usual denotes the interval, i.e., the union of the shortest path between two vertices, and $N(u)$ is the neighborhood of u. An vertex $v \in S(u,x)$ can be viewed as a "road sign" at u indicating the directed towards x. Indeed, *signpost systems* are a much more general setting, requiring only the tree axioms for all $u,v,x \in X$

(A) $v \in S(u,x)$ implies $u \in S(v,u)$ (symmetry)
(B) $v \in S(u,x)$ implies $u \notin S(v,x)$ (no going back along shortest paths)
(H) If $u \neq v$ then there is $z \in X$ such that $x \in S(u,v)$ (existence of a forward step)

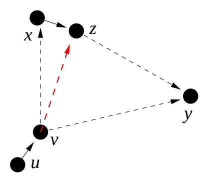
The step systems of connected graphs are signpost systems satisfying an addition set of 5 axioms. It was shown recently that one of them is in fact redundant and thus can be dropped. In the same work is was shown that a connected graph is bipartite if and only if it step system satisfies in addition

(BP) $v \in S(u,v)$ implies $v \in S(u,x)$ or $u \in S(v,x)$.

A property of signpost systems that appears prominently in the literature is the smoothness axiom

(S) $v \in S(u,x)$, $v \in S(u,y)$, and $z \in S(x,y)$ implies $v \in S(u,z)$.

The situtation is depicted on the left.



Theorem 1. [1] *A signpost system S is the step system of a partial cube if and only if satisfies (BP) and (S).*

This result can be taken as a motivation to consider the graphs with a smooth step system in their own right. Note that median graphs are partial cubes, and thus in particular smooth.

It is easy to check that the complete bipartite graph $K_{2,3}$ and the "propeller graph" $K_{1,1,3}$ consisting of three triangle sharing a common edge are not smooth. Moreover, these two graphs must not appear as isometric subgraphs. Since induced subgraphs with diameter 2 are always isometric, it follows that smooth graphs $(K_{2,3}, K_{1,1,3})$-free. The converse is not true, however.

Smoothness has the interesting property of being preserved by many key graph operations:

Theorem 2. *If G and H are smooth, then the Cartesian product $G \Box H$, the strong product $G \boxtimes H$, and any gated amalgam of G and H is smooth.*

It is not difficult to verify that complete graphs, block graphs, and cycles and trees are smooth graphs. From these building blocks, Thm. 2 implies that many well-studied graph classes are smooth in particular quasi-median graphs and more generally, partial Hamming graphs.

A connected graph G with shortest path distance d_G is *embeddable with scale* $\lambda \in \mathbb{N}$ into a graph H with shortest path distance d_H if there exists a mapping $\varphi : V(G) \to V(H)$ such that $d_H(\varphi(x), \varphi(y)) = \lambda d_G(x, y)$ for every pair of vertices $x, y \in V(G)$.

Theorem 3. *Let G be a graph, and let H be a smooth graph such that G has a scale 2 embedding into H. Then G is also a smooth graph.*

As a useful conseqence of Thm. 3 the halve cubes are smooth.

A graph G is a *ℓ_1-graph* if it has a *scale-λ embedding* in a hypercube, i.e., if there an $n \geq 1$ and map $V(G) \to V(Q_n)$ such that $d_{Q_n}(\varphi(x), \varphi(y)) = \lambda d_G(x, y)$ for some fixed positive integer constant λ. Shpectorov [4] proved that a finite graph G is an ℓ_1-graph if and only if it is an isometric subgraph of the Cartesian product of cocktail party graphs (hyperoctahedra), complete graphs, and half-cubes. Since all these building blocks are smooth, we conclude that

Theorem 4. *All ℓ_1-graphs a smooth.*

This in turn establishes smoothness for several large subclasses of weakly modular graphs, including weakly median graphs.

A hint by *Victor Chepoi*, one of the participants in the workshop, just after the meeting uncovered an interesting connection to convexity theory: Denote by $v/u := \{x \in V | v \in \langle\{x, u\}\rangle\}$ the point-shadow of v w.r.t. u and set $v/u := \{x \in V | v \in I(x, u)\}$.

Theorem 5. *G is smooth if and only if $U(u, v)$ is geodesically convex for any two adjacent vertices in G.*

Convexity of $U(u, v)$ appeared as a useful condition in [2]. We note, finally, that there are smooth graphs with non-convex point-shadow of adjacent vertices, such W_4^-, the 4-wheel without a spoke.

Let us call a graph *"strongly prime"* if it is neither a gated amalgam of proper subgraphs nor a nontrivial Cartesian or strong product. It remains an interesting **open problem** to identify the strongly prime smooth graphs.

The work presented here is a collaboration with Boštjan Brešar (U Maribor), Manoj Changat (U Kerala), and Bruno Schmidt (U Leipzig).

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Relative cubulations and random quotients of free products

Markus Steenbock
(joint work with Eduard Einstein†, Suraj Krishna M S, MurphyKate Montee, Thomas Ng)

1. Quotients of free groups

1.1. **The Cayley complex.** Let G be a group given by a group presentation

$$G = \langle X \mid R \rangle,$$

in terms of a finite generating set X and a finite set of relators R, a set of reduced words in $(X \sqcup X^{-1})^*$. In other words, G is a quotient of the free group $F(X)$ by the normal closure $\langle\langle R \rangle\rangle$. Associated to this data is the *Cayley complex*, a 2-complex that can be constructed as follows. Let T be the Cayley graph of $F(X)$, an infinite tree of vertex degree $= 2|X|$. Each non-trivial element g of $F(X)$ acts on T by isometry and fixes a bi-infinite geodesic line, its *axis*, denoted by L_g. Let $\mathcal{L} = \{L_h \mid h \text{ is conjugate into } R\}$. We cone off all the axis in $\mathcal{L}$ to obtain the space

$$\dot{T} = T \bigcup_{L \in \mathcal{L}} (L \times [0,1])/_{(l_1,1)\sim(l_2,1)}.$$

Note that $F(X)$, hence also $\langle\langle R \rangle\rangle$, acts by isometries on $\dot{T}$. The *Cayley complex* $EG := \dot{T}/\langle\langle R \rangle\rangle$. The Cayley complex is a locally finite, connected and simply connected polygonal 2-complex. Morever G acts on EG by automorphism. This action is free and transitive on the vertex set. Note that EG may not be aspherical.

