Mathematisches Forschungsinstitut Oberwolfach

Report No. 15/2026

DOI: 10.4171/OWR/2026/15

# Complex Analysis

Organized by
Eleonora Di Nezza, Roma
Stefan Kebekus, Freiburg
Mihai Păun, Bayreuth
Stefan Schreieder, Hannover

5 April – 10 April 2026

Abstract. The workshop was devoted to analytic and algebraic methods in the theory of Kähler spaces and to their interactions with neighboring areas. The main themes included the interaction of complex geometry with group theory and hyperbolicity, striking recent progress in birational geometry and classification theory, Calabi–Yau and hyperkähler geometry, and analytic aspects of singular Kähler metrics. The program featured nineteen lectures and a session of short presentations by young participants, and emphasized the particularly fruitful exchange between analytic, algebraic, differential-geometric, and metric methods.

*Mathematics Subject Classification (2020):* 14E30, 14J32, 14J40, 14J42, 14J45, 20F65, 20F67, 32J25, 32Q15, 32Q20, 32Q26, 32Q45, 32Q55, 53C55.



## Introduction by the Organizers

The workshop *Komplexe Analysis – Analytic and Algebraic Methods in the Theory of Kähler Spaces*, organised by Eleonora Di Nezza (Rome), Stefan Kebekus (Freiburg), Mihai Păun (Bayreuth), and Stefan Schreieder (Hannover), was held from April 6 to April 10, 2026. It brought together 43 participants, ranging from early career researchers to senior experts. The program consisted of 19 full lectures and, on Thursday evening, a session of five short presentations by young participants. As in previous editions, the schedule deliberately left substantial room for informal discussions and collaborative work in smaller groups.

The workshop covered a broad spectrum of current developments in complex analysis, Kähler geometry, algebraic geometry, and their interactions with neighboring fields. We mention below a few representative talks which illustrate the diversity of techniques and importance of recent developments in the field; the mentioned talks are not exhaustive and the extended abstracts give a more complete account of the individual contributions.

**Group theory, complex hyperbolicity, and complex-analytic methods.** A first theme concerned the interaction of complex geometry with group theory and hyperbolicity questions. The opening lecture by Philippe Eyssidieux discussed algebro-geometric subgroups of the mapping class group, namely images of fundamental groups of curves under complex algebraic maps to the moduli space of smooth curves. Claudio Llosa–Isenrich gave a beautiful lecture explaining how methods from complex analysis, and in particular from complex Morse theory, can be used to settle old conjectures and questions in geometric group theory. This talk also served as a gentle introduction to geometric group theory and the theory of finiteness properties of Kähler groups. Benoît Cadorel's talk about hyperbolicity, specialness and fundamental groups, added another perspective on this important circle of ideas.

**Birational geometry and classification.** A second major theme was birational geometry and classification theory. Ben Bakker gave an online lecture on a spectacular recent joint work with S. Filipazzi, M. Mauri, and J. Tsimerman. They construct Baily–Borel compactifications of period images, using techniques from o-minimality, and apply this construction to prove the $b$-semi-ampleness conjecture in birational geometry. Niklas Müller reported on exciting classification results for minimal varieties for which equality holds in the Miyaoka–Yau inequality. A remarkable feature of his work is that, along the way, he proves the abundance conjecture for all these varieties.

Cécile Gachet's lecture on the Morrison–Kawamata cone conjecture presented recent progress on one of the fundamental problems concerning the structure of cones of divisors and curves on Calabi–Yau varieties, and its relation to automorphism groups and birational transformations, with direct links to finiteness questions for minimal models of such varieties. Sung Gi Park gave a very clear and engaging talk explaining how invariants from singularity theory and $D$-module theory can be used to study GIT-stability questions for hypersurfaces in projective spaces. In particular, he discussed the minimal exponent, which can be read off from the Bernstein–Sato polynomial of the defining equation of the hypersurface, and explained how it characterizes stable and semi-stable hypersurfaces in the classical cases. Philip Engel's lecture on the boundedness of fibered hyperkähler varieties and fibered Calabi–Yau threefolds provided a further highlight within this broad classification-theoretic part of the program.

**Calabi–Yau geometries.** A third central theme was formed by symplectic geometry, hyperkähler geometry, and Calabi–Yau varieties. Jun-Muk Hwang's lecture was centered around elliptically fibered surfaces. Any two such fibrations over the disc are isomorphic, as long as the fibrations are smooth and of maximal variation. In contrast, Hwang explained that under the assumption that the total space carries a symplectic form, the local structure of an elliptic fibration, together with the symplectic form, frequently determines its global geometry. Christian Lehn discussed the non-vanishing conjecture for hyperkähler varieties, with particular emphasis on the case of fourfolds. His lecture connected the birational geometry of hyperkähler manifolds with some of the guiding conjectures of the minimal model program.

Henri Guenancia's lecture was another highlight of the workshop and contributed to the analytic side of the discussion around canonical metrics and Calabi–Yau spaces. More precisely, Guenancia studied the case of log smooth Calabi–Yau pairs $(X, D)$, where $D$ is either smooth or has two components that are proportional in the Néron–Severi group. In this case, $X \setminus D$ is a non-compact Calabi–Yau manifold and Guenancia discussed to which extent classical results and consequences of Yau's celebrated work on the Calabi conjecture, that are well-known for compact Calabi–Yau manifolds, still hold true in this non-compact setting.

**Singular Kähler metrics, curvature bounds, and metric regularity.** A fourth theme concerned analytic and metric aspects of singular Kähler geometry. Hans-Joachim Hein presented a work of his student Klemmensen, who proved a Liouville theorem for constant scalar curvature Kähler metrics on simply-connected Ricci-flat conical Kähler manifolds. In addition to a very clear presentation of Klemmensen's result, Hein's talk served as a beautiful introduction to the field of special metrics on Kähler manifolds.

Yifan Chen reported on recent work showing that a general class of singular Kähler metrics with Ricci curvature bounded from below are actually Kähler currents. This applies in particular to singular Kähler–Einstein metrics on klt pairs, with analogous results for Kähler–Ricci solitons, and also gives criteria under which such metrics define metric-measure spaces satisfying the Riemannian curvature-dimension condition.

**Short talks by young participants.** On Thursday evening, five young participants presented their current work in short ten-minute talks.

- Matthias Paulsen (Paris): Verbal ideals and unobstructed complex parallelisable nilmanifolds.
- Chung-Ming Pan (Montreal): Gauduchon metrics and Hermite–Einstein metrics on non-Kähler varieties.
- William Sarem (Grenoble): Locally symmetric spaces and Stein manifolds.
- Johannes Scheffler (Bayreuth): Auxiliary Hessian equations.
- Simon Pietig (Hannover): Holomorphic one-forms without zeros on Kähler threefolds.

The session provided an excellent opportunity for younger participants to introduce themselves and their research, and it contributed substantially to the lively and open the atmosphere of the meeting.

# Workshop: Complex Analysis

## Table of Contents

Philippe Eyssidieux (joint with Marco Boggi, Louis Funar)
*Algebrogeometric subgroups of mapping class groups* . . . . . . . . . . . . . . . . . . 7

Jun-Muk Hwang (joint with Guolei Zhong)
*Holomorphic symplectic geometry of elliptic surfaces* . . . . . . . . . . . . . . . . . 8

Donu Arapura (joint with Deepam Patel, Botong Wang)
*Euler characteristics of nonpositively curved varieties* . . . . . . . . . . . . . . . . 9

Hans-Joachim Hein
*A Liouville theorem and $C^\alpha$ estimate on Calabi-Yau cones* . . . . . . . . . . . . 12

Claudio Llosa Isenrich (joint with Pierre Py)
*Groups with exotic finiteness properties from complex Morse theory* . . . . 15

Niklas Müller (joint with Masataka Iwai, Shin-ichi Matsumura)
*Inequalities of Miyaoka-Yau-type and Uniformisation of varieties of intermediate Kodaira dimension* . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 18

Benjamin Bakker (joint with Stefano Filipazzi, Mirko Mauri, Jacob Tsimerman)
*Baily–Borel compactifications and b-semiampleness* . . . . . . . . . . . . . . . . . . 21

Botong Wang (joint with June Huh, Mateusz Michałek, and Jian Xiao)
*Volume Inequalities and Volume Polynomials* . . . . . . . . . . . . . . . . . . . . . . . 24

Cécile Gachet (joint with Hsueh-Yung Lin, Isabel Stenger, Long Wang)
*Equivariant descent for the Kawamata–Morrison cone conjecture* . . . . . . 26

Victoria Hoskins (joint with Eloise Hamilton, Joshua Jackson and Tanguy Vernet)
*Relative non-reductive geometric invariant theory and moduli of quivers with multiplicities* . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 29

Christian Lehn (joint with Andreas Höring and Vladimir Lazić)
*Nonvanishing results for compact hyperkähler manifolds* . . . . . . . . . . . . . . 32

Sung Gi Park
*Moduli of hypersurfaces via minimal exponent* . . . . . . . . . . . . . . . . . . . . . . . 33

Henri Guenancia (joint with Tristan Collins)
*On log Calabi-Yau manifolds* . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 35

Benoît Cadorel (joint with Ya Deng, Katsutoshi Yamanoi)
*Hyperbolicity, specialness and fundamental groups* . . . . . . . . . . . . . . . . . . . 39

Philip Engel (joint with Stefano Filipazzi, François Greer, Mirko Mauri, Roberto Svaldi)
*Boundedness of Lagrangian fibrations* . . . . . . . . . . 42

Shih-Kai Chiu (joint with Yang Li, Yu-Shen Lin)
*From tropical curves to special Lagrangians* . . . . . . . . . . 46

Yifan Chen (joint with Shih-Kai Chiu, Max Hallgren, Gábor Székelyhidi, Tat Dat Tô, Freid Tong)
*Geometric behavior of singular Kähler–Einstein metric on klt space* . . . . 48

Mihnea Popa (joint with Andrew Burke, Wanchun Shen)
*Negative $K$-theory and Hodge theory* . . . . . . . . . . 52

Bruno Klingler
*Tautological rings: from $\mathbb{C}$VHS to algebraic cycles* . . . . . . . . . . 55

# Abstracts

## Algebrogeometric subgroups of mapping class groups

Philippe Eyssidieux
(joint work with Marco Boggi, Louis Funar)

The talk was a report on a joint work with Louis Funar [3, 4] and a work in progress with him and Marco Boggi.

Let $g, n \in \mathbb{N}$ be positive integers and assume $2g - 2 + n > 0$. Let $\Sigma_g^n$ be a closed oriented surface of genus $g$ with $n$ marked points. The pure mapping class group $PMod(\Sigma_g^n)$ is the group of connected components of the group of orientation preserving homeomorphisms of $\Sigma_g^n$ fixing the $n$ marked points.

For $c$ an essential closed curve in $\Sigma_g^n$, denote by $T_c \in PMod(\Sigma_g^n)$ the Dehn twist [5] attached to $c$. Dehn twists fall into a finite number of conjugacy classes. An element of each will be denoted by $T_i$, $i = 0, \dots, N$, $T_0$ being separating. Given $\mathbf{k} = (k_0, k_1, \dots, k_N) \in (\mathbb{N}^*)^{N+1}$ a $(N+1)$-uple of positive integers, denote by $PMod(\Sigma_g^n)[\mathbf{k}]$ the normal subgroup of $PMod(\Sigma_g^n)$ generated by $T_0^{k_0}, \dots, T_N^{k_N}$. Then the representations of the groups

$$\Gamma_{g,n,\mathbf{k}} := PMod(\Sigma_g^n)/PMod(\Sigma_g^n)[\mathbf{k}]$$

are precisely the representations of $PMod(\Sigma_g^n)$ such that Dehn twists have finite order images.

Let $\mathcal{M}_{g,n}$ be the moduli stack of smooth genus $g$ curves with $n$ marked points over $\mathbb{C}$. Its fundamental group is naturally isomorphic to $PMod(\Sigma_g^n)$ and the universal covering space of its analytification is the Teichmüller space $\mathcal{T}_{g,n}$, a bounded convex domain in $\mathbb{C}^{3g-3+n}$. It turns out that $\Gamma_{g,n,\mathbf{k}}$ is the fundamental group of a smooth proper Deligne-Mumford stack $\overline{\mathcal{M}_{g,n}}[\mathbf{k}]$ constructed in [1] which comes with a (non representable except if $k_i = 1$ for all $i$) map to the moduli space of stable marked curves $\overline{\mathcal{M}_{g,n}}$ which is an equivalence over $\mathcal{M}_{g,n}$. The moduli space of $\overline{\mathcal{M}_{g,n}}[\mathbf{k}]$ identifies with the moduli space $\overline{M_{g,n}}$ of stable marked curves.

The motivation of the work I reported on is to understand the fundamental group $\Gamma_{g,n,\mathbf{k}}$ and the universal covering stack $\mathcal{T}_{g,n}[\mathbf{k}]$ of $\overline{\mathcal{M}_{g,n}}[\mathbf{k}]$. Introduce the following properties of $\mathbf{k} \in (\mathbb{N}^*)^{N+1}$:

- $I(\mathbf{k})$: $\Gamma_{g,n,\mathbf{k}}$ is an infinite group.
- $S(\mathbf{k})$: the moduli space of $\mathcal{T}_{g,n}[\mathbf{k}]$ is a Stein space.
- $T(\mathbf{k})$: $H^2(\Gamma_{g,n,\mathbf{k}}, \mathbb{Q}) \neq 0$.
- $U(\mathbf{k})$: $\mathcal{T}_{g,n}[\mathbf{k}]$ is a complex manifold.

One has $S(\mathbf{k}) \Rightarrow I(\mathbf{k})$ and if $\mathbf{k}|\mathbf{k}'$ (i.e.: $k_i|k_i'$ for all $i$) $S(\mathbf{k}) \Rightarrow S(\mathbf{k}')$, $I(\mathbf{k}) \Rightarrow I(\mathbf{k}')$, $T(\mathbf{k}) \Rightarrow T(\mathbf{k}')$.

Property $U(\mathbf{k})$ is rather delicate and only settled in special cases. In [3], which restricts to the case $g \geq 2$, it is established for $\mathbf{k} = [p, p, \dots, p]$ under the restrictions $p \equiv 1[2]$, $p \geq 5$, using TQFT representations of the mapping group which are well known to be torsion on Dehn twists. Since the image of these representations is infinite in this case by a classical result of Funar, property $I([p,p,\dots,p])$ is also true.

Given $\rho:\Gamma_{g,n,\mathbf{k}}\to GL_n(\mathbb{C})$ a semisimple representation, still assuming $g\geq 2$, an argument applying to a Shafarevich morphism attached to $\rho$ [2] the results of [6] shows that for every algebraic family $i:C\to\mathcal{M}_{g,n}$ of marked curves, the group $\rho(i_*\pi_1(C))$ is infinite provided $\rho(i_*\pi_1(C'))$ is for $C'$ a fiber of a forgetful map or, when $n=0$, provided the image of $\rho$ is. This is enough, still for $g\geq 2$ to establish $S([p,\dots,p])$ for $p\equiv 1[2], p\geq 7$ and $g\geq 2$ [4]. Another result of [4] that $T(\mathbf{k})$ holds if $g\geq 2$ and $\mathbf{k}$ is divisible enough.

Similar results are available for $\mathbf{k}=[2p,\frac{p}{g.c.d(p,4)},\frac{p}{g.c.d(p,4)},\dots,\frac{p}{g.c.d(p,4)}]$ where $p\equiv 0[2]$.

## References

[1] A. Chiodo, *Stable twisted curves and their r-spin structures*, Annales de l'Institut Fourier, 58 n.5, (2008), 1635–1689.

[2] P. Eyssidieux, *Sur la convexité holomorphe des revêtements linéaires réductifs d'une variété projective algébrique complexe*, Inventiones Mathematicae, 156 (2004), 503–564.

[3] P. Eyssidieux, L. Funar, *Orbifold Kähler Groups related to Mapping Class groups*, arXiv:2112.06726 [math.AG] (2021).

[4] P. Eyssidieux, L. Funar, *Algebrogeometric subgroups of the mapping class groups*, arXiv:2503.22470 [math.AG] (2025).

[5] B. Farb, D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton University Press (2011).

[6] A. Gibney, S. Keel, I. Morrison, *Towards the ample cone of $\overline{M}_{g,n}$*, Journal of the American Mathematical Society, 15, (2002), 273–294.

## Holomorphic symplectic geometry of elliptic surfaces

Jun-Muk Hwang
(joint work with Guolei Zhong)

A symplectic elliptic fibration is a triple $(f\colon M\to C,\sigma,\omega)$, where $M$ is a complex surface, $C$ is a Riemann surface, $f$ is an elliptic fibration, $\sigma$ is a section of $f$ and $\omega$ is a nowhere-vanishing holomorphic 2-form on $M$. An isogeny from $(f\colon M\to C,\sigma,\omega)$ to another symplectic elliptic fibration $(f'\colon M'\to C',\sigma',\omega')$ is a pair $(\phi,\Phi)$ of a surjective holomorphic map $\phi\colon C\to C'$ and a meromorphic map $\Phi\colon M\dashrightarrow M'$ such that $\phi\circ f=f'\circ\Phi$, $\Phi^*\omega'=\omega$ and $\Phi$ sends a general fiber of $f$ isomorphically to a fiber of $f'$.

We give the following classification of isogenies with a fixed target.

**Theorem 1.** *Let $(f\colon M\to C,\sigma,\omega)$ be a symplectic elliptic fibration and let $\phi\colon\widetilde{C}\to C$ be a surjective holomorphic map from a Riemann surface $\widetilde{C}$. Then there exists a symplectic elliptic fibration $(\widetilde{f}\colon\widetilde{M}\to\widetilde{C},\widetilde{\sigma},\widetilde{\omega})$ equipped with an isogeny $(\phi\colon\widetilde{C}\to C,\Phi\colon\widetilde{M}\dashrightarrow M)$, if and only if at each ramification point $x\in\widetilde{C}$ of $\phi$, the quadruple $(\mathrm{mult}_x\phi, f^{-1}(\phi(x)), J(\phi(x)), \mathrm{mult}_{\phi(x)}J)$, where $J\colon C\to\overline{\mathcal{M}}_1\cong\mathbb{P}^1$ is Kodaira's $J$-map of the elliptic fibration $f$ to the compactified moduli of elliptic curves, is one of the following.*

$$\begin{array}{ll}
(2, \mathrm{I}_0^*, J(0), k)) & (2, \mathrm{I}_0^*, 0, 3k) \\
(3, \mathrm{IV}^*, 0, 3k-2) & (6, \mathrm{II}^*, 0, 3k-1) \\
(2, \mathrm{I}_0^*, 1, 2k) & (4, \mathrm{III}^*, 1, 2k-1) \\
(2, \mathrm{I}_k^*, \infty, k) & (3, \mathrm{II}^*, 0, 3k-1) \\
(2, \mathrm{III}^*, 1, 2k-1) & (5, \mathrm{II}^*, 0, 3k-1) \\
(3, \mathrm{III}^*, 1, 2k-1) & (2, \mathrm{IV}^*, 0, 3k-2) \\
(4, \mathrm{II}^*, 0, 3k-1) & (2, \mathrm{II}^*, 0, 3k-1).
\end{array}$$

*Here $k$ is a positive integer or $\infty$, and $\mathrm{mult}_{\phi(x)} J = \infty$ means $J$ is constant.*

We give the following classification of isogenies with a fixed nonisotrivial source.

**Theorem 2.** *Let $(f\colon M \to C, \sigma, \omega)$ be a symplectic elliptic fibration which has non-constant $J$-map. Then there exists a unique symplectic elliptic fibration $(\hat{f}\colon \hat{M} \to \hat{C}, \hat{\sigma}, \hat{\omega})$ and an isogeny $(\xi\colon C \to \hat{C}, \Xi\colon M \dashrightarrow \hat{M})$ with the following universal property: any isogeny $(\phi : C \to C', \Phi\colon M \to M')$ to a symplectic elliptic fibration $(f'\colon M' \to C', \sigma', \omega')$ factorizes $(\xi, \Xi)$, namely, there exists an isogeny $(\xi'\colon C' \to \hat{C}, \Xi'\colon M' \dashrightarrow \hat{M})$ satisfying $\xi = \xi' \circ \phi$ and $\Xi = \Xi' \circ \Phi$.*

These results are proved by employing the classical theory of elliptic surfaces as well as the Arnold-Liouville theorem (the theory of action-angle variables) in symplectic geometry. One application is the following.

**Theorem.** *An isogeny between two elliptic K3 surfaces of Picard number less than 10 must be an isomorphism.*

This is because when $(f\colon M \to C, \sigma, \omega)$ is an elliptic K3 surface with Picard number less than 10, the isogeny $(\xi, \Xi)$ in Theorem 2 must be an isomorphism. In fact, if it is not an isomorphism, then $\xi$ has at least two branch points $x_1, x_2 \in \hat{C}$. Moreover, each of the singular fibers $\hat{f}^{-1}(x_1)$ and $\hat{f}^{-1}(x_2)$ has at least five irreducible components from Theorem 1. This implies that the Picard number of $\hat{M}$ is at least 10 from the classical theory of elliptic surfaces. By a result of Schütt and Shioda, when there is a dominant meromorphic map between two K3 surfaces $M$ and $\hat{M}$, they have the same Picard number. This implies that $M$ has Picard number at least 10, a contradiction to the assumption.

## Euler characteristics of nonpositively curved varieties

Donu Arapura

(joint work with Deepam Patel, Botong Wang)

My talk was largely based on joint work with Botong Wang [1] and Deepam Patel [2]. It ended with a short discussion of some work in progress with Patel.

### 1. The main results

The starting point is an old conjecture of Hopf's from the 1930's that if $X$ is a compact Riemannian manifold of dimension $2d$ with negative (respectively non-positive) sectional curvature, then the Euler characteristic satisfies $(-1)^d \chi(X) > 0$

(respectively $\geq 0$). When $d = 1$ this follows from Gauss-Bonnet. Various other cases are known, but it is still open in general. However, it does hold when $X$ is compact Kähler by work of Gromov [4] and extensions of his results by Eyssideux, Cao-Xavier, and Jost-Zuo. The basic idea is to prove a vanishing theorem for the $L^2$-cohomology of the universal cover $\tilde{X}$ and then apply Atiyah's index theorem. To get strict inequality, Gromov also proves a nonvanishing theorem. The following theorem generalizes these results.

**Theorem 1** (A-Wang, 2025)**.** *If $X$ is compact Kähler with ample (respectively nef) cotangent bundle, then $\chi(X, P) > 0$ (respectively $\geq 0$) for any* perverse sheaf *$P$ on $X$.*

A few comments are useful here. The previous curvature assumptions imply ampleness or nefness of the cotangent bundle. However, as Phillipe Eyssideux observed during the talk, there exist simply connected examples with ample cotangent bundle, so these assumptions are not equivalent. As explained in item (2) below, the sign $(-1)^d$ is absorbed into the definition of a perverse sheaf. The notion of perverse sheaf was introduced by Beilinson-Bernstein-Deligne-Gabber, and the basic properties are summarized as follows.

(1) The category of perverse sheaves $Perv(X, K)$ is a subcategory of the constructible derived category $D^b_c(X, K)$ for any field $K$.
(2) If $Y \subset X$ is a submanifold, or more generally an analytic subvariey with lci singularities, then $P = \mathbb{C}_Y[\dim Y] \in Perv(X, \mathbb{C})$. So in particular, $\chi(X, P) = (-1)^{\dim Y} \chi(Y)$.
(3) If $D_X$ is the sheaf of rings of holomorphic differential operators, a regular holonomic $D_X$-module is a generalization of a regular integrable connection. Given such a $D$-module $M$, one can construct the (shifted) de Rham complex $DR(M) \in Perv(X, \mathbb{C})$. The Riemann-Hilbert correspondence shows that any perverse sheaf with coefficients in $\mathbb{C}$ arises this way from a unique $M$.

