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Low-dimensional Topology and Number Theory

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Abstract. The workshop brought together researchers in low-dimensional topology and number theory to explore the rich and increasingly deep interactions between these fields. Central themes included quantum invariants of knots and 3-manifolds, their asymptotic behavior, and the framework of Arithmetic Topology, all of which reveal striking parallels and connections between geometric, quantum, and arithmetic phenomena.

Mathematics Subject Classification (2020): 11D88, 11F03, 11M06, 57K31, 57K32, 57R56.

Introduction by the Organizers

The workshop Low-Dimensional Topology and Number Theory, organized by Thang T. Q. Lê, Adam S. Sikora, and Don Zagier, was held at MFO from April 19–24, 2026. The meeting continued a long-standing tradition of interaction between researchers in topology and number theory, building on four highly successful earlier workshops on the same theme held at Oberwolfach in 2010, 2012, 2014, and 2017, as well as a smaller-scale meeting organized during the Covid pandemic in 2020.

One of the most active and rapidly developing areas connecting these subjects concerns quantum invariants of knots and 3-manifolds, their asymptotic behavior, and structures arising from the analogies and ideas of Arithmetic Topology. The 2026 meeting highlighted substantial recent progress in these directions, including new developments linking quantum topology, geometry, representation theory, and arithmetic phenomena.

The workshop brought together researchers from around the world, at different stages of their careers, ranging from graduate students and early-career mathematicians to internationally recognized leaders in the field. The participants represented a broad spectrum of perspectives and expertise, fostering lively exchanges across disciplinary boundaries. The program featured 23 talks, complemented by extensive informal discussions and collaborations throughout the week.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Low-dimensional Topology and Number Theory

Table of Contents

Abstracts

A cohomological formula for the central value of symplectic L-functions, for Reidemeister torsion and an arithmetic application

Amina Abdurrahman
(joint work with Akshay Venkatesh)

I presented two works, the first is joint work with A. Venkatesh [1] and gives a cohomological expression for the central value of symplectic L-functions on curves up to squares. The proof relies crucially on a similar formula for the Reidemeister torsion of 3-manifolds.

The second work [2] solves a conjecture contained in [1] and gives a topological perspective on the order of the Tate-Shafarevich group up to squares. Both results rely on an analogous picture in low-dimensional topology that is crucial in the arithmetic proofs and on topological tools. The following presentation is essentially contained in [2].

Consider a smooth projective curve X over a finite field k of cardinality q and characteristic p, and a symplectic local system $\rho : \pi_1^{et}(X) \to \mathrm{Sp}_{2r}(V_\ell)$ of ℓ-vector spaces for ℓ a finite field. Suppose that q has a square root in ℓ and that ℓ has characteristic different from 2 and p. To this pair one attaches an L-function as follows: $L(X, \rho, s) := \prod_{x \in X} \det(1 - t^{\deg(x)}\rho(F_x)|V)^{-1}$, where F denotes a geometric Frobenius, $t = q^{-s}$ and $\deg(x)$ is the degree of the field extension $[k(x) : k]$. The Grothendieck-Lefschetz fixed point formula implies that the L-function just defined is a rational function and satisfies the equality

$$L(X, \rho, s) = \prod_{i=0}^{2} \det(1 - tF|H^i(X_{\bar{k}}, \rho))^{(-1)^{i+1}}.$$

We assume that ρ has no invariants or coinvariants, i.e. both H^0 and H^2 vanish. We are then considering the L-function given by the characteristic polynomial of the Frobenius acting by pullback on H^1. The central value is given by

$$L(X, \rho, \frac{1}{2}) = \det(1 - \bar{F}|H^1)$$

where $\bar{F}$ is the normalized Frobenius $\bar{F} = \frac{1}{\sqrt{q}}F$. Recall that H^1 carries an orthogonal pairing, obtained by composing the cup product, the symplectic form and the trace map. $\bar{F}$ preserves this orthogonal pairing. We are interested in understanding the square class of the central value $L(X, \rho, \frac{1}{2})$.

We prove the following theorem:

Arithmetic theorem: [1, Theorem 3.1] Let X and ρ be as above. Suppose that

(a) ρ is geometrically surjective, i.e. the restriction to $\pi_1(X)^{\mathrm{geom}}$ is surjective,
(b) The order of ℓ is ± 1 modulo 8, the order q of k is 1 modulo 8, a square in the prime field of ℓ, and prime to the order of $\mathrm{Sp}_{2r}(\ell)$.

Let $L(X,\rho) \in \ell^\times/2$ denote the square class of the central value $L(X,\rho,\frac{1}{2})$, assumed nonvanishing. Then we have an equality of square classes

$$L(X,\rho) = \mathrm{tr}_X(\rho^* c_{\mathrm{et}}) \in \ell^\times/2 \tag{1}$$

where

- $c_{\mathrm{et}} \in H^3(\mathrm{Sp}_{2r}(\ell), \ell^\times/2)$ is the $(2,1)$ étale Chern class,
- $\rho^* c_{\mathrm{et}}$ is the pullback of c_{et} to absolute étale cohomology $H^3(X, \ell^\times/2)$,
- $\mathrm{tr}_X : H^3(X, \ell^\times/2) \to \ell^\times/2$ is the trace isomorphism.

The proof of this theorem heavily relies on a theorem for a topological invariant that can be considered analogous to the L-function using ideas from arithmetic topology. The analogy goes as follows. We have a fibration $X_{\bar{k}} \to X_k \to \mathrm{Spec}(k)$. We now replace $X_{\bar{k}}$ by a surface Σ, $\mathrm{Spec}(k)$ by the circle S^1 and X_k by a 3-manifold M_f, which fibers over the circle. It can be constructed as the mapping torus M_f of a mapping class (an orientation preserving diffeomorphism up to isotopy) $f : \Sigma \to \Sigma$ on the surface, which is the 3-manifold obtained by gluing the ends of $\Sigma \times [0,1]$ via f. Under the fundamental group analogy for finite fields above, the Frobenius would then correspond to a simple loop on the circle and its action on the first étale cohomology of the curve $H^1_{et}(X_{\bar{k}}, \rho)$ would correspond to the monodromy action of the circle on $H^1(\Sigma, \rho)$, which is the induced monodromy action $f^* : H^1(\Sigma,\rho) \to H^1(\Sigma,\rho)$ of the mapping class $f : \Sigma \to \Sigma$. Note that f^* is orthogonal. Zassenhaus shows that up to squares $\det(1-f) = $ spinor norm $(f) \in \ell^\times/\ell^{\times 2}$ for any automorphism f of an orthogonal space over ℓ of even dimension, square discriminant and without 1 as an eigenvalue. Hence $L(X,\rho,\frac{1}{2}) = \det(1-\bar{F}) =$ spinor norm of $\bar{F} \in \ell^\times/\ell^{\times 2}$.

The correct topological analogue for the L-function therefore has to be an invariant defined on 3-manifolds, that reduces up to a sign to the spinor norm of f^* in the case of the mapping torus M_f fibered over the circle. This property is satisfied by the Reidemeister torsion. It is an invariant valued in square classes that is attached to a pair (M,ρ) of a smooth 3-manifold M together with a local system ρ. In the case of a mapping torus with fiber the surface Σ and orthogonal monodromy action $f^* : H^1(\Sigma,\rho) \to H^1(\Sigma,\rho)$, and with $H^1(\Sigma,\rho)$ of square discriminant, the Reidemeister torsion $RT(M_f,\rho)$ is equal to the spinor norm of f^* up to a sign.

We can now state the topological theorem, which is used in the proof of the arithmetic theorem:

Topological theorem: [1, Theorem 2.1] Suppose that M is a smooth, oriented 3-manifold, and K a field of characteristic not 2, and $\rho : \pi_1(M) \to \mathrm{Sp}_{2r}(K)$ a symplectic local system on M. Then we have an equality of square classes

$$RT(M,\rho) = (-1)^{\chi_{1/2}(M,\rho)/2}\ \mathrm{tr}_M(\rho^* c_{\mathrm{et}}) \in K^\times/2 \tag{2}$$

where

- $c_{\mathrm{et}} \in H^3(\mathrm{Sp}_{2r}(K), K^\times/2)$ is the étale Chern class as before,
- $\chi_{1/2}(M,\rho)$ is the semicharacteristic $\chi_{1/2}(M,\rho) = \dim H^0(M,\rho) - \dim H^1(M,\rho)$,

- $\mathrm{tr}_M : H^3(M, K^\times/2) \to K^\times/2$ is the isomorphism pairing with the fundamental class of M with $\mathbb{Z}/2$ coefficients.

I sketched the proof strategies for both theorems, especially how the topological theorem is used as a crucial step in the arithmetic proof. I then focused on a conjecture in [1] claiming that the cohomological expression is trivial in the case of a compatible system of ℓ-adic symplectic representations arising from a smooth projective curve family over X.

Let K be the global function field of a smooth projective curve X over the finite field k. Consider an abelian variety A_K defined over K that has good reduction everywhere. Assume that the Hasse-Weil L-function of A_K is non-vanishing at 1. Then it is known that the Tate-Shafarevich group is finite and the Birch-Swinnerton-Dyer conjecture is known [3, Theorem 4.6]:

$$|L(A_K, 1)| = \frac{|Ш(A_K/K)||\det(h)|}{|Tor(A_K(K))|^2} q^*,$$

where q^* is some integer power of q the cardinality fo the field k and where h denotes the height pairing. It is a well-known fact that the order of $Ш(A_K/K)$ is either a square or twice a square. The work of Poonen-Stoll [4] gives a criterion to distinguish these cases. In the case of abelian varieties that are Jacobians of curves one can show that the order of both $\det(h)$ and $Ш(A_K/K)$ is always a square. Note that the Hasse-Weil L-function of the Jacobian A of a family of curves over K can be recovered by studying the central value $L(X, \rho_\ell, \frac{1}{2})$ for ρ_ℓ the family of compatible local systems over X coming from the cohomology of the smooth projective curve family over X.

Theorem: [2, Theorem 3.1] For the universal such compatible local system $\tilde{\rho}_\ell$ on the moduli stack $\mathfrak{M}_g$ of genus g curves over $\mathrm{Spec}(k)$, the class $\tilde{\rho}_\ell^* c_{\mathrm{et}} \in H^3(\mathfrak{M}_g, \ell^\times/2)$ is trivial.

Assuming that the arithmetic theorem holds without assumptions (a) and (b) (the following indicates that this is likely) implies that $L(X, \rho_\ell, \frac{1}{2}) \bmod \ell = \mathrm{tr}_X(\rho^* c_{\mathrm{et}}) \in \ell^\times/{\ell^\times}^2$ is a square. This gives a topological proof that the order of the Tate-Shafarevich group of the associated family of Jacobians is a square in $\mathbb{Q}(\sqrt{q})$, under the above assumption.

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Quantum Invariants, Resurgence, and Borel dual TQFT axioms

Jørgen Andersen
(joint work with Sam Hindson, William Mistegård)

We recall the WRT–TQFT axioms due to Witten–Atiyah–Segal and the explicit construction of Reshetikhin–Turaev of the corresponding coloured Jones polynomial and the quantum invariants of closed oriented 3-manifolds, using explicit formulae for the R-matrices.

Following this, we cover the finite dimensional integral formulae for these quantum invariants obtained in joint work with William Mistegård and Sam Hindson using Faddeev's quantum dilogarithm. We further reported on our asymptotic study of the resulting non-compact functions for link diagrams and described how the leading order term is given by the Volume Conjecture potential and thus, through their work, relates to the complex Chern–Simons functional.

After recalling various aspects of Écalle's approach to Resurgence we formulated the Resurgence Conjecture which implies a number of conjectures concerning the WRT–TQFT.

Based on ongoing joint work with Fantini, Kontsevich and Wheeler, we sketch theoretical descriptions of Resurgent functions, we formulated new Borel dual TFT axioms. These are for an oriented compact 3-manifold with boundary furnished with rings of certain perverse sheaves over the future mirror cover of the pull back of the Chern-Simons line bundle under the restriction map to the boundary together with a certain sheaf map from this sheaf to the sheaf of middle degree holomorphic forms with singularity along the classical Chern-Simons section tensor the space of normal distributions to the image of the moduli space of $\mathrm{SL}_2(\mathbb{C})$-flat connections on 3-manifold in the moduli space of $\mathrm{SL}_2(\mathbb{C})$-flat connections on the boundary of the 3-manifold.

Disjoint union and gluing morphism for such objects was described and the case of a crossing inside a 3-ball and a solid torus with an unknot was discussed.

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Universal link invariants via configuration spaces

Cristina Anghel

My research area is quantum topology, at the intersection between representation theory, low dimensional topology and symplectic topology. Quantum invariants for links and 3-manifolds have origin in representation theory. There are important conjectures predicting the geometric information encoded by asymptotics of these invariants, such as the Volume Conjecture and Gukov-Manolescu's Conjecture.

Motivated by the conjectured geometry that is behind this story, my interest is to study sequences of quantum invariants such as coloured Jones and coloured Alexander polynomials (often called ADO polynomials) from a new topological viewpoint: as graded intersections in configuration spaces. Such models appeared in [1], [2], [3], [4], [6], [7], [8], [9], [12], [13]. The aim of my future research is to:

- understand the geometry and topology encoded in asymptotics of quantum invariants and
- to investigate geometrical type categorifications for quantum invariants.

1) Universal link invariants from configuration spaces ([1],[2])
On the representation theory side, Habiro defined a universal knot invariant providing a unification of all coloured Jones polynomials for knots. This led to his celebrated unification of Witten-Reshetikhin-Turaev invariants for homology spheres ([11]). However, passing from knots to the general link case, such asymptotic results remained open.

Question: Can one construct a unification of coloured Alexander invariants for coloured links? (Parallel to Habiro's famous program unifying coloured Jones polynomials for knots [11]).

In [2] I answered the above open problem that comes from representation theory, using topological tools. Let L be a link seen as the closure of a braid β_n with n strands. Then, for a fixed level $\mathcal{N}$, we construct a weighted Lagrangian intersection: $\Gamma^{\mathcal{N}}(\beta_n) \in \mathbb{L}_{\mathcal{N}} = \mathbb{Z}[w_1^1, ..., w_{\mathcal{N}-1}^l, u_1^{\pm 1}, ..., u_l^{\pm 1}, x_1^{\pm 1}, ..., x_l^{\pm 1}, y^{\pm 1}, d^{\pm 1}]$ in the configuration space of $(n-1)(\mathcal{N}-1)+2$ points in the disc. This intersection $\Gamma^{\mathcal{N}}(\beta_n)$ is parametrised by a set of Lagrangian intersections between submanifolds presented in Figure 1 and weighted via the variables of $\mathbb{L}_{\mathcal{N}}$.

$\mathscr{F}_{\bar{i},\mathcal{N}} \in H_{2n+2,(n-1)(\mathcal{N}-1)+2}$ and $\mathscr{L}_{\bar{i},\mathcal{N}} \in H^{\partial}_{2n+2,(n-1)(\mathcal{N}-1)+2}$

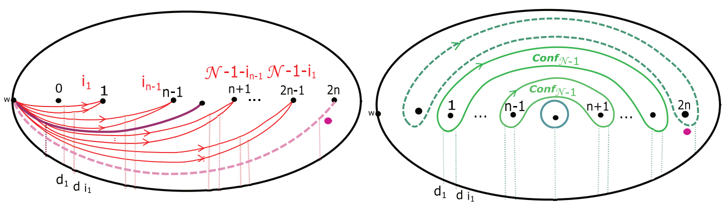

FIGURE 1. Coloured Homology Classes

Our strategy is to build universal link invariants via two sequences of new link invariants, which are defined via the Lagrangian intersection $\Gamma^{\mathcal{N}}(\beta_n)$ and unify more and more coloured Jones and coloured Alexander link polynomials.