1.2. **Hyperbolic and cubulated groups.** Let $\delta \geqslant 0$. A connected and simply connected polygonal 2-complex Y is δ-hyperbolic if for all simple closed curves c in Y there is a disk diagram $D \to Y$ such that its boundary $\partial D = c$ and such that

$$\delta Area(D) < |\partial D|.$$

Here $Area(D)$ is the number of 2-cells in D, and $|\partial D|$ is the number of edges in the boundary of D. We say that G is *δ-hyperbolic* if EG is δ-hyperbolic. If G is hyperbolic, this implies, for instance, that the word problem is solvable. If in addition G is *cubulated*, that is, acting properly and cocompactly by isometries on a $CAT(0)$ cube complex, then G is, for instance, also residually finite and linear.

1.3. **Gromov's model for random groups.** By the Adian-Rabin theorem there is no algorithm to decide whether a group given by a finite presentation is hyperbolic; it is not even possible to decide its triviality. This motivates to study groups given by a random presentation. We fix rules on how to choose such a random presentation. Fix $m > 1$, $d \in (0,1)$, $\ell > 0$ and a set X of cardinality m. Let

$$S_\ell := \{\text{reduced \& cyclically reduced words in } (X \cup X^{-1})^* \text{ of word length} = \ell\}.$$

Let R_ℓ be a collection of S_ℓ^d-many words chosen uniformly, independently with replacement from S_ℓ.

Theorem 1. *Let $G_\ell = \langle X \mid R_\ell \rangle$. With overwhelming probability when $\ell \to \infty$ the following is true.*

(1) *If $d < \frac{1}{2}$, then G_ℓ is torsion-free and $(1-2d)\ell$-hyperbolic* [2, 3].
(2) *If $d > \frac{1}{2}$, then G_ℓ is finite* [2].
(3) *If $d < \frac{1}{6}$, then G_ℓ is cubulated* [4].

The theorem implies, for instance, that if $d < \frac{1}{6}$, then, with overwhelming probability when $\ell \to \infty$, G_ℓ is residually finite and linear.

2. Quotients of free products

We develop a version of Gromov's model for random groups for quotients of a free product of given finitely generated groups.

2.1. **The model space for relative geometry.** Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a collection of finitely generated groups $P_i = \langle X_i, R_i \rangle$, the *factor groups.* We let

$$\mathcal{F}(\mathcal{P}) = P_1 * P_2 \cdots * P_m = \left\langle \bigcup X_i \mid \bigcup R_i \right\rangle$$

be their free product. Let $\mathcal{R} \subset \mathcal{F}(\mathcal{P})$ be a finite set. We represent each $r \in \mathcal{R}$ as a word $r = h_1 h_2 \ldots h_n$ in normal form, that is, each *syllable* h_i is an element of a factor group and no two consecutive syllables h_i and h_{i+1} are in the same factor group. Then $n = |r|_*$ is the *syllable length* of r. We assume, in addition, that $|r|_* \geqslant 2$. We let

$$G = \mathcal{F}(\mathcal{P}) / \langle\langle \mathcal{R} \rangle\rangle.$$

Next, we let T be a Bass-Serre tree for $\mathcal{F}(\mathcal{P})$, more precisely, the universal cover of the star of groups of n edges identified at a central vertex, where the group associated to the i-the leave vertex is P_i, and the trivial group is associated to the central vertex. The pre-images of the leaf vertices are the *factor vertices.* The factor vertices may have infinite edge degree, so T is *not* in general locally finite. The free product $\mathcal{F}(\mathcal{P})$ acts on this tree. The stabiliser of a factor vertex is conjugate to one of the factor groups, the stabilisers of the other vertices and the edges are trivial. Every element of $\mathcal{R}$ stabilises a bi-infinite geodesic line in T. We define $\dot{T}$ as before and let

$$\mathcal{E}G = \dot{T} / \langle\langle \mathcal{R} \rangle\rangle.$$

The space $\mathcal{E}G$ is a connected and simply-connected polygonal 2-complex, but it is usually not locally finite. As before G acts on $\mathcal{E}G$. Each image of a factor vertex in $\mathcal{E}G$ is stabilised by the image of a conjugate of a factor group, the stabiliser of an edge or one of the other vertices is trivial. As before $\mathcal{E}G$ may not be aspherical.

2.2. **Relatively hyperbolic and relatively cubulated groups.** Let $\delta \geqslant 0$. The group G is *δ-hyperbolic relative to $\mathcal{P}$* if

- the projection $\mathcal{F}(\mathcal{P}) \to G$ restricts to an embedding on the factor groups,
- $\mathcal{E}G$ is δ-hyperbolic 2-complex.

Also, G is *cubulated relative to* $\mathcal{P}$ if G is hyperbolic relative to $\mathcal{P}$ and if it acts cocompactly on a CAT(0) cube complex C such that

- each factor group acts elliptically on C, that is, it fixes a cell,
- each infinite cell stabiliser is conjugate to a finite index subgroup of a factor group.

If G is cubulated relative to a collection of residually finite groups, then G itself is residually finite [1]. We don't know in general whether a group cubulated relative to cubulated groups is itself cubulated.

2.3. **The free product density model.** Let $m > 1$, $d \in (0, 1)$ and $\ell > 0$. The set of all elements of syllable length $= \ell$ is an infinite set if a free factor is infinite. To remidiate this, let $B_i \subset P_i$ be balls of a fixed radius. Then let

$$\mathcal{S}_\ell = \{s \mid s = h_1 \dots h_\ell \text{ in normal form such that each syllable } h_j \in \bigcup_{i=1}^{m} B_i\}.$$

Note that $\mathcal{S}_\ell$ is a finite set. Let $\mathcal{R}_\ell$ be a collection of $\mathcal{S}_\ell^d$-many words chosen uniformly, independently with replacement from $\mathcal{S}_\ell$.

