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Group Actions and Harmonic Analysis in Number Theory

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Abstract. Rooted in number theory, this workshop combines recent methodological advances in a variety of adjacent fields including automorphic representation theory, ergodic theory, Fourier analysis and algebraic geometry. It focuses on a selected set of problems that could recently be solved quite surprisingly in a number of cases, but are awaiting a more general framework. This includes in particular equidistribution problems in homogeneous dynamics, automorphic forms in higher rank and exponential sums and arithmetic over function fields.

Mathematics Subject Classification (2020): 11Fxx, 11Lxx, 11Mxx, 22Exx, 37Axx, 42Axx.

Introduction by the Organizers

Although the methods of harmonic analysis and ergodic theory are quite different in nature, they interact very successfully in number theory problems. Forty-six experts, accompanied by five additional online participants (with 15 female participants in total) gathered at the MFO to investigate the fruitful interface of these two areas. A golden thread of the workshop were quantitative equidistribution problems in a variety of situations, with applications to counting problems, quadratic forms, diophantine analysis, Shimura varieties, L-functions and cryptography, drawing on methods from dynamics and measure classification, automorphic forms, exponential sums and p-adic representation theory.

The workshop started with an introductory talk by Elon Lindenstrauss highlighting and explaining the enormous progress in establishing polynomially effective versions of Ratner's theorems, which built among others on work by Mozes-Shah and Einsiedler-Margulis-Venkatesh. Lei Yang presented the details of the most recent breakthrough on polynomially effective equidistribution for unipotent orbits in products of SL_2 factors. Andreas Wieser presented a spectacular application to the representations of binary quadratic forms by quaternary quadratic forms, a scenario that seemed completely out of reach only a few years ago.

Equidistribution of problems of genuine arithmetic flavour were presented by Farrell Brumley, Asbjørn Nordentoft. Brumley explained how the simultaneous equidistribution conjecture of Michel and Venkatesh can be solved in the setting of simultaneous quaternionic embeddings of imaginary quadratic fields having sufficiently many small split primes. Nordentoft gave a new interpretation of an important geometric invariant associated with real quadratic fields, which was introduced by Duke, Imamoglu, and Tóth, and studied equidistribution of partial coverings defined from closed geodesics. Toma investigated the equidistribution of sublattices of a Euclidean lattice with growing index using the theory of higher-rank automorphic forms and geometry of numbers. Complementing such results by algorithms involving Hecke operators and LLL, this proves a worst-case to average-case reduction of the short vector problem relevant to post-quantum cryptography.

The connection of equidistribution to subconvexity for L-function was highlighted by Paul Nelson. That subconvexity can imply equidistribution was known and exploited for a long time. Nelson sketched a long-term program how equidistribution can be turned in subconvexity for higher-rank L-functions in the critical strip. Gergely Harcos reported on effective lower bounds for general families of Rankin-Selberg L-functions at the edge of the critical strip.

The analytic theory of higher-rank automorphic forms has recently seen very strong and unexpected progress in many respects; naturally they were featured in several contexts during this workshop. A more arithmetic viewpoint was taken by Claudia Alfes, Özlem Imamoglu and Abhishek Saha. Alfes presented a rationality result for linear combinations of traces of cycle integrals of certain meromorphic Hilbert modular forms, while Imamoglu proved that twisted traces of special values of Green's functions on hyperbolic 3-space can be expressed in terms of algebraic numbers. Saha showed how to attack the notoriously hard problem of bounding Fourier coefficients of Siegel modular forms. A more analytic point of view was contributed by Edgar Assing, Péter Maga and Anna v. Pippich and Anke Pohl. Maga and v. Pippich reported on new directions in higher-rank versions of the supnorm problem, while Anke Pohl generalized the period function approach for Maaß forms to Jacobi forms. Assing showed how the trace formula leads to a solution of Sarnak's density conjecture for GL_n, with important applications to optimal lifting and the diophantine exponent. Mathilde Gerbelli-Gauthier complemented the investigation of the density hypothesis from a representation theoretic point of view, establishing uniform dimension bounds for p-adic representations. It is worth pointing out that the diophantine exponent can very naturally also be investigated from a dynamical point of view, a breakthrough example of which was presented by Nicolas de Saxcé.

An important input from number theory in many of the above mentioned problems are a careful analysis of arithmetic and exponential sums. Alexandru Pascadi introduced a new non-abelian amplification scheme to bound bilinear forms in classical Kloosterman sums, while Johannes Linn presented power saving bounds for Kloosterman sums for the group $\mathrm{SL}_n(\mathbb{Z})$ in full generality. Katherine Woo applied sieve theory and base change to bound sums of Hecke eigenvalues over polynomial sequences.

The workshop ended with three talks on applications of dynamical methods by Omri Solan, Timothée Bénard and Wenyu Pan. Solan described probability measures invariant under the horocyclic flow, as conjectured by Forni, and presented an application to billiards with rational angles. Bénard showed quantitative measure convergence of random walks on homogeneous spaces under arithmetic conditions – very much in the spirit of this workshop. Pan established local mixing for the diagonal flow and equidistribution results on quasi-Fuchsian manifolds realized as quotients on copies of hyperbolic 3-space.

A lively problem session on Wednesday evening – after an enjoyable hike in excellent weather – rounded the workshop off.

Workshop: Group Actions and Harmonic Analysis in Number Theory

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Abstracts

Effective equidistribution of unipotent flows and representing quadratic forms

Elon Lindenstrauss
(joint work with Manfred Einsiedler, Amir Mohammadi, Andreas Wieser)

Ratner famously classified measures on quotients G/Γ invariant under groups generated by unipotents. As a corollary, Mozes and Shah showed a sequence of periodic orbits on G/Γ that can arise in Ratner's measure classification become equidistributed, unless there is an obvious obstruction. This has been extended to the S-arithmetic context by Gorodnik and Oh.

Building on prior work of Einsiedler–Margulis–Venkatesh [2], Einsiedler–Margulis–Mohammadi–Venkatesh [1], as well as an equidistribution result by Wieser [4], we give an effective, fully adelic version of this result for cocompact quotients $G(\mathbb{A})/G(\mathbb{Q})$ with G a semi-simple $\mathbb{Q}$-group. We make no splitting assumption on the periodic orbits, nor do we assume anything about the centralizer. Our proof relies on an effective version of the Dani–Margulis linearization method by Lindenstrauss–Margulis–Mohammadi–Shah (and an associated closing lemma due to Lindenstrauss–Margulis–Mohammadi–Shah-Wieser), as well as a new effective generation theorem for semisimple groups of independent interest.

This allows us to give new results about a classical number theoretic problem.

A quadratic lattice over $\mathbb{Z}$ of rank m is a pair (Λ, Q) where $\Lambda \cong \mathbb{Z}^m$ and $Q : \Lambda \to \mathbb{Z}$ is a quadratic form. A rank m quadratic lattice (Λ, Q) is *(primitively) representable* by another rank n quadratic lattice (Λ', Q') if we can obtain a quadratic lattice isomorphic to (Λ, Q) by restricting (Λ', Q') to an m-dimensional subspace. A necessary criterion for primitive representability is that there exist local primitive representations, i.e. that $(\Lambda \otimes \mathbb{Z}_p, Q)$ can be obtained from $(\Lambda' \otimes \mathbb{Z}_p, Q')$ by restricting to an m-dimensional space for any prime p as well as the compatibility of the signatures of the real quadratic spaces $(\Lambda \otimes \mathbb{R}, Q)$ and $(\Lambda' \otimes \mathbb{R}, Q')$. In that case, we will say (Λ, Q) is locally primitively representable by (Λ', Q'). When $n \geq m + 3$ and the quadratic form Q' on the rank n lattice is indefinite, strong approximation for spin groups implies a local-to-global principle over $\mathbb{Z}$ for quadratic lattices. When Q' is positive definite, the question is significantly harder. Using the above effective equidistribution theorem, we prove the following:

Theorem 1. *Let m, n be positive integers with $n \geq m + 3$. Then there exist constants $C, A > 0$ depending only on n with the following property.*

Let (Λ, Q) and (Λ', Q') be positively definite quadratic lattices of rank m and n respectively. Suppose that (Λ, Q) is locally primitively representable by (Λ', Q') and that

$$\min_{x \in \Lambda \setminus \{0\}} Q(x) \geq C \operatorname{disc}(\Lambda', Q')^A.$$

Then (Λ, Q) is primitively representable by (Λ', Q').

This improves on a result of Ellenberg and Venkatesh [3] by dropping a splitting assumption when $n - m < 5$, as well as making the required lower bound on $\min_{x\in\Lambda\setminus\{0\}} Q(x)$ explicit and effective.

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Effective Ratner for products of $\mathbf{SL_2}$

Lei Yang

(joint work with Elon Lindenstrauss, Amir Mohammadi)

Ratner's equidistribution theorem [8] is one of the most important theorems in homogeneous dynamics. Let G be a Lie group (or p-adic group), Γ be a lattice in G, and $U = \{u_r : r \in \mathbb{F}\}$ (here $\mathbb{F}$ denotes the corresponding local field) be a one-parameter unipotent subgroup of G. We will consider the dynamical system of the U-action on the homogeneous space $X = G/\Gamma$. Ratner's equidistribution theroem says the following: For any $x \in X$, the orbit closure of Ux is a homogeneous subspace Lx, where L is a Lie subgroup (or p-adic subgroup) of G, and the closed orbit Lx supports a L-invariant probability measure μ_L. Moreover, the U-orbit Ux equidistributes in Lx in the following sense, for any $\varphi \in C_c(X)$,

$$\lim_{T\to\infty} \frac{1}{|u[T]|} \int_{u[T]} \varphi(u_r x)\mathrm{d}r = \int_{Lx} \varphi(y)\mathrm{d}\mu_L.$$

Here $u[T] := \{r \in \mathbb{F} : \|r\| \leq T\}$.

Given this fundamental theorem, it is natural to ask the rate of convergence, namely, for a fixed large $T > 0$, can we estimate

$$D_U(T, x, \varphi) := \left| \frac{1}{|u[T]|} \int_{u[T]} \varphi(u_r x)\mathrm{d}r - \int_X \varphi(y)\mathrm{d}\mu_G \right| ?$$

Results on establishing explicit upper bounds on $D_U(T, x, \varphi)$ are called effective versions of Ratner's theorem. The goal is to show that

$$D_U(T, x, \varphi) = O(T^{-\eta}\|\varphi\|_S)$$

unless the orbit is "close to" a homogeneous subspace Lx'.

Before stating our main result, let me give a brief review of recent results on effective Ratner.

For $G = \mathrm{SL}_2(\mathbb{R})$, U is horospherical in G, namely, there exists $a \in G$ such that

$$U = U(a) := \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}.$$

For this case, there are standard tools from representation theory to establish effective equidistribution with polynomial errors. The following results fit in this setting: Sarnak [9], Burger [1], Flaminio-Forni [2], Strömbergsson [10]. For G being nilpotent, Green and Tao [3] established an effective equidistribution result (with polynomial errors) for polynomial orbits. For $G = \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ and $U \subset \mathrm{SL}_n(\mathbb{R})$ being horospherical, effective equidistribution theorems of U-orbits were established by Strömbergsson [11] for $n = 2$ and by Wooyeon Kim [4] for general n.

In 2023, Lindenstrauss and Mohammadi [5] made a revolutionary breakthrough by introducing tools from harmonic analysis and incidence geometry to the study of unipotent orbits. Following their work, there have been significant developments on this project: Lindenstrauss, Mohammadi, and Wang [6] established effective Ratner for

$$G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{C});$$

the speaker [12] established effective Ratner for singular unipotent subgroups in $\mathrm{SL}_3(\mathbb{R})$; Lindenstrauss, Mohammadi, Wang and the speaker [7] established effective Ratner for generic unipotent subgroups in

$$G = \mathrm{SL}_3(\mathbb{R}), \mathrm{Sp}_4(\mathbb{R}), \mathrm{SU}(2,1), \mathbf{G}_2(\mathbb{R}).$$

In this talk, I will explain a recent work joint with Lindenstrauss and Mohammadi, where we establish effective Ratner for unipotent subgroups in $\prod_{k=1}^n \mathrm{SL}_2$.

Here is the setting: Let $\mathbf{K}$ be a number field and $\mathbf{G}$ be a semisimple $\mathbf{K}$-group such that

$$\mathbf{G}(\mathbb{C}) \cong \prod_{k=1}^{n} \mathrm{SL}_2(\mathbb{C}).$$

Let S be a finite set of places of $\mathbf{K}$ containing all the infinite places, and let $\mathcal{O}_S$ denote the set of S-integers in $\mathbf{K}$. Let $G = \mathbf{G}(\mathbf{K}_S)$ and let Γ be a subgroup of G commensurable with $\mathbf{G}(\mathcal{O}_S)$. Let $X = G/\Gamma$ and m_X denote the probability Haar measure on X. Let d_X denote the metric on X induced by a right invariant metric on G. Let $\mathbf{F}$ be a local field of characteristic zero. Suppose there is an embedding

$$\iota : \mathrm{SL}_2(\mathbf{F}) \to G.$$

Let us denote $H = \iota(\mathrm{SL}_2(\mathbf{F}))$. For $r \in \mathbf{F}$, let us denote

$$u_r := \iota\left(\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}\right),$$

and $U = \{u_r : r \in \mathbf{F}\}$. Let us fix

$$a := \iota\left(\begin{bmatrix} \omega_{\mathbf{F}} & 0 \\ 0 & \omega_{\mathbf{F}}^{-1} \end{bmatrix}\right),$$

where $\|\omega_{\mathbf{F}}\|_{\mathbf{F}} > 1$. For $T > 1$, let $a_T = a^n$ where n denotes the largest integer such that

$$\|\omega_{\mathbf{F}}\|^{2n} \leq T.$$

Here is the main result: For any $\epsilon > 0$, there exist $\eta, A > 0$ such that for any $x \in X$ and $T > 1$, one of the following holds:

(1) For any $\varphi \in C_c^\infty(X)$,

$$\left|\frac{1}{|u[1]|}\int_{u[1]} \varphi(a_T u_r x)\mathrm{d}r - \int_X \varphi \mathrm{d}m_X\right| = O(T^{-\eta}\|\varphi\|_S);$$

(2) There exists $y \in X$ and $H \subseteq L \subseteq G$ such that Ly is closed, $\mathrm{vol}(Ly) \leq T^\epsilon$, and

$$d_X(x,y) \leq T^{-1+A\epsilon}.$$

Note that the orbit $a_T u[1]x == u[T]x'$ where $x' = a_T x$. Thus the main result gives an effective version of Ratner's theorem modulo an almost optimal algebraic obstruction.

The basic strategy of the proof is similar to the ones developed in [6, 12, 7], which is based on a multiscale study of the dimension of the unipotent orbits and a bootstrapping argument. Main novelties of the proof include: (1) a Bourgain type projection theorem that allows us to analyze local obstructions of a bootstrapping step; (2) a new dynamical framework that allows us to combine the local obstructions to a local obstruction coming from an intermediate subgroup.

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Representations of binary by quaternary quadratic forms

Andreas Wieser
(joint work with Wooyeon Kim, Pengyu Yang)

Let q, Q be two positive definite integral quadratic forms in $m < n$ variables. A *representation* of q by Q is a tuple of vectors $(v_1, \ldots, v_m)$ in $\mathbb{Z}^n$ so that

$$q(x_1, \ldots, x_m) = Q(x_1 v_1 + \ldots + x_m v_m).$$

A representation is *primitive* if $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m = (\mathbb{Q}v_1 + \ldots + \mathbb{Q}v_m) \cap \mathbb{Z}^n$. We wish to obtain criteria under which a primitive representation of q by Q exists.

The question is particularly classical if $m = 1$. In this case, we ask for which integers $D \geq 1$ the equation $Q(v) = D$ has a solution $v \in \mathbb{Z}^n$ with $\gcd(v) = 1$. Work of Kloosterman and Tartakovskii from the 1920's proved, assuming $n \geq 4$, that any sufficiently large integer D satisfies

$$\begin{array}{c} Q(v) = D \text{ has a solution } v \in \mathbb{Z}^n \\ \text{with } \gcd(v) = 1 \end{array} \iff \begin{array}{c} Q(v) = D \text{ has a solution } v \in \mathbb{Z}_p^n \\ \text{with } \gcd(v) = 1 \text{ for every prime } p. \end{array}$$

This local-global principle was extended only much later to representations of integers by ternary forms by Duke and Schulze-Pillot [3]. Their work relies heavily on Duke's work [2] establishing bounds for Fourier coefficients of cusp forms of half-integral weight. Note that [3] is not an effective result (due to potential Siegel zeros) as opposed to the earlier work of Kloosterman and Tartakovskii (see e.g. [6]). Furthermore, [3] necessarily needs to include so-called spinor obstructions in addition to the local obstructions.