Theorem 1 (Universal ADO and Jones link invariants [2]). *We construct two geometric link invariants $\hat{\Gamma}(L), \hat{\Gamma}^J(L)$, in two completions of a polynomial ring $\hat{\mathbb{L}}, \hat{\mathbb{L}}^J$,* *that recover all coloured Alexander link invariants and all coloured Jones link invariants respectively. They are defined geometrically, as limits of intersections in configuration spaces:*

$$\hat{\Gamma}(L) := \varprojlim \hat{\Gamma}^{\mathcal{N}}(L) \in \hat{\mathbb{L}} \qquad \hat{\Gamma}^{J}(L) := \varprojlim \hat{\Gamma}^{\mathcal{N},J}(L) \in \hat{\mathbb{L}}^{J} \tag{1}$$

Moreover, $\hat{\Gamma}^{\mathcal{N}}(L)$ is a new link invariant, called the $\mathcal{N}^{th}$ unified ADO invariant, which recovers all ADO invariants at levels bounded by $\mathcal{N}$ through a geometric perspective. The geometric invariant $\hat{\Gamma}^{\mathcal{N},J}(L)$ is a new link invariant, called the $\mathcal{N}^{th}$ unified Jones invariant, which recovers all coloured Jones polynomials for links at multi-colours bounded by the level $\mathcal{N}$.

The geometric origin of the universal invariants gives a new topological perspective for understanding asymptotic behaviour of these non-semisimple invariants, for which a purely topological 3-dimensional description is an important open problem. At the same time, this result opens up a new direction on the representation theory side, concerning the relation between the quantum group that is at the origin of these non-semisimple invariants and the structure of our universal ring, as conjectured in [1].

2) Jones and Alexander polynomials via quantum Heegaard diagrams ([3])
In [3] we showed a unification of the Alexander and Jones polynomials via a new geometric perspective which we call "quantum Heegaard surfaces". We do so by defining "quantum Heegaard diagrams" together with an extra piece of data, given by "quantum Alexander grading". This grading is a refinement of the Alexander grading used in knot Floer homology. Then we define a graded Lagrangian intersection, in two variables, between concrete Lagrangian submanifolds arising from the curves of the diagram in the symmetric power of the quantum Heegaard surface. The two-variable intersection unifies and recovers the Jones and Alexander polynomials as two specialisations of coefficients.

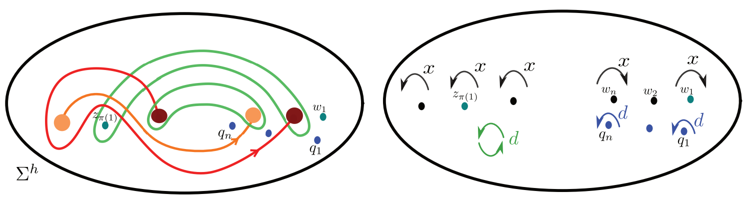

FIGURE 2. Quantum Heegaard diagram and q-Alexander gradings

Let L be an oriented knot and $\beta_n \in B_n$ a braid such that $L = \hat{\beta}_n$. From this we construct explicitly Σ^q, which is a Heegaard surface of genus $n-1$ obtained from the braid action on the punctured disc.

Definition 2 (Quantum Heegaard diagrams and q-Alexander gradings)**.** *Let us define $\mathscr{H}^q_{\beta_n} := (\Sigma^q, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ the Heegaard diagram decorated with an additional set of base points $\overline{q}$ as in Figure 2. Let $\Sigma^q := (\Sigma, \overline{q})$ and call it the* quantum

Heegaard surface. *Then, using this data, we define new gradings* A^{HF} *and* A^{qHF}, *called* q-Alexander gradings, *that provide a two-variable grading using the quantum Heegaard surface* Σ^q.

Definition 3 (Quantum Lagrangian intersection). *We define the* quantum Lagrangian intersection

$$\Omega^q(\beta_n)(x,d) := (d^2x)^{\frac{w(\beta_n)+n-1}{2}} \cdot d^{-(n-1)} \sum_{\bar{z}\in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \varepsilon_{\bar{z}} \cdot x^{A^{HF}(\bar{z})} \cdot d^{A^{qHF}(\bar{z})} \in \mathbb{Z}[x^{\pm\frac{1}{2}}, d^{\pm 1}], \tag{2}$$

where $\mathbb{T}_\alpha$ *and* $\mathbb{T}_\beta$ *are the Lagrangians obtained from the product of* α *(red) and* β *(green) curves from Figure 2.*

Theorem 4 (Jones and Alexander polynomials unified on quantum Heegaard surfaces). *The q-Lagrangian intersection* $\Omega^q(\beta_n)(x,d)$ *defined on the q-decorated Heegaard diagram* $\mathscr{H}^q$ *unifies the Jones and Alexander polynomials as follows:*

$$\Omega^q(\beta_n)|_{x=-d^{-1}} = J_L(x); \qquad \Omega^q(\beta_n)|_{d=1} = \Delta_L(x). \tag{3}$$

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Hopf ideals in quantum groups a roots of 1

Matthew Harper

Quantum groups are a deformation of enveloping algebras of Lie algebras depending on a parameter q. When q is generic, their representation theory is very similar to that of classical Lie algebras, whereas at roots of unity it is closer to Lie algebras in finite characteristic.

Motivated by questions of uniqueness and well-definedness of Reshetikhin–Turaev knot invariants for quantum groups at roots of unity, in this talk we investigate the structure of the ideals and subalgebras generated by power elements in the quantum groups. We show that the choices made in their construction, specifically those related to choices of longest word in the Weyl group, depend only on the underlying group element and not the chosen word. These results extend some work of De Concini–Kac–Procesi on quantum groups at odd roots of unity.

We discuss types A_n and B_2 as specific examples of the general theory and to formulate broader conjectures.

A Trace–Path Integral Formula over Function Fields

Yan Yau Cheng

1. Motivation and Main Result

In a topological quantum field theory, path integrals can often be expressed instead as the trace of a monodromy action on a Hilbert space.

$$\mathrm{Tr}(F|\mathcal{H}) = \int_{\mathcal{F}} e^{i\mathcal{A}(\gamma)} d\gamma.$$

We discuss an arithmetic analogue of this phenomena for function fields, where the phase space is replaced with the ℓ-torsion points of the Jacobian of a curve over a finite field, the path integral is replaced with a sum over the points of $J[\ell]$, and the monodromy is instead replaced with the Frobenius action.

Theorem 1 ([1, Thm 5.1]). *Let J be the Jacobian of a genus g curve X over a finite field $\mathbb{F}_q$. Let ℓ be an odd prime satisfying $q \equiv 1 \pmod{\ell}$. If Fr_q acts semisimply on the $\mathbb{F}_\ell$-vector space $J[\ell]$, then we have the following equality*

$$\mathrm{Tr}(\mathrm{Fr}_q|\mathcal{H}) = \left(\frac{(-1)^g \det'(1-\mathrm{Fr}_q)\det(\mathcal{A})}{\ell}\right) \sum_{\gamma \in J[\ell](\mathbb{F}_q)} e^{2\pi i \mathcal{A}(\gamma)},$$

where $\det'(1-\mathrm{Fr}_q)$ is the regularised determinant of the linear map $1-\mathrm{Fr}_q$ on vector space $J[\ell]$. That is, this is the determinant of $1-\mathrm{Fr}_q$ after quotienting the space $J[\ell]$ by its kernel.

This result is motivated by and adds to the series of analogies between topology and arithmetic. For X a smooth projective curve over a finite field $\mathbb{F}_q$, it is natural to compare $\overline{X} := X \times_{\mathbb{F}_q} \mathrm{Spec}\,\overline{\mathbb{F}}_q$ with a smooth compact Riemann surface Σ. On the other hand, X itself has more in common with a three-manifold M – for instance X has étale cohomological dimension 3 and both sit in Cartesian squares

$$\begin{array}{ccc} \overline{X} & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}\overline{\mathbb{F}}_q & \longrightarrow & \operatorname{Spec}\mathbb{F}_q \end{array} \qquad \begin{array}{ccc} \Sigma & \hookrightarrow & M \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & S^1. \end{array}$$

Just as the manifold M can be viewed as a mapping torus fibred over the circle $M = \Sigma \times [0,1]/F$, for some monodromy action $F\colon \Sigma \to \Sigma$, we can then view the curve X as being analogous to a mapping torus of the form X ‘ = ’ $\overline{X} \times [0,1]/\operatorname{Fr}_q$.

A weaker version of this theorem was initially an unpublished result of Minhyong Kim and Akshay Venkatesh, where they show the formula up to an undetermined sign in the special case where the vector space $J[\ell]$ has an invariant Lagrangian with respect to the Frobenius action.

2. Trace Side

In order to understand the result of theorem 1, one has to define the appropriate arithmetic analogues of the Hilbert space $\mathcal{H}$, and the action functional $\mathcal{A}$. The arithmetic definition of $\mathcal{H}$ can be viewed as an analogue of *geometric quantisation*,

Definition 2. *Define Y to be a lift of $\overline{X}$ to the Witt vectors $W(\overline{\mathbb{F}_q})$. The Jacobian $J_Y = \operatorname{Jac}(Y)$ comes equipped with a canonical principal polarisation. Let $\Theta \to J_Y$ be the theta line bundle associated to this polarisation. Then the arithmetic analogue of the quantisation of $J[\ell]$ is*

$$\mathcal{H}_\ell := \Gamma(J_Y, \Theta^{\otimes \ell}) \otimes \mathbb{C}.$$

Via a finite analogue of the Stone-Von Neumann theorem, one can show that $\mathcal{H}$ is a unique representation of the finite Heisenberg group $H(J[\ell])$. Using machinery from Gurevich-Hadani [2] which gives very explicit descriptions of this representation. By considering indicator functions on a Lagrangian subspace of $J[\ell]$ which forms a basis ofthis representation, we have the key lemma

Lemma 3 ([1, Lem 3.7])**.** *Let $g \in \operatorname{Sp}(V)$ be any symplectomorphism. Given any Lagrangian M, and any complement M' such that $M \oplus M' = V$, let $\mathcal{S}$ be the set*

$$\mathcal{S} := \{x \in M' : gx - x \in M + gM\}.$$

and for each $x \in S$, pick $m_x, n_x \in M$ such that

$$gx - x = m_x + gn_x.$$

Then

$$\operatorname{Tr}(g|C_{M^\circ}) = A_{M^\circ, gM^\circ} \cdot \sum_{x \in \mathcal{S}} \psi\left(\frac{1}{2}\omega(m_x + n_x, x)\right).$$

Then, by decomposing $J[\ell]$ into symplectic subspaces and applying the key lemma repeatedly, we obtain

Theorem 4 ([1, Thm 3.2]). *Suppose that the Frobenius* Fr_q *acts semi-simply on the space* $J[\ell]$. *Then*

$$\mathrm{Tr}(\mathrm{Fr}_q\,|\mathcal{H}) = \left(\frac{(-1)^{g-\dim_\ell J[\ell](\mathbb{F}_q)/2}\det{}'(1-\mathrm{Fr}_q)}{\ell}\right)\sqrt{|J[\ell](\mathbb{F}_q)|}.$$

Where $\det'$ *is the regularised determinant.*

3. Path Integral Side

The arithmetic analogue of the action $\mathcal{A}$ is defined via a pairing from class field theory.

Definition 5. *Suppose* $\mu_\ell \subset \mathbb{F}_q$. *An* ℓ*-torsion point* $\gamma \in J[\ell](\mathbb{F}_q)$ *defines a line bundle* L_γ *with isomorphism* $f : (L_\gamma)^{\otimes \ell} \xrightarrow{\sim} \mathcal{O}_X$. *Then we can define a* $\frac{1}{\ell}\mathbb{Z}/\mathbb{Z} = \mu_\ell$ *torsor* $c_{\gamma,f}$ *via*

$$c_{\gamma,f}(U) \coloneqq \left\{y \in \Gamma(L_\gamma, U) : f(y^{\otimes \ell}) = 1\right\} \in H^1(X, \tfrac{1}{\ell}\mathbb{Z}/\mathbb{Z})$$

for any étale map $U \to X$. *Then we define*

$$\mathcal{A}(\beta,\gamma) \coloneqq c_\gamma(\mathrm{Rec}(\beta))$$

where $\mathrm{Rec}\colon J[\mathbb{F}_q] \to \pi_1^{ab}(X)$ *is the reciprocity map and we view the torsor* c_γ *as a homomorphism* $\pi_1(X)^{ab} \to \frac{1}{\ell}\mathbb{Z}/\mathbb{Z}$. *It can be shown that the pairing is independent of the choice of isomorphism* f. *We also define* $\mathcal{A}(\gamma) \coloneqq \mathcal{A}(\gamma,\gamma)$.

In order to evaluate the path integral, the key ingredient is to prove that the pairing A co-incides with a function field analogue of the abelian arithmetic Chern-Simons action defined in [3]. In particular, one can show that A is a symmetric bilinear pairing, and the theory of quadratic Gauss sums can be employed to prove

Theorem 6 ([1, Thm 4.16]). *The arithmetic path integral evaluates to*

$$\sum_{\gamma \in J[\ell](\mathbb{F}_q)} e^{2\pi i \mathcal{A}(\gamma)} = \sqrt{|J[\ell](\mathbb{F}_q)|}\left(\frac{(-1)^{(\dim_\ell J[\ell](\mathbb{F}_q))/2}\det(\mathcal{A})}{\ell}\right).$$

Finally, combining the results of theorems 4 and 6 yields theorem 1.

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Stated skein TQFTs

Francesco Costantino
(joint work with M. Faitg)

The purpose of this talk is to provide an overview of the construction of a new kind of TQFTs valued in a category of algebras and bimodules and whose origin is to be found in the theory of stated skein algebras and modules.

We will start with a general motivation for our construction which stems from the application of the cobordism hypothesis of Baez and Dolan applied to the 4-category BRTens introduced by Brochier-Jordan-Safronov and Snyder. Indeed we argue that the existence of a putative fully extended 4-TQFT with values in BRTens points at the existence of a TQFT in dimensions 2 and 3 associating to each surface a category and to each 3-manifold a vector space.

We will construct such a TQFT by associating to each surface an algebra (so that the category associated to the surface is implicitly the category of modules over that algebra) and to each 3-manifold M with input boundary $\partial_- M$ and output boundary $\partial_+ M$ a bimodule over the algebras associated to $\partial_\pm M$ (which implicitly provides a functor between the category associated to the surfaces).

We will achieve this only for the so-called Crane-Yetter category of connected surfaces with one boundary component and their connected cobordisms (with side cylindrical boundary), for which a full presentation has been provided independently by Habiro and Bobtcheva-Piergallini.

After presenting this category which turns out to be non trivially braided an balanced, we define stated skeins of 3-manifolds with a marking in their boundary (formed by a finite disjoint union of oriented segments embedded in the boundary). Fixing the algebraic input of a ribbon Hopf algebra, to each such manifold we associate an H-module (possibly infinite dimensional) spanned by the framed oriented tangles in M colored by H-modules and possibly hitting the boundary exactly along the marking where they carry additional combinatorial information (the "state"). The definition of such vector spaces is due originally, for the case of $H = O_q(SL_2)$ to Thang Le for 3-manifolds of the form surface cross interval and was studied in detail by T. Le and the speaker for general 3-manifolds. It was later generalised to SL_3 by Higgins and to general SL_n by T. Le and A. Sikora. As proved by Baseilhac, Faitg Roche for surfaces and general H its gives the algebras formerly known as "moduli algebras" by Alekseev-Schomerus and Buffenoir-Roche.

Exploiting the topological structure of the three manifolds obtained by thickening surfaces, we see that stated skein modules of (thickened) surfaces are algebras and that for a cobordism one gets automatically a bimodule over the stated skein algebras of its boundary. These stated skein modules turn out to have another, crucial, structure, which we called $\mathcal{L}$-linearity where $\mathcal{L}$ is the end of the category of H-modules.