Theorem 2. *Let $G_\ell = \mathcal{F}(\mathcal{P})/\langle\langle \mathcal{R}_\ell \rangle\rangle$. With overwhelming probability when $\ell \to \infty$ the following is true.*

(1) *If $d < \frac{1}{2}$, then G_ℓ is $(1 - 2d)\ell$-hyperbolic relative to $\mathcal{P}$.*
(2) *If $d > \frac{1}{2}$ and $m > 2$, G_ℓ is finite. If $m = 2$, G_ℓ is finite or equal to $\mathcal{F}(\mathcal{P})$.*
(3) *If $d < \frac{1}{6}$, then G_ℓ is cubulated relative to $\mathcal{P}$.*

As a consequence, if $d < \frac{1}{6}$ and if all the factor groups are residually finite, then G_ℓ is residually finite with overwhelming probability when $\ell \to \infty$. If $d < 1/6$ and all the factor groups are cubulated, we prove, in addition, that G_ℓ is cubulated.

2.4. **On the proof.** By a counting argument $|\mathcal{S}_\ell| \sim \lambda^\ell$, for some $\lambda > 0$. Thus $|\mathcal{R}_\ell| \sim \lambda^{d\ell}$, hence, the probability that two 2-cells in $\mathcal{E}G$ share L edges $\lesssim \lambda^{2d\ell - L}$. If $d < \frac{1}{2}\frac{L}{\ell}$ this tends to 0 when $\ell \to \infty$. To prove Theorem 2(1), one elaborates on this idea, following the arguments of [3]. For Theorem 2(3), one proves that the opposite edge hypergraphs make $\mathcal{E}G_\ell$ a wall space of convex walls if $d < 1/6$. In fact, the proof of [4] applies in our situation. The group G_ℓ acts on the dual cube complex, and one proves that this action is a relatively geometric action.

Acknowledment

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Navigating phylogenetic tree space

Karen Vogtmann
(joint work with Lou Billera, Susan Holmes)

Phylogenetics is the branch of biology that studies the evolutionary history of a set of biological taxa. This history is usually encoded as a rooted tree whose leaves are labeled by the taxa. Nodes mark speciation events, and the distance between nodes indicates some measure (such as time) of their difference. Producing phylogenetic trees from data samples is an inexact science for many reasons. The data may be noisy, it may not fit neatly into a tree, different data from the same set of taxa (such as different genes) may produce different trees, different tree-building algorithms or even different runs of the same algorithm on the same data may produce different trees, etc. A strategy for finding the true tree is to use several methods to produce a large set of possible trees and then do a statistical analysis of the set in order to find the most likely true tree.

One approach to this problem is to think of trees as points in a geometric space and use geometric methods to do the statistical analysis. Such a space was proposed by Billera, Holmes and myself in 2001 [1]. We showed that the space we found carries a metric of non-positive curvature, i.e. it satisfies Gromov's CAT(0) condition. Important features of CAT(0) geometry include the fact that there is a unique geodesic between any two trees. Furthermore, the geometry gives a way to define statistical measures on a point set such as a center, spread and density.

There are in fact several natural ways to define a "center" of a finite set of points in a CAT(0) space; in each case the center is unique, but different methods produce different centers. To implement any one of them one must first be able to find the geodesic between two points and its length (i.e. the distance between the points). In the original paper we gave a method for doing this, which I explained in more detail in the unpublished note [4] and which was refined by M. Owen in her thesis in 2009 [2]. This method depends on the fact that the space of trees can be given the structure of a cube complex, and one can explicitly determine the (finite) set of cubes that could possibly carry the geodesic between two trees. Philip Hall's marriage lemma can then be used to isometrically embed this set of cubes as a union of orthants (of various dimensions) in a Euclidean space; we call these *legal* orthants. If the trees being studied have a root and n leaves, then the dimension of the Euclidean space is $n - 2$. The fact that the image is CAT(0) means that once we specify a sequence of legal orthants, standard methods of Euclidean geometry can be used to find the unique shortest path that starts at a specified point in the first orthant and ends at a specified point in the last orthant. To find the geodesic in the space of trees, then, one can list all legal sequences of orthants connecting the first to the last and calculate the shortest path through each of these sequences; the shortest of all these paths is the geodesic.

This algorithm works, but the number of possible legal sequences of orthants is exponential in the number of taxa. In 2011 Owen and J.S. Provan found a more efficient way to determine the best sequence of orthants which is polynomial of degree 4 instead of exponential [3]. They did this by reducing the problem of finding the geodesic to a combinatorial algorithm for analyzing weighted bipartite trees which was known to be $O(n^3)$. In further work, Owen and various coauthors have studied other statistical measures such as the Frechet mean and variance; in this case the center of a finite set is defined as the unique point that minimizes the sum of the squares of the distances to the points.

In the talk I described the space of trees and explained the basics of how to find geodesics, then pointed to the more recent work of M. Owen and others improving and elaborating on this theme.

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Intrinsic geometry from coarse medians

Nick Wright

(joint work with Amina Ladjali, Graham A. Niblo, Jiawen Zhang)

The aim of this talk is to explore and identify the geometry of course median spaces using algebraic and combinatorial approaches.

Median algebras are objects with a notion of “betweenness”: for any triple a, b, c there is a unique point m which is simultaneously between each pair of points. Gromov's δ-hyperbolic spaces (and classical hyperbolic spaces) have an analogous property: for any triple of points a, b, c and geodesic trangle connecting these, each side of the triangle lies in a δ-neighbourhood of the union of the other two. It follows that there exist points which are nearly between each pair in the sense that they are δ-close to any/every geodesic from a to b, b to c and c to a.