Assume now that $m > 1$. We say that q is locally primitively represented by Q if q is primitively represented by Q over $\mathbb{Z}_p$ for every prime p. In a recent breakthrough, Einsiedler, Lindenstrauss, Mohammadi, and the author [5] proved, assuming $n - m \geq 3$, that q is primitively represented by Q if $\min_{0 \neq x \in \mathbb{Z}^m} q(x)$ is sufficiently large (in an effective sense) and q is locally primitively represented by Q. We refer to Lindenstrauss' report on this topic in this issue. The main ingredient is an effective equidistribution results for semisimple adelic periods. In particular, these results are inapplicable when $n - m = 2$.

In work with W. Kim and P. Yang [8, 9], we prove the following:

Theorem 1. *Let p_1, p_2 be two distinct odd primes and let Q be a positive definite quaternary integral quadratic form. Then there exists $C = C(p_1, p_2, Q) > 1$ with the following property.*

Let q be a primitive binary integral quadratic form such that $-\det(q)\det(Q)$ is a non-zero square modulo p_1 and p_2. If q is primitively represented by the spin genus of Q and $\min_{0 \neq x \in \mathbb{Z}^2} q(x) \geq C$ then q is primitively represented by Q.

If e.g. $\det(Q)$ is a cube-free integer, then q is primitively represented by the spin genus of Q if and only if q is locally primitively represented by Q.

Our methods are necessarily starkly different from the approach in [5]. We are strongly inspired by Linnik's ergodic method [10] and rely on a classification result of Einsiedler and Lindenstrauss [4] for measures invariant and ergodic under higher-rank diagonalizable actions on arithmetic quotients of products of SL(2)'s. The result in [4] and the two Linnik-type splitting assumptions in Theorem 1 reduce our main theorem to a concrete counting problem: For an odd prime p estimate for some n

$$\#\left\{(v_1,v_2,w_1,w_2) : \substack{Q(xv_1+yv_2)=Q(xw_1+yw_2)=q(x,y),\, \langle v_1,v_2\rangle\neq\langle w_1,w_2\rangle,\\ k.v_i\equiv w_i \bmod p^n \text{ for an int. rot. } k \text{ in } \langle v_1,v_2\rangle}\right\}. \tag{1}$$

Here, the 'trivial' bound amounts to roughly $\det(q)p^{-2n}$ and any improvement on the exponent of p yields the main theorem. To obtain this improvement, we apply a corollary of the Siegel mass formula which implies that it suffices to count the number of possible inner products $x_{ij} = \langle v_i, w_j\rangle$ arising from (1). Congruence relations mod powers of p obtained from (1) can be seen to yield the trivial bound. In addition to these, we utilize the following simple observation: the determinant of the quadratic form $Q(x_1v_1 + x_2v_2 + x_3w_1 + x_4w_2)$ is the index $y = [\mathbb{Z}^4 : \mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}w_1 + \mathbb{Z}w_2]$ *squared* times the determinant of Q. This is an equation of the shape $\det(Q)y^2 = f(x_{ij})$ where f has degree four (and is explicit). We then obtain the improvement upon the trivial bound to (1) through the determinant method of Bombieri, Pila [1] and Heath-Brown [7].

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Special values of automorphic functions on the hyperbolic 3-space

Özlem Imamoḡlu
(joint work with S. Herrero, A-M von Pippich, M. Schwagenscheidt)

In a seminal paper, Gross and Zagier [4] related the central values of derivatives of L-functions of elliptic curves to heights of Heegner points on modular curves. Moreover, they found a striking factorization of the norms of differences of singular moduli [5]. The proof uses the fact that the function $\log|j(\tau_1) - j(\tau_2)|$ is essentially given by the constant term at $s = 1$ of the automorphic Green's function $G_s^{\mathrm{SL}_2(\mathbb{Z})}(\tau_1, \tau_2)$ on the modular curve $X(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. At the end of [4], Gross and Zagier made a deep conjecture about the algebraicity properties of the values of the level $N \geq 1$ Green's function $G_s^{\Gamma_0(N)}(\tau_1, \tau_2)$ at positive integer values $s = k \geq 2$ and CM points τ_1, τ_2. They predicted that (under some technical conditions) the values at CM points τ_1, τ_2 of the "higher Green's function" $G_k^{\Gamma_0(N)}(\tau_1, \tau_2)$ are essentially given by logarithms of absolute values of algebraic numbers. There has been a lot of interesting work on this conjecture over the last years (see [2, 7, 9, 10, 11]). Major steps in this direction were taken by Bruinier, Ehlen, and Yang in [1], and by Li in [8] who proved the conjecture in the case of level 1 and when one of the discriminants is fundamental. The general case has been solved recently by Bruinier, Li, and Yang [2].

In this talk I presented recent work where we study an analogous problem for Green's functions on the hyperbolic 3-space $\mathbb{H}^3$. A major difference in this case is the fact that $\mathbb{H}^3$ does not have a complex structure. In particular, the theory of complex multiplication, which played an important role in the proof of the Gross–Zagier algebraicity conjecture for the individual values of Green's functions over modular curves is not available on $\mathbb{H}^3$. Hence, it is not clear whether the special values of Green's functions on $\mathbb{H}^3$ still have good algebraic properties. In order to obtain a convenient algebraicity result, we study an averaged version of the problem that can be tackled successfully without the CM theory. This was motivated by the work of Gross, Kohnen and Zagier [3] who proved an averaged version of the algebraicity conjecture over modular curves by summing τ_1 and τ_2 over all classes of CM points of fixed discriminants d_1 and d_2. They showed that these "double traces" are essentially given by logarithms of rational numbers. Inspired by this result we consider the double traces of Green's functions on hyperbolic 3-space. We show that they are given by algebraic linear combinations of logarithms of primes and logarithms of units in real quadratic fields. For the proof of our results we follow ideas of Bruinier, Ehlen, and Yang [1], who proved a partially averaged version of the algebraicity conjecture over modular curves: they fix τ_1 and sum τ_2 over all classes of discriminant d_2.

To describe our results in some more detail. Let $\mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field of discriminant $D < 0$, and let $\mathcal{O}_D$ be its ring of integers. The group $\Gamma = \mathrm{PSL}_2(\mathcal{O}_D)$ acts on the hyperbolic 3-space $\mathbb{H}^3 = \{P = z + rj \,:\, z \in \mathbb{C}, r \in \mathbb{R}^+\}$

by

$$\gamma P = (aP+b)(cP+d)^{-1} \quad \text{if } \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

For $P_1, P_2 \in \mathbb{H}^3$ which are not in the same Γ-orbit, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the *automorphic Green's function* for Γ is defined by

$$G_s(P_1, P_2) = \pi \sum_{\gamma \in \Gamma} \varphi_s\big(\cosh(d(P_1, \gamma P_2))\big),$$

with the function $\varphi_s(t) = \big(t + \sqrt{t^2-1}\big)^{-s} (t^2-1)^{-1/2}$.

We want to investigate the algebraic properties of the Green's function $G_s(P_1, P_2)$ at special points $P_1, P_2 \in \mathbb{H}^3$, and positive integer values $s \geq 2$. The special points we consider are defined as follows. For a number m in $\frac{1}{|D|}\mathbb{N}$ (where $\mathbb{N} = \{1, 2, \ldots\}$) we let

$$L_m^+ = \left\{ X = \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \, : \, a, c \in \mathbb{N}, \, b \in \mathfrak{d}_D^{-1}, \, \det(X) = m \right\}$$

be the set of positive definite integral binary hermitian forms of determinant m over $\mathbb{Q}(\sqrt{D})$. Here $\mathfrak{d}_D^{-1}$ denotes the inverse different of $\mathbb{Q}(\sqrt{D})$. The group Γ acts on L_m^+ by $\gamma.X = \gamma X \overline{\gamma}^t$, and $\Gamma \backslash L_m^+$ has finitely many classes. To a form $X \in L_m^+$ we associate the *special point*

$$P_X = \frac{b}{c} + \frac{\sqrt{m}}{c} j \in \mathbb{H}^3.$$

We may view it as an analogue of a CM point on $\mathbb{H}^3$. Moreover, we define the m-th *trace* of the Green's function by

$$\mathrm{tr}_m(G_s(\,\cdot\,, P)) = \sum_{X \in \Gamma \backslash L_m^+} \frac{1}{|\Gamma_X|} G_s(P_X, P),$$

If we also take the trace in the second variable, we obtain the *double trace*

$$\mathrm{tr}_m \mathrm{tr}_{m'}(G_s) = \sum_{\substack{X \in \Gamma \backslash L_m^+ \\ Y \in \Gamma \backslash L_{m'}^+}} \frac{1}{|\Gamma_X|} \frac{1}{|\Gamma_Y|} G_s(P_X, P_Y).$$

We assume that m and m' are chosen in such a way that we do not evaluate the Green's function at a singularity. Let $D = -4, D = -8$, or $D = -\ell$ for an odd prime $\ell \equiv 3 \pmod 4$. For simplicity we will assume that for an odd positive integer k the space $S_k^+(\Gamma_0(|D|), \chi_D)$ of cusp forms $f = \sum_{n>0} a_f(n) q^n$ of weight k for $\Gamma_0(|D|)$ and character $\chi_D = \left(\frac{D}{\cdot}\right)$ satisfying the "plus space" condition $a_f(n) = 0$ if $\chi_D(n) = -1$ is trivial. We then have the following theorem which is a special case one of the main results in [6].

Theorem 1. *Let $D < 0$ be a prime discriminant and let $n \geq 1$ be a natural number such that $S_{1+2n}^+(\Gamma_0(|D|), \chi_D) = \{0\}$. Let $m, m' \in \frac{1}{|D|}\mathbb{N}$ such that mm' is* *not a rational square. Then the double trace*

$$\frac{1}{\sqrt{|D|mm'}}\mathrm{tr}_m \mathrm{tr}_{m'}(G_{2n})$$

of the Green's function at positive even $s = 2n$ is a rational linear combination of $\log(p)$ *for some primes p and of* $\log(\varepsilon_\Delta)/\sqrt{\Delta}$ *for some fundamental discriminants* $\Delta > 0$*, where ε_Δ denotes the smallest totally positive unit > 1 in* $\mathbb{Q}(\sqrt{\Delta})$*.*

We also compute analogues of twisted traces of CM values of harmonic modular functions on hyperbolic 3-space and show that they are essentially given by Fourier coefficients of the j-invariant. From this we deduce that the twisted traces of these harmonic modular functions are integers.

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Cycle integrals of meromorphic Hilbert modular forms

Claudia Alfes

(joint work with Baptiste Depouilly, Paul Kiefer, Markus Schwagenscheidt)

We present results on cycle integrals of meromorphic Hilbert modular forms. These forms are meromorphic analogues of the Hilbert cusp forms introduced by Zagier in his work on modular forms associated to real quadratic fields [10]. The main result is a rationality theorem for traces of cycle integrals along certain real-analytic cycles on Hilbert modular surfaces.

1. Background and motivation

Let F be a real quadratic field with ring of integers $\mathcal{O}_F$, discriminant $D > 0$, different $\mathfrak{d}_F = (\sqrt{D})$, and norm $N(\nu) = \nu\nu'$. For even integers $k \geq 4$ and $m > 0$, Zagier considered the functions

$$\omega_m^{\mathrm{cusp}}(z_1, z_2) = \sum_{\substack{a,b\in\mathbb{Z},\, \nu\in\mathfrak{d}_F^{-1} \\ N(\nu)-ab=m/D}} \frac{1}{(az_1z_2 + \nu z_1 + \nu' z_2 + b)^k}, \qquad (z_1, z_2) \in \mathbb{H}^2. \tag{1}$$

Zagier showed in [10] that the functions ω_m^{cusp} are Hilbert cusp forms of parallel weight k for $\mathrm{SL}_2(\mathcal{O}_F)$. Moreover, their generating series is a kernel function for the Doi–Naganuma correspondence between elliptic modular forms of weight k for $\Gamma_0(D)$ with Nebentypus $\chi_D = \left(\frac{D}{\cdot}\right)$ and Hilbert modular forms of parallel weight k for $\mathrm{SL}_2(\mathcal{O}_F)$.

If m is allowed to be negative in (1), one obtains meromorphic Hilbert modular forms. We write ω_m^{cusp} for $m > 0$ and ω_n^{mero} for $n < 0$. The goal of this work is to study the cycle integrals of ω_n^{mero} and, in particular, their rationality properties.

This question is motivated by the analogous one-dimensional situation. Zagier also introduced the elliptic modular forms

$$f_{k,d}(z) = \sum_{\substack{a,b,c\in\mathbb{Z} \\ b^2-4ac=d}} \frac{1}{(az^2 + bz + c)^k}.$$

For $d > 0$, their generating series is the kernel for the Shimura–Shintani correspondence [6], and the rationality of the corresponding cycle integrals is due to Kohnen–Zagier [7] and Katok [5]. For $d < 0$, the functions $f_{k,d}$ are meromorphic, and rationality results for their cycle integrals were obtained more recently in [3, 2, 8, 9]. Our work may be viewed as a Hilbert modular analogue of these meromorphic results.

2. Preliminaries

We consider the lattice

$$L = \left\{ \begin{pmatrix} a & \nu' \\ \nu & b \end{pmatrix} : a, b \in \mathbb{Z},\, \nu \in \mathcal{O}_F \right\}, \qquad Q(X) = -\det(X).$$

The dual lattice L' consists of matrices of the same shape with $\nu \in \mathfrak{d}_F^{-1}$. For $m \in \mathbb{Z}$ put

$$L_m = \{X \in L' : Q(X) = m/D\}.$$

The group $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ acts on L by $\gamma.X = \gamma X \gamma'^t$, and for $m \neq 0$ it acts on L_m with finitely many orbits. For $Z = (z_1, z_2) \in \mathbb{H}^2$, $z_j = x_j + iy_j$, and $X = \left(\begin{smallmatrix} a & \nu' \\ \nu & b \end{smallmatrix}\right) \in L'$, define

$$q_Z(X) = -bz_1z_2 + \nu z_1 + \nu' z_2 - a, \qquad p_Z(X) = \frac{1}{y_1}\left(-b\overline{z}_1 z_2 + \nu\overline{z}_1 + \nu' z_2 - a\right).$$

For $X \in L_n$ with $n < 0$ we consider

$$T_n = \bigcup_{X \in L_n} T_X, \qquad T_X = \{Z \in \mathbb{H}^2 : q_Z(X) = 0\}.$$

These are algebraic cycles, also called Hirzebruch–Zagier cycles.

For $Y \in L'$ with $Q(Y) > 0$ we consider the real-analytic cycle

$$C_Y = \{Z \in \mathbb{H}^2 : p_Z(Y) = 0\},$$

which is real two-dimensional.

We now define the meromorphic analogues of the Hilbert modular forms considered by Zagier. For $n < 0$, we let

$$\omega_n^{\mathrm{mero}}(Z) = \sum_{X \in L_n} q_Z(X)^{-k}.$$

This defines a Hilbert modular form of parallel weight k that vanishes at the cusps and has poles along the algebraic cycles T_n.

If G is a Hilbert modular form of weight k for Γ, we define, for $m > 0$, its *m-th trace of cycle integrals* by

$$\mathrm{tr}_m(G) = \sum_{Y \in \Gamma \backslash L_m} \int_{\Gamma_Y \backslash C_Y} G(Z) q_Z(Y)^{k-2}\, dz_1 dz_2, \tag{2}$$

where Γ_Y denotes the stabilizer of Y in Γ and $q_Z(Y)^{k-2}\, dz_1 dz_2$ is the natural differential form making the integrand invariant.

For $G = \omega_n^{\mathrm{mero}}$ we assume that the cycle T_n does not intersect the cycles C_Y occurring in (2).

3. Main results

Theorem 1 (Alfes–Depouilly–Kiefer–Schwagenscheidt [1])**.** *Let $k \geq 4$ be even, let $m > 0$, and let $n < 0$. Suppose that the discriminant D of F is an odd prime and that the space of cusp forms $S_k^+(\Gamma_0(D), \chi_D)$ is trivial. Moreover, assume that T_n does not intersect any of the cycles C_Y with $Y \in L_m$. Then*

$$\mathrm{tr}_m(\omega_n^{\mathrm{mero}}) = \sum_{Y \in \Gamma \backslash L_m} \int_{\Gamma_Y \backslash C_Y} \omega_n^{\mathrm{mero}}(Z) q_Z(Y)^{k-2}\, dz_1 dz_2$$

is a rational multiple of πi.

In [1] we present a more general statement without these simplifying hypothesis. We briefly describe the strategy of the proof:

1. We evaluate the Petersson inner product of ω_n^{mero} and ω_m^{cusp}. We show that on the one hand it vanishes and on the other hand it is given by the sum of $\mathrm{tr}_m(\omega_n^{\mathrm{mero}})$ and a sum of the preimage Ω_m^{cusp} of ω_m^{cusp} under a certain differential operator evaluated at the algebraic cycles T_X. This gives an equality between the cycle integral $\mathrm{tr}_m(\omega_n^{\mathrm{mero}})$ and the second expression.
2. We rewrite the second expression involving Ω_m^{cusp} as a (novel) theta lifting.