The main theorem states that, fixed a ribbon Hopf algebra H, there exists a braided balanced monoidal functor from the Crane-Yetter category to a suitable category of algebras and their bimodules internal to H-mod. The most non trivial part is making sense of the braiding and balancing for the latter category and this requires the introduction of the notion of half braided algebras, independently discovered by Johnson-Fried and Reutter.

In the end of the talk we outline rapidly the links between our construction and the functor of Kerler-Lyubaschenko: when H is a finite dimensional factorizable Hopf algebra the stated skein algebra turns out to be isomorphic to the algebra of endomorphisms of the Kerler-Lyubaschenko vector space and a whole commuting diagram of TQFTs can be established.

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A Langlands duality conjecture for character varieties of 3-manifolds

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(joint work with Léo Bénard)

Let M be a closed orientable 3-manifold. Some of the most natural and well-studied invariants of M are its character varieties. Given a complex semisimple algebraic group G, the G-character variety of M is:

$$X(M,G) = \mathrm{Hom}(\pi_1(M), G)//G$$

where // is the GIT quotient, meaning we identify representations whose orbits under G-conjugation intersect in their closures. We also view $X(M,G)$ as a scheme [9], which in particular implies that points in $X(M,G)$ come with multiplicities.

Some of the interest in character variety (particularly, SL_2-character varieties) come from their relationship with essential surfaces, as follows from the landmark results of Culler and Shalen [2]. A closed orientable surface F embedded in M is said to be essential if either $F \simeq S^2$ and F does not bound a ball B^3 in M, or F is a surface of genus $g \geq 1$ and $\pi_1(F)$ injects in $\pi_1(M)$. By [2], if $X(M, \mathrm{SL}_2(\mathbb{C}))$ is positive dimensional then M contains an essential surface. 3-manifolds without essential surfaces are said to be *small*.

In the following however, we will be interested in G-character varieties for different choices of G; more precisely, for Langlands duals pairs $(G, {}^LG)$. For simplicity, we will restrict our scope to *simple* complex algebraic group; we recall that a complex algebraic group G is called simple if and only if its Lie algebra $\mathfrak{g}$ is simple.

Definition 1. *Let G be a complex algebraic simple group. Then the Langlands dual group ${}^LG$ is the complex algebraic simple group with*

(i) $\pi_1(G) \simeq Z({}^LG)$
(ii) $Z(G) \simeq \pi_1({}^LG)$
(iii) $\mathfrak{g} \simeq {}^L\mathfrak{g}$ *except for* $\mathfrak{g}$ *of type B or C, in which case we have* ${}^L\mathfrak{so}_{2n+1} \simeq \mathfrak{sp}_{2n}$.

Let us note that the notion of Langlands dual group is defined in more generality for any complex reductive group as exchanging the roots and coroots data. We can now state our conjecture:

Conjecture 2. [1] *Let M be a small closed 3-manifold, and let G be a complex reductive group. Then we have*

$$|X(M,G)| = |X(M,{}^LG)|$$

where $|X(M,G)|$ is the number, possibly infinite and counted with multiplicities, of points in the G-character variety $X(M,G)$, and similarly for $X(M,{}^LG)$.

The above conjecture can be considered as the classical version of a conjecture of Ben-Zvi, Gunningham, Jordan and Safronov, concerning G-skein modules. The G-skein module of a 3-manifold is, loosely speaking, a quantization of its G-character variety, and encodes the combinatorics of links (and trivalent graphs) colored by elements of the ribbon category $\mathrm{Rep}_q(G)$, and up to local relations induced by equalities of Reshetikhin-Turaev invariants of tangles. We refer to [8] for details.

Conjecture 3. [8] *Let M be a closed 3-manifold, and let G be a complex reductive group. Then we have*

$$\dim Sk_G(M) = \dim Sk_{{}^LG}(M)$$

where both dimensions are dimensions at a generic parameter $q \in \mathbb{C}$.

Conjecture 3 was in turn motivated by the work of Kapustin and Witten [7] connecting a twisted version of $\mathcal{N} = 4$ super Yang-Mills theory to the Geometric Langlands program. However, there is little direct evidence to Conjecture 3. In [8], it was announced that the conjecture has been verified for $M = \Sigma_g \times S^1$ and $(G, {}^LG) = (\mathrm{SL}_2(\mathbb{C}), \mathrm{PGL}_2(\mathbb{C}))$, however, the computation of $Sk_{PGL_2}(\Sigma_g \times S^1)$ still has not appeared in print t this date.

In constrast, Conjecture 2 can be readily verified for many examples of pairs (M, G). Indeed, many cases follow directly from Poincaré duality:

Proposition 4. *Let M be a closed 3-manifold, let G be a simple complex algebraic group, not of type B or C, and let $A = Z(\widetilde{G})$, where $\widetilde{G}$ is the universal cover of G. Then if $H^1(M, A) = 0$, then $|X(M,G)| = |X(M,{}^LG)|$*

The idea behind the above proposition is very simple. Assume for simplicity that $\pi_1(G) = 1$, so that ${}^LG = G/Z(G)$. Then $X(M,G)$ is endowed with a $H^1(M, Z(G))$-action, which is the multiplication of G-characters by central characters. Orbits of the action are exactly the fibers of the natural map $X(M,G) \longrightarrow X(M, G/Z(G)) = X(M,{}^LG)$. On the other hand, $X(M,{}^LG)$ is decomposed into obstruction classes for lifting to G, which are parametrized by $H^2(M, A)$. The hypothesis then imply that the natural map $X(M,G) \longrightarrow X(M,G/Z(G))$ is a bijection.

Despite the simplicity of Proposition 4, we should highlight that the general situation is more complicated, as the following example shows:

Example 5. *Let $M = \mathbb{R}P^3 \# \mathbb{R}P^3$. Then $|X(M,\mathrm{SL}_2(\mathbb{C}))| = 4$ but $|X(M,\mathrm{PGL}_2(\mathbb{C}))| = \infty$*

The above example also shows that the additional hypothesis in Conjecture 2 compared to Conjecture 3 is necessary. We note that this condition is inspired by the following conjecture of the author and Kalfagianni and Sikora:

Conjecture 6. [5] *Let M be a small 3-manifold, then $\dim Sk_{SL_2}(M) = |X(M,\mathrm{SL}_2(\mathbb{C}))|$.*

We note that in [6], a simple criterion for M to satisfy Conjecture 6 was enounced, and verified for surgeries on the figure eight knot, while Conjecture 6 was also verified for small Seifert manifold in [5].

Let us now turn to the evidence for Conjecture 2. Our first case concerns spherical manifolds, that is, closed 3-manifolds with finite foundamental groups:

Theorem 7. [1] *Conjecture 2 is true for M a spherical 3-manifold, and $(G, {}^LG) = (\mathrm{SL}_n(\mathbb{C}), \mathrm{PGL}_n(\mathbb{C}))$ or $(\mathrm{SO}_{2n+1}(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C}))$*

The proof of Theorem 7 takes advantage of the fact that spherical manifolds have finite foundamental group. In fact, in the process of proving this theorem, we explain how to compute generating series for the numbers $|X(\Gamma, G_n)|$ where Γ is any finite group and $G = SL, PGL, SO$ or Sp.

Our second and third result are concerned with the $(\mathrm{SL}_2(\mathbb{C}), \mathrm{PGL}_2(\mathbb{C}))$-duality, but for more general 3-manifolds. First, we studied the case of Seifert manifolds, whose $\mathrm{SL}_2(\mathbb{C})$-character varieties were studied in full details in [5] and [4]:

Theorem 8. [1] *For any small closed Seifert manifold M such that $X(M,\mathrm{SL}_2(\mathbb{C}))$ and $X(M,\mathrm{PGL}_2(\mathbb{C}))$ are reduced, Conjecture 2 is true for the pair $(G, {}^LG) = (\mathrm{SL}_2(\mathbb{C}), \mathrm{PGL}_2(\mathbb{C}))$.*

Moreover, there are small closed Seifert manifolds M that satisfy Conjecture 2 for this pair, but with a different count of non-reduced points.

Note that the difficulty in establishing Conjecture 2 for all small Seifert manifolds comes from the difficulty in computing the multiplicities of non-reduced points in $X(M,\mathrm{PGL}_2(\mathbb{C}))$. For $X(M,\mathrm{SL}_2(\mathbb{C}))$, the latter was achieved in a recent work of the author [4].

Finally, we consider the case where $M = E_K(p/q)$ is a Dehn filling of a knot K in S^3, and $(G, {}^LG)$ is $(\mathrm{SL}_2(\mathbb{C}), \mathrm{PGL}_2(\mathbb{C}))$. Before stating our result, let us introduce some terminology. For K a knot, we denote by E_K its complement $S^3 \setminus N(K)$, which has boundary $\partial E_K \simeq T^2$. The inclusion $T^2 \hookrightarrow E_K$ induces a map

$$r : X(E_K, \mathrm{SL}_2(\mathbb{C})) \longrightarrow X(T^2, \mathrm{SL}_2(\mathbb{C}))$$

Note that $X(T^2, \mathrm{SL}_2(\mathbb{C})) \simeq (\mathbb{C}^*)^2/\iota$, where $\iota(x,y) = (x^{-1}, y^{-1})$. A theorem of Marché and Maurin [5] implies that if a knot K is such that

(i) K is small, that is, E_K has no closed incompressible non-boundary parallel surfaces
(ii) $X(E_K, \mathrm{SL}_2(\mathbb{C}))$ is reduced
(iii) No singular point of $X(E_K, \mathrm{SL}_2(\mathbb{C}))$ is mapped to a torsion point of $X(T^2, \mathrm{SL}_2(\mathbb{C}))$

then $X(E_K(p/q))$ is reduced for all but finitely many slopes $p/q \in \mathbb{Q} \cup \{\infty\}$.

Finally, we call a character $\chi \in X(E_K, \mathrm{SL}_2(\mathbb{C}))$ *parabolic* if $\chi(\gamma) \in \{\pm 2\}$ for any $\gamma \in \pi_1(\partial E_K)$. Moreover, we call a parabolic character *positive* (resp. *negative*) if $\chi(\lambda) = +2$ (resp. -2), where λ is the conjugacy class $\pi_1(E_K)$ of the zero-framing longitude of K.

Theorem 9. [1] *Let K be a knot satisfying the hypothesis (i)-(iii) above. Then Conjecture 2 for the pair $(G, {}^LG) = (\mathrm{SL}_2(\mathbb{C}), \mathrm{PGL}_2(\mathbb{C}))$ is satisfied for almost all Dehn surgeries on K, if and only if*

$$p_- - p_+ = \frac{\det(K) - 1}{2},$$

where p_+, p_- are the number of positive and negative parabolic characters of K, and $\det(K) = |\Delta_K(-1)|$ is the determinant of K.

We roughly describe how Theorem 9 is proved. We refine the naive idea that the natural map $X(M, \mathrm{SL}_2(\mathbb{C})) \longrightarrow X(M, \mathrm{PGL}_2(\mathbb{C}))$ has fibers which are orbits of the natural $H^1(M, \mathbb{Z}/2\mathbb{Z})$-action on $X(M, \mathrm{SL}_2(\mathbb{C}))$ described above and that obstruction classes are parametrized by $H^2(M, \mathbb{Z}/2\mathbb{Z})$, so that the non-injectivity and non-surjectivity of the map $X(M, \mathrm{SL}_2(\mathbb{C})) \longrightarrow X(M, \mathrm{PGL}_2(\mathbb{C}))$ should exactly compensate thanks to Poincaré duality. Our first additional ingredient is a precise description of the characters $x \in X(M, \mathrm{SL}_2(\mathbb{C}))$ on which the $H^1(M, \mathbb{Z}/2\mathbb{Z})$-action is non-free. Secondly, when M is Dehn-filling of a knot, then $H^1(M, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ at most (when p is even), and we can relate the two obstruction classes in $X(M, \mathrm{PGL}_2(\mathbb{C}))$ as intersections of the A-polynomial curve with two parallel lines.

We note that the numbers p_+, p_- have recently been tabulated for all knots with less than 12 crossings in [3]. Their results seem compatible with the following conjecture:

Conjecture 10. [1] *For any small knot K in S^3 we have*

$$p_- - p_+ = \frac{\det(K) - 1}{2} \quad .$$

We note that the following connection between counts of parabolic representations and the determinant, to the best of the author's knowledge, does not appear anywhere in the litterature. We expect that our Langlands duality conjecture will lead to other interesting new predictions about the structure of the G-character varieties of 3-manifolds.

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The abelian sector of a Chern–Simons TQFT with values in Habiro cohomology

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(joint work with Campbell Wheeler)

We define the abelian sector of a new TQFT that takes values in the Habiro ring of the abelian component of the $\mathrm{SL}_2(\mathbb{C})$-character variety. This sector is a lift (from a power series around $q = 1$ to elements of Habiro rings) of an unpublished manuscript of Rozansky on the $U(1)$-RCC connection of a link with nonzero multivariable Alexander polynomial. We describe the abelian sector of this theory as the unique solution to a simple set of axioms, and define such a theory effectively in a way that leads to explicit computations.

Our purpose is to explain a new example of a TQFT in 2+1 dimensions, where Habiro cohomology contains both exact and perturbative information of (Chern-Simons) quantum field theory. Although there are so technical many problems to overcome to make this work, it is a beautiful story, with axioms, explicit and effective calculations and startling identities.

Let us try to explain the story starting from classical invariants of links and passing to quantum ones. The complement of an oriented link L in S^3 has an $\mathrm{SL}_2(\mathbb{C})$-character variety

$$X_L = \mathrm{Hom}(\pi_1(S^3 \setminus L), \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C}) \tag{1}$$

which is an affine variety (really, stack) defined over $\mathbb{Q}$. A choice of meridians for each components gives a map $X_L \to (\mathbb{C}^*)^{|L|}$ that sends ρ to an eigenvalue m_j of $\rho(\mu_j)$. The Kauffman bracket skein module $\mathrm{Sk}_{S^3\setminus L}$ is in a sense a quantization of X_L. The Habiro cohomology is a different quantization, and the two are related by a map J^{ab}.

To explain this, note that X_L contains a canonical component X_L^{ab} that consists of abelian representations, that is (conjugacy classes of) homomorphisms to a maximal torus of $\mathrm{SL}_2(\mathbb{C})$. Thus, there is a canonical map $X_L^{\mathrm{ab}} \to (\mathbb{C}^*)^{|L|}$ which has degree 1. However, X_L^{ab} has singularities, precisely at the points of its intersection with other components of X_L. And those points are the zeros of the multivariable Alexander polynomial $\Delta_L(t)$ of L. Turning this discussion from spaces into Specs of rings, we arrive at a natural étale map

$$\mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}[t^{\pm 1}, \Delta_L(t)^{-1}] \tag{2}$$

that parametrizes the smooth part of the abelian locus X_L^{ab}. The J^{ab}-map

$$J_L^{\mathrm{ab}} : \mathrm{Sk}_{S^3 \setminus L} \to \mathcal{H}_{\mathbb{Z}[t^{\pm 1}, \Delta_L(t)^{-1}]/\mathbb{Z}[t^{\pm 1}]} \tag{3}$$

associtates an element of the Habiro ring of the map (2) to each element of the skein module $\mathrm{Sk}_{S^3 \setminus L}$. In fact, it associates to the trivial element of $\mathrm{Sk}_{S^3 \setminus L}$ (namely the empty link), an element that determines (and is determined by) none other than the colored Jones polynomial of L. It is an unfortunate fact of life that the Alexander polynomial can be identically zero–this never happens for knots, but happens for example for the disjoint union of links– and for the moment we will restrict to links L with $\Delta_L(t) \neq 0$. In a later publication, we will follow the geometry along with further ideas, to overcome this issue.