Motivated by this Bowditch [1] introduced the concept of coarse median spaces: metric spaces with a coarse median operation. In addition to the majority vote axiom $\langle a, a, b\rangle = a$, and symmetry of $\langle a, b, c\rangle$ under permutation of the arguments, a coarse median is required to satisfy the following:

- there exist $K, H_0 > 0$ such that if $a \sim_C a'$ then $\langle a, b, c\rangle \sim_{KC+H_0} \langle a', b, c\rangle$.
- for every integer p there is a constant H_p such that if $F \in X$ has cardinality p then there is a median algebra Π and a function $\lambda : \Pi \to X$ such that
  - $\lambda(\Pi) \supseteq F$
  - $\lambda(\langle a, b, c\rangle) \sim_{H_p} \langle \lambda(a), \lambda(b), \lambda(c)\rangle$.

The first condition is a large-scale Lipschitz condition on the operation, while the second says roughly that finite sets of points can, uniformly for a given size, be approximated by points in a median algebra.

Now any identity which must hold in a median algebra, for instance the 4-point "associativity" $\langle a, b, \langle c, b, d\rangle\rangle = \langle\langle a, b, c\rangle, b, d\rangle$, or the 5-point "distributivity" $\langle a, b, \langle c, d, e\rangle\rangle = \langle\langle a, b, c\rangle, \langle a, b, d\rangle, e\rangle$, must be approximately true in a coarse median space, with the error bounded independent of the specific tuple of points to which it is applied. With Niblo and Zhang [3], we gave a converse to this:

Theorem 1 (Niblo-W-Zhang)**.** *A metric space equipped with a ternary operation satisfying the majority vote, symmetry and large-scale Lipschitz axioms, is a coarse median space* ***if and only if*** *there is a constant κ_4 such that*

$$\langle a, b, \langle c, b, d\rangle\rangle \sim_{\kappa_4} \langle\langle a, b, c\rangle, b, d\rangle.$$

To show this, for each p-tuple in X we must find a median algebra and a map from this to X whose image includes these points.

The median algebra we take is the free median algebra Π_p on the alphabet $\{\alpha_1, \dots, \alpha_p\}$. One way to construct this is to consider well-formed ternary expressions in these letters and impose the equivalence relation that two expressions are equal if for every median algebra X and for every tuple $(a_1, \dots, a_p)$, the two expressions evaluate to the same point of X. These ternary expressions are a recursively defined language in the alphabet $\{\alpha_1, \dots, \alpha_p, \langle, \rangle\}$ and the evaluation at a tuple $(a_1, \dots, a_p)$ is likewise defined recursively.

While Π_p will not admit a morphism to the putative coarse median space, the ternary expressions *can* be evaluated. We obtain the map $\Pi_p \to X$ by selecting representative expressions for each element of Π_p, and evaluating these. We then argue that this map satisfies Bowditch's axiom by showing that the equivalence relation defining Π_p is generated by elementary transformations corresponding to the majority vote and symmetry axioms along with the 4-point condition.

For a median algebra which is the vertex set of a CAT(0) cube complex, not only does the metric determine the median, but conversely the median also determines the metric. Two natural questions arises for *coarse* median spaces: does the metric determine the coarse median, and does the coarse median determine the metric?

For the first question the answer is no: in the coarse world, quasi-isometric metrics should be viewed as equivalent and therefore the edge metric on $\mathbb{Z}^n$ is equivalent to the Euclidean metric. However the latter does not distinguish axis directions and therefore many coarse medians are possible here.

For the second the answer is yes: with Niblo and Zhang [4] we showed that for a bounded geometry quasi-geodesic coarse median space, the metric is determined up to quasi-isometry by the median. The geometry is intrinsic to the median, indeed we can give a formula defining an intrinsic metric from the median.

Our starting point is to return to the idea of "betweenness". The interval from a to b, denoted $[a,b]$ is the set of points $\langle a,b,x\rangle$ as x ranges over the space. We think of the interval as the points between a and b, and indeed the triple intersection $[a,b]\cap[b,c]\cap[c,a]$ is a set of bounded diameter containing the median $\langle a,b,c\rangle$.

The cardinality of the interval is a proxy for the metric which is entirely determined by the median operation: letting $D(a,b) = \#[a,b]-1$ we have $D(a,a)=0$ and $D(a,b)=D(b,a)$. We can build a metric from D using the following definition:

$$d_{\langle\rangle}(a,b) = \min\{\sum_{i=1}^{k} D(x_{i-1},x_i) : x_0 = a, x_k = b\}.$$

Theorem 2 (Niblo-W-Zhang)**.** *For a bounded geometry quasi-geodesic coarse median space $(X,d,\langle\ \rangle)$, the metric d is the same as $d_{\langle\rangle}$ up to quasi-isometry.*

In the case of a CAT(0) cube complex our definition recovers the edge metric.

For the purpose of bounding distances, the function D is sufficient. We use this to define a coarse median **algebra**. This consists of a set X and a ternary operation such that the majority rule and symmetry axioms hold along with the following uniform bound:

$$\exists K>0 \text{ such that } D(\langle a,b,\langle c,d,e\rangle\rangle,\langle\langle a,b,c\rangle,\langle a,b,d\rangle,e\rangle) < K \quad \forall a,b,c,d,e\in X.$$

A metric is no longer part of the required data. The 5-point condition here provides a proxy for the large-scale Lipschitz condition as well as yielding the 4-point condition. Thus a coarse median algebra, when equipped with its intrinsic metric, gives a coarse median space in the sense of Bowditch.

We conclude with a question about the internal structure of intervals, which is joint work with Ladjali [2]. In a CAT(0) cube complex, intervals are "flat" in the sense that they can be embedded as median subalgebras of $\mathbb{Z}^n$.

In a median algebra, points in an interval $[a,b]$ have a *partial ordering*

$$x\le y \iff x\in[a,y] \iff y\in[x,b]$$

making the interval into a *lattice*. The meet and join are defined using the median:

$$x\wedge y = \langle a,x,y\rangle \qquad x\vee y = \langle x,y,b\rangle.$$

We define a point u in $[a,b]$ to be **edge-maximal** if

$$x,y\in[u,b] \text{ and } u = x\wedge y \implies x=u \text{ or } y=u.$$

In a CAT(0) cube complex these are points u for which there is at most one edge (u,v) with $u<v$.

