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Cohomology of Finite Groups: Interactions and Applications

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Abstract. This workshop on cohomology of finite groups has traditionally a special emphasis on interactions with neighboring fields, such as group theory, algebraic topology, commutative algebra, and representation theory. This subject has been the topic of five workshops held at Oberwolfach in the last twenty-five years. The workshop aims to highlight recent progress and explore new connections, fostering the continued development of the interactions that have been so productive in recent years. Special emphasis was given to topics which have recently seen important breakthroughs, such as higher representation theory, homotopy theory of permutation modules, and the solution of a conjecture of Quillen.

Mathematics Subject Classification (2020): 20J06.

Introduction by the Organizers

The workshop *Cohomology of finite groups: Interactions and Applications* had 45 in-person participants, and two who participated remotely via Zoom. All three organisers were physically present. All career stages were represented among participants, from senior researchers to PhD students. And also there was a broad variety of participants from neighboring fields: group theory, algebraic topology, commutative algebra, and representation theory.

Many participants submitted an abstract to give a talk. The schedule accomodated 25 talks ranging in length from 20 minutes to an hour. Moreover, on Wednesday evening a gong show was scheduled with short 10-minute talk+questions which was very successful and lasted around an hour and a half. The weather cleared on Wednesday after some rainy days, allowing for the traditional afternoon hike to St. Roman.

The workshop started on Monday with a special schedule: four talks on four different highlighted topics in the style of survey or state of the art. Since participants were from different backgrounds, those talks aimed to introduce the relevant questions in different fields. Julia Pevtsova gave a state of the art talk on finite generation of cohomology, from Golod to van der Kallen. The talk focused on recent results and stating open questions related to the finite generation of the cohomology of finite group schemes over Verlinde categories. Paul Balmer surveyed recent progress with Gallauer and others on the homotopy category of permutation modules, describing advances in understanding its tensor-triangular spectrum, and its relation to Picard groups. Dan Nakano's talk on the cohomology of algebraic groups went through the historical context for Donkin's conjectures, and stated a revised version of the tilting module conjecture. Finally, Peter Symonds described recent progress on the cohomology of profinite groups, and approached the question of concretely describing the cohomology of a profinite group considered as a discrete group, if one can predict which groups in the limit can have the same cohomology in finite coefficients.

The rest of the week then proceeded with research talks presenting recent breakthroughs in the different fields of the workshop, with main themes including rank conjectures, higher representation theory, and cohomology of generalizations of finite groups, as detailed in the extended abstracts to follow. For instance Sam Miller explained the role of the Picard group of endotrivial complexes of permutation modules, and the connection to the recent work of Balmer and Gallauer; John Greenlees presented recent results on the singularity category of the cochains on the classifying space of a finite group, and Ian Hambleton uncovered the relationship between Tate cohomology and the existence of finite 2-complexes and 4-manifolds with certain properties related to the difference between stable and unstable classification. We also included a special block of short talks on Thursday morning on recent developments to settle Quillen's conjecture on the poset of p-subgroups of a finite group.

As always in Oberwolfach, the environment led to many discussions among participants during breaks, meals, and walks, and we received very positive unsolicited feedback from several participants after the conference.

Cohomology of Finite Groups: Interactions and Applications

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Abstracts

Finite generation of cohomology: from Golod to van der Kallen

Julia Pevtsova

This was a survey talk. One of the goals was to attract attention to a recent set of notes posted on the arxiv in October of 2025: J.O. Gomez and C. Parker, "Notes on Cohomological finite generation for finite group schemes" which explain van der Kallen's proof of cohomological finite generation for finite group schemes over an arbitrary ring [10].

1. Conjectures

Definition 1.1 ([6])**.** *A finite tensor category over a field k is a k-linear tensor category $\mathcal{C}$ such that*

(1) *$\mathcal{C}$ is abelian and rigid,*
(2) $\dim_k \operatorname{Hom}_{\mathcal{C}}(X, Y) < \infty$ *for all X, Y,*
(3) *every object has finite length,*
(4) *$\mathcal{A}$ has finitely many simple objects,*
(5) *the tensor unit $\mathbf{1}$ is simple.*

Conjecture 1.2 (Etingof-Ostrik'04)**.** *Let $\mathcal{C}$ be a finite tensor category.*

(1) $H^*(\mathcal{C}) = \operatorname{Ext}^*_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ *is Noetherian;*
(2) *For any $M, N \in \mathcal{C}$, $\operatorname{Ext}^*_{\mathcal{C}}(M, N)$ is a finite module over $\operatorname{Ext}^*_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$.*

When conjecture holds, we say that $\mathcal{C}$ satisfies *CFG (Cohomological Finite Generation)*. This is an essential property to be able to study *tensor triangular geometry* for the category $\mathcal{C}$ and its ind-completion. In particular, as an "application" to the Etingof-Ostrik conjecture one can state another conjecture [12], in terms of the stable category of $\mathcal{C}$ which we denote $\operatorname{stab}\mathcal{C}$.

Conjecture 1.3 (Nakano-Vashaw-Yakimov, 2021)**.** *Let $\mathcal{C}$ be a finite tensor category. Then the Balmer spectrum of the stable category of $\mathcal{C}$ can be identified with a certain subring $Z_c(H^*(\mathcal{C}))$ of $H^*(\mathcal{C})$, called the "categorical center".*

Remark 1.4. *When the category $\mathcal{C}$ is symmetric, the "categorical center" is the entire cohomology ring; and the conjecture simplifies to the following statement:*

$$\operatorname{Spec}(\operatorname{stab}\mathcal{C}) \cong \operatorname{Proj} H^*(\mathcal{C}).$$

The same simplification holds in a braided situation. Even for non-braided categories, such as representations of a quantum borel for a semisimple Lie algebra, the conjectured calculation of the Balmer spectrum for the stable category involves the entire cohomology ring.

2. Cohomology of Hopf algebras

Let A be a finite dimensional Hopf algebra over a field k. Then finite dimensional representations of A form a finite tensor category $\operatorname{rep} A$, which is a major source of examples of finite tensor categories. The tensor structure in this case is determined by the coproduct on A. The CFG conjecture is wide open for Hopf algebras except when A is cocommutative. We recall the classical case when $A = kG$, a group algebra for a finite group G [1], [14], [7].

Theorem 2.1 (Golod'58, Venkov'59, Evens'61)**.** *Let G be a finite group, and k a field of positive characetristic p. Then the cohomology algebra $H^*(G,k)$ is a finitely generated k-algebra; and for any finite kG-module M, the cohomology $H^*(G,M)$ is a finite module over $H^*(G,k)$. In other words, $\operatorname{rep}_k G$ satisfies cohomological finite generation.*

There is a generalization to finite group schemes [9].

Theorem 2.2 (Friedlander-Suslin, 1995)**.** *Let G be a finite group scheme over a field k. Then $\operatorname{rep}_k G$ satisfies cohomological finite generation.*

The "tensor triangular geometry" conjecture is also known in this case, thanks to the work of Benson-Carlson-Rickard [4] in the finite group case and Friedlander-Pevtsova [8] and Benson-Iyengar-Krause-Pevtsova [5] in the finite group scheme case:

Theorem 2.3.

$$\operatorname{Spec}(\operatorname{stab} kG) \cong \operatorname{Proj} H^*(G,k)$$

3. van der Kallen's theorem: group schemes over rings

In December of 2022 van der Kallen announced a generalization of the Friedlander-Suslin finite generation theorem to (flat) finite group schemes over arbitrary coefficients [13]. We'll state the result equivalently in terms of Hopf algebras.

Theorem 3.1. *[W. van der Kallen, 2025] Let R be a commutative Noetherian ring, and A be a finite projective cocommutative Hopf algebra over R. Then A satisfies "cohomological finite generation" property; in particular, $H^*(A,R)$ is Noetherian.*

In this case, the tensor triangular application is also known, due to a recent work of Barthel, Benson, Iyengar, Krause, Pevtsova [3]; see also Lau [11].

Theorem 3.2. *Let R be a commutative Noetherian ring, and A be a finite projective cocommutative Hopf algebra over R. Then the tensor triangulated category* $\operatorname{Stab}(A,R)$ *(the stable module category of lattices of A-modules over R) is stratified by* $\operatorname{Proj} H^*(A,R)$.

In particular, there is a homeomorphism

$$\operatorname{Spec}(\operatorname{stab}(A,R)) \cong \operatorname{Proj} H^*(A,R).$$

The proof of Theorem 3.1 takes only 3 pages in van der Kallen's paper but the "real" proof is outlined in the notes [10]. It consists of three steps:

(1) Reduce to $\mathbb{GL}_n$ (easy)
(2) Main step: reduction of the CFG to the statement that $\{H^n(\mathbb{GL}_n, B) \mid n \geq 1\}$ has uniformly bounded torsion for any finitely generated $\mathbb{GL}_n$-algebra B. This reduction takes the most effort, putting together many papers written over the last 30 years, including Friedlander-Suslin's work.
(3) Proving bounded torsion. This is what is effectively done in van der Kallen's 2025 paper; the notes of Gomez-Parker offer a different proof.

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Survey of the geometry of permutation modules

Paul Balmer

This talk was a survey of work undertaken with Martin Gallauer on the tensor-triangular geometry of permutation modules. We saw: for G a finite group and k a field characteristic p dividing $|G|$. We have a full subcategory of p-permutation modules:

$$\mathrm{mod}(kG) \hookleftarrow \mathrm{perm}(kG) = \{kX \mid X \text{ is a finite } G\text{-set}\}$$

that yields a Verdier localization on derived categories:

$$D_b(kG) \twoheadleftarrow K_b(\mathrm{perm}(kG))^\natural =: \mathcal{K}(G).$$

Applying the spectrum, we get the top inclusion below:

$$\begin{array}{ccc} Spc(D_b(kG)) & \hookrightarrow & Spc(\mathcal{K}(G)) \\ \text{B.C.R.}\ \downarrow \cong & & \downarrow \\ Spec^h(H^\bullet(G,k)) & \longrightarrow & Spec^h(k) = * \end{array}$$

where we identify the left-hand side by Benson-Carlson-Rickard, using the comparison map. Note that the comparison map for $\mathcal{K}(G)$ is not very useful.

The reduction to elementary abelian groups E, while involved $E \leq G$ in the classical story, now involves *subquotients* $E = H/K$ for $K \trianglelefteq H \leq G$ in the case of $\mathcal{K}(G)$.

One key ingredient that is specific to $\mathcal{K}(G)$ and does not appear for $D_b(kG)$: The *geometric fixed pts* (a.k.a. *Brauer quotients*, in the module case):

$$\Phi^H : \mathcal{K}(G) \longrightarrow \mathcal{K}(G//H)$$

where $G//H = N_G(H)/H$ is the Weyl gp of H in G, when H is a p-subgroup. (This allows reduction to elementary abelian groups as above.)

For elementary abelian groups, the topology is controlled thanks to $\otimes$-invertible objects that are locally trivial or, somewhat equivalently, via a multigraded endomorphism ring.

We refer to Sam Miller's abstract for some very interesting extension of the twisted cohomology beyond elementary abelian.

Cohomology for Algebraic Groups and Donkin's Conjectures

Daniel K. Nakano

(joint work with Christopher P. Bendel, Cornelius Pillen, Paul Sobaje)

In the study of modular representations of Lie algebras arising from algebraic groups, a fundamental problem is determining when such representations lift to the ambient algebraic group. In 1960, Curtis [C60] established that if G is a simple, simply connected algebraic group over an algebraically closed field k of characteristic $p > 0$, the simple restricted representations for $\mathfrak{g} = \text{Lie } G$ lift uniquely to G. This result became a cornerstone for the theory of reductive groups; it enabled Steinberg [St63] to prove that irreducible rational G-modules can be constructed via twisted tensor products of simple $\mathfrak{g}$-modules lifted to G. Since restricted representations for $\mathfrak{g}$ are equivalent to representations for the first Frobenius kernel G_1, one naturally extends this study to higher Frobenius kernels G_r. The modules for G_r coincide with those of the finite-dimensional cocommutative Hopf algebra $\text{Dist}(G_r)$.

Within this framework, one considers the projective covers (or equivalently, injective hulls) of simple G_r-modules. In 1973, the Humphreys-Verma Question ([HV-Quest], see [HV73], [Hum06, 10.4]) asked whether the G_r-structure on such projective modules necessarily lifts to a G-structure. Affirmative answers were provided by Ballard [B78] for $p \geq 3h-3$ and subsequently by Jantzen [Jan80], who lowered the bound to $p \geq 2h-2$.

Building on these developments, Donkin [Don93] proposed his celebrated Tilting Module Conjecture ([DTilt]) in 1990. The conjecture asserts that these G_r-structures arise from tilting modules for G. While [DTilt] was long believed to hold for all p, and was verified for $p \geq 2h-2$, the authors [BNPS20] discovered the first counterexample in 2019. Subsequent counterexamples have been identified in root systems of types B_n ($n \geq 3$), C_n ($n \geq 3$), D_n ($n \geq 4$), E_6, E_7, E_8, F_4, and G_2 [BNPS24a]. Interestingly, the conjecture holds for all primes in types A_n ($n = 1, 2, 3$) and B_2, and no counterexamples have yet been found for type A_n in general. These counterexamples were constructed by analyzing the structure of $\mathrm{Ext}^1_{G_1}(L(\lambda), L(\mu))^{(-1)}$ for p-restricted weights λ and μ.

The following statements are the new state of the art Donkin Tilting Module Conjectures:

Revised Conjecture A: Let G be a simple simply connected algebraic group scheme with k an algebraically closed field of characteristic $p > 0$. Then Donkin's Tilting Module Conjecture holds for all p if and only if the root system Φ is of type A_n or B_2.

Revised Conjecture B: Let G be a simple simply connected algebraic group scheme with k an algebraically closed field of characteristic $p > 0$. Then Donkin's Tilting Module Conjecture holds for all $p > h$.

In 1990, Donkin also proposed the (p, r)-Filtration Conjecture ([DFilt $\Leftrightarrow$]), building upon a question posed by Jantzen [Jan80]. This conjecture provides a necessary and sufficient condition for a rational G-module to admit a good (p, r)-filtration. Recent progress in verifying or disproving the Tilting Module Conjecture has relied heavily on the work of Kildetoft, Nakano, and Sobaje [KN15, So18], who established critical links between [DTilt] and good (p, r)-filtrations.

For the entire up to date collection of recent works on the subject, the reader is referred to [BNPS19, BNPS20, BNPS22, BNPS24a, BNPS24b, BNPS25] .

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Cohomology of Profinite Groups

Peter Symonds

Why? Here are some motivations for studying the cohomology of profinite groups, particularly pro-p groups.

Cohomology is a substitute for geometry when studying profinite groups. For example, a profinite group is not naturally the fundamental group of a manifold, but one can define Poincaré duality groups. There is no geometric description of the number of ends of a profinite group, but one can still use the formula that the number of ends of G is equal to $1 + \dim H^1(G, \mathbb{F}_p G)$.

There is a theory of group actions on pro-p trees, where one constructs a group action on a chain complex that mimics the chain complex of an action on a tree.

We do not have surface groups, but Demushkin groups, defined in terms of the cup product pairing $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$, have similar properties.

In homotopy theory, the Morava stabilizer groups are profinite groups and their cohomology is of fundamental importance.

In low-dimensional topology one studies the profinite completion of the fundamental group.

Profinite Groups. A profinite group is an inverse limit of finite groups and a pro-p group is an inverse limit of finite p-groups. They are given the inverse limit topology. For example, the p-adic integers $\hat{\mathbb{Z}}_p$ or $\mathrm{Gl}_n(\hat{\mathbb{Z}}_p)$. There are also such things as free profinite groups and pro-p groups.

For homological algebra we need a category of modules. Let $\mathcal{C}(G)$ be the category of profinite modules and $\mathcal{D}(G)$ the category of discrete torsion modules. In both cases the group G is required to act continuously. $\mathcal{C}(G)$ and $\mathcal{D}(G)$ are Pontryagin dual. $\mathcal{C}(G)$ has enough projectives and $\mathcal{D}(G)$ has enough injectives, so we can consider $\mathrm{Ext}^*(M,N)$ for $M \in \mathcal{C}(G)$ and $N \in \mathcal{D}(G)$. This is very well behaved and includes $H^*(G,\mathbb{F}_p) = \mathrm{Ext}^*(\mathbb{F}_p,\mathbb{F}_p)$. It does not include $H^*(G,\hat{\mathbb{Z}}_p)$; this can still be defined, by taking a projective resolution of $\mathbb{Z}_p$, but the result might be neither profinite nor torsion. This causes problems with such things as the LHS spectral sequence.

Similarly, one can define homology groups.

The familiar Mackey double coset formula does not hold unless one of the subgroups is open. Instead one has a continuously parametrized family of submodules that can be used to produce another module. This can be difficult to use in practice.

If the group G is of finite virtual cohomological dimension then one can produce complete resolutions in $\mathcal{C}(G)$ in the same way as usual and hence produce Gorenstein projective modules and a stable category. This is stratified by $H^*(G)$.

In such a stable category one can define endotrivial modules as those with an inverse under tensor product. It is even possible to calculate some small examples. However, the dual module is not defined in $\mathcal{C}(G)$, so there is no candidate for this inverse and the usual proofs do not carry over.

One can also define blocks and fusion systems for profinite groups.

Comparison with the cohomology as an abstract group. A profinite group is still a group, so we can compare the homology or cohomology defined above with that as an abstract group. We will only consider $\mathbb{F}_p$ coefficients and we will write H_*^{cts} and H_*^{abs} to distinguish the two cases. If G is poly-$\mathbb{Z}_p$ by finite then $H_*^{\mathrm{cts}}(G) \cong H_*^{\mathrm{abs}}(G)$.

Letting F_r denote the free pro-p group of rank r we have $H_n^{\mathrm{cts}}(F_r) = 0$. However, Bousfield showed that $H_2^{\mathrm{abs}}(F_r) \oplus H_3^{\mathrm{abs}}(F_r)$ is uncountable when $r \geq 2$. In 2017, Ivanov and Mikhailov showed that in fact $H_2^{\mathrm{abs}}(F_r)$ is uncountable.

In a similar vein, we can consider a pro-p group G and $H_n^{\mathrm{abs}}(G,\mathbb{F}_q)$ for $q \neq p$. Is this 0 for $n > 0$, as it is for H_n^{cts}?

What this shows is that we have a limited understanding of profinite groups as abstract groups.

Powerful p-groups. We fix a prime p and assume that $p \neq 2$ (the theory can be made to work for $p = 2$, but the statements are more complicated). A finitely-generated pro-p group is powerful if $[G,G] \leq G^p$, where G^p is the subgroup generated by pth powers. Let $G_1 = G^p$, $G_{i+1} = [G,G_i]G_i^p$. The p-power map gives a surjection $G_i/G_{i+1} \to G_{i+1}/G_{i+2}$, which is an isomorphism if G is torsion-free.

For example, if G is the first congruence subgroup in $\mathrm{Gl}_n(\hat{\mathbb{Z}}_p)$ then G is powerful and G_i is the $(i+1)$st congruence subgroup.

G is Ω-extendible if there is a pro-p group H such that $G \cong H/\Omega_1(H)$, where $\Omega_1(H)$ is the subgroup generated by elements of order p.

If G is powerful and Ω-extendible then $H^*(G,\mathbb{F}_p) \cong \Lambda(H^1(G,\mathbb{F}_p)) \otimes \mathbb{F}_p[y_1,\ldots,y_r]$, where the $\mathrm{res}^G_{\Omega_1(G)}(y_i)$ form a basis for $\beta H^1(\Omega_1(G))$. This can be shown for G/G_i by induction on i and using a simple spectral sequence argument. For profinite G one then takes the colimit. For powerful torsion-free G, the polynomial part vanishes and this is a result of Lazard, by very different methods.

The rank of a p-group is the maximum of the number of generators needed for a subgroup. It is then a theorem that if the rank of G is bounded by r then G has a powerful Ω-extendible subgroup of index bounded by $p^r(\lceil \log_2 r \rceil + 2)$.

In other words, deep down a group of bounded rank is nice.

Using the fact that the regularity of the cohomology ring of a finite group is bounded by 0, it can be deduced that among groups of rank bounded by r there are only finitely many cohomology rings up to isomorphism.

Example. Suppose that N is an open normal subgroup of G that is powerful and Ω-extendible.

If G acts trivially on N/N_1 then $H^*(G/N_i) \cong H^*(G/N_1)/\langle y_1,...,y_j\rangle \otimes \mathbb{F}_p[y'_1,...,y'_j]$, with $i=j$ if N is torsion free.

If G permutes a basis of N/N_1 and the extension associated to $N \trianglelefteq G$ is split then the spectral sequence associated to this extension collapses at the E_2 page and in fact $E_2 \cong H^*(G)$ as rings. This can be thought of as a generalization of Nakaoka's theorem.

If we apply this result to G/N_i instead of G/N we see that $H^*(G/N_i)$ is independent of i since $H^*(N/N_i)$ is so.

For a specific case, let G be the Sylow 3-subgroup of $\mathrm{Gl}_2(\hat{\mathbb{Z}}_3)$ and N be the first congruence subgroup. We find that the cohomology ring of the Sylow 3-subgroup of $\mathrm{Gl}_2(\mathbb{Z}/3^n)$ is independent of n for $n \geq 2$. The case $n=2$ was calculated by Green and King, so we know the cohomology for all n.