Putting this issue aside for the moment, note that X_L has other components, too. For example, when L is hyperbolic, there is a component X_L^{geom} that contains a lift of the discrete faithful $\mathrm{PSL}_2(\mathbb{C})$-representation. The map $X_L^{\mathrm{geom}} \to (\mathbb{C}^*)^{|L|}$ has finite degree degree. Just as with the abelian component, X^{geom} has singularities, in fact at the zeros of the torsion polynomial $\delta^{\mathrm{geom}}(t)$. Thus, we have another natural étale map

$$\mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}[t^{\pm 1}, \delta_L^{\mathrm{geom}}(t)^{-1/2}] \tag{4}$$

and this, too, should give a J^{geom}-map

$$J_L^{\mathrm{geom}} : \mathrm{Sk}_{S^3 \setminus L} \to \mathcal{H}_{\mathbb{Z}[t^{\pm 1}, \delta_L^{\mathrm{geom}}(t)^{-1/2}]/\mathbb{Z}[t^{\pm 1}]} \,. \tag{5}$$

In fact, already here there is a lie. We need a line bundle over the said Habiro ring, and the map should take values in sections of this line bundle. This is due to the fact that there is a classical (complexified) volume map

$$X_L/(\mathbb{C}^*)^{|L|} \to \mathbb{C}/(4\pi^2\mathbb{Z}) \tag{6}$$

which although vanishes at the abelian representations, it does not vanish at the geometric representation. So, we need the line bundle, and need to construct sections.

In a sense the abelian and the geometric components are two sectors of this TQFT. But this is not all. As is well-known, the character variety X_L may have components of dimension more than the number of components of a link. A nice example is the knot 10_{98} of Yoon [1]. Then, the corresponding map of coordinate rings is no longer étale. What to do in that case? The answer is to use Habiro cohomology instead. In other words, the aim is to define a single map

$$J_L : \mathrm{Sk}_{S^3\setminus L} \to H^\bullet_{\mathrm{Hab}}(X_L/(\mathbb{C}^*)^{|L|}, \mathcal{L}) \tag{7}$$

that specializes to the previous maps J_L^{ab}, J_L^{geom}, etc of the various components of X_L. As a further wish, one may want to define all these maps J_L simultaneously, and to give axioms that uniquely characterize such a theory, much in the spirit of TQFT in 2+1 dimensions, or of the homology theories of algebraic topology. Apparently, such a theory is predicted to exist by P. Scholze, in which case the wish is to complement these abstract higher categorical definitions by explict, effective ones, as well as to give axioms of that theory.

The above discussion paints in broad strokes the landscape of such a Chern–Simons TQFT with values in Habiro cohomology. The aim of our paper is to describe the abelian sector J^{ab} of this theory as the unique solution to a simple set of axioms, and to define such a theory effectively in a way that leads to explicit computations.

The colored Jones polynomial and its loop expansion. But before we present the axioms for J^{ab} and its properties, let us motivate it by recalling how to lift one of the best pieces of Chern–Simons theory, namely the colored Jones polynomial of a link in 3-space.

The colored Jones polynomial $J_{K,n}(q)$ of a knot K in S^3 is a sequence of polynomials in $\mathbb{Z}[q^{\pm 1}]$ indexed by $n \in \mathbb{Z}_{>0}$, or alternatively, an element of the ring $\mathbb{Q}[n][\![q-1]\!]$ which in fact is a power series in $q-1$ with coefficients integer-valued functions on the integers (whose argument is the color n).

In a seminal paper [4], Rozansky lifted the colored Jones polynomial of a knot K to an element $J_K^{\mathrm{loop}}(t,q)$ of the ring $\mathbb{Z}[t^{\pm 1}, \Delta_K(t)^{-1}][\![q-1]\!]$ where $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ is the Alexander polynomial satisfying $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$. This so-called loop expansion of the colored Jones polynomial is related to the colored Jones polynomial, i.e., determines, and is determined by

$$J_{K,n}(q) \;=\; J_K^{\mathrm{loop}}(q^n, q) \in \mathbb{Q}[n][\![q-1]\!], \qquad (n \in \mathbb{Z}_{>0})\,. \tag{8}$$

In a sense, the loop expansion is a resummation that expresses power series in n in terms of rational functions in q^n whose denominator is a power of the Alexander polynomial. Among other things, this resummation allows to define algebraic operations on power series that commute with the Dehn surgery operations on the knot, as we will see later.

From the point of view of physics, the loop expansion is the contribution of the Chern-Simons path integral at the abelian representations. Although there are hightly nontrivial components of the $\mathrm{SL}_2(\mathbb{C})$-character variety of the knot complement that contain nonabelian representations, in the case of a knot complement, the trivial representation lies only on the abelian component and this explains the equality (8).

This key property however fails for links. With considerable effort, in an unpublished preprint [5], Rozansky extended the results from knots in S^3 to links L

whose multivariable Alexander polynomial is nonzero. From the point of view of physics, this is the contribution of Chern-Simons theory to the abelian character variety of the link complement. Rozansky formulated his results using some axioms whose uniqueness implied topological invariance, but whose existence was a difficult problem dealt with by suitable braid presentations of the links.

A lift of the loop expansion of a knot. Recently, a further lift of the colored Jones polynomial was given in [2], namely to an element of the Habiro ring $\mathcal{H}_{\mathbb{Z}[t^{\pm 1},\Delta_K(t)^{-1}]/\mathbb{Z}[t^{\pm 1}]}$ of the étale map $\mathbb{Z}[t^{\pm 1},\Delta_K(t)^{-1}]/\mathbb{Z}[t^{\pm 1}]$. It turns out that Rozansky's aim can be formulated in a clean fashion using the theory Habiro ring of étale maps that was introduced in the lectures of Scholze [6] and further studied in [3, 2]. A key part of these axioms, the leading term (so-called leading term) of the expansion of these invariants, is the inverse multivariable Alexander polynomial of an oriented link in S^3.

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Skein algebra and double affine Hecke algebra

Kazuhiro Hikami

The double affine Hecke algebra (DAHA) was introduced by Cherednik in the context of Macdonald polynomials. Cherednik also pointed out [1] that the Macdonald polynomial is related to the refined invariant for torus knots. Our motivation is to construct the refined invariant for hyperbolic knots via DAHA.

The correspondence between the DAHA and the refined invariant makes sense via the skein algebra $\mathrm{Sk}_A(\Sigma)$ on surface Σ. Take the A_1-type for example. The A_1-DAHA $\mathcal{H}_{q,t}$ is generated by $\mathsf{X}^{\pm 1}$, $\mathsf{Y}^{\pm 1}$ and T with the following relations:

$$(\mathsf{T}+t)(\mathsf{T}-t^{-1})=0, \quad \mathsf{TXTX}=\mathsf{TY}^{-1}\mathsf{TY}^{-1}=1, \quad \mathsf{XY}=q^{-1}\mathsf{YXT}^2.$$

The spherical DAHA is $\mathcal{SH}_{q,t}=\mathsf{e}\mathcal{H}_{q,t}\mathsf{e}$, where $\mathsf{e}=\frac{\mathsf{T}+t}{t+t^{-1}}$ is the idempotent. There exists the isomorphism $\mathcal{A}:\mathrm{Sk}_{A=q^{-\frac{1}{2}}}(\Sigma_{1,1})\to\mathcal{SH}_{q,t}$ such that

$$\mathcal{A}:(\mathbf{x},\mathbf{y},\mathbf{b})\mapsto \mathsf{e}(\mathsf{X}+\mathsf{X}^{-1},\mathsf{Y}+\mathsf{Y}^{-1},-(t^2/q+q/t^2))\mathsf{e},$$

where $\mathbf{x}$, $\mathbf{y}$ are the generators of the skein algebra on the once-punctured torus $\Sigma_{1,1}$, and $\mathbf{b}$ denotes the boundary curve around the puncture. Then the $SL_2(\mathbb{Z})$ actions on the DAHA

$$\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) : (\mathsf{X}, \mathsf{Y}, \mathsf{T}) \mapsto (q^{-\frac{1}{2}}\mathsf{Y}\mathsf{X}, \mathsf{Y}, \mathsf{T}), \quad \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) : (\mathsf{X}, \mathsf{Y}, \mathsf{T}) \mapsto (\mathsf{X}, q^{\frac{1}{2}}\mathsf{X}\mathsf{Y}, \mathsf{T}),$$

induce the Dehn twists $D_{\mathbf{y}}$ and $D_{\mathbf{x}}^{-1}$ respectively. Correspondingly the refined invariant for the torus knot $T_{r,s}$ is proposed as

$$P_n(K = T_{r,s}; x, q, t) = \gamma_{r,s}(M_{n-1}(\mathsf{Y}; q, t))1.$$

where $\gamma = \left(\begin{smallmatrix} * & r \\ * & s \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and $M_n(x; q, t)$ is the A_1-Macdonald polynomial.

In [2, 3, 4, 5], we have proposed the generalized DAHA for $\mathrm{Sk}_A(\Sigma_{1,2})$ and $\mathrm{Sk}_A(\Sigma_{2,0})$ using the $C^\vee C_1$-DAHA which denotes $\mathrm{Sk}_A(\Sigma_{0,4})$. Regarding the automorphisms of the generalized DAHA as the Dehn twists on the surfaces, we have constructed the q-difference operators assigned to simple closed curves on $\Sigma_{1,2}$ and $\Sigma_{2,0}$.

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Analytic invariants from the coloured Jones polynomials

Rinat Kashaev

(joint work with Vladimir Mangazeev, Marcos Mariño)

The family of coloured Jones polynomials

$$\{J_{K,n}(q)\}_{n \in \mathbb{Z}_{\geq 1}} \subset \mathbb{Z}[q, q^{-1}]$$

of a knot $K \subset \mathbb{R}^3$ arises from finite-dimensional irreducible representations of the quantum group $U_q(\mathfrak{sl}_2)$ [5, 6]. Owing to the interesting geometric and asymptotic properties of these invariants, the problem of converting this family into analytic functions has received considerable attention. In particular, the problem of interpolating coloured Jones polynomials to non-integer colours was initiated by K. Habiro in [4] and subsequently studied by S. Willetts in [7].

Using the q-hypergeometric notation of [3], Habiro's formula

$$J_{K,n}(q) = \sum_{m=0}^{n-1} (-1)^m q^{-m(m+1)/2} (q^{1+n}, q^{1-n}; q)_m H_{K,m}(q),$$

where $H_{K,m}(q)$ are the Habiro polynomials[1], allows one to introduce the formal power series

$$J_{K,x}(q,z) = \sum_{m=0}^{\infty} (-1)^m q^{-m(m+1)/2} (q^{1+x}, q^{1-x}; q)_m H_{K,m}(q) z^m,$$

which specializes to the coloured Jones polynomials for $x \in \mathbb{Z}_{\neq 0}$ and $z = 1$.

Another possibility is to consider the generating formal power series

$$V_K(q,z) := \sum_{n=0}^{\infty} J_{K,n+1}(q) z^n.$$

Unfortunately, both of the above series often have zero radius of convergence, in particular, in the case of the figure-eight knot with $H_{4_1,m}(q) = 1$ for all $m \geq 0$.

With the goal of converting these series into analytic expressions, we treat them in the spirit of Borel resummation [2], using its q-deformed version.

q-Borel resummation

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a formal power series whose *q-Borel transform*

$$(\mathcal{B}_q f)(t) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} a_n t^n$$

converges and admits analytic continuation along the ray from the origin through a given point $z \in \mathbb{C}$. We define the *q-Borel resummation* of f by

$$\hat{f}(z) = \frac{-1}{\log q} \int_0^{\infty} \frac{(\mathcal{B}_q f)(zt)}{(-t;q)_{\infty}}\, dt.$$

This resummation method is based on Ramanujan's q-beta integral

$$\int_0^{\infty} t^{s-1} \frac{(-at;q)_{\infty}}{(-t;q)_{\infty}}\, dt = \frac{(a, q^{1-s}; q)_{\infty}}{(q, aq^{-s}; q)_{\infty}} \frac{\pi}{\sin(\pi s)}, \quad s > 0,\ |a| < q^s,$$

see [1, (2.8), (2.9)].

Results for the figure-eight knot

Theorem 1. Let $0 < q < 1$, $x \in \mathbb{R}$, $n \in \mathbb{Z}_{>|x|-1}$ and $z \in \mathbb{C} \setminus (q^{\mathbb{Z}} \cup \mathbb{R}_{\leq 0})$. Then

$$\hat{J}_{4_1,x}(q,z) = \sum_{k=0}^{n-1} (-1)^k q^{-k(k+1)/2} (q^{1+x}, q^{1-x}; q)_k z^k + z^n \frac{\big(q^{1+x}, q^{1-x}; q\big)_{\infty}}{(q;q)_{\infty}^3}$$
$$\times \sum_{m \in \mathbb{Z}} \frac{\big(\frac{\log z}{\log q} + m\big) q^{mn} (-1)^k q^{k(k+1)/2} \big(q^{1-m-x}; q\big)_{\infty}}{1 - zq^m} {}_1\phi_1\big(0; q^{1-m-x}; q, q^{1-m+x}\big).$$

[1] In [4], Habiro uses the notation $a_m(K)$ for $H_{K,m}(q)$.

Theorem 2. Let $0 < q < 1$ and $z \in \mathbb{C} \setminus (-q^{\mathbb{Z}} \cup \mathbb{R}_{\geq 0})$. Then

$$\hat{V}_{4_1}(q,z) = \frac{(q;q)_\infty}{2(q^2;q^2)_\infty^3} \times \sum_{\substack{k,m,n\in\mathbb{Z}\\ n\geq \max(2k,-m)}} A_q(q^{1+n}) \frac{z^{n-2k}(-1)^k q^{k(k-1)+(1+n)(m+n)}}{(q;q)_{n-2k}} \frac{\frac{\log(-z)}{\log q}+m}{1+zq^m},$$

where

$$A_q(x) := \sum_{l\geq 0} \frac{q^{l^2}}{(q;q)_l}(-x)^l.$$

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Transverse intersection algebra and fractional linking

Ruth Lawrence-Naimark

(joint work with Dennis Sullivan, Daniel An, Ofir Aharoni, Alice Kwon)

The motivation for this work is an attempt to construct natural finite-dimensional models of PDEs such as Euler's equation for incompressible fluid flow using ideas from algebraic topology. A problem which arises when naïvely discretising PDEs is that discretisations of equivalent formulations of a continuum model may lead to inequivalent discrete models (for example [5]), even in so far as one discretisation may blow up and the other not with the same initial conditions (see [3]). This phenomenon is due to the fact that vector calculus identities used to identify equivalence of the continuum models, fail under discretisation where a partial derivative is replaced by a divided difference. Typically conserved quantities in the continuum model may not always discretise to conserved quantities in the discrete model.

A *fluid algebra* [4] is a vector space V along with

1. a positive definite inner product $(\ ,\ )$ (the *metric*)
2. a symmetric non-degenerate bilinear form $\langle\ ,\ \rangle$ (the *linking form*)
3. an alternating trilinear form $\{\ ,\ ,\ \}$ (the *triple intersection form*)

Given a fluid algebra, the *associated Euler equation* is an evolution equation for $X(t) \in V$ given implicitly by

$$(\dot{X}, Z) = \{X, DX, Z\} \text{ for all test vectors } Z \in V \tag{1}$$

where $D : V \to V$ is the operator defined by $\langle X, Y\rangle = (DX, Y)$ for all $X, Y \in V$. This formulation automatically guaranteed two conserved quantities, energy (X, X) and helicity (X, DX).

Example 1: This is the classical infinite-dimensional example [4]. Let V consist of coexact 1-forms on a 3-dimensional closed oriented Riemannian manifold M,

$$V = \left\{d^*\omega \middle| \omega \in \Omega^2(M)\right\} \subset \Omega^1(M)$$

The fluid algebra structures on V are given by

$$(a, b) = \int_M a \wedge *b, \quad \langle a, b\rangle = \int_M a \wedge db, \quad \{a, b, c\} = \int_M a \wedge b \wedge c$$

Symmetry of $\langle\ ,\ \rangle$ follows from Leibniz $d(a \wedge b) = da \wedge b - a \wedge db$ and Stokes' theorem. Then $D = *d = d^**$. The implicit evolution equation (1) is equivalent to the explicit evolution equation

$$\dot{X} = \phi\big(*(X \wedge *dX)\big) \tag{2}$$

where $\phi : \Omega^1 \to V$ is the projection on the $\Im d^*$ part of the Hodge decomposition. On the 3-torus, this is equivalent (up to some considerations on the homology) to the usual formulation of the Euler equation for incompressible fluids,

$$\partial_t u_i + u_j \partial_j u_i = \partial_i p, \qquad \partial_i u_i = 0$$

where $X = u_1 dx + u_2 dy + u_3 dz$ with periodic boundary conditions.

