How big is my slice of cheese?

Chiara Meroni

In this snapshot, we introduce the study of slices of polytopes – geometric shapes with flat sides – and examine the area of these slices. This is connected to combinatorics and polynomials and is surprisingly complex, even in three dimensions. Since we are greedy humans, we conclude by finding the largest possible slice of cheese.

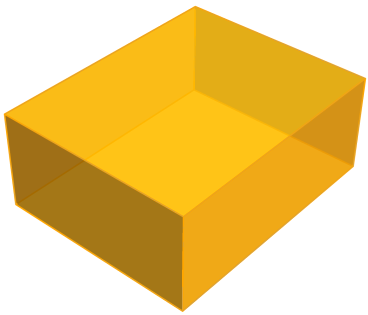

Figure 1: Polytope made of Cheddar.

When I was a kid, my parents would cut the cheese in interesting shapes to make me eat it: cubes, pyramids, ... Throughout this snapshot, let's assume to be my four-year-old self, believing that cheese naturally comes in these shapes, which mathematicians call polytopes.

1 Polytopal cheese

A *polytope* is a geometrical object with vertices, edges, and flat faces that has no dents, holes, or bumps. Figure 1 exhibits an example. Polytopes are a generalization of the famous “Platonic solids”, whose discovery goes back to the ancient Greeks.

A precise mathematical definition relies on the concept of convexity: A set S in three-dimensional space $\mathbb{R}^3$ is *convex* if for every pair of points x, y belonging to S, the line segment connecting them is entirely contained in S. Then, a polytope is the smallest convex set containing a bunch of given points. In particular, a polytope is the smallest convex set containing its vertices. Formally, a *vertex* or corner of a polytope $P \subset \mathbb{R}^3$ is a point v of P for which there are planes in space that intersect the polytope only in the point v. This definition makes sense because every plane that intersects the polytope must either cut through its interior, contain one of its faces or edges completely, or only touch a single corner.

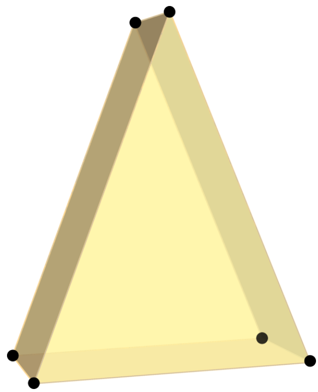

Figure 2: Polytopal Parmesan in yellow, with vertices marked in black.

There is another way to define a polytope, which fits better to the cheese analogy we are exploring: Imagine you have an infinite amount of cheese filling the whole space. Take a knife and make perfectly straight cuts all the way through. After each cut, you choose to keep just one of the two sides of the cut. After some steps, you are going to get a finite piece of cheese, which is actually a polytope! For instance, with six cuts we can get a piece of Cheddar, as in Figure 1. With five cuts we can get a piece of Parmesan, as in Figure 2. From a mathematical perspective, each cut with the knife corresponds to a plane, and a three-dimensional polytope is the intersection of finitely many half-spaces, provided that this set has a finite volume.

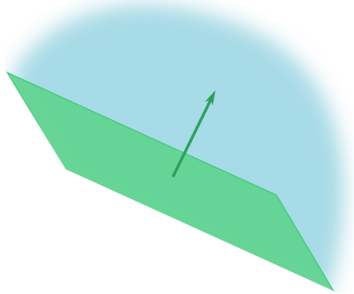

Figure 3: A green vector (a, b, c), its associated green plane $a\,x + b\,y + c\,z = 0$ for $d = 0$, and the blue half-space $a\,x + b\,y + c\,z \geq 0$.

A plane in $\mathbb{R}^3$ can be identified by a vector (a, b, c) in $\mathbb{R}^3$ and a real number d, via the equation

$$a\,x + b\,y + c\,z = d$$

that is satisfied for all points (x, y, z) in the plane. The vector (a, b, c) points in the direction perpendicular to the cutting plane, while d indicates how far from the origin the plane is. A half-space is one of the two parts separated by the plane, namely the set of points (x, y, z) in $\mathbb{R}^3$ that satisfy one of the following *affine inequalities*:

$$a\,x + b\,y + c\,z \geq d, \qquad a\,x + b\,y + c\,z \leq d.$$

A polytope P is then the set of all points (x, y, z) that satisfy a bunch of affine inequalities

$$\begin{aligned} a_1\,x + \ b_1\,y + \ c_1\,z &\geq d_1, \\ a_2\,x + \ b_2\,y + \ c_2\,z &\geq d_2, \\ &\cdots \\ a_m\,x + b_m\,y + c_m\,z &\geq d_m \end{aligned}$$

for some vectors (a_i, b_i, c_i) in $\mathbb{R}^3$ and real numbers d_i. We also require that this set has a finite volume. We just saw two equivalent definitions of a polytope, namely its v-description, as a convex hull of its *vertices*, and its h-description, as an intersection of *half-spaces*.