I will give a brief sketch of the proof of Theorem 1. Instead of a vanishing theorem, the proof is based on intersection theory. Given a regular holonomic $D$-module $M$, the associated graded $Gr_F M$ with respect to a good filtration can be viewed as a coherent sheaf on the cotangent bundle $T^*X \cong \operatorname{Specan} Gr_F D_X$. The support is the so called *characteristic variety* $Char(X)$. Holonomcity means that the scheme theoretic support is a Lagrangian conical subscheme. Therefore, it determines an effective cycle $CC(M) \in CH^d(T^*X)$, called the *characteristic cycle*. Here $d = \dim_{\mathbb{C}} X$. Theorem 1 follows from two additional facts:

(1) Given a regular holonomic $D$-module $M$ corresponding to $P \in Perv(X, \mathbb{C})$, the *Dubson-Kashiwara*, or DK, formula says that
$$\chi(X, P) = CC(M) \cdot (\text{zero section})$$
(2) The work of Fulton-Lazarsfeld, and its extension to the analytic setting by Demailly-Peternel-Schneider, implies that the right side of the DK formula is nonnegative (respectively positive) when $T^*X$ is nef (respectively ample).

Since the moduli stack $\mathscr{A}_g$ of $g$-dimensional principally polarized abelian varieties is a locally symmetric hermitian orbifold, it satisfies the above curvature assumptions. This suggests the first part of the next theorem.

**Theorem 2** (A-Patel, 2026). *Let $X$ be either $\mathscr{A}_g$ or the moduli stack $\mathscr{M}_{g,n}$ of smooth genus-$g$ curves with $n$ marked points, where $2g-2+n>0$. Let $P \in Perv(X,\mathbb{C})$. Then the orbifold Euler characteristic $\chi(X,P)\geq 0$.*

I will just indicate the proof when $X=\mathscr{A}_g$; the second case of $\mathscr{M}_{g,n}$ is considerably more complicated. First, we choose a smooth toroidal compactification $\overline{\mathscr{A}}_g$ such that the boundary divisor $D$ has normal crossings. The argument follows the same outline as above, with two modifications.

(1) A result of Faltings-Chai shows that $\Omega^1_{\overline{\mathscr{A}}_g}(\log D) = S^2F^1$, where $F^1$ is the Hodge bundle of the universal family of abelian varieties. When combined with results of Fujita-Kawamata, we can conclude that the log cotangent bundle is nef.
(2) A log version of the Dubson-Kashiwara formula due to Wu and Zhou expresses the Euler characteristic as an intersection number. This is seen to be nonnegative exactly as before.

## 2. Conjectural refinements

Returning to the $L^2$-vanishing theorem of Gromov et. al. described above, it is worth noting that since $L^2$ harmonic forms can be decomposed into $(p,q)$-type, we obtain an inequality $(-1)^p\chi(\Omega^p_X)\geq 0$ for nonnegatively curved compact Kähler manifolds. This along with other evidence led L. Maxim, B. Wang and I [3] to conjecture this inequality holds for manifolds with nef cotangent bundle. In fact, we made a much stronger conjecture in this case involving Hodge, and more generally, mixed Hodge modules. A Hodge module is, roughly speaking, a variation of Hodge structures with singularities. To be a bit more precise, it consists of a perverse sheaf $P\in Perv(X,\mathbb{Q})$, and a filtered holonomic $D$-module $(M,F)$ with an isomorphism $\alpha: P\otimes\mathbb{C}\cong DR(M)$. For a mixed Hodge module, there is an additional filtration $W$, but it will not play a role here. These form an abelian category $MHM(X)$. The definition is due to M. Saito.

**Conjecture.**

(1) *(A-Maxim-Wang) Suppose that $X$ is a compact Kähler manfold with nef cotangent bundle. If $(M,F,\ldots)\in MHM(X)$, then for any $p$*

$$\chi(X, Gr^p_F DR(M))\geq 0$$

(2) *(A) Suppose that $X$ is $\mathscr{A}_g$ or $\mathscr{M}_{g,n}$ with $2g-2+n>0$, and let $j: X\to\overline{X}$ be either a good toroidal or Deligne-Mumford-Knudsen compactification, if $(M,F,\ldots)\in MHM(X)$, then*

$$\chi(X, Gr^p_F j_* DR(M))\geq 0$$

In current work in progress with D. Patel, we found a strategy to attack these conjectures. The first step is to refine the DK formula to express the coefficients of the Laurent polynomial

$$p_M(t) = \sum_p (-1)^p t^{-p} \chi(X, \Omega_X^p \otimes Gr_F M)$$

as intersection numbers on the cotangent bundle. It is useful to think of $p_M(t)$ as an element of the representation ring of $\mathbb{G}_m$. This suggests upgrading everything to the $\mathbb{G}_m$-equivariant setting. Using Laumon's proof of DK using Riemann-Roch as a model, we define the refined characteristic cycle $\widetilde{CC}(M)$ of $M \in MHM(X)$ as its image under the composition

$$K_0(MHM(X)) \xrightarrow{Gr} K_0^{\mathbb{G}_m}(T^*X) \xrightarrow{\tau} \widehat{CH}^*_{\mathbb{G}m}(T^*X)_{\mathbb{Q}}$$

where the group in the middle is the equivariant Grothendieck group and $\tau$ is the Edidin-Graham Riemann-Roch map to the completed equivariant Chow ring. Our refined DK-formula would say that

$$p_M(\exp(u)) = \widetilde{CC}(M) \cdot (\text{zero section})$$

in $\widehat{CH}^*_{\mathbb{G}m}(pt)_{\mathbb{Q}} = \mathbb{Q}[[u]]$. To finish the proof, this would need to be combined with an appropriate positivity statement. We are currently working on this.

## References


[1] D. Arapura, B. Wang, *Perverse sheaves on varieties with large fundamental groups*, J. Diff. Geom. **129** (2025), no. 1, 1–15.

[2] D. Arapura, D. Patel, *Nonnegativity of signed Euler characteristics of moduli of curves and abelian varieties*, Math. Annalen **394**, 34 (2026).

[3] D. Arapura, L. Maxim, B. Wang, *Hodge-theoretic variants of the Hopf and Singer conjectures*, Proceedings on Geometry and Topology of Aspherical Manifolds, Contemp. Math. AMS (2025).

[4] M. Gromov, Kähler hyperbolicity and L2-Hodge theory, J. Differential Geom. **33** (1991), no. 1, 263–292.


## A Liouville theorem and $C^\alpha$ estimate on Calabi-Yau cones

Hans-Joachim Hein

This is a report on the paper [10] by my student Johan Klemmensen, who is now working in industry. The paper forms part of his PhD thesis (Münster, 2025).

**Introduction to Calabi-Yau metrics.** Let $B = B_1(0) \subset \mathbb{C}^n$ $(n \geq 2)$ denote the unit ball with its standard coordinates $z_1, \dots, z_n$. A smooth function $\varphi : B \to \mathbb{R}$ is called plurisubharmonic (psh) if the Hermitian $n \times n$ matrix $(\varphi_{j\bar{k}}) = (\partial_{z_j} \partial_{\bar{z}_k} \varphi)$ is strictly positive definite at every point of $B$. Equivalently, the closed $(1,1)$-form $\omega = i\partial\overline{\partial}\varphi$ is a Kähler form, i.e. the symmetric $(0,2)$-tensor $g = \omega(\cdot, J\cdot)$ is positive definite everywhere, i.e. $g$ is a Riemannian metric. Any Riemannian metric $g$ has associated curvature quantities $\mathrm{Rm}_g$ (the curvature tensor), $\mathrm{Ric}_g$ (the Ricci tensor) and $\mathrm{Scal}_g$ (the scalar curvature). Kähler [9] discovered the miracle that, in our case,

$\rho = \mathrm{Ric}_g(J\cdot,\cdot)$ is again a closed $(1,1)$-form; more precisely, $\rho = -i\partial\overline{\partial}\log\det(\varphi_{j\overline{k}})$. Thus, if $\varphi$ solves the complex Monge-Ampère equation $\det(\varphi_{j\overline{k}}) = 1$, then $g$ is a Ricci-flat (Einstein) Riemannian metric. Finding new examples of Einstein metrics actually seems to have been Kähler's primary motivation in [9]. A trivial solution is $\varphi(z) = |z|^2$, which generates the Euclidean metric $\omega_{\mathbb{C}^n}$ ($\mathrm{Rm}_{g_{\mathbb{C}^n}} = 0$). Since the complex Monge-Ampère equation is a second-order elliptic PDE, one would expect there to exist plenty of other solutions at least locally. This is indeed true:

**Theorem** (Caffarelli-Kohn-Nirenberg-Spruck [2])**.** *For all $f \in C^\infty(\partial B)$ there is a unique $\varphi \in C^\infty(\overline{B})$ such that $\varphi|_{\partial B} = f$ and $\varphi$ is psh with $\det(\varphi_{j\overline{k}}) = 1$ on $B$.*

The proof in [2] has much in common with the proof of the Aubin-Yau theorem [1, 13] on compact Kähler manifolds. Instead of focusing on the differences we will briefly sketch those steps that are common to both proofs and relevant to this talk. The basic point is to set up a family of equations $\det(\varphi_{t,j\overline{k}}) = (1-t)\det(\varphi_{0,j\overline{k}})+t$, to be solved for a (necessarily unique) $\varphi_t \in C^\infty(\overline{B})$, psh on $B$, with $\varphi_t|_{\partial B} = f$ for all $t \in [0,1]$. Starting with any psh extension $\varphi_0$ of $f$ with $\inf_B \det(\varphi_{0,j\overline{k}}) > 0$, it is not hard to see using the inverse function theorem and some linear PDE theory that the set $\mathcal{T} = \{t \in [0,1] : \varphi_t \text{ exists}\}$ is open as well as nonempty. Thus, the closedness of $\mathcal{T}$, hence the fact that $\mathcal{T} = [0,1]$ and in particular that $1 \in \mathcal{T}$, will follow from the so-called a priori estimates, the main difficulty in this business: for all $k$ there is a $C$ such that for all $t \in [0,1]$, if $t \in \mathcal{T}$ then $\|\varphi_t\|_{C^k} \leq C$. For $k \geq 3$ this again follows from linear PDE theory if the cases $k \leq 2+\alpha$ are known. In our setting $k = 0$ is simple, $k = 1$ is similar to $k = 2$, and the cases $k = 2$ and $k = 2+\alpha$ should be viewed as the two key steps. The case $k = 2$ is equivalent to proving that there exists a $C$ such that for all $t \in [0,1]$, if $t \in \mathcal{T}$ then $\omega_t = i\partial\overline{\partial}\varphi_t$ satisfies $\frac{1}{C}\omega_{\mathbb{C}^n} \leq \omega_t \leq C\omega_{\mathbb{C}^n}$, i.e. $\omega_t$ is uniformly bi-Lipschitz to $\omega_{\mathbb{C}^n}$. Equivalently, the equation is not just elliptic but uniformly elliptic, and then the step $k = 2+\alpha$ is the so-called Evans-Krylov estimate in the theory of nonlinear elliptic PDEs.

**Statement and context of Klemmensen's theorem.** To state the theorem, we need to recall the notion of a Calabi-Yau cone. This is done in three steps:

- Let $(L, g_L)$ be a compact Riemannian manifold without boundary. The cone with link $(L, g_L)$ is the Riemannian manifold $(\mathcal{C}, g_{\mathcal{C}})$, $\mathcal{C} = \mathbb{R}^+ \times L$, $g_{\mathcal{C}} = dr^2 + r^2 g_L$, where $r$ denotes the standard coordinate on the $\mathbb{R}^+$ factor. Then $r$ is the $g_{\mathcal{C}}$-metric distance to the apex $o$ of the cone. For the sake of exposition it is often convenient to pretend that $o$ is included into $\mathcal{C}$, even though formally it never is.
- We assume that $g_{\mathcal{C}}$ is Kähler with respect to a complex structure $J_{\mathcal{C}}$ on $\mathcal{C}$, i.e. $J_{\mathcal{C}}$ is orthogonal and parallel with respect to $g_{\mathcal{C}}$.
- A Kähler cone $(\mathcal{C}, g_{\mathcal{C}}, J_{\mathcal{C}})$ as above is called a Calabi-Yau cone if there exists a $J_{\mathcal{C}}$-holomorphic volume form $\Omega_{\mathcal{C}}$ that is parallel with respect to $g_{\mathcal{C}}$. In particular, by Kähler's magic formula, the cone metric $g_{\mathcal{C}}$ is then Ricci-flat.

**Examples:** (1) $\mathcal{C} = \mathbb{C}^n/\Gamma$, where $\Gamma$ is a finite subgroup of $\mathrm{SU}(n)$ acting freely on the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$. We take $\omega_{\mathcal{C}}$ to be the pushdown of $\omega_{\mathbb{C}^n}$ and $\Omega_{\mathcal{C}}$ to be the pushdown of $\Omega_{\mathbb{C}^n} = dz_1 \wedge \ldots \wedge dz_n$. These are exactly the flat Calabi-Yau cones ($\mathrm{Rm}_{g_{\mathcal{C}}} = 0$), and in dimension $n = 2$ they are the only Calabi-Yau cones.

(2) $\mathcal{C} = \{z \in \mathbb{C}^{n+1} : f(z) = z_0^2 + \cdots + z_n^2 = 0\}$, the ordinary double point, together with the Stenzel metric $\omega_{\mathcal{C}} = i\partial\overline{\partial}|z|^{2(n-1)/n}$ and the holomorphic volume form $\Omega_{\mathcal{C}}$ given by $\Omega_{\mathbb{C}^{n+1}} = df \wedge \Omega_{\mathcal{C}}$ along $\mathcal{C}$. For $n = 2$ this is Example (1) with $\Gamma = \{\pm \mathrm{Id}\}$, but the Ricci-flat cone metric $\omega_{\mathcal{C}}$ is not the restriction of the ambient flat metric $\omega_{\mathbb{C}^{n+1}}$ even in this case. For $n \geq 3$ the Stenzel metric $\omega_{\mathcal{C}}$ is not flat, and so its sectional curvatures diverge like $\pm r^{-2}$ as $r = |z|^{(n-1)/n} \to 0$.

We also require one non-trivial fact: for any Kähler cone $(\mathcal{C}, g_{\mathcal{C}}, J_{\mathcal{C}})$, the transverse automorphism group $\mathrm{Aut}^T(\mathcal{C}, g_{\mathcal{C}}, J_{\mathcal{C}}) = \{F \in \mathrm{Aut}(\mathcal{C}, J_{\mathcal{C}}) : F_*(r\partial_r) = r\partial_r\}$ is a finite-dimensional complex Lie group, and it is reductive (in fact, semi-simple) if the cone is Calabi-Yau. For example, whereas $\mathrm{Aut}(\mathbb{C}^n)$ is infinite-dimensional, we have that $\mathrm{Aut}^T(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C})$ with respect to the standard cone structure.

**Theorem** (Klemmensen [10])**.** *Let $(\mathcal{C}, g_{\mathcal{C}}, J_{\mathcal{C}})$ be a Calabi-Yau cone. Then there is an $\alpha > 0$, and for all $C, D > 0$ there is a $C' > 0$, such that the following holds. Let $\omega$ be a Kähler metric on the unit ball $B_1(o) \subset \mathcal{C}$ centered at the apex $o$ such that $\frac{1}{C}\omega_{\mathcal{C}} \leq \omega \leq C\omega_{\mathcal{C}}$ and $|\mathrm{Scal}_\omega| \leq D$. Then there exists an $F \in \mathrm{Aut}^T(\mathcal{C}, g_{\mathcal{C}}, J_{\mathcal{C}})$ such that for all $r \in (0, \frac{1}{2}]$ we have that $\sup_{B_r(o)} |F^*\omega - \omega_{\mathcal{C}}|_{\omega_{\mathcal{C}}} \leq C' r^\alpha$.*

Some remarks to put this result in context:

• This is an Evans-Krylov type estimate. However, since the standard method for proving such an estimate does not seem to work in our setting, we instead use an indirect approach as in the work of H-Tosatti [6, 8] on collapsing holomorphic fibrations, which was in turn inspired by the indirect approach of Chen-Wang [3] to the Evans-Krylov estimate on $B = B_1(0) \subset \mathbb{C}^n$ (but with a crucial new ingredient to deal with the singularity of the background geometry).

• Donaldson-Sun theory yields statements similar to the above (e.g. see H-Sun [7] or Chiu-Székelyhidi [4] based on Donaldson-Sun [5]), although as far as we can tell there is no direct implication in either direction. Let us simply point out the main difference: Donaldson-Sun theory requires no bi-Lipschitz assumption, but our approach requires no input from Cheeger-Colding theory or from $K$-stability of valuations; it is completely elementary.

**Proof of the theorem.** The proof factors into two steps:

(a) Liouville-type rigidity: If $\omega$ is a Ricci-flat Kähler metric on the entire cone $\mathcal{C}$, uniformly bi-Lipschitz to $\omega_{\mathcal{C}}$, then $F^*\omega = \omega_{\mathcal{C}}$ for some $F \in \mathrm{Aut}^T(\mathcal{C}, g_{\mathcal{C}}, J_{\mathcal{C}})$.

(b) From Liouville to Evans-Krylov via blow-up and contradiction.

Here we only note that (a) requires various special geometric arguments based on the behavior of the heat kernel of the Kähler space $(\mathcal{C}, \omega)$ and on some differential-geometric properties of Calabi-Yau cones. On the other hand, (b) is a universal real analysis argument going back to work of Leon Simon [12] in the analysis of elliptic PDEs. However, the implementation of this argument in our context is quite technical (40 pages in [10] as compared to 20 pages for part (a)) and requires the introduction of a $C^\alpha$ seminorm tailored to match the Liouville theorem of (a). On a technical level (b) is probably the more interesting part of the proof.

**Remark.** It is enough to assume $\mathrm{Ric}_\omega = 0$ in (a) even though we only assume $|\mathrm{Scal}_\omega| \leq D$ in the theorem. This is because after blowing up we obtain an entire scalar-flat metric $\omega$ on $\mathcal{C}$, and then by definition $\Delta_\omega u = 0$, where $u = \log(\omega^n/\omega_{\mathcal{C}}^n)$. But $u \in L^\infty(\mathcal{C})$ by the bi-Lipschitz assumption, entire $\omega_{\mathcal{C}}$-harmonic functions in $L^\infty(\mathcal{C})$ are necessarily constant, and the latter property is stable under bi-Lipschitz perturbations of the metric according to Saloff-Coste [11].

## References


[1] T. Aubin, *Equations du type Monge-Ampère sur les variétés Kähleriennes compactes*, Bull. Sci. Math. **102** (1978), 63–95.

[2] L. Caffarelli, J.J. Kohn, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations, II: Complex Monge-Ampère, and uniformly elliptic, equations*, Comm. Pure Appl. Math. **38** (1985), 209–252.

[3] X. Chen, Y. Wang, *$C^{2,\alpha}$-estimate for Monge-Ampère equations with Hölder-continuous right hand side*, Ann. Glob. An. Geom. **49** (2016), 195–204.

[4] S.-K. Chiu, G. Székelyhidi, *Higher regularity for singular Kähler-Einstein metrics*, Duke Math. J. **172** (2023), 3521–3558.

[5] S. Donaldson, S. Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, II*, J. Diff. Geom. **107** (2017), 327–371.

[6] H.-J. Hein, *A Liouville theorem for the complex Monge-Ampère equation on product manifolds*, Comm. Pure Appl. Math. **72** (2019), 122–135.

[7] H.-J. Hein, S. Sun, *Calabi-Yau manifolds with isolated conical singularities*, Publ. Math. IHES **126** (2017), 73–130.

[8] H.-J. Hein, V. Tosatti, *Higher-order estimates for collapsing Calabi-Yau metrics*, Camb. J. Math. **8** (2020), 683–773.

[9] E. Kähler, *Über eine bemerkenswerte Hermitesche Metrik*, Abh. Math. Sem. Univ. Hamb. **9** (1932), 173–186.

[10] J. Klemmensen, *A Liouville theorem and $C^\alpha$ estimate for Calabi-Yau cones*, preprint, arXiv:2502.02361.

[11] L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds*, J. Diff. Geom. **36** (1992), 417–450.

[12] L. Simon, *Schauder estimates by scaling*, Calc. Var. PDE **5** (1997), 391–407.

[13] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–411.


## Groups with exotic finiteness properties from complex Morse theory


Claudio Llosa Isenrich

(joint work with Pierre Py)


Algebraically, a group $G$ can be described as a quotient $G \cong \langle X \mid R \rangle := F_X / \langle\langle R \rangle\rangle$ of a free group $F_X$ on a generating set $X$ for $G$ by a normal subgroup $\langle\langle R \rangle\rangle$ generated by a set of relations $R \subset F_X$ that describe equations between generators. Two of the most basic invariants are finite generation, which means that we can choose $X$ to be finite, and finite presentability, which means that we can choose $X$ and $R$ to be finite. These finiteness properties have a topological interpretation in terms of classifying spaces. A classifying space for a group $G$ is a connected CW-complex $X = K(G,1)$ with $\pi_1(X) \cong G$ which is aspherical (i.e. its universal cover is contractible). It is not hard to see that every group admits a classifying space.

One starts from a vertex and attaches one loop for every generator. Then one attaches one 2-cell for every relation along the word describing the relation. This yields a space with fundamental group $G$. Finally, one attaches higher dimensional cells to make the universal cover contractible. This shows that finite generation (resp. finite presentability) of $G$ can be characterised by the existence of a $K(G,1)$ with finitely many cells of dimension $\leq 1$ (resp. $\leq 2$), motivating the

**Definition** (C.T.C Wall, 1965)**.** A group $G$ is of finiteness type $\mathcal{F}_n$ if and only if it admits a $K(G,1)$ with finitely many cells of dimension $\leq n$.

Finiteness properties, or rather the failure thereof, provide interesting geometric obstructions. If a group is not $\mathcal{F}_n$ for some $n \in \mathbb{N}$, then it cannot be the fundamental group of a closed aspherical manifold, since by real Morse theory such a manifold admits a finite CW-complex structure. Moreover, finiteness properties are invariant under quasi-isometries [1], making them natural invariants to consider in geometric group theory. The first groups of type $\mathcal{F}_{n-1}$ and not $\mathcal{F}_n$ for every $n \in \mathbb{N}$ have been constructed by Stallings ($n = 3$) [18] and Bieri ($n \geq 4$) [3]. A common construction method is Morse theory, which appears in different incarnations (discrete, real, complex). Most examples in the literature were constructed using versions of discrete Morse theory and this approach succeeded in producing a variety of examples with interesting additional properties, e.g. subgroups of non-positively curved groups [2, 3, 18] or simple groups [17]. However, this approach relies on combinatorial arguments that reach their limits in certain situations.

The goal of this talk is to explain how complex Morse theory can be used to overcome these limitations and produce new interesting examples of groups with exotic finiteness properties (i.e. $\mathcal{F}_{n-1}$ and not $\mathcal{F}_n$) that answered open questions. The strength of complex Morse theory has long been recognized and it forms a key ingredient for many results in complex geometry, such as the Lefschetz Hyperplane Theorem [13], the study of isolated singularities [14], and work of Simpson [16].

To build groups with exotic finiteness properties one often starts with a fundamental group $G = \pi_1(X)$ of some compact aspherical CW complex $X$ (e.g. a closed aspherical manifold). Then one constructs a circle-valued Morse function $f : X \to S^1$ and takes the lift $f_{\mathbb{Z}} : X_{\mathbb{Z}} \to \mathbb{R}$ of $f$ to the cyclic covering induced by the kernel of the morphism $f_* : \pi_1(X) \to \pi_1(S^1) = \mathbb{Z}$. If the singularities of $f$ are sufficiently "nice", then Morse theory implies that $X_{\mathbb{Z}}$ has a CW-complex structure with finitely many cells below some dimension. Since $X_{\mathbb{Z}}$ is aspherical, one concludes that $\pi_1(X_{\mathbb{Z}}) = \ker(f_*)$ has good finiteness properties. To show that the group under consideration is not $\mathcal{F}_n$ for some $n$, one exploits obstructions coming from (co)homology or $\ell^2$-Betti numbers, see e.g. [2, 12].