With Ladjali we showed that width of the poset M of edge-maximal points equals the rank of the interval. By Dilworth's Lemma this poset therefore splits as a union of n chains.

We view these chains as providing coordinates on the interval: For a point $x\in[a,b]$ we define the coordinates of x to be the tuple $(u_1,\ldots,u_n)$, where u_i

is the least element of the ith chain such that $x \leq u_i$. We can recover x as $x = u_1 \wedge \cdots \wedge u_n$.

Extending this to *coarse* median spaces is more difficult. We would like to say $x \leq_C y$ if $x \wedge y := \langle a, x, y\rangle$ is C-close to x, however this relation will in no longer transitive. We generalise the notion of edge-maximality defining $u \in [a, b]$ to be (D, E) edge-maximal if

$$u \sim_D x_1 \wedge x_2 \wedge \cdots \wedge x_n \implies u \sim_E x_i \text{ for some } i$$

Theorem 3 (Ladjali-W)**.** *There exists a constant D such that for any E there exists C such that the $\leq_C$-antichains in the (D, E) edge-maximal set have cardinality bounded by the (coarse) rank of $[a, b]$.*

As a consequence we deduce that the relation $\leq_C$ restricted to a sufficiently separated subset of the edge-maximal set is loop-free. Hence $\leq_C$ extends to a partial-order and once again we can split into chains and construct tuples of coordinates $(u_1, \ldots, u_n)$. These coordinates coarsely respect the median structure:

Theorem 4 (Ladjali-W)**.** *The map $x \mapsto (u_1, \ldots, u_n)$ is a quasi-morphism.*

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Problem session

This section contains a collection of open problems, some of which are well-known. They were contributed by the workshop participants during the problem session.

Problem 1. The Cannon conjecture, *Chatterji*
The *Cannon conjecture* states that if G is a hyperbolic group whose boundary at infinity is homeomorphic to the 2-sphere, i.e., $\partial G \cong S^2$, then G admits a discrete, faithful, and cocompact action on hyperbolic 3-space $\mathbb{H}^3$. Equivalently, G is isomorphic to a Kleinian group, i.e., $G < PSL_2(\mathbf{C})$. Markovic [1] and Haissinsky [2] showed that this holds under the assumption that G is cubulated. The following set of questions are rephrased from a conversation with Agol in 2019.

Question i) Can you replace "cubulated" by "medianized" (= act geometrically on a median space) in Markovic result?

Question ii) Can you medianize a hyperbolic group with $\partial G \cong S^2$? That is potentially much easier than cubulate.

Question iii) Can we even prove that there is any effective action on a median space for such a hyperbolic group? That amounts to proving that it doesn't have Kazhdan's property (T). Is it related to the previous question?

Question iv) Is every medianized hyperbolic group cubulable? Can you even prove residually finite?

Question v) Is every group that is cocompact median of finite rank cubulable?

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Problem 2. Symmetric groups and wall structures, *Arzhantseva*
Let S_n denote the permutation group on a finite set of n elements.

Question i) Does there exist a sequence of generating sets X_n of S_n such that the sizes $|X_n|$ are uniformly bounded and the corresponding sequence of Cayley graphs $(Cay(S_n, X_n))_n$ admits a wall structure whose wall metric is coarsely equivalent to the word length metric (i.e., to the graph metric) on $(Cay(S_n, X_n))_n$?

The existence of a wall structure is an obstruction to expansion: such a structure implies a coarse embedding of the sequence $(Cay(S_n, X_n))_n$ into a Hilbert space. While a celebrated result of Kassabov provides generating sets Y_n such that the resulting Cayley graphs $(Cay(S_n, Y_n))_n$ form a family of expanders, it is well known that other families, such as $(Cay(S_n, \{(12), (12\dots n)\}))_n$ fail to expand.

Beyond mere existence, one may ask about the prevalence of such generating sets.

Question ii) For a fixed $k \geqslant 2$, does the proportion of generating sets X_n of S_n of size k whose Cayley graphs admit a wall structure coarsely equivalent to the word length metric remain bounded away from zero as $n \to \infty$?

Positive answers to these questions would stand in sharp contrast to the expansion properties proved by Kassabov.

Problem 3. Coarse median spaces, *Arzhantseva*
A coarse median space is a metric space endowed with a ternary operator that satisfies axioms of a median algebra only up to a bounded error. Introduced by Brian Bowditch, examples of coarse median spaces include Gromov hyperbolic spaces, $CAT(0)$ cube complexes, mapping class groups, Teichmüller spaces of compact surfaces, and hierarchically hyperbolic spaces.

Question. Does every coarse median space coarsely embed into a Hilbert space?

An affirmative answer would provide a far-reaching generalisation of the well-known positive result on median spaces.

Problem 4. Single-peakedness, *Chatterji*
Can singled peakedness preferences be understood in terms of normal cube path, the space of orders on X and measures on X? What happens when we allow X

to be infinite countable? Once the peak has been chosen, the minimum might be on the boundary, is it well-defined?

Problem 5. Partial data, *Huber, interpreted by Chatterji & Niblo*
Suppose that in constructing the Buneman graph we encounter some uncertainties in determining which half spaces some vertices lie in. This can occur because of missing data, or measurement errors for example. Is there a coarse variant of the Buneman graph that can model this data, allowing us to compute medians up to controlled error?

One possibility is to define a partial wall space as sets of probability measures on pairs of partitions that differ at one single point (or a small proportion of points), so that $\mathcal{H} \subseteq [0,1]^X$ and define $H^c = \mathbf{1}_X \setminus H$. We still have a map

$$\pi\colon \mathcal{H} \to \mathcal{W}$$

and look at sections. Now when 2 sections differ we no longer get an integer value but we should still get a metric space, which may not be median, but might have controlled geometry.