Example (Parmesan cheese)**.** We can use the two definitions to describe Parmesan cheese. Consider the six points $(\pm 1, \pm 2, 0)$ and $(\pm 1, 0, 5)$. They are represented as the black dots in Figure 2 and their convex hull is the yellow polytope. On the other hand, the same object is also cut out by the following affine inequalities:

$$x \geq -1, \qquad -x \geq -1, \qquad z \geq 0,$$
$$5y - 2z \geq -10, \qquad -5y - 2z \geq -10.$$

Now that we know what a polytopal cheese is, we can move on to our problem: slicing it. We want to cut very very thin, that is, two-dimensional, slices of our cheese. Formally, a *slice* of cheese is the intersection of our original polytopal cheese P with a plane H, which we denote by $Q = P \cap H$. Using the h-description of P, and noticing that a plane is the intersection of the two (closed) half-spaces it defines, we conclude that Q is also a polytope! However, Q lives in a plane and therefore it is a two-dimensional polytope, commonly known as a (convex) *polygon*.

A given polytope P may have infinitely many slices, but many of them look alike, namely they have the same number of vertices and edges. A vertical slice of Parmesan is going to be a triangle; by moving the vertical plane a little bit, we will obtain other triangular slices. On the other hand, a horizontal plane will produce a rectangular slice. Technically, we will say that these different types of slices have different *combinatorial types*. You can check that triangles, quadrilaterals, and pentagons are actually the only possible combinatorial types of slices of Parmesan cheese, as shown in Figure 4. How many different combinatorial types of slices does Cheddar have? (See Figure 8 for the answer.)

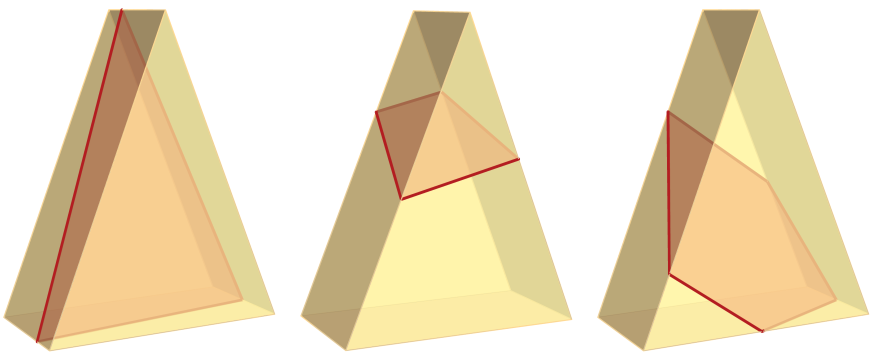

Figure 4: All the possible combinatorial types of slices of the Parmesan.

2 Triangles and areas

Not only do we want to slice our cheese, but also to compute how big the slice is. This translates into the computation of the *area* of the slices. While some slices seem nice and regular, such as the rectangular vertical slice of Cheddar, others, such as the slice on the right side of Figure 4, are not. How can we compute the area of a polygon? For regular polygons there are well known formulas, but what if the polygon is not regular at all?

One of the possible strategies is to reduce the question to the computation of areas of triangles. In order to do this, we have to subdivide the polygon into a finite number of triangles; this is known as a *triangulation*. More precisely, a triangulation $\mathcal{T}$ of a polygon Q is a set of triangles such that

1. every triangle in $\mathcal{T}$ is a subset of Q;
2. any two triangles in $\mathcal{T}$ have exactly one common edge, one common vertex, or do not intersect at all;
3. the union of all triangles in $\mathcal{T}$ is the polygon Q.

For instance, in order to construct a triangulation of a polygon Q we can use the following procedure: We pick a point v inside Q (not on its boundary). Then, the triangles in our triangulation are exactly those with one edge of Q and the vertex v. In this way, we obtain a triangulation $\mathcal{T}$ with as many triangles as edges of Q. This is shown on the left side of Figure 5.