For arbitrary discrete or real Morse functions, it is difficult to ensure that the singularities are "nice". In contrast, in complex Morse theory this comes for free. Indeed, if $X$ is a complex $n$-manifold and $f$ is (locally) the real part of a complex Morse function, then all singularities are of index $n$. This implies that one can construct a CW-structure on $X_{\mathbb{Z}}$ from a CW-structure on a (closed) generic fibre of $f$ by attaching $n$-cells and thus that $\ker(f_*)$ is of type $\mathcal{F}_{n-1}$.

By combining techniques from Simpson's work [16] with a theorem of Eyssidieux [6], we use this approach to prove:

**Theorem 1** ([10, 11]). *For every integer $n \geq 1$ and every cocompact arithmetic lattice $\Lambda < PU(n,1)$ with $b_1(\Lambda) > 0$ there is a finite index subgroup $\Lambda_0 \leq \Lambda$ so that there exists a Morse function $f : \mathbb{H}^n_{\mathbb{C}}/\Lambda_0 \to S^1$ with all singularities of index $n$. In particular, $\ker(f_* : \Lambda_0 \to \mathbb{Z})$ is $\mathcal{F}_{n-1}$ and not $\mathcal{F}_n$.*

Since such lattices are hyperbolic groups, we deduce:

**Corollary 2** ([10]). *For every integer $n \geq 2$ there is a hyperbolic group $G$ with a subgroup $H \leq G$ of type $\mathcal{F}_{n-1}$ and not $\mathcal{F}_n$.*

This answers a longstanding question of Gersten [7] and Brady [4] in geometric group theory. Previously only the cases $n = 2$ [15], $n = 3$ [4] and $n = 4$ [9] were known. Besides the homotopical finiteness properties $\mathcal{F}_n$ there are also their homological counterparts $FP_n(R)$, where $R$ is a unital ring; see [2] for their definition. A natural next question is then:

**Question.** Can non-finitely presented subgroups of hyperbolic groups attain the same range of homological finiteness properties as the Bestvina–Brady groups?

An answer might be beyond the realm of complex Morse theory; partial results were obtained by Kropholler [8], using discrete Morse theory.

Finiteness properties also have interesting applications within complex geometry. In particular, Dimca, Papadima and Suciu [5] used complex Morse theory to prove that fundamental groups of smooth complex projective varieties can be of finiteness type $\mathcal{F}_{n-1}$ and not $\mathcal{F}_n$ for every $n \geq 3$. This negatively answered Kollár's question if every fundamental group of a smooth complex projective variety has a classifying space which is an aspherical quasi-projective variety. The examples in [5] have since been generalized. We refer to [11] for an overview of known examples, as well as for further applications of complex Morse theory in the context of finiteness properties.

## References

[1] J.M. Alonso, *Finiteness conditions on groups and quasi-isometries*, J. Pure Appl. Algebra **95**, no. 2, 121–129, 1994.

[2] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** , no. 3, 44–470, 1997.

[3] R. Bieri, *Homological dimension of discrete groups*, Mathematics Department, Queen Mary College, London, Queen Mary College Mathematics Notes, 1976.

[4] N. Brady, *Branched coverings of cubical complexes and subgroups of hyperbolic groups*, J. London Math. Soc. (2) **60**, no. 2, 461–480, 1999.

[5] A. Dimca, S. Papadima, and A.I. Suciu, *Non-finiteness properties of fundamental groups of smooth projective varieties*, J. Reine Angew. Math. **629**, 89–105, 2009.

[6] P. Eyssidieux, *Orbifold Kähler groups related to arithmetic complex hyperbolic lattices*, arXiv preprint arXiv:1805.00767, 2018.

[7] S. Gersten, *Finiteness properties of asynchronously automatic groups*, Geometric group theory (Columbus OH, 1992), vol. 3, OSU Math. Res. Inst. Publ., deGruyter, Berlin, pp. 121–133, 1995.

[8] R. Kropholler, *Hyperbolic groups with almost finitely presented subgroups*, with an appendix by R. Kropholler and F. Vigolo, Groups Geom. Dyn. **16**, no. 1, 153–178, 2022.
[9] C. Llosa Isenrich, B. Martelli, and P. Py, *Hyperbolic groups containing subgroups of type $F_3$ not $F_4$*, J. Differential Geom. **127**, no. 3, 1121–1147, 2024.
[10] C. Llosa Isenrich, P. Py, *Subgroups of hyperbolic groups, finiteness properties and complex hyperbolic lattices*, Invent. Math. **235**, no. 1, 233–254, 2024.
[11] C. Llosa Isenrich, P. Py, *Groups with exotic finiteness properties from complex Morse theory*, J. Topol. **18**, no. 1, Paper No. e70013, 29 pp, 2025.
[12] W. Lück, *$L^2$-invariants: theory and applications to geometry and $K$-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 44, Springer-Verlag, Berlin, 2002.
[13] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Ann. of Math. Stud., No. 51, Princeton University Press, Princeton, NJ, vi+153 pp, 1963.
[14] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Stud., No. 61, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, iii+122 pp, 1968.
[15] E. Rips, *Subgroups of small cancellation groups*, Bull. London Math. Soc. **14**, no. 1, 45–47, 1982.
[16] C. Simpson, *Lefschetz theorems for the integral leaves of a holomorphic one-form*, Compositio Math. **87**, no. 1, 99–113, 1993.
[17] R. Skipper, S. Witzel, M.C.B. Zaremsky, *Simple groups separated by finiteness properties*, Invent. Math. **215** (2019), no. 2, 713–740.
[18] J.R. Stallings, *A finitely presented group whose 3-dimensional integral homology is not finitely generated*, Amer. J. Math. **85**, 541–543, 1963.


## Inequalities of Miyaoka-Yau-type and Uniformisation of varieties of intermediate Kodaira dimension


Niklas Müller

(joint work with Masataka Iwai, Shin-ichi Matsumura)


Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$. We denote by $\Theta_\omega(X) \in \mathcal{A}^2(\mathrm{End}(\mathcal{T}_X))$ the Chern curvature tensor of the induced Hermitian metric $h$ on the tangent bundle $\mathcal{T}_X$. Then it has long been known that the differential forms

$$\mathrm{Tr}\left(\Theta_\omega(X)^{\wedge i}\right) \in \mathcal{A}^{2i}(X, \mathbb{C}), \quad i = 1, \dots, n$$

are $d$-closed. The associated cohomology classes $c_i(X) \in H^{2i}_{dR}(X, \mathbb{C})$ are independent of the choice of $\omega$ and they are called the *Chern classes* of $X$. Concretely, $c_1(X) = [-K_X]$ is the negative of the canonical divisor, while $c_n(X) = \chi^{\mathrm{top}}(X)$ is the topological Euler number.

It has long been understood that the Chern classes of smooth projective varieties of nonpositive curvature - i.e. with semiample canonical divisor $K_X$ - satisfy certain inequalities. For example, the following classical result grew out of the classification and study of minimal projective surfaces:

**Theorem 1.** (Enriques, Kodaira, Miyaoka, Yau) *Let $X$ be a minimal smooth projective surface which is not uniruled. Then*

$$3c_2(X) \geq c_1(X)^2.$$

*Moreover, the equality is attained if and only if there exists a finite étale cover $\pi\colon X' \to X$ by one of the following surfaces:*

*(1)* $X' = A$ *is an Abelian surface.*
*(2)* $X' \cong E \times C$ *is the product of an elliptic curve* $E$ *and a curve* $C$ *of genus* $g(C) \geq 2$.
*(3)* $X' \cong \mathbb{B}^2/\Lambda$ *is a ball quotient, where* $\mathbb{B}^2 := \{z \in \mathbb{C}^2 \mid \|z\|^2 < 1\}$.

Note that the cases (1), (2) and (3) in Theorem 1 are mutually exclusive and can be distinguished, for example, according to the numerical dimension $\nu(X)$ of $X$, which is defined as follows:

$$\nu(X) := \max\left\{k \mid c_1(X)^k \neq 0 \in H^*(X, \mathbb{C})\right\}.$$

The past years have seen a surge of interest in extending Theorem 1 to higher dimensional spaces. The minimal model program indicates that the higher dimensional analogue of minimal smooth projective surfaces are mildly singular, e.g. log terminal, complex projective varieties whose canonical divisor $K_X$ is nef. The following result summarises these efforts:

**Theorem 2.** *Let* $X$ *be a log terminal projective variety of dimension* $n$*. Assume that* $K_X$ *is nef, of numerical dimension* $\nu$*. Let* $H$ *be an ample Cartier divisor on* $X$*. Then there exists a number* $\varepsilon_0 > 0$ *such that*

$$(\star) \qquad \Big((2\nu+1)\hat{c}_2(X) - \nu\hat{c}_1(X)^2\Big) \cdot (K_X + \varepsilon H)^{n-2} \geq 0$$

*for all* $0 < \varepsilon < \varepsilon_0$*. Moreover, the equality holds if and only if* $X$ *admits a finite quasi-étale Galois cover* $\pi\colon X' \to X$ *by one of the following varieties:*
*(1)* $X'$ *is an Abelian variety.*
*(2)* $X'$ *admits a smooth fibration* $f\colon X' \to C$ *whose fibres are Abelian varieties and whose base* $C$ *is a smooth projective curve of genus* $g(C) \geq 2$.
*(3)* $X' \cong A \times B$ *is the product of an Abelian variety* $A$ *and a smooth ball quotient variety* $B \cong \mathbb{B}^\nu/\Lambda$.

*In particular, in this situation* $X$ *has only finite quotient singularities. Moreover,* $K_X$ *is semiample.*

Here, $\hat{c}_i(X)$, $i = 1, 2$ denote the *orbifold Chern classes* of $X$ introduced in [2] and $\mathbb{B}^\nu = \{z \in \mathbb{C}^\nu \mid \|z\|^2 < 1\}$.

Theorem 2 contains contributions by many authors. Indeed, for threefolds, Theorem 2 was proved by Peternell–Wilson [10]. For varieties of any dimension, the case $\nu(X) = 0$ was treated by Lu–Taji [7] and the case $\nu(X) = \dim X$ was treated by Guenancia–Taji [4] and Greb–Kebekus–Peternell–Taji [3]. When $\nu \leq 2$, the inequality $(\star)$ is due to Miyaoka [8], while the equality case was treated in joint work of the author with Iwai–Matsumura [6]. Finally, Hao–Schreieder [5] studied varieties with $\nu = n - 1$ satisfying equality in $(\star)$. The remaining cases were treated by the author in [9].

**Strategy of the proof.** After presenting the main result and explaining some of its history the rest of the talk was spent with sketching the proof of Theorem 2 which consists in two parts: the proof of inequality $(\star)$ and the treatment of the equality case. To prove $(\star)$, following the philosophy of Miyaoka [8], it suffices to prove the following semistability-type inequality for all subsheaves $\mathcal{F}$ of the cotangent sheaf $\Omega^1_X$ of $X$:

$$\frac{c_1(\mathcal{F})\cdot K_X^{\nu-1}(K_X+\varepsilon H)^{n-\nu}}{\mathrm{rk}\mathcal{F}} \le \frac{K_X^{\nu}\cdot(K_X+\varepsilon H)^{n-\nu}}{\nu}. \tag{1}$$

In turn, the key ingredients in the proof of (1) are the semistability of the logarithmic cotangent sheaf of a klt pair of log general type due to Guenancia–Taji [4] and a trick tracing back to work of Enoki [1].

Regarding the equality case, using some standard arguments, one can show that, up to finite quasi-étale cover, $X$ is smooth, the cotangent bundle $\Omega^1_X$ is nef and there exists a short exact sequence of vector bundles

$$0 \to \mathcal{F} \to \Omega^1_X \to \mathcal{Q} \to 0$$

such that

(1) $\mathrm{rk}(\mathcal{F}) = \nu$,
(2) $\mathcal{F}$ is $K_X^{\nu-1}\cdot H^{n-\nu}$ semistable and $\mathcal{Q}$ is $H^{n-1}$-semistable,
(3) $c_1(\mathcal{F}) \equiv K_X$ and $c_1(\mathcal{Q}) \equiv 0$, and
(4) $(2(\nu+1)c_2(\mathcal{F}) - \nu c_1(\mathcal{F})) \equiv c_2(\mathcal{Q}) \equiv 0$.

The main challenge, compared with previous results, is then to show that in the above situation $K_X$ is semiample. To this end, one observes that it follows from (1)-(4) and the Non-Abelian Hodge correspondence that the vector bundles $\mathcal{E}\mathrm{nd}(\mathcal{F}\oplus\mathcal{O}_X)$ and $\mathcal{Q}$ both admit a flat $C^\infty$-connection. The key idea of the proof is then to show that the Shafarevich morphism for the monodromy representation of these connections coincides with the Iitaka fibration.

## References


[1] I. Enoki, *Stability and negativity for tangent sheaves of minimal Kähler spaces*, Geometry and analysis on manifolds (Katata/Kyoto, 1987). Lecture Notes in Math., vol. 1339, Springer, Berlin, 1988.

[2] D. Greb, S. Kebekus, T. Peternell, and B. Taji, *The Miyaoka-Yau inequality and uniformisation of canonical models*, Ann. Sci. Ec. Norm. Supér. (4) **52** (2019), no. 6, 1487–1535.

[3] D. Greb, S. Kebekus, T. Peternell, and B. Taji, *Harmonic metrics on Higgs sheaves and uniformization of varieties of general type*, Math. Ann. **378** (2020), no. 3-4, 1061–1094.

[4] H. Guenancia and B. Taji, *Orbifold stability and Miyaoka-Yau inequality for minimal pairs*, Geom. Topol. **26** (2022), no. 4, 1435–1482.

[5] F. Hao and S. Schreieder, *Equality in the Bogomolov-Miyaoka-Yau inequality in the non-general type case*, J. Reine Angew. Math. **775** (2021), 87–115.

[6] M. Iwai, S. Matsumura, and N. Müller, *Minimal projective varieties satisfying Miyaoka's equality*, 2024. Preprint arXiv:2404.07568.

[7] S. Lu and B. Taji, *A characterization of finite quotients of abelian varieties*, Int. Math. Res. Not. IMRN **1** (2018), 292–319.

[8] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic geometry, Sendai, 1985, 1987, pp. 449–476.

[9] N. Müller, *Inequalities of Miyaoka-Yau type & Uniformisation of varieties of intermediate Kodaira Dimension*, 2026. Preprint arXiv:2601.15138.

[10] T. Peternell and P. Wilson, *Threefolds with extremal Chern classes*, Higher dimensional complex varieties. Proceedings of the international conference, Trento, Italy, June 15–24, 1994, 1996, pp. 357–378.

**Baily–Borel compactifications and b-semiampleness**

Benjamin Bakker

(joint work with Stefano Filipazzi, Mirko Mauri, Jacob Tsimerman)

The Siegel modular variety $A_g := \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ starts off life as a complex analytic variety and parametrizes principally polarized abelian varieties. Motivated by ideas from reduction theory, Satake [11] introduced a topological compactification

$$\overline{A}_g := A_g \sqcup A_{g-1} \sqcup \cdots A_1 \sqcup A_0$$

in 1956 and suggested it was the topological space underlying a complex analytic variety. This expectation was confirmed in 1958 by Baily [2], who furthermore showed $\overline{A}_g$ naturally underlies a projective complex algebraic variety, realized as $\operatorname{Proj} R$ for a finitely generated graded ring of automorphic forms $R$. Shortly thereafter, following further work of Satake [13, 12], a celebrated paper of Baily–Borel [3] constructed the corresponding projective algebraic compactifications of any locally symmetric variety in 1966. These compactifications enjoy many important properties, and have found many applications.

The general theory of variations of Hodge structures was developed at roughly the same time by Griffiths and Deligne. Roughly speaking, to any smooth projective family $f : X \to Y$, the local system $V_\mathbb{Z} := R^k f_*\mathbb{Z}_X$ together with the natural filtration on the associated flat vector bundle $R^k f_*\Omega^\bullet_{X/Y}$ gives such a variation. In particular, choosing a basepoint $y \in Y$ and letting $\Gamma$ be the image of the monodromy representation on $V_{\mathbb{Z},y}$, there is a natural map

$$\phi : Y \to \Gamma\backslash\mathbb{D}$$

where $\mathbb{D}$ is a period domain—a certain open subset of the variety of flags on $V_{\mathbb{C},y}$ of the appropriate type. In his 1970 ICM address, Griffiths [7] suggested that the closure of the image of the period map $\phi$ (which he proved was a complex analytic variety) should be algebraic, and that it should admit a compactification closely analogous to the Baily–Borel compactifications described above.

The algebraicity (in fact quasiprojectivity) of the closure $Z$ of the image of $\phi$ was established by the author with Y. Brunebarbe and J. Tsimerman in [4] using o-minimal GAGA. In more recent work with S. Filipazzi, M. Mauri, and J. Tsimerman [5] we confirm the remainder of Griffiths' conjecture.

Specifically, for a polarizable $\mathbb{Z}$-variation of Hodge structures $V$ on $Y$ with underlying local system $V_\mathbb{Z}$ and filtration $F^\bullet V_\mathcal{O}$ on the associated flat vector bundle $(V_\mathcal{O}, \nabla)$, let

$$L_Y := \otimes_p \det F^p V_\mathcal{O}$$

be the Griffiths line bundle. We define a section of $L_Y^k$ to have *moderate growth* if its Hodge norm grows sub-polynomially, or equivalently if its pullback $L^k_{Y'}$ to a resolution $Y'$ extends to a section of the natural extension[1] $(L^k_{Y'})_{\overline{Y}'}$ to a log

[1] If the local monodromy is unipotent there is no ambiguity; otherwise, some care must be taken in making this precise. In general one must also be careful with distinctions between stacks and coarse spaces, and one must take a sufficiently divisible power of the line bundle.

smooth compactification $\overline{Y}'$. We define the ring of moderate growth sections $B_Y := \bigoplus_k H^0_{\mathrm{mg}}(Y, L_Y^k)$.

**Theorem 1** (B–Filipazzi–Mauri–Tsimerman)**.** *Let $Z$ be the closure of the image of a period map. Then*

*(1) The ring $B_Z$ is finitely generated, $Z^{\mathrm{BB}} := \operatorname{Proj} B_Z$ is a projective compactification of $Z$, and the ample bundle $\mathcal{O}_{Z^{\mathrm{BB}}}(n)$ (for sufficiently divisible $n$) naturally restricts to $L_Z^n$.*

*(2) Any analytic morphism $(\Delta^*)^\ell \to Z$ for which the resulting morphism $(\Delta^*)^\ell \to \Gamma\backslash\mathbb{D}$ is locally liftable[2] extends to a morphism $\Delta^\ell \to Z^{\mathrm{BB}}$. Moreover, $\mathcal{O}_{Z^{\mathrm{BB}}}(n)$ (for sufficiently divisible $n$) pulls back to $(L^n_{(\Delta^*)^\ell})_{\Delta^\ell}$.*

**Corollary 2.** *Let $(\overline{Y}, D)$ be a log smooth algebraic space and $V$ a polarizable integral pure variation of Hodge structures on $\overline{Y}\backslash D$ with unipotent local monodromy. Then the Griffiths bundle $L_{\overline{Y}}$ of $V$ is semiample.*

**Corollary 3.** *Let $\mathcal{Y}$ be a reduced separated Deligne–Mumford stack equipped with a quasifinite period map. Then the coarse space $Y$ has a compactification $Y^{\mathrm{BB}}$ for which some power $L_Y^n$ of the Griffiths bundle extends to an ample bundle $\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ and such that for any analytic morphism $g : (\Delta^*)^\ell \to \mathcal{Y}$, the map on coarse spaces extends to $\bar{g} : \Delta^\ell \to Y^{\mathrm{BB}}$ with the property that $\bar{g}^*\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ and $(L^n_{(\Delta^*)^\ell})_{\Delta^\ell}$ are naturally identified.*

Corollary 3 for instance applies to any moduli stack of polarized varieties with an infinitesimal Torelli theorem, such as Calabi–Yau manifolds and most hypersurfaces.

We also prove a version of Corollary 2 for *any* line bundle arising from Hodge theory. We say a variation is a CY variation if the deepest piece of the Hodge filtration is a line bundle, and in this case we call this line bundle the Hodge bundle. The Griffiths bundle of $V$, for example, is the Hodge bundle of the variation $\otimes_p \wedge^{\operatorname{rk} F^p V_{\mathcal{O}}} V$. We then have the following:

**Theorem 4** (B–Filipazzi–Mauri–Tsimerman)**.** *Let $(\overline{Y}, D)$ be a proper log smooth algebraic space, $V$ a polarizable integral pure CY variation of Hodge structures on $\overline{Y}\backslash D$ with unipotent local monodromy, and $M_{\overline{Y}}$ the Hodge bundle on $\overline{Y}$. If $M_{\overline{Y}}$ is integrable with torsion combinatorial monodromy, then it is semiample.*

The integrability condition means that for any analytic germ $T \subset \overline{Y}$ for which the Hodge bundle is flat, it is flat on the Zariski closure $T^{\mathrm{Zar}}$ as well. The torsion combinatorial monodromy condition means that for any nodal curve $g\colon C \to \overline{Y}$ for which $g^*M_{\overline{Y}}$ is numerically trivial (hence torsion on each component), it is in fact torsion. Both conditions are needed in the statement of Theorem 4.

Theorem 4 is a step in the proof of Theorem 1: the Griffiths bundle has torsion combinatorial monodromy by a theorem of Green–Griffiths–Robles [6] (the integrability is standard). We also use Theorem 4 to resolve the b-semiampleness

[2] The local liftability is equivalent to the condition that $(\Delta^*)^\ell \to \Gamma\backslash\mathbb{D}$ be the period map of a variation. If $\Gamma$ is torsion-free the local liftability condition is automatic.

conjecture, formulated in increasing generality by Mori, Kawamata, Shokurov, Ambro, and Prokhorov–Shokurov; see [9, 1, 10] and references therein.

**Theorem 5** (B–Filipazzi–Mauri–Tsimerman)**.** *Let $f\colon (X,\Delta)\to Y$ be an lc-trivial fibration from a pair $(X,\Delta)$ such that $\Delta$ is effective over the generic point of $Y$. Then the moduli part $\mathbf{M}$ is b-semiample.*

There is a version of Theorem 1 for the ring of moderate growth sections of a Hodge bundle satisfying the two conditions of Theorem 4, which in particular implies:

**Corollary 6.** *Let $\mathcal{Y}$ be a moduli stack of polarized smooth Calabi–Yau varieties. Then the coarse space $Y$ has a unique normal compactification $Y^{\mathrm{BBH}}$ for which some power $M_Y^n$ of the Hodge bundle of the variation of Hodge structures on middle cohomology extends to an ample bundle $\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ and such that for any family $g:(\Delta^*)^\ell\to\mathcal{Y}$, the map on coarse spaces extends to $\bar{g}:\Delta^\ell\to Y^{\mathrm{BBH}}$ with the property that $\bar{g}^*\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ pulls back to $(M^n_{(\Delta^*)^\ell})_{\Delta^\ell}$.*

In brief, for the proof of Theorem 4 we first use the result of [4] to construct a map $q:\overline{Y}\to\overline{Z}$ on the level of definable topological spaces such that $\overline{Z}$ has a locally closed definable stratification $\overline{Z}^i$ by subspaces (with decreasing dimension) for which $\overline{Y}^i := q^{-1}(\overline{Z}^i)\to\overline{Z}^i$ is algebraic and for which $M_{\overline{Y}}|_{\overline{Y}^i}$ descends to an ample bundle. Letting $\overline{Z}^{<i}$ be the union of the first $i$ strata, we proceed by induction as follows. Assuming $\overline{Z}^{<i}$ is algebraic, we use Hodge theory to glue the definable analytic structures of $\overline{Z}^{<i}$ and $\overline{Z}^i$ into a definable analytic variety $\overline{Z}^{\leq i}$. We then use o-minimal GAGA to algebraize $\overline{Z}^{\leq i}$. For the proof of Theorem 5, we identify the moduli part with the Hodge bundle on the variation coming from middle cohomology of the minimal lc center of a sufficiently nice model. We then verify the two conditions in Theorem 4 using the immersivity of the Kodaira–Spencer map (for the integrability) and results of Kollár [8] on the $\mathbb{P}^1$-linking of minimal lc centers (for the torsion combinatorial monodromy).