(3) By splitting the kernel function of this theta lifting along the algebraic cycles we can interpret the second expression as a regularized Petersson inner product involving harmonic Maass forms and unary theta functions.
(4) Stokes' theorem, in the spirit of Bruinier–Ehlen–Yang [4], then reduces the expression to Fourier coefficients of harmonic weak Maass forms and unary theta series, whose algebraicity implies the rationality statement.

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Shears and moments of L-functions in higher rank

Paul D. Nelson
(joint work with Subhajit Jana)

Subconvex bounds for L-functions and equidistribution results for arithmetic objects have long been known to be intertwined – in many interesting cases, each can be deduced from the other. In one direction, the Duke–Linnik problems concerning equidistribution for ternary quadratic forms reduce to subconvex bounds for GL_1 twists of GL_2 L-functions, and effective forms of the Rudnick–Sarnak arithmetic quantum unique ergodicity conjecture reduce to subconvex bounds for GL_3 twists of GL_2 L-functions. In the other direction, the works of Michel and Venkatesh establish subconvex bounds using quantitative equidistribution as an input. In work in progress with Subhajit Jana, we aim to reduce many cases of the subconvexity problem (e.g., for GL_m twists of GL_n L-functions) to a conjecture of "equidistribution" flavor.

We recall that the subconvexity problem asks for power-saving improvements over the convexity bound for automorphic L-functions at the center of the critical strip. For the sake of focus and simplicity, we restrict to the case of full level over $\mathbb{Q}$. Let π be a unitary cuspidal automorphic representation of GL_n over $\mathbb{Q}$, with archimedean parameters $\mu_1, \dots, \mu_n$. The convexity bound gives $L(\frac{1}{2}, \pi) \ll_\varepsilon C(\pi)^{1/4+\varepsilon}$, where $C(\pi) := \prod_j (1 + |\mu_j|)$ is the analytic conductor. It is a fundamental open problem to replace the exponent $1/4$ by $1/4 - \delta$ for some $\delta > 0$. This has been achieved for $n \leq 2$ by Michel–Venkatesh [3], and for $n \geq 3$ under the *uniform parameter growth* hypothesis $|\mu_j| \asymp |\mu_k|$ in [4, 5]. What is still missing is a method that works for $n \geq 3$ without imposing such uniformity. Extending subconvexity beyond the uniform parameter growth case would have interesting applications, including a polynomially effective form of arithmetic quantum unique ergodicity.

Our approach is modeled on the method of Michel–Venkatesh [3] for subconvexity for GL_1 L-functions (and their fixed GL_2-twists). A key idea in their work is to reduce subconvexity to a problem of equidistribution. One takes an automorphic form φ_0 on GL_2 and translates it by the unipotent element

$$u_T = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix},$$

then studies the L^2-norm of the restriction of the resulting translate to GL_1. Via Parseval's identity, this yields a weighted fourth moment of GL_1 L-functions concentrated on conductors in a dyadic range determined by T. An asymptotic formula for this moment, combined with amplification, delivers a subconvex bound.

We work over a number field F with adele ring $\mathbb{A}$, and write $[G] := G(F)\backslash G(\mathbb{A})$. We use the Rankin–Selberg integral representation for $\mathrm{GL}_m \times \mathrm{GL}_n$ $(n < m)$ due to Jacquet–Piatetski-Shapiro–Shalika [1]. We embed $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_m$ in the upper-left block. Given $\varphi \in \pi$ on GL_m and $\Psi \in \sigma$ on GL_n, one has

$$\int_{[\mathrm{GL}_n]} P\varphi \cdot \Psi = L(\tfrac{1}{2}, \pi \times \sigma) \int W_\varphi W_\Psi, \tag{1}$$

where P denotes a certain projection operator involving integration over a unipotent radical. We generalize the Michel–Venkatesh construction to this setting. We fix an automorphic form φ_0 on GL_m and form its translate $\varphi_0(\bullet\, u_Q)$ by a unipotent element u_Q, where $Q \in \mathbb{A}$ is a large parameter and u_Q is supported in a single off-diagonal entry in the "middle" of the matrix. For example, when $(m, n) = (4, 2)$,

$$u_Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & Q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We then study the L^2-norm of the restriction of $P(\varphi_0(\sqrt{Q} \bullet u_Q))$ to $[\mathrm{GL}_n]$. As in the $n = 1$ case, this L^2-norm gives a weighted moment of $\mathrm{GL}_m \times \mathrm{GL}_n$ L-functions. A key lemma is that the weights in this moment are bounded from below by their average size over some dyadic range.

Let P_{n+1} denote the mirabolic subgroup of GL_{n+1}, consisting of matrices whose bottom row is $(0, \ldots, 0, 1)$. For example, when $(m, n) = (4, 2)$,

$$P_3 = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We formulate the following:

Conjecture 1 ($\mathcal{C}(m,n)$)**.** *For $\varphi \in C_c^\infty([\mathrm{GL}_m])$ and $|Q|_\mathfrak{p} \geq 1$ at each place $\mathfrak{p}$,*

$$\int_{[\mathrm{GL}_n]} |P\varphi(\sqrt{Q}\, g\, u_Q)|^2 \, dg = \int_{[P_{n+1}]} |P\varphi(\sqrt{Q}\, g\, u_Q)|^2 \, dg + \mathrm{O}(|Q|^{-\delta}). \tag{2}$$

Since $u_Q \in P_{n+1}(\mathbb{A})$, the shift may be absorbed on the right-hand side; the first term then turns out to be a reasonably explicit main term. The conjecture thus asserts that translates $\sqrt{Q}\,[\mathrm{GL}_n]\, u_Q$ become equidistributed inside $[P_{n+1}]$ when tested against $|P\varphi|^2$. In the case $(m, n) = (2, 1)$, this equidistribution is established in [3] (see also [2]).

This conjecture has the following consequence:

Theorem 1 (Jana–N., work in progress)**.** *Assume $\mathcal{C}(2n, n)$ for all n. Then subconvexity holds for the standard L-function on GL_n and for Rankin–Selberg L-functions with one factor fixed, at least in the depth aspect where ramification is restricted to a fixed finite set of places.*

Subconvexity for $\mathrm{GL}_m \times \mathrm{GL}_n$ with the GL_n factor fixed follows in the case $m = 2n$ from the conjecture via the amplification method, as in the work of Michel–Venkatesh. The general case reduces to this one by considering suitable Eisenstein series. The main evidence for $\mathcal{C}(m, n)$ is that it likely follows from standard conjectures for moments of L-functions in families. One perspective on the difficulty in verifying it when $m - n \geq 2$ is that the projected automorphic forms $P\varphi$ are automorphic with respect to P_{n+1} but not with respect to GL_{n+1}, so the standard spectral expansion over GL_{n+1} does not apply.

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Asymptotic properties of representations of p-adic GL_N

Mathilde Gerbelli-Gauthier
(joint work with Rahul Dalal, Simon Marshall)

Let F_v be a p-adic field with maximal ideal $\mathfrak{p}$, and π_v a smooth irreducible representation of $GL_N(F_v)$. We consider two asymptotic invariants of π_v:

- The rate of growth of $\dim \pi_v^{K(\mathfrak{p}^i)}$ for $K(\mathfrak{p}^i)$ the principal congruence subgroup of level $\mathfrak{p}^i$,
- The rate of decay of matrix coefficients of π_v.

Our motivation is global: we think of $GL_N(F_v)$ as $G(F_v)$ for a reductive group G over a global field F of which v is a place. The group G could be GL_N, but also $U_{N,E/F}$ for E/F a quadratic extension such that v splits in E. In this case, the naïve Ramanujan conjecture predicts that if π is a cuspidal automorphic representation of G, then all local factors of π are tempered.

- When $G = GL_N$, the naïve Ramanujan conjecture is unknown in general.
- When $G = U_N$, the conjecture is known to be false, starting with counterexamples constructed by Howe and Piatetski-Shapiro [10].

In this context, the Sarnak-Xue Density Hypothesis (SXDH) can be understood as either a fix or an approximation sufficiently good for many applications. It predicts that automorphic representations with non-tempered local factors should be rare, and that the rarity of representations with a factor π_w at a place w is governed by

$$p(\pi_w) = \inf\{p \geq 2 : \pi_w \hookrightarrow L^{p+\epsilon}(G(F_w)),\ \forall \epsilon > 0\}.$$

There has been much work on the Sarnak-Xue Density Hypothesis. Some results from the last decade include [8, 16, 15, 6, 2, 9, 7, 3, 4, 14, 11].

Our motivation, as was the case initially for Sarnak-Xue [19], is the question of growth of cohomology of arithmetic groups, in which one takes w to be an archimedean place and π_w a cohomological representation. A sample result in this direction is:

Theorem 1 (Dalal–G-G., [6]). *Let $G = U_{N,E/F}$ with E/F a CM extension. Let π_0 be a cohomological unitary irrep of $G_\infty = G(F_\infty)$, $\mathfrak{p}$ a prime of F that splits in E, v the corresponding place, and $K(\mathfrak{p}^i) \subset G(\mathcal{O}_F)$ the principal congruence subgroup. Then for a suitable Haar measure μ on $G(\mathbb{A}_F)$, we have*

$$\dim \mathrm{Hom}_{G_\infty}(\pi_0, L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F)/K(\mathfrak{p}^i))) \ll_\epsilon \mu(K(\mathfrak{p}^i))^{1-2/p(\pi_0)+\epsilon}.$$

In particular, the SXDH holds for these representations.

The proof of the theorem relies on the endoscopic classification of representations [1, 17, 12] and in particular on the stabilization of the trace formula. An alternative approach would be to split the count into a local and a global component: count first automorphic representations with a fixed $K(\mathfrak{p}^i)$ vector, and then, for each of these, estimate the dimension of the $K(\mathfrak{p}^i)$-fixed vectors.

It is known that

$$\dim \pi_v^{K(\mathfrak{p}^i)} \asymp |\mathfrak{p}|^{i\, d_{GK}(\pi_v)} \tag{1}$$

where $d_{GK}(\pi_v)$ is the Gelfand-Kirillov dimension of π_v, equal to half the dimension of the wavefront set of π_v, and related to its failure to be generic [18]. It follows from the general theory of endoscopy that if π is an automorphic representation of U_N, a large value of $p(\pi_\infty)$ for cohomological π_∞ forces a small $d_{GK}(\pi_v)$ for split finite v. However, the coefficients implicit in (1) depend on π_v, and for global applications to the SXDH, it is desirable to bound them uniformly. This is our first theorem:

Theorem 2 (Dalal-G-G–Marshall, [5])**.** *Let $\epsilon > 0$. There exists $C_\epsilon = C_{\epsilon,N,F_v}$ such that for any smooth irreducible representation π_v of $GL_N(F_v)$, we have*

$$\dim \pi_v^{K(\mathfrak{p}^i)} \leq C_\epsilon |\mathfrak{p}|^{i\cdot d_{GK}(\pi_v)+\epsilon}.$$

The proof is inductive, via Zelevinsky's classification of irreducible representations of GL_N in terms of multisegments. The result for cuspidals, already present in [16], uses a result of Lapid [13] on support of functions in the Whittaker model to reduce the problem to a double coset count. The result is then extended to Speh representations via the theory of degenerate Whittaker models, and finally to general irreducible representations.

To recap: in a possible global application, the failure of temperedness of a local representation at infinity would imply, via the endoscopic classification, a bound on the Gelfand-Kirillov dimension of the other local factors π_v, and we have now promoted this bound to a uniform bound on the dimension of fixed vectors.

A natural question arises: does the failure of temperedness of π_v influence the growth of fixed vectors of π_v directly? We give a quantitative answer:

Theorem 3 (Dalal-G-G–Marshall, [5])**.** *Let π_v be a smooth, unitarizable, irreducible representation of $GL_N(F_v)$. Then*

$$1 - \frac{2d_{GK}(\pi_v)}{N(N-1)} \leq 1 - \frac{2}{p(\pi_v)} \leq 1 - \frac{2d_{GK}(\pi_v)}{N(N-1)} + \frac{2}{N}.$$

At the extreme values of $d_{GK}(\pi_v) = \frac{N(N-1)}{2}$ and $d_{GK}(\pi_v) = 0$, we recover the well-known statements that tempered representations of GL_N are generic (but not the converse), and that the matrix coefficients of one-dimensional representations are not in $L^p(GL_N(F_v))$ for any finite p. The novelty of the result is, as was the case for the SXDH, in the intermediate cases, and we view this result as a local analogue of the SXDH.

The proof of this result goes through Tadić's classification of the unitary dual. We use it to attach to every representation an "Arthur-SL_2", and we compute both invariants $d_{GK}(\pi_v)$ and $p(\pi_v)$ explicitly in terms of the Arthur SL_2. We then compare these quantities explicitly to derive the theorem.

It would be interesting, and facilitate the extension of the result to other groups, to have a more direct proof of this that does not depend on a classification of unitary representations of GL_N.

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Revisiting Sarnak's density hypothesis

Edgar Assing

Let $G = \mathrm{SL}_n(\mathbb{R})$ and write $\Pi(G)$ for its unitary dual. This space is equipped with the Plancherel measure μ_{Pl}, whose support is the so called tempered dual denoted by $\Pi_{\mathrm{temp}}(G)$. Further, denote by $\Pi_{\mathrm{sph}}(G) \subseteq \Pi(G)$ the subset of spherical representations.

Given $\pi \in \Pi(G)$ we associate an invariant $\mu(\pi)$, which we simply call the spectral parameter. For the purposes of this note it is sufficient to treat this as a n-tuple of complex numbers $\mu(\pi) = (\mu_1(\pi), \dots, \mu_n(\pi)) \in \mathbb{C}$. We refer to [3, Section 2.3] as well as [15, Section 7.2] for precise definitions and content ourselves with the following remarks:

- For $\pi \in \Pi(G)$ one attaches the local L-factor $L(\pi, s) = \prod_{j=1}^{n} \Gamma_{\mathbb{R}}(s + t_j(\pi))$ (with $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)$). After re-arranging the product, if necessary, one has $\mu_i(\pi) = t_i(\pi) + O(1)$ for $1 \le i \le n$.
- Let $\pi_{\mathrm{triv}} \in \Pi(G)$ denote the trivial representation. Then we have
$$\mu(\pi_{\mathrm{triv}}) = \left(-\frac{n-1}{2}, \dots, \frac{n-1}{2}\right).$$
- If $\pi \in \Pi_{\mathrm{sph}}(G)$, then $\mu(\pi)$ agrees with the classical spectral parameter as used in [1, 2] for example. In particular, for $\pi \in \Pi_{\mathrm{sph}}(G)$, we have the important equivalence $\pi \in \Pi_{\mathrm{temp}}(G)$ if and only if $\Re(\mu(\pi)) = 0$.

The preceding discussion leads to the definition of the following invariant, which turns out to be a convenient measure for non-temperedness.

Definition. *For $\pi \in \Pi_{sph}(\Pi)$ we define*
$$\sigma(\pi) = \max_{i=1,\dots,n} \Re(\mu_i(\pi)).$$

We turn towards the global picture and let F be a real quadratic field with ring of integers $\mathcal{O}$. Given an ideal $\mathfrak{q} \subseteq \mathcal{O}$ we define
$$\Gamma(\mathfrak{q}) = \ker(\mathrm{SL}_n(\mathcal{O}) \to \mathrm{SL}_n(\mathcal{O}/\mathfrak{q})).$$
This defines a discrete subgroup of $G \times G$. Let $R_{\mathfrak{q}}$ denote the right regular representation of $G \times G$ on $L^2(\Gamma(\mathfrak{q})\backslash G \times G)$. We have the sequence of (stable) subspaces
$$L^2_{\mathrm{cusp}}(\Gamma(\mathfrak{q})\backslash G \times G) \subseteq L^2_{\mathrm{disc}}(\Gamma(\mathfrak{q})\backslash G \times G) \subseteq L^2(\Gamma(\mathfrak{q})\backslash G \times G)$$
and we write $R_{\mathfrak{q},\mathrm{cusp}}$ and $R_{\mathfrak{q},\mathrm{disc}}$ for the corresponding restrictions of $R_{\mathfrak{q}}$. Given $\pi_1, \pi_2 \in \Pi(G)$, we define the multiplicity
$$m_{\mathfrak{q},\mathrm{cusp}}(\pi_1 \otimes \pi_2) = \dim \mathrm{Hom}_{G\times G}(\pi_1 \otimes \pi_2, R_{\mathfrak{q},\mathrm{cusp}}).$$

A consequence of the generalized Ramanujan conjecture (GRC) is the following (natural) extension of Selberg's eigenvalue conjecture.