Problem 6. Graphs of hyperplanes, *Genevois*
Let X be a median graph.

- Its *transversality graph* $\mathcal{T}(X)$ is the graph whose vertices are the hyperplanes of X and whose edges connect any two transverse hyperplanes.
- Its *contact graph* $\mathcal{C}(X)$ is the graph whose vertices are the hyperplanes of X and whose edges connect any two hyperplanes that are either transverse or tangent.

One easily shows that not every graph can be described as the contact graph of some median graphs, e.g. cycles of length ≥ 4. Hence:

Question i) Which graphs can be described as contact graphs of median graphs?

On the other hand, it is known that every graph can be realised as the transversality graph of some median graph. But the degree of the median graph one has to consider may explode. For instance, it can be shown that the transversality graph of a median graph X is $(3 - \deg(X)/2)$-hyperbolic. This motivates the following definition: Given an arbitrary graph G, its *transversality degree* $\mathrm{TD}(G)$ is the smallest degree of a median graph whose transversality graph is isomorphic to G.

Question ii) How can one compute the transversality degree of a graph?

Notice, for instance, that the transversality degree of a grid $[0,n] \times [0,n]$ tends to $+\infty$ as n grows (as a consequence of the previous comment on hyperbolicity of transversality graphs).

Problem 7. Travelling polytopes, *Genevois*
Given a graph X, two vertices $x, y \in V(X)$, and a finite set of vertices $F \subset V(X)$, define

Travelling salesman: $\mathrm{TS}(x, F, y)$ as the shortest length of a path connecting x to y and passing through all the vertices in F;

Travelling polytope: $\mathrm{TP}(x, F, y)$ as the shortest length of a path of polytopes $P_0 := \{x\}, P_1, \ldots, P_{k-1}, P_k := \{y\}$ such that $F \subset \bigcup_i V(P_i)$.

Here, a polytope refers to the convex hull of finitely many vertices. A path of polytopes is a sequence such that, given two consecutive polytopes, one (say P) can be obtained from the other (say Q) as the convex hull of $Q \cup \{u\}$ for some vertex $u \in \partial Q$.

The problem is to compare the solutions to the travelling salesman and polytope problems. It is clear that $\mathrm{TP} \leq 2\mathrm{TS}$. For the reverse equality, we introduce the quantity

$$\mathrm{TP}(X) := \sup\{\alpha \mid \exists C > 0, \forall x, y, \forall F, \mathrm{TP}(x, F, y) + |F| \geq C(\mathrm{TS}(x, F, y) + |F|)^{\alpha}\}.$$

It is known that, given an unbounded median graph X with bounded degree:

- $\mathrm{TP}(X) \in [1/2, 2/3] \cup \{1\}$;
- $\mathrm{TP}(X) = 1$ if and only if X is hyperbolic;
- $\mathrm{TP}(X) = 1/2$ if X has infinite cubical dimension;
- $\mathrm{TP}(\mathbb{E}^d) = \frac{d}{2d-1}$.

A natural question is:

Question. Which values can take $\mathrm{TP}(\cdot)$ on median graphs?

For instance, it would be interesting to compute the travelling polytope constant of a product of (regular) trees.

Problem 8. Metric actions on median graphs, *Genevois*

An action $G \curvearrowright X$ on a graph is *proper* if vertex-stabilisers are finite and *metrically proper* if $\{g \in G \mid d(o, go) \leq R\}$ is finite for every $R \geq 0$ (given an arbitrary basepoint $o \in V(X)$). An interesting problem is to understand the difference between acting properly on a median graph and acting metrically properly on a median graph.

For instance, consider a permutational wreath product $\mathbb{Z}_2 \wr_{G/N} G$. On the one hand, if G acts properly on a median graph X, then $\mathbb{Z}_2 \wr_{G/N} G$ acts properly on $\left(\bigoplus_{G/N} \mathbb{Z}_2\right) \times X$, where the direct sum is thought of as a hypercube. On the other hand, a theorem of Chifan and Ioana shows that $\mathbb{Z}_2 \wr_{G/N} G$ is not a-T-menable whenever G is a-T-menable but G/N is not. Thus, $\mathbb{Z}_2 \wr_{\mathbb{F}_n/N} \mathbb{F}_n$ admits a proper action on a median graph but not a metrically properly action on a median graph (as a consequence of a theorem of Niblo and Reeves). It would be interesting to understand such examples from the point of view of median geometry.

Question i) If $\mathbb{F}_n/N$ does not act metrically properly on a median graph (e.g. $\mathrm{F}_n/N \simeq \mathrm{BS}(1, 2)$), can $\mathbb{Z}_2 \wr_{\mathbb{F}_n/N} \mathbb{F}_n$ act metrically properly on a median graph?

For another example of interest, we can mention:

Question ii) Does the Grigorchuk group act metrically properly on a median graph?

A construction of Schneeberger shows that a proper action does exist, but this specific action is not metrically proper.

Problem 9. Fixed-point properties, *Genevois*
Inspired by Thompson's group V as a group of homeomorphisms of the Cantor set $\mathfrak{C}$, Brin defined a *higher-dimensional Thompson group* $2V$ as a group of homeomorphisms of $\mathfrak{C}^2$. Interestingly, despite the fact that V acts metrically properly on a locally finite median graph, $2V$ does not (as it contains distorted elements).

Question i) Does $2V$ admit an action with unbounded orbits on some median graph? Does it satisfy Kazhdan's property (T)?

There exist rather few interesting examples of groups with the bounded-orbit property on median graphs (a.k.a. property (FW)). Finding new examples would be interesting. For instance:

Question ii) Construct explicit examples of hyperbolic groups with property (FW) but without property (T).

Such examples are known to exist, for instance by applying the Rips construction obtained by Ollivier and Wise to $\mathrm{SL}(2, \mathbb{Z}[\sqrt{2}])$.