## References

[1] F. Ambro, *The adjunction conjecture and its applications*, Thesis (Ph.D.)–The Johns Hopkins University, ProQuest LLC, Ann Arbor, MI, 1999.

[2] W. L. Baily, Jr., *Satake's compactification of $V_n$*, Amer. J. Math. **80** (1958), 348–364.

[3] W. L. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Annals of Mathematics **84** (1966), no. 3, 442–528.

[4] B. Bakker, Y. Brunebarbe, and J. Tsimerman, *o-minimal GAGA and a conjecture of Griffiths*, Inventiones Mathematicae **232** (2023), no. 1, 163–228.

[5] B. Bakker, S. Filipazzi, M. Mauri, and J. Tsimerman, *Baily–Borel compactifications of period images and the b-semiampleness conjecture*, arXiv:2508.19215, 2025.

[6] M. Green, P. Griffiths, and C. Robles, *Analog of Satake–Baily–Borel for period maps*, arXiv preprint arXiv:2010.06720, 2025.

[7] P. A. Griffiths, *Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems*, 1970.

[8] J. Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013.

[9] S. Mori, *Classification of higher-dimensional varieties*, in *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, Proc. Sympos. Pure Math., vol. 46, Part 1, Amer. Math. Soc., Providence, RI, 1987, pp. 269–331.
[10] Yu. G. Prokhorov and V. V. Shokurov, *Towards the second main theorem on complements*, J. Algebraic Geom. **18** (2009), no. 1, 151–199.
[11] I. Satake, *On the compactification of the Siegel space*, J. Indian Math. Soc. (N.S.) **20** (1956), 259–281.
[12] I. Satake, *On compactifications of the quotient spaces for arithmetically defined discontinuous groups*, Ann. of Math. (2) **72** (1960), 555–580.
[13] I. Satake, *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math. (2) **71** (1960), 77–110.

## Volume Inequalities and Volume Polynomials

Botong Wang
(joint work with June Huh, Mateusz Michałek, and Jian Xiao)

The talk presented a unified framework for understanding and generalizing several fundamental inequalities in convex geometry, algebraic geometry, linear algebra, and combinatorics. The guiding principle is that many of these inequalities can be viewed as inequalities among coefficients of *volume polynomials*. This point of view relates the Loomis–Whitney inequality, the Alexandrov–Fenchel inequality, the Khovanskii–Teissier inequality, the reverse Khovanskii–Teissier inequality, and certain negative correlation inequalities for bases of vector configurations.

**Volume polynomials.** For convex bodies $K_1, \dots, K_m \subset \mathbb{R}^n$, Minkowski's theorem states that

$$\mathrm{Vol}(x_1K_1 + \cdots + x_mK_m)$$

is a homogeneous polynomial of degree $n$ in $x_1, \dots, x_m$. Its coefficients are the mixed volumes. Similarly, for positive semidefinite Hermitian matrices $M_1, \dots, M_m$, the polynomial

$$\det(x_1M_1 + \cdots + x_mM_m)$$

is a homogeneous polynomial of degree $n$, whose coefficients are mixed discriminants. Finally, if $X$ is a projective variety, or more generally a compact Kähler manifold, and if $D_1, \dots, D_m$ are nef classes, then

$$\int_X (x_1D_1 + \cdots + x_mD_m)^n$$

is a homogeneous polynomial whose coefficients are intersection numbers.

The above three families of polynomials are called the *Convex body, Hermitian and projective (or Kähler) volume polynomials*, respectively. Convex body volume polynomials can be realized, up to limits, as projective volume polynomials using toric varieties. Hermitian volume polynomials can be realized as projective volume polynomials over $\mathbb{C}$ using abelian varieties and parallel semipositive $(1,1)$-forms. Thus intersection-theoretic inequalities imply the corresponding inequalities for mixed volumes and mixed discriminants.

**Main inequality.** The main result discussed in the talk is the following general inequality. Let $S$ be a multiset on $\{1,\ldots,s\}$ of cardinality $n$, and let $Q_1,\ldots,Q_k$ be an $r$-cover of $S$. Then, for nef classes $B_1,\ldots,B_s,A,C_1,\ldots,C_m$ on a projective variety $X$ of dimension $m+n$, one has

$$\prod_{i=1}^{k}\int_X B_1^{Q_i(1)}\cdots B_s^{Q_i(s)}A^{n-|Q_i|}C_1\cdots C_m \geq c\left(\int_X B_1^{S(1)}\cdots B_s^{S(s)}C_1\cdots C_m\right)^r\left(\int_X A^nC_1\cdots C_m\right)^{k-r},$$

where $c$ is an explicit constant depending only on $S,Q_1,\ldots,Q_k$. In the compact Kähler setting, the same inequality is proved when $m=0$.

This theorem contains several classical inequalities as special cases. When $n=2$, $S=\{1^2\}$, and $Q_1=Q_2=\{1\}$, it recovers the Khovanskii–Teissier inequality

$$\left(\int_X ABC_1\cdots C_m\right)^2\geq\int_X B^2C_1\cdots C_m\int_X A^2C_1\cdots C_m.$$

The corresponding mixed volume inequality is the Alexandrov–Fenchel inequality. When $S=\{1,\ldots,n\}$ and $Q_1=\{1,\ldots,k\}$, $Q_2=\{k+1,\ldots,n\}$, one recovers the reverse Khovanskii–Teissier inequality

$$\int_X B_1\cdots B_kA^{n-k}\int_X B_{k+1}\cdots B_nA^k\geq\frac{k!(n-k)!}{n!}\int_X B_1\cdots B_n\int_X A^n.$$

When $S=\{1,\ldots,n\}$ and $Q_i=\{i\}$, the corresponding mixed volume inequality gives the Loomis–Whitney inequality.

**A negative correlation inequality for vector configurations.** The theorem also implies a new negative correlation inequality for vector configurations. Let $V$ be a finite-dimensional vector space, and let $E\subset V$ be a finite generating set. For $R\subset E$, let $p(R)$ denote the probability that a uniformly random basis contained in $E$ contains $R$. If $B\subset E$ is a basis and $R_1,\ldots,R_r$ is a partition of $B$, then

$$p(R_1)p(R_2)\cdots p(R_r)\geq p(B).$$

The proof uses the arrangement Schubert variety $Y_E$ associated to the vector configuration. On $Y_E$ there are nef divisors $D_i$, indexed by the elements of $E$, such that the top intersection number of $D_{i_1},\ldots,D_{i_n}$ is 1 precisely when the corresponding vectors form a basis, and is 0 otherwise. The main theorem applied to these divisors gives the desired probabilistic inequality.

**Ideas of proof and further directions.** The proof has two complementary approaches. First, using linear operators on volume polynomials, the general multiset case can be reduced to the case in which $S$ is a set. In that case, one proves a corresponding mixed discriminant inequality for matrices and then globalizes it to Kähler geometry using Yau's theorem. Second, in the projective setting, there is an algebraic proof using the volume function on the pseudo-effective cone defined by global sections.

The talk concluded with two directions for further study. Since the Chow ring of a matroid is a combinatorial analogue of the cohomology ring of a smooth projective variety and satisfies Poincaré duality, hard Lefschetz, and the Hodge–Riemann relations, it is natural to ask whether the main inequality extends to the Chow ring of a matroid. Such a result would imply the analogous negative correlation inequality for arbitrary matroids. It is also natural to ask whether the full projective statement, with arbitrary auxiliary nef classes $C_1, \dots, C_m$, extends to compact Kähler manifolds.

## Equivariant descent for the Kawamata–Morrison cone conjecture

Cécile Gachet

(joint work with Hsueh-Yung Lin, Isabel Stenger, Long Wang)

The 1980s–2000s saw some striking finiteness results for the birational geometry of mildly singular Fano varieties.

**Theorem 1.** *Let $X$ be a terminal $\mathbb{Q}$-factorial variety with $K_X$ ample. Then*

1) *(Mori cone theorem, 1980s; see [7]) The nef cone $\mathrm{Nef}(X)$ is spanned by finitely many semiample Cartier divisor classes. In particular, it is a rational polyhedral cone.*
2) *(see [6, 1]) The movable cone $\overline{\mathrm{Mov}}(X)$, that is the closure of the convex cone spanned by all classes of divisors $D$ such that the base locus of $|D|$ has codimension at least 2, is rational polyhedral. Moreover, it decomposes into finitely many chambers with disjoint interiors, as*
$$\overline{\mathrm{Mov}}(X) = \bigcup_{i=1}^{k} \alpha_k^* \mathrm{Nef}(X_k),$$
*where the $\alpha_k \colon X \dashrightarrow X_k$, for $1 \leq i \leq k$, represent all isomorphisms in codimension 1 from $X$ to a $\mathbb{Q}$-factorial target.*
3) *(see [6]) The convex cone $\mathrm{Eff}(X)$ spanned by effective divisor classes is closed and rational polyhedral. As an aside, it also admits a finite chamber decomposition of geometric meaning.*

As one expects, these results dramatically fail when the ampleness condition is dropped. Let us explain how. An easy obstruction to rational polyhedrality of cones is given by the following observation:

**Lemma 2.** *Let $V = V_{\mathbb{Z}} \otimes \mathbb{R}$ be a finite dimensional real vector space with a preferred a lattice $V_{\mathbb{Z}}$ and let $\mathcal{C}$ be a rational polyhedral cone in $V$. Then the group $\Gamma$ of linear transformations preserving $V_{\mathbb{Z}}$ and $\mathcal{C}$ is finite.*

**Example 3.** This observation applies in many geometric situations.

1) If the group of components $\pi_0\mathrm{Aut}(X)$ is infinite, then $\mathrm{Nef}(X)$ is not rational polyhedral. For instance, take $n \geq 2$ and pick a general elliptic curve $E$. Then
$$\mathrm{Aut}(E^n) = E^n \rtimes \mathrm{GL}_n(\mathbb{Z})$$

has infinitely many components. The nef cone $\mathrm{Nef}(E^n)$ can be identified with the cone of $n$ by $n$ positive-definite symmetric real matrices.

2) Denote by $\mathrm{PsAut}(X)$ the group of birational automorphisms of $X$ that are isomorphisms of codimension one. If $\pi_0\mathrm{PsAut}(X)$ is infinite, then $\overline{\mathrm{Mov}}(X)$ is not rational polyhedral. For instance, Cantat–Oguiso prove in [2] that a general anticanonical threefold $X$ in $(\mathbb{P}^1)^4$ has
$$\mathrm{Aut}(X) = \{1\}, \quad \mathrm{PsAut}(X) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$
Its nef cone is simplicial, but its movable cone is of fractal nature.

3) Lemma 2 also applies to other group actions. Stenger–Xie use this fact to prove in [10] that the movable cone of the blow-up $X$ of $\mathbb{P}^3$ at 8 very general points has infinitely many extremal rays. Note that in this case, $\mathrm{PsAut}(X)$ is trivial ([3]).

Despite these obstructions, Morrison and then Kawamata proposed the following conjecture. In the stronger form popularized by Totaro [11], it goes:

**Conjecture 4.** *(Kawamata–Morrison cone conjecture) Let $(X, \Delta)$ be a klt Calabi–Yau pair, that is a klt pair with $K_X + \Delta \equiv 0$.*

1) *(nef) There is a rational polyhedral fundamental domain (RPFD) for the action of* $\mathrm{Aut}(X, \Delta)$ *on* $\mathrm{Nef}^+(X)$.
2) *(movable) There is a RPFD for the action of* $\mathrm{PsAut}(X, \Delta)$ *on* $\mathrm{Mov}^+(X)$.

Here, for a convex cone $\mathcal{C}$, we denote by $\mathcal{C}^+$ the convex hull of the set of rational points in the closure of $\mathcal{C}$. One cannot expect the action of $\mathrm{Aut}(X, \Delta)$ on the **whole** nef cone $\mathrm{Nef}(X)$ to have a RPFD, since the boundary of $\mathrm{Nef}(X)$ might contain extremal irrational rays; see Example 3 1) for $n = 2$.

During the talk, I spent some time explaining interesting consequences of the cone conjecture. I also surveyed families of varieties and a few sporadic examples for which the cone conjecture is known. This part of the talk is already well covered by the excellent survey [8], so I will only mention one recent result, which allows to focus on the movable cone conjecture:

**Theorem 5.** *([9, 12, 5]) Let $(X, \Delta)$ be a klt Calabi–Yau pair. If $\dim X = 3$, if $X$ is a projective hyperkähler variety, or if more generally, the chamber decomposition of $\mathrm{Mov}^+(X)$ generalizing Theorem 1 1) is locally finite, the following are equivalent:*

- *$(X, \Delta)$ satisfies the movable cone conjecture;*
- *$(X, \Delta)$ has finitely many minimal models, each of which satisfies the nef cone conjecture.*

It is worth emphasizing that in most known cases, the movable cone conjecture follows from a **structural description** of the movable cone: typically, $\overline{\mathrm{Mov}}(X)$ is clipped out by hyperplanes inside a homogeneous cone $\mathcal{H}$, and $\mathrm{PsAut}(X, \Delta)$ compares to an arithmetic subgroup of the linear algebraic group $\mathrm{Aut}^\circ(\mathcal{H})$. This is not always the case, notably not in Example 3 3). Nevertheless, this begets the question: What kind of description can one plausibly conjecture for the movable cone of a Calabi–Yau variety $X$? To be sure, it seems hard to gain much structural insight from Conjecture 4.

Another issue with Conjecture 4 is that it behaves quite poorly under equivariant descent. For instance, the following question is open:

**Question 6.** Let $(X, \Delta)$ be a klt Calabi–Yau pair and $G$ be a finite group acting on $X$ while preserving $\Delta$. The Riemann–Hurwitz formula lets us define a quotient Calabi–Yau pair $(X/G, \Delta_G)$, which is still klt. If the movable cone conjecture holds for $(X, \Delta)$, does it descend to $(X/G, \Delta_G)$?

This question is notably motivated by the singular Beauville–Bogomolov decomposition theorem and by index-one covers. One can ask the same question for a Galois group action: a positive answer would allow to descend the cone conjecture from an algebraically closed field to suitable subfields. Finally, this question is motivated by many interesting Calabi–Yau varieties built *à la Kummer*, by taking a finite singular quotient of a $K$-trivial variety, and then resolving it. During the talk, I described such a construction due to Oguiso and based on the Jacobian of the Klein quartic curve; see [4, Section 6] for more details.

Let us explain the main challenge of Question 6. The movable cone of a finite quotient $X/G$ naturally identifies with the $G$-invariant part of the movable cone of $X$. Meanwhile, the pseudoautomorphism group of the quotient pair $(X/G, \Delta_G)$ naturally inherits a subgroup of the form $C_{\mathrm{PsAut}(X,\Delta)}(G)/G$, where $C_{\mathrm{PsAut}(X,\Delta)}(G)$ denotes the centralizer of $G$ in $\mathrm{PsAut}(X, \Delta)$. Here, it is important to note that the size of the groups $G$ and $\mathrm{PsAut}(X, \Delta)$ do not allow to control the size of the centralizer $C_{\mathrm{PsAut}(X,\Delta)}(G)$ easily: for example, each of the four generating involutions of the pseudoautomorphism group of Example 3 3) has trivial centralizer. In short, to answer Question 6 affirmatively, one needs to control the discrepancy between the groups $C_{\mathrm{PsAut}(X,\Delta)}(G)$ and $\mathrm{PsAut}(X, \Delta)$. This can be done here:

**Theorem 7.** *(*[4, Theorem 1.4]*) Consider a klt Calabi–Yau pair $(X, \Delta)$ of the form*

$$A \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{m} (S_j, \Delta_j),$$

*where $A$ is an abelian variety, $Y_i$ is a primitive symplectic variety of dimension $2$ or of second Betti number at least $5$, and $(S_j, \Delta_j)$ is a klt Calabi–Yau pair on a smooth rational surface. Then, any finite quotient pair of $(X, \Delta)$ satisfies the movable cone conjecture.*

I then presented several results that had previously been obtained in this direction (following [4, Section 1.C]) and said a few words about the key ingredients of the proof, notably about how well self-dual homogeneous cones and arithmetic groups behave under taking invariant slices and centralizers, respectively.

## References


[1] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc.**23(2)** (2010), 405–468.

[2] S. Cantat and K. Oguiso, *Birational automorphism groups and the movable cone theorem for Calabi-Yau manifolds of Wehler type via universal Coxeter groups*, Amer. J. Math **137(4)** (2015), 1013–1044.

[3] C. Gachet, *The pseudoautomorphism group of $\mathbb{P}^3$ blown-up at 8 very general points*, arXiv:2509.12977.
[4] C. Gachet, *Well-clipped cones under finite quotient and applications to the cone conjecture*, arXiv:2504.01753.
[5] C. Gachet, H.-Y. Lin, Stenger I., and L. Wang, *The effective cone conjecture for Calabi–Yau pairs*, arXiv:2406.07307. To appear in J. Reine Angew. Math.
[6] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Mich. Math. J. **48** (2000), 331–348.
[7] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics **130** (1998).
[8] V. Lazić, K. Oguiso, and T. Peternell, *The Morrison-Kawamata cone conjecture and abundance on Ricci flat manifolds* in Uniformization, Riemann-Hilbert correspondence, Calabi-Yau manifolds and Picard-Fuchs equations (2018), 157–185.
[9] Z. Li and H. Zhao, *On the relative Morrison-Kawamata cone conjecture*, Proc. London Math. Soc. **131(5)** (2025).
[10] I. Stenger and Z. Xie, *Cones of divisors on $\mathbb{P}^3$ blown up at eight very general points*, arXiv:2303.12005.
[11] B. Totaro, *The cone conjecture for Calabi–Yau pairs in dimension 2*, Duke Math. J. **154(2)** (2010), 241–263.
[12] F. Xu, *On the cone conjecture for certain pairs of dimension at most 4*, arXiv:2405.20899.

## Relative non-reductive geometric invariant theory and moduli of quivers with multiplicities

Victoria Hoskins

(joint work with Eloise Hamilton, Joshua Jackson and Tanguy Vernet)

We begin by listing some key examples to motivate our work.

**Motivating examples of non-reductive actions.**

(1) Moduli of weighted projective hypersurfaces involves actions of non-reductive automorphism groups; see [3]. For example,
$$\mathrm{Aut}(\mathbb{P}(1,1,2)) = ((\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times) \ltimes (\mathbb{C}^+)^3)/\mathbb{C}^\times$$
where $(\mathbb{C}^+)^3$ indexes automorphisms $z \mapsto z + ax^2 + bxy + cy^2$.
(2) Bérczi and Kirwan [2] obtain spectacular results on hyperbolicity of a general projective hypersurface $X$ using moduli spaces of jets in $X$ constructed as quotients by a reparametrisation group of regular jets.
(3) Moduli of unstable objects (e.g. vector bundles of a fixed Harder–Narasimhan type) involve parabolic group actions.
(4) Moduli of representations of a quiver with multiplicities, involve a product of jet groups of general linear groups.

This last example is the central application of the talk, where our motivation comes from both geometry and representation theory.

**Representations of a quiver with multiplicities.** A representation of a quiver $Q$ is a tuple of vector spaces and linear maps between them indexed by the vertices and arrows of $Q$ respectively. This generalises to a representation of a quiver with multiplicities by replacing the vector spaces at each vertex by free modules over the truncated polynomial rings $\mathbb{C}_m = \mathbb{C}[\epsilon]/\epsilon^m$, where the order of truncation is determined by multiplicities at each vertex. For the one loop Jordan quiver with multiplicity $m$, the isomorphism classes of rank $r$ representations correspond to the orbits of the non-reductive action

$$\mathrm{G}L_{m,r} := \mathrm{G}L_r(\mathbb{C}_m) \curvearrowright \mathrm{Mat}_{r\times r}(\mathbb{C}_m).$$

Our motivation comes from both geometry and representation theory. Yamakawa originally introduced quivers with multiplicities to study moduli spaces of irregular connections on $\mathbb{P}^1$. They are also related to open de Rham spaces, which are symplectic reductions of products of coadjoint orbits for jet groups of $\mathrm{G}L_r(\mathbb{C})$.

There is a long and rich history connecting the representation theory of a quiver $Q$ over a field and an associated symmetric Kac–Moody Lie algebra $\mathfrak{g}_Q$. Work of Geiss, Leclerc and Schröer [4] shows that to generalise this to symmetrisable Kac–Moody Lie algebras, one needs quivers with multiplicities $(Q, \mathbf{m})$. For example, finite-representation type quivers without multiplicities are classified by the ADE Dynkin diagrams, and by allowing multiplicities one additionally obtains non-simply laced BCFG Dynkin diagrams. The representation theory of (quantum groups associated to) $\mathfrak{g}_Q$ can often be geometrically realised using the (co)homology of moduli spaces attached to $Q$. Nakajima [7] used moment map techniques to construct (non-compact hyperkähler) quiver varieties with interesting Lagrangian subvarieties, called nilpotent quiver varieties, whose top homology groups geometrically realise highest weight representations of $\mathfrak{g}_Q$. It is intriguing to ask if these geometric interpretations might also extend to quivers with multiplicities.

**Our relative perspective.** Often a group $G = U \rtimes R$ with unipotent radical $U$ acting on a variety $X$ fits into an equivariant action

$$\begin{array}{ccc} G & \curvearrowright & X \\ \downarrow & & \downarrow f \\ R & \curvearrowright & Z, \end{array}$$

where we know how to take a reductive GIT quotient of the $R$-action on $Z$. Our idea is to construct a quotient of the $G$-action on $X$ relative to a reductive GIT quotient $Z/\!/R$ by locally taking projective spectrum of rings of (semi)-invariant functions and gluing.

For vector bundles of fixed Harder–Narasimhan type, the morphism $f$ is the associated graded map for the Harder–Narasimhan filtration, and $Z/\!/R$ is a product of moduli spaces for the semistable factors. For quivers with multiplicities, $f$ is a truncation map sending a representation with multiplicities to a representation without multiplicities, and $Z/\!/R$ is King's moduli space of semistable quiver representations.

**Non-reductive geometric invariant theory.** Many difficulties of non-reductive group actions (see [5, §5.1]) have been overcome by using a 'grading' to control the unipotent radical; see [1] and the references therein. One of their key insights gives a way to construct slices of unipotent group actions on affine varieties provided (i) there is a grading of the action by $\mathbb{C}^\times$ and (ii) $\mathbb{C}^\times$-fixed points have trivial unipotent stabilisers. In moduli-theoretic set-ups, there is often a natural grading, but the unipotent stabiliser assumption (ii) can be trickier.

**Our relative approach.** With E. Hamilton and J. Jackson, we observe that the key insight mentioned above naturally generalises to actions on an affine morphism. This led us to construct relative quotients for equivariant actions on affine morphisms. Our relative results generalise (and recover) previous constructions of non-reductive GIT quotients, whilst being conceptually simpler and better suited to various applications. Moreover our relative perspective enables a direct proof of the non-reductive Hilbert–Mumford criterion.