Conjecture (Generalized Selberg Eigenvalue Conjecture)**.** *Let $\pi_1 \in \Pi(G)$. If $\max_{\mathfrak{q},cusp}(\pi_1 \otimes \pi_2) \neq 0$ for some $\pi_2 \in \Pi(G)$ and some $\mathfrak{q} \subseteq \mathcal{O}$, then $\pi_1 \in \Pi_{temp}(G)$.*

While this conjecture appears to be out of reach of current technology, much work has been done to establish good approximations. One approach is to establish bounds of the form
$$\sup\{\sigma(\pi_1)\colon \pi_1 \in \Pi(G) \text{ for } m_{\mathfrak{q},\mathrm{cusp}}(\pi_1 \otimes \pi_2) \neq 0 \text{ for some } \mathfrak{q} \subseteq \mathcal{O},\ \pi_2 \in \Pi(G)\} \le \theta,$$
for some (hopefully) small $\theta \ge 0$. For general n the best known bound is still $\theta \le \frac{1}{2} - \frac{1}{n^2+1}$ established in [12]. We refer to [5] and the references within for more discussion.

Another approach, spelled out by Sarnak in his 1990 ICM address [17], is to establish that the contribution of highly non-tempered representations is negligible. To make this precise we first recall the following version of Weyl's law

$$\sum_{\substack{\pi_1\in\Pi_{\mathrm{sph}(G)},\\ \|\mu(\pi_1)\|\leq M,\\ \pi_2\in\Pi(G),\\ \|\mu(\pi_2)\|\leq X}} X^{\frac{n(n-1)}{2}}\cdot m_{\mathfrak{q},\mathrm{cusp}}(\pi_1\otimes\pi_2)\asymp_{M,F,n} X^{n^2-1}[\Gamma(1)\colon\Gamma(\mathfrak{q})],$$

which can be compared to the results in [14]. It is a consequence of the limit multiplicity property, which was established in a series of works culminating in [8], that non-tempered representations appear rarely in the cuspidal spectrum. More precisely, we have

$$\sum_{\substack{\pi_1\in\Pi_{\mathrm{sph}(G)},\\ \|\mu(\pi_1)\|\leq M,\,\sigma(\pi_1)\geq\sigma,\\ \pi_2\in\Pi(G),\\ \|\mu(\pi_2)\|\leq X}} X^{\frac{n(n-1)}{2}}\cdot m_{\mathfrak{q},\mathrm{cusp}}(\pi_1\otimes\pi_2)=o_{X,M,F,n}([\Gamma(1)\colon\Gamma(\mathfrak{q})])$$

for $\sigma>0$ as $\mathrm{Nr}(\mathfrak{q})\to\infty$. Sarnak's philosophy tells us, that we should expect a quantitative version of this estimate, which interpolates between Weyl's law for $\sigma=0$ and multiplicity one (in the discrete spectrum) for the trivial representation for $\sigma=\frac{n-1}{2}$, to hold. The following result combines the results from [2] and [3] establishing a hybrid sub-convex version of Sarnak's density hypothesis in the setting introduced above.

Theorem (A.-Blomer-Nelson & A.-Jana). *Let $\epsilon,M>0$ and $X\geq 1$ as well as $\mathfrak{q}\subseteq\mathcal{O}$. Then, for $\sigma\geq 0$, we have*

$$\sum_{\substack{\pi_1\in\Pi_{sph(G)},\\ \|\mu(\pi_1)\|\leq M,\,\sigma(\pi_1)\geq\sigma,\\ \pi_2\in\Pi(G),\\ \|\mu(\pi_2)\|\leq X}} X^{\frac{n(n-1)}{2}}\cdot m_{\mathfrak{q},cusp}(\pi_1\otimes\pi_2)$$

$$\ll_{M,F,n,\epsilon}(X^{n^2-1}\cdot[\Gamma(1)\colon\Gamma(\mathfrak{q})])^{1-\frac{2}{n-1}\sigma(1+\frac{1}{n+1})+\epsilon}.$$

The set-up for this density theorem was motivated by possible applications to approximation exponents for orbits on homogeneous spaces. Indeed, we expect that this results (or co-compact versions thereof) can be useful in extending [7, Theorem 2.2] to higher rank.

Finally, let us note that Sarnak's density hypothesis in the broad sense has seen a great amount of activity in the past 50 years. Let us mention the following highlights:

- First breakthroughs towards the density hypothesis were made using Selberg's trace formula in [10, 16]. We refer to [9] for a modern version of this approach as well as for very general results in the GL(2) setting.
- Another approach towards the density hypothesis, going back to [11], is to use the Kuznetsov formula. This idea was extended to GL(n) in [4, 1, 2, 3]. It is this approach that underlies the proof of the theorem above.

- Finally, there are other incarnations of the density hypothesis that can be approached using theory of endoscopy (for groups such as $U(n)$). We refer to [13, 6] and the references within for a discussion of this approach.

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Joint Linnik Problems

Farrell Brumley

(joint work with Valentin Blomer, Maksym Radziwiłł)

Let $d > 0$ denote a squarefree integer satisfying $d \not\equiv 7 \bmod 8$. We call such d admissible and associate to each of them the non-empty set

$$R_3(d) = \{x \in \mathbb{Z}^3 : \|x\|^2 = d\},$$

consisting of integral representations of d as the sum of three squares. For every $x \in R_3(d)$ we consider the oriented lattice $x^\perp \cap \mathbb{Z}^3$, equipped either with the restriction of the norm form, if $d \equiv 1, 2 \bmod 4$, or half of that, if $d \equiv 3 \bmod 8$. In either case, this defines an integral binary quadratic form ϕ_x of discriminant

$-D = \mathrm{disc}(\mathbb{Q}(\sqrt{-d}))$. Let $[\phi_x]$ denote its $\mathrm{SL}_2(\mathbb{Z})$-equivalence class. Writing $\mathcal{Q}_{-D}$ for the set of all such equivalence classes, this yields a map

$$\mathrm{Orth}: R_3(d) \to \mathcal{Q}_{-D}, \quad x \mapsto [\phi_x],$$

first considered by Gauss in the Disquisitiones Arithmeticae, and nowadays referred to as the Gauss orthogonal lattice procedure.

We can associate to every $x \in R_3(d)$ its Linnik point on the sphere, obtained by projection $\mathrm{Lin}(x) = d^{-1/2}x \in S^2$. Likewise, to every $[\phi] \in \mathcal{Q}_{-D}$ we can associate its CM point on the modular curve $z_{[\phi]} \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, defined as the $\mathrm{SL}_2(\mathbb{Z})$-orbit of the unique solution to $\phi(z,1) = 0$ in the upper half-plane. The famous Duke equidistribution theorem states that Linnik points on the sphere, as well as CM points on the modular curve, equidistribute according to the respective uniform measures, as $d \to \infty$ along admissible squarefree integers in the first case, and as $-D \to -\infty$ along fundamental discriminants in the second.

The Gauss orthogonal lattice procedure allows one to consider both equidistribution problems at the same time. In joint work with V. Blomer and M. Radziwiłł [4], we show (among other results) that Linnik points on the sphere paired with their orthogonal complement equidistribute simultaneously on the product space $S^2 \times \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, provided that the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ admits enough small split primes.

More precisely, we establish the following result.

Theorem 1 (Blomer–B.–Radziwiłł). *As $d \to \infty$ along admissible squarefree integers, the set of pairs*

$$\{(\mathrm{Lin}(x), z_{\mathrm{Orth}(x)}) : x \in R_3(d)\}$$

equidistributes in $S^2 \times \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ according to the product of the uniform measures, provided $\zeta_{\mathbb{Q}(\sqrt{-d})}(s)$ has no zero in the region $|s-1| = 1/o(\log D)$.

By comparison, in [1], it was shown that the same equidistribution statement holds without the assumption on $\zeta_{\mathbb{Q}(\sqrt{-d})}(s)$ but subject to the Linnik-style condition that $-d$ is a non-zero square modulo two distinct odd primes. Moreover, in earlier work with V. Blomer [2], we proved the equidistribution statement under a variety of stronger assumptions, including the Generalized Riemann Hypothesis for a bevy of higher rank automorphic L-functions, but only when tested against the cuspidal spectrum of $L^2(S^2 \times \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$.

The methods in [4] are inspired by the mollification scheme of Radziwiłł–Soundararajan [5]. Our variant of this allows us to reduce the analysis to two separate instances of the mixing conjecture of Michel–Venkatesh, for self-products of the sphere S^2 and of the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, in easy parameter ranges. These instances were proven unconditionally in [3]. A final step exploits the full strength of the Sym^2, Sym^3, and Sym^4 functorial lifts, due to Gelbart–Jacquet, Kim–Shahidi, and Kim, along with the accompanying cuspidality criteria, to verify that the mollifier contains sufficient arithmetic information to ensure decay of the Weyl sums. This last step is the source of the no-Siegel-zero assumption.

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Sums of Hecke eigenvalues along polynomials and base change for GL(2)

Katharine Woo

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ that satisfies the Ramanujan-Petersson conjecture; let λ_π denote the normalized Hecke eigenvalues of π. Fix an irreducible nonconstant solvable polynomial $P(x) \in \mathbb{Z}[x]$. In this talk, we focus on the following correlation sum:

$$\sum_{n \le X} \lambda_\pi(|P(n)|).$$

While it is expected that a power-saving upper bound should hold for most tuples of (π, P), the only known bounds of such strength exist when $P(x)$ is either linear or quadratic. To gain flexibility in the polynomial argument, we place absolute values around the Hecke eigenvalues. In particular, we show that if the base change π_K to $\mathrm{GL}_2(\mathbb{A}_K)$ is cuspidal, then the following bound holds:

$$\sum_{n \le X} |\lambda_\pi(|P(n)|)| \ll_{\pi,P} X \log(X)^{-0.066}.$$

Further, one can derive explicit dependency on $\|P\|$ (that is, the size of the coefficients of P) in the bound above. This upper bound builds on the work of Elliot, Moreno, and Shahidi in [2]; similar bounds also exist in Chiriac and Yang [1] for non-CM holomorphic cups form coefficients.

As part of the proof of the above result, we demonstrate the following classification of the L^1-norm of certain automorphic representations. Let K be a number field and π an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ that is tempered at all finite places of K. Then there exists a positive constant $c_\pi > 0$ such that

$$\sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ N(\mathfrak{a}) \le X}} |\lambda_\pi(\mathfrak{a})| \ll_\pi X \log(X)^{-c_\pi}$$

if and only if π is cuspidal. Moreover, if π is cuspidal, then there is a universal lower bound $c_\pi > 0.066$.

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Tatuzawa's theorem for Rankin–Selberg L-functions

Gergely Harcos

(joint work with Jesse Thorner)

Establishing zero-free regions for automorphic L-functions is a central problem of number theory. In 2023, we established [7] a new zero-free region for all GL_1-twists of $\mathrm{GL}_n \times \mathrm{GL}_m$ Rankin–Selberg L-functions, generalizing Siegel's celebrated work [25] on Dirichlet L-functions. We report about our recent strengthening of this result [8], which generalizes Tatuzawa's refinement [26] of Siegel's theorem. Throughout, F will be a number field with adele ring $\mathbb{A}_F$. We denote by $\mathfrak{F}_n$ the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_F)$, and for each $\pi \in \mathfrak{F}_n$, we denote by $C(\pi) \geq 3$ the analytic conductor of π as defined in [7, §3.1] and [8, §2.1].

We recall first some classical results for Dirichlet L-functions (corresponding to $n = m = 1$ and $F = \mathbb{Q}$). The standard zero-free region for these L-functions and the rarity of exceptions is due to de la Vallée Poussin [20], Gronwall [6], Landau [15], and Titchmarsh [28]. We state the result with explicit constants, as can be derived from the work of McCurley [18, Theorem 1].

Theorem 1. *If χ is a primitive Dirichlet character, then $L(\sigma + it, \chi)$ has at most one zero (necessarily real and simple) in the region*

$$\sigma \geq 1 - \frac{1}{10 \log(C(\chi)(|t| + 3))}.$$

If the exceptional zero exists, then $\chi^2 = 1$. Moreover, if χ_1 and χ_2 are two distinct primitive Dirichlet characters, then $L(s, \chi_1)L(s, \chi_2)$ has at most one real zero (counted with multiplicity) in the interval

$$\sigma \geq 1 - \frac{1}{10 \log(C(\chi_1)C(\chi_2))}.$$

Concerning the location of the possible exceptional zero, we can derive an elegant result from Tatuzawa's theorem [26, Theorem 1] with the help of Rademacher's convexity bound [21, Theorem 3] and Cauchy's formula for $L'(s, \chi)$.

Theorem 2. *If $\varepsilon \in (0,1)$, then the following holds for all but one primitive quadratic Dirichlet character χ:*

$$L(\sigma,\chi) \neq 0, \qquad \sigma \geq 1 - \frac{\varepsilon^3}{240}C(\chi)^{-\varepsilon}.$$

Zero-free regions have been established for various families of automorphic L-functions [13, 24, 19, 9, 10, 1, 22, 23, 2, 4, 5, 3, 11, 14, 17, 30, 12, 29, 27]. In order to state some of these results, we need further notation. Given $(\pi,\chi) \in \mathfrak{F}_n \times \mathfrak{F}_1$, we denote by $\pi \otimes \chi$ the (unique) element of $\mathfrak{F}_n$ whose space consists of the functions $g \mapsto f(g)\chi(\det g)$, where $g \mapsto f(g)$ is from the space of π. It is straightforward to check that $\pi \otimes \chi$ is isomorphic to the representation $g \mapsto \pi(g)\chi(\det g)$. It is convenient to introduce $\mathfrak{F}_n^*$, the set of those elements of $\mathfrak{F}_n$ whose central character is trivial on the diagonally embedded positive reals. For each $\pi \in \mathfrak{F}_n$, there exists a unique pair $(\pi^*, t_\pi) \in \mathfrak{F}_n^* \times \mathbb{R}$ such that $\pi = \pi^* \otimes |\cdot|^{it_\pi}$.

The next theorem combines some important results of Brumley, Humphries, and Thorner (see [2], [16, Theorem A.1], [3, Theorem A.1], [11, Theorem 2.1]).

Theorem 3. *There exists an effective constant $c_1 = c_1(n,m,[F:\mathbb{Q}]) > 0$ such that if $(\pi,\rho) \in \mathfrak{F}_n^* \times \mathfrak{F}_m^*$, then $L(\sigma+it,\pi\times\rho)$ has no zero in the region*

$$\sigma \geq 1 - c_1(C(\pi)C(\rho))^{-n-m}(|t|+1)^{-nm}.$$

Moreover, if $\pi = \tilde{\pi}$ or $\rho = \tilde{\rho}$ or $\rho = \tilde{\pi}$, then $L(\sigma+it,\pi\times\rho)$ has at most one zero (necessarily real and simple) in the region

$$\sigma \geq 1 - c_1/\log(C(\pi)C(\rho)(|t|+3)).$$

If the exceptional zero exists, then $(\pi,\rho) = (\tilde{\pi},\tilde{\rho})$ or $\rho = \tilde{\pi}$.

Our first new result generalizes Theorem 1, [11, Theorem 1.1], [12, Theorem 8.2].

Theorem 4 ([8, Theorem 1.6])**.** *If $(\pi,\chi) \in \mathfrak{F}_n \times \mathfrak{F}_1^*$, then $L(\sigma+it,\pi\times(\tilde{\pi}\otimes\chi))$ has at most one zero (necessarily real and simple) in the region*

$$\sigma \geq 1 - \frac{1}{903\log(C(\pi)^{2n}C(\chi)^{n^2}(|t|+3)^{n^2[F:\mathbb{Q}]})}.$$

If the exceptional zero exists, then $\pi\otimes\chi^2 = \pi$. Moreover, if $(\pi,\chi_1,\chi_2) \in \mathfrak{F}_n \times \mathfrak{F}_1^* \times \mathfrak{F}_1^*$ satisfies $\pi\otimes\chi_1 \neq \pi\otimes\chi_2$, then $L(s,\pi\times(\tilde{\pi}\otimes\chi_1))L(s,\pi\times(\tilde{\pi}\otimes\chi_2))$ has at most one real zero (counted with multiplicity) in the interval

$$\sigma \geq 1 - \frac{1}{158\log(C(\pi)^{4n}C(\chi_1)^{n^2}C(\chi_2)^{n^2})}.$$

Our second new result generalizes Theorem 2 and refines the zero-free region of [7, Theorem 1.1].