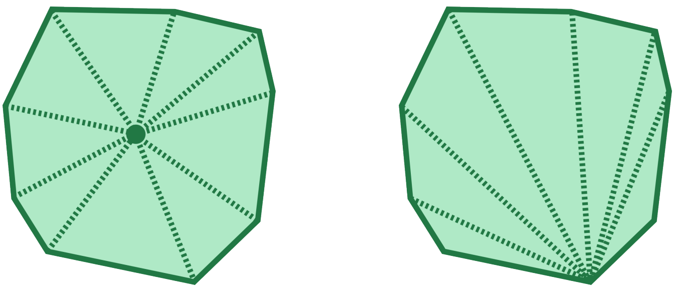

Figure 5: Two possible triangulations of a polygon.

However, there are many other possible triangulations of the same polygon Q. We can add some requirement that we want $\mathcal{T}$ to satisfy. A common one is the following:

All the vertices of all the triangles in $\mathcal{T}$ are also vertices of Q.

Our previous triangulation does not satisfy this property because the additional point v inside Q is a vertex of every triangle, but not of Q. We can construct another triangulation satisfying the extra condition as follows: For a start, we enumerate all the vertices of Q counterclockwise and call them $v_1, v_2, \dots, v_n$, starting with our favourite vertex v_1. Then, the triangles in $\mathcal{T}$ will be those with vertices v_1, v_i, and v_{i+1}, where i goes from 2 to $n-1$. This triangulation is represented on the right side of Figure 5, where v_1 is the lower right vertex. Now, the triangulation $\mathcal{T}$ has two triangles less than the previous one, and all the vertices of the triangles in $\mathcal{T}$ are vertices of Q.

At this point, we can finally compute the area of our polygon Q. Indeed, consider any triangulation $\mathcal{T}$ of Q. Then, the area of Q is just the sum of the areas of all the triangles in $\mathcal{T}$. And we know at least one formula to compute the area of a triangle! For example, we can take half the product of a base length and the corresponding height.

Example. Suppose that Q is the quadrilateral with vertices

$$v_1 = (0,0), \quad v_2 = (2,2), \quad v_3 = (1,5), \quad v_4 = (-4,4),$$

as shown in Figure 6. Then the area of Q is not very easy to compute at first sight. However, we can divide Q into two triangles:

$$\begin{aligned} &\Delta_1 \qquad \text{with vertices } v_1, v_2, v_3, \\ &\Delta_2 \qquad \text{with vertices } v_1, v_3, v_4, \end{aligned}$$

Now, any formula for the area of a triangle will tell us that $\text{area}(\Delta_1) = 4$ and $\text{area}(\Delta_2) = 12$. Therefore,

$$\text{area}(Q) = 4 + 12 = 16.$$

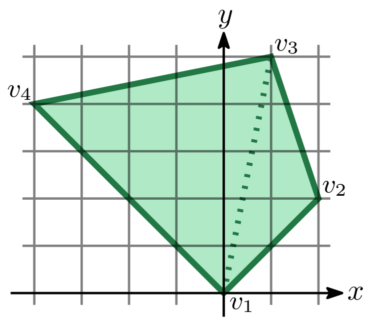

Figure 6: A possible triangulation of a quadrilateral.

3 The biggest slice please, I am hungry!

We learned how to slice some polytopal cheese P and how to measure how big the slice is. A natural question arises:

How can we find the largest slice?

Since there are infinitely many slices, we cannot hope to compute the area of every single one of them and compare the numbers. The solution is to *parametrize* the area of the slices. In other words, we would like to write the area of the slice $Q = P \cap H$ as a function of the position of the plane H. Is it possible? What kind of function is it?

In order to answer this question, let us use the observation we made earlier: There are infinitely many slices, but many of them look alike. Indeed, the slice of the Parmesan cheese in Figure 7 is very similar, though not identical, to the right-most slice displayed in Figure 4. Both slices are pentagons, but this is not the only thing that they have in common. The vertices of the two polygons lie on the same edges of the Parmesan P. Moreover, one can prove that for any generic plane H, all vertices of Q are precisely the points in which the plane H defining Q intersects the edges of P. By *generic*, we mean that H does not contain vertices of the polytope P. In Figure 7, the green plane H is generic and intersects exactly five edges of P. Hence, the slice Q is a pentagon.

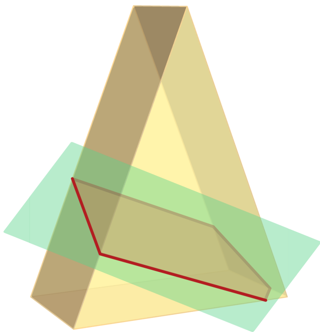

Figure 7: Another slice (red) of Parmesan, of the same combinatorial type as the one on the right side of Figure 4. The plane that cuts out this slice is shown in green.