Problem 10. Covers of cube complexes, *Shepherd*
The following question has been posed a while ago:

Question i) *(Haglund)*: If X_1 and X_2 are finite special cube complexes with a common universal cover, do they necessarily have a common finite cover?

Leighton's theorem says that the answer is YES if X_1 and X_2 are graphs. More generally, the answer is YES if X_1 and X_2 have free fundamental groups (Bridson–Shepherd). The answer is also YES if the common universal cover is the right-angled building associated to a graph product of finite groups (Shepherd). The specialness assumption is necessary, indeed without it one can construct counter-examples where the common universal cover is a product of trees (using examples of Wise or Burger–Mozes).

Question ii) Is there an algorithm that takes as input a pair of finite non-positively curved cube complexes and determines whether they have isomorphic universal covers?

Such an algorithm is known for the case of graphs (Angluin), it is based on the well-known colour refinement procedure from graph theory.

Problem 11. Median/cubical actions of hyperbolic groups, *Hagen*
Let G be a hyperbolic group.

(1) Suppose that for all infinite-order $g \in G$, there exists a cobounded (not necessarily proper) action of G on a CAT(0) cube complex X_g such that the hyperplane-stabilisers are all quasiconvex subgroups of G, and $\langle g \rangle$ has unbounded orbits. Is G cocompactly cubulable? Or, at least, is G a-T-menable?

(2) Let G be hyperbolic and act properly and cocompactly on a (finite rank?) median space. Is G necessarily residually finite? Or, in fact, linear? (This is in principle easier than the well-known question about cocompactly cubulating hyperbolic median groups.)

Problem 12. Cartan-Hadamard theorem for median spaces, *Bowditch*
One can show that any complete simply connected path-metric space which is uniformly locally median is in fact median. Here, "uniformly locally median" means that there is some $\epsilon > 0$ such that any triple of points of diameter less than epsilon has a unique median.

Question. Can one remove the word "uniform" from this statement, i.e., instead just suppose that every point has a neighbourhood in which every triple has a unique median?

This is elaborated on in some (too much!) detail in my preprint cited below:

References

[1] B. Bowditch. *A Cartan-Hadamard theorem for median metric spaces*, `https://bhbowditch.com/papers/ch-median.pdf`

Problem 13. Universal real cubings, *Casals-Ruiz (from discussions with M. Hagen and I. Kazachkov)*
In his study of asymptotic invariants, Gromov proved that each non-principal asymptotic cone of a hyperbolic group is a universal real tree and that this completely characterises the class of hyperbolic groups: a finitely generated group is hyperbolic if and only if each (non-principal) asymptotic cone is a universal real tree.

Question i) Which finitely generated groups have all non-principal asymptotic cones bilipschitz equivalent to a universal real cubing?

Bowditch proved that non-principal asymptotic cones of coarse median groups are median algebras. Can we characterise groups quasi-isometric to an HHS within the class of coarse median groups in terms of their asymptotic cones? Namely,

Question ii) Is a coarse median group quasi-isometric to an HHS if and only if each (non-principal) asymptotic cone is (bi-Lipschitz equivalent to) a universal real cubing?

Problem 14. Graphical apiculate algebras, *Chepoi & Bandelt*
Every connected graph $G = (V, E)$ can be turned into a ternary algebra, called an *apex algebra* of G: an *apex operation* $(\dots) : V^3 \to V$ maps any triplets u, v, w and

u, w, v to a vertex $x = (uvw) = (uwv) \in I(u,v) \cap I(u,w)$, called a u-apex relative to v and w, such that $I(u,x)$ is maximal with respect to inclusion. A graph G is called *apiculate* if the u-apex relative to v and w is unique for all $u, v, w \in V$ [1]. Equivalently, G is apiculate if for each u the basepoint order $(V, \leq_u)$ (in which $x \leq_u y$ if and only if $x \in I(u,y)$) is a meet-semilattice.

A *metric triangle* xyz is a triplet of vertices x, y, z such that $I(x,y) \cap I(x,z) = \{x\}, I(y,x) \cap I(y,z) = \{y\}$, and $I(z,x) \cap I(z,y) = \{z\}$ [4]. A metric triangle $y_1y_2y_3$ is called a *quasi-median* of a triplet $x_1, x_2, x_3 \in V$ if for any $i, j \in \{1,2,3\}, i \neq j$, the vertices y_i, y_j lie on a common shortest path between x_i and x_j. Any triplet of vertices in any connected graph admits at least one quasi-median. A graph $G = (V, E)$ is called a *generalized median graph* if any triplet $x_1, x_2, x_3 \in V$ has a unique quasi-median. Each generalized median graph is apiculate but the converse is not true, see [1]. We will call the apex algebra of a generalized median graph a *generalized median algebra.*

A ternary algebra $(V, (\ldots))$ is called *apiculate* if there exists an apiculate graph $G = (V, E)$ such that $(\ldots)$ is the apex operation of G endowed with the standard graph-distance d and standard interval function I. A ternary algebra $(V, (\ldots))$ is called a *graphical apiculate algebra* if $(V, (\ldots))$ and **all its subalgebras** are apiculate algebras.

Discrete median, quasi-median [3], and weakly median algebras [2] are graphical apiculate algebras. In general, an apex algebra of a graph is not necessarily graphical. The following questions naturally arise from the work [1, 2]:

Question i) Characterize the apiculate graphs having graphical apiculate algebras.

Question ii) Characterize the generalized median graphs having graphical generalized median algebras.

References

[1] H.-J. Bandelt and V. Chepoi, *The algebra of metric betweenness I: subdirect representation and retracts*, Europ. J. Combin. **28** (2007), 1640–1661.

[2] H.-J. Bandelt and V. Chepoi, *The algebra of metric betweenness II: axiomatics of weakly median graphs*, Europ. J. Combin. **29** (2008), 676–700.

[3] H.-J. Bandelt, H. M. Mulder and E. Wilkeit, *Quasi-median graphs and algebras*, J. Graph Theory **18** (1994), 681–703.