According to folklore, it is impossible to construct a homologically faithful finite-dimensional algebraic model of differential forms preserving graded commutativity, associativity and Leibniz.

In [1], we show how to extend the standard cubical complex by adjoining infinitesimal elements to define a transverse intersection product on cells; the original subcomplex is not closed under the intersection product in the case of the intersection pair of cells which are not in general position.

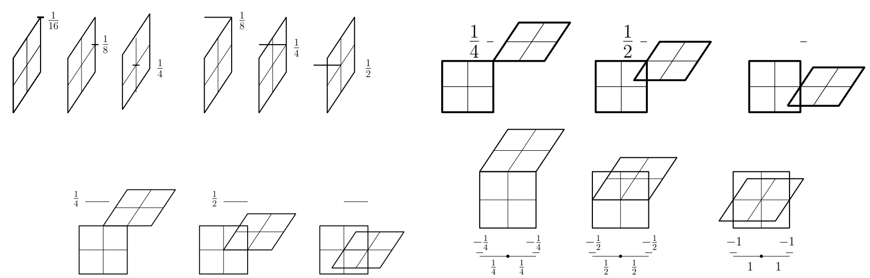
Theorem [1]**.** There exists a chain complex EC_* enlarging the usual triply periodic cubical h-complex C_*, on which there is a local transverse intersection product $\pitchfork$: $EC_* \otimes EC_* \to EC_*$ (local means that $\mathrm{Supp}(X \pitchfork Y) = \mathrm{Supp}X \cap \mathrm{Supp}Y$ on basis elements X, Y) such that

(A) graded commutativity: $a \pitchfork b = (-1)^{c_a c_b} b \pitchfork a$ for all $a, b \in EC_*$, where each cell a is graded by its codimension c_a;

(B) associativity: $(a \pitchfork b) \pitchfork c = a \pitchfork (b \pitchfork c)$ for all $a, b, c \in EC_*$;

(C) the product rule (Leibniz) on the original complex: $\partial(a \pitchfork b) = (\partial a) \pitchfork b + (-1)^{c_a} a \pitchfork (\partial b)$ for all $a, b \in C_*$
(D) $\pitchfork$ is invariant under the symmetries of the cubical lattice
(E) $a \pitchfork b \neq 0$ for basis elements a, b with $\mathrm{Supp} a \cap \mathrm{Supp} b \neq \emptyset$
(F) for cuboidal cells in general position a, b, $a \pitchfork b$ agrees with the natural intersection product (signed geometric intersection of closed cells);
(G) the augmentation map (counting points) in degree zero defines a non-degenerate pairing $C_1 \times C_2 \to \mathbb{Q}$ when the lattice periods in all directions are odd;
(H) on the subalgebra FC_* of EC_* generated as an algebra by $2h$-squares, the product rule holds without reservation.
(I) There is a unique minimal extension of C^* with a product satisfying (A)-(F).
(J) The natural (crumbling) chain mapping from a coarse periodic cubical complex to a finer periodic cubical complex commutes with the intersection product satisfying (A)-(F).

The algebra FC_* of (H) contains only objects of dimensions zero (h-lattice points), one (directed sticks parallel to the axes of length 0 or h) and two ($2h$-squares parallel to the coordinate planes) but is graded commutative, associative and satisfies Leibniz. The products $\pitchfork$: $FC_1 \times FC_2 \to FC_0$ and $\pitchfork$: $FC_2 \times FC_2 \to FC_1$ are explicitly



Example 2: Using the transverse intersection algebra in place of differential forms, we construct [2] a finite dimensional fluid algebra on the vector space

$$V = \left\{ *\,\partial x \middle| x \in FC_2 \right\}$$

The structure maps in the fluid algebra are given by

$$(a, b) \equiv \#(a \pitchfork *b), \qquad \langle a, b \rangle \equiv \#(a \pitchfork \partial b), \qquad \{a, b, c\} \equiv \#(a \pitchfork b \pitchfork c)$$

where $\# : FC_0 \to \mathbb{Q}$ is the augmentation map which counts points with weighting. Note that the linking numbers generated in this fluid algebra are fractional.

The Euler equation defined by the implicit evolution equation (1) is now a finite dimensional evolution (with conserved quantities) and is equivalent to the explicit evolution equation

$$\dot{X} = *\pi\big(i(X \pitchfork *\partial X)\big) \tag{3}$$

where $\pi : C_1^{2h} \to \Im(\partial : C_2^{2h} \to C_1^{2h})$ is the projection onto boundary 1-chains and $i : FC_1 \to C_1^{2h}$ is uniquely determined by the condition $\#(x \pitchfork y) = \#(i(x) \pitchfork y)$

for all $y \in FC_2$. An explicit formula for a component of $i(X \pitchfork *\partial X)$ is a sum of 428 elementary products.

This is our current proposal for a natural finite dimensional model for the Euler equation on a cubic lattice. It naturally possesses two invariant quantities, energy and helicity, as do all Euler evolutions coming from fluid algebras. Current numerical simulations seem not to be stable, but this is possibly due to some numerical instability in the implementation.

From the start of the subject, knot theory and fluid dynamics seem to have been close. Indeed, after the work of Gauss and Listing, the beginning of the systematic study of knot theory can be traced to the work of Maxwell, Tait and Thompson (later Lord Kelvin) motivated by their proposed vortex theory of matter in which atoms were to be corresponded to vortices in the motion of the æther. The formulation of Euler's equation in the form that vorticity is transported by the flow means that link invariants of vortex lines are preserved. Knot theory indeed can be said to have been one of the impetuses for the development of algebraic topology and now we come full circle, here using ideas from algebraic topology back again in a formulation of a natural discretisation of Euler's equation.

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On signed Verlinde Algebras

Julien Marché

(joint work with Seokbeom Yoon)

Let $0 < q < p$ be two coprime odd integers and set $\zeta = \exp(\frac{i\pi q}{p})$. The SU_2-TQFT associated to the root ζ is a collection of complex vector spaces $\mathcal{V}_\zeta(S_{g,n}, \lambda)$ associated to a surface $S_{g,n}$ of genus g with n marked points. The data $\lambda = (\lambda_1, \dots, \lambda_n)$ is a collection of colors $\lambda_i \in \{0, \dots, p-2\}$. These vector spaces are endowed with a pseudo-hermitian form and support a projective unitary action of the mapping class group, in formulas they provide representations

$$\rho_{g,n}^\lambda : \mathrm{Mod}_{g,n} \to \mathrm{PU}(\mathcal{V}_\zeta(S_{g,n}, \lambda)).$$

Moreover these vector spaces satisfy compatibility conditions by cutting surfaces along simple curves. In [1], the authors defined by pulling back cohomological invariants of projective unitary groups a family of cohomological classes

$$\mathrm{sch}(\rho^\lambda_{g,n}) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

which satisfy the axioms of a Cohomological Field Theory. The problem of computing thes invariants for a general ζ is still wide open. Here we focus on the 0-dimensional part, which is often called TFT and reduces to a Frobenius algebra denoted by $V_{q/p}$. We will denote by $\Omega \in V_{q/p}$ the invertible element such that one has $\epsilon(x) = \mathrm{Tr}_{V_{q/p}}(\Omega^{-1}x)$ for all $x \in V_{q/p}$ where ϵ is the co-unit.

The purpose of this talk is to study these Frobenius algebras and show that they are isomorphic to the algebra of regular functions on the parabolic character variety of the 2-bridge knot $K(p,q)$. To be more precise, define $\epsilon_n = (-1)^{\lfloor nq/p \rfloor}$ so that the fundamental group of $S^3 \setminus K(p,q)$ is given by

$$G = \langle u, v | wu = vw \rangle \text{ where } w = u^{\epsilon_1} v^{\epsilon_2} \cdots v^{\epsilon_{p-1}}$$

A parabolic representation of G can be put up to conjugation in the form

$$\rho_z(u) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \rho_z(v) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

One check that ρ_z defines a representation of G if and only if $P_{q/p}(z) = 0$ where $P_{q/p}$ is the characteristic polynomial of the matrix

$$M = \begin{pmatrix} 0 & -\epsilon_1\epsilon_2 & & & \\ 1 & 0 & -\epsilon_2\epsilon_3 & & \\ & & \ddots & & \\ & & 1 & 0 & -\epsilon_{p-2}\epsilon_{p-1} \\ & & & 1 & 0 \end{pmatrix}$$

This gives an explicit expression for the algebra $V_{q/p} = \mathbb{Q}[x]/P_{q/p}(x)$: we also show that $\Omega = -\iota P'_{q/p}(z)$ where $\iota^2 = -1$, this is proven in [2].

We end this talk by explaining that the element Ω definining the Frobenius algebra is related to the Reidemeister torsion of $S^3 \setminus K(p,q)$ in the adjoint representation, yielding a completely knot-theoretical description of the TFT. We use it to prove a reciprocity formula relating the Frobenius algebras $V_{q/p}$ and $V_{q^*/p}$ where $qq^* = \pm 1$ modulo p.

We end our presentation by explaining the heuristic behind the proof given in [3] of an analogue of the asymptotics of the Verlinde algebras, replacing dimensions by signatures, precisely we prove that

Theorem: Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$, set $q_n = a + bn, p_n = c + dn$ and ζ_n the corresponding root. Then $\mathrm{Sign}\,\mathcal{V}_{\zeta_n}(S_{g,0})$ is a polynomial in n of degree $3g-3$ (with an explicit leading term).

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Signatures in TQFT : Asymptotics and Modularity

Gregor Masbaum
(joint work with Julien Marché)

This talk was a report on joint work [5] with Julien Marché. I began with the following concrete problem. Let $0 < q < p$ be odd coprime integers and define for any integer n the sign $\varepsilon_n = (-1)^{\lfloor nq/p \rfloor}$. Let

$$T_p = \Theta_{p-2} \cap (\mathbb{Z}^3)^{ev} \ ,$$

where $\Theta_n \subset \mathbb{R}^3$ is the tetrahedron with vertices $(0,0,0)$, $(n,n,0)$, $(n,0,n)$, and $(0,n,n)$, and $(\mathbb{Z}^3)^{ev} \subset \mathbb{Z}^3$ is the lattice of triples of integers with even sum. Our goal is to study the following integer-valued sum :

$$\sigma_2(q/p) = \sum_{(j,k,\ell)\in T_p} \varepsilon_{j+1}\varepsilon_{k+1}\varepsilon_{\ell+1} \ . \tag{1}$$

More precisely, we wish to study the asymptotics of $\sigma_2(q/p)$ as q/p goes to some irrational number $\theta \in [0,1]$. We will use this to find a transformation law for $\sigma_2(q/p)$ showing that the sum (1) of signs over lattice points in the tetrahedron has modular properties.

One possible motivation from number theory for studying this sum could be that the similar, but simpler, sum

$$\sum_{n=1}^{p-1} \varepsilon_n \tag{2}$$

is closely related (see [5, Section 6.1]) to the Dedekind sum $s(q,p)$ which is Example 0 in Zagier's Quantum Modular Forms [8]. In particular $s(q,p)$ has modular properties, as it satisfies a reciprocity formula relating $s(q,p)$ with $s(p,q)$. So one might hope that $\sigma_2(q/p)$ also has interesting modular properties.

Our original motivation was, however, the connection with Topological Quantum Field Theory. Indeed, $\sigma_2(q/p)$ is the signature of the natural Hermitian form on the SU(2)-TQFT vector space $\mathcal{V}_p(S_2)$ associated to the closed genus 2 surface S_2, where $\mathcal{V}_p$ is SU(2)-TQFT "at the p-th root of unity" (equivalently, at level $p-2$). The vector space $\mathcal{V}_p(S_2)$ and its Hermitian form can be defined over the cyclotomic field $\mathbb{Q}(\zeta)$, where $\zeta = e^{i\pi q/p}$. (Note that $e^{i\pi q/p}$ has order precisely $2p$, as the coprime integers q and p are both asssumed to be odd.) The connection with the set T_p of lattice points in the tetrahedron Θ_{p-2} is the following: The vector space $\mathcal{V}_p(S_2)$ has a basis b_σ indexed by $\sigma \in T_p$. In particular, its dimension is

$$\dim \mathcal{V}_p(S_2) = |T_p| = \binom{p+1}{3} \ . \tag{3}$$

Moreover, this basis is orthogonal for the Hermitian form, and when $\sigma = (j,k,\ell)$, the sign of the Hermitian form on the basis vector b_σ is $\varepsilon_{j+1}\varepsilon_{k+1}\varepsilon_{\ell+1}$. (See [1, Remark 4.12] but notice that our $\mathcal{V}_p$ corresponds to the V_{2p} of [1].) Thus Formula (1) gives indeed the signature of this Hermitian form.

It is worth observing that when $q = 1$ the Hermitian form is positive definite for all p, as $\varepsilon_n = 1$ for $1 \le n < p$ when $q = 1$. Thus $\sigma_2(1/p) = \dim \mathcal{V}_p(S_2)$.

Our first result is the following

Theorem 1. *(Asymptotics)* [5, Theorem 4.1] *For almost all irrational $\theta \in [0,1]$, if we denote by q_k/p_k the sequence of convergents of the continued fraction expansion of θ, then one has*

$$\lim_{k\to\infty} \frac{\sigma_2(q_k/p_k)}{p_k^2} = \Lambda(\theta)$$

where

$$\Lambda(\theta) = \frac{16}{\pi^3} \sum_{n\ge 1,\,\mathrm{odd}} \frac{1}{n^3 \sin(n\pi\theta)}.$$

Here, we take the limit only over those k such that both q_k and p_k are odd, as we were assuming this to define $\sigma_2(q_k/p_k)$. For almost all irrational θ, the set of those k is infinite (see [5, Remark 4.4]) so that the limit makes sense.

Notice in particular that the signature $\sigma_2(q_k/p_k)$ grows like p_k^2 when $q_k/p_k \to \theta$, whereas the dimension of $\mathcal{V}_p(S_2)$ grows like p^3 (see Formula (3).) In higher genus $g > 2$, numerical experiments seem to indicate that the signature grows like p^{2g-2} whereas the growth rate of the dimension of the TQFT vector spaces is well-known to be p^{3g-3} (see [7, Section 3].)

The key ingredient in the proof of Theorem 1 is Brion's formula [2] for enumerating the lattice points in a lattice polytope as a sum of rational functions indexed by the vertices of that polytope. After applying a finite Fourier transform to the expression (1), we use Brion's formula to get an explicit trigonometric expression for $\sigma_2(q/p)$ (see [5, Section 3]) from which we can then extract a limit when q/p goes to an irrational number θ as stated above.

Let us now discuss a modularity property of the signature which, as far as we know, has not been observed before. We found it after first observing the following modularity properties of the limit function $\Lambda(\theta)$.

Theorem 2. *(Modularity)* [5, Corollary 6.11] *The function $\Lambda(\theta)$ satisfies the transformation laws $\Lambda(\theta + 2) = \Lambda(\theta)$ and*

$$\Lambda\Big(\frac{\theta}{2\theta+1}\Big)(2\theta+1)^2 - \Lambda(\theta) = 2\theta^2 + 2\theta + 1\ . \tag{4}$$

The first transformation law is, of course, obvious from the definition of $\Lambda(\theta)$. For the second one, that is, Formula (4), the proof starts with observing that $\Lambda(\theta)$ is the boundary value of an Eichler integral of a certain modular form of weight 4 for the level 2 congruence subgroup $\Gamma(2)$ (see [5, Proposition 6.9].) Recall that $\Gamma(2)$ is generated by the two transformations $\phi_1(\tau) = \tau + 2$ and $\phi_2(\tau) = \frac{\tau}{2\tau+1}$. The transformation law (4) for $\Lambda(\theta)$ is then obtained by showing that the period polynomial of our Eichler integral for the transformation ϕ_2 is given (up to an appropriate scalar) by $2\tau^2 + 2\tau + 1$.