Let us put together all the planes that intersect the same edges of P. Consider k edges of P, which we call $e_1, \ldots, e_k$, and define $\mathcal{H}(e_1, \ldots, e_k)$ as the set of all planes that intersect these edges. Then, as we already observed, all the slices Q cut out by generic planes H in $\mathcal{H}(e_1, \ldots, e_k)$ are going to have exactly k vertices. If we call v_i the vertex of Q that lies on the edge e_i of P, we can use the triangulation introduced in section 2 where the triangles have vertices $\{v_1, v_i, v_{i+1}\}$. In this way, with only one choice of triangulation, we managed to triangulate all the generic slices $Q = P \cap H$ where H belongs to the set $\mathcal{H}(e_1, \ldots, e_k)$. Using this fact, we can then find a formula for the area of every such slice Q.

To keep track of a plane H, remember that it can be described by an equation $a\,x + b\,y + c\,z = d$ with real numbers a, b, c, d. We call $Q_{a,b,c,d}$ the slice of P cut out by that plane H. Then, one can prove that there is a function f with

$$\text{area}(Q_{a,b,c,d}) = f(a, b, c, d) \qquad \text{for all } H \text{ in } \mathcal{H}(e_1, \ldots, e_k).$$

Luckily for us, the function f is *rational*, which means it is the quotient of two polynomials, and also describes the area of the non-generic slices correctly. This is very convenient when we want to find its maximum. The proof, which involves some symbolic determinant computation, is omitted here for simplicity.

Finding the biggest slice of a polytope P is more challenging from a computational point of view, so we will use the help of computer algebra software. The idea is as follows: Consider all possible tuples $(e_1, \ldots, e_k)$ of edges of P, for all k. For each one, compute the set $\mathcal{H}(e_1, \ldots, e_k)$ and the rational function $f(a, b, c, d)$. Then, an algorithm can find the maximum of the function f and hence the maximum of the area of Q, among the slices arising from planes in $\mathcal{H}(e_1, \ldots, e_k)$. Now repeat the process for all other tuples of edges. As the number of tuples is not infinite anymore, you will eventually have a list of slices that have the maximal area for each type of plane. Choose the largest and you have the answer!

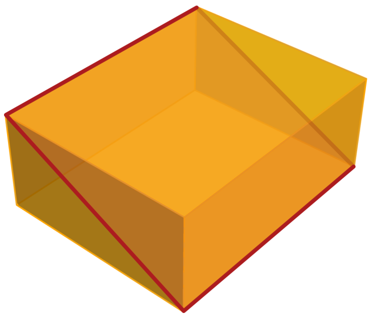

Figure 8: A slice of Cheddar can be a triangle, quadrilateral, pentagon or hexagon. The red slice shown here is the biggest one.

4 Higher-dimensional cheese and beyond

The theory explained in the previous sections is based on discrete geometry, semialgebraic geometry, and optimization. One of the topics of discrete geometry is the study of polytopes (even when they do not represent cheese!). For more details about that we refer to [8] or to other Oberwolfach snapshots like [2, 3, 4]. Semialgebraic geometry studies sets defined by polynomial equations and inequalities. These include polytopes, since they are defined by linear inequalities, which are inequalities involving polynomials of degree one. However, semialgebraic sets are not limited to polytopes: they can assume many more shapes, while still having useful properties. Their features make them suitable for optimization problems and algorithms. This interplay is explained, for instance, in [1]. The more specific topic of this snapshot is based on the results in [5] and the code for the algorithm in `SageMath` is available at [6].

We have been discussing polytopes in three dimensions and their two-dimensional slices, but why not consider *higher-dimensional* cheese? In fact, all the statements in the previous sections have a counterpart in higher dimensions. This setting involves a polytope P of dimension d that lives in $\mathbb{R}^d$ and planes are substituted by $(d-1)$-dimensional *hyperplanes*. In this case, like in three dimensions, we can implement an algorithm that answers the question:

What is the biggest slice of P?

The steps of the algorithm rely on the higher-dimensional version of a triangle, called a *simplex*, on higher-dimensional triangulations, and on the symbolic computation of determinants of some matrices. See [7] for the shape of all slices of cheddar in dimensions four, five, and six.

Using the same methods, we can answer similar questions and find the best slice of a polytope, for other notions of "best". For instance, what is the slice with the largest number of vertices? Or what if your parents cut the cheese in more curvy shapes, like a moon or a ball? What is the best slice of a Mozzarella?

There are indeed many more generalizations of this problem that wait to be investigated and better understood.

...Yet another Italian talking about food!

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Image credits

All figures were created by the author using `Mathematica` or `Inkscape`.

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