Endotrivial complexes and remixed twisted cohomology

Sam K. Miller

Let G be a finite group and k a field of prime characteristic p. Recall a kG-module is *permutation* if it admits a G-stable k-basis, and *p-permutation* if it is a direct summand of a permutation module. Set

$$\mathbf{K}(G) = \mathbf{K}(G;k) := \mathrm{K}_b(\mathsf{perm}(G;k)^\natural),$$

then $\mathbf{K}(G)$ is the compact part of a rigidly-compactly generated (i.e. 'big') tensor-triangulated category

$$\mathbf{D}(G) = \mathbf{D}(G;k) := \mathrm{DPerm}(G;k),$$

the *derived category of permutation modules.* This big tt-category is equivalent to a wide range of tt-categories arising in geometry and topology; see [BG23b, Fuh25] for details.

Balmer–Gallauer gave an extension of the classical *Brauer quotient* (see e.g. [Bro85]), termed the *modular fixed points functor*, for big tt-categories. For every p-subgroup $H \leq G$, one has a tt-functor

$$\Psi^H\colon \mathbf{D}(G) \to \mathbf{D}(N_G(H)/H)$$

satisfying $\Psi^H(k(X)) \cong k(X^H)$ for every G-set X [BG25, Proposition 2.7]. Moreover, we have a Verdier quotient $\Upsilon_G\colon \mathbf{K}(G) \twoheadrightarrow \mathrm{D}_b(kG)$, and this is the compact part of a corresponding finite localization $\Upsilon_G\colon \mathbf{D}(G) \twoheadrightarrow \mathrm{KInj}(kG)$ [BG23a, Theorem 1.5]. We set $\hat{\Psi}^H := \Upsilon_{N_G(H)/H} \circ \Psi^H$.

Theorem 1. [BG25, Theorems 6.12 and 7.16] *The family of functors*

$$\{\hat{\Psi}^H\colon \mathbf{D}(G) \to \mathrm{KInj}(k(N_G(H)/H))\}_{\mathrm{Sub}_p(G)/G}$$

indexed by conjugacy classes of p-subgroups of G, is jointly conservative, and the induced map on Balmer spectra

$$\bigsqcup_{H\in s_p(G)/G} \mathrm{Spc}(\mathrm{D}_b(k(N_G(H)/H))) \to \mathrm{Spc}(\mathbf{K}(G))$$

is a bijection of sets.

Endotrivial complexes. An *endotrivial complex* is a invertible object of $\mathbf{K}(G)$, i.e., an element $C \in \mathbf{K}(G)$ satisfying

$$C^* \otimes C \cong k[0] \text{ in } \mathbf{K}(G).$$

Since tt-functors preserve endotriviality, for all p-subgroups $H \leq G$, $\hat{\Psi}^H(C) \in \mathrm{D}_b(N_G(H)/H)$ is isomorphic to a shift of a k-dimension 1 kG-module concentrated in degree $h_C(H) \in \mathbb{Z}$. This datum defines a *superclass function* $h_C \in \mathrm{CF}(G,p)$. Furthermore, the assignment $C \mapsto h_C$ induces a group homomorphism $\mathrm{Pic}(\mathbf{K}(G)) \to \mathrm{CF}(G,p)$ [Mil24, Definition 3.5], the *h-mark homomorphism.*

Theorem 2. *Let G be a finite group and $S \leq G$ be a Sylow p-subgroup.*

(1) *Restriction induces a short exact sequence of abelian groups*

$$0 \to \mathrm{Hom}(G, k^\times) \to \mathrm{Pic}(\mathbf{K}(G)) \xrightarrow{\mathrm{Res}^G_S} \mathrm{Pic}(\mathbf{K}(S))^G \to 0,$$

where $\mathrm{Pic}(\mathbf{K}(S))^G$ denotes the subgroup of endotrivial kS-complexes with G-stable h-mark functions. [Mil25c, Theorem 12.6]

(2) *The h-mark homomorphism induces an isomorphism*

$$\mathrm{Pic}(\mathbf{K}(S)) \cong \mathrm{CF}_b(S),$$

where $\mathrm{CF}_b(S)$ denotes the group of Borel-Smith *superclass functions (see e.g.* [tD87, Page 210]*).* [Mil25a, Theorem 4.6]

Borel-Smith functions arise in multiple contexts; historically they arise from dimension functions of representation spheres [tD87] and they play a key role in the classification of endopermutation modules for p-groups by Bouc et al. [Bou06]. [Mil25b, Theorem 4.6] is proven by recovering a short exact sequence constructed by Bouc–Yalçin [BY07], but as a result, if G is a p-group, we conclude that endotrivials arise from reduced Bredon homology of representation spheres.

Proposition 3. [Mil25b, Theorem 2.12] *Let G be a p-group. There is a surjective group homomorphism* $\mathrm{RO}(G) \to \mathrm{Pic}(\mathbf{K}(G))$ *induced via taking the reduced Bredon homology of a representation sphere. In particular,* $\mathrm{Pic}(\mathbf{K}(G))$ *has a canonical basis $\mathcal{B}(G)$ induced from the irreducible real representations of G.*

In particular, if an endotrivial $C \in \mathrm{Pic}(\mathbf{K}(G))$ for a p-group G has decreasing h-marks, then there exists a representation sphere whose Bredon homology identifies with C. We call such endotrivials *effective*. A bit more technical work allows us to deduce the *global sections* of an effective endotrivial C, in the sense of [Gal25].

Theorem 4. [Mil25b, Proposition 3.10, Theorem 4.3] *Let G be a p-group, let $C \in \mathbf{K}(G)$ be an indecomposable effective endotrivial, and let $s \in \mathbb{Z}$. Then*

$$\mathrm{Hom}_{\mathbf{K}(G)}(k[0], C[s]) = \mathrm{Hom}_{\mathsf{mod}(kG)}(k, C_s).$$

In particular, for every subgroup $H \leq G$, there exists a forerunner *homomorphism $\iota_C^H \colon k[0] \to C[-h_C(H)]$ for which $\hat{\Psi}^H(\iota_C^H)$ is an isomorphism in* $\mathrm{D}_b(k(N_G(H)/H))$.

The topology of the spectrum. To deduce the topology of $\mathrm{Spc}(\mathbf{K}(G))$, Balmer–Gallauer prove a colimit theorem reducing to the case of G an elementary abelian p-group, then develop a a range of new techniques in that case. Given our understanding of endotrivials for p-groups, we aim to extend their techniques and results for arbitrary p-groups. First, we construct an open cover of $\mathrm{Spc}(\mathbf{K}(G))$.

Theorem 5. [Mil25b, Theorem 4.12] *Let G be a p-group. Set*

$$U(H) := \bigcap_{C \in \mathcal{B}(G)} \mathrm{supp}(\iota_C^H)^c.$$

Then the collection of quasi-compact open subsets of $\mathrm{Spc}(\mathbf{K}(G))$

$$\{U(H)\}_{H \in \mathrm{Sub}(G)/G}$$

is an open cover of $\mathrm{Spc}(\mathbf{K}(G))$*. Moreover, the closed point* $\mathfrak{m}(H) \in \mathrm{Spc}(\mathbf{K}(G))$ *satisfies $\mathfrak{m}(H) \in U(K)$ if and only if K and H are conjugate subgroups of G.*

Corollary 6. [Mil25b, Theorem 4.13] *Let G be a p-group. Then every endotrivial complex $C \in \mathrm{Pic}(\mathbf{K}(G))$ is a line bundle. Explicitly, we have an isomorphism $C \cong k[h_C(H)]$ in the localization $\mathbf{K}(G)|_{U(H)}$.*

This result does not hold for non-p-groups, as not all endotrivials have locally trivial homology. However, there is always a full-rank subgroup of $\mathrm{Pic}(\mathbf{K}(G))$ consisting of line bundles. This result motivates the next definition, building from [BG25, Definition 12.16].

Definition 7. [Mil25b, Definition 5.1] Let $\tilde{\mathcal{B}}(G) = \mathcal{B}(G) \setminus \{k[1]\}$. Let $\mathbb{N}^{\tilde{\mathcal{B}}(G)} = \{q\colon \tilde{\mathcal{B}}(G) \to \mathbb{N}\}$. The *(remixed) twisted cohomology ring* is the following $(\mathbb{Z} \times \mathbb{N}^{\tilde{\mathcal{B}}(G)})$-graded ring:

$$\mathrm{H}^{\bullet\bullet}(G) = \mathrm{H}^{\bullet\bullet}(G;k) := \bigoplus_{s\in\mathbb{Z}} \bigoplus_{q\in\mathbb{N}^{\tilde{\mathcal{B}}(G)}} \mathrm{Hom}_{\mathbf{K}(G)}\left(k[0], \bigotimes_{C\in\tilde{\mathcal{B}}(G)} C^{\otimes q(C)}[s]\right).$$

We have a canonical *comparison map*

$$\mathrm{comp}_G\colon \mathrm{Spc}(\mathbf{K}(G)) \to \mathrm{Spec}^h(\mathrm{H}^{\bullet\bullet}(G)).$$

Since every endotrivial is locally trivial, under localization to each $U(H)$, the twisted cohomology ring identifies with the usual graded cohomology ring of $\mathbf{K}(G)|_{U(H)}$, and the comparison map identifies with Balmer's standard comparison map [Bal10]. In a sense, the twisted cohomology ring is a "global" cohomology ring for $\mathbf{K}(G)$. If the twisted cohomology ring is Noetherian, this enables us to deduce the topology of the spectrum, first locally, then globally.

Theorem 8. [Mil25b, Theorem 6.2, Corollary 6.6] *Let G be a p-group. The comparison map is injective with dense image. Furthermore, if* $\mathrm{H}^{\bullet\bullet}(G)$ *is Noetherian, the following hold.*

(1) *For every subgroup $H \leq G$, the comparison map restricts to a homeomorphism*

$$\mathrm{comp}_{U(H)}\colon U(H) \cong \mathrm{Spec}^h(\mathrm{End}^{\bullet}_{\mathbf{K}(G)|_{U(H)}}(k[0])).$$

In particular, $\mathrm{Spc}(\mathbf{K}(G))$ *is a Dirac scheme.*

(2) *The comparison map is an open immersion.*

Noetherianity holds for all Dedekind p-groups (groups with every subgroup normal) and all p-groups of order at most p^3 [Mil25b, Theorem 7.4, Corollary 7.9].

Conjecture 9. *Let G be a finite p-group. Then* $\mathrm{H}^{\bullet\bullet}(G)$ *is Noetherian.*

Independent of Noetherianity, we obtain that the above theorem holds at the trivial subgroup of G. As a result, we obtain a nice alternative characterization of the usual cohomology ring of a p-group.

Theorem 10. [Mil25b, Theorem 4.19, Construction 5.7(a)] *The open subset $U(1)$ is the* cohomological open

$$\mathrm{Spc}(\mathrm{D}_b(kG)) \subseteq \mathrm{Spc}(\mathbf{K}(G)).$$

In particular, the following holds; let S_1 denote the multiplicative subset of $\mathrm{H}^{\bullet\bullet}(G)$ *generated by all forerunners of the form ι_C^1, with $C \in \mathcal{B}(G)$. Then*

$$\mathrm{H}^{\bullet}(G) \cong \left(\mathrm{H}^{\bullet\bullet}(G)[S_1^{-1}]\right)_{0\text{-twist}}$$

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Local periods in group cohomology and beyond

Martin Gallauer

Let k be a (commutative, unital) ring. Its *characteristic* is the non-negative integer satisfying

$$\langle \mathrm{char}(k) \rangle = \ker\left(\mathbb{Z} \xrightarrow{\mathrm{can}} k\right).$$

This invariant constitutes a rough yet useful organizing principle in algebraic geometry: 'mixed characteristic', 'characteristic p' or '(equi-)characteristic 0' describe subdomains with somewhat distinct tools and methods.

Now let K be a (small) stably symmetric monoidal ∞-category. Direct translation leads to considering the kernel of the unique map $\mathrm{Sp}^\omega \to K$ from finite spectra. We pass to invertible objects and define the *period* of K as the non-negative integer satisfying

$$\langle \mathrm{per}(K) \rangle = \ker\left(\mathbb{Z} = \mathrm{Pic}(\mathrm{Sp}^\omega) \xrightarrow{\mathrm{can}} \mathrm{Pic}(K)\right).$$

Hence we may view the period as a 'mutiplicative characteristic' of stably symmetric monoidal ∞-categories.

Remark 1. More explicitly, $\mathrm{per}(K)$ is the minimal positive integer d such that $\Sigma^d \mathbb{1} \simeq \mathbb{1}$, or 0 if no such d exists.

We may then ask about similarities with, and differences to, its additive cousin. For example, just as the points $\mathfrak{p} \in \mathrm{Spec}(k)$ come with local characteristics, namely the characteristic of the stalk $\mathcal{O}_{k,\mathfrak{p}}$, so do the points $\mathcal{P} \in \mathrm{Spec}(K)$ come with associated *local periods*

$$\mathrm{per}_{\mathcal{P}} \;:=\; \mathrm{per}_{\mathcal{P}}(K) \;:=\; \mathrm{per}(\mathcal{O}_{K,\mathcal{P}}) = \mathrm{per}(K/\mathcal{P}).$$

In the talk I discussed mainly two families of examples of local periods.

Example 2. Let G be a (pro-)finite group and k a field. The space underlying $\mathrm{Spec}(\mathrm{D}^b(kG))$ is computed by Benson–Carlson–Rickard [BCR97] as

$$|\mathrm{Spec}(\mathrm{D}^b(kG))| \xrightarrow{\sim} |\mathrm{Spec}(\mathrm{H}^*(G,k))|,$$

the graded spectrum of its cohomology. If $\mathcal{P} \leftrightarrow \mathfrak{p}$ under this identification then

$$\mathrm{per}_{\mathcal{P}} = \gcd\{\, |f| > 0 \mid f \in \mathrm{H}^*(G,k), f(\mathfrak{p}) \neq 0 \,\}.$$

In other words, to compute these local periods one needs to have rather precise information about the cohomology algebra of the group (modulo nilpotents).

Remark 3 (Group cohomology)**.** The result of [BCR97] just mentioned can be understood as consisting of two steps:

(1) The object Σk is an *ample line bundle* on $\mathrm{Spec}(\mathrm{D}^b(kG))$.
(2) The embedding to the Dirac affine scheme defined by its global sections

$$\oplus_n \Gamma(\mathrm{Spec}(\mathrm{D}^b(kG)), \Sigma^n k) = \mathrm{H}^*(G,k)$$

is surjective.

The second step is actually not necessary in order to compute the local periods. That is, to determine local periods in other contexts, one can try to find a convenient ample (family of) line bundle(s) and thereby reduce to studying the associated Dirac scheme.

Example 4. Let G be a (pro-)finite group and k a field of characteristic $p > 0$. Now consider the refinement of $\mathrm{D}^b(kG)$ studied in collaboration with Balmer in [BG25] (and elaborated on in his talk at this workshop):

$$\mathrm{DPerm}(G;k)^\omega = \mathrm{Perf}_{H\underline{k}}(\mathrm{Sp}^G),$$

the (compact objects in the) derived category of permutation modules. The points of its spectrum are pulled back through modular fixed points from $\mathrm{D}^b(k(G/\!\!/H))$ for the Weyl group $G/\!\!/H$ of (closed pro-)p-subgroups $H \le G$. We may therefore label them as $\mathcal{P}(H,\mathfrak{p})$. We may then ask about the relation:

$$\mathrm{per}_{\mathcal{P}(H,\mathfrak{p})} \overset{?}{\leftrightsquigarrow} \mathrm{per}_{\mathfrak{p}} \tag{5}$$

(The right-hand side divides the left-hand side for trivial reasons.)

The following is relatively straight-forward.

Proposition 6. *If $H \trianglelefteq G$ is normal then the two sides in* (5) *are equal.*

We don't know whether such an equality should be expected in general. On the other hand, we *do* believe that the following weaker relation should. Call a point $\mathcal{P} \in K$ *periodic* if $\mathrm{per}_{\mathcal{P}} > 0$.

Conjecture 7. *Let G be a (pro-)finite group, $H \leq G$ a (pro-)p-subgroup. Then $\mathcal{P}(H, \mathfrak{p})$ is periodic iff $\mathfrak{p}$ is.*

In other words, the periodic locus *of* $\mathrm{Spec}(\mathrm{DPerm}(G; k)^\omega)$ *is precisely the complement of the closed points.*

Remark 8 (Twisted group cohomology)**.** The natural proof strategy would follow the one of Example 2 and Remark 3. Unfortunately, the line bundle $\Sigma\mathbb{1}$ is very much not ample (except if $p \nmid \#G$) and it seems difficult to prove ampleness of *any* family. In [BG25] we succeeded for elementary abelian p-groups, and we called the associated global sections the *(permutation) twisted cohomology.* Recently, their construction was generalized to p-groups by Miller [Mil25] (see his talk at this workshop). While he doesn't establish ampleness, he does show that the comparison map to the Dirac scheme is injective and he gives good control on the behaviour of sections "p-locally" (that is, after applying modular fixed points). With this it is possible to mimick the natural proof strategy and establish:

Theorem 9 ([Gal25])**.** *Conjecture 7 holds when G is a (pro-)p-group.*

I refer to [Gal25] for applications in motivic tt-geometry. These are what prompted the study of periods in the first place.

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The Segal conjecture and H_∞ ring structures

Achim Krause

1. Spectra and group cohomology

In stable homotopy theory, spectra with G-action, that is objects of $\mathrm{Fun}(BG, \mathrm{Sp})$, can be viewed as a generalization of complexes of $\mathbb{Z}G$-modules. For such an $X \in \mathrm{Fun}(BG, \mathrm{Sp})$, its homotopy fixed points

$$X^{hG} := \lim_{BG} X$$

can be thought of as a version of "group cohomology with coefficients in X". In fact, if $X = HA$ is an Eilenberg–MacLane spectrum, $\pi_{-n}(HA^{hG}) = H^n(G; A)$.

Analogously, one has $X_{hG} := \mathrm{colim}_{BG} X$ which relates to group homology in a similar way. The norm map $H_0(G; M) \to H^0(G; M)$ used to define Tate cohomology refines to a norm map $X_{hG} \to X^{hG}$ whose cofiber X^{tG} is the Tate construction. Again, if $X = HA$, the homotopy groups of the Tate construction specialize to Tate cohomology. We can think of this norm map as arising from the transfer map from cohomology of the trivial subgroup, and X^{tG} as obtained from X^{hG} by killing these transfers. The $(-)_{hG}$ arises from additional Weyl group symmetry of the transfer.

More generally, there is a "family Tate construction" $X^{t_{\mathcal{F}}G}$ for each family of subgroups $\mathcal{F}$, obtained from X^{hG} by killing transfers from subgroups $H \in \mathcal{F}$. For $\mathcal{F} = \{e\}$, this specializes to the usual Tate construction, while for $\mathcal{F}$ the family of all proper subgroups of G, it is the *proper Tate construction*

$$X^{\varphi G} := X^{t_{\mathcal{F}_{\mathrm{proper}}} G}.$$

With Eilenberg–MacLane coefficients, this "proper Tate construction" always yields a localisation of group cohomology:

Proposition 1.

$$\pi_*(HA^{\varphi G}) = \begin{cases} A \textit{ in degree } 0 \textit{ if } G \textit{ is trivial} \\ 0 \textit{ if } G \textit{ not elementary abelian} \\ H^*(G; A)[e_i^{-1}] \textit{ if } G \textit{ is elementary abelian} \end{cases}$$

where the e_i are a family of elements which come from the canonical generator of $H^2(C_p; \mathbb{Z})$ under all maps $G \to C_p$.

For families $\mathcal{F}$ and $\mathcal{F}'$ where $\mathcal{F}'$ differs from $\mathcal{F}$ by a single conjugacy class of subgroups H, the proper Tate construction measures the difference between family Tate constructions, in the form of a cofiber sequence

$$X^{\varphi H}_{hW_G(H)} \to X^{t_{\mathcal{F}}G} \to X^{t_{\mathcal{F}'}G}.$$

Passing from the empty family to the full family one conjugacy class at a time, this yields a filtration (the "isotropy separation filtration") on X^{hG} whose associated graded is $\bigoplus_H X^{\varphi H}_{hW_G(H)}$. So the proper Tate constructions control the entire homotopy fixed points. (In the case of Eilenberg-MacLane spectra, this is related to a long history of results about how elementary abelian subgroups control group cohomology, for example Quillen's F-isomorphism.)

2. Sphere coefficients and the Segal conjecture

With $\mathbb{S}$ instead of Eilenberg-MacLane spectra, the situation is dramatically different. One has the following deep result of stable homotopy theory. Despite the name, it was fully proven by Carlsson in 1984, building on previous work by many others.

Theorem 2 ("Segal conjecture", [1])**.** *We have*

$$\mathbb{S}^{\varphi G} \simeq \begin{cases} \mathbb{S} \text{ if } G \text{ is trivial} \\ 0 \text{ if } G \text{ is not a } p\text{-group} \\ \mathbb{S}_p^\wedge \text{ if } G \text{ is a nontrivial } p\text{-group} \end{cases}$$

Remark 3. The usual way to express the Segal conjecture involves genuine-equivariant homotopy theory. This is simply a way to organize fixed points and all transfers and restrictions between them into Mackey functor-like objects, but is equivalent to the form given here.

Instead of periodic, the proper Tate constructions here all end up connective. Also, $(-)^{\varphi G}$ is nontrivial not only for elementary abelian p-groups, but all p-groups.