**Applications to quivers with multiplicities.** Moduli of representations of a quiver $Q$ (over $\mathbb{C}$) of dimension vector $\mathbf{r}$ are constructed by King as reductive GIT quotients of the conjugation action

$$G_{\mathbf{r}} := \prod_{i} \mathrm{G}L_{r_i}(\mathbb{C}) \curvearrowright R_{\mathbf{r}} := \prod_{a\colon i\to j} \mathrm{Mat}_{r_j\times r_i}(\mathbb{C})$$

using a stability parameter $\theta$. For a quiver with multiplicities $(Q,\mathbf{m})$, this extends to an analogous action which fits into an equivariant action

$$\begin{array}{ccc} G_{\mathbf{m},\mathbf{r}} & \curvearrowright & R_{\mathbf{m},\mathbf{r}} \\ \downarrow & & \downarrow^{\tau} \\ G_{\mathbf{r}} & \curvearrowright & R_{\mathbf{r}}, \end{array}$$

on a truncation morphism $\tau$, which forgets the positive powers of $\epsilon$ for constant $\mathbf{m}$ . With J. Jackson and T. Vernet [6], we find a grading of this action and under certain unipotent stabiliser assumptions (which hold if $\theta$-semistability and $\theta$-stability coincide on $R_{\mathbf{r}}$), we build new moduli spaces for $(Q,\mathbf{m})$ depending on an additional stability parameter $\rho$ with a projective-over-affine morphism

$$\mathcal{M}^{\theta,\rho-\mathrm{ss}}_{Q,\mathbf{m};\mathbf{r}} := R_{\mathbf{m},\mathbf{r}}/\!/_{\theta,\rho}\, G_{\mathbf{m},\mathbf{r}} \to \mathcal{M}^{\theta-\mathrm{ss}}_{Q;\mathbf{r}} := R_{\mathbf{r}}/\!/_{\theta}\, G_{\mathbf{r}}$$

to King's moduli space. Moreover, we moduli-theoretically describe $(\theta,\rho)$-semistability using the non-reductive Hilbert–Mumford criterion, and construct analogous versions of Nakajima quiver varieties for $(Q,\mathbf{m})$.

This opens up many interesting questions about the geometry (and cohomology) of these new moduli spaces, and in particular, their relationship with representation theory. We are only just scratching the surface: so far we have established cohomological purity of these moduli spaces, but we hope to establish a rich geometric story and find deep connections with representation theory.

## References

[1] G. Bérczi, F. Kirwan, *Moment maps and cohomology of non-reductive quotients*, Invent. math. **235** (2014), 1–79.

[2] G. Bérczi, F. Kirwan, *Non-reductive geometric invariant theory and hyperbolicity*, Invent. math. **235** (2024), 81–127.

[3] D. Bunnett, *On the moduli of hypersurfaces in toric orbifolds*, Proc. Edinburg Math. Soc. **67(2)** (2024), 577–616.

[4] C. Geiss, B. Leclerc, J. Schröer, *Quivers with relations for symmetrizable Cartan matrices I: Foundations*, Invent. math. **209** (2017), 61–158.
[5] V. Hoskins, *Moduli spaces and geometric invariant theory: old and new perspectives* in *Moduli Spaces and Vector Bundles – New Trends*, Contemp. Math. **803** (2024) AMS.
[6] V. Hoskins, J. Jackson and T. Vernet, *Moduli spaces of representations of quivers with multiplicities via non-reductive GIT*, arXiv preprint 2510.24559.
[7] H. Nakajima, *Instantons on ALE spaces, quiver varieties and Kac-Moody algebras*, Duke Math. J. **2** (1994), 365–416.

## Nonvanishing results for compact hyperkähler manifolds

Christian Lehn
(joint work with Andreas Höring and Vladimir Lazić)

In a joint work with A. Höring and V. Lazić, we address the Semiampleness Conjecture for compact hyperkähler manifolds. In this context, it essentially specializes to the Hyperkähler SYZ Conjecture, which can be formulated as follows.

**Conjecture 1.** *Let $X$ be a compact hyperkähler manifold of dimension $2n$, and let $L$ be a nef line bundle on $X$ satisfying $q_X(L) = 0$, where $q_X$ denotes the Beauville–Bogomolov–Fujiki form. Then $L$ is semiample.*

Note that any such morphism would be a Lagrangian fibration by Matsushita's theorem [4]. A common strategy for conjectures of this kind is to divide and conquer:

(1) Prove that a multiple of $L$ has a section, i.e. that there exists $m > 0$ such that $H^0(X, mL) \neq 0$ (equivalently $\kappa(X, L) \geq 0$).
(2) Suppose that some multiple of $L$ has a nontrivial section. Then show that $L$ is semiample.

Part 1 is called the Nonvanishing Conjecture. We refer to part 2 as the (proper) Semiampleness Conjecture. Both are wide open in general.

Let us describe our results. Concerning the Nonvanishing Conjecture, we build on ideas of Verbitsky [5]. This generalizes work of Liu–Matsumura [3] in dimension four to arbitrary dimension. We show in [2] that in the situation of Conjecture 1 either $\kappa(X, L) \geq 0$ or the Lelong sets of a current in $\mathrm{c}_1(L)$ have a highly constrained geometry. Among other things, they are shown to be coisotropic for the holomorphic symplectic form on $X$. This makes them amenable e.g. to methods from deformation theory.

In an ongoing joint work with the same authors, we aim to establish Part 2, i.e. the semiampleness of $L$, when the dimension of $X$ is four. Given $D \in |mL|$ for some $m \gg 0$, we first show that $K_D = mL|_D$ is semiample. Note that $D$ might be highly singular, even nonnormal, but is always Gorenstein since $X$ is smooth. Thus, $K_D$ is a line bundle. The typical strategy from birational geometry would then be to lift sections of $\ell K_D$ on $D$ to sections of $\ell mL$ on $X$. This strategy works well in some cases, using among other things the plt extension theorem of [1]. In the dlt case, we instead use deformation theory of the Lagrangian surfaces which appear as fibers of the fibration $D \to \Gamma$ induced by $\ell K_D$. All of this relies on a detailed analysis of the geometry of $D$ and its fibration.

## References


[1] J.-P. Demailly, C. D. Hacon and M. Păun, *Extension theorems, non-vanishing and the existence of good minimal models*, Acta Math. **210** (2013), no. 2, 203–259.

[2] A. Höring, V. Lazić and C. Lehn, *Nonvanishing results for Kähler varieties*, preprint, arXiv:2508.14634 (2025).

[3] H. Liu and S.-i. Matsumura, *Strictly nef divisors on K-trivial fourfolds*, Math. Ann. **387** (2023), no. 1–2, 985–1008.

[4] D. Matsushita, *On fibre space structures of a projective irreducible symplectic manifold*, Topology **38** (1999), no. 1, 79–83.

[5] M. Verbitsky, *Parabolic nef currents on hyperkaehler manifolds*, preprint, arXiv:0907.4217 (2009).


## Moduli of hypersurfaces via minimal exponent

Sung Gi Park

This talk is based on the paper [3], which establishes a new GIT stability criterion for hypersurfaces in projective space and describes the indeterminacy locus of the period map in terms of the minimal exponent.

Geometric invariant theory provides a way to construct a compact GIT moduli space $\overline{\mathcal{M}}^{\mathrm{GIT}}_{(n,d)}$ of degree $d$ hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$. Concretely, this moduli space is obtained by taking the quotient of the Hilbert scheme of degree $d$ hypersurfaces by the action of linear changes of coordinates. However, this GIT moduli space does not parametrize all hypersurfaces in $\mathbb{P}^n$, but only those that are GIT semistable.

From the viewpoint of singularities, a number of sufficient criteria for GIT (semi)stability of a hypersurface $X \subset \mathbb{P}^n$ were previously known: $X$ is stable when $X$ is smooth and $d \geq 3$ (Mumford); $X$ is stable (resp. semistable) when $\mathrm{lct}(\mathbb{P}^n, X) > \frac{n+1}{d}$ (resp. $\geq$) and $d \geq n+1$ (Hacking, Kim–Lee); and $X$ is stable when $X$ has canonical singularities and $d = n+1$ (Tian). Our new stability criterion generalizes these results and is especially effective for Fano hypersurfaces, that is, when $d \leq n$:

**Theorem 1.** *A hypersurface $X \subset \mathbb{P}^n$ of degree $d \geq 3$ is GIT stable (resp. semistable) if its minimal exponent satisfies $\widetilde{\alpha}(X) > \frac{n+1}{d}$ (resp. $\geq$).*

The global minimal exponent $\widetilde{\alpha}(X)$ is defined as the minimum of the local minimal exponents $\widetilde{\alpha}_x(X)$ over all points $x \in X$, namely $\widetilde{\alpha}(X) := \min_{x \in X} \widetilde{\alpha}_x(X)$. If $f = 0$ is a local defining equation of $X$ at a point $x$, then $\widetilde{\alpha}_x(X)$ is defined as the negative of the largest root of the reduced Bernstein–Sato polynomial of $f$.

The minimal exponent $\widetilde{\alpha}(X)$ is a positive rational number unless $X$ is smooth, in which case one sets $\widetilde{\alpha}(X) = \infty$. Moreover, we have $\mathrm{lct}(\mathbb{P}^n, X) = \min\{\widetilde{\alpha}(X), 1\}$, and $X$ has canonical singularities if and only if $\widetilde{\alpha}(X) > 1$. Thus, the theorem recovers the previously known criteria mentioned above. As an immediate consequence, every nodal hypersurface is GIT stable when $d \geq 3$ and $n \geq 3$. This was also obtained recently by Mordant and He by completely different methods.

The converse statement, namely a meaningful lower bound on the minimal exponent for GIT semistable hypersurfaces, is difficult to expect in general when $d \geq 4$, since there exist reducible GIT semistable hypersurfaces. By contrast, for cubic hypersurfaces, the GIT moduli space is conjecturally equivalent to the K-moduli space. Combined with Odaka's theorem that K-semistable varieties have klt singularities, this conjectural equivalence suggests that GIT semistable cubic hypersurfaces should have canonical singularities, a question raised by Spotti and Sun [4, Question 5.8]. By proving a uniform lower bound for the minimal exponent, we obtain a positive answer to this question:

**Theorem 2.** *For $n \geq 3$, every GIT semistable cubic hypersurface $X \subset \mathbb{P}^n$ satisfies*

$$\widetilde{\alpha}(X) \geq \max\left\{\frac{4}{3}, \frac{n+1}{9}\right\}.$$

*If $n \geq 6$, then $\widetilde{\alpha}(X) \geq \frac{5}{3}$ and $X$ has terminal singularities.*

For some low-dimensional and low-degree hypersurfaces, a complete classification of GIT (semi)stable hypersurfaces is known. We summarize below the analyses of low-dimensional cubic hypersurfaces, in comparison with Theorem 1.
(1) $n = 3$: cubic surfaces (Hilbert).

$$\text{stable} \Longleftrightarrow \text{at worst } A_1\text{-singularities} \Longleftrightarrow \widetilde{\alpha}(X) > \tfrac{4}{3}$$
$$\text{semistable} \Longleftrightarrow \text{at worst } A_1, A_2\text{-singularities} \Longleftrightarrow \widetilde{\alpha}(X) \geq \tfrac{4}{3}$$

(2) $n = 4$: cubic threefolds (Allcock, Yokoyama). Let $T$ denote the chordal cubic, which is known to be GIT polystable.

$$\text{stable} \Longleftrightarrow \text{at worst } A_1, A_2, A_3, A_4\text{-singularities} \Longleftrightarrow \widetilde{\alpha}(X) > \tfrac{5}{3}$$
$$\text{semistable} \not\sim_{GIT} T \Longleftrightarrow \text{at worst } A_1, A_2, A_3, A_4, A_5, D_4\text{-singularities}$$
$$\Longleftrightarrow \widetilde{\alpha}(X) \geq \tfrac{5}{3}$$

Here, "semistable $\not\sim_{GIT} T$" means "semistable but not GIT equivalent to $T$."
(3) $n = 5$: cubic fourfolds (Yokoyama, Laza). Let $\chi$ denote the one-parameter family of GIT polystable cubic fourfolds introduced in [2, Theorem 2.6].

$$\text{stable, isolated singularities} \Longleftrightarrow \text{simple } ADE\text{-singularities} \Longleftrightarrow \widetilde{\alpha}(X) > 2$$
$$\text{semistable} \not\sim_{GIT} \chi \Longrightarrow \widetilde{\alpha}(X) \geq 2 \Longrightarrow \text{semistable}$$

In summary, on the 10-dimensional moduli space of cubic threefolds and the 20-dimensional moduli space of cubic fourfolds, the only exceptions not captured by our sufficient criterion for stability in Theorem 1 are $T$ and $\chi$, respectively. These exceptional loci are precisely the indeterminacy loci of the period map to the Baily–Borel compactification of the period domain, by results of Shah, Kondō, Allcock, Carlson, Toledo, Looijenga, Swierstra, Laza, and Artebani:

**Theorem 3.** *For each pair $(n, d) \in \{(2,4), (2,6), (3,3), (4,3), (5,3)\}$, there exists a Baily–Borel compactification $(\Gamma\backslash D)^*$ of the period domain $\Gamma\backslash D$ associated to degree $d$ hypersurfaces in $\mathbb{P}^n$, together with the period map*

$$\mathcal{P} : \overline{\mathcal{M}}^{\mathrm{GIT}}_{(n,d)} \dashrightarrow (\Gamma\backslash D)^*.$$

*Moreover, the domain of definition of $\mathcal{P}$ is precisely $\mathcal{M}_{\widetilde{\alpha} \geq \frac{n+1}{d}}$, and the restriction of $\mathcal{P}$ to $\mathcal{M}_{\widetilde{\alpha} > \frac{n+1}{d}}$ is an open embedding into $\Gamma\backslash D$.*

Here, $\mathcal{M}_{\widetilde{\alpha}\geq\frac{n+1}{d}}$ (resp. $\mathcal{M}_{\widetilde{\alpha}>\frac{n+1}{d}}$) denotes the locus in the GIT moduli space consisting of hypersurfaces satisfying $\widetilde{\alpha}(X) \geq \frac{n+1}{d}$ (resp. $>$).

Motivated by this observation, and by the recent work of Bakker, Filipazzi, Mauri, and Tsimerman [1] proving the existence of a Hodge-theoretic compactification for moduli spaces of polarized Calabi–Yau varieties, we propose the following question.

**Question.** For a pair $(n, d)$, does there exist a Hodge-theoretic compactification $\overline{M}^{\mathrm{BBH}}$ of the moduli space $\mathcal{M}_{\widetilde{\alpha}>\frac{n+1}{d}}$, together with a period map

$$\mathcal{P} : \overline{\mathcal{M}}^{\mathrm{GIT}}_{(n,d)} \dashrightarrow \overline{M}^{\mathrm{BBH}},$$

whose domain of definition is exactly $\mathcal{M}_{\widetilde{\alpha}\geq\frac{n+1}{d}}$?

The results of [1] imply the existence of $\overline{M}^{\mathrm{BBH}}$ for $d \geq n+1$, though the precise domain of definition of the resulting period map is still unclear. For Fano hypersurfaces, i.e. when $d \leq n$, this question appears to be widely open, apart from a few low-dimensional and low-degree cases.

## References


[1] B. Bakker, S. Filipazzi, M. Mauri, and J. Tsimerman, *Baily–Borel compactifications of period images and the b-semiampleness conjecture*, preprint arXiv:2508.19215.

[2] R. Laza, *The moduli space of cubic fourfolds via the period map*, Ann. of Math. (2) **172** (2010), no. 1, 673–711.

[3] S. G. Park, *The GIT stability and Hodge structures of hypersurfaces via minimal exponent*, preprint arXiv:2510.14352.

[4] C. Spotti and S. Sun, *Explicit Gromov-Hausdorff compactifications of moduli spaces of Kähler-Einstein Fano manifolds*, Pure Appl. Math. Q. **13** (2017), no. 3, 477–515.


# On log Calabi-Yau manifolds


Henri Guenancia

(joint work with Tristan Collins)


## 1. Introduction

Let $X$ be a compact Kähler manifold such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. A seminal result of Yau [9] shows the existence in any given Kähler class of a unique Kähler metric $\omega$ which is Ricci flat, i.e. $\mathrm{Ric}\,\omega = 0$. This result is the cornerstone behind the celebrated Beauville-Bogomolov decomposition theorem [1], asserting that there is a finite étale cover $X' \to X$ which splits as a product $X' \simeq T \times \prod_{i\in I} Y_i \times \prod_{j\in J} Z_j$ where $T$ is a complex torus, the $X_i$ are irreducible Calabi-Yau manifolds and the $Y_j$ are irreducible holomorphic symplectic manifolds. Among some consequences of the decomposition theorem (or its proof) we have

(1) *Bochner principle:* holomorphic tensors are parallel with respect to $\omega$.
(2) *Stability:* the tangent bundle $T_X$ is polystable with respect to any Kähler class.

(3) *Albanese:* the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is a locally trivial fibration, which becomes trivial after a finite base change.
(4) *Universal cover:* the universal cover $\widetilde{X}$ is isomorphic to $\mathbb{C}^r \times Y$ where $r \in \mathbb{N}$ and $Y$ is compact Kähler. In particular, $\widetilde{X}$ is an analytic Zariski open set in a compact Kähler manifold.

It is natural to wonder to which extent the above results hold true in the setting of log Calabi-Yau manifolds, i.e. for pairs $(X, D)$ where $X$ is a compact Kähler manifold and $D = \sum_{i=1}^k D_i$ is a divisor with simple normal crossings such that $K_X + D \simeq \mathcal{O}_X$.

The first question to address is about the existence of a complete Ricci flat Kähler metric on the complement of $D$. This turns out to be a quite thorny question and very few cases are known. E.g. the answer is positive if $D$ is smooth and ample [8], if $D = D_1 + D_2$ with $D_i$ ample mutually proportional [3], or if $D$ is a fiber of an elliptic fibration from a rational elliptic surface [7]. In general, it is expected that there is no complete Ricci flat Kähler metric on $X \setminus D$.

In the article [2], we focus on the first two cases to showcase how these two similarly looking types of log Calabi-Yau manifolds actually behave in a drastically different way.

## 2. The case of one ample divisor

Here, we assume that $X$ is a compact Kähler manifold of dimension $n \geq 2$ and that $D$ is a *smooth, ample* divisor such that $K_X + D \simeq \mathcal{O}_X$. We let $\omega$ denote the complete Ricci flat Kähler metric on $X \setminus D$ constructed by [8]. Since $D$ is ample, one can see that $\pi_1(X \setminus D)$ is finite so that, in particular, the quasi-Albanese variety of $(X, D)$ is a point. It remains to investigate the analogue of Bochner principle and the polystability of the tangent bundle to this logarithmic setting. This is the content of the first result of [2] below.

**Theorem 1.** *In the setup above, we have*
*(1) Every holomorphic logarithmic tensor on $(X, D)$ is parallel with respect to $\omega$ on $X \setminus D$.*
*(2) The logarithmic tangent bundle $T_X(-\log D)$ is stable with respect to $c_1(D)$.*

By definition, the logarithmic tangent bundle $T_X(-\log D)$ is the dual of the vector bundle of the logarithmic differentials $\Omega^1_X(\log D)$, and a holomorphic logarithmic tensor is a global section of the holomorphic vector bundle $T_X(-\log D)^{\otimes p} \otimes \Omega^1_X(\log D)^{\otimes q}$ for some integers $p, q \geq 0$.

A word about the proof of (1). Given a holomorphic tensor $\sigma$, we have on $X \setminus D$

$$\Delta_\omega |\sigma|^2_\omega = |\nabla \sigma|^2_\omega$$

since $\omega$ is Ricci flat, and where $\nabla$ is the Chern connection associated to $\omega$. The main observation that allows one to derive the vanishing $\nabla\sigma = 0$ is that $\omega$ has strictly subquadratic volume growth at infinity, so that one can justify the formal identity $\int_{X \setminus D} \Delta_\omega(|\sigma|^2_\omega)\, \omega^n = 0$ using suitable cut-off functions.

## 3. The case of two linearly equivalent ample divisors

Let us now consider what is probably one of the simplest examples involving two linearly equivalent ample divisors. Namely, we fix an odd integer $n \geq 3$, set $X := \mathbb{P}^n$ and pick two general hypersurfaces $D_1, D_2$ of degree $\frac{n+1}{2}$ so that $K_X + D \simeq \mathcal{O}_X$ with $D = D_1 + D_2$. We denote by $\omega$ the complete Ricci flat Kähler metric on $X \setminus D$ constructed in [3]. The second main result of [2] is the following

**Theorem 2.** *In the setup above, the following properties hold.*

*(1) The Bochner principle for holomorphic logarithmic tensors fails.*
*(2) The quasi-Albanese map* $\mathrm{alb}_{(X,D)} : X \setminus D \to \mathrm{Alb}(X, D)$ *is not locally trivial.*
*(3) The logarithmic tangent bundle* $T_X(-\log D)$ *is not polystable.*
*(4) The universal cover of* $X \setminus D$ *is not biholomorphic to an analytic Zariski open subset of a compact complex manifold.*

Some explanations are in order. Given a log smooth pair $(X, D)$, the quasi-Albanese variety of $X$ is the quotient $\mathrm{Alb}(X, D) := H^0(X, \Omega_X^1(\log D))^*/\mathrm{Im}(H_1(X \setminus D, \mathbb{Z}))$ where $[\gamma] \in H_1(X \setminus D, \mathbb{Z})$ acts on $\alpha \in H^0(X, \Omega_X^1(\log D))$ by $\gamma \cdot \alpha = \int_\gamma \alpha$. When $X$ is projective, $\mathrm{Alb}(X, D)$ has a structure of quasi-abelian variety (i.e. an extension of an abelian variety by $(\mathbb{C}^*)^d$ for some $d \geq 0$). Moreover, there is a holomorphic map $\mathrm{alb}_{(X,D)} : X \setminus D \to \mathrm{Alb}(X, D)$. We refer to [6] for a comprehensive picture.

The key to understanding the failure of the four properties (Bochner principle, local triviality of Albanese, polystability of the tangent bundle, quasi-compactness of the universal cover) in our example is provided by the function

$$f : X \setminus D \to \mathbb{C}^*$$

defined by $f = \frac{s_1}{s_2}$ where $s_i \in H^0(X, \mathcal{O}_X(D_i))$ is "the" canonical section, i.e. $(s_i = 0) = D_i$.

In other words, $f$ is just the Lefschetz pencil associated to $\{D_1, D_2\} \subset |\mathcal{O}_{\mathbb{P}^n}(\frac{n+1}{2})|$. It is a classical fact that when $D_i$ are general, $f$ has always at least one critical value in $\mathbb{C}^*$ and the fiber of $f$ over such a point has a single singular point which is an ordinary double point. That is, there are coordinates near that point such that $f(z) = \sum_{i=1}^n z_i^2$.

Let us now give a brief breakdown of how one can prove the theorem.

*Bochner.* Consider the logarithmic one-form $\alpha := \frac{df}{f} \in H^0(X, \Omega_X^1(\log D))$. One can check directly from the asymptotics of $\omega$ that the norm of $\alpha$ with respect to $\omega$ is unbounded near $D$. In particular $\alpha$ cannot be parallel.

*Albanese.* With little work, one can identify $f$ with $\mathrm{alb}_{(X,D)}$. Since $f$ has some singular fibers, it follows that the quasi-Albanese map cannot be locally trivial. Actually much more can be said using the deep results of Schreieder (see [4] and [5]): when $n \geq 9$ and $D_i$ are very general, then two very general fibers of $\mathrm{alb}_{(X,D)}$ are not birational to each other.