[4] V. Chepoi, *Classifying graphs by metric triangles (in Russian)*, Metody Diskretnogo Analiza **49** (1989), 75–93.

Problem 15. Corners in subcomplexes of $\mathbb{Z}^3$, *V. Chepoi (from discussions with J. Chalopin and M. Kokkou)*

Let G be a finite connected (partial) subgraph of $\mathbb{Z}^3$. Let $X(G)$ be the cube complex of G, i.e., a square (respectively, a 3-cube) of $\mathbb{Z}^3$ belongs to $X(G)$ if and only if all 4 vertices and 4 edges of this square belong to G (respectively, all 8 vertices and 12 edges of this cube belong to G). Then G is the 1-skeleton of $X(G)$. A *corner* of $X(G)$ is a vertex v belonging to a unique maximal cell of $X(G)$. A *corner peeling* of G (or of $X(G)$) is a total ordering of its vertices $v_1, \ldots, v_n$ such that v_i is a corner of $G_i = G[v_1, \ldots, v_i]$ (clearly, $G_n = G$).

By a *coordinate hyperplane* we will mean any hyperplane of $\mathbb{R}^3$ of the form $x_i = \alpha$ for some $i = 1, 2, 3$ and $\alpha \in \mathbb{R}$. A *section* of $X(G)$ is a nonempty intersection of $X(G)$ with a coordinate hyperplane. A *piece* is a connected component of a section. An *integer piece* (respectively, a *half-integer piece*) is any piece defined by a coordinate hyperplane $x_i = \alpha$, where α is an integer (respectively, where α is a half-integer, i.e., $\alpha = \alpha_0 + \frac{1}{2}$ with $\alpha_0 \in \mathbb{Z}$).

The following conjecture was formulated in connection with leader election (designing a unique global leader by local decisions, see [2] for results relating leader election with metric graph theory) via corner peelings:

Conjecture. *Let $G = (V, E)$ be a finite connected subgraph of $\mathbb{Z}^3$ satisfying the following conditions:*

(1) *the cube complex $X(G)$ of G is simply connected;*
(2) *each piece of $X(G)$ is simply connected.*

Then G admits a corner peeling.

Clearly, it is sufficient to require condition (2) only for integer and half-integer pieces. We can prove this conjecture with the following additional condition. For each half-integer piece H (which can be viewed as an analog of the mid-hyperplane in CAT(0) cube complexes), consider the carrier $N(H)$ of H in $X(G)$. Let $N^+(H)$ and $N^-(H)$ are the two half-carriers of H. Both $N^+(H)$ and $N^-(H)$ are isomorphic to H (and $N(H)$ is the product of the segment $[0, 1]$ with H). Both $N^+(H)$ and $N^-(H)$ are subcomplexes of the integer-pieces H^+ and H^- of $X(G)$, respectively. Then, consider the following condition:

(3) $N^+(H)$ is an isometric subgraph/subcomplex of H^+ and $N^-(H)$ is an isometric subgraph/subcomplex of H^-.

Then one can show that if G satisfies the conditions (1),(2),(3), then G admits a corner peeling (in fact, one can ask (3) only for coordinate hyperplanes of one direction).

On the other hand, there exists an isometric subgraph G of $\mathbb{Z}^{12}$ (in fact of a 12-dimensional hypercube) without any corner, which has a contractible cube complex $X(G)$, and whose each section is an isometric subgraph [1].

References

[1] J. Chalopin, V. Chepoi, S. Moran and M. Warmuth, *Unlabeled sample compression schemes and corner peelings for ample and maximum classes*, J. Comput. Syst. Sci. **127** (2022), 1–28.

[2] J. Chalopin, V. Chepoi and M. Kokkou, *Distance-based certification for leader election in meshed graphs and local recognition of their subclasses*, arXiv:2602.12894v1 (2026).

Problem 16. The free product density model, *Steenbock (from discussions with Einstein, Montee, Ng, Suraj Krishna)*

In Gromov's random model for random groups, if the density $d > 1/3$, then by a result of Zuk, the random group has Kazhdan's Property (T) with overwhelming probability.

Question. Does this remain true for a random quotient of a free product of at least 3 factors in the free product density model at density $d > 1/3$?

As Property (T) is closed under quotients, it would be sufficient to prove that a random quotient of a free product of free groups in the free product density model at density $d > 1/3$ has Property (T) with overwhelming probability.

At least it would be good to show the following.

Conjecture. *A random quotient of a free product of at least 3 factors in the free product density model at density* $d > 1/3$ *is not cubulated relative to the free factors.*

Dictionary across disciplines

In a number of contexts CAT(0) cube complexes, and their duals, discrete median algebras, provide a natural model for binary choices and their interactions. As a result, the concept has appeared in a range of disciplines including economics, mathematical biology and computer science.

For some of these areas, there is a clear correspondence between the concepts, in others, there is instead a fruitful analogy. This is particularly the case with social choice theory and the theory of concurrency, where the analogies become precise only for suitably restricted classes of domains. In both cases, the lack of a common vocabulary was a significant barrier to communication and inter-disciplinary collaboration. The following table was compiled by workshop participants in an effort to overcome that obstruction. The completed entries played an important role in supporting inter-disciplinary dialog throughout the workshop, while the gaps suggest areas for future fruitful cross-disciplinary collaboration.

Geometric group theory	**Discrete mathematics**	**Computer science**	**Phylo-genomics**	**Economics**
CAT(0) cube complex	median algebras/median graphs	certain coherent event structures	split system	property space
wallspace	cut system			
wall/hyperplane	cut	event	split	issue
halfspace	prime ideal/prime convex set		side	binary property
crossing walls	crossing cuts	concurrent events	incompatible splits	independent choices
Sageev's construction	Stone/Roller duality	Buneman graph		
principal ultrafilter	vertex	configuration	S-map/split-map	consumption bundle
convex set	ideal in a distributive lattice/median subalgebra	configuration domain		consistent set of choices
gate-convex set	Chebyshev ideal/retract	retract/projection-stable subdomain		

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