The isotropy separation filtration on $\mathbb{S}^{hG}$ has associated graded given by $(\mathbb{S}^{\varphi H})_{hW_G(H)}$, i.e. a copy of $\mathbb{S}_{hG}$, and $(\mathbb{S}_p^\wedge)_{hW_G(P)}$ where P ranges over all the p-Sylow subgroups. In fact, in this case the filtration splits, and $\mathbb{S}^{hG} \simeq \mathbb{S}_{hG} \oplus \bigoplus_P (\mathbb{S}_p^\wedge)_{hW_G(P)}$.

Corollary 4. *If T is a finite G-set and G a finite p-group,*

$$(\mathbb{S}T)^{\varphi G} \simeq \mathbb{S}_p^\wedge T^G$$

This means that one has a similar diagram like for modular fixed points

$$\begin{array}{ccc} \mathrm{FinSet}^G & \xrightarrow{(-)^H} & \mathrm{FinSet}^{W_G(H)} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{\mathbb{S}G} & \xrightarrow{(-)^{\varphi H}} & \mathrm{Mod}_{\mathbb{S}_p^\wedge W_G(H)}, \end{array}$$

only that one recovers more than just $\mathbb{F}_p T^H$, since $\pi_0(\mathbb{S}_p^\wedge T^H) = \mathbb{Z}_p T^H$.

3. Compatibility with infinite products and sums

The fact that $\mathbb{S}^{hG}$, and more generally $(\mathbb{S}T)^{hG}$, is connective and has p-completely finitely generated homotopy groups, means that in the complicated homotopy fixed point spectral sequence, the E^∞ page has a certain finiteness property. More precisely, if we reduce mod p, $(\mathbb{S}/p)^{hG}$ literally is connective with finite homotopy groups. The "abutment" filtration on these therefore has to be finite, and the whole E_∞ page is finite along each codiagonal.

In this case, the homotopy fixed point spectral sequence of $(\bigoplus_I \mathbb{S})^{hG}$ must have the same shape, and one gets that

$$\bigoplus_I \mathbb{S}^{hG} \to (\bigoplus_I \mathbb{S})^{hG}$$

becomes an equivalence after completion at any prime p, i.e. in this specific case the *limit* that gives $(-)^{hG}$ actually commutes with infinite direct sums. This is expected for cohomology with coefficients in ordinary modules (due to $H_*(BG;\mathbb{Z})$ being finitely generated in each degree), but certainly not with coefficients in a general spectrum!

More generally, this works for sums of the form $\bigoplus_i \mathbb{S}T_i$ for T_i transitive G-sets, since there are only finitely many different isomorphism classes and so one still has uniform behavior in the homotopy fixed point spectral sequences.

A similar but easier argument also allows one to commute homotopy orbits with certain infinite products. Together, these imply the following:

Theorem 5. *For X, Y bounded below spectra with bounded homology, and G a finite p-group with map to $\Sigma_n \times \Sigma_m$, we have an equivalence*

$$\mathrm{map}(X^{\otimes n}, Y^{\otimes m})^{\varphi G} \simeq \mathrm{map}(X^{\otimes n/G}, Y^{\otimes m/G})^{\wedge}_p$$

Proof. Using a Goodwillie calculus argument, one can reduce to the case where $X = \bigoplus_I \mathbb{S}$ and $Y = \bigoplus_J \mathbb{S}$. In that case, we may write $X^{\otimes n} = \bigoplus_{I^{\times n}} \mathbb{S}$, $Y^{\otimes m} = \bigoplus_{J^{\times m}} \mathbb{S}$, and then

$$\mathrm{map}(X^{\otimes n}, Y^{\otimes m}) \simeq \prod_\alpha \bigoplus_\beta \mathbb{S}T_{\alpha\beta}$$

where the $T_{\alpha\beta}$ arise from decomposing $I^{\times n} \times J^{\times m}$ into G-orbits. Then, $(-)^{\varphi G}$ commutes with both the product and the coproduct by expressing it in terms of homotopy fixed points and homotopy orbits, and kills all nontrivial orbits since these are induced from subgroups. We are left with the p-completion of

$$\prod_{I^{\times n/G}} \bigoplus_{J^{\times m/G}} \mathbb{S} \simeq \mathrm{map}(X^{\otimes n/G}, Y^{\otimes m/G}).$$

□

Remark 6. One can think of this statement as a bivariate version of Nikolaus-Scholze's Tate diagonal [2] for C_p. Their statement that $Y \to (Y^{\otimes p})^{tC_p}$ is an equivalence after p-completion for every bounded below Y is however not a special case, since there Y does not have bounded homology. We expect a similar strengthening of the above statement as long as the G-action on m is free, note that $Y \to Y^{tC_p}$ is definitely not an equivalence after p-completion for general connective Y, like $H\mathbb{Z}$.

To summarize, even if we allow infinitely generated homology, $(-)^{\varphi G}$ for p-groups seems to extract some sort of diagonal out of equivariant maps $X^{\otimes n} \to Y^{\otimes m}$.

4. Commutative ring spectra and Frobenius lifts

The correct ∞-categorical notion of commutative algebras in Sp is that of "E_∞ ring spectra". The free E_∞ ring on X has the form $\bigoplus X^{\otimes n}_{h\Sigma_n}$, so part of the data of an E_∞ algebra structure on X is n-fold multiplication maps $X^{\otimes n}_{h\Sigma_n} \to X$, or equivalently Σ_n-equivariant maps $X^{\otimes n} \to X$. These need to satisfy composition relations in a complicated homotopy coherent way. The notion of H_∞ ring spectra is a sort of less coherent compromise: An H_∞ ring structure simply consists of homotopy classes of equivariant maps $X^{\otimes n}_{h\Sigma_n} \to X$, equivalently elements of

$$\pi_0 \operatorname{map}(X^{\otimes n}, X)^{h\Sigma_n},$$

which satisfy composition relations only up to homotopy.

The following observation goes back to Nikolaus-Scholze [2], and is an important ingredient in the theory of topological cyclic homology and prismatic cohomology:

Proposition 7. *Assume A is an E_∞ ring spectrum which is* flat *(over $\mathbb{S}$): $\pi_0(A)$ is a flat $\mathbb{Z}$-module, and $\pi_n(A) \cong \pi_0(A) \otimes_{\pi_0(\mathbb{S})} \pi_n(\mathbb{S})$. Then there is a canonical map*

$$\varphi_p : A \to (A^{\otimes p})^{tC_p} \to A^{tC_p} \simeq A^\wedge_p,$$

which on $\pi_0(A) \to \pi_0(A)^\wedge_p$ is a Frobenius lift*: Mod p, it agrees with $x \mapsto x^p : \pi_0(A) \to \pi_0(A)/p$.*

Here, the first map is Nikolaus-Scholze's Tate diagonal, the second is induced by the equivariant multiplication map which comes as part of the ring structure on A, and the third is a general fact about these flat spectra (which is also a special case of Theorem 5 above).

The Frobenius lift is responsible for various surprising rigidity phenomena in this world. For example, there is a flat "polynomial ring" $\mathbb{S}[x] := \bigoplus_{\mathbb{N}} \mathbb{S}$. It turns out this has much fewer endomorphisms than polynomial rings in algebra. Indeed, the Frobenius lift on its π_0 turns out to be $\varphi_p : \mathbb{Z}[x] \to \mathbb{Z}[x]^\wedge_p$, $x \mapsto x^p$, and since this structure is determined naturally by the commutative ring structure, every ring endomorphism has to preserve it. For example, there cannot be an endomorphism f which on π_0 takes $f(x) = x + 1$, since then

$$\varphi_p(f(x)) = x^p + 1 \neq (x+1)^p = f(\varphi_p(x)).$$

A more detailed version of this same argument shows that every commutative ring map $\mathbb{S}[x] \to \mathbb{S}[x]$ on π_0 must take x to 0 or x^k for some k.

So these flat ring spectra come intrinsically with a Frobenius lift on their π_0 (after p-completion). In some examples, it also seems that this Frobenius lift controls the ring structure rather rigidly (for example [3]). So a natural question is the following:

Question 8. How much of the E_∞ ring structure is determined by the Frobenius lifts $\varphi_p : \pi_0(A) \to \pi_0(A)^\wedge_p$?

We don't give a fully satisfactory answer to this, but give the following (already surprising) partial answer:

Theorem 9. *If A is a flat spectrum, H_∞ structures on A are fully determined by the following data:*

1. *A commutative ring structure on $\pi_0(A)$ (extending the given abelian group structure).*
2. *For each p, a Frobenius lift $\varphi_p : \pi_0(A) \to \pi_0(A)^\wedge_p$.*

One can think of this as possibly the first step in an obstruction-theoretic approach to the above question, but we have no idea what to expect further on. Specifically, it is not clear to us whether all of these H_∞ structures extend to E_∞ structures, and how uniquely they do so. Various flat rings with Frobenius lifts would maybe be unreasonable to expect to come from flat E_∞ rings, for example the free δ-ring

$$\mathbb{Z}[x, \delta x, \delta^2 x, \dots]_p^\wedge$$

with Frobenius lift $\varphi_p(\delta^n x) = (\delta^n x)^p + p\delta^{n+1}x$.

The proof of the theorem is based on exploiting Theorem 5. There is a fracture square argument to reduce to the case of p-complete A. From the isotropy separation filtration, one then gets that the maps

$$\pi_0 \operatorname{map}(A^{\otimes n}, A)^{h\Sigma_n} \to \prod_{P \subseteq \Sigma_n} \pi_0 \operatorname{map}(A^{\otimes n/P}, A) \cong \operatorname{Hom}(\pi_0(A)^{\otimes n/P}, \pi_0(A)),$$

obtained by restriction to P and passing to $(-)^{\varphi P}$, are injective. Here the latter identification comes from flatness. So an H_∞ structure is determined by operations $\pi_0(A)^{\otimes n/P} \to \pi_0(A)$ for all n and p-subgroups $P \subseteq \Sigma_n$, satisfying some composition relation. In analogy with Witt vectors, we think of these elements as the *ghosts* of an H_∞ structure.

Of course, not every collection of ghosts (with composition relation) is a priori in the image. We look at what the ghosts encode: For $n < p$, there are no nontrivial p-subgroups, and the ghosts for $P = e$ just amount to a commutative ring structure on $\pi_0(A)$. For $n = p$, we also have $P = C_p$ and the corresponding ghost is a homomorphism $\pi_0(A) \to \pi_0(A)$. In this case, one can actually give a pullback square which explains that in order to lift to $\pi_0 \operatorname{map}(A^{\otimes p}, A)^{h\Sigma_p}$, this ghost needs to exactly be compatible with the p-th power map $\pi_0(A) \to \pi_0(A)/p$, so it needs to be a Frobenius lift! The composition relations in $n = 2p$ imply that this Frobenius lift is also multiplicative. Using the fact that the p-Sylow subgroup of Σ_n is built from C_p's under products and wreath products, one can express all the elements of $\operatorname{Hom}(\pi_0(A)^{\otimes n/P}, \pi_0(A))$ arising from an H_∞ structure in terms of this Frobenius lift, and so the Frobenius lift together with the underlying ring structure on $\pi_0(A)$ completely determines the H_∞ structure.

Conversely, if we start with a ring structure and a Frobenius lift, this gives an element of $\pi_0 \operatorname{map}(A^{\otimes p}, A)^{h\Sigma_p}$. Restricting to C_p and iterating this, we can construct elements in $\pi_0 \operatorname{map}(A^{\otimes n}, A)^{hP}$ for P the p-Sylow subgroup of Σ_n. Their ghosts (for subgroups $Q \subseteq P$) can be expressed in terms of the Frobenius, and are conjugacy invariant. So, by the stable element formula, we actually get elements on $\pi_0 \operatorname{map}(A^{\otimes n}, A)^{h\Sigma_n}$. So a Frobenius lift also determines an H_∞ structure.

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Equivariant Poincaré duality and Smith-theoretic methods

Kaif Hilman
(joint work with Dominik Kirstein, Christian Kremer)

Classically, for a commutative ring R, we say that a topological space X satisfies *R-Poincaré duality* if there is a local coefficient system $\mathcal{L} \in \mathrm{Fun}(\Pi_1 X, \mathrm{Mod}_R)$ of R-modules and an integer $n \geq 0$ participating in an isomorphism

$$H^*(X; R) \cong H_{n-*}(X; \mathcal{L})$$

of graded abelian groups. This was formalised by Wall [8] in his notion of Poincaré complexes, and these objects constitute the basic input to surgery theory. On the other hand, the need and desire for equivariant versions thereof have arisen naturally from many geometric contexts, for example in studying group actions on manifolds. As such, equivariant Poincaré duality is an old subject that has received much attention in homotopy theory and geometric topology, and traces its history all the way back to the birth of genuine equivariant stable homotopy theory.

However, many new subtleties and structures appear in this setting. Roughly speaking, the classical version of equivariant (co)homology theory on a topological space X with G-action is that of its Borel (co)homology $H_*(X_{hG}; \mathbb{Z})$ and $H^*(X_{hG}; \mathbb{Z})$ respectively. While often computable, one can never hope to articulate with this invariant that smooth closed G-manifolds satisfy some sort of an equivariant Poincaré duality: for example, already for the point with the trivial G-action, $H_*(*_{hG}; \mathbb{Z}) \cong H_*(BG; \mathbb{Z})$ is not bounded above and $H^{-*}(*_{hG}; \mathbb{Z}) \cong H^{-*}(BG; \mathbb{Z})$ is not bounded below when G is finite, precluding any reasonable form of Poincaré duality. What is more, from the perspective of Atiyah duality on ordinary Poincaré duality, the dimension shift in Eq (1) in the case when X is a closed smooth manifold is implemented by the stable normal bundle of X. Thus, it is natural to expect that the equivariant tangent bundle of an equivariant smooth manifold should feature somewhere in a putative duality. The complication here is that there is not just one dimension shift that needs to happen since the fixed points of a smooth G-manifold are submanifolds of smaller dimensions, and this should be reflected by the varying dimensions of the tangent representations. This naturally led to a notion of $RO(G)$-graded homology theories, and Costenoble and Waner [3] have set up a theory of equivariant Poincaré duality on this basis.

In joint work with Dominik Kirstein and Christian Kremer [6], we establish an independent theory of equivariant Poincaré duality spaces for arbitrary compact Lie groups G using the formalism of parametrised ∞-categories. Examples thereof include smooth closed G-manifolds, but also more exotic objects coming purely from homotopy theory such as tom Dieck's generalised homotopy representations. A key feature here is that the ∞-categorical treatment affords excellent interactions between an equivariant space and its fixed points, which we use as a fundamental manoeuvre in all our applications. More precisely, we prove a Smith-theoretic principle that if X is a G-Poincaré space in our sense, then for all subgroups $H \leq G$, the space X^H is a Poincaré space in the classical sense of Wall. The converse to this result is false, as we shall soon explain.

To illustrate that there is some geometric mileage to be had from the abstract theory, we prove a strong rigidity result which says that when G is a solvable finite group, a G-Poincaré space whose underlying space is contractible is already equivariantly contractible. This result is sharp in that there are counterexamples when G is not solvable, for instance, when $G = A_5$. Using a famous example of Lowell Jones in his work on the converse to Smith theory [7], we also found a compact C_p-space whose underlying space is contractible and whose C_p-fixed points space is a noncontractible Poincaré space. By the rigidity result, this space cannot be C_p-Poincaré. In particular, this shows that there are G-spaces all of whose fixed points spaces are Poincaré in the classical sense, but without itself being G-Poincaré. Nonetheless, by identifying the "missing piece" governing the interaction between the fixed points space and the underlying space, we give in a companion work [5] a complete characterisation for when a compact C_p-space is itself C_p-Poincaré given that its underlying and fixed points spaces are Poincaré.

Apart from that, we also use this theory to generalise and give a new proof of Atiyah–Bott's [1] solution to a conjecture of Conner and Floyd. In slightly more detail, Atiyah and Bott employed a Lefschetz fixed points argument powered by Atiyah–Singer's index theory to show that the fixed points of an odd cyclic p-group action on a closed, connected, orientable, and positive-dimensional smooth manifold cannot just be a single point. Subsequently, Conner and Floyd [2] supplied their own proof using bordism spectra. By using our theory of equivariant Poincaré duality, we extend these results to arbitrary G-Poincaré spaces whose underlying spaces are connected, oriented, and positive-dimensional, for odd cyclic p-groups G. This is achieved, once the abstract theory of Poincaré duality is set up, without recourse to any deep geometric input and instead only uses elementary facts from group cohomology. Philosophically, this also suggests that the result is in fact a purely global, homotopical phenomenon and not a local one of geometry.

As yet another application, we also apply the theory above to the Nielsen realisation problem. More specifically, for an extension $1 \to \pi \to \Gamma \to G \to 1$ of discrete groups where G is finite, Davis and Lück [4] provided a list of conditions ensuring that there exists a cocompact topological Γ-manifold model for the universal space $E_\Gamma\mathrm{Fin}$. Given such a manifold model, we may then pass to the quotient $\pi\backslash E_\Gamma\mathrm{Fin}$ to obtain a G-manifold model for the aspherical space $B\pi$, whence its connection to the aforementioned realisation problem. Among the Davis–Lück conditions is a homological one which is necessary, but rather technical and mysterious to check in general. By way of introducing the notion of *genuine virtual Poincaré duality groups* built upon equivariant Poincaré duality and exploiting the associated theory of fundamental classes, we show in [5] that when $G = C_p$ where p is odd, the homological condition is actually automatic under some assumptions on the group extension.

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Free Actions on Products of Real Projective Spaces

Ergün Yalçin

I gave a talk on recent work in which the following theorem was proved:

Theorem 1 (Yalçın [8]). *Let $G = (\mathbb{Z}/2)^r$ and X be a finite-dimensional, free G-CW-complex homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$ such that the induced G-action on the mod-2 cohomology of X is trivial. Then*

$$r \leq \mu(n_1) + \cdots + \mu(n_k),$$

where for each positive integer n,

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ 2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

This settles the homotopy-theoretic version of a conjecture by Cusick [4] from 1983. Cusick conjectured that the conclusion of the above theorem should still hold under the assumption that X is a finite free G-CW-complex such that

$$H^*(X;\mathbb{F}_2) \cong H^*(\prod_{i=1}^k \mathbb{R}P^{n_i};\mathbb{F}_2).$$

There are examples of free actions of elementary abelian 2-groups on products of real projective spaces which show that the upper bound in Theorem 1 is sharp. To see this, note that the quaternion group Q_8 of order 8 acts freely on S^3 via its identification as the group of unit quaternions. By taking joins of this action, we obtain a Q_8-action on S^{4m+3} for all $m \geq 0$. This induces a free $\mathbb{Z}/2\times\mathbb{Z}/2$-action on $\mathbb{R}P^{4m+3}$ for all $m \geq 0$. There is also a free $\mathbb{Z}/2$-action on $\mathbb{R}P^{2m+1}$ for any $m \geq 0$, induced by the free $\mathbb{Z}/4$-action on S^{2m+1} defined by $(z_0, \ldots, z_m) \to (iz_0, \ldots, iz_m)$. Taking products of these actions, we obtain a free action of $G = (\mathbb{Z}/2)^{k+2l}$ on $X = \prod_{i=1}^{k} \mathbb{R}P^{2m_i+1} \times \prod_{i=1}^{l} \mathbb{R}P^{4m_i+3}$.

Studying restrictions on groups that can act freely on a given topological space has a long history. The most well-known problem in this area is the rank conjecture:

Conjecture 2. *If $G = (\mathbb{Z}/p)^r$ acts freely on a finite-dimensional CW-complex X homotopy equivalent to a product of k spheres, then $r \leq k$.*

The rank conjecture is still open in its full generality. Special cases of this conjecture were proved by Carlsson [2], [3], Adem and Browder [1], Yalçın [7], Hanke [5], Okutan and Yalçın [6], and others. For free actions of $G = (\mathbb{Z}/p)^r$ on $X \simeq (S^n)^k$, the conjecture is completely settled except for the cases $p = 2$ and $n = 3, 7$.

We prove Theorem 1 using the methods introduced in [4] and [7]. The new input comes from a closer look at the interaction between the Serre spectral sequences with integer and mod-2 coefficients. Recall that for a G-CW-complex X, there is an associated Borel fibration

$$X \to EG \times_G X \to BG.$$

For a commutative ring R, the Serre spectral sequence for the Borel fibration with coefficients in R is of the form

$$E_2^{p,q} = H^p(BG; H^q(X; R)) \Rightarrow H^{p+q}(EG \times_G X; R).$$

The Serre spectral sequence has a multiplicative structure which makes it easier to calculate differentials at higher dimensions. Moreover, if $R = \mathbb{F}_2$ and the induced G-action on $H^*(X; \mathbb{F}_2)$ is trivial, then there is an isomorphism of bigraded $\mathbb{F}_2$-algebras

$$E_2^{*,*} \cong H^*(BG; \mathbb{F}_2) \otimes H^*(X; \mathbb{F}_2).$$

If X is a free G-CW-complex, then $EG \times_G X$ is homotopy equivalent to the orbit space X/G, and if X is a finite-dimensional G-CW-complex, X/G is a finite-dimensional CW-complex. It follows that $H^*(EG \times_G X; R)$ vanishes above degree $\dim X$, and hence

$$E_\infty^{p,q} = 0 \quad \text{for } p + q > \dim X.$$

This gives restrictions on the differentials of the Serre spectral sequence for a free action on a finite-dimensional complex.

