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Reductive Algebraic Groups

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SPHERICAL SUBGROUPS IN REDUCTIVE ALGEBRAIC GROUPS

FRIEDRICH KNOP AND GERHARD RÖHRLE

*Dedicated to Professor Jean-Pierre Serre on the occasion of his 100th birthday
with our deepest admiration and respect.*

ABSTRACT. In [KR15], we determined all spherical affine homogeneous varieties for simple algebraic groups in arbitrary characteristic. The present paper extends this classification to semisimple groups. This generalizes work done independently by Brion and Mikityuk in characteristic zero. Our primary approach relies on a lifting lemma to characteristic zero, enabling us to directly apply Brion’s and Mikityuk’s results.

1. INTRODUCTION

Let G be a reductive algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$. A closed subgroup H of G is called *spherical* if it has a dense orbit on the flag variety G/B of G . Alternatively, B acts on G/H with an open orbit. Accordingly, a G -variety with this property is also called *spherical*. If H is spherical in G , then we also speak of (G, H) as a *spherical pair*.

The purpose of this paper is to classify all connected reductive spherical subgroups of reductive groups in arbitrary characteristic, thereby generalizing the classifications of Brion [Bri87] and Mikityuk [Mik86, §5] to positive characteristic.

The classification of such subgroups in the case when G is simple was obtained by Krämer in [Kr79]. In [KR15], we extended Krämer’s classification to arbitrary positive characteristic.

The class of reductive spherical subgroups is of particular importance. This is shown by the fact that Krämer’s list permeates much of the theory of spherical varieties in characteristic zero. In particular, these kinds of subgroups provide many of the building blocks for arbitrary spherical subgroups (see, e.g., Bravi–Pezzini [BP16]). We expect reductive spherical subgroups to play a similar role for arbitrary p . For instance, the results from [KR15] were already used in [Kn14] to list all spherical subgroups of rank 1, which is crucial for the theory of general spherical varieties.

For $p \neq 2$, the class of reductive spherical subgroups includes all symmetric subgroups, i.e., subgroups that are fixed point sets of an involutive automorphism of G (see e.g., Springer [Sp85]). Nevertheless, for $p = 2$, symmetric subgroups are ill-behaved. Thus, reductive spherical subgroups seem to be the correct replacement.

As a consequence of the Bruhat decomposition, any reductive group G gives rise to a spherical pair – this is one of the most fundamental examples of a spherical subgroup (indeed, of a

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symmetric subgroup when $p \neq 2$) in the non-simple case. For example, let G be a reductive group and B a Borel subgroup of G . By the Bruhat decomposition, the orbits of the diagonal action of G on $G/B \times G/B$ are in bijective correspondence with the Weyl group of G . In particular, there is a dense G -orbit in $G/B \times G/B$, and consequently, the diagonal subgroup in $G \times G$ is spherical in $G \times G$.

Note that the requirement of having an open orbit in G/B implies that H has, in fact, only finitely many orbits on G/B (see, e.g., [Kn95]). Therefore, our classification theorem can also be viewed as a contribution to the program by Seitz [Se97] to classify all pairs of subgroups X, Y of a reductive group G such that there are only finitely many (X, Y) -double cosets in G (see also [Br98]).

The most important previous works are the aforementioned classifications due to Mikityuk [Mik86, §5] and Brion [Bri87]. Not only do we use their lists as a guideline, but, more importantly, they enter crucially into our computations even in positive characteristic. This is because we extensively employ the technique of reduction mod p developed in [KR15]. Here, we show that all instances from the Brion–Mikityuk list descend to arbitrary positive characteristic.

In [KR15], we have already observed that for semisimple groups G in positive characteristic, infinitely many “new” spherical subgroups can arise in connection with a finite orbit module involving Frobenius twists, cf. [GLMS97, Lem. 2.6]. Specifically, consider the irreducible representation $\Delta_q : \mathrm{SL}(2) \rightarrow \mathrm{GL}(4)$ of $\mathrm{SL}(2)$ with highest weight $(q+1)\omega_1$ with $q = p^m > 1$. Since Δ_q is self-dual, its image lies in $\mathrm{SO}(4)$.

Note that for the purpose of classifying spherical subgroups, we may replace G with an isogenous group (using Lemma 2.7). Therefore, the simply connected Spin groups do not appear in Table 1, for instance, but rather their isogenous counterparts do.

As the case when G is simple is handled in [KR15], in Theorem 1.1 we restrict ourselves to the non-simple case. Thanks to Lemma 2.6, we can reduce to the instance when G is semisimple. For the concepts of an indecomposable spherical pair and for an isogeny between spherical pairs, see Definition 2.1.