The above modularity properties of the limit function $\Lambda(\theta)$ suggest that a similar integral version holds for the signatures themselves. This is clear for the transformation $\phi_1(\tau) = \tau+2$ as the signature only depends on the root of unity $e^{i\pi q/p}$, and hence $\sigma_2((q+2p)/p) = \sigma_2(q/p)$. Concerning the transformation $\phi_2(\tau) = \frac{\tau}{2\tau+1}$, we found by numerical experiments (guided by the transformation law (4)) that the signature transforms as follows :

$$\sigma_2\Big(\frac{q}{2q+p}\Big) - \sigma_2\Big(\frac{q}{p}\Big) = 2q^2 + 2qp + p^2 - 1 \ . \tag{5}$$

Taking limits as in Theorem 1 it is easy to check that Formula (5) implies the transformation law (4) for $\Lambda(\theta)$. But the converse is not true.

In the first version of our paper [5] (of December 15, 2025), we stated Formula (5) as a conjecture which we had checked by computer for all $0 < q < p < 100$, but which we were unable to prove in general at that time. Less than two months later, in early February 2026, Y. Murakami [6] obtained a proof of Formula (5) starting from our trigonometric formula for $\sigma_2(q/p)$. A nice feature of his proof is that he makes $\sigma_2(q/p)$ itself (not just its limit $\Lambda(\theta)$) appear as the boundary value of a (linear combination of) Eichler integrals.

A similar but simpler story exists for the signature of the TQFT vector space $\mathcal{V}_p$ of a four-punctured sphere with insertions $(p-1)/2$ at the punctures, as this signature is twice the sum (2) which, as already mentioned, is closely related to Dedekind sums. In particular, this signature also has modular properties.

A natural question is, then, what happens in higher genus ? Note that one can write down a formula for the signature of the TQFT vector space associated to any surface as a sum of signs indexed by lattice points inside some polytope depending on the surface (see [1, Remark 4.12].) But in general the signs are much more complicated than in Formula (1) and we don't know yet if (or how) our results generalize to higher genus. Presumably one needs to employ the Frobenius algebra techniques for computing signatures developed by Deroin and Marché in [3], see also [4] and Marché's talk at this workshop.

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On a relation between Deninger's foliated dynamical systems and Connes-Consani's adelic spaces

Masanori Morishita

In the 1990s, Deninger and Connes proposed new approaches to study number-theoretic zeta functions. Although their geometric models (phase spaces) for a number ring have similar structures of foliation and dynamical system, their approaches seem deeply different and their relation has been unknown for a long time. Recently we found a relation between them and we report it here. A key observation is that the arithmetic linking homomorphism of a prime p with other primes

$$\mathrm{lk}_p : p^{\hat{\mathbb{Z}}} \longrightarrow \hat{\mathbb{Z}}^\times_{(p)} := \prod_{q \neq p} \mathbb{Z}_q^\times,$$

plays roles in both Deninger's theory and Connes-Consani's theory as the monodromies in the coverings of phase spaces associated to abelian extensions of $\mathbb{Q}$. In this sense, arithmetic topology provides a bridge between them, which highlights the geometric view of class field theory.

• **Connes-Consani's adelic spaces.** Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and let $\hat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$. $\mathbb{Q}^\times$ acts diagonally on $\mathbb{A}$ and $\hat{\mathbb{Z}}^\times$ acts on the finite part of $\mathbb{A}$ by multiplication. Then we let

$$\mathscr{X}_{\mathbb{Q}} := \mathbb{Q}^\times \backslash \mathbb{A}, \quad \mathscr{X}_{\mathbb{Q}}^{\mathrm{ab}} := \mathbb{Q}^\times \backslash \mathbb{A} / \hat{\mathbb{Z}}^\times,$$

which may be regarded as noncommutative spaces corresponding to crossed C^*-algebras. Here $\mathbb{Q}^{\mathrm{ab}}$ denotes the maximal abelian extension of $\mathbb{Q}$ so that $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ $= \hat{\mathbb{Z}}^\times$. More generally, for an abelian extension$F$ of $\mathbb{Q}$, we may consider the noncommutative space

$$\mathscr{X}_F := \mathbb{Q}^\times \backslash \mathbb{A} / \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/F),$$

which we call Connes-Consani's adelic space [2]. $\mathbb{R}_+$ acta on the infinite component of $\mathbb{A}$ by multiplication and thus $\mathscr{X}_F$ is equipped with the dynamical system, where prime p is visualized as a closed orbit in $\mathscr{X}_F$

$$C_p := \mathbb{Q}^\times \{(a_v) \in \mathbb{A} \;|\; a_\infty > 0, a_q = 1 (q \neq p), a_p = 0\} \hat{\mathbb{Z}}^\times,$$

which is homeomorphic to a circle $\mathbb{R}_+/p^{\mathbb{Z}} \simeq \mathbb{R}/(\log p)\mathbb{Z}$ of length $\log p$. Let $\pi : \mathscr{X}_{\mathbb{Q}}^{\mathrm{ab}} \to \mathscr{X}_{\mathbb{Q}}$ be a natural covering. Then $\pi^{-1}(C_p)$ is the mapping torus $(\hat{\mathbb{Z}}^\times_{(p)} \times \mathbb{R}_+)/(u,s) \sim (pu, sp)$ $(\hat{\mathbb{Z}}_{(p)} := \prod_{q \neq p} \mathbb{Z}_q)$ and the monodromy around C_p is described by the arithmetic linking between p and othe all primes: $p^{\mathbb{Z}} \to \prod_{q \neq p} \mathbb{Z}_q^\times$ [2].

• **Deninger's foliated dynamical systems.** Let K be a number field (an algebraic extension of $\mathbb{Q}$) and let $\mathcal{O}_K$ be the ring of integers of K, and set $X_K = \mathrm{Spec}(\mathcal{O}_K)$. We define the rational Witt ring of a unital ring R by $W_{\mathrm{rat}}(R) :=$ $\{P(t)/Q(t) \,|\, P(t), Q(t) \in R[t],\ P(0) = Q(0) = 1\}$, and define the rational Witt space for X_K by the ringed space

$$W_{\mathrm{rat}}(X_K) = (X_K, W_{\mathrm{rat}}(\mathcal{O}_{X_K})).$$

Then the Frobenius endmorphisms on rational Witt rings induce the Frobenius endmorphisms $F_n (n \in \mathbb{N})$ on $W_{\mathrm{rat}}(X_K)$. The $\mathbb{C}$-valued points on $W_{\mathrm{rat}}(X_K)$ are described by the following:

$$\dot{X}_K(\mathbb{C}) := W_{\mathrm{rat}}(X_K)(\mathbb{C}) = \{(\mathfrak{p}, P) \mid \mathfrak{p} \in X_K, P : W_{\mathrm{rat}}(\kappa(\mathfrak{p})) \to \mathbb{C} \, \text{is a ring hom.}\}.$$

Here $\kappa(\mathfrak{p})$ is the residue field of X_K at $\mathfrak{p}$. By taking the colimit with respect to F_n on $\dot{X}_K(\mathbb{C})$, we let $\check{X}_K(\mathbb{C}) := \varinjlim_{n \in \mathbb{N}} \dot{X}_K(\mathbb{C})$, which is equipped with the inverted Frobenius $\mathbb{Q}_+$-action F_r $(r \in \mathbb{Q}_+)$. Let

$$\mathfrak{X}_K := \check{X}_K(\mathbb{C}) \times_{\mathbb{Q}_+} \mathbb{R}_+ = (\check{X}_K(\mathbb{C}) \times \mathbb{R}_+)/(x, u) \sim (F_{r^{-1}}(x), ur) \ \ \exists r \in \mathbb{Q}_+$$

be the suspension and define the $\mathbb{R}$-action (suspended flow) by $t.[x, s] := [x, ue^t]$. Namely, Frobenius $\mathbb{N}$-action is extended by the suspended flow. The leaves are defined by the images of $\check{X}_K(\mathbb{C}) \times \{u\}$ $(u \in \mathbb{R}_+)$. Under the composition of the projections $\mathfrak{X}_K \to \check{X}_K(\mathbb{C}) \to X_K$, the $\mathbb{R}_+$-orbit in $\mathfrak{X}_K$ over a closed point $\mathfrak{p} \in X_K$ is given as follows(p is the characteristic of $\kappa(\mathfrak{p})$:

$$\Gamma_{K,\mathfrak{p}} = \left\{ \begin{array}{ll} \hat{\mathbb{Z}}_{(p)} \times_{p^{\mathbb{Z}}} \mathbb{R}_+ & K \supset \mathbb{Q}^{\mathrm{ab}}, \\ \hat{\mathbb{Z}}_{(p)}/(\mathrm{N}\mathfrak{p})^{\hat{\mathbb{Z}}} \times_{p^{\mathbb{Z}}} \mathbb{R}_+ & [K : \mathbb{Q}] < \infty . \end{array} \right\}$$

For a Galois extension L/K of number fields, There is a natural bijection $\check{X}_K(\mathbb{C}) \simeq \check{X}_L(\mathbb{C})/\mathrm{Gal}(L/K)$ and we have a covering $\mathfrak{X}_L \to \mathfrak{X}_K$.

Theorem. There is a canonical $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/F)$-equivariant and $\mathbb{R}_+$-anti equivariant surjective continuous map $\Phi_{\mathbb{Q}^{\mathrm{ab}}} : \mathfrak{X}_{\mathbb{Q}^{\mathrm{ab}}} \to \mathscr{X}_{\mathbb{Q}}^{\mathrm{ab}}$, which induces a map $\Phi_F : \mathfrak{X}_F \to \mathscr{X}_F$ for each finite abelian extension F of $\mathbb{Q}$, so that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{X}_{\mathbb{Q}^{\mathrm{ab}}} & \xrightarrow{\Phi_{\mathbb{Q}^{\mathrm{ab}}}} & \mathscr{X}_{\mathbb{Q}}^{\mathrm{ab}} \\ \downarrow & & \downarrow \\ \mathfrak{X}_F & \xrightarrow{\Phi_F} & \mathscr{X}_F \\ \downarrow & & \downarrow \\ \mathfrak{X}_{\mathbb{Q}} & \xrightarrow{\Phi_{\mathbb{Q}}} & \mathscr{X}_{\mathbb{Q}} \end{array}$$

Here any closed orbit in $\Gamma_{\mathbb{Q},p}$ is sent to C_p under $\Phi_{\mathbb{Q}}$.

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Volume Conjecture for Pretzel knots and links

Jun Murakami

We prove the volume conjecture for the Pretzel knots and links by using the quantum $6j$ symbols and the related complexified tetrahedra. The volume conjecture was first proposed for hyperbolic knots by R. Kashaev [1]. It was generalized for every knots and links by H. Murakami and the author [3], then complexified by H. Murakami et. al. in [4].

Complexified Volume Conjecture. Let K be a knot or a link, then the following holds.

$$\lim_{N\to\infty} \frac{2\pi}{N} \log J_N(K) = \mathrm{Vol}(\mathrm{S}^3 \setminus \mathrm{K}) + \sqrt{-1}\mathrm{CS}(\mathrm{S}^3 \setminus \mathrm{K}),$$

where $J_N(K)$ is the colored Jones polynomial corresponding to the N-dimensional irreducible representation of $\mathcal{U}_q(sl_2)$ at $q = e^{\pi i/N}$.

This conjecture was proved for some simple knots and links. For example, it is proved easily for the figure eight knot and the Borromean ring, and proofs for prime knots with up to seven crossings is given by T. Ohtsuki and Y. Yokota [6], [7], [8]. In these proofs, the following strategy is used. Please remind that the colored Jones polynomial is a sum of terms of the product of quantum factorials. At first, the quantum factorials in the colored Jones polynomial is replaced by the quantum dilogarithm function, then consider the large N limit, where the quantum dilogarithm function converge to the dilogarithm function. Next, apply the Poisson sum formula to the sum to replace sum by integral. At last, apply the saddle point method to investigate the integral and obtain the limit. In this argument, the dilogarithm functions corresponds to the ideal triangulation of the complement of $S^3 \setminus K$ and the value of each dilogarithm function at the saddle point somehow matches to the hyperbolic volume of the corresponding tetrahedron. The difficult part of the proof is to check the condition to apply the saddle point methods. The shape of the range of sum differs for every knot and any general method is not known to check the condition.

Here, by taking a slight different approach, a method to prove the volume conjecture for double twist knots, pretzel knots and two-bridge knots is proposed. At first, express the colored Jones polynomial in terms of the quantum $6j$ symbols introduced by Kirillov-Reshetikhin [2] instead of the quantum R matrices and quantum factorials. As the quantum factorial corresponds to the ideal tetrahedron, we introduce the complexified tetrahedron as the corresponding geometric object of the quantum $6j$ symbol. In the expression by quantum $6j$ symbols, the formula becomes $0/0$ and the l'Hopital's row is applied to get the invariant. Next, apply the Poisson sum formula to convert the sum to integral as before. In this step, integral by part is used to eliminate the differential for the l'Hopital's row, and the central term of the Poisson sum disappear by this operation. Such vanishing explains the big cancelation we observed at the study of such asymptotics for the Turaev-Viro invariant. At last, apply the saddle point method to get the limit. In this case, the ranges of sum and integral are all hyper-cubes and the ranges are the same for various cases. For example, the condition to apply the saddle point method is satisfied for double twist knot case [5].

For pretzel knot, the colored Jones polynomial is explained by the quantum $6j$ symbols as follows. Let $K_{a_1,a_2,\cdots,a_n}$ be the pretzel knot with a_1, a_2, $\cdots$, a_n half twist. By using the quantum spin network developed in [2], by combining the twisted edges, $J_N(K_{a_1,a_2,\cdots,a_n})$ is given by a sum of phase factors corresponds to a_1, a_2, $\cdots$, a_n and the spin network of n-prism. Now apply the I-H move of the spin network which corresponds to the quantum $6j$ symbol. In this operation, one new edge, say e is added. Geometrically, e is considered as the common perpendicular of the top and the bottom faces of the n prism. Then, by replacing triangles in the spin network by quantum $6j$ symbols, we get an expression of $J_N(K_{a_1,a_2,\cdots,a_n})$ in terms of the quantum $6j$ symbols, which corresponds to the tetrahedron given by the edge corresponding to a_i, and its four adjacent edges, and e.

At the saddle point of the large N limit of $J_N(K_{a_1,a_2,\cdots,a_n})$, the parameters of the quantum $6j$ symbols corresponds to complex numbers. If these parameters are real or modulus 1 complex numbers respectively, then it is known to relate the length or angles respectively of the edges of a tetrahedron. The geometric object corresponds to the tetrahedron with complexified lengths and angles is realized as a part of the fundamental domain of the action of the fundamental group $\pi_1(S^3 \setminus K)$ to the hyperbolic space $\mathbb{H}^3$. The edges of the complexified tetrahedron are given by the fixed points or axes of some elements of $\pi_1(S^3 \setminus K)$, and the complexified edge parameters are given by the eigenvalues of the element. The condition for the saddle point method should be satisfied if $|a_1|$, $|a_2|$, $\cdots$, $|a_n|$ are large enough.

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Gauge theory and skein modules

Du Pei

In this talk, based on [1], we investigate skein modules of 3-manifolds, using supersymmetric gauge theories and their infrared dynamics as our primary tools.

