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The Kakeya conjecture

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The geometry of lines is a fundamental part of mathematics and the way we interact with the physical world. Core concepts such as distance and angle, and the accompanying theory of trigonometry, have been studied since antiquity and taught to countless generations of students. However, there are simple questions about lines which have stumped some of the greatest minds in mathematics over the last fifty years. One notable example is the Kakeya conjecture, which asks how lines which point in different directions can be packed together in a small space.

1 Kakeya's question

A *Kakeya set* is a shape that can fit a straight line segment of length 1 in every possible direction. In Figure 1, we illustrate three simple examples: a disc, an equilateral triangle, and a deltoid (a curved triangular shape). The equilateral triangle is the most informative. Imagine fixing one end of a needle of length 1 at the top corner of the triangle. You can then rotate the needle inside the triangle along an angle of 60° pointing upward. By symmetry, you also get another 60° pointing downward. Altogether, that gives 120° of directions from

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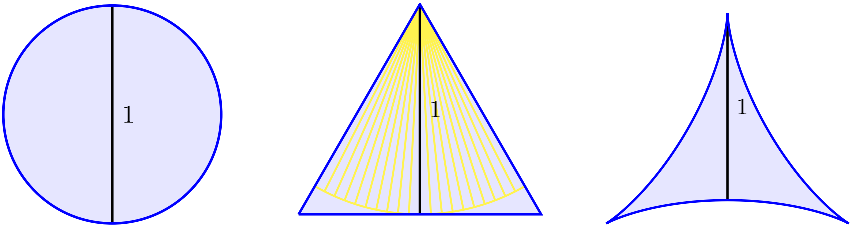

(a) Area: $\pi/4$. (b) Area: $1/\sqrt{3}$. (c) Area: $\pi/8$.

Figure 1: Three examples of Kakeya sets.

this one corner. Repeating the same idea at the other two corners covers all the possible directions (a full 360°).

If we compare the areas of the three Kakeya sets in Figure 1, we see that they get successively smaller. In fact, the deltoid has half the area of the disc. This means that, within the deltoid, we have managed to pack the line segments much more tightly.

In the early 20th century, the Japanese mathematician Sōichi Kakeya (1886–1947) was interested in these shapes and understanding different ways to pack lines into small spaces. From his work, a natural question arose: *what is the smallest possible area of a Kakeya set?*

2 Besicovitch's surprising answer

Around the same time, Abram Besicovitch, an influential mathematician working in the field of analysis, was independently thinking about similar questions. He discovered new and surprising methods to fit line segments into shapes, and created Kakeya sets with much, much smaller area than any example seen before. In fact, Besicovitch was able to show that

there exist Kakeya sets of *arbitrarily small* area!

That is, if you want a Kakeya set with area less than 1/100, then Besicovitch's method can construct one for you. The same is true if you want a Kakeya set of area less than 1/1000000000000: no matter how small a number you choose, Besicovitch's method can construct a Kakeya set with area smaller than that number!

How is this possible? How can we pack so many line segments into such a tiny space?[2] To understand this, we go back to the example of the equilateral

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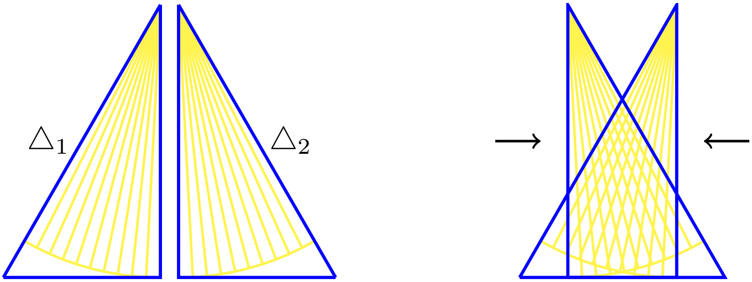
[2] Here we sketch a simplification of Besicovitch's argument. A detailed history of the problem, together with the precise description of the method, can be found in [1, Chapter 7].



(a) Cut $\triangle$ into subtriangles $\triangle_1$ and $\triangle_2$. (b) Horizontally shift $\triangle_1$ and $\triangle_2$.