*Stability.* The function $f$ extends to a rational map $X \dashrightarrow \mathbb{P}^1$ and induces an exact sequence

$$0 \to T_{X/\mathbb{P}^1}(-\log D) \to T_X(-\log D) \to f^*T_{\mathbb{P}^1}(-\log(0+\infty)) \to 0.$$

The rightmost term is trivial (trivialized by the logarithmic vector field $z\frac{\partial}{\partial z}$ generating the standard $\mathbb{C}^*$ action on $\mathbb{P}^1$ preserving 0 and $\infty$). If $T_X(-\log D)$ were polystable, then the map $T_X(-\log D) \to \mathcal{O}_X$ should split, yielding a vector field $v \in H^0(X, T_X(-\log D))$ lifting $z\frac{\partial}{\partial z}$. From here, there are at least two ways to conclude. Either one can argue directly that $H^0(X, T_X(-\log D))$ using Kodaira vanishing theorem (see [2, Proposition 2.7]). Or one can consider the holomorphic function $\alpha(v)$ which has to be constant. Since $df$ vanishes at some point in $X \setminus D$, we infer $\alpha(v) \equiv 0$, i.e. $v$ is vertical for $f$, hence it cannot lift $z\frac{\partial}{\partial z}$.

*Universal cover.* Set $M := X \setminus D$ and define

$$\widehat{M} := M \times_{\mathbb{C}^*} \mathbb{C} = \{(x,w) \in M \times \mathbb{C};\ e^{2\pi i w} = f(x)\}$$

which sits in the commutative diagram

$$\begin{tikzcd} \widehat{M} \arrow[r, "\widehat{f}"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "e^{2\pi i \cdot}"] \\ M \arrow[r, "f"] & \mathbb{C}^* \end{tikzcd}$$

It can be checked that $\widehat{M}$ is connected and that the map $\widetilde{M} \to \widehat{M}$ from the universal cover of $M$ is finite. Therefore it is enough to show that $\widehat{M}$ is not quasi-compact. Actually one proves the stronger property that $H_n(\widehat{M}, \mathbb{Z})$ is not finitely generated. The two key points are that $f$ has an ODP singularity and that $\mathbb{Z}$ acts on $\widehat{M}$. The first fact allows one to construct an $n$-cycle $L$ on $\widehat{M}$ by connecting the two vanishing cycles associated to a singular fiber of $\widehat{f}$ and its translate by $1 \in \mathbb{Z}$. The second fact allows one to translate $L$ by the elements of the Galois group $\mathbb{Z}$ of $\widehat{M} \to M$ in order to create infinitely many cycle classes in $H_n(\widehat{M}, \mathbb{Z})$ whose span cannot be finitely generated.

## References


[1] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **4** (1984), 755–782.

[2] T. Collins and H. Guenancia, *Log Calabi–Yau manifolds: holomorphic tensors, stability and universal cover*, arXiv:2509.07508.

[3] T.C. Collins and Y. Li, *Complete Calabi-Yau metrics in the complement of two divisors*, Duke Math. J. **173** (2024), no. 18, 3559–3604.

[4] S. Schreieder, *Stably irrational hypersurfaces of small slopes*, J. Am. Math. Soc. **32** (2019), no. 4, 1171–1199.

[5] E. Shinder, *Variation of stable birational types of hypersurfaces*, Recent developments in algebraic geometry. To Miles Reid for his 70th birthday. Cambridge: Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. 478, 296-313 (2022)

[6] O. Fujino, *On quasi-Albanese maps*, arXiv:2402.04595.

[7] H.-J. Hein, *Gravitational instantons from rational elliptic surfaces*, J. Am. Math. Soc. **25** (2012), no. 2, 355–393.

[8] G. Tian and S.-T Yau, *Complete Kähler manifolds with zero Ricci curvature. I*, J. Am. Math. Soc. **3** (1990), no. 3, 579–609.
[9] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Commun. Pure Appl. Math. **31** (1978), 339–411.

## Hyperbolicity, specialness and fundamental groups

Benoît Cadorel
(joint work with Ya Deng, Katsutoshi Yamanoi)

### 1. The conjectures

1.1. **Varieties of general and special types.** *Complex hyperbolicity*, in a somewhat narrow sense, aims at understanding the behaviour of entire curves in complex varieties. Here, an *entire curve* on a given variety $X$ should be understood as a non-constant holomorphic map $f : \mathbb{C} \to X$; if $X$ does not admit any such map, it is said to be *(Brody) hyperbolic*. The main conjecture in the field is the *Green-Griffiths-Lang conjecture*:

**Conjecture 1** (Green-Griffiths [4], Lang [5]). *Let $X$ be a complex projective manifold. The following are equivalent:*

*(1) $X$ is of* general type*;*
*(2) $X$ is* Brody hyperbolic *modulo a proper algebraic subset $Z \subsetneq X$. In other words, any entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Z$.*

Let us recall that $X$ is of *general type* if its canonical bundle $K_X := \bigwedge^{\dim X} \Omega_X$ is *big*, i.e. we have the maximal growth

$$h^0(X, K_X^{\otimes m}) \geq C m^{\dim X} \quad \text{as } m \longrightarrow +\infty,$$

for some $C > 0$.

Today, the Green-Griffiths-Lang conjecture is still largely open, even though many partial results have been obtained for specific classes of varieties (e.g. hypersurfaces, varieties of high irregularity, varieties supporting variations of Hodge structures, etc.)

As advocated by Campana [1], the statement above should be seen as one part of a diptych of conjectures: on the one hand, the Green-Griffiths-Lang conjecture predicts that varieties of *general type* admit "few" entire curves, while on the other hand varieties of *special type* should contain "many" such curves:

**Conjecture 2** (Campana [1]). *Let $X$ be a complex projective manifold. The following are equivalent:*

*(1) $X$ is of* special type*;*
*(2) $X$ admits a* dense entire curve *(either for the Euclidean or the Zariski topology).*

Loosely speaking, a projective manifold $X$ is of *special type* if it does not admit any rational fibration $f : X \dashrightarrow Y$ where $Y$, endowed with its ramification divisor $\Delta_Y$, forms a pair $(Y, \Delta_Y)$ of general type. In other words, $K_Y + \Delta_Y$ cannot be big[1].

Note that all the conjectures above admit analogues in the quasi-projective setting. It is possible to define a quasi-projective manifold $U$ as being of general (resp. special) type, by requiring a suitable property for a log-compactification $X = U \sqcup D$. The two conjectures above are expected to hold without change if we replace *projective* by *quasi-projective.*

1.2. **A conjectural Liouville property for special manifolds.** Note that the second item of Conjecture 2 could be stated even more optimistically, and be replaced by the following:

*(2')* the universal cover $\widehat{X}$ admits a dense entire curve for the Euclidean topology.

If this holds, this would imply that universal covers of special manifolds satisfy some sort of Liouville property:

**Conjecture 3.** *Let $X$ be a projective manifold of special type. Then any bounded holomorphic map $f : \widetilde{X} \to \mathbb{C}$ is constant.*

Again, this conjecture makes sense in the quasi-projective setting as well. For example, this conjecture should apply to log-Calabi-Yau quasi-projective manifolds, which are special by a theorem of Campana.

## 2. An isotriviality result

Conjecture 3 seems quite difficult to show in general. If one adds more conditions on the map $f$, we can still hope to obtain partial results; the next statement adds an equivariance condition on this map:

**Theorem 4.** *Let $U$ be a quasi-projective manifold of special type. Assume that we are given:*

(1) *a holomorphic map $f : \widetilde{U} \to \mathbb{B}^n$, where $\mathbb{B}^n$ is the complex ball;*
(2) *a representation $\rho : \pi_1(U) \to \mathrm{Aut}(\mathbb{B}^n)$;*

*such that $f$ is $\rho$-equivariant. Then $f$ is constant.*

This theorem is actually a consequence of the following more general result:

**Theorem 5** ([2])**.** *Let $U$ be a quasi-projective manifold of special type. Let $\mathbb{V} := (\mathbb{V}, F^\bullet\mathcal{V}, h)$ be a polarized $\mathbb{C}$-VHS on $U$. Then $\mathbb{V}$ is* isotrivial, *i.e. its period map is constant.*

[1] There are several caveats about this definition. First of all, the construction of $\Delta_Y$ should follow Campana's definition, where *minima* of multiplicities are used instead of *gcd*s. Also, one may need to replace $f : X \dashrightarrow Y$ by another birational model before checking the base is not of general type.

Recall that a polarized $\mathbb{C}$-VHS as above is essentially equivalent to the data of a *period map* $\phi : \widetilde{U} \to \mathcal{D}$, equivariant under some representation $\pi_1(U) \to \mathcal{D}$. The theorem can be seen as an analogue of an earlier result of Taji [6], who showed the isotriviality of families of canonically polarized varieties on varieties of special type.

To prove Theorem 5, we can use the classical fact that the period map $\phi : \widetilde{U} \to \mathbb{D}$ factors through a subdomain $\mathcal{D}' \subset \mathcal{D}$, homogeneous under the action of a *semi-simple* real algebraic group $H$. This group turns out to be trivial, because of the following:

**Theorem 6** ([3]). *Let $U$ be a quasi-projective manifold, and let $\rho : \pi_1(U) \to G$ be a Zariski dense, big*[2] *representation with values in a semi-simple complex algebraic group. Then $U$ is of general type.*

Note that this last theorem is actually a small part of a more general hyperbolicity statement, which generalizes several earlier results of Yamanoi and Campana-Claudon-Eyssidieux to the quasi-projective setting.

## 3. An application to the structure of cusps in a ball quotient

Perhaps surprisingly, Theorem 5 can be used to give a criterion for a given isolated singularity to be a local model for the cusps appearing in a Baily-Borel-Mok compactification of a ball quotient.

3.1. **Baily-Borel-Mok model for compactifications of ball quotients.** Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a lattice, and let $X := {}_{\Gamma}\backslash^{\mathbb{B}^n}$ be the quotient. Baily-Borel and Mok have shown that $X$ can be endowed with the structure of a quasi-projective variety, compactifiable into a projective variety by adding a finite number of points

$$X^* = X \sqcup \{p_1, \dots, p_n\}.$$

Perhaps after replacing $\Gamma$ by a smaller lattice[3], each point $p_i$ has a neighborhood which is locally analytically a cone over an abelian variety. After blowing-up these points, we can obtain a log-crepant resolution $p : \overline{X} \to X^*$, so that if $D$ is the boundary divisor (made of finitely many abelian varieties), we have $K_{\overline{X}} + D \sim p^* K_{X^*}$.

3.2. **A structure result.** Let $X^* = X \sqcup \{p\}$ be a germ of isolated singularity, such that the smooth part $X$ admits

(1) a map $f : \widetilde{X} \to \mathbb{B}^n$ étale at any point;
(2) a representation $\rho : \pi_1(X) \to \mathrm{Aut}(\mathbb{B}^n)$ making $f$ equivariant.

---

[2]This "bigness" condition means essentially for a general subvariety $V \subset U$ the image of $\pi_1(V)$ *via* $\rho$ is infinite. In our setting, this can always be achieved after performing a "Shafarevich reduction" on $U$.

[3]So as to make it a *neat* lattice.

One can ask some necessary conditions for $X^*$ to be a cusp as described above; this is essentially equivalent to the metric $f^*h_{\mathbb{B}^n}$ being complete. In general, there is no reason for this to be the case: it is easy to construct counterexamples by contracting to a point a totally geodesic hypersurface in a suitable ball quotient. Note however that these counterexamples do not have log-canonical singularities, and thus certainly do not admit any log-crepant resolution.

The last result shows that the existence of this log-resolution actually ensures that $X^*$ has the required form:

**Theorem 7** ([2])**.** *Let $X^*$ be a germ of isolated singularity admitting (1) and (2) as above. Assume that there exists a log-crepant resolution $p : \overline{X} \to X^*$. Then*

*(1) $f^*h_{\mathbb{B}^n}$ is complete on $\widetilde{X}$;*
*(2) $X^*$ is a cone over an abelian variety.*

To prove this result, an important step is to "extend" $f$ to the smooth strata of the exceptional divisor of $p$. Because $p$ is log-crepant, these strata are log-CY varieties, hence are special; this is where Theorem 5 comes into play.

## References


[1] F. Campana, *Orbifolds, special varieties and classification theory.* Annales de l'Institut Fourier **54** (2004), no. 3, 499-630.

[2] B. Cadorel, *Uniformization of varieties with log-canonical singularities.* arXiv:2408.17120 (2024).

[3] B. Cadorel, Y. Deng, K. Yamanoi, *Hyperbolicity and fundamental groups of complex quasi-projective varieties (III): applications.* arXiv:2512.20360 (2025).

[4] M. Green, P. Griffiths, *Two Applications of Algebraic Geometry to Entire Holomorphic Mappings.* In: Hsiang, WY., Kobayashi, S., Singer, I.M., Wolf, J., Wu, HH., Weinstein, A. (eds) The Chern Symposium 1979. (1980) Springer, New York, NY

[5] S. Lang, *Introduction to complex hyperbolic spaces*, (1987), Springer-Verlag, New York.

[6] B. Taji, *The isotriviality of smooth families of canonically polarized manifolds over a special quasi-projective base.* Compositio Mathematica **152(7)** (2016), 1421-1434.


## Boundedness of Lagrangian fibrations

Philip Engel

(joint work with Stefano Filipazzi, François Greer, Mirko Mauri, Roberto Svaldi)

A well-known open question is whether, in a fixed dimension, there are finitely many deformation classes of smooth, projective, $K$-trivial varieties.

Fixing a dimension $g$, all abelian varieties lie in a single deformation class. The components $\mathcal{A}_{g,\mathbf{d}}$ of their moduli are arithmetic quotients of the Siegel upper half-space, indexed by a sequence of positive integers $\mathbf{d} = (d_1 \mid \cdots \mid d_g)$ successively dividing each other, encoding the polarization type of an ample line bundle.

The moduli spaces of hyperkähler varieties are also well understood in a fixed deformation class [4], each being an arithmetic quotient of a Type IV Hermitian symmetric domain. The pressing question is whether there are finitely many deformation classes in each dimension, and if so, how to classify them.

In joint work with Filipazzi, Greer, Mauri, and Svaldi [3], we prove:

**Theorem 1.** *There are only finitely many deformation classes of compact hyperkähler manifolds, of fixed dimension, admitting a Lagrangian fibration.*

More generally, we prove the same result for *primitive symplectic varieties*: Projective varieties $X$ with canonical singularities and $h^1(X, \mathcal{O}_X) = 0$, admitting a unique symplectic form on the smooth locus, up to scale.

The hyperkähler SYZ conjecture asserts that any hyperkähler manifold with $b_2(X) \geq 5$ deforms to one admitting a Lagrangian fibration. Thus, Theorem 1 shows that if the hyperkähler SYZ conjecture holds, there are only finitely many deformation types of hyperkähler manifolds with $b_2(X) \geq 5$ in each dimension.

## 1. Proof of the main theorem

Let $f\colon X \to Y$ be a Lagrangian fibration of a hyperkähler variety. Then $f$ is an abelian fibration (in the sense that the very general fiber of $f$ is an abelian variety). Let $Y^o \subset Y$ be the open set over which $f$ is smooth. We proceed as follows:

(1) Bound the base $Y$, together with two $\mathbb{Q}$-divisors $B_Y$ and $M_Y$ encoding, respectively, the singularities of $f$ in codimension 1, and the variation in moduli of the family $f$ of abelian varieties.
(2) Bound the classifying map $\Phi^o\colon Y^o \to \mathcal{A}_{g,\mathbf{d}}$ assigning to a point $y \in Y^o$ the abelian variety $\mathrm{Aut}^0(X_y)$, i.e. the group of translations of the fiber over $y$.
(3) Understand the collection of Lagrangian fibrations $f\colon X \to Y$ which have a specified classifying map $\Phi^o\colon Y^o \to \mathcal{A}_{g,\mathbf{d}}$ and divisor $B_Y$.

Here boundedness means that there is a family over a **finite type** base containing all instances of the objects in question. For example, a collection of triples $\{(Y_i, B_i, M_i)\}_{i\in I}$ of projective varieties $Y_i$ with two $\mathbb{Q}$-divisors $B_i, M_i$ is *bounded* if there exists a family $(\mathcal{Y}, \mathcal{B}, \mathcal{M}) \to \mathcal{S}$, with $\mathcal{B}, \mathcal{M}$ two $\mathbb{Q}$-divisors on $\mathcal{Y}$, over a finite type base $\mathcal{S}$ such that: for all $i \in I$, there exists some closed point $s \in \mathcal{S}$ for which $(Y_i, B_i, M_i) \simeq (\mathcal{Y}_s, \mathcal{B}_s, \mathcal{M}_s)$. Notions of boundedness extend to morphisms (and rational maps) by considering the graphs (and their closures).

**Bounding the base.** Given an abelian fibration $f\colon X \to Y$ with $K_X \sim_{\mathbb{Q},f} 0$, there is a *canonical bundle formula* for $K_X$ in terms of $K_Y$ and two corrections:

$$K_X \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y).$$

Here $M_Y$ is the *moduli divisor*, defined as $\Phi^*(\lambda_g)$ where $\Phi\colon Y \dashrightarrow \overline{A}_{g,\mathbf{d}}$ is the period map to the Baily–Borel compactification of the coarse space $A_{g,\mathbf{d}}$ and $\lambda_g$ is the Hodge bundle. In particular, $M_Y \sim_{\mathbb{Q}} 0$ if and only if $f$ is an isotrivial abelian fibration. So $M_Y$ measures if and how much the abelian fibers of $f$ vary. And $B_Y$ is the *boundary divisor*, defined as $\sum_{P\subset Y \text{ prime}}(1 - \mathrm{lct}_P(f)) \cdot P$ where $\mathrm{lct}_P(f) \in (0, 1] \cap \mathbb{Q}$ is the log canonical threshold of the central fiber of $f$, localized at the prime $P \subset Y$. The coefficient of $P$ is positive iff the generic fiber over $P$ is not slc.

By [1], the triple $(Y, B_Y, M_Y)$ is bounded in codimension 1 (i.e. up to birational maps which are isomorphisms in codimension 1) whenever

(1) $Y$ is rationally connected, and
(2) one can bound the integer $c \in \mathbb{N}$ for which $cM_Y$ is *b-free*.

Property (1) is granted by [5]—the base of a Lagrangian fibration is a Fano variety with the $\mathbb{Q}$-homology of projective space. Even better, such a variety $Y$ admits no nontrivial flops, and so boundedness in codimension 1 implies boundedness.

Thus, we are reduced to showing Property (2). Here *b-freeness* means that on a birational modification $Y' \to Y$ eliminating indeterminacy of the period map $\Phi$, the moduli divisor $M_{Y'}$ is free. This $b$-freeness is granted once $c\lambda_g$ is a free linear series on $\overline{A}_{g,\mathbf{d}}$. Thus, our question reduces to understanding why the integer $c$ can be chosen independently of the infinitely many possible polarization types $\mathbf{d}$.

Here we introduce a key tool, the *Zarhin trick*. It is a map

$$Z\colon \mathcal{A}_{g,\mathbf{d}} \to \mathcal{A}_{8g}$$

sending $A \mapsto Z(A) = A^{\oplus 4} \oplus (A^*)^{\oplus 4}$, but with a nonstandard, principal polarization. Passing to coarse spaces and compactifying, we get $Z^*(\lambda_{8g}) \sim 8\lambda_g$. Thus, the integer $c$ can indeed be chosen independently of $\mathbf{d}$, for instance, by taking $c = 8N$ for $N$ a multiple of the Hodge bundle $N\lambda_{8g}$ which is very ample on $\overline{A}_{8g}$.

This concludes the argument that the triple $(Y, B_Y, M_Y)$ is bounded, for any Lagrangian fibration $f\colon X \to Y$ of fixed dimension.

**Bounding the classifying map.** The next step is to bound the classifying map $\Phi^o\colon Y^o \to \mathcal{A}_{g,\mathbf{d}}$. This first requires bounding the open set $Y^o \subset Y$ over which $f$ is smooth. We again use the Zarhin trick. Consider the composition

$$Z \circ \Phi\colon Y \dashrightarrow \overline{A}_{8g}.$$

As $M_Y$ has already been bounded in the previous step and $(Z \circ \Phi)^*(\lambda_{8g}) \equiv 8M_Y$, a Chow scheme argument bounds the space of possible composed maps $Z \circ \Phi$.

In this way, the sublocus of $Y$ which is sent into the Hodge-theoretic boundary $\overline{A}_{8g} \setminus A_{8g}$ (and in turn $\overline{A}_{g,\mathbf{d}} \setminus A_{g,\mathbf{d}}$) can be bounded. Additionally removing the singular locus of $Y$, the indeterminacy locus of $Z \circ \Phi$, and the support of the (bounded) divisor $B_Y$ we produce a bounded Zariski open subset $Y^o \subset Y$ over which the fibration $f$ is necessarily smooth.

Then, we may bound the lifts of $Z \circ \Phi|_{Y^o}$ from the coarse space $A_{8g}$ to the orbifold $\mathcal{A}_{8g}$ and thus produce a bounded family of maps $Z \circ \Phi^o\colon Y^o \to \mathcal{A}_{8g}$ which contain the Zarhin trick of the classifying map of any Lagrangian fibration. Next, we must somehow "unwind" the Zarhin trick.

We pull back along $Z \circ \Phi^o$ the tautological $\mathbb{Z}^{16g}$-local system $\mathbb{V}_{16g}$ on $\mathcal{A}_{8g}$. Since $Z \circ \Phi^o$ has been bounded, these local systems come in only finitely many topological types, on the fibers of the (now) bounded family of open sets $Y^o \subset Y$. Via an argument involving traces of monodromies similar to [2], we in turn bound the pullback $(\Phi^o)^*\mathbb{V}_{2g}$ of the tautological $\mathbb{Z}^{2g}$-local system $\mathbb{V}_{2g}$ on $\mathcal{A}_{g,\mathbf{d}}$.

It follows from generalities on Lagrangian fibrations that the global sections of the second exterior power $H^0(Y^o, \wedge^2(\Phi^o)^*\mathbb{V}_{2g}) = \mathbb{Z}\psi_{\mathbf{d}}$ are 1-dimensional, and that the generator is a primitive polarization on a general fiber of $f$ of some type $\mathbf{d}$. As $\mathbb{V}_{2g}$ has been bounded, we deduce a bound on $\mathbf{d}$ itself. Thus, we have reduced the targets of the classifying map $\Phi^o\colon Y^o \to \mathcal{A}_{g,\mathbf{d}}$ to a finite number of possibilities. Re-applying the earlier arguments used for $Z \circ \Phi^o$, we can now bound $\Phi^o$.

**Bounding the Tate–Shafarevich group.** We have bounded the classifying map

$$\Phi^o\colon Y^o \to \mathcal{A}_{g,\mathbf{d}}$$
$$y \mapsto \mathrm{Aut}^0(X_y)$$

where $Y^o \subset Y$ is the open set over which $f$ is smooth. One may wonder why aren't we done; after all, $X_y \simeq \mathrm{Aut}^0(X_y)$ for any $y \in Y^o$. Can we just pull back the universal family $\mathcal{X}_{g,\mathbf{d}} \to \mathcal{A}_{g,\mathbf{d}}$ to deduce that $X$ is bounded? We cannot, as this pullback is only fiberwise isomorphic to $f\colon X \to Y$ (over $Y^o$). Indeed, the pullback admits a rational section, whereas $f\colon X \to Y$ may well admit no rational section. Thus, we define the *Albanese fibration*

$$f^{\mathrm{Alb}}\colon X^{\mathrm{Alb}} \to Y$$

to be (a primitive symplectic compactification of) the pullback of $\mathcal{X}_{g,\mathbf{d}}$ along $\Phi^o$.