1. Skein modules

Three-manifold skein modules were first introduced in the context of the skein relations for the Kauffman bracket polynomial [2, 3]. This construction has a generalization for an arbitrary braided ribbon category. What concerns us in this paper is the case of $\mathcal{C} = \mathrm{Rep}_q(G)$, where G is a connected reductive algebraic group over $\mathbb{C}$, and we study the corresponding skein module $\mathrm{sk}(M_3; G)$. The main focus of this paper is on the case with the value of $q \in \mathbb{C}^*$ being generic, enabling one to regard $\mathrm{sk}(M_3; G)$ as a vector space over $\mathbb{C}$.

One of our goals is to clarify the connection between skein modules and physics so that we can then use quantum field theories to make mathematical predictions about the structure and properties of skein modules.

2. Puzzles

There is a tempting identification,

$$\mathrm{sk}(M_3; G) \overset{?}{\simeq} \mathcal{H}(M_3; G), \tag{1}$$

where $\mathcal{H}(M_3; G)$ denotes the space of states (a.k.a. the Hilbert space) of the topologically twisted 4d $\mathcal{N} = 4$ super–Yang–Mills theory. However, these two vector spaces are almost polar opposites, with the skein module being always finite dimensional [4], while $\mathcal{H}(M_3; G)$ is almost always infinite dimensional.

One might then hope that the dimension on the right-hand side can be regularized. When $M_3 = T^3$, one natural way to regularize is to replace the dimension with the "Witten index" of a massive deformation of the theory.

Below, we compare the two quantities:

G	SL(2)	SL(3)	SL(4)	SL(5)	GL(5)	
$\dim \mathrm{sk}(T^3; G)$	9	29	75	131	7	...
Witten index	10	30	84	130	0	

In this table, one finds that the two rows are almost never identical.

For $M_3 = \Sigma \times S^1$, with Σ a genus-g Riemann surface, the mismatch becomes worse. On the side of the skein module, one has [6, 7]

$$\dim \mathrm{sk}(\Sigma \times S^1; \mathrm{SL}(2)) = 2^{2g+1} + 2g - 1, \tag{2}$$

while the Witten index is given by

$$\mathcal{I}(\Sigma \times S^1) = 2^{2g+1} + 2. \tag{3}$$

Conceptually, what is more concerning is not the fact that the difference between the two formulae grows with the genus, but rather the non-TQFT behavior of (2), due to the presence of the term that is linear in g. In other words, while (3) is of the form expected from the partition function of a TQFT, the formula in (2) is not, and one should not expect to obtain it from a TQFT in a simple fashion.

3. The solution

These puzzles are resolved by combining three ingredients, starting with a more precise understanding of the relation between the skein module and the Hilbert space of the 4d gauge theory, which will ultimately allow us to generalize (2) to other groups. Schematically, the dimension formula for a simply connected G takes the following form,

$$\dim \mathrm{sk}(\Sigma \times S^1; G) \sim \sum_{\lambda} |C_\lambda|^{2g+1} \cdot \Gamma\left(G_\lambda, c_\lambda \cdot (2g-2)\right), \tag{4}$$

which is a sum over conjugacy classes λ of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ (or, equivalently, nilpotent orbits $\mathcal{O}_\lambda \subset \mathfrak{g}$), with the summand being a product of two factors. The first factor counts the number of flat C_λ-connections over $\Sigma \times S^1$ with $C_\lambda := \pi_0(G_\lambda)$ denoting the group of components of the centralizer subgroup $G_\lambda \subset G$ of the triple. The second factor $\Gamma\left(G_\lambda, c_\lambda \cdot (2g-2)\right)$ counts the number of dominant weights of G_λ at level $c_\lambda \cdot (2g-2)$—with c_λ related to the embedding index of $G_\lambda \subset G$—modulo the action of the center symmetry $Z(G)$.

To obtain the actual dimension formula and to incorporate cases with non-simply connected G, there are a few modifications to (4). For now, we will give some examples for $G = \mathrm{SL}(N)$ or $\mathrm{GL}(N)$, for which no modification of (4) is needed.

3.1. Examples.

When the genus $g = 1$, we have $\Gamma(G_\lambda, 0) = 1$. For $G = \mathrm{SL}(N)$ or $\mathrm{GL}(N)$, the nilpotent orbits are labeled by partitions of N. The formula (4) becomes

$$\dim \mathrm{sk}(\Sigma \times S^1; G) = \sum_{\lambda \vdash N} |C_\lambda|^3. \tag{5}$$

While the group C_λ is trivial for $\mathrm{GL}(N)$, for $G = \mathrm{SL}(N)$ it is given by $C_\lambda = \mathbb{Z}_{\gcd(\lambda)}$, a cyclic group of order equal to the greatest common divisor of all parts of the partition, and thus (4) recovers the results in [8].

On the other hand, for general g, the formula reproduces (2) when $G = \mathrm{SL}(2)$, but yields new predictions for other Lie groups. For example, when $G = \mathrm{SL}(3)$ or $\mathrm{PSL}(3)$, the formula gives

$$\dim \mathrm{sk}(\Sigma \times S^1; \mathrm{SL}(3)) = 3^{2g+1} + 3(2g+1)(g-1) + 1 + \delta_{g,1}. \tag{6}$$

3.2. Strategy.

In deriving (4), we utilize the following three main ideas.

- **Embedding into gauge theory.** We propose an embedding of $\mathrm{sk}(M_3; G)$ into the dual of $\mathcal{H}(M_3; G)$,

$$\mathrm{sk}(M_3; G) \subset \mathcal{H}(M_3; G)^\vee, \tag{7}$$

  utilizes a setup similar to those in [9, 10, 11, 12], which we believe clarifies the relation between them.

- **An $\mathcal{N}=1$ deformation.** When an M_3 has reduced holonomy, a powerful computational tool for the dimension of $\mathrm{sk}(M_3;G)$ is the $\mathcal{N}=1$ deformation studied in [13, 14]. Combined with the proposed embedding (7), this gives the dimension of $\mathrm{sk}(T^3)$ for any G. Furthermore, it leads to a surprising finding that the dimensions are often different for Langlands dual pairs,

$$\dim \mathrm{sk}(T^3;G) \neq \dim \mathrm{sk}(T^3;{}^{\mathrm{L}}G) \tag{8}$$

  once we go beyond the A-series. When $M_3 = \Sigma \times S^1$, the deformation will give rise to "cosmic strings"—surface defects with massless degrees of freedom on their worldsheet, and one technical challenge to overcome is to understand how they contribute to $\mathrm{sk}(M_3)$. This leads to the next point.
- **Bulk–string coupling.** It turns out that we will need to understand the action of the bulk Wilson lines on the Hilbert space of the cosmic string, which we argue can be modeled by the coupled system of a Chern–Simons bulk and a WZW/free-fermion boundary theory after compactifying on the "meridian circle." This allows us to obtain the dimension formula (4) and to give an algorithm to produce the set of generators.

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A relationship between topological gauge theories with finite and continuous gauge groups

Pavel Putrov

(joint work with Thomas Nicosanti, Johann Quenta-Raygada)

We explore relations between two families of 3-dimensional topological quantum field theories (TQFTs) and verify it in various examples. The first family is the Reshetikhin-Turaev TQFTs $Z^{\mathrm{RT}}_{U_\xi(\mathfrak{sl}_2)}$ associated with the category of representations of the quantum group $U_\xi(\mathfrak{sl}_2)$ at roots of unity $\xi = e^{\frac{2\pi i s}{k}}$, $(s,k)=1$ [1]. It provides a mathematical description of the $SU(2)$ Chern-Simons gauge theory [3]. The second family is the Dijkraaf-Witten TQFTs $Z^{\mathrm{DW}}_{SL(2,\mathbb{F}_q),\omega_q}$ associated with the finite groups $SL(2,\mathbb{F}_q)$ and a certain universal choice of the third cohomology class $\omega_q \in H^3(SL(2,\mathbb{F}_q),\mathbb{Q}/\mathbb{Z})$ [2]. Alternatively, it can be understood as the Turaev-Viro TQFT associated with the spherical fusion category $\mathrm{Vec}^{\omega_q}_{SL(2,\mathbb{F}_q)}$, or the Reshetikhin-Turaev TQFT associated with the twisted quantum double of $SL(2,\mathbb{F}_q)$. It provides a mathematical description of $SL(2,\mathbb{F}_q)$ gauge theories.

Conjecture [4]**.**

(i) For a closed oriented 3-manifold M, the values of TQFTs in those two families have expansions of the following form:

$$Z^{RT}_{U_{e^{\frac{2\pi i s}{k}}}(\mathfrak{sl}_2)}(M) \underset{k\to\infty}{\simeq} \sum_{Q\in S} e^{\frac{2\pi i k}{s}Q}\, k^{-\Delta_Q/2}\, I_Q^{(s,\,k \bmod s)}\left(\frac{1}{k}\right)$$

where $Q\subset\mathbb{C}$ is a finite set and $I_Q^{(s,\,k \bmod s)}\left(\frac{1}{k}\right)\in\mathbb{C}[[1/k]]$ has a non-vanishing constant term for at least some s;

$$Z^{DW}_{SL(2,\mathbb{F}_q),12r\omega_q}(M) = \sum_{Q\in S_q} e^{24\pi i r Q}\, T_Q(q)$$

where $S_q\subset\mathbb{Q}$ is a finite set and $0\le T_Q(q)\le \mathrm{const}\, q^{\tilde{\Delta}_Q}$

(ii) There is a bijection $f: S \xrightarrow{\cong} S_q$ such that $\tilde{\Delta}_{f(Q)} = \Delta_Q$.

(iii) Moreover, if M does not have any hyperbolic components in its Thurston's geometric decomposition, one can choose $f=\mathrm{id}$.

Remark. The first formula in the conjecture above can be understood as a certain version of the asymptotic expansion conjecture originating from [3].

In [4] we verify the conjecture in various examples and prove a slightly weaker version of it in the spherical case.

Theorem. *The conjecture is true for M spherical, $q=p^n$ for infinitely many primes, and n a multiple of a sufficiently large number.*

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Linear recurrent sequences in arithmetic topology

Honami Sakamoto

(joint work with Daichi Matsuzuki, Ryoto Tange, Jun Ueki)

This is an extended abstract of my talk based on the papers [STU26, MSU26].

1. Liminal $\mathrm{SL}_2\mathbb{Z}_p$-characters and cyclic covers of knots

Let p be a prime number, and let $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ be the ring of p-adic integers.

The analogies between knots and primes, or 3-manifolds and number rings have played important roles since the era of Gauss (cf.[Mor12]). In modern times, among other things, the analogy between the Alexander–Fox theory of $\mathbb{Z}$-covers and the Iwasawa theory of $\mathbb{Z}_p$-extensions of number fields, and that between deformation theories of knot group representations (eg. Thurston's hyperbolic deformation) and Galois representations (eg. due to Hida–Mazur) have been pointed out.

There are special interests in irreducible $\mathrm{SL}_2\mathbb{Z}_\mathrm{p}$-representations whose residual representations are reducible. In our study, following Mazur [Maz11], we aimed to“go the other way”.

Definition 1. *Let π be a group.*
(1) A function $\chi : \pi \to \mathbb{Z}_p$ is called an $\mathrm{SL}_2\mathbb{Z}_p$-character *if there exists an SL_2-representation ρ over an integral extension of $\mathbb{Z}_p$ such that $\chi = \mathrm{tr}\,\rho$ holds.*
(2) An $\mathrm{SL}_2\mathbb{Z}_p$-character is said to be liminal *if it is reducible and its every open neighborhood contains an irreducible $\mathrm{SL}_2\mathbb{Z}_p$-character.*

Theorem 2 ([STU26, Theorem1.1])**.** *Let $K = J(2k, 2l)$ be a genus-one two-bridge knot in S^3. For $p \neq 2$ (resp. $p = 2$), if p (resp. 2^3) divides the size of the 1st homology group of some odd-th cyclic branched cover of K, then its group $\pi_1(S^3 \setminus K)$ admits a liminal $\mathrm{SL}_2\mathbb{Z}_p$-character.*

In the case of genus two two-bridge knot $K = 6_2, 6_3$, we may also observe by calculation that a similar assertion holds outside a small exceptional set, which we expext to be finite.

2. Positive characteristic analogues of finite algebraic numbers

Consider the ring

$$\mathcal{A} = \frac{\prod_p \mathbb{Z}/p\mathbb{Z}}{\bigoplus_p \mathbb{Z}/p\mathbb{Z}},$$

where p runs through all rational primes. This ring was introduced by Ax [Ax68] and later studied by Kontsevich [Kon09]. It plays a significant role as the natural habitat for finite multiple zeta values (MZVs) proposed by Kaneko–Zagier (cf. Kaneko [Kan19]).

J. Rosen [Ros20, Theorem 1.1] introduced the algebra of finite algebraic numbers $\mathcal{P}^0_{\mathcal{A}} \subset \mathcal{A}$, characterized in three ways, namely, by using (1) linear recurrent sequences, (2) Frobenius evaluation map, and (3) 0-dimensional finite periods. In addition, let $\mathcal{C}_{\mathcal{A}}$ denote the set of all $\alpha \in \mathcal{A}$ that is algebraic over $\mathbb{Q}$. Then the set $\mathcal{P}^0_{\mathcal{A}}$ is a $\mathbb{Q}$-subalgebra of $\mathcal{A}$, and there are proper inclusions

$$\mathbb{Q} \subsetneq \mathcal{P}^0_{\mathcal{A}} \subsetneq \mathcal{C}_{\mathcal{A}} \subsetneq \mathcal{A}.$$

Example 3 (Rosen [Ros20, Example 1.5])**.** *Let $(F_n)_n$ denote the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$. Then the Legendre symbol satisfies $F_p \equiv \left(\frac{5}{p}\right) \mod p$. The Dirichlet density theorem assures that the element $\alpha = [(F_p \mod p)_p] \in \mathcal{P}^0_{\mathcal{A}}$ is not in the image of $\mathbb{Q}$ in $\mathcal{A}$. We find that $f(x) = x^2 - 1$ satisfies $f(\alpha) = 0$, so $\alpha \in \mathcal{C}_{\mathcal{A}}$.*

Now let q be a power of a prime number p and let θ be an indeterminate element. We aim to replace $\mathbb{Z}$ and $\mathbb{Q}$ in Rosen's theory by the polynomial ring $R = \mathbb{F}_q[\theta]$ and the rational function field $K = \operatorname{Frac} R = \mathbb{F}_q(\theta)$. Chang–Mishiba's ring is defined by

$$\mathcal{A}_K = \frac{\prod_P R/(P)}{\bigoplus_P R/(P)},$$

where P runs through all primes (monic irreducible elements) of R. Via the diagonal embedding $K \hookrightarrow \mathcal{A}_K$, we often assume $K \subset \mathcal{A}_K$.

We may prove that, under the following natural modification, the positive characteristic analogues of all assertions on $\mathcal{P}^0_{\mathcal{A}}$ in [Ros20] hold true.

linear recurrence formula	linear recurrence formula with a separable eigen polynomial
$a_p \mod p$	$a_{q^{\deg P}} \mod P$
finite extension	finite separable extension

Here, $f(x) \in K[x]$ is said to be *separable* if it has no multiple roots. An eigen polynomial is said to be separable if it is a product of separable polynomials.