Figure 2: The cut-and-shift procedure.

triangle, which we denote by $\triangle$. In Figure 2, we carry out the following "cut-and-shift" procedure to arrive at a new shape:

(a) Cut $\triangle$ into two "subtriangles" $\triangle_1$ and $\triangle_2$ along the vertical line through the top corner.
(b) Shift $\triangle_1$ and $\triangle_2$ horizontally in opposite directions so that they overlap.

The key observation is that, as we cut-and-shift, we do not lose any of the line segments swept out from the top corner. Furthermore, because of the overlap between the shifted subtriangles, the new shape has smaller area than $\triangle$. The procedure therefore results in a shape with smaller area than the equilateral triangle $\triangle$, but which still contains line segments in 120° of directions. This on its own is not very impressive. However, if we apply the procedure over and over many times, then we can amplify its effect and obtain a Kakeya set with a small area as desired.

Rather than cutting $\triangle$ into just two subtriangles, in Figure 3 we cut it into many subtriangles. As illustrated, we then shift the subtriangles horizontally, first to form pairs, and then to form pairs of pairs, and then pairs of pairs of pairs, and so on. By doing so, we create a huge amount of overlap, which means that the final shape K has much, much smaller area than the original triangle. By choosing the number of subtriangles to be very large, and carefully controlling the shifts, we can ensure that the area of K is as small as we wish. As before, K contains (shifted versions of) all the line segments spanned by the top corner of $\triangle$, corresponding to lines in 120° of directions. We can then take three rotated copies of K to form a true Kakeya set containing line segments in all directions, see Figure 3.[3]

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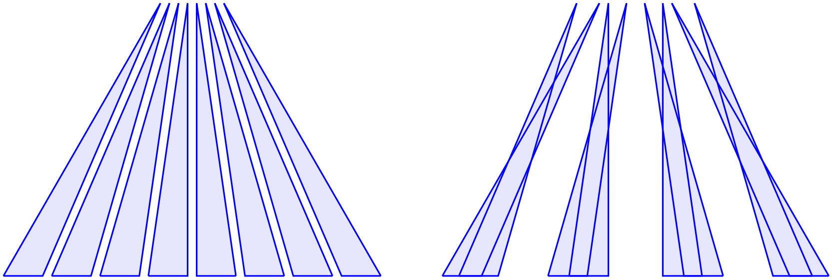
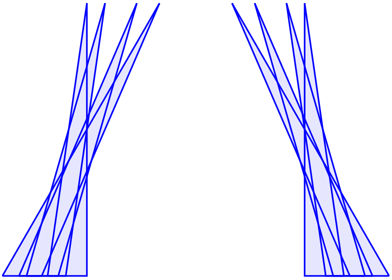
[3] Taking three copies of K will increase the area by a factor of 3, but we can compensate for this by ensuring that the area of K is three times smaller than our target value.



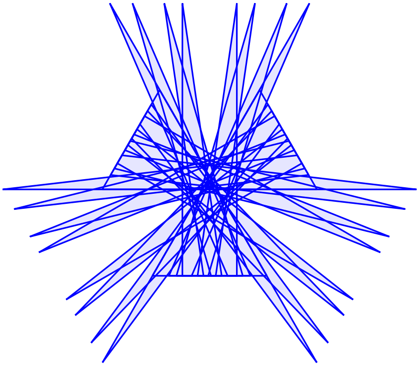
(a) Cut $\triangle$ into many subtriangles.

(b) Horizontally shift to form pairs.



(c) Form pairs of pairs...

(d) The final shape K.



(e) Take three rotated copies of K to form a true Kakeya set.

Figure 3: Construction of a Kakeya set with very small area.

The above procedure describes a method of constructing a Kakeya set with area smaller than any prescribed target value. In fact, using the same ideas, Besicovitch was able to go one step further and showed that Kakeya sets of zero area exist!