The only novelty is that, up to isogeny, there is essentially only one series (case (S9) in Table 2) that is genuinely unique to positive characteristic, namely in characteristic 2.

Theorem 1.1. *Let G be a connected semisimple algebraic group and let $H \subset G$ be a spherical connected reductive subgroup of G . Assume that G is not simple and that the pair (G, H) is indecomposable. Then (G, H) is isogenous to one of the items of Table 2.*

Here, “isogeny” does not only mean only central isogenies but also outer automorphisms, Frobenius morphisms, and exceptional isogenies. Taking this into account, in positive characteristic a given group G might contain an infinitude of spherical connected reductive subgroups.

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2. PRELIMINARIES

2.1. Notation. Throughout, G is a connected reductive algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$. Let B be a Borel subgroup of G . By $\text{rk } G$ we denote the rank of G .

Unless stated otherwise, all subgroups are assumed to be closed and reduced. Let $H \subseteq G$ be a subgroup. Then H^0 denotes the identity component of H . Moreover, $\Delta H \subset H \times H$ is the diagonal embedding of H into $H \times H$. If H acts on the variety X , we write Hx for the H -orbit of $x \in X$ and $C_H(x)$ for its (reduced) stabilizer in H .

A homomorphism $\varphi : H_1 \rightarrow H_2$ between connected groups is an *isogeny* if it is surjective with finite kernel. We say that H_1 and H_2 are isogenous and denote this by $H_1 \sim H_2$, if there is a connected group H_0 and isogenies $H_1 \leftarrow H_0 \rightarrow H_2$. More generally:

Definition 2.1. Let G and H be connected and reductive.

- (i) A *homogeneous pair* $(G, H; \varrho)$ is given by a homomorphism $\varrho : H \rightarrow G$ such that $H \rightarrow \varrho(H)$ is an isogeny. We usually suppress ϱ if no confusion can arise. The homogeneous variety $G/\varrho(H)$ is also denoted by G/H . The pair (G, H) is *trivial* if $\varrho(H) = G$.
- (ii) An *isogeny between two homogeneous pairs* $(G_1, H_1; \varrho_1)$ and $(G_2, H_2; \varrho_2)$ is an isogeny $\varphi : G_1 \rightarrow G_2$ which maps $\varrho_1(H_1)$ onto $\varrho_2(H_2)$. This induces a surjective finite morphism $G_1/H_1 \rightarrow G_2/H_2$. Two homogeneous pairs are said to be *isogenous* if there is a homogeneous pair (G_0, H_0) and isogenies $(G_1, H_1) \leftarrow (G_0, H_0) \rightarrow (G_2, H_2)$.
- (iii) A pair (G, H) is *decomposable* if it is isogenous to a product pair $(G_1 \times G_2, H_1 \times H_2)$ where both pairs (G_1, H_1) and (G_2, H_2) are non-trivial. A pair which is neither trivial nor decomposable is called *indecomposable*.

Remark 2.2. The advantage of this notion is that it allows us to assume that both G and H are the Cartesian product of a torus and simple factors.

2.2. On some subgroups of simple groups. The following auxiliary results are used in the proof of Lemma 4.5 below.

Lemma 2.3. *Let G be simple and let $H \subsetneq G$ be a proper subgroup of maximal dimension. Then H is a parabolic subgroup of G .*

Proof. We claim that there is a finite-dimensional G -module V such that $\mathbb{P}(V)^G = \emptyset$ and $\mathbb{P}(V)^H \neq \emptyset$. Thanks to Chevalley's Theorem, see [Bo91, Ch. II, Thm. 5.1], there is a finite-dimensional G -module $\tilde{V}$ and $y \in \mathbb{P}(\tilde{V})$ such that $C_G(y) = H$. Let $V := \tilde{V}/\tilde{V}^G$. Since $\text{Ext}_G^1(k, k) = 0$, e.g., see [Ja03, (2.12(1))], we have $V^G = 0$. But then $\mathbb{P}(V)^G = \emptyset$, since the character group of G is trivial. Now observe that $y \notin \mathbb{P}(\tilde{V}^G)$, since $C_G(y) = H \subsetneq G$. Therefore, the image of y in $\mathbb{P}(V)$ is an H -fixed point, proving the claim.