Theorem 4 ([MSU26, Theorem 1.13])**.** *Let $\alpha \in \mathcal{A}_K$. The following conditions are equivalent.*

(1) *There is a linear recurrent sequence* $(a_n)_n$ *over* K *whose eigen polynomial is a product of separable polynomials in* $K[x]$ *such that* $\alpha = [(a_{q^{\deg P}} \mod P)_P]$ *holds.*

(2) *There exist a finite Galois extension* L/K *and a map* $g : \mathrm{Gal}(L/K) \to L$ *satisfying "*$g(\sigma\tau\sigma^{-1}) = \sigma(g(\tau))$ *for every* $\sigma, \tau \in \mathrm{Gal}(L/K)$*" such that* $\alpha = [(g(\varphi_P) \mod P)_P]$ *holds, where* $\varphi_P \in \mathrm{Gal}(L/K)$ *denotes "the Frobenius at* P*".*

(3) *There exists a finite Galois extension* L/K *such that, for an arbitrarily chosen basis of* L *over* K*,* α *is a* K*-linear combination of the matrix coefficients of "the* $\mathcal{A}_K$*-valued Frobenius automorphism"* $F_{\mathcal{A}_K} : L \otimes \mathcal{A}_K \to L \otimes \mathcal{A}_K$*.*

Definition 5. *An element* $\alpha \in \mathcal{A}_K$ *is said to be a* finite separable element *over* K *if* α *satisfies the equivalent three conditions in 4. The set of all finite separable elements over* K *is denoted by* $\mathcal{P}^0_{\mathcal{A}_K}$*. In addition, the set of all* $\alpha \in \mathcal{A}$ *that is separable (resp. algebraic) over* K *is denoted by* $\mathcal{C}^{\mathrm{sep}}_{\mathcal{A}_K}$ *(resp.* $\mathcal{C}^{\mathrm{alg}}_{\mathcal{A}_K}$*).*

Theorem 6 ([MSU26, Theorem 1.16])**.** (1) *The set* $\mathcal{P}^0_{\mathcal{A}_K}$ *is a* K*-subalgebra of* $\mathcal{A}_K$*.*

(2) *There are proper inclusions*

$$K \subsetneq \mathcal{P}^0_{\mathcal{A}_K} \subsetneq \mathcal{C}^{\mathrm{sep}}_{\mathcal{A}_K} \subsetneq \mathcal{C}^{\mathrm{alg}}_{\mathcal{A}_K} \subsetneq \mathcal{A}_K.$$

Example 7. *(1) Suppose* $q = 2^r$ *with* $r \in \mathbb{Z}_{>0}$*. Define a linear recurrent sequence* $(F_n)_n \in \mathbb{F}_2^{\mathbb{N}} \subset R^{\mathbb{N}}$ *by* $F_0 = F_1 = \cdots = F_{q-1} = 1$ *and* $F_{n+q} = F_n + F_{n+1} + \cdots + F_{n+q-1}$*, so we have* $F_q = 0$ *and* $F_{n+q+1} = F_n$*. By* $q^n \equiv (-1)^n \mod (q+1)$*, we obtain* $F_{q^n} = F_{(-1)^n} = 1 + n \mod 2$*.*

(2) Suppose $2 \nmid q$*. Define* $(F_n)_n \in R^{\mathbb{N}}$ *by* $F_1 = 1$*,* $F_2 = 0$*, and* $F_{n+2} = \theta F_n$*. By* $R/(P)^\times \cong \mathbb{Z}/(q^{\deg P} - 1)\mathbb{Z}$*, the Legendre symbol satisfies* $F_{q^{\deg P}} = \theta^{(q^{\deg P}-1)/2} \equiv \left(\frac{\theta}{P}\right) \mod P$*.*

In both cases, the element $\alpha = [(F_{q^{\deg P}} \mod P)_P] \in \mathcal{P}^0_{\mathcal{A}_K}$ *is not in the image of* K *in* $\mathcal{A}_K$*. whereas* $f(x) = x^2 + x$ *or* $x^2 - 1$ *satisfies* $f(\alpha) = 0$*, so* $\alpha \in \mathcal{C}^{\mathrm{sep}}_{\mathcal{A}_K}$*.*

Remark 8. *As suggested from a viewpoint of anabelian geometry, in the condition (2) of the definitions of* $\mathcal{P}^0_{\mathcal{A}}$ *and* $\mathcal{P}^0_{\mathcal{A}_K}$*, the target* L *of a map* $g \in A(L)$ *may be replaced by the abelianization* $\mathrm{Gal}(\overline{L}/L)^{\mathrm{ab}}$ *of the absolute Galois group, using the conjugate action* $\mathrm{Gal}(L/\mathbb{Q}) \curvearrowright \mathrm{Gal}(\overline{L}/L)^{\mathrm{ab}}$ *and the Artin reciprocity isomorphism* $\mathrm{Gal}(\overline{L}/L)^{\mathrm{ab}} \overset{\cong}{\to} C_L = I_L/P_L$ *to the idele class group.*

Remark 9. *We find that the studies of* $\mathcal{P}^0_{\mathcal{A}}$ *and* $\mathcal{P}^0_{\mathcal{A}_K}$ *come close to the heart of the analogy between knots and primes in arithmetic topology (cf.* [Mor12]*), especially in light of the ramification theories and the Chebotarev density theorems (cf.* [Uek21, GUW26]*). Among other things, finding correct analogues of periods of motives for knots and 3-manifolds in this context would be an important problem.*

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Combinatorial description of closed 3-manifolds via ordered ideal triangulations

Sakie Suzuki

(joint work with Kohei Muramatsu, Koki Taguchi, Stavros Garoufalidis, Rinat Kashaev)

An ideal triangulation of a compact 3-manifold M with non-empty boundary is a decomposition of $\mathrm{int}(M)$ into a collection of ideal tetrahedra with their faces glued in pairs, where an ideal tetrahedron is obtained by removing the four vertices of a tetrahedron. A fundamental problem in this approach is to determine which local moves relate ideal triangulations representing the same 3-manifold. In particular, it is known that any two ideal triangulations of the same 3-manifold with at least two ideal tetrahedra are related by a sequence of Pachner 2–3 moves and their inverses [6]; see Figure 1.

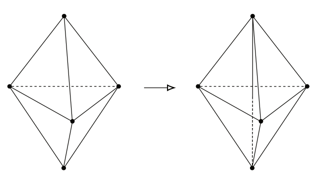

Figure 1. Pachner 2–3 move.

An *ordered ideal triangulation* of compact oriented 3-manifolds is an ideal triangulation in which each tetrahedron is equipped with a total ordering of its vertices and the face identifications preserve this ordering. In this report, we introduce a combinatorial description of closed 3-manifolds via ordered ideal triangulations.

Ordering structures on triangulations arise naturally in several contexts in quantum topology, where one assigns an operator to each tetrahedron using the ordering structure [1, 3, 7, 8]. Such operators satisfy pentagon-type relations corresponding to the Pachner 2–3 move, just as the Yang–Baxter equation corresponds to the Reidemeister III move in quantum link invariants.

An ordered tetrahedron has two possible types, as shown in Figure 2, according to whether the vertex ordering is compatible with the orientation of the 3-manifold.

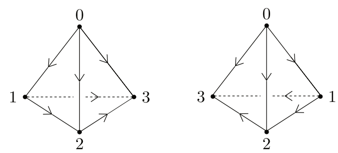

FIGURE 2. Two types of ordered ideal tetrahedra.

An ordered ideal triangulation naturally induces a nowhere-vanishing vector field on 3-manifolds, whose flow follows the ordering of the vertices of each tetrahedron. To describe a closed connected 3-manifold M, we consider ideal triangulations of $M \setminus \mathrm{int}(B^3)$. In this setting, we restrict to ordered ideal triangulations satisfying a *closedness condition* ([2]): the boundary of the triangulation is required to be S^2, and the induced vector field on the boundary divides the sphere into two connected regions, ingoing and outgoing, separated by a circle along which the vector field is tangent to the boundary; see Figure 3. Under this condition, the nowhere-vanishing vector field extends uniquely from $M \setminus \mathrm{int}(B^3)$ to M.

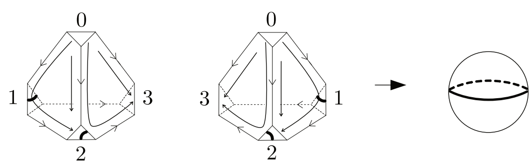

FIGURE 3. Closedness condition. To describe the boundary of a 3-manifold, we use truncated tetrahedra, where the triangular faces form the boundary. On the boundary, the thick lines indicate the tangency locus of the vector field induced by the ordering structure. The closedness condition requires that the boundary be S^2 and that the tangency locus form an S^1.

An *ordered* 2–3 *move* is the Pachner 2–3 move on the underlying triangulations, such that the ordering of the vertices on each of the six boundary faces is preserved;

see Figure 4. There are 20 types of ordered 2–3 moves. There are 16 combinatorial patterns before the move, and for 4 of them there are exactly two distinct possible results.

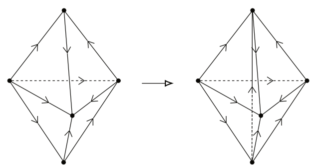

FIGURE 4. An example of an ordered 2–3 move.

An *ordered* 0–2 *move* inserts, along a pair of faces that share an edge, a ball triangulated by two ordered tetrahedra sharing two faces; see Figure 5. The ordering structure on the four triangles at the boundary of the inserted ball must agree with that of the corresponding faces. For a given pair of faces, the result of the ordered 0–2 move is unique, and there are 6 combinatorial patterns of ordered 0–2 moves.

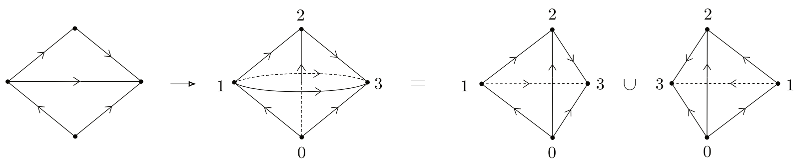

FIGURE 5. An example of an ordered 0–2 move. The right-hand side consists of two ordered tetrahedra glued along two faces with matching labels.

We arbitrarily choose one ordered 2–3 move and call it the preferred move.

Theorem 1 ([7, 5]). *Each ordered* 2–3 *move can be realized as a sequence consisting of a single preferred ordered* 2–3 *move (or its inverse) together with ordered* 0–2 *moves and their inverses.*

Theorem 2 ([5]). *The equivalence classes of ordered ideal triangulations satisfying the closedness condition, up to the preferred ordered* 2–3 *move and ordered* 0–2 *moves, are in one-to-one correspondence with the homeomorphism classes of closed* 3*-manifolds.*

Theorem 2 refines [4, Theorem 3.1] for closed 3-manifolds, and gives an affirmative answer to a restricted version of [2, Question 9.1.3] for closed normal o-graphs.

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Theta for fibered knots

Roland van der Veen
(joint work with Dror Bar-Natan)

In [1] we introduced a two variable knot polynomial Θ. It is a Laurent polynomial with S_3-symmetry that is both efficient to compute and seems to have remarkable properties and connections to topology. Conjecturally Θ equals the two-loop invariant of Ohtsuki [2] a powerful and rather complicated reduction of the Kontsevich integral that distinguishes some mutant knots. From our point of view however Θ is like an older sibling to the Alexander polynomial that is almost as easy to compute, allowing us to explore knots with hundreds of crossings.

In this talk we illustrate the connection of Θ to topology by formulating a conjectural expression for the leading coefficient of for fibered knots. Recall a fibered knot is a knot whose complement is swept out by a family of Seifert surfaces emanating from the knot. More concretely its commutator subgroup is a free group with g generators where g is the genus of the knot.

We conjecture that just like the ADO invariants of fibered knots the Hopf invariant plays a key role in the expression for Θ of a fibered knot [3]. Roughly speaking the Hopf invariant h_K of a fibered knot compares the plane field coming from the Seifert surfaces in the complement to the standard plane fields obtained from the Seifert surfaces of the unknot. It also shows up as the top coefficient of the Heegaard-Floer homology of a fibered knot and counts the number of negative Hopf-links one needs to plumb the fiber surface.

A precise version of our conjecture that was verified for all fibered knots up to 13 crossings is the following:

Conjecture. For a fibered knot of genus g we have

$$\Theta_K = (g - h_K)\Delta_K(T_1)T_2^{2g} + \mathcal{O}(T_2^{2g-1})$$

where h_K is the Hopf invariant of K and $\Delta_K \in \mathbb{Z}[T_1]$ is the Alexander polynomial normalized by $\Delta_K(1) = 1$.

Analogues of fibered knots and their Alexander polynomials exist in arithmetic topology extending the dictionary to include parts of Θ might be an interesting next step. Another connection to arithmetic comes from Lehmer's problem. The polynomial with minimal Mahler measure happens to be the Alexander polynomial of the (-2,3,7) pretzel knot (a fibered knot). Does the Θ of this knot also have special properties?

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Stokes constants and the 3d-index

Campbell Wheeler

(joint work with Jørgen Andersen, Veronica Fantini, Maxim Kontsevich)

I discussed a sheaf-theoretic approach to Borel resummation. This is used in [2] and work in progress [3] to prove that the asymptotic series that comes from the volume conjecture is resummable and compute the associated Stokes matrices.

For the lecture, I focussed on asymptotic series that come from integrals of the form

$$\int_{\mathcal{C}} e^{f/\hbar}\nu \sim \Phi(\hbar)$$

where X is an algebraic variety of dimension n, $f : X \to \mathbb{C}$ is regular, $\nu \in H^n_{dR}(X)$, and $\mathcal{C} \in H_n(X; f/\hbar)$. I described the sheaf theoretic approach to the resummation of these series, which has recently be summarised in [1].

The main idea, we use, is to consider the Borel transform of $\hbar^{\log(\alpha)/2\pi i+1}e^{f/\hbar}$, which is simply given by $(f-\eta)^{\log(\alpha)/2\pi i}$ for some $\alpha \in \mathbb{C}^\times \smallsetminus \{1\}$. The monodromy of this function on $X \times \mathbb{C}$ can be encoded in a constructible sheaf $\mathcal{F}_\alpha$. Then the monodomy of the Borel transform of $\mathcal{B}\Phi(\eta)$ is encoded in the sheaf $(\mathrm{pr}_2)_!\mathcal{F}_\alpha$.

Therefore, in this algebraic setting, the fact that the Borel transform has endless analytic continuation follows from resolution of singularities, which provides smooth compactifications, compatible with the sheaves.

This leads to a definition of resurgent structures. We say that a perverse sheaf $\mathcal{F} \to \mathcal{O}_{X\times\mathbb{C}}$ is an algebraic *family of resurgent structures* if $(\mathrm{pr}_1)_!\mathcal{F} = 0$. Then an immediate consequence is that family of resurgent structures are preserved under, proper pushforwards, pullback, and convolution.

These geometric operations allow for the computation of Stokes constants. It was explained when X is 2-dimensional, that the Stokes phenomenon of $\Phi(\hbar)$ could be computed by fist projecting to a one dimensional variety and then finally to a point. The final computation can then be explicitly done by drawing pictures, which encode the geometry of 2-dimensional real contours in a 2-dimensional complex space. This is pictured in Figure 1.

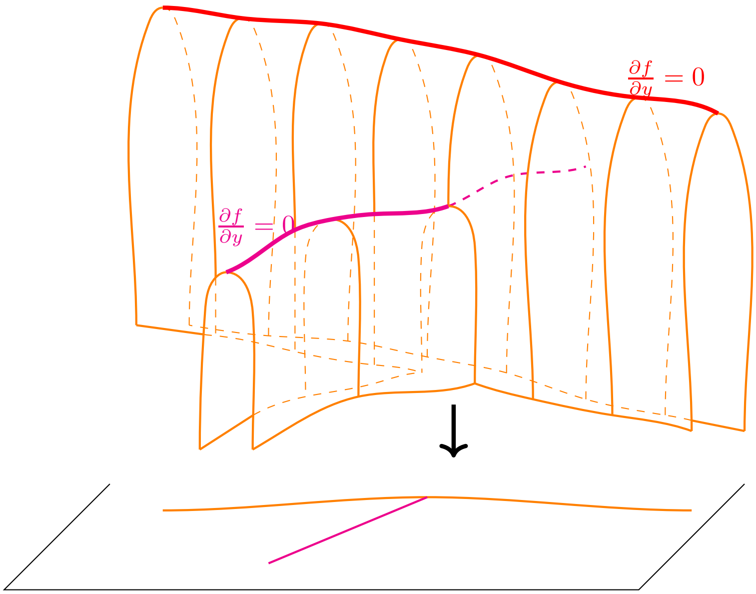

FIGURE 1. Picture of the limiting steepest descent contours in (x, y)-plane. The y-axis is in the vertical direction. Includes the projection down to x-plane.

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