Let X be a finite-dimensional free G-CW-complex homotopy equivalent to $\prod_{i=1}^{k} \mathbb{R}P^{n_i}$ and $G \cong (\mathbb{Z}/2)^r$. Then we have

$$H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[t_1, \ldots, t_k]/(t_1^{n_1+1}, \ldots, t_k^{n_k+1})$$

where $|t_i| = 1$ for all i, and

$$H^*(G; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_r]$$

where $|x_i| = 1$ for all i. Let $d_2 : E_2^{0,1} \to E_2^{2,0}$ be the differential on the E_2-page, and $\alpha_1, \ldots, \alpha_k \in H^2(G; \mathbb{F}_2)$ be the cohomology classes such that for each i,

$d_2(1 \otimes t_i) = \alpha_i \otimes 1$. These cohomology classes are called the **k-invariants** of the G-action on X.

Since the restriction of the free G-action on X to any cyclic subgroup C of G is also free, we see that the restrictions of the k-invariants to C cannot be all equal to zero. It follows that the k-invariants $\alpha_1, \dots, \alpha_k$, viewed as quadratic polynomials in $\mathbb{F}_2[x_1, \dots, x_r]$, have no common zero in $(\mathbb{F}_2)^r$. This implies that $r \leq 2k$ by a standard result on zeros of polynomials with $\mathbb{F}_2$-coefficients.

To obtain a sharper upper bound for r, we also need to consider the Steenrod operations on the k-invariants. In [4], Cusick proves that for every i such that n_i is even, $\alpha_i = 0$, and for every i such that $n_i \equiv 1 \pmod 4$, $Sq^1(\alpha_i)$ lies in the ideal generated by $\alpha_1, \dots, \alpha_k$ in $H^*(G; \mathbb{F}_2)$. Using this, Cusick concludes that if $n_i \not\equiv 3 \pmod 4$ for all i, then the inequality in Theorem 1 holds. Note that in this case the k-invariants form an ideal closed under Steenrod operations. For such an ideal whose generators have no common zero in $(\mathbb{F}_2)^r$, we can apply a theorem of Serre to conclude that $r \leq$ the number of nonzero α_i (see [3]).

In our work, we extend Cusick's observations on the k-invariants and prove that for each i, the k-invariant α_i generates an ideal closed under Steenrod operations. We prove the following:

Proposition 3. *Let G and X be as in Theorem 1, and $\alpha_1, \dots, \alpha_k$ be the k-invariants of the action. Assume that $n_i \geq 2$ for all i. Then for every i such that $n_i \equiv 1 \pmod 4$, $\alpha_i = l_i(l_i + l_i')$ for some $l_i, l_i' \in H^1(G; \mathbb{F}_2)$. Moreover, if the G-action on the integral cohomology is trivial, then for every i such that $n_i \equiv 1 \pmod 4$, $\alpha_i = l_i^2$ for some $l_i \in H^1(G; \mathbb{F}_2)$.*

The case where some of the n_i is equal to 1 is handled separately using a covering space argument. Proposition 3, together with the fact that the k-invariants $\alpha_1, \dots, \alpha_k$ have no common zero, completes the proof of Theorem 1.

We prove Proposition 3 in two steps. First we consider the case where the induced action on integral cohomology is trivial and prove the last statement of Proposition 3. For this proof we compare the Serre spectral sequences in integral and mod-2 coefficients.

A similar but more complicated argument gives a proof for the general case where the induced action on the integral cohomology may be nontrivial. We also construct some examples of free actions with nontrivial induced action on integral cohomology to illustrate how the spectral sequences behave in this case.

The talk focused on the main ideas of the proofs of Theorem 1 and Proposition 3 with particular emphasis on the relationship between the differentials in the Serre spectral sequences with integer and mod-2 coefficients.

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Bounds for Rouquier dimension using regular sequences

Janina C. Letz

Generation plays an important role in the study of of triangulated categories. A *generator* of a triangulated category $\mathcal{T}$ is an object G such that $\mathcal{T} = \operatorname{thick}(G)$ is the smallest thick subcategory containing G. This means every object in $\mathcal{T}$ can be obtained from G by taking finitely many extensions, suspensions, desuspensions and summands. We can filter $\operatorname{thick}(\mathcal{T})$ by successively adding extensions:

- $\operatorname{thick}^1(G)$ is the smallest strictly full subcategory containing G that is closed under finite coproducts, rectracts, suspensions and desuspensions;
- $\operatorname{thick}^n(G)$ as the smallest strictly full subcategory of objects M for which there exists an exact triangle

$$L \to M \oplus M' \to N \to \Sigma L \quad \text{with } L \in \operatorname{thick}^1(G) \text{ and } M \in \operatorname{thick}^{n-1}(G)\,.$$

This construction is due to [BvdB03, Section 2.2]. An object $G \in \mathcal{T}$ is a *strong generator* if $\mathcal{T} = \operatorname{thick}^n(G)$ for some $n \geq 1$. This means every object in $\mathcal{T}$ can be obtained from G in at most n steps. The generation time n depends on the generator. Their infimum

$$\operatorname{Rdim}(\mathcal{T}) := \inf\{n | \mathcal{T} = \operatorname{thick}^{n+1}(G) \text{ for some } G \in \mathcal{T}\}$$

is the *Rouquier dimension*; it was first introduced and studied in [Rou08]. In general, the Rouquier dimension is hard to compute.

Theorem ([Let25, Theorem B])**.** *Let R be a graded-commutative ring and $\mathcal{T}$ an Ext-noetherian R-linear triangulated category. Let $M \in \mathcal{T}$ and $\mathfrak{a} \subseteq R$ a homogeneous ideal with $\mathfrak{a}\operatorname{End}^*_{\mathcal{T}}(M) \neq \operatorname{End}^*_{\mathcal{T}}(M)$. Then*

$$\operatorname{depth}_R(\mathfrak{a}, \operatorname{End}^*_{\mathcal{T}}(M)) \leq \operatorname{Rdim}(\mathcal{T})\,.$$

The $\mathfrak{a}$-depth of a graded module $\mathcal{M}$ is the maximal length of a $\mathcal{M}$-sequence in $\mathfrak{a}$. By [KLS25, Proposition 7.18] a sequence $x_1, \dots, x_t \in R$ is $\operatorname{End}^*_{\mathcal{T}}(M)$-regular if and only if

(1) $\Sigma^{-1} M /\!\!/ (x_1, \dots, x_i) \to \Sigma^{-i+1} M /\!\!/ (x_1, \dots, x_{i-1})$ is M-ghost for all $1 \leq i \leq t$; and

(2) the composition

$$\Sigma^{-t} M/\!\!/(x_1, \dots, x_t) \to \Sigma^{-(t-1)} M/\!\!/(x_1, \dots, x_{t-1}) \to \cdots \to \Sigma^{-1} M/\!\!/x_1 \to M$$

is non-zero.

Here $N/\!\!/x$ denotes the Koszul object of $N \in \mathcal{T}$ with respect to $x \in R$. That is, it is defined by the exact triangle

$$N \xrightarrow{x(N)} \Sigma^{|x|} N \to N/\!\!/x \to \Sigma N\,.$$

The main idea of the proof is the construction of an G-ghost map for any $N \in \mathcal{T}$ and any $x \in R$. Explicitly, when $\operatorname{Hom}^*_{\mathcal{T}}(G, N)$ is a noetherian graded R-module, then there exists $n \gg 0$, depending on N and x, such that the composition

$$\Sigma^{-1} N/\!\!/x^{n+1} \to N \xrightarrow{x^n} \Sigma^{n|x|} N$$

is G-ghost. This allows one to construct a sequence of G-ghost morphisms of the same length as the regular sequence $x_1, \dots, x_t$. The regularity of this sequence ensures that the composition is non-zero.

Applying the Theorem to a commutative regular noetherian ring, we can determine the Rouquier dimension for the bounded derived category of a commutative regular noetherian ring.

Theorem ([Let25, Theorem A])**.** *Let A be a commutative regular noetherian ring. Then*

$$\dim(A) = \operatorname{Rdim}(\mathrm{D}_b(\operatorname{mod}(A)))\,,$$

where $\dim(A)$ *is the Krull dimension of A.*

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Finite permutation resolutions for infinite groups

Luca Pol
(joint work with J.O. Gómez)

This talk is based on current work in progress [GP26] with J.O. Gómez.

Recall that for a discrete group G, there is a G-CW-complex $\underline{E}G$ whose homotopy type is determined by the requirement that $\underline{E}G^H \simeq *$ if $H \subseteq G$ is finite, and $\underline{E}G^H = \emptyset$ otherwise. The space $\underline{E}G$ is often referred to as the *classifying space of proper actions*. In this talk, we will only work with groups G that admit a finite dimensional model for $\underline{E}G$. This is a large class of groups containing any group of virtual finite cohomological dimension. We will also fix a commutative Noetherian ring k of finite global dimension. In this setting, one may ask:

Question 1. *To what extent the representation theory of G is controlled by the finite subgroups?*

In order to answer this question we consider:

$$\mathrm{K}(\mathrm{Proj}(kG)) \quad \text{and} \quad \mathrm{D}(kG)$$

the homotopy category of projective kG-modules, and the derived category of kG-modules. Following [MS19] one can then define the stable module category

$$\mathrm{StMod}(kG) := \mathrm{Ac}(\mathrm{K}(\mathrm{Proj}(kG)))$$

as the subcategory of acyclic complexes, and also consider a variant of the derived category of permutation modules [BG23]:

$$\mathrm{DPerm}^{\mathrm{fin}}(G;k) := \mathrm{Loc}_{\mathrm{K}(\mathrm{Mod}(kG))}(k[G/H] \mid H \subseteq G \text{ finite}).$$

All these categories are functorial in the group variable, and hence define functors

$$\mathrm{Orb}_{G,\mathrm{fin}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{\mathrm{st}})$$

from the G-orbit category with finite isotropy to the ∞-category of presentably symmetric monoidal stable ∞-categories. We then give an answer to our question in the following form:

Theorem 2 ([GP26])**.** *Let G and k as above and let $\mathcal{C}$ denote any of the above functors. Then there are symmetric monoidal equivalences:*

$$\mathcal{C}(G) \simeq \lim_{G/H \in \mathrm{Orb}_{G,\mathrm{fin}}^{\mathrm{op}}} \mathcal{C}(H)$$

As an application of this descent result, we prove a version of the finite permutation resolution of Balmer-Gallauer [BG23]. Recall that a kG-module is of type FP_∞ if it admits a resolution of finitely generated projective kG-modules.

Theorem 3 ([GP26])**.** *Let G and k be as above. Any kG-module of type FP_∞ is a retract of a module which admits a finite resolution by (retracts of) permutation modules with finite isotropy.*

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Commuting varieties and S_n-cofixed quotients of polynomial algebras

Simon Gritschacher

Let k be a commutative ring and $G \subseteq \mathrm{GL}_n(k)$ a subgroup. Then G acts on the polynomial ring $R := k[x_1, \dots, x_n]$ by linear substitution. One may consider the k-algebra of invariants R^G and the k-module of coinvariants R_G (also called the *G-cofixed quotient of R*). The latter is naturally a module over the former. If G is finite and k has characteristic zero, then R_G identifies with the image of the transfer map

$$\mathrm{tr}^G\colon\ R \to R^G, \quad f \mapsto \textstyle\sum_{g\in G} g \cdot f\,.$$

If moreover $|G| \in k^\times$, then R_G is a free R^G-module of rank one. In general, very little is known about the structure of R_G as an R^G-module.

Pevzner [4] studied this question for symmetric groups. Let $G = S_n$, let $p \le n$ be a prime, write $r = n-p$ and take $k = \mathbb{Z}_{(p)}$. In the range $p \le n < 2p$ she showed that the image of the transfer $\mathrm{Im}(\mathrm{tr}^{S_n}) \subseteq R^{S_n} = \mathbb{Z}_{(p)}[e_1, \dots, e_n]$ (where e_i is the i-th elementary symmetric polynomial) is a complete intersection ideal:

$$\mathrm{Im}(\mathrm{tr}^{S_n}) = (p, e_1e_p - e_{p+1}, e_2e_p - e_{p+2}, \dots, e_re_p - e_{p+r}, e_{r+1}, \dots, e_{p-1})\,.$$

Equipping R with its standard grading, one can define the graded Betti numbers of R_{S_n} by

$$\beta_{ij}(R_{S_n}) = \dim_{\mathbb{F}_p} \mathrm{Tor}_i^{R^{S_n}}(R_{S_n}, \mathbb{F}_p)_j\,.$$

From the above description, Pevzner observed that for all $i, j_0 \ge 0$ the sum $\sum_{j\equiv j_0 \bmod p} \beta_{ij}(R_{S_n})$ is independent of n in the range $p \le n < 2p$. She conjectured a more general stability phenomenon for the graded Betti numbers of R_{S_n} for arbitrary n.

A similar type of stability appears in preliminary calculations of the mod-p cohomology of a certain space $C_2(\mathfrak{u}_n)^+$, which we now describe. Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$. For $m \ge 0$, define the *commuting variety*

$$C_m(\mathfrak{k}) = \{(X_1, \dots, X_m) \in \mathfrak{k}^m \mid \forall i,j : [X_i, X_j] = 0\}\,.$$

In algebraic geometry the commuting variety has been studied for many years [5]. From the viewpoint of topology, the one-point compactification $C_m(\mathfrak{k})^+$ is the more interesting space. Some basic properties of these spaces were established in [1]. For example, if the order of the Weyl group of K is invertible in k, then $C_m(\mathfrak{k})^+$ is a k-homology sphere, of dimension $m\,\mathrm{rk}(\mathfrak{k})$ for m even, and $\dim(\mathfrak{k}) + (m-1)\mathrm{rk}(\mathfrak{k})$ for m odd.

The group K acts on $C_m(\mathfrak{k})^+$ via the adjoint representation, and the Borel equivariant cohomology $\mathrm{H}^*_K(C_m(\mathfrak{k})^+;k)$ is therefore a module over $\mathrm{H}^*(BK;k)$. In the talk, I explained a connection, for $\mathfrak{k}=\mathfrak{u}_n$, between the cohomology of the commuting variety and the cofixed quotient R_{S_n}. Identifying $\mathrm{H}^*(BU(n);\mathbb{Z})$ with $R^{S_n}=\mathbb{Z}[e_1,\dots,e_n]$ our first main result is the following:

Theorem. *For each n, there is an isomorphism of R^{S_n}-modules*

$$\tilde{\mathrm{H}}^*_{U(n)}(C_2(\mathfrak{u}_n)^+;\mathbb{Z})\cong R_{S_n}[-2n]\,.$$

By the Eilenberg-Moore spectral sequence, this suggests a close relationship between the graded Betti numbers of R_{S_n} and the ranks of the ordinary cohomology groups $\mathrm{H}^*(C_2(\mathfrak{u}_n)^+;\mathbb{F}_p)$. A full computation of the latter, however, remains open.

I also described an application to the representation variety $\mathrm{Hom}(\mathbb{Z}^2,U(n))$. (It is perhaps a little surprising, but the cohomology of this representation variety is still unknown.) In joint work with A. Adem and J. Gómez [1] we construct a filtration of $\mathrm{Hom}(\mathbb{Z}^2,U(n))$ whose associated graded is described in terms of Thom spaces of bundles of commuting varieties over flag manifolds. The equivariant cohomology ring $\mathrm{H}^*_{U(n)}(\mathrm{Hom}(\mathbb{Z}^2,U(n));\mathbb{Z})$, for which a presentation was given in [3], inherits a filtration by R^{S_n}-submodules

$$0=\mathcal{F}^{2n}\subseteq\mathcal{F}^{2n-1}\subseteq\cdots\subseteq\mathcal{F}^{-1}=\mathrm{H}^*_{U(n)}(\mathrm{Hom}(\mathbb{Z}^2,U(n));\mathbb{Z})\,.$$

Its associated graded module $\mathrm{gr}^{\mathcal{F}}=\bigoplus_{k\geq 0}\mathcal{F}^{k-1}/\mathcal{F}^k$ satisfies the following.

Theorem. *For each n, there is an isomorphism of R^{S_n}-modules*

$$\mathrm{gr}^{\mathcal{F}}\cong\bigoplus_{k\geq 0}\bigoplus_{\substack{(a,b,c,d)\vdash n\\ b+c+2d=k}}(R^{S_a}\otimes R^{S_b}\otimes R^{S_c}\otimes R_{S_d})[-b^2-c^2-2d]\,.$$

I concluded the talk by explaining two approaches to the cohomology of $C_2(\mathfrak{u}_n)^+$:

(1) For fixed n and varying m, the spaces $C_m(\mathfrak{u}_n)^+$ assemble into a prespectrum whose stable homotopy type agrees with the n-th subquotient of the stable rank filtration of the connective K-theory spectrum ku [2]. This is used to prove the first theorem.

(2) If the action of K on $C_m(\mathfrak{k})^+$ has connected isotropy groups of maximal rank (this happens when K is $U(n)$, $SU(n)$, $Sp(n)$ or a finite product thereof), then there is a spectral sequence

$$\mathrm{H}^{*,\mathrm{Bredon}}_W(S^{\mathfrak{t}^m},\mathcal{M})\Longrightarrow\mathrm{H}^*_K(C_m(\mathfrak{k})^+;\mathbb{Z})\,,$$

where W is the Weyl group, $\mathfrak{t}$ is the Lie algebra of the maximal torus, and $\mathcal{M}$ is the coefficient system $W/H\mapsto\mathrm{Sym}(\mathfrak{t}^*)^H$.

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Swan modules and homotopy types after a single stabilisation

John Nicholson
(joint work with Tommy Hofmann)

A *Swan module* over a finite group G is a $\mathbb{Z}G$-module defined by a left ideal $(N, r) \leq \mathbb{Z}G$ where $N = \sum_{g \in G} g$ and $r \in \mathbb{Z}$ is an integer coprime to $|G|$. Swan modules play an important role in topology through the classification of spherical space forms and more general CW-complexes. A classical question, which arose in work of Dyer [Dye76, Note (c) p276] and Plotnick [Plo82, p98], is whether every stably free Swan module is free. The case where G has periodic cohomology, which is most relevant for applications to topology, appeared in C. T. C. Wall's problem list [Wal79, Problem A4]. In joint work with Tommy Hofmann, we show:

Theorem. *There exists a finite group G and $r \in (\mathbb{Z}/|G|)^\times$ such that (N, r) is a non-free stably free $\mathbb{Z}G$-module. Furthermore, G can be taken to have periodic cohomology.*

In particular, we can take $G = Q_{56}$ to be the quaternion group of order 56 and $r = 15$. The main step in the proof is showing that $(N, 15)$ is not free. This is achieved as a result of extensive computations of unit groups of rings which arise in two pullback squares.

We say two finite n-complexes X and Y are *stably equivalent* if $X \vee kS^n \simeq Y \vee kS^n$ for some $k \geq 0$. Is one stabilisation always enough, i.e. does $k = 1$ always suffice? This question, with various conditions on the CW-complexes, appeared in [Dye79, Problem (c)], [HAMS93, p124] and [Nic26, Problem B2]. The analogous question for simple homotopy equivalence ($\simeq_s$) appeared in [Dye81, Question 1]. Using the examples of non-free stably free Swan modules in the theorem above, we show:

Theorem. *Let $n \geq 1$ with $n \equiv 3 \bmod 4$. Then there exist finite n-complexes X and Y such that $X \vee 2S^n \simeq Y \vee 2S^n$ but $X \vee S^n \not\simeq Y \vee S^n$. Furthermore, we can assume that X and Y have finite fundamental group, $(n-1)$-connected universal covers, and satisfy $X \vee 2S^n \simeq_s Y \vee 2S^n$.*

These examples are obtained by taking X to be the closed n-manifold S^n/Q_{56} and defining $Y = X_{15}$ via a general construction which is described in [HN25, Construction 1.1]. The fact that G can be taken to be finite is of interest since it was shown by Browning that, for $n \geq 2$ even, one stabilisation is enough for finite n-complexes with finite fundamental group and $(n-1)$-connected universal covers.

We now give our third main result. Recall that a finite group G has *free period* k if there exists a k-periodic resolution of finitely generated free $\mathbb{Z}G$-modules. Such groups necessarily have k-periodic cohomology. The question of whether or not the converse holds originated in work of Swan and was a major motivating question during the early development of algebraic K-theory, particularly in light of the fact that finite 3-manifold groups necessarily have free period 4. The question featured in C. T. C. Wall's problem list [Wal79, Problem A3] and was eventually resolved by Milgram who showed that certain groups with 4-periodic cohomology of the form $Q(2^n a, b, c)$ do not have free period 4.