3 Why are Kakeya sets important?

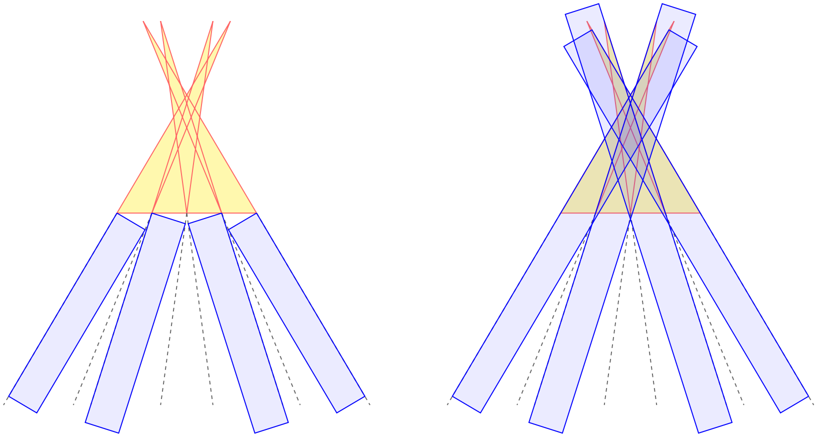
Besicovitch's construction is undoubtedly striking and beautiful. However, around the time of its discovery, it is perhaps fair to say that the existence of Kakeya sets of zero area was something of a curiosity. Besicovitch applied his construction to solve a problem about integrals, but beyond this, it did not appear to have much use.

Mathematical ideas have a wonderful habit of reappearing in new and seemingly unrelated contexts. What was once thought a curiosity can later emerge as a profoundly important idea. Kakeya sets are a striking example of this. Fifty years after Besicovitch's work, startling discoveries led mathematicians to realise their true significance, particularly in Fourier analysis.

Fourier analysis is an important branch of mathematics which, at first sight, has little to do with the kind of geometric problems we discussed. It instead deals with the *Fourier transform*. This is a mathematical process which takes an input function (which, for instance, can be interpreted as a signal, such as a sound recording) and expresses it in terms of basic sine and cosine waves, each with a fixed fundamental frequency. In this way, the Fourier transform is the mathematical equivalent of sheet music. The same way sheet music describes the individual notes which make up a chord, the Fourier transform describes the frequencies of the individual waves which make up a function.

Fourier analysis is essential to vast swathes of mathematics, physics, and engineering. Many physical processes are much easier to understand when described in terms of fundamental frequencies using the Fourier transform. The same is true for pure mathematics: many famous problems in number theory, including the ternary Goldbach problem [3], have been solved using the Fourier transform.

Here, we are interested in Fourier analysis in two dimensions. The fundamental building blocks are plane waves, which can be thought of as idealised wave patterns extending across a flat surface. Each plane wave has a fixed amplitude, frequency, and direction. Now imagine the function, to which we want to apply the Fourier transform, describes the surface of a choppy sea. For instance, we could imagine a harbour where the waves bounce off the shore and end up travelling in lots of different directions. The Fourier transform allows us to understand the choppy water in terms of underlying plane waves, each with a fixed amplitude, frequency and direction. This decomposition reveals



(a) Before filtering, the function is nicely spread out over the blue rectangles.

(b) Filtering causes the blue rectangles to double and the output function has an enormous peak.

Figure 4: Fefferman's example.

how seemingly complicated behaviour emerges from the superposition of very simple waves.

In the early 1970s, Charles Fefferman studied a concept in Fourier analysis, called *(low pass) filter*. A filter removes the high-frequency parts of a function, leaving behind the smoother, low-frequency components. Filters play an important role in signal processing. As a simple example, imagine a poor-quality sound recording with lots of clicks, cracks and glitches. Since this unwanted noise is typically made up of high frequencies, applying a filter removes it and improves the quality.

Fefferman was interested in how filters behave in two dimensions, a problem that was not well understood at the time. We would expect the filtered function (with all the noise removed) to be smoother and better behaved than the original input. This was known to be true in one dimension. Surprisingly, Fefferman showed that this was not always the case in two dimensions! There were functions which, before applying the filter, were nicely spread out in space; however, after filtering, they focused into an enormous peak [2].