Theorem 5 ([8, Theorem 1.1])**.** *Let $(\pi,\rho,\chi) \in \mathfrak{F}_n \times \mathfrak{F}_m \times \mathfrak{F}_1$ and $\varepsilon > 0$. There exist an effective constant $c_2 = c_2(n,m,[F:\mathbb{Q}],\varepsilon) > 0$ and a character $\psi = \psi_{\pi,\rho,\varepsilon} \in \mathfrak{F}_1$ such that if $L(s,\pi\times(\rho\otimes\chi))$ differs from $L(s,\pi\times(\rho\otimes\psi))$, then*

$$L(\sigma,\pi\times(\rho\otimes\chi)) \neq 0, \qquad \sigma \geq 1 - c_2(C(\pi)C(\rho)C(\chi))^{-\varepsilon}.$$

Moreover, $L(s,\pi\times(\rho\otimes\psi))$ *has at most one real zero (necessarily simple) in the interval* $\sigma\geq 1-c_2(C(\pi)C(\rho)C(\psi))^{-\varepsilon}$.

The proofs of Theorems 4 and 5 are based on the zero repulsion technique of Hoffstein–Lockhart [9, Lemma 3.3] and Goldfeld–Hoffstein–Lieman [9, Appendix, Lemma]. Interestingly, Theorem 4 is used in the proof of Theorem 5.

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Thin subgroup, partial coverings and equidistribution

Asbjørn Nordentoft

(joint work with Peter Humphries, Ser Peow Tan, Anders Södergren)

The overall goals of the line of work presented in this talk is to get a more geometric understanding of an extremely rich paper of Duke, Imamoḡlu, and Tóth [1]. The setting is as follows: Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ denote the modular group and $Y_0(1) := \Gamma\backslash\mathbb{H}$ the modular curve equipped with its hyperbolic (orbifold) structure. It is a classical result that (primitive, oriented) closed geodesics on $Y_0(1)$ can be parametrized by ideal classes of (narrow) class groups of orders in real quadratic fields such that the geodesic length equals the corresponding regulator [5]. We will denote by $\mathcal{C}_A$ the closed geodesic on $Y_0(1)$ corresponding to an ideal class $A \in \mathrm{Cl}_D^+$.

In the paper [1] the authors introduce, using minus continuous fractions, a geometric refinement of this invariant, namely a *thin subgroup* $\Gamma_A \leq \Gamma$ (Zariski dense and of infinite index) such that the infinite volume hyperbolic orbifold $Y_A := \Gamma_A\backslash\mathbb{H}$ has a unique boundary component. By general theory this is (freely) homotopic to a unique closed geodesic on Y_A and it is proved in [1] that the image under the natural projection to $Y_0(1)$ is $\mathcal{C}_A$. Cutting Y_A at the closed geodesic one obtains a finite volume hyperbolic orbifold with geodesic boundary with a map to $Y_0(1)$. Counting the fibers of this map yields a function

$$\mathcal{P}_A : Y_0(1) \to \mathbb{Z}_{\geq 0},$$

which we refer to as the *partial covering associated to A*. The main result obtained in [1] is the following equidistribution result regarding these.

Theorem 1 (Duke–Imamoḡlu–Tóth, '16)**.** *For each fundamental discriminant $D > 0$ choose a genus $G_D \in \mathrm{Cl}_D^+/(\mathrm{Cl}_D^+)^2$. Then the partial coverings $\sum_{A\in G_D} \mathcal{P}_A$ equidistribute on $Y_0(1)$ as $D \to \infty$, i.e. for any continuous and bounded function $f : Y_0(1) \to \mathbb{C}$ it holds that*

$$(1)\qquad \frac{\sum_{A\in G_D} \int_{Y_0(1)} \mathcal{P}_A(z) f(z) \frac{dxdy}{y^2}}{\sum_{A\in G_D} \int_{Y_0(1)} \mathcal{P}_A(z) \frac{dxdy}{y^2}} \to \frac{\int_{Y_0(1)} f(z) \frac{dxdy}{y^2}}{\int_{Y_0(1)} 1 \frac{dxdy}{y^2}}, \quad \text{as } D \to \infty.$$

Here it is important to restrict to genera of the class group since the partial covering for all of Cl_D^+ trivially equidistributes (i.e. one has *equality* in (1)).

To summarize, we can divide the content of [1] into three parts:

(1) construction of thin subgroups with specific boundary
(2) construction of partial coverings associated to closed geodesics
(3) equidistribution of partial coverings

In joint work with Peter Humphries [3] we extended (1)-(3) above to modular curves $Y_0(q)$ of prime level q. This forced us to use more geometric tools and we encountered a new phenomenon; *topological terms* in the Weyl sums.

I will now go on to report on progress to obtain a geometric reinterpretation/generalization of (1)-(3). This is joint work (partially in progress) with Ser Peow Tan.

Our first contribution [2] is to circumvent (1) in the construction of (2): Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a discrete and cofinite subgroup and let $\mathcal{F} \subset \mathbb{H}$ be a fundamental polygon. Then we associated to each (oriented) closed geodesic $\mathcal{C}$ on $Y_\Gamma := \Gamma\backslash\mathbb{H}$ a partial covering

$$\mathcal{P}(\mathcal{C}, \mathcal{F}) : Y_\Gamma \to \mathbb{Z}_{\geq 0},$$

by "painting to the left": we identify Y_Γ with $\mathcal{F}$, then cut up $\mathcal{C}$ according to intersections with $\partial\mathcal{F}$ and then add 1 (to the value of the function we are constructing) to all points in $\mathcal{F}$ above $\mathcal{C}$ in terms of the orientation. This recovers[1] the partial coverings in [1] by considering the fundamental polygon with vertices $\{0, \infty, e^{\pi i/3}\}$.

Secondly we reduce the equidistribution of the partial coverings (3) to that of the closed geodesics.

Theorem 2 (Nordentoft–Tan '26). *Let $(\mathbf{C}_n)_{n\geq 1}$ be a sequence of collections of (oriented) closed geodesics that equidistribute in the unit tangent bundle of Y_Γ as $n \to \infty$. Then the associated partial coverings*

$$\mathcal{P}(\mathbf{C}_n, \mathcal{F}) := \sum_{\mathcal{C}\in\mathbf{C}_n} \mathcal{P}(\mathcal{C}, \mathcal{F}),$$

equidistribute on Y_Γ as $n \to \infty$.

The proof is completely geometric in nature. Combining with [4] this yields a new proof of Theorem 1 above and furthermore this leads to many new applications.

This leaves (1), the construction of thin subgroups. We will interpret this as understanding the image of the following map:

$$\partial : \{\text{thin subgroups of } \Gamma\} \to \{\sum_{\mathcal{C}} n(\mathcal{C})\mathcal{C} : \text{multi-curve on } Y_\Gamma\}, \tag{2}$$

mapping a thin subgroup to the multi-curve (a formal $\mathbb{Z}_{\geq 0}$-linear combination of oriented closed geodesics on Y_Γ) consisting of its boundary components (i.e. the images under the canonical projection to Y_Γ of the unique closed geodesics

[1] Özlem Imamoglu added that they actually also started with (2) and then constructed thin subgroups by gluing together these different pieces.

associated to each boundary component). As was already observed in [3], this map is not surjective in general.

Proposition 1. *The image of ∂ is contained in the subset of homologically trivial multi-curves, i.e. multi-curves for which $\sum_{\mathcal{C}} n(\mathcal{C})[\mathcal{C}] = 0 \in H_1(\overline{Y_\Gamma}, \mathbb{Z})$, where $\overline{Y_\Gamma}$ denotes the underlying topological space obtained by adding in the cusps and $[\mathcal{C}]$ denotes the associated homology class.*

In certain cases we can show that this is *exactly* the image.

Theorem 3 (Nordentoft–Tan, 26+)**.** *Assume that Y_Γ has at least one cusp and at most one cone point of order > 2. Then the image of ∂ is exactly the homologically trivial multi-curves.*

In particular, this applies to the modular curve and the thin subgroups we construct recovers the construction in [1]. Note that in this case since the modular curve has genus 0 there is no homological obstruction and so ∂ is surjective. Our proof works by resolving the cone point of order > 2 (if it exists) and then constructing explicitly a fundamental polygon for the sought-after thin subgroup using geometric coding for the closed geodesics combined with the date of a cutting sequence. When specializing to the modular group, this gives a different (and geometric) perspective on the construction in [1].

Finally, due to the geometric (and soft) nature of our construction of partial coverings as in (2) we can generalize to many other context. In work-in-progress with Anders Södergren we consider the case of partial coverings associated with growing horoballs for higher dimensional non-compact hyperbolic manifolds $\Gamma\backslash\mathbb{H}^n$. We are hopefull that we can obtain a result in the vein of Theorem 3 in this setting.

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Non-abelian amplification and bilinear forms with Kloosterman sums

Alex Pascadi

We discuss a new approach to bound 'Type-II' bilinear forms with Kloosterman sums of the shape

$$\sum_{\substack{m\le M\\(m,n,c)=1}}\sum_{n\le N}\alpha_m\beta_n S(m,n;c),\qquad S(m,n;c):=\sum_{x\in(\mathbb{Z}/c\mathbb{Z})^\times} e\left(\frac{mx+n\bar{x}}{c}\right),$$

where c is a positive integer, $x\bar{x}\equiv 1 \pmod{c}$, $M,N\le c$, and (α_m), (β_n) are arbitrary complex sequences. Such bilinear forms arise in several problems across analytic number theory, for instance in estimating moments of twisted cuspidal L-functions, or in the spectral large sieve for GL_2 Maass forms.

To state a version of our main result, we focus on the case $M=N=\sqrt{c}$, which is critical for certain applications. The 'trivial' bound here, coming from either the pointwise Weil bound or a simple Fourier-analytic argument, is

$$\sum_{\substack{m,n\le\sqrt{c}\\(m,n,c)=1}}\sum\alpha_m\beta_n S(m,n;c)\ll\|\alpha\|\|\beta\|\,c^{1+o(1)},$$

where $\|\alpha\|$, $\|\beta\|$ denote the ℓ^2 norms of the sequences (α_m), (β_n).

Obtaining a power-saving over this bound has been a longstanding open problem. Blomer–Milićević [1] achieved this unless c is close to a prime or a product of two primes of the same size, leveraging a suitable factorization of c through a q-analogue of the van der Corput method. Kowalski–Michel–Sawin [2] (see also [3]) used the shift-by-ab trick of Vinogradov and Karatsuba (or Burgess), combined with deep inputs from the theory of algebraic trace functions, to achieve a power-saving for prime moduli. Shkredov [5] gave an independent solution (with a weaker saving in this range) for the case of prime moduli $c=p$, relying on a connection to the distribution of words in $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and on L^2-flattening results. Our work settles the case of arbitrary moduli.

Theorem 1 (P. '25+ [6])**.** *With $\delta:=\frac{1}{700}$, one has*

$$\sum_{\substack{m,n\le\sqrt{c}\\(m,n,c)=1}}\sum\alpha_m\beta_n S(m,n;c)\ll\|\alpha\|\|\beta\|\,c^{1-\delta+o(1)},$$

for any positive integer c. The value of δ can be improved depending on the factorization of c.

Our proof of Theorem 1 uses the representation theory of $\mathrm{SL}_2(\mathbb{Z}/c\mathbb{Z})$ and a novel amplification argument with non-abelian characters. We also briefly mention work in progress, joint with Valentin Blomer, which leads to an improved saving when the modulus c is a prime; this relies on a connection to quadratic character sums.

We also point the reader to recent work of Milićević–Qin–Wu [4], who simultaneously and independently achieved a result similar to Theorem 1 using different methods. The two results perform differently depending on the ranges of M,N

and the factorization of c. A highlight of our result is that if $c = pq$ where $p \asymp q$ are primes (which was a bottleneck of previous methods), then one can take

$$\delta = \frac{1}{12}$$

in Theorem 1.

The first step in our proof of Theorem 1 relies on a connection to words in $\mathrm{SL}_2(\mathbb{Z}/c\mathbb{Z})$, similar to that observed by Shkredov [5]. Specifically, we view the bilinear form as (bounded by) the spectral norm of a Kloosterman matrix, and we use the trace method to pass to a moment of its singular values. By Fourier analysis, we are left to estimate an expression of the form (up to smooth weights)

$$\sum_{\substack{|h_1|,|h_3|,\ldots,|h_{2k-1}|\le \frac{c}{M} \\ |h_2|,|h_4|,\ldots,|h_{2k}|\le \frac{c}{N}}} \chi\left(T^{h_1}S\cdots T^{h_{2k}}S\right), \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where χ is a certain character of the non-abelian group $\mathrm{SL}_2(\mathbb{Z}/c\mathbb{Z})$. By decomposing χ, we may assume without loss of generality that χ is irreducible (the initial restriction that $(m, n, c) = 1$ turns out to be useful in eliminating small-dimensional irreducible characters during this step).

The key new step in our work is an amplification argument which exploits a non-trivial factorization of c. We pass to a sum over all irreducible characters χ' of $\mathrm{SL}_2(\mathbb{Z}/c\mathbb{Z})$ of the expression above (with χ replaced by χ'), weighted by a non-abelian amplifier $A(\chi, \chi')$ which should be large when $\chi = \chi'$. The construction of the amplifier depends on a normal subgroup of $\mathrm{SL}_2(\mathbb{Z}/c\mathbb{Z})$, specifically the kernel

$$\ker\left(\mathrm{SL}_2(\mathbb{Z}/c\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/d\mathbb{Z})\right),$$

where d is a suitable divisor of c. After evaluating the sum over χ' by character orthogonality, we are left to bound the number of solutions in $h_1, \ldots, h_{2k}$ to congruences of the shape

$$T^{h_1}S\cdots T^{h_{2k}}S \equiv \pm I \pmod{d},$$

where I is the identity. We take $2k = 6$, and obtain an upper bound for this counting problem using explicit combinatorial arguments. When $M = N = \sqrt{c}$, this leads to a non-trivial final bound unless c has a prime factor $p > c^{1-\epsilon}$; combining this with previous results for prime moduli completes the proof of Theorem 1.

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Bounds for Kloosterman sums for $\mathbf{GL}_n$

JOHANNES LINN

The classical Kloosterman sum

$$S(m,n;c) := \sum_{\substack{x \mod c \\ (x,c)=1}} e\left(\frac{mx+n\overline{x}}{c}\right)$$

appearing among many places in the geometric side of the Kuznetsov formula is a well studies object in number theory. Its size, distribution and structure is of relevance especially in the theory of automorphic forms on GL_2.

Going to the higher rank setting of autmorphic forms for GL_n we encounter more general exponential sums so called Kloosterman sums for GL_n. These can be defined as follows:

Let $G := \mathrm{GL}_n$ and let U be the upper triangular matrices with diagonal equal to 1, $U_w := U \cap w^{-1}U^T w$. Fix a rational diagonal matrix $\mathfrak{c}$ with determinant one and an element w of the Weyl group, which in this case is the group of permutations of n elements, write $\mathfrak{n} := \mathfrak{c} \cdot w$. We define the Kloosterman set as the double quotient

$$X(\mathfrak{n}) := U(\mathbb{Z})\backslash(U(\mathbb{Q})\mathfrak{n}U(\mathbb{Q}) \cap G(\mathbb{Z}))/U_w(\mathbb{Z}).$$

Taking two characters ψ and ψ' of $U(\mathbb{Q})$, which are trivial on $U(\mathbb{Z})$ we define the generalized Kloosterman sum as follows:

$$\mathrm{Kl}(\psi,\psi';\mathfrak{c}\cdot w) := \sum_{u\mathfrak{n}u' \in X(\mathfrak{n})} \psi(u)\psi'(u').$$

This sum is not well defined for all choices of w, ψ and ψ' and we set it to 0 if it is not.

Replacing $\mathbb{Q}$ with $\mathbb{Q}_p$ gives the local Kloosterman sum, which is going to be the object of interest for this talk.

There has previously been work on these Kloosterman sums in small dimension for example for GL_3 [2] or in higher dimension but only for specific Weyl elements. In [4] the Voronoi element is treated by writing the corresponding sum as a hyper Kloosterman sum. In [1] and [6] bounds for the long Weyl element and in the first case also for a special Weyl element w_* are shown using different parametrizations for the double quotient.

In the general setting the previously best known bound is the size of $X(\mathfrak{n})$ computed in [3].

Our main result is a parametrization and for all Weyl elements and all n and bounds resulting from this parametrization.

The following is an example of such a sum for GL_3 with moduli $c_1 = c_2 = p$.

$$\sum_{x_1 \in \mathbb{Z}/p\mathbb{Z}} \sum_{x_2 \in (\mathbb{Z}/p\mathbb{Z})^*} \exp(2\pi i(x_1 \bar{x}_2 p^{-1} + \bar{x}_1 p^{-1})) + \sum_{x_1 \in (\mathbb{Z}/p\mathbb{Z})^*} \sum_{x_3 \in (\mathbb{Z}/p\mathbb{Z})^*} \exp(2\pi i(x_1 p^{-1} + \overline{x}_3 p^{-1})) \ .$$

Using this we can make the following observations:

(1) We can directly see the size of $X(\mathfrak{n})$
(2) The individual sums resemble classical or GL_2 Kloosterman sums.
(3) We can use the Weil-bound to bound individual sums and obtain power savings for the whole sum. The structure also enables us to do this for multiple variables at the same time.
(4) Not all variables are invertible modulo p, like x_1 in the first sum. This enables exact calculations as these full sums can lead to total cancellation. The above sum is for example equal to 0.
(5) These exact calculations and vanishing is new in higher rank and can be used to gain further insight about sums appearing in the Kuznetsov formula.