If G is a finite group with k-periodic cohomology, then $H^k(G;\mathbb{Z}) \cong \mathbb{Z}/|G|$ and the functor $H^k(-;\mathbb{Z})$ induces a map $\psi_k : \mathrm{Aut}(G) \to (\mathbb{Z}/|G|)^\times$. It was shown independently by Dyer [Dye76, Note (b) p276] and Davis [Dav83] that if G has free period k, then $(N, \psi_k(\theta))$ is stably free for all $\theta \in \mathrm{Aut}(G)$. Plotnick [Plo82, p98] asked whether this holds in general (see also [Dav83, p488], [Nic24, Question 7.3]). By work of Bentzen–Madsen, this holds for $Q(8,p,q)$ where p, q are distinct odd primes. We show:

Theorem. *Let $k \geq 1$ with $k \equiv 4 \bmod 8$. Then there exists a finite group G with k-periodic cohomology and an automorphism $\theta \in \mathrm{Aut}(G)$ such that $(N, \psi_k(\theta))$ is not stably free.*

In particular, for each k, we can take $G = Q(16,5,1)$ and $\theta \in \mathrm{Aut}(G)$ such that $\psi_k(\theta) = 9$. By combining this with Dyer's result that $(N, \psi_k(\theta))$ is stably free for all $\theta \in \mathrm{Aut}(G)$ provided G has free period k, we obtain:

Corollary. *Let $k \geq 1$ with $k \equiv 4 \bmod 8$. Then there exists a finite group with k-periodic cohomology which does not have free period k.*

This was previously established by Milgram who used that, if G has k-periodic cohomology, there exists an element

$$\sigma_k(G) \in \widetilde{K}_0(\mathbb{Z}G)/\{[(N,r)] : r \in (\mathbb{Z}/|G|)^\times\}$$

known as the *Swan finiteness obstruction* which vanishes if and only if G has free period k. He used pullback squares to show that $\sigma_4(Q(8,p,q)) \neq 0$ for certain primes p, q. Using an analogous approach, Davis showed that $\sigma_4(Q(16,p,1)) \neq 0$ for certain primes p, including the case $p = 5$. The extension to the case $k = 4i$ for i odd follows from Wall's theorem that $2 \cdot \sigma_4(G) = 0$ whenever G has 4-periodic cohomology, combined with the standard fact that $\sigma_{4i}(G) = i \cdot \sigma_4(G)$.

In contrast, we make no use of the Swan finiteness obstruction or Wall's theorem and instead we need only show that $[(N,9)] \neq 0 \in \widetilde{K}_0(\mathbb{Z}Q(16,5,1))$. Whilst it is possible to prove this by hand, we use algorithms of Bley–Boltje and Bley–Wilson which explicitly compute $\widetilde{K}_0(\mathbb{Z}G)$ for a finite group G. Both algorithms have been implemented in Magma.

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The exotic Picard group of a $K(n)$-local algebra not too far from the sphere

Vesna Stojanoska

(joint work with Alicia Lima)

The K(n)-local categories, at an implicit prime p and a height n, are the building blocks of the stable homotopy category. They are best studied via descent for the Morava stabilizer group $\mathbb{G}_n$ acting on the Lubin-Tate spectrum $\mathrm{E_n}$. This is made possible by the fact that $\mathrm{E_n}$ is cohomologically easy (it is an even-periodic ring spectrum whose π_0 is a complete local ring), and $\mathbb{G}_n$ is a compact p-adic analytic group of virtual cohomological dimension n^2. However, the story is complicated by the fact that the action of $\mathbb{G}_n$ on $\pi_*\mathrm{E_n}$ is not very explicit.

An important invariant of the K(n)-local category is its Picard group, Pic_n. It sits in an exact sequence

$$0 \to \kappa_n \to \mathrm{Pic}_n \xrightarrow{\epsilon} H^1(\mathbb{G}_n, \pi_* E^\times) \to 0,^1$$

where the quotient is the algebraic Picard group of $\pi_*\mathrm{E_n}$-modules with a compatible $\mathbb{G}_n$-action. This algebraic Picard group was recently computed by Barthel, Schlank, Stapleton, and Weinstein [BSSW25], using methods from p-adic geometry.

My focus is on the exotic Picard group κ_n, which consists of those invertible K(n)-local spectra whose $\mathrm{E_n}$-homology looks like $\pi_*\mathrm{E_n}$, even with its $\mathbb{G}_n$-action. The group κ_n can be non-trivial only when $p-1$ divides n or $n^2 \geq 2p-1$ [HMS94],

[1]except possibly at $n = p = 2$, when it is not known whether ϵ is surjective, or has image of index 2

and it is a measure of the non-algebraicity of the K(n)-local category itself. When non-trivial, κ_n for $n > 2$ is completely unknown, though it is fully computed when $n = 1$ ([HMS94]) and $n = 2$ ([GHMR15] for $p = 3$ and [BBG$^+$26] for $p = 2$).

The case when $n = p - 1$ is special, because $\mathbb{G}_n$ has only C_p as its p-torsion, so its cohomology beyond dimension n^2 is well-understood due Hopkins and Miller [Sym04, Nav10]. This is the choice of height we work with. In [BLL$^+$25], we determined a bound on the descent filtration of κ_n, and a major tool we used was a comparison of the cohomology of $\mathbb{G}_n$ with the cohomology of its subgroup N that is the normalizer of the p-torsion. The group N has virtual cohomological dimension n, but the restriction $H^*(\mathbb{G}_n) \to H^*(N)$ is an isomorphism in dimensions above n^2 [Sym04].

This leads us to studying exotic Picard elements that can be seen by N, or the related group $\kappa(N)$ of exotic invertible modules over the algebra $\mathrm{E}_{\mathrm{n}}^{N}$. We show that $\kappa(N)$ is an elementary abelian group of a rank equal to the solution of a specific combinatorial problem. For example, when $p = 5$ it has rank 4, when $p = 7$ it has rank 8, while when $p = 11$, it has rank 56.

There are two main steps in proving this theorem: finding an upper bound for $\kappa(N)$, and showing that every element in the upper bound is in fact realized. The upper bound is determined using the Picard descent spectral sequence for the N-action on E_{n} ([BLL$^+$25]). Controlling large-scale phenomena in this spectral sequence using various available tools, leads to the conclusion that $\kappa(N)$ is the subgroup of $H^{2n+1}(N, \pi_{2n}\mathrm{E}_{\mathrm{n}})$ that is the kernel of a few differentials we cannot a priori determine.

Thus we turn to constructing invertible $\mathrm{E}_{\mathrm{n}}^{N}$-modules that are detected by these classes. This is done by twisting the action on E_{n} of various subgroups of the centralizer of C_p, each isomorphic to $\mathbb{Z}_p$, and then taking homotopy fixed points for this twisted action. The upshot is that all classes in $H^{2n+1}(N, \pi_{2n}\mathrm{E}_{\mathrm{n}})$ are realizable as invertible $\mathrm{E}_{\mathrm{n}}^{N}$-modules.

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A stable splitting for spaces of commuting elements in unitary groups

Alejandro Adem
(joint work with José Manuel Gómez, Simon Gritschacher)

We prove an analogue of Miller's stable splitting of the unitary group $U(m)$ for spaces of commuting elements in $U(m)$. After inverting $m!$, the space $\mathrm{Hom}(\mathbb{Z}^n, U(m))$ splits stably as a wedge of Thom-like spaces of bundles of commuting varieties over certain partial flag manifolds. Using Steenrod operations we prove that our splitting does not hold integrally. Analogous decompositions for symplectic and orthogonal groups as well as homological results for the one-point compactification of the commuting variety in a Lie algebra are also provided.

In [3] Miller showed that the unitary group $U(m)$ admits a stable splitting

$$U(m)_+ \simeq \bigvee_{0\le k\le m} \mathrm{Gr}_k(\mathbb{C}^m)^{\mathfrak{u}_k}$$

where

$$\mathrm{Gr}_k(\mathbb{C}^m)^{\mathfrak{u}_k} = (U(m)/U(m-k))_+ \wedge_{U(k)} \mathfrak{u}_k^+$$

is the Thom space of the adjoint bundle, i.e., of the vector bundle over the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^m)$ associated with the adjoint representation of $U(k)$. In this paper we prove an analogous stable splitting for the space of commuting n-tuples in $U(m)$,

$$\mathrm{Hom}(\mathbb{Z}^n, U(m)) = \{(x_1,\dots,x_n)\in U(m)^n \mid x_ix_j = x_jx_i \text{ for all } 1\le i,j\le n\},$$

which holds after inverting $m!$. In our splitting the role of the adjoint bundle is played by a bundle of commuting varieties over a generalised Grassmannian. We denote by

$$C_n(\mathfrak{u}_k) = \{(X_1,\dots,X_n)\in \mathfrak{u}_k^n \mid [X_i,X_j]=0 \text{ for all } 1\le i,j\le n\}$$

the space of commuting n-tuples in the Lie algebra $\mathfrak{u}_k$. The group $U(k)$ acts on $C_n(\mathfrak{u}_k)$ diagonally by the adjoint representation, and this action extends to one on the one-point compactification $C_n(\mathfrak{u}_k)^+$. Our splitting is indexed by a poset of partitions of m denoted $\mathcal{P}$ in which partitions are indexed by the set $I=\{0,1\}^n$ of binary sequences of length n. Given $a\in I$ we write $|a| = \sum_{i=1}^n a(i)$.

Theorem A. *After inverting $m!$ there is a stable splitting for all $n\in\mathbb{N}$,*

$$\mathrm{Hom}(\mathbb{Z}^n, U(m))_+ \simeq \bigvee_{(\lambda_a)_{a\in I}\in\mathcal{P}} \left(U(m)_+ \wedge_{\prod_{a\in I} U(\lambda_a)} \bigwedge_{a\in I} C_{|a|}(\mathfrak{u}_{\lambda_a})^+ \right).$$

The stable summands are Thom-like spaces for bundles of commuting varieties. These commuting varieties are irreducible, and in the non-abelian case the bundles that appear are not vector bundles, but rather bundles of infinite subspace arrangements.

Our methods apply to prove an analogous stable splitting for spaces of commuting elements in the compact symplectic group $Sp(m)$, but not for $SU(m)$. This is because our proof uses a stable splitting of a maximal torus which is equivariant for the Weyl group action, and we do not know if such a splitting exists in the case of $SU(m)$. For orthogonal groups our methods apply; but because the orthogonal groups have abelian subgroups that are not contained in a maximal torus, we obtain a stable splitting only of the path-component of $\mathrm{Hom}(\mathbb{Z}^n, O(m))$ containing the trivial homomorphism.

To put our theorem in context we recall that the first author and F. Cohen [2] have previously proved a stable splitting of the space $\mathrm{Hom}(\mathbb{Z}^n, G)$ for any compact Lie group G. Let $S_n(G) \subseteq \mathrm{Hom}(\mathbb{Z}^n, G)$ be the subspace of those n-tuples of commuting elements of which at least one coordinate is the neutral element $1_G \in G$. Then there is a stable splitting

$$\mathrm{Hom}(\mathbb{Z}^n, G)_+ \simeq \bigvee_{0 \le r \le n} \left(\bigvee^{\binom{n}{r}} \mathrm{Hom}(\mathbb{Z}^r, G)/S_r(G) \right).$$

This splitting holds without inverting primes, but with the exception of the trivial cases and the cases $G = SU(2)$ and $G = SO(3)$, the stable summands have not been identified geometrically. For $G = U(m)$ and with $m!$ inverted, our decomposition is finer than the above, in that there is a stable equivalence

$$\mathrm{Hom}(\mathbb{Z}^n, U(m))/S_n(U(m)) \simeq \bigvee_{(\lambda_a)_{a\in I} \in \mathcal{S}} \left(U(m)_+ \wedge_{\prod_{a\in I} U(\lambda_a)} \bigwedge_{a \in I} C_{|a|}(\mathfrak{u}_{\lambda_a})^+ \right)$$

for a certain subset $\mathcal{S} \subseteq \mathcal{P}$.

Miller's stable splitting is modelled on a filtration of $U(m)$ in which the top filtration quotient is $\mathfrak{u}_m^+$. Similarly, our splitting is obtained from a filtration with top filtration quotient $C_n(\mathfrak{u}_m)^+$. The next theorem shows that when $n \ge 2$, certain primes *must* be inverted for the top filtration quotient to split off stably. Using Steenrod operations we prove:

Theorem B. *Let p be a prime and let $n \ge 2$ be an integer. The projection*

$$\mathrm{Hom}(\mathbb{Z}^n, U(p))_+ \to C_n(\mathfrak{u}_p)^+$$

does not have a stable section up to homotopy at the prime p.

These results have been published in [1].

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Quillen's conjecture and unitary groups

Antonio Díaz Ramos

We prove that the Quillen p-subgroup complexes $\mathcal{A}_p(G)$ of p-extensions G of simple unitary groups have non-zero homology in the largest possible dimension, for odd primes p and with just a few exceptions. This establishes a conjecture raised by Aschbacher–Smith in 1992. In particular, by their work and a more recent article by Piterman–Smith, Quillen's conjecture on the p-subgroup complex holds for odd primes.

With more detail, for any prime p, Quillen's conjecture for the p-subgroup complex states that, if the largest normal p-subgroup of a finite group G is trivial, then its Quillen p-subgroup complex $\mathcal{A}_p(G)$ is not contractible. If p is odd, by the work of Aschbacher-Smith [1] and the recent work of Piterman-Smith [5], to show that Quillen's conjecture is valid, it is enough to prove that [1, Conjecture 4.1] holds.

This conjecture [1, Conjecture 4.1] states that the Quillen p-subgroup complexes $\mathcal{A}_p(G)$ of p-extensions G of the simple unitary group $\mathrm{PSU}_n(q)$ have non-zero homology in the largest possible dimension, for odd primes p and with just a few exceptions. So far, [1, Conjecture 4.1] has been known to hold but for the case $p \mid (q+1)$, and in recent work we have proven that the conjecture also holds in this case, so that Quillen's conjecture on the p-subgroup complex holds for odd primes.

Our solution to [1, Conjecture 4.1] for p odd and $p \mid (q+1)$ consists in constructing explicit non-zero homology classes in the largest possible dimension for $\mathcal{A}_p(G)$, where G is one of the groups described below. To construct such classes, we first consider an elementary abelian p-subgroup $E \leq G$ of maximum p-rank, i.e., with $\mathrm{m}_p(E) = \mathrm{m}_p(G)$, and then define a simplicial chain $\mathbf{C}_E$ that corresponds to the barycentric subdivision of an $(\mathrm{m}_p(G)-1)$-simplex with E as barycenter. In turn, we find a subset X of G such that a linear combination of the X-conjugates of $\mathbf{C}_E$ is a non-trivial cycle of $\widetilde{H}_{\mathrm{m}_p(G)-1}(|\mathcal{A}_p(G)|;\mathbb{Z})$. We have developed several approaches

(a) $G = \mathrm{PSU}_n(q)$ or $G = \mathrm{PGU}_n(q)$, E consists of diagonal elements, and X is a subset of a subgroup of G generated by non-commuting transvections, and which is a homomorphic image of the braid group,
(b) $G = \mathrm{PGU}_n(q)$, E consists of diagonal elements, and X contains permutation matrices and a quasi-reflection, and
(c) G equals $\mathrm{PGU}_n(q)$ extended by field automorphisms of order p, E consists of diagonal elements and the generator of the field automorphisms of order p, and X contains permutation matrices, a quasi-reflection, and a diagonal element that that does not centralize the field automorphisms of order p.

In cases (a) and (b), the resulting complex is a triangulation of the sphere similar to the Coxeter complex of the symmetric group Σ and has $\mathrm{m}_p(G)+1$ $(\mathrm{m}_p(G)-1)$-simplices. This contrasts with previous works, [2, 4], where X was a subset of an abelian subgroup of G and the resulting complex has $2^{\mathrm{m}_p(G)}$ $(\mathrm{m}_p(G)-1)$-simplices. In case (c), the resulting complex is a suspension of the complex constructed in (b) for $\mathrm{PGU}_n(q^{1/p})$. In case (a), none of the X-conjugates of the $(\mathrm{m}_p(G)-1)$-simplices making up $\mathbf{C}_E$ are equal, while for cases (b,c), such identifications might occur for some such simplices. Cases (b,c) suffice to prove [1, Conjecture 4.1] by an argument that traces back to the work of Aschbacher-Smith [1].

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Synthetic Buildings for Finite Groups

Peter Webb

(joint work with Emily Gullerud)

Let G be a finite group and p a prime. A *G-simplicial complex* means a finite simplicial complex Δ with a simplicial action of G, so that the stabilizer of each simplex fixes the simplex pointwise. We consider two conditions that Δ might satisfy:

(1) the fixed point space Δ^H is contractible whenever $H \leq G$ with $O_p(H) \neq 1$, and

(2) every simplex stabilizer has order divisible by p.

If Δ satisfies (1) and (2) we call it a *synthetic building for G at p.* If Δ only satisfies (1) (and possibly not (2)) we call it a *weak synthetic building for G at p.*

The first examples of synthetic buildings are as follows.

(1) When $p \mid |G|$, a single point $\bullet$ with trivial action of G is a synthetic building. This we call the *trivial synthetic building.*

(2) When p does not divide $|G|$ the empty set is a synthetic building, and is the only synthetic building for G at p.

(3) The non-identity p-subgroups complex $\Delta(\mathcal{S}_p(G))$ of K.S. Brown is a synthetic building for G at p.

(4) The building a group of Lie type in characteristic p is a synthetic building for G at p.

The following properties of weak synthetic buildings exemplify their use, and give strong reasons for studying them.

Theorem 1 ([2] [3]). *If Δ is a weak synthetic building for G at p and M is a finitely generated representation of G then the following hold:*

- *Cohomology formula: $\hat{H}^n(G,M)_p = \sum_{\sigma \in G\backslash\Delta} (-1)^{\dim \sigma} \hat{H}^n(G_\sigma, M)_p$ in the Grothendieck group of finite abelian groups with relations given by direct sum decompositions..*
- *Over a p-local ring, the augmented chain complex $\tilde{C}_\bullet(\Delta)$ is chain homotopy equivalent to a complex of projective modules.*
- *The reduced Lefschetz module of $\tilde{C}_\bullet(\Delta)$ is a virtual projective module.*

Synthetic buildings seem rather hard to construct and they carry important information about G. By introducing synthetic buildings we provide a framework for studying them all. What do they look like? How many of them are there? The following are some initial observations.

Theorem 2. (1) *If G is a p-group then $\Delta \simeq_G *$ is the only possibility for a synthetic building.*
(2) *If Δ is a synthetic building for G at p, so is the suspension $\Sigma\Delta$.*
(3) *There exists an infinite family of A_5-graphs that are synthetic buildings for A_5 at $p = 2$.*

We define a synthetic building to be *exotic* if it is not G-homotopy equivalent to a single point or $\Delta(\mathcal{S}_p(G))$. The infinite families of synthetic buildings just described provide examples of exotic synthetic buildings.

Although the one-dimensional synthetic buildings in the infinite family for A_5 are exotic, they all have a subcomplex that is a smaller synthetic building: either the 5 points permuted like the cosets of A_4 in A_5, or a single fixed point. In cohomology formulas such subcomplexes give shorter expressions for cohomology than the full expression, and we are really interested in the shorter expressions. We define a synthetic building for G at p to be *minimal* if it has no subspace that is also a synthetic building for G at p. Thus the one dimensional buildings for A_5 are not minimal. What are the minimal synthetic buildings? Some exotic ones have been found previously:

Theorem 3 ([1]). *At $p = 2$ both the alternating group A_7 and the Mathieu group M_{11} have two minimal synthetic buildings of dimension 1, one of which is exotic.*

We show that groups may have an arbitrarily large finite number of exotic minimal synthetic buildings. We also show that groups may have exotic minimal synthetic buildings of different dimensions.

We present a number of conjectures, some of which are as follows.

Conjecture 4. *For each finite group and prime p, there are only finitely many equivariant homotopy types of minimal synthetic buildings.*

Conjecture 5. *For each finite group and prime p, the p-subgroups complex $\Delta(\mathcal{S}_p(G))$ always has the equivariant homotopy type of a minimal synthetic building.*

Conjecture 6. *Minimal synthetic buildings are all equivariantly homotopy equivalent to homomorphic images of $\Delta(\mathcal{S}_p(G))$.*

Conjecture 7. *For groups of Lie type in characteristic p the only non-trivial minimal synthetic buildings are equivariantly homotopy equivalent to the Tits building.*

Theorem 8. *Conjecture 7 is true for groups of Lie type of Lie rank at most 2.*

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Group actions on compact contractible complexes

Kevin Ivan Piterman
(joint work with Ivan Sadofschi Costa)

A natural question in transformation theory is whether a given group action on a topological space has a global fixed point. In full generality, the answer is negative even in very simple situations: for instance, one may consider the translation action of $\mathbb{Z}$ on the real line, the antipodal action on spheres, or the free action of a group on the universal cover of its classifying space. To exclude such examples, we restrict our attention to finite groups acting cellularly on compact CW-complexes. We also impose strong homotopy-theoretic conditions on the space, asking it to be acyclic or even contractible. This leads to the following question:

Question. Let G be a finite group acting on a compact CW-complex X, which is either acyclic over some ring or contractible. Can we say whether X^G, the fixed point set, is non-empty?

For example, by Smith theory, a finite p-group acting on a mod-p acyclic finite-dimensional simplicial complex has a fixed point. We also know by Brouwer's fixed-point theorem that a cyclic group acting on a disc has a fixed point. Hence, p-groups and cyclic groups cannot act fixed-point-freely on discs. In the 1970s, B. Oliver classified the finite groups that can act without fixed points on discs [6]. In fact, he showed that a finite group G acts without fixed points on a disc if and only if G does not contain subgroups $P \trianglelefteq H \trianglelefteq G$ such that P is a p-group, H/P is cyclic, and G/H is a q-group, where p, q are primes.

On the other hand, a well-known theorem by J.P. Serre in the 1970s states that a finite group acting on a tree has a fixed point. However, in higher dimensions, we encounter compact contractible spaces with fixed-point-free actions by finite groups. Concretely, E. Floyd and R.W. Richardson showed in [5] that there is an action of the alternating group A_5 on the 2-skeleton X_P of the Poincaré homology 3-sphere, and X_P is acyclic and fixed-point-free. Viewing A_5 as a discrete space with the regular action of A_5, the join $X = A_5 * X_P$ is a 3-dimensional compact and contractible complex with $X^{A_5} = \emptyset$.