Up to technical issues, we may view $f\colon X \to Y$ as a *twist* of $f^{\mathrm{Alb}}\colon X^{\mathrm{Alb}} \to Y$, in the sense that, fiberwise, and hence étale locally, $f$ and $f^{\mathrm{Alb}}$ are isomorphic.[1] Choosing such isomorphisms $\varphi_i\colon X^{\mathrm{Alb}}|_{U_i} \to X|_{U_i}$ for an open cover $\{U_i\}_{i\in I}$ of $Y$, the transition functions $t_{ij} = \varphi_j \circ \varphi_i^{-1}$ define vertical translations of $X^{\mathrm{Alb}}|_{U_i\cap U_j}$ and in turn, an element of the cohomology group $t(f) \in H^1_{\text{ét}}(Y, X^{\mathrm{Alb}})$. Here we view $X^{\mathrm{Alb}}$ as a group scheme hence sheaf of abelian groups over $Y$ in the étale topology.

GAGA- and Hartogs-type results show that $H^1_{\text{ét}}(Y, X^{\mathrm{Alb}}) = H^1_{\mathrm{an}}(Y, X^{\mathrm{Alb}})_{\mathrm{tors}}$. By analyzing the exponential exact sequence, one deduce that a certain subgroup $\mathrm{Sha} \subset H^1_{\mathrm{an}}(Y, X^{\mathrm{Alb}})$ containing the torsion fits into an exact sequence

$$0 \to \mathbb{C}/\mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_r \to \mathrm{Sha} \to N \to 0$$

with $N$ finite abelian and $\alpha_i \in \mathbb{C}$ rationally independent. Thus, $H^1_{\text{ét}}(Y, X^{\mathrm{Alb}})$ is an extension of a finite group by an infinite, divisible group $(\mathbb{Q}/\mathbb{Z})^r$.

So there are infinitely many abelian fibrations $f\colon X \to Y$ with the same Albanese fibration. But the twists parameterized by the complex line $\mathbb{C} \supset \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_r$ form a family of analytic hyperkähler manifolds in a fixed deformation class. So despite the infinitude of these twists, only finitely many deformation classes arise, completing the proof of Theorem 1.

## References

[1] C. Birkar, G. Di Cerbo, and R. Svaldi. *Boundedness of elliptic Calabi–Yau varieties with a rational section*, J. Differential Geom. 128 (2024), no. 2, 463–519.

[2] P. Deligne. *Un théorème de finitude pour la monodromie*, Discrete groups in geometry and analysis, Pap. Hon. G. D. Mostow 60th Birthday, Prog. Math. 67, 1–19 (1987).

[3] P. Engel, S. Filipazzi, F. Greer, M. Mauri, and R. Svaldi, *Boundedness of some fibered $K$-trivial varieties*, arXiv: 2507.00973.

[1] Additional details arise here; this is only true over a big open subset $\widetilde{Y}^+ \subset \widetilde{Y}$ of a Galois $G$-cover $\widetilde{Y} \to Y$ over which the pullback of $f$ admits a semiabelian reduction in codimension 1.

[4] D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. 135 (1999), no. 1, 63–113.
[5] D. Matsushita. *On base manifolds of Lagrangian fibrations*, Sci. China Math. 58 (2015), no. 3, 531–542.

## From tropical curves to special Lagrangians

Shih-Kai Chiu
(joint work with Yang Li, Yu-Shen Lin)

Motivated by the Strominger–Yau–Zaslow (SYZ) conjecture [11] and by the tropical-to-holomorphic correspondence established by Mikhalkin [9], one expects tropical objects in the base of the SYZ torus fibration to lift to geometric objects on the mirror side. In the symplectic setting, such philosophy is reflected by the tropical-to-Lagrangian correspondences of Mikhalkin [10] and Matessi [8], which associated tropical curves or hypersurfaces with Lagrangian submanifolds. Since Lagrangian submanifolds are flexible objects, it is natural to ask whether such lifts can be made special Lagrangians, which are calibrated and hence volume-minimizing in their homology classes.

Motivated by the above picture, in [2] we establish examples of a tropical-to-special-Lagrangian correspondence. Let us consider the simplest case of the SYZ torus fibration

$$\pi : X \to B, \qquad X = T^*T^n, \quad B = (\mathbb{R}^n)^*.$$

We equip $X$ with the Euclidean Calabi-Yau structure

$$\omega = \sum_i d\theta_i \wedge d\mu_i, \qquad \Omega = \bigwedge_i \Big( d\theta_i + \sqrt{-1} \sum_j g_{ij}\, d\mu_j \Big),$$

and

$$g = \sum_{i,j} g_{ij} d\mu_i\, d\mu_j + \sum_{i,j} g^{ij} d\theta_i\, d\theta_j .$$

in terms of the action-angle coordinates $\mu_i, \theta_i$ and a positive definite $n \times n$ constant matrix $(g_{ij})$ with inverse $(g^{ij})$. Thus $B$ is naturally equipped with an integral affine structure. Analogous to the tropical-to-holomorphic correspondence, Our goal is to lift tropical objects in $B$ to special Lagrangians in $X$. The main result of [2] is the following:

**Theorem 1.** *Let $n \geq 2$, and let $\Gamma \subset B$ be the image of a locally planar tropical curves. Then for all sufficiently large $T > 0$, there exists a special Lagrangian submanifold $L_T \subset X$ such that $(1/T)\pi(L_T)$ converges in the Hausdorff distance to $\Gamma$ as $T \to +\infty$.*

This result opens several further directions. More generally, one expects a correspondence between tropical varieties $\Gamma \subset B$ of dimension $2 \leq m \leq n-1$, possibly with multiplicities along their top-dimensional strata, and special Lagrangian submanifolds in $X$. When $m = n-1$, a local model corresponding to the standard tropical hypersurface has recently been constructed by Y. Li [7]. A more ambitious direction is to seek an analogous correspondence between tropical curves in the SYZ base and special Lagrangians in a compact Calabi-Yau manifold near a large complex structure limit.

The tropical-to-holomorphic correspondence has also motivated developments in manifolds with special holonomy. Donaldson [4] and later Donaldson-Scaduto [5] conjectured that, near an adiabatic limit, counts of associative submanifolds in a compact $G_2$ manifold with a Kovalev-Lefschetz fibration should be governed by counts of gradient cycles in the base. A dimensional reduction of this picture leads one to look for special Lagrangians in K3-fibered Calabi-Yau 3-folds. See, e.g., [3], [6], and [1] for recent progress in this direction.

Theorem 1 is proved using a gluing construction. Given a locally planar tropical curve $\Gamma \subset B$, one associates to each edge $e$ a $T^{n-1}$-invariant special Lagrangian cylinder $L_e$ together with a translation constant $\exp(\sqrt{-1}\theta_e) \in U(1)$, which projects to the line containing $e$. To each vertex $v$, one associates a $T^{n-2}$-invariant asymptotically cylindrical special Lagrangian submanifold $L_v$, whose asymptotic ends are given by the cylinders $L_e$ corresponding to edges adjacent to $v$. Such a vertex model exists provided that the adjacent translation constants $\theta_e$ satisfy the balancing condition

$$\sum_{e \text{ adjacent to } v} \theta_e = N\pi \qquad (\mathrm{mod}\, 2\pi\mathbb{Z}),$$

where $N$ is the number of adjacent edges of $v$. This condition is equivalent to prescribing the asymptotics of a hypersurface in $(\mathbb{C}^*)^2$. The linear system of balancing conditions is solvable whenever the number of edges is greater than the number of vertices, which is the case when the valency of each vertex is at least three. By stretching the tropical curve by a large constant $T > 0$, the edge and vertex models can be glued to obtain a Lagrangian submanifold whose special Lagrangian error is exponentially small in $T$.

The perturbation to a special Lagrangian submanifold follows from a more general gluing scheme involving matching data, generalizing the vertex and cylinder models described above. From the PDE gluing perspective, the main analytical difficulty is that the linearized special Lagrangian operator $d + d^*$ acting on 1-forms, or equivalently, after restricting to exact 1-forms, the corresponding scalar Laplacian $d^*d$ acting on potentials, does not admit a right inverse whose norm is uniformly bounded in the gluing parameter $T$. Moreover, localizing the problem to the local model special Lagrangians further introduces finite-dimensional obstructions to solving the linearized equation. A key point in our parametrix construction is to remove these local obstructions by first solving a discretized version of the linearized equation. One limitation of our method is that, when the tropical curve contains loops, the resulting perturbation need not be Hamiltonian. Nevertheless, this gluing scheme may be useful for the related problems mentioned above.

## References

[1] S.-K. Chiu and Y.-S. Lin, *Special Lagrangian submanifolds in K3-fibered Calabi–Yau 3-folds*, arXiv:2410.17662 (2024).

[2] S.-K. Chiu, Y. Li, and Y.-S. Lin, *From tropical curves to special Lagrangians*, arXiv:2509.04843 (2025).

[3] A. Doan and T. Walpuski, *On counting associative submanifolds and Seiberg–Witten monopoles*, Pure Appl. Math. Q. **15** (2019) 1047–1133.

[4] S. Donaldson, *Adiabatic limits of co-associative Kovalev–Lefschetz fibrations*, in *Algebra, Geometry, and Physics in the 21st Century*, Progr. Math. **324**, Birkhäuser, Cham, 2017, pp. 1–29.

[5] S. Donaldson and C. Scaduto, *Associative submanifolds and gradient cycles*, Surv. Differ. Geom. **24** (2019) 39–65.

[6] S. Habibi Esfahani and Y. Li, *On the Donaldson–Scaduto conjecture*, arXiv:2401.15432 (2024).

[7] Y. Li, *Special Lagrangian pair of pants*, Comm. Pure Appl. Math. **78** (2025) 1320–1356.

[8] D. Matessi, *Lagrangian submanifolds from tropical hypersurfaces*, Internat. J. Math. **32** (2021) 2150046.

[9] G. Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005) 313–377.

[10] G. Mikhalkin, *Examples of tropical-to-Lagrangian correspondence*, Eur. J. Math. **5** (2019) 1033–1066.

[11] A. Strominger, S.-T. Yau, E. Zaslow, *Mirror symmetry is T-duality*, Nucl. Phys. B **479** (1996) 243–259.


## Geometric behavior of singular Kähler–Einstein metric on klt space


Yifan Chen

(joint work with Shih-Kai Chiu, Max Hallgren, Gábor Székelyhidi, Tat Dat Tô, Freid Tong)


Ricci curvature lies at the intersection of Riemannian geometry, Kähler geometry, and algebraic geometry, as illustrated by the following formula on a Kähler manifold $(X, g, J)$ with Kähler form $\omega = g(J\cdot, \cdot)$:

$$\mathrm{Ric}_\omega := \mathrm{Ric}_g(J\cdot, \cdot) = -\sqrt{-1}\partial\bar{\partial}\log\det\omega.$$

This defines a closed $(1,1)$-form representing $c_1(-K_X)$. It is a longstanding problem to find Einstein metrics $g$. In the Kähler setting, finding a Kähler–Einstein metric $\omega_{KE} = \omega_X + \sqrt{-1}\partial\bar{\partial}u$, with $\lambda\omega_X \in c_1(-K_X)$, reduces to solving a complex Monge–Ampère equation for $u$. The existence of Kähler–Einstein metrics on compact Kähler manifolds was established by Aubin and Yau [1, 26] for $c_1(-K_X) < 0$, by Yau [26] for $c_1(-K_X) = 0$, and by Chen–Donaldson–Sun [7, 8, 9] for $c_1(-K_X) > 0$ via the $K$-stability criterion, with the implication from existence to $K$-polystability due to Berman [2].

As a natural generalization, singular Kähler–Einstein metrics are defined as closed positive $(1,1)$-currents with bounded potentials whose restrictions to the regular locus of $X$ are smooth Kähler–Einstein metrics. When $X$ has log terminal singularities, Eyssidieux–Guedj–Zeriahi [15] constructed these canonical metrics by solving a degenerate complex Monge–Ampère equation for $\lambda \in \{-1, 0\}$:

$$\omega_{KE}^n = (\omega_X + \sqrt{-1}\partial\bar{\partial}u)^n = e^{-\lambda u}\mu,$$

where $\mu$ is the adapted measure associated to a smooth Hermitian metric on $K_X$ with prescribed curvature $-\lambda\omega_X \in c_1(K_X)$. The mildness of log terminal singularities ensures that the adapted measure is well-defined, has finite mass, and admits an $L^p$ density:

$$\mu = e^F \omega_X^n, \qquad e^F \in L^p(\omega_X), \quad p > 1.$$

This allows one to apply pluripotential theory to the Monge–Ampère equation. For $\lambda = 1$, the relation between singular Kähler–Einstein metrics and $K$-stability was established in [2, 3, 23] and the references therein. Regularity of the singular Kähler potential has been studied in [12, 11, 19]; in general, it is expected to be at most Hölder continuous, as suggested by two-dimensional orbifold examples. Since understanding the geometric behavior near the singular locus typically requires second-order estimates of the potential, such behavior remains largely unexplored. The local behavior of singular Kähler–Einstein metrics with isolated singularities was studied in [20]. Motivated by Ricci limit spaces in Cheeger–Colding theory [4, 5, 6] and by Gromov–Hausdorff convergence of Kähler–Einstein manifolds in Donaldson–Sun theory [13, 14], a folklore conjecture asserts that the metric completion $\hat{X}$ of $(X_{\mathrm{reg}}, \omega_{KE})$ is homeomorphic to $X$ and admits a stratification and algebraic structure. Székelyhidi [25] confirms the homeomorphism between $\hat{X}$ and the underlying variety $X$ under the following assumptions:

(1) strict positivity: $\omega_{KE} \geq c\,\omega_X$ for some $c > 0$,
(2) $\hat{X}$ is a non-collapsed RCD$(\lambda, 2n)$ space.

Property (1) was previously known in the smoothable case and in complex dimension three with nonpositive Ricci curvature; see [17]. Adapting an argument of Székelyhidi [25] with the uniform heat kernel estimate of Guo–Phong–Song–Sturm [18], we prove:

**Theorem 1.** [10, Theorem 1.1]**.** Singular Kähler–Einstein metrics are always strictly positive.

In fact, property (1) holds in a more general setting: it suffices for the singular Kähler metric $(X, \omega)$ to have an $L^p$ density and a Ricci lower bound on the regular part. The same method also applies to Kähler–Ricci solitons and klt pairs.

**Theorem 2.** [10, Theorem 1.2]**.** Suppose that $(X, \omega_X)$ is a compact normal Kähler space, with smooth metric $\omega_X$. Suppose that $\omega = \omega_X + i\partial\bar{\partial}u$ is a positive current with bounded potential satisfying the following properties:

i. $u$ is smooth on $X^{\mathrm{reg}} \backslash D$ for a divisor $D$,
ii. $\omega^n = e^F \omega_X^n$, where $e^F \in L^p(\omega_X)$ for some $p > 1$, and $F \in L^1(\omega_X)$.
iii. $\mathrm{Ric}(\omega) \geq -A(\omega + \omega_X)$ on $X^{\mathrm{reg}}$ in the sense of currents, for some $A > 0$.

Then there is a constant $C > 0$, depending on $A, p$, an upper bound for $\left\| e^F \right\|_{L^p}$, and $(X, \omega_X)$, such that $\omega > C^{-1}\omega_X$.

In the general case, we construct a resolution $(Y, \omega_\varepsilon)$ of $X$, where $\omega_\varepsilon = \pi^*\omega_X + \varepsilon\omega_Y + \sqrt{-1}\partial\bar{\partial}u_\varepsilon$ are Kähler metrics on $Y$ such that $u_\varepsilon$ are uniformly bounded and approximate $u$ locally uniformly on $X^{\mathrm{reg}}$. Furthermore, $\omega_\varepsilon^n = e^{f_\varepsilon}\omega_Y^n$ with $\|e^{f_\varepsilon}\|_{L^p(Y,\omega_Y)}$ uniformly bounded. We call such $(Y, \omega_\varepsilon)$ a *tame approximation* of $(X, \omega)$. Despite the lack of pointwise Ricci lower bound on $X$, we apply the Schwarz lemma to these tame approximations $(Y, \omega_\varepsilon)$. Thanks to the uniform heat kernel estimate in [18], together with a uniform $L^1$ estimate for $(\mathrm{Ric}\,\omega_\varepsilon + A(\omega_\varepsilon + \omega_X))_-$, we obtain the strict positivity of $\omega$.

By a criterion of Honda [21], property (2) can be reduced to gradient estimates for $W^{1,2}$ eigenfunctions of the Laplacian on $\hat{X}$. Székelyhidi [25] verified this when $(X, \omega_{KE})$ admits a constant scalar curvature Kähler approximation. In [10] we establish property (2) by requiring a slightly weaker approximation condition on a resolution of $X$:

**Theorem 3.** [10, Theorem 1.5, simplified]**.** Suppose that $\omega$ is a singular Kähler-Einstein metric on $X$, and that $\omega$ has a tame approximation $\omega_\varepsilon$ with

$$\|(\mathrm{Ric}\,\omega_\varepsilon - \lambda\,\omega_\varepsilon)_-\|_{L^p(Y,\omega_\varepsilon)} < C$$

for $C$ independent of $\varepsilon$, and $p > \frac{2n-1}{n}$. Then the metric measure space $\hat{X}$ is a non-collapsed RCD space.

More generally, the singular Kähler–Einstein condition can be replaced by a lower bound for $\mathrm{Ric}\,\omega$ on $X^{\mathrm{reg}} \setminus D$, for some divisor $D \subset X$; see [10, Theorem 5.1] for details.

Whether property (2) holds for arbitrary singular Kähler–Einstein metrics remains open in general, but has recently been confirmed in complex dimension three by [16]. Once property (2) is known, tangent cones exist at every point of $\hat{X}$ and are metric cones. It is expected from the Donaldson–Sun theory [13, 14] that these tangent cones are algebraic with log terminal singularities and are uniquely determined by the analytic germ $(X, x)$ as shown in [22]. These tangent cones may be viewed as first-order approximations of singular Kähler–Einstein metrics, and further higher-order information could be detected via Gromov–Hausdorff limits arising from bubbling phenomena, as studied in [24]. This provides strong motivation for studying complete Calabi–Yau metrics on noncompact Kähler manifolds and their singular analogues. This direction is also related to ongoing joint work with Chung-Ming Pan on the construction of such metrics.

## References


[1] T. Aubin. *Equations du type Monge–Ampère sur les variétés kählériennes compactes.* C. R. Acad. Sci. Paris Sér. A-B **283** (1976), 119–121.

[2] R. J. Berman. *K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler–Einstein metrics.* Inventiones mathematicae **203.3** (2016), 973–1025.

[3] R. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. *Kähler–Einstein metrics and the Kähler–Ricci flow on log Fano varieties.* Journal für die reine und angewandte Mathematik (Crelles Journal) **2019.751** (2019), 27–89.

[4] J. Cheeger, and T. Colding. *On the structure of spaces with Ricci curvature bounded below. I.* Journal of Differential Geometry **46.3** (1997), 406–480.

[5] J. Cheeger, and T. Colding. *On the structure of spaces with Ricci curvature bounded below. II.* Journal of Differential Geometry **54.1** (2000), 13–35.

[6] J. Cheeger, and T. Colding. *On the structure of spaces with Ricci curvature bounded below. III.* Journal of Differential Geometry **54.1** (2000), 37–74.

[7] X. Chen, S. Donaldson, and S. Sun. *Kähler–Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities.* Journal of the American Mathematical Society **28.1** (2015), 183–197.

[8] X. Chen, S. Donaldson, and S. Sun. *Kähler–Einstein metrics on Fano manifolds. II: Limits with cone angle less than* $2\pi$*.* Journal of the American Mathematical Society **28.1** (2015), 199–234.

[9] X. Chen, S. Donaldson, and S. Sun. *Kähler–Einstein metrics on Fano manifolds. III: Limits as cone angle approaches* $2\pi$ *and completion of the main proof.* Journal of the American Mathematical Society **28.1** (2015), 235–278.

[10] Y. Chen, S. Chiu, M. Hallgren, G. Székelyhidi, T. Tô, F. Tong. *On Kähler–Einstein Currents.* arXiv preprint arXiv:2502.09825 (2025).

[11] Y. Cho, and Y. Choi. *Continuity of solutions to complex Monge–Ampère equations on compact Kähler spaces.* Mathematische Annalen **393.1** (2025), 807–830.

[12] D. Coman, V. Guedj, and A. Zeriahi. *Extension of plurisubharmonic functions with growth control.* Journal für die reine und angewandte Mathematik (Crelles Journal) **2013.676** (2013), 33–49.

[13] S. Donaldson, and S. Sun. *Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry.* Acta Mathematica **213.1** (2014), 63–106.

[14] S. Donaldson, and S. Sun. *Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry, II.* Journal of Differential Geometry **107.2** (2017), 327–371.

[15] P. Eyssidieux, V. Guedj, and A. Zeriahi. *Singular Kähler–Einstein metrics.* Journal of the American Mathematical Society **22.3** (2009), 607–639.

[16] X. Fu, B. Guo, and J. Song. *RCD structures on singular Kähler spaces of complex dimension three.* arXiv preprint arXiv:2503.08865 (2025).

[17] V. Guedj, H. Guenancia, and A. Zeriahi. *Strict positivity of Kähler–Einstein currents.* Forum of Mathematics, Sigma. Vol. **12**. Cambridge University Press, 2024.

[18] B. Guo, D. Phong, J. Song, and J. Sturm. *Sobolev inequalities on Kähler spaces.* arXiv preprint arXiv:2311.00221 (2023).

[19] B. Guo, S. Kolodziej, J. Song, J. Sturm. *Hölder estimates for degenerate complex Monge–Ampère equations.* arXiv preprint arXiv:2508.20933 (2025).

[20] H. Hein, and S. Sun. *Calabi–Yau manifolds with isolated conical singularities.* Publications mathématiques de l'IHÉS **126** (2017), 73–130.

[21] S. Honda. *Bakry–Émery conditions on almost smooth metric measure spaces.* Analysis and Geometry in Metric Spaces, vol. **6**, no. **1**, 2018, pp. 129–145.

[22] C. Li, X. Wang, and C. Xu. *Algebraicity of the metric tangent cones and equivariant K-stability.* Journal of the American Mathematical Society **34.4** (2021), 1175–1214.

[23] C. Li. *G-uniform stability and Kähler–Einstein metrics on Fano varieties.* Inventiones mathematicae **227.2** (2022), 661–744.

[24] S. Sun. *Bubbling of Kähler-Einstein metrics.* Pure and Applied Mathematics Quarterly **21.3** (2025), 1317–1348.

[25] G. Székelyhidi. *Singular Kähler–Einstein metrics and RCD spaces.* Forum of Mathematics, Pi. Vol. **13**. Cambridge University Press, 2025.

[26] S. Yau. *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I.* Communications on Pure and Applied Mathematics **31.3** (1978), 339–411.

## Negative $K$-theory and Hodge theory

Mihnea Popa

(joint work with Andrew Burke, Wanchun Shen)

The talk was devoted to a number of conjectures on the behavior of the negative $K$-theory of singular complex varieties, which will appear in upcoming work [1] of the speaker, joint with Andrew Burke and Wanchun Shen.

The negative $K$-groups of a variety $X$, $K_{-i}(X)$ for $i > 0$, can be defined inductively as follows. Assuming $K_{-i+1}$ has already been defined, we set

$$K_{-i}(X) = \operatorname{coker}\Big(K_{-i+1}(X[t]) \oplus K_{-i+1}(X[t^{-1}]) \xrightarrow{\pm} K_{-i+1}(X[t,t^{-1}])\Big). \tag{1}$$

This starts with $i = 1$, in which case $K_0(\cdot)$ represents the stadard Grothendieck group of vector bundles on the respective variety; as such, $K_{-1}$ represents the obstruction for lifting vector bundle classes on $X \times \mathbf{C}^*$ to differences of classes on the two copies of $X \times \mathbf{A}^1$ that provide the standard cover of $X \times \mathbf{P}^1$.