Furthermore, we can prove the following bound for these Kloosterman sums in general:

$$\mathrm{Kl}(\psi, \psi'; \mathfrak{c} \cdot w) \ll_\varepsilon C|X(\mathfrak{c} \cdot w)|^{1-\frac{1}{4l(w)}+\varepsilon} \ ,$$

where $n-1 \leq n(n-1)/2$ is the length of the Weyl element, and C depends on the characters.

The proof consists of three main ingredients: A stratification by Dabrowski and Reeder [3], the idea by Blomer and Man to make this explicit [1], and a way to treat all Weyl elements at the same time using a clever induction and diagrams to capture the structure of the sums [5]

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Diophantine approximations of linear subspaces

Nicolas de Saxcé

In 1967, Schmidt suggested to view the classical results of Diophantine approximation as more general problems about rational approximations of linear subspaces of the Euclidean space. More precisely, he observed that the relative position of two subspaces A and B of respective dimensions d and e in $\mathbb{R}^n$ is completely described by the sines successive angles

$$0 \leq \psi_1(A,B) \leq \psi_2(A,B) \leq \cdots \leq \psi_t(A,B) \leq 1, \qquad t = \min(d,e).$$

Given a d dimensional subspace A in $\mathbb{R}^n$, this naturally leads one to define a family of diophantine exponents $\beta_{n,d,e,j}(A)$, for $e = 1,\dots,n$ and $j = 1,\dots,\min(d,e)$ by the formula

$$\beta_{n,d,e,j}(A) = \sup\left\{\beta > 0 \;\middle|\; \exists (B_k)_{k\geq 1} \in \mathrm{Gr}_{n,e}(\mathbb{Q}) : \begin{array}{l} \psi_j(B_k,A) \to_{k\to\infty} 0 \\ \text{and } \forall k,\ \psi_j(B_k,A) \leq H(B_k)^{-\beta} \end{array}\right\},$$

where $H(B)$ denotes the height of the rational subspace B given by the Plücker embedding $\mathrm{Gr}_{n,e} \hookrightarrow \mathbb{P}^{\binom{n}{e}}$. Two problems are then to determine

- the almost sure value of $\beta_{n,d,e,j}(A)$ when A is chosen randomly according to the Lebesgue measure on the Grassmann variety $\mathrm{Gr}_{n,d}(\mathbb{R})$;
- the minimal value $\inf_{A\in \mathrm{Gr}_{n,d}(\mathbb{R})} \beta_{n,d,e,j}(A)$.

We explained in the first part of the talk how this problem can be translated to a problem of effective density of translates of unipotent orbits under a well-chosen diagonal flow. In the case $j = \min(d,e)$, this allowed us to show that the almost sure and the minimal value of the exponents are equal.

Interestingly, this is not the case when $(n,d,e,j) = (4,2,2,1)$. Indeed, Elio Joseph constructed a two-dimensional subspace $A_{\min} < \mathbb{R}^4$ such that $\beta_{4,2,2,1}(A_{\min}) = 3$. By a result of Schmidt, this is the minimal possible value of $\beta_{4,2,2,1}$. On the other hand, in joint work with Uri Shapira, we use a ubiquity transference approach to show that for almost every A, one has $\beta_{4,2,2,1}(A) = 4$.

The non-spherical sup-norm problem on $\mathbf{GL}_n$

Péter Maga

(joint work with Valentin Blomer, Gergely Harcos, Djordje Milićević)

The classical sup-norm problem of automorphic forms asks for pointwise bounds on L^2-normalized spherical Hecke–Maaß cusp forms. More specifically, it was shown by Sarnak [5] that if ϕ is an L^2-normalized eigenfunction of the invariant differential operators on a compact locally symmetric Riemannian space X, then we have the baseline bound

$$\|\phi\|_\infty \ll_X \lambda_\phi^{\frac{\dim - \mathrm{rk}}{4}}, \tag{1}$$

where dim and rk stand for dimension and the rank of the space in question, and λ_ϕ is the Laplace eigenvalue of ϕ. If X is non-compact, then (1) might fail [3], but it is still true when restricted to some compact $\Omega \subset X$, and in this case, the implied constant in $\ll$ also depends on Ω. Note also that (1) is known to be sharp in this broad generality, that is, there are spaces where high frequency eigenwaves exhibit these large peaks.

In the arithmetic situation, we assume that ϕ is further an eigenfunction of the Hecke operators, and the arithmetic sup-norm problem asks for power-savings on the right-hand side of (1): is it true that

$$\|\phi\|_\infty \ll_X \lambda_\phi^{\frac{\dim - \mathrm{rk}}{4} - \delta},$$

for some $\delta > 0$ depending only on X?

The answer is known to be affirmative in many cases, as it was pioneered by Iwaniec and Sarnak [4] for the setup $\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})/\mathrm{PO}_2(\mathbb{R})$, and followed by many.

However, many automorphic forms on GL_n do not contain a spherical vector, the basic examples are holomorphic forms and their symmetric powers. Therefore it is natural to ask what the baseline bound (that uses no arithmetic) is and if a saving over that can be achieved in the presence of Hecke operators. We fix the notation $G := \mathrm{GL}_n(\mathbb{R})$, $\Gamma := \mathrm{GL}_n(\mathbb{Z})$, $K := \mathrm{O}_n(\mathbb{R})$. Let π be a cuspidal automorphic representation appearing in $L^2(\Gamma\backslash G, \omega)$ with a unitary central character ω. Then, on the one hand, π has a Plancherel content

$$\Delta(\pi) := \text{volume of the 1-ball centered at } \pi \text{ in the unitary dual of } G,$$

which essentially mimics the Plancherel density at π (but it is slightly more general, and takes care of the situation when π is off-tempered, or its Plancherel density accidentally vanishes). On the other hand, π decomposes to K-types, and assume τ is the minimal dimensional K-type of positive multiplicity, then this multiplicity is necessarily one. Consequently, we have a well-defined $\dim\tau$-dimensional subspace $V_{\pi,\tau}$ of $L^2(\Gamma\backslash G, \omega)$, which is isomorphic to τ as a K-representation, and generates π. Take an orthonormal basis $\phi_1, \dots, \phi_{\dim\tau}$ in $V_{\pi,\tau}$, and consider the vector-valued form

$$\Phi := (\phi_1, \dots, \phi_{\dim\tau}),$$

and its size defined as

$$\|\Phi(g)\| := \sqrt{\sum_{j=1}^{\dim\tau} |\phi_j(g)|^2}.$$

Then the baseline bound, the counterpart of (1) (and in fact, reproducing (1) when τ is the trivial representation, that is, π is spherical) is

$$\|\Phi|_\Omega\| \ll_\Omega (\Delta(\pi)\cdot\dim\tau)^{\frac{1}{2}}.$$

Theorem. There exists some $\delta > 0$ depending only on n such that

$$\|\Phi|_\Omega\| \ll_\Omega (\Delta(\pi)\cdot\dim\tau)^{\frac{1}{2}-\delta}.$$

Ingredients of the proof include a construction of a Paley–Wiener type function tailored to the problem, estimates on the non-spherical generalization of the spherical trace functions (a generalization of [2]) and a matrix-counting (in the spirit of [1]).

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L^∞-norm bounds for Jacobi cusp forms

Anna-Maria Pippich

(joint work with Anilatmaja Aryasomayajula, Jürg Kramer)

1. Introduction and Motivation

The study of the supremum norm (or L^∞-norm) of automorphic forms is a well-established problem at the intersection of harmonic analysis and number theory. For classical elliptic cusp forms, optimal upper bounds with respect to the weight k have been extensively studied since the seminal work of Iwaniec and Sarnak.

Jacobi forms, systematically introduced by Eichler and Zagier [3], generalize classical modular forms by introducing an additional spatial variable $z \in \mathbb{C}$, reflecting the geometry of abelian varieties. These forms transform naturally under the group action of the semi-direct product $\Gamma_0 \ltimes \mathbb{Z}^2$, where $\Gamma_0 = \mathrm{SL}_2(\mathbb{Z})$ denotes the full modular group.

An interesting extension of the classical sup-norm problem is to establish sharp bounds for Jacobi cusp forms simultaneously in terms of both the weight k and the index m. While Anamby and Das [1] recently established the baseline bound $\|\varphi\|_{L^\infty} = O_\epsilon((km)^{1+\epsilon})$, the aim of our work is to present an alternative geometric and analytic proof method that links the supremum norm of Jacobi forms explicitly to the geometry of the Bergman kernel.

To control these norms, we first establish L^∞-norm bounds for cusp forms of any positive real weight $k \geq 5$ and character χ for cofinite Fuchsian subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$. Utilizing the representation of Jacobi cusp forms as linear combinations of half-integral weight modular forms and Jacobi theta functions, we then derive our bound. Although our resulting dependency on the index m is slightly weaker than the aforementioned baseline, this structural approach has the distinct advantage of extending naturally to higher genus.

2. Main results

Let $\mathbb{H}$ denote the upper half-plane. For $k, m \in \mathbb{N}$, we denote the complex vector space of Jacobi cusp forms of weight k and index m for Γ_0 by $J^{\mathrm{cusp}}_{k,m}(\Gamma_0)$. For $\tau = \xi + i\eta \in \mathbb{H}$ and $z = x + iy \in \mathbb{C}$, the pointwise Petersson norm of a Jacobi form $\varphi \in J^{\mathrm{cusp}}_{k,m}(\Gamma_0)$ is defined by

$$\|\varphi(\tau, z)\|^2_{\mathrm{Pet}} := |\varphi(\tau, z)|^2 \, \eta^k e^{-\frac{4\pi m y^2}{\eta}}.$$

This defines a $\Gamma_0 \ltimes \mathbb{Z}^2$-invariant function on $\mathbb{H} \times \mathbb{C}$. We normalize φ with respect to the Petersson inner product such that

$$\|\varphi\|^2_{L^2} = \int_{\Gamma_0 \ltimes \mathbb{Z}^2 \backslash \mathbb{H} \times \mathbb{C}} \|\varphi(\tau, z)\|^2_{\mathrm{Pet}} \, \frac{d\xi \wedge d\eta \wedge dx \wedge dy}{\eta^3} = 1.$$

The first main theorem of our recent work [2], concerning Jacobi cusp forms, is formulated as follows:

Theorem 1. *Let $k \in \mathbb{Z}_{\geq 5}$ and $m \in \mathbb{Z}_{\geq 1}$. For any L^2-normalized Jacobi cusp form $\varphi \in J^{\mathrm{cusp}}_{k,m}(\Gamma_0)$ and a given $\epsilon > 0$, the L^∞-norm bound*

$$\|\varphi\|_{L^\infty} = \sup_{(\tau, z) \in \mathbb{H} \times \mathbb{C}} \|\varphi(\tau, z)\|_{\mathrm{Pet}} = O_{\Gamma_0, \epsilon}\left(k \cdot m^{7/4 + \epsilon}\right)$$

holds, where the implied constant depends on Γ_0 and the choice of $\epsilon > 0$.

Theorem 1 relies on a general bound for the Bergman kernel, which is defined for the space $S_{k,\chi}(\Gamma)$ of holomorphic cusp forms of weight k and character χ for Γ. Given an orthonormal basis $\{f_1, \ldots, f_d\}$ of $S_{k,\chi}(\Gamma)$ with respect to the Petersson inner product, the Bergman kernel is defined by

$$B_{k,\chi}(\tau, \tau') := \sum_{j=1}^{d} f_j(\tau) \overline{f_j(\tau')}.$$

By extending the techniques of Friedman, Jorgenson, and Kramer [4] to arbitrary real weights and characters, we prove the following result of independent interest:

Theorem 2. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a cofinite Fuchsian subgroup, $k \in \mathbb{R}_{\geq 5}$, and χ a character. The pointwise Petersson norm of the Bergman kernel $B_{k,\chi}(\tau, \tau)$ associated to the space of cusp forms of weight k and character χ satisfies*

$$\sup_{\tau \in \mathbb{H}} \|B_{k,\chi}(\tau, \tau)\|_{\mathrm{Pet}} = O_\Gamma(k^{3/2}).$$

If Γ is cocompact without elliptic elements, this bound improves to $O_\Gamma(k)$.

3. Sketch of methods and higher genus

The proof relies on a decomposition strategy combined with sharp estimates for the Bergman kernel and classical Jacobi theta functions. By a fundamental result of Eichler and Zagier [3], any Jacobi cusp form $\varphi \in J^{\mathrm{cusp}}_{k,m}(\Gamma_0)$ can be uniquely decomposed as a finite linear combination

$$\varphi(\tau, z) = \sum_{\mu=0}^{2m-1} h_\mu(\tau)\vartheta_{\mu,m}(\tau, z),$$

where $\vartheta_{\mu,m}$ are classical Jacobi theta functions of index m. The coefficients h_μ are cusp forms of half-integral weight $k - 1/2$ for the congruence subgroup $\Gamma_1 = \Gamma_{0,1}(4m) \subseteq \Gamma_0(4m)$. Crucially, the L^2-normalization of φ implies that $\sum_{\mu=0}^{2m-1} \|h_\mu\|^2_{L^2} = O_\epsilon(m^{5/2+\epsilon})$.

To bound the components $h_\mu(\tau)$, we invoke Theorem 2 for the Bergman kernel $B_{k-1/2}(\tau, \tau')$ associated to the space of modular forms of half-integral weight. The crucial polynomial dependence on k emerges from analyzing the forms near the cusp. Applying the maximum principle to the associated subharmonic functions, the bound is dominated by the peak of $\eta^{k+1/2}e^{-4\pi\eta}$, which occurs at a height proportional to k. Truncating the fundamental domain at this height and applying the Bergman kernel bounds on the boundary yields the uniform supremum bound

$$\sup_\tau \left(\|h_\mu(\tau)\|^2_{\mathrm{Pet}}\, \eta^{1/2} \right) = O_{\Gamma_0}(k^2)\, \|h_\mu\|^2_{L^2}\,.$$

Independently, we evaluate the growth of the theta functions. By expressing $\vartheta_{\mu,m}$ via the standard theta function $\vartheta_{0,m}$ and applying an integral test to the resulting exponential series, we prove that the sum of their squared Petersson norms satisfies

$$\sum_{\mu=0}^{2m-1} \|\vartheta_{\mu,m}(\tau, z)\|^2_{\mathrm{Pet}} \leq 2m\eta^{1/2} + O(m^{1/2})$$

for all relevant (τ, z). The final synthesis involves substituting the decomposition into the pointwise Petersson norm of φ and applying the Cauchy-Schwarz inequality. Factoring the estimation into the sum over the modular forms h_μ and the sum over the theta functions $\vartheta_{\mu,m}$, we insert the respective bounds derived from the Bergman kernel and the theta series. The aggregation of the $O(m^{5/2+\epsilon})$ contribution from the L^2-normalization of the components, the $O(m)$ contribution from the theta functions, and the algebraic manipulations of the invariant height η yields the final squared exponent of $7/2$. Taking the square root concludes the proof, giving the stated index dependency of $m^{7/4+\epsilon}$.

Work in progress: Higher Genus. The geometric framework presented here naturally generalizes to Jacobi forms of higher genus $g \geq 1$. For Jacobi cusp forms of weight $k \geq g + 1$ and index m for the Siegel modular group $\mathrm{Sp}_{2g}(\mathbb{Z})$, the decomposition into Siegel modular forms of weight $k - 1/2$ and higher-dimensional theta functions allows us to extract explicit bounds. To bound the resulting higher degree Bergman kernels, we adapt recent uniform sup-norm bounds for Siegel cusp forms derived by Kramer and Mandal [5]. Ongoing work indicates a provisional squared L^∞-norm bound exhibiting polynomial growth of the form

$$\|\varphi\|^2_{L^\infty} = O_{\Gamma_0,\epsilon}\left(k^{(3g^2+5g)/4}\, m^{2g^2+5g/2+\epsilon}\right),$$

where the precise numerical constants in the exponents are currently being finalized. Furthermore, establishing such sharp L^∞-bounds for individual Fourier–Jacobi coefficients is a crucial prerequisite for investigating the convergence properties of symmetric Fourier–Jacobi series.