In dimension 2, it was conjectured by C. Casacuberta and W. Dicks [4], and independently raised as a question by M. Aschbacher and Y. Segev [2], that a finite group acting on a compact contractible 2-complex has a fixed point. The conjecture was established for solvable groups using Smith theory in [4]. Invoking the classification of finite simple groups, Aschbacher and Segev show that a finite group acting without fixed points on a compact acyclic 2-complex must have a composition factor isomorphic to the Janko group J_1 or to one of the simple groups of Lie type and Lie rank 1. In 2002, B. Oliver and Y. Segev achieved substantial progress on this problem by classifying finite groups acting without fixed points on finite acyclic 2-complexes [7]. One of their main theorems states, in essence, that if a finite group G acts without fixed points on a finite acyclic 2-complex then a certain quotient of G is one of the simple groups $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(2^{2k+1})$, with some restrictions on q and k. In fact, they construct examples of compact acyclic 2-complexes without fixed points for each of these simple groups. We refer to the seminal exposition by A. Adem [1] for more details.

The Casacuberta–Dicks conjecture for compact complexes was settled in work of I. Sadofschi Costa [9] and in joint work [8]. These works show that if X is a compact contractible 2-dimensional G-CW-complex, then G has a fixed point on X.

The idea of the proof is to start with a fixed-point-free compact *acyclic* 2-complex X, and show that the fundamental group $\pi_1(X)$ must be non-trivial. Based on a previous article [10], which establishes the case $G = A_5$, Sadofschi Costa reduced the study of the conjecture to the simple groups G in the Oliver–Segev theorem (namely, the 2-dimensional finite linear groups and Suzuki groups). It is also shown in [9] that it is enough to establish $\pi_1(X) \neq 1$ for X in the family of examples constructed in [7]. Once we have these reductions, we construct a manifold M encoding unitary representations of the group extension Γ of $\pi_1(X^1)$ by G, obtained by lifting the maps $g \in G$ to the universal cover of X^1. Using a theorem by K. Brown [3], we obtain an explicit description of Γ in terms of generators, relators and the action of G, and this yields an extension $\widetilde{\Gamma}$ of $\pi_1(X)$ by G after quotienting by certain words corresponding to the attachment of the 2-cells. Then we construct a smooth map $f : M \to N$, where N is a product of unitary groups $U(m)$, such that if $f(x) = 1_N$ (the identity matrix), then x corresponds to a unitary representation of Γ that factors through $\widetilde{\Gamma}$. Moreover, M arises as a quotient of a product of centralisers in N, and hence contains the class of the identity matrix $\overline{1}_M$. This point $\overline{1}_M \in M$ corresponds to a unitary representation of $\widetilde{\Gamma}$ that factors through G, which means that $\ker(\overline{1}_M) \supseteq \pi_1(X)$. Indeed, $\overline{1}_M$ is an irreducible representation of G that restricts to an irreducible representation of a Borel subgroup of G, and the construction of f strongly depends on choosing such a representation in the first place. Moreover, $\overline{1}_M \in f^{-1}(1_N)$ and it is the unique representation in this preimage that contains $\pi_1(X)$ in its kernel. Therefore, to show that $\pi_1(X) \neq 1$, we show that the preimage $f^{-1}(1_N)$ contains at least two points. For that, we use a degree argument. Our construction of M and the acyclic hypothesis on X imply that M is an orientable compact connected manifold of the same dimension as N, 1_N is a regular value, and f is homotopic to a non-surjective map. Thus $\deg(f) = 0$, which implies that $f^{-1}(1_N)$ has an even number of points. In particular, there is a point $x \in f^{-1}(1_N)$ with $x \neq \overline{1}_M$ that corresponds to a unitary representation of $\widetilde{\Gamma}$ which does not factor through G, so it restricts to a non-trivial representation of $\pi_1(X)$. Therefore, X is not contractible.

Finally, the non-compact case of the Casacuberta–Dicks conjecture remains open.

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Smith theory revisited

Vignesh Subramanian

(joint work with Robert Burklund)

Given a finite $\mathbb{Z}/p$-CW complex X, we have

$$\chi(X) \equiv \chi(X^{\mathbb{Z}/p}) \ mod\ p$$

since there is only one type of isotropy and every non-fixed cell has orbit of size p. This leads us to ask the following question:

Question 1. How much of the homological information about $X^{\mathbb{Z}/p}$ can be recovered from that of X?

In [Smi38, Smi39], P.A. Smith initiated this program and analysed actions of groups of prime order, in particular involutions on spheres and projective spaces. These ideas were further explored by [Bor60, AS65, Qui71a, Qui71b] via the localisation theorem:

Theorem 2. Let X be a $\mathbb{Z}/p$-CW complex, then we have equivalence of rings:

$$H^*(X_{h\mathbb{Z}/p};\mathbb{F}_p)[t^{-1}] \cong H^*(X^{\mathbb{Z}/p};\mathbb{F}_p) \otimes H^*(B\mathbb{Z}/p;\mathbb{F}_p)[t^{-1}]$$

where t is the polynomial generator of $H^*(B\mathbb{Z}/p;\mathbb{F}_p)$.

The localised algebra $H^*(X_{h\mathbb{Z}/p};\mathbb{F}_p)[t^{-1}]$ can be given an $\mathcal{A}_p$-module structure (see [Wil77]), where $\mathcal{A}_p$ is the $\mathbb{F}_p$-based Steenrod algebra. In their seminar paper [DW88], Dwyer and Wilkerson proved:

Theorem 3. The cohomology $H^*(X^{\mathbb{Z}/p};\mathbb{F}_p) \otimes H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ is the largest unstable $\mathcal{A}_p$-submodule of $H^*(X_{h\mathbb{Z}/p};\mathbb{F}_p)[t^{-1}]$.

In joint work with Burklund [BS], we study the above theorem in a highly structured way using the theory of Dyer-Lashof operations for $\mathbb{E}_\infty$-k-algebras, where k is a field of characteristic p. In order to state our results we introduce some definitions.

Fix an $\mathbb{E}_\infty$-k-algebra A. Let Q^0 be the zeroth Dyer-Lashof operation, which is an additive map $Q^0 : \pi_n(A) \to \pi_n(A)$ that enjoys the following properties:

(1) Q^0 is the zero map on $\pi_n(A)$ for $n > 0$.
(2) Acts by Frobenius $x \mapsto x^p$ on $\pi_0(A)$.

Definition 4. [BS] An $\mathbb{E}_\infty$-k-algebra A is called ***perfect*** if Q^0 acts by isomorphism for all π_n for all $n \in \mathbb{Z}$. We denote the category of $\mathbb{E}_\infty$-k-algebras by CAlg_k and the full subcategory of perfect algebras by $\mathrm{CAlg}_k^\sharp$.

Theorem 5. The category of perfect algebras $\mathrm{CAlg}_k^\sharp$ is a presentable category and the inclusion to the category of $\mathbb{E}_\infty$-k-algebras CAlg_k, admits both adjoints. We call the left adjoint $(-)^\sharp$ as ***perfection*** and the right adjoint $(-)^\flat$ as ***tilting***.

Remark 6. Note that since Q^0 acts by isomorphism on all homotopy groups π_n for $n > 0$, we have that all perfect algebras are coconnective. This makes the theory of perfect algebras particularly well-behaved.

We can further describe the homotopy groups of tilt:

Theorem 7. [BS] Let A be an $\mathbb{E}_\infty$-k-algebra. The tilt $A^\flat$ is coconnective. The remaining homotopy groups of $A^\flat$ sit in short exact sequences

$$0 \to \lim_F{}^1 \pi_{n+1} A \to \pi_n A^\flat \to \lim_F \pi_n A \to 0.$$

Let k be a perfect field. Then, for the $n = 0$ case, the $\lim^1$ term vanishes, providing an isomorphism between $\pi_0 A^\flat$ and the inverse limit of $\pi_0 A$ along the p'th power map. If A is of finite type,[1] then the $\lim^1$ term vanishes.

[1] meaning that $\pi_n A$ is finite dimensional for all n

We can now state a version of the Dwyer-Wilkerson theorem (see 3).

Theorem 8. [BS] Let X be a $\mathbb{Z}/p$ - CW complex. Let k be a perfect field. Then we have an equivalence of $\mathbb{E}_\infty$-k-algebras:

$$C^*(X^{\mathbb{Z}/p} \times B\mathbb{Z}/p; k) \simeq (C^*(X; k)^{t\mathbb{Z}/p})^\flat$$

where $(-)^{t\mathbb{Z}/p}$ is the Tate construction and $C^*(X; k)$ has the natural $\mathbb{Z}/p$-action coming from the action on X.

Remark 9. We view tilting as a coherent way to pick unstable cohomological classes. From the above theorem, by tensoring $(C^*(X; k)^{t\mathbb{Z}/p})^\flat$ along the map $C^*(B\mathbb{Z}/p; k) \to k$, we can recover the commutative algebra $C^*(X^{\mathbb{Z}/p}; k)$. Moreover, when k is an algebraically closed field, we can appeal to Mandell's model for p-adic homotopy theory [Man01] to recover the p-adic homotopy type of $X^{\mathbb{Z}/p}$. In this way, we have answered the question 1.

This is rather surprising, as we have recovered the genuine fixed points from the Borel data, that is, only the data of $C^*(X; k)$ with a $\mathbb{Z}/p$-action. This is in the spirit of the Sullivan conjecture, which asserts that the map $X^{\mathbb{Z}/p} \to X^{h\mathbb{Z}/p}$ is a p-adic equivalence.

The Sullivan conjecture was a very sought-after problem in algebraic topology during the late 20th century; Miller, Carlsson, and Lannes settled it [Mil84, Car91, Lan92] independently. In our work [BS], we give an alternative proof of the Sullivan conjecture, which does not appeal to the unstable Adams spectral sequence or the theory of unstable modules over the Steenrod algebra.

The process of perfection and tilting kills a lot of classes in the homotopy groups of an algebra A. This also leads to an alternative and less computational proof of Mandell's p-adic model [Man01] and a version of Yuan's model for spherical cochains [Yua23].

Finally, we would like to conclude by commenting on analogous definitions for derived algebras in the sense of Raksit [Rak20] and Brantner-Mathew [BM25]. Let us denote the category of derived k-algebras by DAlg_k, and we call a derived algebra perfect if the underlying commutative algebra is perfect.

Theorem 10. [BS] The forgetful functor $U : \mathrm{DAlg}_k \to \mathrm{CAlg}_k$ restricts to an equivalence on the category of perfect algebras.

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Decomposition numbers of symmetric groups and Verlinde categories

Chris Hone

(joint work with Finn Klein, Bregje Pauwels, Alexander Sherman, Oded Yacobi, Victor L. Zhang)

Representation theory in characteristic $p > 0$ is incredibly rich and complicated. For a finite group G, representations need not be semisimple, and many approaches have been developed to study the homological properties of $\mathrm{Rep}_{\Bbbk} G$. Local representation theory and support theory have been particularly successful and continue to attract widespread interest ([1], [4]). In a separate vein, computing the decomposition numbers (or even the dimensions of irreducible representations) of the symmetric group remains one of the largest open problems in modular representation theory ([7], [9]), and has important connections to the modular representation theory of the algebraic group GL_n. In this work we introduce *One Tree Island (OTI) functors* for studying $\mathrm{Rep}_{\Bbbk} G$ and categorical Heisenberg actions in characteristic p. Such functors were already introduced and studied in [2] in the setting of algebraic groups, where a (conjectural) connection was found with the Finkelberg–Mirkovic (FM) conjecture. The themes of [2] and this summary are that OTI functors:

(1) have explicit, computable definitions,
(2) are compatible with categorical structures (e.g. they are symmetric monoidal and/or define morphisms of categorical actions),
(3) and provide extensions of known functors of interest (hypercohomology composed with FM equivalence in the setting of [2], and periodic equivalences of symmetric group representations in the setting of this summary).

Using OTI functors we obtain explicit consequences for representations of symmetric groups, which we believe further motivate the study of OTI functors in modular representation theory and categorical Heisenberg actions.

What is an OTI functor? The case of finite groups. Let G be a finite group such that $p \,\big|\, |G|$. Let $\Bbbk$ be an algebraically closed field of characteristic p, and let $\operatorname{Rep}_{\Bbbk} G$ be the category of finite dimensional representations of G over $\Bbbk$.

By Cauchy's theorem, there exists a cyclic subgroup $H \subseteq G$ with $H \cong C_p$. One may then consider the functor Φ_H defined by the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Rep}_{\Bbbk} G & \xrightarrow{\operatorname{Res}_H} & \operatorname{Rep}_{\Bbbk} C_p \\ & \searrow^{\Phi_H} & \downarrow{\scriptstyle ss} \\ & & \operatorname{Ver}_p \end{array}$$

Here, Ver_p is a semisimple, symmetric tensor category defined as the *semisimplification* of the tensor category $\operatorname{Rep} C_p$. As an abelian category, this category is just $\prod_{i=1}^{p-1} \operatorname{vec}_{\Bbbk}$. The semisimplication functor ss is symmetric monoidal, but not exact, and in this case admits an explicit, computable description.

Thus the functor Φ_H is defined by restricting, then semisimplifying. The centralizer subgroup $C_G(H)$ naturally acts on $\Phi_H(M)$ for any module M, and H acts trivially. Hence we obtain a functor

$$\Phi_H : \operatorname{Rep}_{\Bbbk} G \to \operatorname{Rep}_{\operatorname{Ver}_p} C_G(H)/H,$$

where $\operatorname{Rep}_{\operatorname{Ver}_p} C_G(H)/H$ denotes the representations of $C_G(H)/H$ in Ver_p. The functor Φ_H is our first example of an OTI functor, and it satisfies many desirable properties.

As an application in the finite groups setting, we consider the situation where we have a group H of order p acting on G. The Glauberman correspondence is a canonical bijection between the irreducible representations of G fixed by H and the irreducible representations of the invariants G^H. By Clifford theory, an irreducible representation of G that is fixed by H extends uniquely to an irreducible representation of $G \rtimes H$. Observing that $C_{G\rtimes H}(H)/H \cong G^H$, we obtain an OTI functor:

$$\Phi_H : \operatorname{Rep}_{\Bbbk}(G \rtimes H) \to \operatorname{Rep}_{\operatorname{Ver}_p}(G^H).$$

Theorem. *The OTI functor Φ_H above categorifies the Glauberman correspondence.*

One novel aspect of this categorification is that it explains the sign that appears in the Glauberman correspondence.

Periodic equivalences for symmetric groups. It was first observed in [6] (see also [8], and [5] for a more recent treatment) that there is an equivalence between certain full abelian subcategories of $\operatorname{Rep}_{\Bbbk} S_n$ and $\operatorname{Rep}_{\Bbbk} S_{n-p^r}$, corresponding to truncating the first row of a partition by p^r.

For the purposes of this introduction, we say a partition λ of n is p^r-stable if $\lambda_1 \gg p^r$ and $\sum_{i>1} \lambda_i < p^r$. Let $\mathcal{S}_{n,p^r}$ be the smallest abelian subcategory of $\operatorname{Rep}_{\Bbbk} S_n$ containing all permutation modules M^λ, where λ is p^r-stable. The category $\mathcal{S}_{n,p^r}$ contains all Specht modules for p^r-stable partitions and all simple modules for p^r-stable, p-regular partitions. Then [5, 6] produce equivalences $\mathcal{S}_{n,p^r} \xrightarrow{\sim} \mathcal{S}_{n-p^r,p^r}$, which on simple, Specht and permutation modules correspond to taking the partition $\lambda = (\lambda_1, \lambda_2, \dots)$ to $(\lambda_1 - p^r, \lambda_2, \dots)$. We call these "periodic equivalences".

The equivalences of [5, 6] have useful applications in computing decomposition numbers and constructing Deligne interpolation categories, but they are inexplicit and difficult to compute directly. Both functors are only defined on subcategories of $\operatorname{Rep} S_n$, and require passage through Schur–Weyl duality amongst other operations to define them.

Our first theorem improves upon [5, 6] by providing explicit, global functors $\operatorname{Rep}_{\Bbbk} S_n \to \operatorname{Rep}_{\Bbbk} S_{n-p^r}$ which restrict to the periodic equivalences. Furthermore, our functors commute with the well-known categorical action of $\widehat{\mathfrak{sl}_p}$ on $\bigoplus_n \operatorname{Rep}_{\Bbbk} S_n$.

To state our theorem, let $S_{p^r} \subseteq S_n$ denote the natural subgroup determined by the permutations of $\{n - p^r + 1, \dots, n\}$. Then S_{p^r} contains a transitive, elementary abelian subgroup A of order p^r. Write $\sigma_1, \dots, \sigma_r$ for generators of A. Let $a_1, \dots, a_r \in \Bbbk$ be linearly independent over $\mathbb{F}_p$, and set

$$z_{\mathbf{a}} = a_1(\sigma_1 - 1) + \dots + a_r(\sigma_r - 1) \in \Bbbk A.$$

Note that $z_{\mathbf{a}}$ defines an operator on any $V \in \operatorname{Rep}_{\Bbbk} S_n$, which commutes with S_{n-p^r}. Hence $\ker(z_{\mathbf{a}}), \operatorname{im}(z_{\mathbf{a}})$, etc. can be viewed as functors from $\operatorname{Rep}_{\Bbbk} S_n \to \operatorname{Rep}_{\Bbbk} S_{n-p^r}$.

Theorem A. *The functors* $\operatorname{Rep}_{\Bbbk} S_n \to \operatorname{Rep}_{\Bbbk} S_{n-p^r}$ *given by*

$$\frac{\ker(z_{\mathbf{a}})}{\ker(z_{\mathbf{a}}) \cap \operatorname{im}(z_{\mathbf{a}})} \quad \text{and} \quad \frac{\ker(z_{\mathbf{a}})}{\operatorname{im}(z_{\mathbf{a}}^{p-1})}$$

have the following properties:

(1) *they restrict to the periodic equivalences* $\mathcal{S}_{n,p^r} \xrightarrow{\sim} \mathcal{S}_{n-p^r,p^r}$, *and*
(2) *they commute with the* $\widehat{\mathfrak{sl}_p}$ *categorical action.*

Theorem A provides many distinct functors which globalize our equivalence of interest: we can choose any generic tuple $\mathbf{a}$ and either functor given in the statement. In the next section we will generalise the construction of OTI functors, and the functors above will be specific examples of this general construction.

An OTI functor from categorical actions. Let us distill the definition of the OTI functor from the finite group case as follows. We may replace $\operatorname{Rep}_{\Bbbk} G$ and $\operatorname{Rep}_{\Bbbk} H$ by (nice) abelian categories $\mathscr{C}, \mathscr{D}$, and replace $\operatorname{Res}_H : \operatorname{Rep}_{\Bbbk} G \to \operatorname{Rep}_{\Bbbk} H$ by a functor $R : \mathscr{C} \to \mathscr{D}$ with an action of C_p. Then we obtain a functor

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\quad R \quad} & \mathscr{D} \boxtimes \operatorname{Rep}_{\Bbbk} C_p \\ & \underset{\Phi}{\searrow} & \big\downarrow{\scriptstyle ss} \\ & & \mathscr{D} \boxtimes \operatorname{Ver}_p . \end{array}$$

Here, $\mathscr{D} \boxtimes \operatorname{Rep}_{\Bbbk} C_p$ is the category of objects in $\mathscr{D}$ with a C_p-action, and $\mathscr{D} \boxtimes \operatorname{Ver}_p$ is equivalent to $p-1$ copies of $\mathscr{D}$. Note that the Frobenius functor is of the above form, where $R(M) = M^{\otimes p}$.

Our primary case of interest comes from the categorical action of the Heisenberg algebra. Recall that a degenerate categorical Heisenberg action on $\mathscr{C}$ is the data of a pair of biadjoint functors $E, F : \mathscr{C} \to \mathscr{C}$ and natural transformations $x : E \to E$ and $T : E^2 \to E^2$ satisfying certain compatibilities. One such requirement is that T generates an action of S_n on E^n, which in particular provides an action of C_p on E^p. This provides the functor $R = E^p$ above.

We also generalise the semisimplification functor $ss : \operatorname{Rep} C_p \to \operatorname{Ver}_p$ to any $\Bbbk$-linear functor $\varphi : \operatorname{Rep}_{\Bbbk} S_{p^r} \to \mathscr{A}$ such that $\varphi \circ \operatorname{Ind}_H^{S_{p^r}} = 0$ for all non-transitive subgroups $H \subseteq S_{p^r}$. We call such a functor φ a *CF functor*, a modest generalisation of the notion of $\mathbf{V}$-functor introduced in [3]. CF functors satisfy the minimal conditions necessary for our machinery to behave well.

Now given a category $\mathscr{C}$ with a degenerate Heisenberg action and a CF functor φ, we define the OTI functor $\Phi_\varphi : \mathscr{C} \to \mathscr{C} \boxtimes \mathscr{A}$ by the following (now familiar) diagram:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\quad E^{p^r} \quad} & \mathscr{C} \boxtimes \operatorname{Rep}_{\Bbbk} S_{p^r} \\ & \underset{\Phi_\varphi}{\searrow} & \big\downarrow{\scriptstyle 1 \boxtimes \varphi} \\ & & \mathscr{C} \boxtimes \mathscr{A} \end{array}$$

There are many interesting, well-studied examples of degenerate categorical Heisenberg actions, the most famous being on the category $\mathcal{S}ym := \bigoplus_{n \in \mathbb{N}} \operatorname{Rep}_{\Bbbk} S_n$. In this case, Φ_φ will be symmetric monoidal whenever φ is. For example, if $p^r = p$ and $\varphi : \operatorname{Rep}_{\Bbbk} S_p \to \operatorname{Ver}_p$ is given by restriction to C_p and semisimplification, we obtain an OTI functor $\operatorname{Rep}_{\Bbbk} S_n \to \operatorname{Rep}_{\Bbbk} S_{n-p} \boxtimes \operatorname{Ver}_p$. On the other hand, the functors introduced in the statement of Theorem A are also examples of OTI functors.