To see this, Fefferman constructed a function which lives on rectangles placed outside a Kakeya set. These rectangles are all spaced apart, so the function is nicely spread out. When the filter is applied, he showed it has the effect of doubling each of the rectangles. The geometry of the Kakeya set forces the doubled rectangles to overlap, creating an enormous peak (see Figure 4).

Fefferman's work revealed that the geometry of Kakeya sets plays a fundamental role in Fourier analysis in two and higher dimensions. Since Fourier analysis is so central to the study of many physical processes, it soon became clear that Kakeya sets are deeply linked to the study of important equations of nature. These include the wave equation, which models water, sound, and seismic waves, as well as Schrödinger's famous equation from quantum mechanics [6, 8]. Kakeya sets, which were once a curiosity, suddenly became one of the most important objects in pure mathematics! For further details of the connection between Fourier analysis and Kakeya sets, we refer the reader to [9].

4 The Kakeya conjecture

After it became clear that Kakeya sets are fundamental to many different mathematical problems, researchers aimed to develop a deeper understanding of their geometry. In addition to the planar Kakeya sets that we have discussed so far, mathematicians became increasingly interested in their solid counterparts in 3-dimensional space. The definition of a Kakeya set makes perfect sense for solid shapes, but the situation becomes more complicated in three dimensions because there are far more possible directions to consider.

As discussed earlier, Besicovitch showed that there exist Kakeya sets in the plane with zero area. Using Besicovitch's ideas, it is not difficult to show that Kakeya sets also exist in 3-dimensional space with zero volume. Despite this, Kakeya sets should still be, in some sense, reasonably large and fill out space. The precise formulation of this statement is known as the 3–dimensional Kakeya conjecture.

To understand the Kakeya conjecture, we return to the 2-dimensional plane. We take a step back and think a little bit about what we mean by *area.*

In Figure 5, we illustrate a simple way to approximate the area of a shape S. For a small number $\delta > 0$, we place S on a grid of squares of side length δ. We then count the number of squares which intersect S and denote this number by $N(S, \delta)$. For example, in the case shown in Figure 5, we have $N(S, \delta) = 44$. We then have the simple approximation

$$\mathrm{Area}(S) \approx \underbrace{N(S,\delta)}_{\text{Number of squares}} \times \underbrace{\delta^2}_{\text{Area of a square}} .$$

This approximation becomes more and more accurate as δ gets smaller and smaller, which corresponds to taking a finer and finer grid of squares. We express this succinctly by saying that Area(S) is given by the *limit* of $N(S, \delta) \times \delta^2$ as δ tends to 0. In symbols, this is written as

$$N(S,\delta) \times \delta^2 \to \mathrm{Area}(S) \qquad \text{as} \qquad \delta \to 0.$$

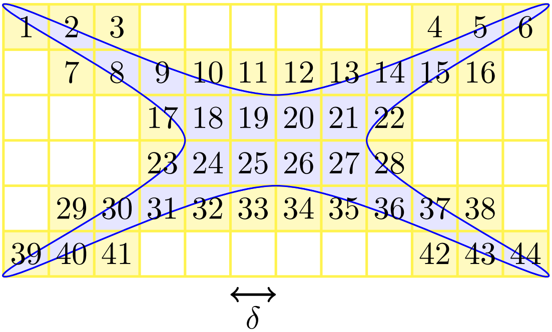


Figure 5: For the above choice of $\delta > 0$, we approximate the area of the blue shape S by $\text{Area}(S) \approx N(\delta, S) \times \delta^2 = 44 \times \delta^2$.

Besicovitch's construction shows that there exists a Kakeya set K satisfying

$$N(K, \delta) \times \delta^2 \to 0 \quad \text{as} \quad \delta \to 0.$$

This is precisely what it means for a set K to have zero area.

We can get more information about the size of a shape S by studying the approximate areas $N(S, \delta) \times \delta^2$ for different values of δ. To illustrate this:

(1) *Single point:* Suppose S is just a single point. Then S intersects only one square, so $N(S, \delta) = 1$. The approximate area is therefore

$$N(S, \delta) \times \delta^2 = \delta^2.$$

As δ gets smaller, δ^2 gets smaller at a fast rate. We can think of the approximate area as shrinking to zero very quickly.