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Period functions for Jacobi Maass cusp forms and vector-valued Maass cusp forms

Anke Pohl

(joint work with Roelof Bruggeman, YoungJu Choie)

In [5], Lewis and Zagier established a notion of period functions for the Maass cusp forms of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, extending earlier work of Lewis [3]. Here, a period function for the parameter $s \in \mathbb{C}$ is a real-analytic function $f\colon (0,\infty) \to \mathbb{C}$ that satisfies the functional equation

$$f(t) = f(t+1) + (t+1)^{-2s} f\left(\frac{t}{t+1}\right) \qquad \text{for } t \in (0,\infty) \tag{1}$$

and the condition on growth and regularity that the function

$$t \mapsto \begin{cases} f(t) & \text{for } t \in (0,\infty) \\ -|t|^{-2s} f\left(-\frac{1}{t}\right) & \text{for } t \in (-\infty,0) \end{cases}$$

extends smoothly to 0 and hence then also to ∞, defining a function on all of $P^1_{\mathbb{R}}$. By means of analytic number theory Lewis and Zagier provided a linear isomorphism between the space of Maass cusp forms for $\mathrm{PSL}_2(\mathbb{Z})$ with spectral parameter s and the space of the period functions with parameter s. An alternative proof based on cohomology was presented by Bruggeman in [1]. The functional equation in (1) can be deduced from a transfer operator for the geodesic flow on the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$, thereby showing an instance of the deep relation between the dynamical entities of the modular surface (in this case, the geodesic flow) and its spectral entities (in this case, the Maass cusp forms). The seminal work on these transfer operator aspects is due to Mayer [6, 7]. An alternative proof, somewhat more direct, is obtained in [11, 8]. Generalizations of such a relation between Maass cusp forms, period functions and the geodesic flow on hyperbolic orbisurfaces to a considerably large class of Fuchsian groups is provided by the combination of [2] and [10]; surveys are provided by [4, 13]. In each case, the passage from Maass cusp forms to period functions is given by an integral transform. The proof is based on cohomology and transfer operator techniques. The necessary functional equations are determined by discretizations of the geodesic flow and eigenfunction equations of the associated transfer operators. All results in this realm of research previous to the result presented in this talk were for Fuchsian groups and automorphic forms and functions of weight 0.

With this talk we presented the first result in this research field for a non-Fuchsian group and for non-zero weight situations, namely for Jacobi Maass cusp forms of any real weight, any positive integer index and any parameter in $\mathbb{Z}/12\mathbb{Z}$ as well as for vector-valued Maass cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ of any real weight and with any finite-dimensional unitary representation, obtained in [12] in joint work with Bruggeman and Choie.

For a more detailed explanation we set $\Gamma := \mathrm{SL}_2(\mathbb{Z})$, and we let Γ^J denote the discrete (integral) Jacobi group of level 1, which is the semi-direct product of Γ and the integer points $\mathrm{Hei}(\mathbb{Z})$ of the Heisenberg group. Let $k \in \mathbb{R}$, $s \in \mathbb{C}$ and suppose that $\varrho : \Gamma \to \mathrm{U}(n)$ is a finite-dimensional unitary representation of Γ. Let v_k denote the standard multiplier system of weight k for Γ, and let $\mathcal{A}_k^0(s, \varrho v_k)$ be the space of vector-valued Maass cusp forms for Γ of weight k and with representation ϱ, i.e., the space of functions $u\colon \mathbb{H} \to \mathbb{C}^n$ which satisfy the equivariance

$$u|_{\varrho v_k, k} = u\,,$$

where

$$u|_{\varrho v_k k} g(\tau) = \varrho(g)^{-1} v_k(g)^{-1} e^{-ik\arg(c\tau+d)} u(g.\tau) \qquad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,,$$

that are eigenfunctions with spectral parameter s of the differential operator $\Delta_k = -y^2(\partial_x^2 + \partial_y^2) + iky\partial_x$ (with $z = x + iy \in \mathbb{H}$), and that have exponential decay towards the cusp of the modular surface $\Gamma\backslash\mathbb{H}$. Let further $m \in \mathbb{N}$ and $a \in \mathbb{Z}/12\mathbb{Z}$, and let $\mathcal{A}_{k,m}^{J,0}(s, \varphi_a v_k)$ denote the space of Jacobi Maass cusp forms of weight k, index m, parameter a and spectral parameter s. We refer to [12] for precise definitions.

Theorem (Bruggeman–Choie–P., [12])**.** *Suppose* $\mathrm{Re} s \geq 0$. *Then* $\mathcal{A}_{k,m}^{J,0}(s, \varphi_a v_k)$ *and* $\mathcal{A}_{k'}^0(s', \varrho_{a,m} v_{k'})$ *are linear isomorphic, where* $s' = (s+1)/2$ *and* $k' = k - 1/2$*, and* $\varrho_{a,m}$ *is a certain explicit* $2m$*-dimensional unitary representation of* Γ*, depending on* a *and* m*.*

This theorem allows us to transfer the quest for period functions for Jacobi Maass cusp forms to a quest for period functions for vector-valued Maass cusp forms for Γ. The proof of this theorem is based on a generalization of Pitale's theta decomposition in [9].

For vector-valued Maass cusp forms for Γ, the underlying geometry and dynamics is the same as for classical (i.e., weight 0, scalar) Maass cusp form for Γ. Staying true to the heuristics that a suitable transfer operator approach determines the functional equation for the (candidates of the) period functions, the functional equation for elements in $\mathcal{A}_k^0(s, \varrho v_k)$ should be of a similar structure as in (1). Using the standard slash action $|_s$, we may rewrite (1) as

$$f = f|_s T + f|_s T^t\,,$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad T^t = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{the transpose of } T.$$

Indeed, with an adapted action $|^{\mathrm{ps}}_{\varrho v_k, s, k}$, we may define the space $\mathrm{FE}^{\omega}_{\varrho v_k, s, k}$ of period functions for vector-valued Maass cusp forms as the real-analytic functions $f\colon (0,\infty) \to \mathbb{C}^n$ satisfying the functional equation

$$f = f|^{\mathrm{ps}}_{\varrho v_k, s, k} T + f|^{\mathrm{ps}}_{\varrho v_k, s, k} T^t$$

and obeying certain growth and regularity properties in analogy to above. By developing a certain, highly explicit cohomological interpretation of vector-valued Maass cusp forms we can establish the following result.

Theorem (Bruggeman–Choie–P., [12]). *If* $\mathrm{Re}s \in (0,1)$ *and* $s \not\equiv \pm k/2 \mod 1$*, then* $\mathcal{A}_k^0(s, \varrho v_k)$ *and* $\mathrm{FE}^{\omega}_{\varrho v_k, s, k}$ *are linear isomorphic.*

The isomorphism in this theorem can be given as an integral transform, generalizing the results mentioned above. We refer to [12] for all details.

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Fourier-Jacobi coefficients of degree 2 Siegel cusp forms and the refined GGP conjecture

Abhishek Saha
(joint work with Biplab Paul, Ameya Pitale, Ralf Schmidt)

Let F be a Siegel cusp form of degree 2 and even weight k for the group $\mathrm{Sp}_4(\mathbb{Z})$. Then F has a Fourier expansion

$$F(Z) = \sum_S a(F,S) e^{2\pi i\,\mathrm{tr}(SZ)},$$

where the Fourier coefficients $a(F,S)$ are indexed by matrices S of the form

$$(1)\qquad S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}, \qquad a,b,c \in \mathbb{Z}, \qquad a > 0, \qquad \mathrm{disc}(S) := b^2 - 4ac < 0.$$

For each positive integer m, we can construct a half-integral weight cusp form given by
(2)

$$f_m(z) = \sum_{n=1}^{\infty} a_m(n) e^{2\pi i n z}, \quad \text{where} \quad a_m(n) = \sum_{\substack{0 \le r \le 2m-1 \\ r^2 \equiv -n \pmod{4m}}} a\left(F, \begin{pmatrix} \frac{n+r^2}{4m} & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}\right)$$

It is known that $f_m \in S_{k-\frac{1}{2}}(4m)$ (see [1, Theorem 5.6]). The half-integral weight forms f_m are of particular interest for studying F, since their Fourier coefficients, as is evident from (2), are closely related to those of F. It is therefore natural to ask about the relationship between the Petersson norms of f_m and F.

For Saito–Kurokawa lifts there is a classical answer. If F is a Saito–Kurokawa lift and a Hecke eigenform, and $m = 1$, then Kohnen and Skoruppa [2] proved

$$(3)\qquad \frac{\langle f_1, f_1\rangle}{\langle F, F\rangle} = \frac{24\pi^k}{\Gamma(k) L(3/2, f)},$$

where $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ is the classical Hecke eigenform on the upper half-plane whose Saito–Kurokawa lift is F. Analogous formulas in this Saito–Kurokawa case for $\frac{\langle f_m, f_m\rangle}{\langle F, F\rangle}$ can likely be derived from (3) for all squarefree m, using existing expressions (e.g., [1, Theorem 6.2]) for the mth Fourier–Jacobi coefficient in terms of the first.

The natural problem is to understand what replaces (3) when F is *not* a Saito–Kurokawa lift. We prove the following conditional identity which gives a clean and rather striking answer: the above ratio is a weighted average of central values of degree 10 L-functions.

Theorem 1. *Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform, and assume that k is even and that F is not a Saito–Kurokawa lift. Let m be an odd squarefree integer. Assume the refined Gan–Gross–Prasad conjecture for Fourier–Jacobi periods on Sp_4. Then*

$$\frac{\langle f_m, f_m\rangle}{\langle F, F\rangle} = C_k \sum_{q|m} r_q \sum_{f\in\mathcal{B}^{\mathrm{new}}_{2k-2}(\Gamma_0(q))} \frac{L(\frac{1}{2}, F\times f)}{L(1, F, \mathrm{Ad})L(1, f, \mathrm{Ad})}. \tag{4}$$

Here $\mathcal{B}^{\mathrm{new}}_{2k-2}(\Gamma_0(q))$ is an orthogonal Hecke basis of holomorphic newforms of weight $2k-2$ and level q, $C_k \asymp \frac{(4\pi)^k}{\Gamma(k)(2k-2)}$ is an explicit constant depending only on k, and $r_q \asymp q^{-1+o(1)}$ is an explicit product of local factors at the primes dividing m, with the local factor depending on whether the prime divides q or not.

Remark 2. *In the case of Saito–Kurokawa lifts, $L(s, F, \mathrm{Ad})$ has a pole at $s = 1$, and an analogue of (4) for $m = 1$ would reduce to a single nonzero term corresponding to the unique f (which lifts to F) for which $L(1/2, F \times f)$ also has a pole. Simplifying the resulting expression gives us (3) up to a constant.*

Remark 3. *As a consequence of Theorem 1, we obtain an essentially optimal upper bound for $\frac{\langle f_m, f_m\rangle}{\langle F,F\rangle}$ under GRH.*

The key input in the above theorem is an explicit computation of certain ramified local factors appearing in Xue's refined Gan–Gross–Prasad conjecture for Fourier–Jacobi periods on symplectic groups [3]. In the case relevant here we take $G = \mathrm{Sp}_4$. Let $H \subset G$ be the Heisenberg subgroup

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

and embed SL_2 into G by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $\widetilde{\mathrm{SL}}_2$ denotes the metaplectic double cover, we set $R = \mathrm{SL}_2 \cdot H$ and $\widetilde{R} = \widetilde{\mathrm{SL}}_2 \cdot H$. For a non-trivial additive character ψ of $\mathbb{Q}\backslash\mathbb{A}$ and $m \in \mathbb{Q}^\times$, let ω_{ψ^m} be the Weil representation of $\widetilde{R}(\mathbb{A})$ on $\mathcal{S}(\mathbb{A})$, and define the theta function

$$\Theta_{\psi^m}(r, \phi) = \sum_{x\in\mathbb{Q}} \omega_{\psi^m}(r)\phi(x), \qquad r \in \widetilde{R}(\mathbb{A}),\ \phi \in \mathcal{S}(\mathbb{A}).$$

Now let (π, V_π) be a cuspidal automorphic representation of $\mathrm{Sp}_4(\mathbb{A})$ and (σ, V_σ) a genuine cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. For $\Psi \in V_\pi$, $\Lambda \in V_\sigma$, and $\phi \in \mathcal{S}(\mathbb{A})$ one defines the global Fourier–Jacobi period

$$\mathcal{FJ}(\Psi, \Lambda, \phi; \psi^m) = \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \Psi(ng)\Lambda(g)\Theta_{\psi^m}(ng, \phi)\, dn\, dg.$$

For factorizable data there are corresponding local linear forms $\alpha_v(\Psi_v, \Lambda_v, \phi_v; \psi_v^m)$, and after dividing by suitable local norms and local zeta factors one obtains normalized local periods $\alpha_v^{\#}(\Psi_v, \Lambda_v, \phi_v; \psi_v^m)$ that are equal to 1 for almost all v.

Xue's conjecture is an exact analogue of the Ichino–Ikeda formalism [4]. In our setting it predicts that, under temperedness hypotheses,

$$(5)\quad \frac{|\mathcal{FJ}(\Psi, \Lambda, \phi; \psi^m)|^2}{\langle \Psi, \Psi\rangle\langle \Lambda, \Lambda\rangle\langle \phi, \phi\rangle} = C \cdot \frac{L_{\psi^m}(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \mathrm{Ad})L_{\psi^m}(1, \sigma, \mathrm{Ad})} \prod_v \alpha_v^{\#}(\Psi_v, \Lambda_v, \phi_v; \psi_v^m).$$

In our setting, we let $\Psi = \Phi_F$ be the adelization of the Siegel cusp form F, and we choose σ and the test data so that the left-hand side becomes a multiple of

$$\frac{|\langle f_m, h\rangle|^2}{\langle F, F\rangle\langle h, h\rangle}$$

for a half-integral weight cusp form $h \in S_{k-\frac{1}{2}}(\Gamma_0(4m))$. In other words, the global Fourier–Jacobi period admits a classical interpretation in terms of Petersson inner products.

At this point one sums over an orthogonal Hecke basis of the relevant half-integral weight space and applies Parseval. The Shimura–Waldspurger correspondence then converts the representation σ into an automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A})$, with f a newform of weight $2k-2$ and level dividing m. This produces the inner sum in (4).

The technically hardest part of the proof is local. At each prime $p \mid m$ one must compute the normalized local factors $\alpha_p^{\#}(\Psi_p, \Lambda_p, \phi_p; \psi_p^m)$ explicitly. After several reductions, one arrives at integrals on the Jacobi group involving matrix coefficients associated to induced representations of the Jacobi group. The local integral breaks naturally into many valuation regions and the final weights r_q in (4) are precisely the products of these ramified local contributions.

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An application of equidistribution to cryptography

Radu Toma

(joint work with Koen de Boer, Aurel Page, Benjamin Wesolowski)

The shortest vector problem for Euclidean lattices is a central hard problem in complexity theory with applications to post-quantum cryptography. More generally, given a lattice $L \subset \mathbb{R}^n$, one may look for a non-zero lattice element whose norm is within a factor $\gamma > 1$ of the smallest such norm (γ-SVP) or for a collection of n linearly independent vectors (v_i) that are short in a similar way: $\|v_i\| \leq \gamma \cdot \lambda_n(L)$ (this is called γ-SIVP). Such computational problems are expected to be hard in the regime where the approximation factor $\gamma = \mathrm{poly}(n)$ is polynomial in main complexity parameter n.

However, applications often require the problem to be hard on *random* instances and they mobilise lattices with additional symmetries given by number fields (module lattices). The most useful notion of randomness may vary and needs to be studied. Ideally, one would like to prove a *worst-case to average-case reduction*: a proof that if random instances of the problem can be solved efficiently (with good probability), then arbitrary instances can also be solved efficiently. We achieve such a reduction in our recent paper [1] for module lattices of fixed rank over varying number fields, defining the average case of the problem using the *invariant probability measure*.

To state an informal version of the result, recall that a module lattice is a module over the ring of integers $\mathcal{O}_K$ of a number field K. Thus, its rank over $\mathbb{Z}$, i.e. its rank as a lattice, is equal to $r \cdot d$, where r is its rank over $\mathcal{O}_K$ and $d = [K : \mathbb{Q}]$. Such module lattices are parametrised by a standard automorphic quotient of $\mathrm{GL}(r)$ over K. The space of lattices now inherits the Haar measure of $\mathrm{GL}(r, \mathbb{A}_K)$.

Consider the rank r fixed. We prove that, if there exists a polynomial-time algorithm for solving γ-SIVP for Haar-random module lattices of rank r over K, then there exists a polynomial-time algorithm for solving γ'-SIVP for arbitrary such module lattices, where $\gamma' = \mathrm{poly}_r(|\Delta_K|^{1/d}, d) \cdot \gamma$. Here, Δ_K is the discriminant of K and poly_r means polynomial behaviour where the coefficients of the polynomial and its degree may depend on r. Note that, for the prime-power cyclotomic fields and small ranks used in standardised protocols (e.g. ML-KEM), we have $\mathrm{poly}_r(|\Delta_K|^{1/d}, d) = \mathrm{poly}(n)$ for $n = dr$. However, as of now, the dependence on r is exponential, suggesting that the rank aspect is significantly more difficult.