To state our next theorem, note that $\mathscr{C} \boxtimes \mathscr{A}$ inherits a degenerate categorical Heisenberg action from $\mathscr{C}$. Furthermore, recall that a degenerate categorical Heisenberg action on $\mathscr{C}$ induces an action of an integral form of $\widehat{\mathfrak{sl}_p}$ on $K_0(\mathscr{C})$, where $e := [E]$ is the sum of positive Chevalley generators.

Theorem B. *Let φ be a CF functor.*

(1) *The OTI functor $\Phi_\varphi : \mathscr{C} \to \mathscr{C} \boxtimes \mathscr{A}$ defines a morphism of degenerate categorical Heisenberg actions.*
(2) *If $\varphi : \operatorname{Rep}_{\Bbbk} S_p \to \operatorname{Ver}_p$ is given by restriction to C_p followed by semisimplification, then Φ_φ induces an endomorphism of the mod p Grothendieck*

group $K_0(\mathscr{C})/(p)$. This endomorphism is given by the central element e^p in the mod p enveloping algebra of $\widehat{\mathfrak{sl}_p}$.

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The singularity category and duality for complete intersection groups

J.P.C. Greenlees

1. Context

If G is a finite group, the structure of the modular representation theory depends on the cochains $C^*(BG;k)$, viewed as a commutative ring spectrum. We consider here its singularity category ([7]) and show that the singularity category is the bounded derived category of the Ω-Tate spectrum (the k-nullification of the Koszul dual connective ring spectrum $C_*(\Omega BG_p)$). We establish a form of Gorenstein duality for $C_*(\Omega BG_p)$, and show that the Ω-Tate spectrum enjoys a form of Tate duality. Under a complete intersection hypothesis we give a method for calculating the Ω-Tate homology. More details may be found in [6].

1.1. **The enhanced group cohomology ring.** Many structural features of the representation theory of a finite group G over a field k of characteristic p are reflected in the cohomology ring $H^*(BG;k) = \mathrm{Ext}^*_{kG}(k,k)$, starting with Quillen's theorem that the Krull dimension is the p-rank of G. This is a Noetherian ring (Venkov) and very special structurally: for example if it is Cohen-Macaulay, it is automatically Gorenstein (Benson-Carlson). However the structural features are more clearly reflected if we consider an enrichment: we consider the cochains $C^*(BG) = C^*(BG;k)$ rather than the cohomology ring $H^*(BG) = \pi_*(C^*(BG))$.

In fact we take the commutative ring model $C^*(BG) = \text{map}(BG, Hk)$ (ring spectrum of maps from BG into the Eilenberg-Maclane spectrum Hk) so as to apply methods of commutative algebra.

1.2. **The spectrum of behaviour.** A massive benefit of working with cochains is that $C^*(BG)$ is Gorenstein [5, 10.3] for all finite groups G without exception. At the other extreme, following Auslander, Buchsbaum and Serre in classical commutative algebra, one may define regular local rings in a homotopy invariant way, and it turns out that $C^*(BG)$ is regular if and only if G is p-nilpotent [4, 7.3]. It is then natural to consider the spread of behaviour on the spectrum between Gorenstein and regular, and to use the singularity category to place groups along the range.

Results of [7] allow the apparatus of the singularity category to be applied for $C^*(BG)$. Some specific calculations have been made in [3, 1].

1.3. **Koszul duality.** Morita theory allows us to take a kG-module M and obtain the module $C^*(BG; M) := \text{Hom}_{kG}(k, M)$ over the Koszul dual ring $(kG)^! := \text{Hom}_{kG}(k, k) \simeq C^*(BG)$. If we attempt to return to kG-modules we obtain an action of the ring $\mathcal{E}$ of $C^*(BG)$-endomorphisms of k.

The Eilenberg-Moore spectral sequence arises from an equivalence

$$\text{Hom}_{C^*X}(k, k) \simeq C_*(\Omega X),$$

provided X is connected, p-complete and $\pi_1(X)$ is a finite p-group. Since the Bousfield-Kan p-completion $BG \longrightarrow (BG)^\wedge_p$ induces an isomorphism in $H^*(\cdot; k)$, we see

$$\mathcal{E} = C^*(BG)^! = \text{Hom}_{C^*(BG)}(k, k) \simeq C_*(\Omega(BG^\wedge_p)).$$

For brevity we write $C_*(\Omega BG_p) = C_*(\Omega(BG^\wedge_p))$ from now on. Since $C_*(\Omega BG_p) = ((kG)^!)^!$ is the double Koszul dual of kG, we have a double-centralizer completion map $kG \longrightarrow C_*(\Omega BG_p)$. This is an equivalence if G is a p-group, but generally very far from it.

1.4. **Morita equivalence.** The advantage of working with $C_*(\Omega BG_p)$ is that we do get a precise Morita equivalence [7, 9.1] between appropriate categories of $C^*(BG)$-modules and $C_*(\Omega BG_p)$-modules.

$$\mathsf{D}^b(C^*(BG)) \simeq \mathsf{D}^b(C_*(\Omega BG_p)).$$

(It is shown in [7] how to define a bounded derived category with good formal properties).

2. Results

2.1. **Classical Tate duality for finite groups.** For a finite group G we have the norm sequence

$$C_*(BG) \xrightarrow{\nu} C^*(BG) \longrightarrow \hat{C}^*(BG).$$

The only degree in which ν may be non-zero is degree zero; since G acts trivially on coefficient group k, the norm is multiplication by the group order. If p does not divide the group order then it is an isomorphism and $1 = 0$ in Tate cohomology so the Tate cohomology is zero. If p does divide the group order $\nu_* = 0$ and we have a short exact sequence

$$0 \longrightarrow H^*(BG) \longrightarrow \hat{H}^*(BG) \longrightarrow \Sigma H_*(BG) \longrightarrow 0.$$

Since homology and cohomology are dual, we obtain the Tate duality statement that the positive codegrees are dual to negative degrees with a shift.

Similarly for a compact Lie group G of dimension d with k-orientable adjoint representation, where $C_*(BG)$ is replaced by $\Sigma^d C_*(BG)$; if d is positive ν_* is automatically zero and

$$\hat{H}^*(BG) \simeq \Sigma^{d+1}(\hat{H}^*(BG))^\vee.$$

2.2. **Gorenstein duality for $C_*(\Omega BG_p)$.** The argument of [5, 8.5] shows that $C_*(\Omega BG_p)$ is also Gorenstein of shift d, and effective constructibility gives the desired conclusion.

Theorem 2.1. *If G the adjoint representation is k-orientable, then*

$$\Gamma_k C_*(\Omega BG_p) \simeq \Sigma^d C^*(\Omega BG_p).$$

2.3. **Anderson-Tate duality for the Koszul dual.** In terms of ring spectra, if R is an augmented k-algebra, we may take the cofibre sequence

$$\Gamma_k R \longrightarrow R \longrightarrow L_k R.$$

If $R = C^*(BG)$ we recover the discussion of Subsection 2.1 with $L_k C^*(BG)$ the classical Tate spectrum. We now repeat this with the Koszul dual ring $R = C_*(\Omega BG_p)$ obtaining the Ω-Tate spectrum $L_k C_*(\Omega BG_p)$. Since the ring $R = C_*(\Omega BG_p)$ again has Gorenstein duality of shift d, this is

$$\Sigma^d C^*(\Omega BG_p) \xrightarrow{\nu} C_*(\Omega BG_p) \longrightarrow L_k C_*(\Omega BG_p),$$

but now the suspension means that ν_* is potentially non-zero in degrees between 0 and d. One may use Benson's squeezed resolutions [2] to show that for finite groups ν_* is either an isomorphism (if G is p-nilpotent) or zero.

Question 2.2. If G is a compact Lie group of dimension $d > 0$ with $\pi_0(G)$ not p-nilpotent, does it follow that the map $\pi_*(\Gamma_k C_*(\Omega BG_p)) \longrightarrow \pi_*(C_*(\Omega BG_p))$ is zero?

2.4. **The singularity category as a bounded derived category.** The main result is that the singularity category is the bounded derived category of the Ω-Tate spectrum.

Theorem 2.3.

$$\mathsf{D}_{sg}(C^*(BG)) \simeq \mathsf{D}_{csg}(C_*(\Omega BG_p)) \simeq \mathsf{D}^b(L_k C_*(\Omega BG_p)).$$

2.5. **Complete intersections.** Under a complete intersection hypotheses, the duality properties for $C_*(\Omega BG_p)$ can be formulated as a local cohomology theorem, showing that $H_*(\Omega BG_p)$ is a very special ring.

To explain the assumption, recall that in the commutative algebra of Noetherian rings Gulliksen has shown that ci rings are precisely those for which the Ext algebra $\mathrm{Ext}^*_R(k,k)$ has polynomial growth. Without such an assumption, the ring $H_*(\Omega BG_p) \simeq \pi_*(\mathrm{Hom}_{C^*(BG)}(k,k))$ has no hope of good Noetherian behaviour, so it is reasonable to make a ci assumption. Under this assumption, the Ω-Tate spectrum can be constructed using a stable Koszul complex inverting central elements, giving a convenient method of calculation.

Example 2.4. Suppose Γ is a simply connected compact Lie group for which p is not a torsion prime. The classifying space of the finite Chevalley groups $G = \Gamma(q)$ for q prime to p fits into a p-adic homotopy pullback square

$$\begin{array}{ccc} BG & \longrightarrow & B\Gamma \\ \downarrow & & \downarrow \\ B\Gamma & \xrightarrow{\{1,\Psi^q\}} & B\Gamma \times B\Gamma \end{array}$$

Accordingly, there is a fibration $\Gamma \longrightarrow BG \longrightarrow B\Gamma$. Since Γ is connected and the fibre is finite, this is a normalization, and since p is not a torsion prime $H^*(\Gamma;\mathbb{F}_p)$ is exterior. This shows that G satisfies a suitable ci condition.

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Lie (super)algebras generated by reflections in finite reflection groups

Christopher M. Drupieski
(joint work with Jonathan R. Kujawa)

Given a finite reflection group W, let $\mathrm{Lie}(\mathbb{C}W)$ be the group algebra of W over the field of complex numbers, considered as a Lie algebra via the commutator bracket $[x,y] = xy - yx$, and let $\mathfrak{s}$ be the Lie subalgebra of $\mathrm{Lie}(\mathbb{C}W)$ generated by the set of reflections in W. In 2007, motivated by questions from the representation theory of the braid group and the Iwahori–Hecke algebras of type A [5, 6], Marin determined the structure of $\mathfrak{s}$ in the case $W = \mathfrak{S}_n$ of the symmetric group on n letters [7]. In this talk I reported on work with Jonathan Kujawa extending Marin's calculations to all irreducible finite reflection groups [1]. Our results currently cover all classical Coxeter types (A_r, B_r, C_r, D_r) and some exceptional types (F_4, $I_2(r)$, H_3, H_4) with substantial numerical evidence pointing toward (if not definitively establishing) specific answers in the remaining exceptional types (E_6, E_7, E_8).

In parallel to Marin's work, the structure of the Lie algebra $\mathfrak{s}$ is determined by way of the Artin–Wedderburn decomposition of the algebra $\mathbb{C}W$. The Lie algebra $\mathfrak{s}$ is reductive, with a center $Z(\mathfrak{s})$ that is either one- or two-dimensional (depending on the number of root lengths in W), and with a semisimple part $\mathfrak{s}' = [\mathfrak{s}, \mathfrak{s}]$ that is a product of factors of the forms $\mathfrak{sl}(V)$, $\mathfrak{so}(V)$, or $\mathfrak{sp}(V)$, for V ranging over a subset of $\mathrm{Irr}(G)$, the ordinary irreducible representations of W. Typically, a factor of the form $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$ in $\mathfrak{s}'$ arises from each $V \in \mathrm{Irr}(G)$ such that $V \cong V \otimes \mathrm{sgn}$, where sgn denotes the one-dimensional sign (or determinant) representation of W, while a factor of the form $\mathfrak{sl}(V)$ arises from each pair $\{V, V'\} \subset \mathrm{Irr}(W)$ such that $V \not\cong V'$ but $V \otimes \mathrm{sgn} \cong V'$. Exceptions to the last rule may occur when V is an exterior power of the reflection representation of W, or (especially in type BCD) when V arises via inflation from an irreducible representation of a lower rank reflection group, in which case the corresponding $\mathfrak{sl}(V)$ is omitted. In general, the determination of $\mathfrak{s}$ seems to require a significant understanding of the ordinary representation theory of W. Our proofs rely on a combination of inductive arguments (arguing by induction on the rank of the reflection group, and making heavy use of the explicit branching rules to restrict a representation $V \in \mathrm{Irr}(W)$ to a subgroup $W' \subset W$ of smaller rank) and explicit computer calculations in GAP [4] to help deal with some low-rank and exceptional types.

One can also consider $\mathbb{C}W$ as a Lie superalgebra, by considering the reflections in W to be of odd superdegree and by using the graded commutator bracket $[x, y] = xy - (-1)^{\overline{x} \cdot \overline{y}} yx$. One can then ask for the structure of the Lie sub-superalgebra $\mathfrak{s}$ generated by the set of reflections in W. In the last part of the talk, I summarized our work answering this question [2, 3]. In the super setting, the description of $\mathfrak{s}$ is much more uniform: one has $\mathfrak{s} = \mathfrak{D}(\mathbb{C}W) + \mathbb{C} \cdot \mathcal{C}_\ell + \mathbb{C} \cdot \mathcal{C}_s$, where $\mathfrak{D}(\mathbb{C}W)$ is the derived Lie superalgebra of $\mathbb{C}W$, and $\mathcal{C}_\ell$ and $\mathcal{C}_s$ are the class sums in $\mathbb{C}W$ of the long and short reflections, respectively (where $\mathcal{C}_s$ is taken to be zero if W has only one root length). In turn, $\mathfrak{D}(\mathbb{C}W)$ is a direct sum of Lie superalgebras of the forms $\mathfrak{sq}(W)$ and $\mathfrak{sl}(W)$ as W ranges over the simple $\mathbb{C}W$-supermodules of types Q and M, respectively. The proofs in the super setting proceed along lines roughly parallel to those from the non-super setting, but with a dominant role now played by the ordinary representation theory of the index-2 even subgroup $W_{\overline{0}} = \ker(\mathrm{sgn}) \subset W$.

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Free symmetries of products of spheres and Steenrod closed parameter ideals

Marc Stephan
(joint work with Henrik Rüping, Ergün Yalçın)

A fundamental question in the theory of transformation groups is: Which finite groups can act freely on a given product of spheres? Benson and Carlson conjectured in [BC87] that a finite group G can act freely on a finite CW-complex X of the homotopy type of a product of r spheres $S^{n_1} \times \ldots \times S^{n_r}$ if and only if G does not contain an elementary abelian subgroup of rank $r+1$. This was already known for $r = 1$ by the work of Swan [Swa60]. For $r = 2$, the rank condition on the group is necessary, but it is open whether it is sufficient. By the works of Adem and Smith [AS01] and of Jackson [Jac07], the case $r = 2$ is reduced to constructing free actions for groups that involve $\mathrm{Qd}(p) = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathrm{SL}_2(p)$ as a subquotient for an odd prime p. In particular, it is open whether $\mathrm{Qd}(p)$ can act freely on a finite CW-complex $X \simeq S^n \times S^m$ for p odd. For $p = 2$, the group $\mathrm{Qd}(2)$ is isomorphic to the symmetric group S_4, and free actions on a finite CW-complex $X \simeq S^n \times S^m$ are known to exist for some n, m. It is interesting to determine the possible dimensions of the spheres. Oliver [Oli79] showed that the case $m = n$ is not possible. In fact, already the alternating group $A_4 \subset S_4$ cannot act freely on a product of equidimensional spheres and Blaszczyk asked explicitly for the possible dimensions m, n for free A_4-actions.

Steenrod operations provide strong restrictions in this setting. Let X be a finite, free A_4-CW-complex of the homotopy type of a product of spheres $S^n \times S^m$ for some $n, m \geq 1$. Then the induced homomorphism $H^*(BA_4;\mathbb{F}_2) \to H^*(X/A_4;\mathbb{F}_2)$ is surjective. Its kernel I is a homogeneous ideal generated by a regular sequence of two parameters of degrees $n+1$, $m+1$, and the ideal I is closed under Steenrod operations. Such ideals can be classified. The description uses the calculation

$$H^*(BA_4;\mathbb{F}_2) \cong \mathbb{F}_2[a,b]^{C_3} \cong \mathbb{F}_2[u,v,w]/\langle u^3 + v^2 + vw + w^2 \rangle$$

with $u = a^2 + ab + b^2$, $v = a^2b + ab^2$, and $w = a^3 + a^2b + b^3$.

Theorem 1 ([RSY25, Theorem 1.2]). *The set of Steenrod closed parameter ideals in $H^*(BA_4;\mathbb{F}_2)$ consists of*

(1) *the* fibered ideals $\langle v^k, u^l\rangle$ *with* $l \geq 1$ *and* $1 \leq k \leq 2^t$*, where* 2^t *is the largest power of* 2 *dividing* l*;*
(2) *the* twisted ideals $\langle x_n, \mathrm{Sq}^1(x_n)\rangle$ *for* $n \geq 2$*, where* x_n *is recursively defined as* $x_1 = u$ *and* $x_{n+1} = ux_n^2 + \mathrm{Sq}^1(x_n)^2$*;*
(3) *and* the mixed ideals $\langle v^i x_n^{2^m}, x_{n+1}^{2^{m-1}}\rangle$*, where* $m \geq 1$*, and either* $n = 1$ *and* $1 \leq i < 2^{m-1}$*, or* $n \geq 2$ *and* $0 \leq i < 2^{m-1}$*.*

Moreover, for a given pair of natural numbers there is at most one Steenrod closed parameter ideal with parameters of these degrees.

Corollary 2. *If* A_4 *acts freely on a finite CW-complex* $X \simeq S^n \times S^m$*, then its equivariant Borel cohomology* $H^*(X/A_4; \mathbb{F}_2)$ *only depends on* n *and* m*.*

Examining the degrees of the parameters in Theorem 1 eliminates almost all possible dimensions m, n in the following sense.

Corollary 3. *For* $r \geq 1$*, the percentage of those pairs* (n, m) *with* $n, m \leq r$ *for which there exists a finite, free* A_4*-CW-complex* $X \simeq S^n \times S^m$ *tends to zero as* $r \to \infty$*.*

Remark 4. *Corollary 3 holds more generally. It suffices that* X *is a finite* A_4*-CW-complex with four-dimensional mod* 2 *cohomology such that the restricted action by* $\mathbb{Z}/2 \times \mathbb{Z}/2$ *is free; see* [RS25].

Classifying Steenrod closed parameter ideals is of independent interest for invariant rings in general. For example, for the invariant ring $\mathbb{F}_2[a, b]^{\mathrm{SL}_2(2)} \cong \mathbb{F}_2[u, v]$, we obtain the same list as in Theorem 1.

Working towards restrictions for free $\mathrm{Qd}(p)$-actions for odd primes p, Rüping and I have classified the Steenrod closed parameter ideals in the invariant ring $\mathbb{F}_p[a, b]^{\mathrm{SL}_2(p)}$ for $p = 3$. This is a polynomial ring in the Dickson invariant $u = \sum_{i=0}^{p}(a^{p-1})^{p-i}(b^{p-1})^i$ and $v = ab^p - ba^p$; see [Wil83].

Theorem 5. *Let* $p = 3$*. The set of Steenrod closed parameter ideals in the invariant ring* $\mathbb{F}_p[a, b]^{\mathrm{SL}_2(p)} = \mathbb{F}_p[u, v]$ *consists of*

(1) *the fibered ideals* $\langle u^l, v^k\rangle$ *with* $l \geq 1$ *and* $1 \leq k \leq (p-1) \cdot p^t$ *where* p^t *is the largest power of* p *dividing* l*;*
(2) *the twisted ideals* $\langle x_n, P^1(x_n)\rangle$ *for* $n \geq 2$*, where* x_n *is recursively defined by* $x_1 = u$ *and* $x_{n+1} = ux_n^p - P^1(x_n)^p$*;*
(3) *the mixed ideals* $\langle v^i x_n^{p^m}, x_{n+1}^{p^{m-1}}\rangle$*, where* $m \geq 1$ *and, either* $n = 1$ *and* $0 < i < (p-1) \cdot p^{m-1}$*, or* $n \geq 2$ *and* $0 \leq i < (p-1) \cdot p^{m-1}$*.*

We conjecture that Theorem 5 extends to every prime p.

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The Green biset functor of complex characters

Serge Bouc

Let $\mathcal{C}$ denote the biset category of finite groups (see [1, Chapter 3]), where the objects are finite groups, and for finite groups G and H, the hom set $\mathrm{Hom}_{\mathcal{C}}(G,H)$ is the Grothendieck group $B(H,G)$ of (finite) (H,G)-*bisets*. The direct product of finite groups endows $\mathcal{C}$ with a *symmetric monoidal* structure, for which the identity object is the trivial group $\mathbf{1}$.