(2) *Line or curve:* Suppose S is a line or a curve of length 1. Each square can cover only a segment of S of length roughly δ,[4] so in total we need roughly $1/\delta$ squares to cover S. In other words,

$$N(S, \delta) \approx 1/\delta.$$

Therefore, the approximate area is

$$N(S, \delta) \times \delta^2 \approx \delta.$$

This still shrinks to zero as δ gets smaller, but at a slower rate than in Example (1).

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[4] If the curve bends a lot in the square, then the square will cover more of the length of the curve. However, the exact length is not important for us: we just need to know it is *roughly* δ.

Both the point in Example (1) and the line or curve in Example (2) have zero area. However, the *rate* at which the approximate area $N(S, \delta) \times \delta^2$ shrinks to zero distinguishes their size. In this way, the rate indicates the *dimension* of the underlying shape. For the 0-dimensional point in Example (1), the rate is very fast. For the 1-dimensional line or curve in Example (2), the rate is much slower. Pushing this a little further, it is possible for a shape S in the plane to be 2-dimensional and have zero area. This means that the rate at which

$$N(K, \delta) \times \delta^2$$

shrinks to 0 is very, very slow.

For planar shapes, the Kakeya conjecture asserts that

> a Kakeya set K in the plane always has dimension 2.

In other words, the approximate area $N(K, \delta) \times \delta^2$ can only shrink to zero at a very slow rate. In this sense, although Kakeya sets can have zero area, they must still be reasonably large and fill out space.

The same idea can be applied to solid Kakeya sets in 3-dimensional space. For this, we consider the *approximate volume* $N(S, \delta) \times \delta^3$ of a solid shape S. This is computed by drawing a grid of $\delta \times \delta \times \delta$ cubes and counting the number $N(S, \delta)$ of cubes which intersect S. In three dimensions, the Kakeya conjecture asserts that

> a Kakeya set K in 3-dimensional space always has dimension 3.

Analogously to the planar case, this means that the approximate volume of K can only shrink to zero at a very slow rate.

It turns out that the Kakeya conjecture in the plane is not too difficult. Multiple proofs were found in the 1970s, each showing that any planar Kakeya set has dimension 2.[5] However, the problem in 3-dimensional space is much, much harder, as the additional directions make the situation considerably more complicated.

Understanding whether the Kakeya conjecture holds in three dimensions is one of the most important problems in modern mathematics. If it were false, then that would mean that there exist extremely small Kakeya sets that could be used to construct functions which behave extremely badly under filtering. These functions would be far worse behaved than those featured in Fefferman's original argument. Consequently, they would also disprove a large number of important conjectures in Fourier analysis and related areas. For this reason, there are many questions we cannot hope to answer without first determining whether or not the Kakeya conjecture is true.

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[5] See, for instance, [8] and [5].

5 Spectacular progress

Over the last 50 years, many mathematicians have worked on the Kakeya conjecture in three and higher dimensions. Through their work, deep connections were discovered between the problem and questions in arithmetic, algebra, and other areas of mathematics [4, 8]. However, a solution always appeared far out of reach. The Kakeya conjecture achieved a certain notoriety, and within the field, became a byword for a difficult, nearly impossible challenge.

This remained the state of affairs until February 2025, when Hong Wang and Joshua Zahl announced a spectacular full solution of the 3-dimensional problem [7]. Their monumental proof is hundreds of pages long and is spread over multiple papers. It combines a rich array of ideas, and relies on contributions from many different mathematicians over the last 30 years. However, their proof also incorporates a wealth of new and deep insights and marks a major leap in our understanding. The work of Wang and Zahl is certainly a great and historic achievement in modern mathematics.

Mathematicians are now working hard to fully digest the innovations introduced by Wang and Zahl. It is likely that their work will lead to breakthroughs on a wide range of important problems in the geometry of lines, Fourier analysis, and beyond. The next chapter of this story will be an exciting one!

Image credits

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