A result of Quillen says that if $X$ is smooth, then $K_{-i}(X) = 0$ for all $i > 0$. Therefore the negative $K$-groups are invariants of singularities. They have been rather poorly understood, besides the case of curves and surfaces, and even those in a rather ad hoc manner not related to Hodge theory. The main purpose of this work is to provide a general conjectural picture for their vanishing (mostly over $\mathbb{Q}$), or sometimes concrete formulas, in terms of Hodge-theoretic and singularity invariants.

This is split into two parts: first we try to understand the "regularity" of $K_{-i}(X)$, i.e. under what conditions we have

$$K_{-i}(X) \simeq K_{-i}(X \times \mathbf{A}^r) \quad \text{for all} \quad r \geq 0.$$

If this is the case, then the natural morphism

$$K_{-i}(X) \to KH_{-i}(X)$$

to the so-called homotopy $K$-theory of $X$ is an isomorphism. We then focus on the $KH$-groups. These are sheaves in the cdh topology, and therefore can (in theory) be computed from a hyperresolution of $X$. This turns out to make them amenable to methods coming from Hodge theory.

The regularity problem has essentially been solved in [9]. Here is a selection of the results in that paper.

**Theorem 1** ([9, Proposition C, Theorem D])**.** *Let $X$ be a complex variety of dimension $n$ with $\dim X_{\mathrm{sing}} = s$.*
*(i) If $X$ has Du Bois singularities, then it is $K_{-n+1}$-regular.*
*(ii) If $X$ has pre-m-Du Bois singularities, then it is $K_{-t}$-regular for $t \geq \max\{1, n-2m-2+s\}$. If $X$ is in addition seminormal, then the conclusion extends to $t = 0$.*
*(iii) If $X$ is a local complete intersection and $X$ is $K_{-n+2m+1+s}$-regular, then it is pre-m-Du Bois.*

*In particular, if $X$ is a local complete intersection with isolated singularities, then*

$$X \text{ is } m\text{-Du Bois} \iff X \text{ is } K_{-d+2m+2}\text{-regular}$$

Here $m$-Du Bois singularities are a numerical refinement of the standard notion of Du Bois singularities, introduced recently in [4], [3], [5]. If $X$ is a local complete intersection (LCI) variety, we say that it has such singularities if the canonical morphisms

$$\Omega_X^p \to \underline{\Omega}_X^p$$

are isomorphisms for all $0 \leq p \leq m$, where $\underline{\Omega}_X^p$ is the p-th Du Bois complex of $X$. In general, the definition has to be modified appropriately; see [10].

Given the above, the main point becomes to understand the $KH$-groups. In [1] we make the following:

**Conjecture 2.** *Fix an integer $i > 0$. If $X$ is projective and $W_{j-i}H^j(X, \mathbb{Q}) = 0$ for all $j \leq 2n - i$, then*

$$KH_{-i}(X)_{\mathbb{Q}} = 0,$$

*or in other words $KH_{-i}(X)$ is torsion.*

The notation in the hypothesis refers to the weight filtration on the singular cohomology of $X$, a part of the data of its mixed Hodge structure. In particular, Conjecture 2 predicts that if the rational Hodge structure on singular cohomology of $X$ is pure in each degree, then *all* negative $K$-theory is torsion. The most obvious such case is that of rational homology manifolds, but there are other situations when this holds without having Poincaré duality. For example, by a result from [8], a threefold with isolated rational singularities satisfies this property if and only if the global $\mathbb{Q}$-factoriality defect is equal to the sum of the local analytic $\mathbb{Q}$-factoriality defects.

The key point is that the hypotheses of both Theorem 1 and Conjecture 2 are satisfied by spaces assumed to have $m$-rational singularities, another recently introduced (in [2], [6]) sibling of the notion of $m$-Du Bois singularities discussed above. Restricting again here to the LCI setting for simplicity, *$m$-rational singularities* if the canonical compositions

$$\Omega_X^p \to \underline{\Omega}_X^p \to \mathbf{D}_X(\underline{\Omega}_X^{n-p})$$

are isomorphisms for all $0 \leq p \leq m$, where $\mathbf{D}_X(\cdot) := R\mathcal{H}om(\,\cdot\,, \omega_X)$. For $m = 0$, this coincides with the classical notions of rational singularities. Crucially, the $m$-Du Bois and $m$-rationality conditions can be concretely checked numerically, in terms of the minimal exponent of the singularity, determined by a root of a Bernstein-Sato-type polynomial.

Roughly speaking, it is known that $X$ has $m$-rational singularities if and only if it has $m$-Du Bois singularities and satisfies condition $D_m$. Here condition $D_m$, introduced under this name in [7], stands for duality up to level $m$; concretely, it means that the canonical duality morphisms

$$\underline{\Omega}_X^p \to \mathbf{D}_X(\underline{\Omega}_X^{n-p})$$

are isomorphisms for $p \leq m$. (Note that, unlike for smooth varieties, in general this is not usually the case.) On the other hand, an equivalent formulation of one of the main results of [7] says that if $X$ satisfies condition $D_m$, then $H^i(X, \mathbb{Q})$ has weights $\geq \min\{2m+2, i\}$ if $i \leq n$, and $H^i(X, \mathbb{Q})$ has weights $\geq \min\{2i-2n+2m+2, i\}$ if $i \geq n$.

Putting all of this together, we are led to formulating the following:

**Conjecture 3.** *Let $X$ be a projective variety with $m$-rational singularities, for some $0 \leq m \leq (n-2)/2$, and let $s = \dim X_{\mathrm{sing}}$. Then:*

*(i) If $s \geq 1$, then*

$$K_{-i}(X)_{\mathbb{Q}} = 0 \quad \text{for all} \quad i \geq n - 2m - 1 + s.$$

*(ii) If $s = 0$, or if $m = 0$, then*

$$K_{-i}(X)_{\mathbb{Q}} = 0 \quad \text{for all} \quad i \geq n - 2m - 1.$$

*(ii) If $s = 0$ and $m < \frac{(n-2)}{2}$, then*

$$K_{-n+2m+2}(X)_{\mathbb{Q}} \simeq \mathrm{gr}^W_{2m+2} H^n(X, \mathbb{Q}).$$

As for results, in [1] we are able to prove the conjectures above for the groups $K_{-i}$ and $KH_{-i}$ when $i = n, n-1$ and $n-2$. In particular, this almost covers the case of threefolds, in the sense that we obtain unconditional results for threefolds with klt singularities, but in the general rational singularities case we need to assume Bloch's conjecture on 0-cycles on smooth projective surfaces. As suggested, for these results we need to apply methods related to higher codimension cycles, as well as cdh-motivic cohomology. We are also able to prove the main conjecture for fourfolds with isolated klt singularities, employing in addition methods from the minimal model program. Finally, we show that our conjectures are fully implied by a generalized version of the Bloch-Beilinson conjectural filtration on (higher) Chow groups.

## References


[1] A. Burke, M. Popa, and W. Shen, *Negative $K$-theory and Hodge theory*, in preparation.

[2] Robert Friedman and Radu Laza, *Higher Du Bois and higher rational singularities*, Duke Math. J. **173** (2024), no. 10, 1839–1881.

[3] S.-J. Jung, I.-K. Kim, M. Saito, and Y. Yoon, *Higher Du Bois singularities of hypersurfaces*, Proc. Lond. Math. Soc. **125** (2022), no. 3, 543–567.

[4] M. Mustaţă, S. Olano, M. Popa, and J. Witaszek, *The Du Bois complex of a hypersurface and the minimal exponent* Duke Math. J. **172** (2023), no. 7, 1411–1436.

[5] M. Mustaţă and M. Popa, *Hodge filtration on local cohomology, Du Bois complex and local cohomological dimension*, Forum Math. Pi **10** (2022), Paper No. e22, 58.

[6] M. Mustaţă and M. Popa, *On k-rational and k-Du Bois local complete intersections*, Algebr. Geom. **12** (2025), no. 2, 237–261.

[7] S. G. Park and M. Popa, *Hodge symmetry and Lefschetz theorems for singular varieties*, preprint arXiv:2410.15638.

[8] S. G. Park and M. Popa, $\mathbb{Q}$-*factoriality and Hodge-Du Bois theory*, preprint arXiv:2508.17748.

[9] W. Shen, *On higher Du Bois singularities and $K$-regularity*, preprint arXiv:2504.12402.

[10] W. Shen, S. Venkatesh, and A. D. Vo, *On k-Du Bois and k-rational singularities*, to appear in Ann. Inst. Fourier.

## Tautological rings: from $\mathbb{C}$VHS to algebraic cycles

Bruno Klingler

We work over the field of complex numbers. Given a smooth quasi-projective variety $S$, its rational Chow ring $\mathrm{CH}^\bullet(S)_\mathbb{Q}$ parametrizing the algebraic cycles on $S$ (with rational coefficients) up to rational equivalence, and endowed with the intersection product, is a fundamental invariant of $S$. One way of studying it is through the (singular) cohomology of the complex manifold $S^{\mathrm{an}}$: there exists a cycle class morphism of graded rings $\mathrm{cl} : \mathrm{CH}^\bullet(S)_\mathbb{Q} \to H^{2\bullet}(S^{\mathrm{an}})$. In general however, the $\mathbb{Q}$-algebra $\mathrm{CH}^\bullet(S)_\mathbb{Q}$ is wild, the cycle class $\mathrm{cl} : \mathrm{CH}^j(S)_\mathbb{Q} \to H^{2j}(S)$ is far from injective, and captures only a small shadow of the Chow ring, namely cycles modulo homological equivalence. For instance, a famous result of Mumford [4] states that when $S$ is a smooth projective surface, $\mathrm{CH}^2(S)_\mathbb{Q}$ is never "finite-dimensional" in any reasonable sense as soon as the Hodge number $h^{2,0}(S) > 0$. In this situation, one would like to construct "natural", finitely-generated subalgebras of $\mathrm{CH}^\bullet(S)_\mathbb{Q}$, big enough however to encode an interesting part of the geometry of $S$. This is the idea of tautological rings. Ideally, one could even hope that these tautological rings inject in singular cohomology, in particular are finite dimensional $\mathbb{Q}$-vector spaces.

How can we generate candidates for such rings? Except in the case of very specific geometries, most varieties do not, at first glance, exhibit algebraic cycles that are "more natural" than others. Fortunately, the ring $\mathrm{CH}^\bullet(S)_\mathbb{Q}$ can also be studied using objects maybe less elusive than algebraic cycles: algebraic vector bundles on $S$. Let $\mathrm{VB}(S)$ be the exact tensor category of algebraic vector bundles on $S$ and $K_\bullet(S)$ its algebraic $K$-theory. The Grothendieck-Riemann-Roch theorem and the Atiyah-Hirzebruch theorem state that the diagram of rings

$$\begin{tikzcd} K_0(S)_\mathbb{Q} \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d] & \mathrm{CH}^\bullet(S)_\mathbb{Q} \arrow[d, "\mathrm{cl}"] \\ K_0^{\mathrm{topo}}(S^{\mathrm{an}})_\mathbb{Q} \arrow[r, "\sim", "\mathrm{ch}^{\mathrm{topo}}"'] & H^{2\bullet}(S^{\mathrm{an}}, \mathbb{Q}), \end{tikzcd} \tag{1}$$

is commutative (where ch denotes the Chern character). Thus, one way of defining "natural" subrings of $\mathrm{CH}^\bullet(S)_\mathbb{Q}$ consists in exhibiting "natural" collections of vector bundles on $S$ and studying the subrings of $K_0(S)_\mathbb{Q}$ they generate.

We propose a construction coming from Hodge theory. Let $\mathbb{V}$ be a polarized $\mathbb{C}$VHS on the complex manifold $U^{\mathrm{an}}$. Thus $\mathbb{V}$ specifies a complex local system $\mathbb{V}_\mathbb{C}$ on $U^{\mathrm{an}}$, whose monodromy we denote $\rho : \pi_1(U^{\mathrm{an}}, s_0) \to \mathrm{G}L(V)$; and a decreasing filtration $F^\bullet$ (the Hodge filtration) by holomorphic vector bundles of $\mathcal{V}^{\mathrm{an}}$, where $(\mathcal{V}^{\mathrm{an}}, \nabla^{\mathrm{an}})$ is the flat holomorphic connection on $U^{\mathrm{an}}$ associated to $\mathbb{V}_\mathbb{C}$ under the Riemann-Hilbert correspondence. The simplest polarizable $\mathbb{C}$VHS on $U^{\mathrm{an}}$ are those obtained as complexifications of the classical polarizable $\mathbb{Z}$VHS introduced by Griffiths, in particular those of geometric origin, that is, of the form $R^i f_* \mathbb{C}$ for $f : Y \to U$ a smooth projective morphism. Let $\mathrm{G} \subset \mathrm{G}L(V)$ be the algebraic monodromy group of the complex local system $\mathbb{V}_{\mathbb{C}}$ (i.e. the connected component of the Zariski closure of $\rho(\pi_1(Y, y_0)$ in $\mathrm{G}L(V)$); and let $\mathrm{Q} \subset \mathrm{G}$ be the parabolic subgroup defined by the filtration $F^\bullet$. We denote by $\check{X}$ the G-flag variety G/Q. We say that $\mathbb{V}$ is non-trivial if $\check{X}$ is not a point.

Given a complex algebraic group H, let Rep(H) be the abelian tensor category of its finite dimensional algebraic representations and $R(\mathrm{H})$ be its rational representation ring $K_0(\mathrm{Rep}(\mathrm{H}))_{\mathbb{Q}}$. This is a finitely generated $\mathbb{Q}$-algebra easily described in terms of root systems. With this notation our main result regarding the construction of tautological rings is the following:

**Theorem 1.** *Let $S$ be a smooth irreducible quasi-projective variety over $\mathbb{C}$, and $D \subset S$ a (possibly empty) simple normal divisor on $S$. Let $\mathbb{V}$ be a non-trivial polarizable complex variation of Hodge structure on the analytification $U^{\mathrm{an}}$ of $U := S - D$, and let $\mathrm{Q} \subset \mathrm{G}$ be the algebraic groups associated to $\mathbb{V}$ as above. Assume that $\mathbb{V}$ is quasi-unipotent around $D$. These data define a commutative diagram of $\mathbb{Q}$-algebras with non-trivial horizontal maps*

$$\begin{tikzcd} R(\mathrm{Q}) \arrow[r] \arrow[d] & K_0(S)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d] & \mathrm{CH}^\bullet(S)_{\mathbb{Q}} \arrow[d, "\mathrm{cl}"] \\ H^{2\bullet}(\check{X}, \mathbb{Q}) \simeq R(\mathrm{Q}) \otimes_{R(\mathrm{G}),\varepsilon} \mathbb{Q} \arrow[r] & K_0^{\mathrm{topo}}(S^{\mathrm{an}})_{\mathbb{Q}} \arrow[r, "\sim", "\mathrm{ch}^{\mathrm{topo}}"'] & H^{2\bullet}(S^{\mathrm{an}}, \mathbb{Q}), \end{tikzcd} \tag{2}$$

*where $\varepsilon : R(\mathrm{Q}) \to \mathbb{Q}$ is the natural augmentation map.*

With these notation, we define the tautological ring $\mathrm{Taut}^\bullet(S, \mathbb{V}) \subset \mathrm{CH}^\bullet(S)_{\mathbb{Q}}$ as the finitely generated $\mathbb{Q}$-subalgebra image of

$$R(\mathrm{Q}) \to K_0(S)_{\mathbb{Q}} \overset{\mathrm{ch}}{\simeq} \mathrm{CH}^\bullet(S)_{\mathbb{Q}}.$$

Theorem 1 is a generalization to arbitrary $\mathbb{C}$VHS (quasi-unipotent at infinity) of classical results of Matsushima [1], Mumford [5], Van der Geer [6] and Wedhorn-Ziegler [7] for Shimura varieties. It relies on an idea of Milne [2] and results of Mochizuki [3].

## References


[1] Y. Matsushima, *On Betti numbers of compact, locally symmetric Riemannian manifolds*, Osaka Math. J. **1**4 (1962), 1–20.

[2] J. S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, in *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, 283–414, Perspect. Math., 10, Academic Press, Boston, MA.

[3] T. Mochizuki, *Kobayashi-Hitchin correspondence for tame harmonic bundles and an application*, Astérisque No. 309 (2006), viii+117 pp.

[4] D. B. Mumford, *Rational equivalence of* 0*-cycles on surfaces*, J. Math. Kyoto Univ. **9** (1968), 195–204.

[5] D. B. Mumford, *Hirzebruch's proportionality theorem in the noncompact case*, Invent. Math. **42** (1977), 239–272.

[6] G. van der Geer, *Cycles on the moduli space of abelian varieties*, in *M*oduli of curves and abelian varieties, 65–89, Aspects Math., E33, Friedr. Vieweg, Braunschweig.

[7] T. Wedhorn and P. Ziegler, *Tautological rings of Shimura varieties and cycle classes of Ekedahl-Oort strata*, Algebra Number Theory **17** (2023), no. 4, 923–980.

*Reporter: Niklas Müller*

# Participants

**Prof. Dr. Donu Arapura**
Department of Mathematics
Purdue University
West Lafayette, IN 47907-1395
UNITED STATES

**Dr. Benjamin Bakker**
Department of Mathematics
University of Illinois at Chicago
851 S. Morgan St.
Chicago IL 60607
UNITED STATES

**Prof. Dr. Damian Brotbek**
Institut Elie Cartan,
Université de Lorraine
P.O. Box 70239
54506 Vandoeuvre-lès-Nancy
FRANCE

**Dr. Benoît Cadorel**
Université de Lorraine
Institut Elie Cartan de Lorraine
Faculté des sciences
P.O. Box 239
54506 Vandoeuvre-lès-Nancy
FRANCE

**Dr. Junyan Cao**
Laboratoire J.-A. Dieudonné, Université
Côte d'Azur
Parc Valrose
06108 Nice Cedex 2
FRANCE

**Dr. Yifan Chen**
Department of Mathematics
University of Rome, Tor Vergata
Roma 310027
ITALY

**Dr. Shih-Kai Chiu**
Department of Mathematics
University of California, Irvine
Irvine CA 92697-3875
UNITED STATES

**Dr. Ronan Conlon**
Department of Mathematical Sciences
The University of Texas at Dallas
800 W. Campbell Rd.
Richardson TX 75080
UNITED STATES

**Dr. Ya Deng**
CNRS, IMJ-PRG
Sorbonne Université
4, place Jussieu
75252 Paris Cedex 05
FRANCE

**Prof. Dr. Eleonora Di Nezza**
Dipartimento di Matematica
Universita di Roma Tor Vergata
00131 Roma
ITALY

**Prof. Dr. Yajnaseni Dutta**
Mathematisch Instituut
Universiteit Leiden
Einsteinweg 55
2333CC Leiden
NETHERLANDS

**Prof. Dr. Philip M. Engel**
Dept. of Mathematics, Statistics
and Computer Science
University of Illinois Chicago
2028 W Le Moyne 3W
Chicago IL, 60622
UNITED STATES

**Prof. Dr. Philippe Eyssidieux**
Institut Fourier
UMR 5582 du CNRS
Université de Grenoble Alpes
BP 74
100, Rue des Maths
38610 Gières Cedex
FRANCE

**Prof. Dr. Cécile Gachet**
Fakultät für Mathematik
Ruhr-Universität Bochum
44801 Bochum
GERMANY

**PD Dr. Patrick Graf**
Fakultät für Mathematik, Physik
und Informatik
Universität Bayreuth
95440 Bayreuth
GERMANY

**Prof. Dr. Daniel Greb**
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

**Dr. Henri Guenancia**
Institut de Mathématiques de Bordeaux
Université de Bordeaux
351 cours de la Libération
33405 Talence
FRANCE

**Prof. Dr. Hans-Joachim Hein**
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

**Dr. Jaehyun Hong**
Center for Complex Geometry
Institute for Basic Science
55, Expo-ro, Yuseong-gu,
Daejeon 34126
KOREA, REPUBLIC OF

**Prof. Dr. Victoria Hoskins**
Fakultät für Mathematik
Universität Duisburg-Essen
Thea-Leymann-Straße 9
Postfach 9010
45127 Essen
GERMANY

**Prof. Dr. Klaus Hulek**
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

**Prof. Dr. Jun-Muk Hwang**
IBS Center for Complex Geometry
55, Expo-ro, Yuseong-gu,
Daejeon 34126
KOREA, REPUBLIC OF

**Prof. Dr. Stefan Kebekus**
Mathematisches Institut
Universität Freiburg
Ernst-Zermelo-Straße 1
79104 Freiburg i. Br.
GERMANY

**Prof. Dr. Bruno Klingler**
Fachbereich Mathematik
Humboldt Universität Berlin
Unter den Linden 6
10099 Berlin
GERMANY

**Prof. Dr. Christian Lehn**
Fakultät für Mathematik
Ruhr-Universität Bochum
44801 Bochum
GERMANY

**Prof. Dr. Claudio Llosa Isenrich**
Université du Luxembourg,
Department of Mathematics
6 Avenue de la Fonte
4364 Esch-sur-Alzette
LUXEMBOURG

**Niklas Maximilian Müller**
Institute of Mathematics
EPFL SB MATH
MA A2 383 (Bâtiment MA)
Station 8
1015 Lausanne
SWITZERLAND

**Dr. Chung-Ming Pan**
CIRGET
Université du Québec à Montréal
201 avenue du Président Kennedy
Montréal QC H2X 3Y7
CANADA

**Dr. Sung Gi Park**
Department of Mathematics
Princeton University
Fine Hall
Princeton NJ 08544-1000
UNITED STATES

**Dr. Matthias Paulsen**
Département de mathématiques et applications
École Normale Supérieure
45 Rue d'Ulm
75230 Paris
FRANCE

**Prof. Dr. Mihai Păun**
Mathematik
Universität Bayreuth
Postfach 101251
95412 Bayreuth
GERMANY

**Prof. Dr. Dr. h.c. Thomas Peternell**
Fakultät für Mathematik, Physik und Informatik
Universität Bayreuth
95440 Bayreuth
GERMANY

**Simon Pietig**
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

**Prof. Dr. Mihnea Popa**
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

**Prof. Dr. Pierre Py**
Université Grenoble Alpes
Institut Fourier
CS 40700
38058 Grenoble Cedex 09
FRANCE

**Prof. Dr. Erwan Rousseau**
Dept. de Mathématiques
Université de Bretagne Occidentale
6, Avenue Victor Le Gorgeu
29238 Brest Cedex 3
FRANCE

**Willliam Sarem**
Institut Fourier
UMR 5582, CNRS/UJF
Université de Grenoble I
P.O. Box 74
38402 Saint-Martin-d'Hères
FRANCE

**Johannes Scheffler**
Lehrstuhl Mathematik VIII
Mathematisches Institut
Universität Bayreuth
Universitätsstraße 30
95447 Bayreuth
GERMANY

**Prof. Dr. Stefan Schreieder**
Institut für Algebraische Geometrie
Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

**Prof. Dr. Isabel Stenger**
Institut für Algebraische Geometrie
Fakultät für Mathematik und Physik
Leibniz Universität Hannover
30167 Hannover
GERMANY

**Prof. Dr. Botong Wang**
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison 53706-1388
UNITED STATES

**Prof. Dr. Julie Tzu-Yueh Wang**
Institute of Mathematics
Academia Sinica
No.1 Roosevelt Rd. Sc.4
Taipei 10617
TAIWAN

**Dr. Rosa Winter**
Mathematisches Institut
Albert-Ludwigs-Universität Freiburg
Ernst-Zermelo-Str. 1
79104 Freiburg i. Br.
GERMANY

**Zhixin Xie**
Institut Élie Cartan de Lorraine
Université de Lorraine
P.O. Box 70239
54506 Vandoeuvre-lès-Nancy
FRANCE