The only previous result of this type was obtained in [2] for $r = 1$, i.e. in the case of ideal lattices. Our work is the first to use the theory of higher rank automorphic forms to study computational lattice problems. The main idea is to use random walks in the space of lattices, based on taking sublattices.[1] It is well known that sublattices of any given lattice are equidistributed according to the

[1]The theory of rank one automorphic forms was used in a different type of post-quantum cryptography, based on supersingular elliptic curves. In that case, one is lead to study random walks in a finite regular graph, and these walks equidistribute rapidly as a consequence of bounds towards Ramanujan for automorphic forms on definite quaternion algebras. See more in [5], for instance.

invariant measure. Furthermore, finding a short vector in a sublattice provides a vector in the original lattice; however, the quality of the solution for the original lattice may decrease as the index of the sublattice increases. This leads to the loss in the approximation factor.

Consequently, one of the main ingredients of the method is a fine control over the rate of equidistribution of Hecke points, i.e. of lattices. Crucially, compared to other treatments of this equidistribution problem, we require an explicit dependence on the starting lattice. We provide such a result using the explicit spectral decomposition due to Moeglin–Waldspurger (a method used also in [3]) and tools from the geometry of numbers. The dependence on the starting lattice is measured in terms of a *balancedness* parameter α, which bounds all the quotients $\lambda_{j+1}^K/\lambda_j^K$, where λ_j^K is a generalised j-th successive minimum of the module lattice, defined with respect to K.

Theorem 1 (Hecke equidistribution theorem, simplified)**.** *Let $X_r(K)$ be the space of rank r module lattices over a number field K of degree d, equipped with the invariant probability measure μ. Let ϕ_{L_0} be the bump function on $X_r(K)$ centered around a lattice L_0 defined in* [1, Sec. 4.1].[2] *For a large parameter $B \gg d\log d$, let*

$$T_{\mathcal{P}} = \frac{1}{|\mathcal{P}|}\sum_{\mathfrak{p}\in\mathcal{P}} T_{\mathfrak{p}},$$

where $\mathcal{P}$ is the set of primes of norm at most B. Finally, assume ERH for the L-function of every Hecke character of K of trivial modulus. Then, for any $\varepsilon > 0$, if L is α-balanced, we have

$$\|T_{\mathcal{P}}\phi_{L_0} - 1\| = O\left(\alpha^{O(r^3d)} r^{O(r^2d)} \Delta_K^{O(r^2)} d^{O(d)} B^{-3/8+\varepsilon}\right)$$

where the implied constants depend only on ε.

In the simplest settings, we obtain satisfactory approximation factor losses only for $\alpha = \mathrm{poly}(d)$. The set of such balanced lattice is called the *bulk*, and we distinguish two other regions of the space: the *flare* and the *cusp*. Different algorithms are used for these regions to reduce the problem to the bulk. In the flare, we use the full Hecke algebra to balance lattices. In the cusp, we use LLL and what one may call lattice surgery. Conversely, we also study the balancedness of random lattices, generalising work of Shapira and Weiss on stable lattices [4].

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[2] This bump function is constructed using a "splitting" of GL(r) into SL(r) and GL(1) and taking a characteristic of a ball and a Gaussian on the respective components. There is an implied choice of parameters, e.g. the size of the support.

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Periodic billiards and invariant measures on translation surfaces

Omri Solan

A billiard trajectory in a polygonal table is a straight-line motion which is reflected at the boundary according to the rule that the angle of incidence equals the angle of reflection. When all angles of the polygon are rational multiples of π, the table is called a rational billiard table. The unfolding construction replaces the reflected billiard trajectory by a straight-line trajectory on a compact translation surface: instead of reflecting the path at an edge, one reflects the polygon itself and continues the path in a straight line. Thus questions about billiards in rational polygons can be translated into questions about straight-line flows on translation surfaces.

A striking example of this connection is due to Veech. He considered the right triangle with angles

$$\frac{\pi}{2}, \qquad \frac{\pi}{5}, \qquad \frac{3\pi}{10}.$$

By unfolding this triangle one obtains the translation surface formed from two regular pentagons. Veech showed that this surface has a closed $\mathrm{SL}_2(\mathbb{R})$-orbit in the moduli space of translation surfaces. This dynamical property naming the translation surfaces “Veech surfaces”, allowed him to compute precise quadratic asymptotics for the number of periodic billiard trajectory families of bounded length in this triangle.

The aim of the talk was to explain a generalization of this picture. For an arbitrary rational polygon, the unfolding construction again produces a translation surface. The surface’s $\mathrm{SL}_2(\mathbb{R})$-orbit is most likely not periodic, so Veech’s original argument does not apply directly. Instead, Eskin and Masur presented a substitute for the orbit being periodic. A measure classification of horocycle invariant measure on the moduli space is sufficient. I have proved this measure classification result.

The proof continues a line of works by Eskin-Mirzakhani, Eskin-Mirzakhani-Mohammadi, and Eskin-Filip-Wright, but its core is inspired by Ratner’s measure classification theorem for unipotent flows on homogeneous spaces. In Ratner’s setting, a central role is played by the polynomial drift argument. If two nearby points are followed under a unipotent flow, their relative displacement is described by a polynomial map. Thus two generic orbits drift apart at a controlled polynomial rate. Using Birkhoff’s ergodic theorem, one may choose two nearby points whose orbits equidistribute for the same invariant measure. Since the two orbits differ for long intervals by an almost constant polynomial drift, the limiting measure must be invariant under this drift. Repeating the argument produces enough additional invariance to force the measure to be homogeneous.

The difficulty in applying this idea to the moduli space of translation surfaces is that this space is not homogeneous. The relative drift between nearby surfaces is governed by the Kontsevich–Zorich cocycle, whose Lyapunov behavior varies along the orbit. Consequently we expect the drift to vary wildly. It may be very small at one time, become large later, and then become small again. Therefore the direct polynomial drift argument breaks down: one cannot simply compare two long unipotent orbit segments and deduce invariance from an almost constant drift.

The replacement is the conditional drift argument. Rather than using the whole unipotent orbit at once, one conditions on the times at which the relative drift has a prescribed size. On these conditional sets the drift is controlled well enough to recover the invariance mechanism from Ratner's proof. The ergodic input is a version of the Hopf ratio ergodic theorem which studies infinite measure dynamical systems. It states (up to mild restrictions) that almost every orbit equidistributes when restricted / conditioned on being in a finite measure set. It allows one to average only over those parts of the orbit satisfying the required drift condition. This makes it possible to extract nontrivial invariance from orbit pairs even though the unconditional drift fluctuates rapidly.

With this machinery to produce extra-invariance, one can classify all horocycle-invariant ergodic probability measures on the moduli space, and thus conclude the quadratic asymptotics for periodic-trajectory families in rational billiard tables.

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Random walks on homogeneous spaces

Timothée Bénard
(joint work with Weikun He)

Let G be a connected noncompact real Lie group with simple Lie algebra and finite center, let Λ be a discrete subgroup of G with finite covolume (a lattice), and denote by $X = G/\Lambda$ the quotient space. We call X a *finite-volume homogeneous space.*

Let μ be a finitely supported probability measure on G. Set $\Gamma_\mu = \langle \mathrm{supp}\mu \rangle_{\mathrm{grp}}$, the subgroup generated by the support of μ, and assume Γ_μ is Zariski-dense in G. For $x \in X$, define $\mu^{*n} * \delta_x$ to be the law of $g_n \dots g_1 x$ as the g_i vary independently with law μ. It is a probability measure on X, which can be interpreted as the *n-step distribution of the random walk on X driven by μ.*

With Weikun He, we show the following in [1, 2].

Theorem 1. *For every $x \in X$, we have, as n goes to infinity:*

$$\mu^n * \delta_x \to \mathrm{Haar}_X \tag{1}$$

unless the orbit $\Gamma_\mu x$ is finite.

In the above, Haar_X refers to the unique G-invariant probability measure on X, and the arrow $\to$ refers to weak-$*$ convergence.

Under extra arithmetic assumptions, we also obtain a rate of equidistribution.

Theorem 2. *Assume Λ is arithmetic, and μ is algebraic with respect to Λ.*

There exists a constant $A = A(X, \mu) > 1$ such that for all $x \in X$, $n \in \mathbb{N}$, $f \in \mathrm{Lip}(X)$, $R \geq 2$, we have

$$|\mu^n * \delta_x(f) - \mathrm{Haar}_X(f)| \leq R^{-1} \|f\|_{\mathrm{Lip}}$$

as soon as $n \geq A \log R + A \max\{|\log \mathrm{dist}(x, W_{\mu,R^A})|, \mathrm{dist}(x, x_0)\}$.

In their breakthrough work [4, 5], Benoist–Quint established (1) in Cesàro average only. They asked in their Takagi lectures [6] whether the Cesàro average could be removed. Theorem 1 answers positively. Moreover, our method of proof is disjoint from the work of Benoist–Quint.

Theorem 2 can be seen as a noncommutative extension of the work of Bourgain–Furman–Lindenstrauss–Mozes [3], who dealt with random walks on the torus $\mathbb{T}^d$, driven by a probability measure on $\mathrm{SL}_d(\mathbb{Z})$. The proof of BFLM relies strongly on Fourier analysis, and the question of more general homogeneous spaces remained open for nearly twenty years, due to the lack of a suitable analogue of Fourier analysis on semisimple homogeneous spaces.

The proofs of Theorems 1 and 2 rely on a new multislicing framework that extends Bourgain's celebrated sum–product theorem [7]. At a heuristic level, our main multislicing theorem shows that a probability measure ν on $\mathbb{R}^d$ of dimension α with respect to balls exhibits an enhanced dimension $\alpha + \varepsilon$ when viewed through sufficiently generic anisotropic ellipsoidal scales.

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Precise local mixing for self-joinings of cusped surfaces via infinite coding

Wenyu Pan

(joint work with Dongryul Kim and Hee Oh)

Let S be an oriented punctured hyperbolic surface of finite area. We consider k-tuples $\rho_1, \ldots, \rho_k$ of pairwise non-conjugate, type-preserving, geometrically finite representations of $\pi_1(S)$ into $\mathrm{SO}(n_i, 1)$ for $n_i \geq 2$. The diagonal product $\rho = \prod_{i=1}^k \rho_i$ defines a self-joining subgroup $\Gamma_\rho = \rho(\pi_1(S))$ of $G = \prod_{i=1}^k \mathrm{SO}(n_i, 1)$. We establish a precise local mixing result for the diagonal flow on $\Gamma_\rho \backslash G$, obtaining an asymptotic expansion of every order for the correlation function. To handle cusps, our proof proceeds in two parts. Geometrically, we construct an infinite countable Markov coding. Analytically, we establish the spectral properties of the associated transfer operators, which are given by vector-valued cocycles and act on appropriate Banach spaces. This is a joint work with Dongryul Kim and Hee Oh.

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Problem session

1. Radu Toma. There are several examples of equidistribution problems in which spectral methods are currently successful only on a part of the spectrum (e.g. in problems such as the Mixing Conjecture). The hope is that perhaps this gives enough information to classify the potential measures that can appear in the limit. This would be similar to the approach of homogeneous dynamics, except we do not assume there is any dynamics. The remaining measures could then be excluded by other means.

For example, take a compact arithmetic hyperbolic surface X with uniform probability measure μ and $\mathcal{B}$ a basis of Hecke–Maaß forms. We can define a basis of $L^2(X \times X, \mu \otimes \mu)$ by $\mathcal{B} \otimes \mathcal{B}$. Suppose we have a probability measure ν on $X \times X$ such that $\nu(f \otimes g) = 0$ for all $f, g \in \mathcal{B}$ such that $f \neq g$. Note that this means that the marginals of ν (the projections to the two X-components) are μ. Show that ν is a convex combination of the uniform measure $\mu \otimes \mu$ and measures supported on Hecke correspondences in $X \times X$, e.g. the diagonal or homogenenous Hecke sets (see the Mixing Conjecture literature).

Another example, although I am less confident about this one, would be $X = X_0(1)$ and a probability measure ν on X such that $\nu(f) = 0$ for all Hecke–Maaß cusp forms f. Can we show that ν must be a convex combination of the uniform measure μ and Lebesgue measures supported on $\{x + iy \mid x \in [0, 1]\}$?

2. Philippe Michel. Let $q = p^n$ be a prime power and f, g two cusp forms. We think of f, g, p as fixed and n growing to infinity. Consider the second moment

$$\sum_{\chi (\mathrm{mod}\ q)} L(\tfrac{1}{2}, f \otimes \chi^{p-1}) \overline{L(\tfrac{1}{2}, g \otimes \chi^{p-1})}. \tag{1}$$

Show that this mean value is non-zero, ideally by proving an asymptotic formula. The main term in such a formula is of size $q \log q$ if $f = g$ and of size q if $f \neq g$. The best known error term is of size $O(q(\log q)^{1-\delta})$ for some small $\delta > 0$, hence the problem of showing non-vanishing for the second moment for $f \neq g$ is open.

3. Nihar Gargava. Let $[K : \mathbb{Q}] = 3$ be totally real. For each order $\mathcal{O} \subseteq \mathcal{O}_K$, the unit group $\mathcal{O}^\times$ has $\mathbb{Z}$-rank 2. Its image $\mathcal{O}^\times \hookrightarrow \mathbb{R}^2$ under the logarithm embedding is a lattice. Let $z_{\mathcal{O}} \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ denote the corresponding shape point. What can we say about the distribution of $z_{\mathcal{O}}$ in the following three scenarios?

- Fix K, consider $\{z_{\mathcal{O}}\}_{\mathcal{O}}$. Nothing is known.
- Fix $\{z_{\mathcal{O}}\}_{\mathcal{O},K}$ varying $K, \mathcal{O}$. Density is known, equidistribution is not known. This question depends on the ordering of the fields.
- $\{z_{\mathcal{O}_K}\}$, nothing is known.

Even proving unboundedness of the set or positive Hausdorff measure of the closure is interesting for any of the above cases.

4. Elon Lindenstrauss. There has been a lot of progress recently on effective versions of Ratner's theorem. Give nice applications of effective equidistribution of unipotent flows.

5. Paul Nelson. Referring to the equidistribution conjecture $\mathcal{C}(m,n)$ introduced in joint ongoing work of Jana and Nelson, show a weak form of $\mathcal{C}(k,3)$ for $k \leq 6$ as large as possible, e.g. $k = 4$ or 5, ideally $k = 6$. A concrete incarnation is as follows: let q be a large prime. Establish the upper bound

$$\sum_{\substack{\pi:\mathrm{GL}_3/\mathbb{Q}\text{ cuspidal}\\ C(\pi_\infty)\ll 1,\\ C(\pi_{\mathrm{fin}})=q}} \int \left|\Lambda\Big(\frac{1}{2}+it,\pi\Big)\right|^{2k} dt \ll q^{3+\varepsilon}. \tag{2}$$

Ideally, give an asymptotic formula. Classically, this moment is over cusp forms with respect to $\Gamma_1(q) \subseteq \mathrm{SL}_3(\mathbb{Z})$ with bounded spectral parameter.

6. Nicolas de Saxcé. Take an irreducible projective variety X over $\mathbb{Q}$. Assume that $X(\mathbb{Q}) \subseteq X(\mathbb{R})$ is dense and that $X(\mathbb{R})$ is connected. Assume X is endowed with a height and a Riemannian metric, and define the Diophantine exponent of a point x in $X(\mathbb{R})$ by

$$\beta(x) = \sup\left\{\beta > 0 \;\middle|\; \exists (v_k)_{k\geq 1} \text{ in } X(\mathbb{Q}) \text{ such that } \begin{array}{l} d(v_k,x) \to_{k\to\infty} 0 \\ \text{and } \forall k,\ d(v_k,x) \leq H(v_k)^{-\beta} \end{array}\right\}.$$

Is $\beta(x)$ almost surely constant when x is chosen randomly according to the Lebesgue measure class on $X(\mathbb{R})$?

7. Paul Nelson. Establish some example of non-abelian Fourier uniqueness. A possible starting point can be as follows: take $G = \mathrm{SL}_2(\mathbb{R})$ and $K = \mathrm{SO}(2)$. For a bi-K-invariant function f let $\hat{f}$ be the spherical transform f. For which sets $S \subseteq K\backslash G/K$, $\widehat{S} \subseteq \widehat{K\backslash G/K}$ can we conclude that $f|_S = 0$, $\widehat{f}|_{\widehat{S}} = 0$ implies $f = 0$?

8. Paul Nelson. Adapt the method of Nelson-Venkatesh on moments of Gan–Gross–Prasad type Rankin–Selberg L-functions to a non-compact setting. This may involve ruling out exceptional measures after the application of Ratner theory.

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