By Day's convolution, the category $\mathcal{F}$ of *biset functors* (i.e. additive functors from $\mathcal{C}$ to abelian groups) becomes symmetric monoidal: Two biset functors M and N admit a tensor product $M \otimes N$ in $\mathcal{F}$. This product is commutative and associative. The *Burnside functor* B, i.e. the representable functor $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, -)$, is the identity object for the tensor product in $\mathcal{F}$.

A *Green biset functor* A is by definition a *monoid* in $\mathcal{F}$: It is a biset functor, equipped with a product morphism $\mu_A : A \otimes A \to A$ and an identity morphism $e_A : B \to A$ in $\mathcal{F}$, which are associative and left-right-unital in the obvious sense. Equivalently, for any finite groups G and H, there is an external product

$$\begin{aligned} A(G) \times A(H) &\xrightarrow{\times} A(G \times H) \\ (\alpha, \beta) &\longmapsto \alpha \times \beta \end{aligned}$$

which is bilinear, bifunctorial, associative, and admits a left-right-identity element $\varepsilon_A \in A(\mathbf{1})$. Examples of Green biset functors are:

1. *The Burnside Green biset functor B*: For finite groups G and H, the external product $B(G) \times B(H) \to B(G \times H)$ comes from the product sending a G-set X and an H-set Y to the $(G \times H)$-set $X \times Y$, and the identity element ε_B is the one element set $\bullet \in B(\mathbf{1})$.

2. *The Green biset functor of complex characters $R_{\mathbb{C}}$*: It sends a finite group G to its group $R_{\mathbb{C}}(G)$ of complex characters: For finite groups G and H, the external product $R_{\mathbb{C}}(G) \times R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G \times H)$ is the external product of characters, and the identity element $\varepsilon_{R_{\mathbb{C}}}$ is the trivial character of the trivial group.

For a Green biset functor A, a *left A-module* M is a biset functor endowed with a morphism $A \otimes M \to M$, which is associative and left-unital. Equivalently, for any finite groups G and H, there are bilinear bifunctorial maps

$$\begin{aligned} A(G) \times M(H) &\xrightarrow{\times} M(G \times H) \\ (\alpha, m) &\longmapsto \alpha \times m \end{aligned}$$

which are associative and left-unital. One defines right A-modules similarly, as well as (A, C)-bimodules, for Green biset functors A and C. There is a natural notion of opposite Green functor C^{op} of a Green biset functor C, and a right C-module is nothing but a left C^{op}-module. The functor C is called *commutative*[1] if $C = C^{\mathrm{op}}$. The tensor product of two Green biset functors is a Green biset functor, and an (A, C)-bimodule is nothing but an $(A \otimes C^{\mathrm{op}})$-module.

A morphism of Green biset functors $\varphi : A \to C$ is a morphism in $\mathcal{F}$ such that $\varphi_{\mathbf{1}}(\varepsilon_A) = \varepsilon_C$ and $\varphi_G(\alpha) \times \varphi_H(\beta) = \varphi_{G\times H}(\alpha \times \beta)$, for any finite groups G and H and any $(\alpha, \beta) \in A(G) \times A(H)$. With this definition, Green biset functors form a category GBF. The Burnside functor B is initial in GBF.

If A is a Green biset functor, a morphism of (left) A-modules $\psi : M \to N$ is a morphism in $\mathcal{F}$ such that $\alpha \times \psi_H(m) = \psi_{G\times H}(\alpha \times m)$, for any finite groups G and H and any $(\alpha, m) \in A(G) \times M(H)$. With this definition, left A-modules form an abelian category A-Mod (for example B-$\mathsf{Mod} = \mathcal{F}$). One defines similarly categories of right modules and bimodules over Green biset functors.

Notation.

- *For positive integers n and m with $n|m$, let $\pi_{m,n}$ be the projection map $(\mathbb{Z}/m\mathbb{Z})^\times \to (\mathbb{Z}/n\mathbb{Z})^\times$. Let $\hat{\mathbb{Z}}^\times = \varprojlim_{n>0} (\mathbb{Z}/n\mathbb{Z})^\times$, that is*
$$\hat{\mathbb{Z}}^\times = \big\{ s = (s_n)_{n>0} \mid s_n \in (\mathbb{Z}/n\mathbb{Z})^\times,\ n \mid m \Rightarrow \pi_{m,n}(s_m) = s_n \big\}.$$
- *For a finite group G, and $s \in \hat{\mathbb{Z}}^\times$, let $\Psi^s_G : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G)$ be defined by $\Psi^s_G(\chi)(g) := \chi(g^{\tilde{s}_n})$ for $g \in G$, where $n \equiv 0\ (\exp(G))$, and $\tilde{s}_n \in \mathbb{Z}$ lifts $s_n \in \mathbb{Z}/n\mathbb{Z}$. Simply set ${}^s\chi := \Psi^s_G(\chi)$ when G is understood.*

It is straightforward to check that the maps Ψ^s_G, for finite groups G, form a well defined automorphism Ψ^s of the Green biset functor $R_{\mathbb{C}}$. In fact:

Theorem 1.

(1) $\mathrm{End}_{\mathsf{GBF}}(R_{\mathbb{C}}) = \mathrm{Aut}_{\mathsf{GBF}}(R_{\mathbb{C}})$.
(2) *The map $s \in \hat{\mathbb{Z}}^\times \mapsto \Psi^s \in \mathrm{Aut}_{\mathsf{GBF}}(R_{\mathbb{C}})$ is a group isomorphism.*

Proof (sketch): The proof of Assertion 2 proceeds by reduction to cyclic groups, using Brauer induction theorem. The proof of Assertion 1 uses the following:

[1] The two above examples B and $R_{\mathbb{C}}$ are commutative.

Theorem 2 (B. - Romero [2, Theorem 4.1]). *Let A be a Green biset functor. The following are equivalent:*

(1) *The A-module A is a generator*[2] *of A-*Mod.
(2) *For all finite groups G and H, the map $\times$ induces an isomorphism from $A(G) \otimes_{A(\mathbf{1})} A(H)$ to $A(G \times H)$.*
(3) *$M \in A\text{-}\mathsf{Mod} \mapsto M(\mathbf{1}) \in A(\mathbf{1})\text{-}\mathsf{Mod}$ is an equivalence of categories*[3]. □

A consequence of Theorem 2 is that evaluation at $\mathbf{1}$ is an equivalence of categories from $R_{\mathbb{C}}$-Mod to the category of abelian groups. Now in order to consider $(R_{\mathbb{C}}, R_{\mathbb{C}})$-bimodules, we introduce the following;

Notation.

- *We denote by Γ the ring $(R_{\mathbb{C}} \otimes R_{\mathbb{C}})(\mathbf{1})$.*
- *We denote by $\mathcal{L}oc(\hat{\mathbb{Z}}^\times, \mathbb{Z})$ the ring of locally constant maps $\hat{\mathbb{Z}}^\times \to \mathbb{Z}$.*

Recall ([1, Section 8.4.8]) that $\Gamma \cong \bigoplus_{X \in [\mathsf{GR}]} \big(R_{\mathbb{C}}(X) \otimes_{\mathbb{Z}} R_{\mathbb{C}}(X)\big)/\mathcal{R}$, where [GR] is a set of representatives of isomorphism classes of finite groups, and $\mathcal{R}$ is generated by all differences $\lfloor \varphi(m) \otimes n \rfloor_Y - \lfloor m \otimes \varphi^{\mathrm{op}}(n) \rfloor_X$, for $X, Y \in [\mathsf{GR}]$, $m \in M(X)$, $n \in N(Y)$, and $\varphi \in B(Y, X)$ (where $\varphi^{\mathrm{op}} \in B(X, Y)$ is the opposite of φ, and $\lfloor \varphi(m) \otimes n \rfloor_Y$ is in the component of the above direct sum indexed by Y).

Theorem 3.

(1) *There is a well defined ring isomorphim $\Xi : \Gamma \xrightarrow{\cong} \mathcal{L}oc(\hat{\mathbb{Z}}^\times, \mathbb{Z})$ sending $\lfloor \alpha \otimes \beta \rfloor_X \in \Gamma$ to the map $\big(s \in \hat{\mathbb{Z}}^\times \mapsto \langle \alpha, {}^s\beta \rangle_X \in \mathbb{Z}\big)$.*
(2) *Evaluation at $\mathbf{1}$ is an equivalence from the category of $(R_{\mathbb{C}}, R_{\mathbb{C}})$-bimodules to Γ-*Mod.

Definition 4 (Romero [3, Proposition 4.2]). *Let A be a Green biset functor. For $1 \le i < n$, let $d_n^i : A^{\otimes n} \to A^{\otimes(n-1)}$ be the map $Id_A^{\otimes(i-1)} \otimes \mu_A \otimes Id_A^{\otimes(n-i-1)}$, and $d_n := \sum_{i=1}^{n-1} (-1)^{i-1} d_n^i$. Then*

$$\dots \xrightarrow{d_{n+1}} A^{\otimes n} \xrightarrow{d_n} A^{\otimes(n-1)} \longrightarrow \dots \xrightarrow{d_3} A^{\otimes 2} \xrightarrow{\mu_A} A \longrightarrow 0$$

is an exact complex of (A, A)-bimodules, called the bar complex *of A. The truncated complex*

$$\mathcal{B}_A =: \ \dots \xrightarrow{d_{n+1}} A^{\otimes n} \xrightarrow{d_n} A^{\otimes(n-1)} \longrightarrow \dots \xrightarrow{d_3} A^{\otimes 2} \longrightarrow 0$$

is called the bar resolution *of A.*

For a biset functor M and a finite group X, the *shifted* biset functor M_X is the functor obtained by precomposition with the functor $(- \times X)$ in $\mathcal{C}$. If M is an A-module, then M_X is again an A-module. For two A-modules M and N, the biset functor $\mathcal{H}_A(M, N)$ is defined by $\mathcal{H}_A(M, N)(X) = \mathrm{Hom}_{A\text{-}\mathsf{Mod}}(M, N_X)$.

[2] Recall that an object L of an abelian category M is a *generator* of M if any object of M is a quotient of a direct sum of copies of L.

[3] A quasi-inverse is $V \mapsto \big(G \mapsto A(G) \otimes_{A(\mathbf{1})} V\big)$.

Definition 5 (Romero [3, Definition 4.3]). *Let A be a Green biset functor, and set $A^e = A \otimes A^{\mathrm{op}}$. Let M be an (A, A)-bimodule. The cohomology objects of the (cochain) complex $\mathcal{H}_{A^e}(\mathcal{B}_A, M)$ are called the* Hochschild cohomology (biset) functors *of A with values in M, and denoted by $\mathcal{H}H^n(A, M)$.*

Using the equivalence $(R_{\mathbb{C}})^e$-Mod $\cong \Gamma$-Mod of Theorem 3, we now get:

Theorem 6.

(1) *Let $n \in \mathbb{N}$ with $n \geq 2$. Then $R_{\mathbb{C}}^{\otimes n}(\mathbf{1})$ is isomorphic to $\Gamma^{\otimes(n-1)}$ as Γ-module. Moreover $\Gamma^{\otimes(n-1)}$ identifies with $\mathcal{L}oc\big((\hat{\mathbb{Z}}^\times)^{n-1}, \mathbb{Z}\big)$, the module of locally constant maps from $(\hat{\mathbb{Z}}^\times)^{n-1}$ to $\mathbb{Z}$.*

(2) *The bar complex of $R_{\mathbb{C}}$, evaluated at $\mathbf{1}$, is isomorphic to the complex of (right) Γ-modules*

$$(*):\ \dots \xrightarrow{\delta_{n+1}} \Gamma^{\otimes n} \xrightarrow{\delta_n} \Gamma^{\otimes(n-1)} \longrightarrow \dots \xrightarrow{\delta_2} \Gamma \xrightarrow{\delta_1} \mathbb{Z} \longrightarrow 0,$$

where $\mathbb{Z}$ is a Γ-module via $\gamma \cdot 1 = \gamma(1)$, for $\gamma \in \Gamma$. Here for $n \geq 2$, $\delta_n(\gamma_1 \otimes \ldots \otimes \gamma_n) = \gamma_1(1)(\gamma_2 \otimes \ldots \otimes \gamma_n) - (\gamma_1\gamma_2 \otimes \gamma_3 \otimes \ldots \otimes \gamma_n) + \ldots + (-1)^{n-1}(\gamma_1 \otimes \ldots \otimes \gamma_{n-1}\gamma_n)$, and $\delta_1(\gamma) = \gamma(1)$.

(3) *The ring Γ is a* Hopf algebra, *with coproduct $\Delta : \Gamma \to \Gamma^{\otimes 2} \cong \mathcal{L}oc\big((\hat{\mathbb{Z}}^\times)^2, \mathbb{Z}\big)$ given by $\Delta(\gamma)(s,t) = \gamma(st)$, for $\gamma \in \Gamma$, counit $\delta_1 : \gamma \mapsto \gamma(1)$, and antipode $(S\gamma)(s) = \gamma(s^{-1})$. Moreover the complex (*) is a projective resolution of $\mathbb{Z}$ as Γ-module, so*

$$\forall n \geq 0,\ \mathcal{H}H^n(R_{\mathbb{C}}, M)(\mathbf{1}) \cong \mathrm{Ext}^n_\Gamma\big(\mathbb{Z}, M(\mathbf{1})\big).$$

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Four-manifolds, two-complexes and Tate cohomology

Ian Hambleton

(joint work with John Nicholson)

This is a report on my talk in the workshop, based on joint work [5] with John Nicholson (University of Glasgow).

We are interested in finding closed smooth 4-manifolds M, M' such that M and M' are not homotopy equivalent ($M \not\simeq M'$), but

$$M \# r(S^2 \times S^2) \cong M' \# r(S^2 \times S^2)$$

are diffeomorphic, for some $r \geq 0$. In this case, we say that M and M' are stably diffeomorphic.

The existence of exotic smooth structures shows that simply-connected oriented 4-manifolds which are stably diffeomorphic need not be diffeomorphic, but it follows from results of Donaldson [2] and Wall [9] that such 4-manifolds are h-cobordant and hence homotopy equivalent.

Possible examples arise from a "thickening" construction $X \mapsto M(X)$, where X is a finite 2-complex and $M(X) = \partial N(X)$ is the boundary of a smooth compact 5-manifold $N(X) \subset \mathbb{R}^5$, which is a thickening of an embedding $X \hookrightarrow \mathbb{R}^5$.

Question. *If $X \simeq Y$, then $M(X) \simeq M(Y)$, but what happens if $X \not\simeq Y$?*

If $\chi(X) = \chi(Y)$ and $\pi_1(X) \cong \pi_1(Y)$, it follows from work of J. H. C. Whitehead [10] that

$$X \vee r(S^2) \simeq Y \vee r(S^2)$$

are (simple) homotopy equivalent, for some $r \geq 0$, and hence $M(X)$ and $M(Y)$ are stably diffeomorphic. Kreck's modified surgery [7] gives techniques to classify 4-manifolds up to stable diffeomorphism, and these methods have been applied to study manifolds over a range of fundamental groups [4, 3, 6].

Kreck and Schafer produced the first examples of stably diffeomorphic closed smooth 4-manifolds $M(X)$ and $M(Y)$ which are not homotopy equivalent. These are thickenings of pairs X, Y of finite 2-complexes such that $X \not\simeq Y$, and $\pi_1(X) \cong \pi_1(Y)$ with fundamental groups restricted to certain elementary abelian p-groups with $p \equiv 1 \pmod 4$, for example $G = (\mathbb{Z}/5)^3$.

By extending their methods, we formulate a new homotopy invariant, called the *quadratic bias invariant*, on the class of 4-manifolds arising as thickenings of 2-complexes with finite fundamental group. As an application we show that, for any $k \geq 2$, there exist a family of k closed smooth 4-manifolds which are all stably diffeomorphic but are pairwise not homotopy equivalent.

1. The bias invariant

Let G be a finite group and let $X = X(\mathcal{P})$ be a 2-complex arising from a finite presentation of $\pi_1(X) \cong G$. Assume that $\mathcal{P}$ is an *efficient* presentation, meaning that $\chi(X) = 1 + d(H_2(G;\mathbb{Z}))$, where $d(H_2(G;\mathbb{Z}))$ denotes the minimal number of generators of

$$H_2(G;\mathbb{Z}) \cong \mathbb{Z}/m_1 \oplus \cdots \oplus \mathbb{Z}/m_r$$

where $m_i \mid m_{i+1}$ for all $i \geq 1$, and set $m_G = m_1$. It is known that finite abelian groups are efficient, but Swan constructed a group of the form $G \cong (\mathbb{Z}/7)^3 \rtimes \mathbb{Z}/3$ which is not efficient.

Let $\mathrm{HT}_{min}(G)$ denote the set of homotopy types of finite 2-complexes X with $\pi_1(X) = G$, and minimal possible Euler characteristic. The *bias invariant* $\beta(X)$ defined by Metzler [8] gives a map

$$\beta\colon \mathrm{HT}_{min}(G) \to B(G) := (\mathbb{Z}/m)^\times/\langle \pm D(G)\rangle$$

to an abelian group $B(G)$, where $m = m_G$ and $D(G)$ is the image of a certain map $\varphi\colon \mathrm{Aut}(G) \to (\mathbb{Z}/m)^\times/\{\pm 1\}$.

The bias invariant is an obstruction to the existence of a chain map $h\colon C(\widetilde{X}) \to C(\widetilde{Y})$ over the universal coverings, which induces an isomorphism $h_*\colon H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ on homology.

It follows from work of Metzler [8] and Browning [1] that the bias invariant completely classifies minimal 2-complexes up to homotopy equivalence for any finite abelian group G.

2. The quadratic bias invariant and Tate cohomology

The quadratic bias invariant $\beta_Q\colon \mathscr{M}_4(G) \to B_Q(G)$ is defined for a thickening $M(X)$ by an invariant in the isometry group of the *hyperbolic* Tate form

$$\widehat{e}_G\colon ((\widehat{L})^* \oplus \widehat{L}) \times ((\widehat{L})^* \oplus \widehat{L}) \to \mathbb{Z}/|G|$$

associated to the equivariant intersection form $s_{M(X)}$ on $\pi_2(M(X))$.

To define the Tate form, note that $s_{M(X)}$ is a metabolic form with respect to the decomposition $\pi_2(M(X)) \cong \pi_2(X)^* \oplus \pi_2(X)$. We apply Tate cohomology $\widehat{H}^0(G;-)$ to $s_{M(X)}$, and use the canonical identification $(L^*)\hat{}\oplus \widehat{L} \cong (\widehat{L})^* \oplus \widehat{L}$, where $\widehat{L} = \widehat{H}^0(G;\pi_2(X))$.

The module $L = \pi_2(X)$ has the property that $\widehat{H}^0(G;L) \cong H_2(G;\mathbb{Z})$ by dimension shifting. If $H_2(G;\mathbb{Z})$ has a certain special form we can explicitly compute the quadratic bias obstruction group $B_Q(G)$.

Theorem A ([5])**.** *Let G be a finite group such that $H_2(G;\mathbb{Z}) \cong (\mathbb{Z}/m)^d$ for some $m \geq 1$, $d \geq 3$. If G is efficient, then there is an isomorphism*

$$B_Q(G) \cong \frac{(\mathbb{Z}/m)^\times}{\pm(\mathbb{Z}/m)^{\times 2} \cdot D(G)}$$

where $D(G) = \mathrm{im}(\varphi_G\colon \mathrm{Aut}(G) \to (\mathbb{Z}/m)^\times/\{\pm 1\})$.

Remark 2.1. We would expect the structure of $B_Q(G)$ to be much more complicated in general. Its definition involves new ideas related to the unitary isometries of multi-scaled hyperbolic forms arising from the decomposition of $H_2(G;\mathbb{Z})$ into cyclic factors.

Theorem B ([5])**.** *For each $k \geq 2$, there exist closed smooth 4-manifolds $M_1, M_2, \cdots, M_k$ which are all stably diffeomorphic but not pairwise homotopy equivalent.*

The quadratic bias invariant also leads to examples over non-abelian fundamental groups.

Theorem C ([5])**.** *Let $G = Q_8 \times (\mathbb{Z}/p)^3$ where p is a prime such that $p \equiv 1 \mod 8$. Then:*

1. *There exist minimal finite 2-complexes X, Y with fundamental group G which are homotopically distinct.*
2. *There exist closed smooth 4-manifolds M, N with fundamental group G which are stably diffeomorphic but not homotopy equivalent.*

A number of interesting questions remain concerning the doubles $M(X)$. If $X \simeq Y$, then $M(X)$ and $M(Y)$ are h-cobordant. More generally, we ask:

Question 2.2. *If $X \simeq Y$, then are $M(X)$ and $M(Y)$ diffeomorphic?*

An important special case is when X is a point and Y is the presentation 2-complex of a potential counterexample to the Andrews-Curtis conjecture. In this case we have $X \simeq Y$, $M(X) = S^4$, $M(Y) \simeq S^4$ and so 2.2 is equivalent to the question of whether $M(Y)$ is an exotic 4-sphere.

See [5] for full details about the construction of the quadratic bias invariant, and how to calculate $B_Q(G)$ via isometries of hermitian forms. In the paper, we also generalize the invariant to obtain similar examples in all dimensions $4n \geq 4$.

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