Now let V be as in the claim and choose $y \in \mathbb{P}(V)^H$. Then the orbit map induces a map $\varphi : X := G/H \rightarrow \mathbb{P}(V)$ mapping $x := 1H \in X$ to y . If $\dim \varphi(X) < \dim X$, then $\dim C_G(y) >$

$\dim H$, contradicting the maximality of $\dim H$. Otherwise, $\dim \varphi(X) = \dim X$. In this case, $C_G(y)$ contains $H = C_G(x)$ as a subgroup of finite index. Suppose H is not parabolic in G . Then $C_G(y)$ is not parabolic either. Hence $\varphi(X)$ is not complete and therefore not closed in $\mathbb{P}(V)$. Pick $z \in Z := \overline{\varphi(X)} \setminus \varphi(X)$ in $\mathbb{P}(V)$. Then $\dim Gz \leq \dim Z < \dim \varphi(X) = \dim Gx$. Therefore, $\dim C_G(z) > \dim C_G(x) = \dim H$, again contradicting the maximality of $\dim H$. Thus H is parabolic after all. $\square$

Lemma 2.4. *Let G be simple and let $H \subsetneq G$ be a proper subgroup. Then $\dim G/H \geq \operatorname{rk} G$.*

Proof. Let $T \subseteq G$ be a maximal torus. By the rigidity of tori, there exists a subgroup S of T such that $S = C_T(x)$ for a generic $x \in G/H$. Then S acts trivially on G/H and so S lies in the kernel of the action of G on G/H . Since G is simple and $H \subsetneq G$, it follows that $S^0 = 1$. Consequently, there is an x in G/H such that $\dim G/H \geq \dim Tx = \dim T = \operatorname{rk} G$, and the result follows. $\square$

2.3. Reduction results for spherical subgroups. The first well-known observation, which is immediate from the definition, allows us to reduce to the case when H is connected.

Lemma 2.5. *Let $H \subseteq G$ be a subgroup of G . Then H is spherical in G if and only if the identity component H^0 is spherical.*

The next observation allows us to reduce to the case when G is semisimple, cf. [Kr79, §4], which we assume henceforth.

Lemma 2.6. *Let G be reductive and let Z be its connected center. Then H is spherical in G if and only if ZH/Z is spherical in G/Z .*

The next lemma states that the set of spherical subgroups of G depends only on its isogeny class.

Lemma 2.7. *Let $\varphi : G_1 \rightarrow G_2$ be an isogeny between connected reductive groups. Then φ induces a bijection between the sets of (conjugacy classes of) connected (reductive) spherical subgroups of G_1 and G_2 .*

Definition 2.8. A homogeneous pair (G, H) is called *spherical* if G/H is spherical.

The lemma implies that this property is isogeny-invariant. It is clear that a product of homogeneous pairs is spherical if and only if each factor is spherical. Thus, we may restrict our attention to indecomposable spherical pairs.

2.4. Deformation of spherical subgroups. In this subsection, we recall some results from [KR15, Thm. 3.4] which enables us to compare spherical subgroups in positive characteristic to those in characteristic zero. This approach reduces most of the classification work to the results of Brion–Mikityuk.

Let $\mathcal{G}$ be a split reductive group scheme over $\mathbb{Z}$ (this entails connected geometric fibers), e.g., see [SGA3, Exp. I, 4.2], and $\mathcal{H} \subseteq \mathcal{G}$ a closed flat subgroup scheme. Then for any

algebraically closed field k , we obtain the geometric fibers $H_k \subseteq G_k$ by base change and the homogeneous variety $X_k = G_k/H_k$. We say that a homogeneous pair (G, H) defined over k *lifts to characteristic zero* if it is isogenous to X_k for some pair $(\mathcal{G}, \mathcal{H})$. In this case, the lift is $X_{\mathbb{C}}$, where $\mathbb{C}$ is the field of complex numbers. If $k = \mathbb{C}$, we also say that $X_{\mathbb{C}}$ is defined over $\mathbb{Z}$. Now it follows from [KR15, Thm. 3.4]:

Lemma 2.9. *Assume that the homogeneous G -variety X defined over k lifts to characteristic zero. Then X is spherical if and only if its lift $X_{\mathbb{C}}$ is spherical.*

Remark 2.10. An inspection of the tables by Brion and Mikityuk shows that all indecomposable spherical pairs (G, H) defined over $\mathbb{C}$ with G not simple have a $\mathbb{Z}$ -form. Their k -forms comprise items (S0)–(S8) of Table 2.

2.5. Some sphericity criteria. The following numerical criterion is fundamental in identifying or ruling out candidates for spherical subgroups.

Lemma 2.11. *Let $H \subseteq G$ be spherical in G . Then*

$$(2.12) \quad \dim H \geq \dim G/B = \frac{1}{2}(\dim G - \operatorname{rk} G).$$

Proof. By definition, B has an open orbit in G/H . Hence $\dim B \geq \dim G/H$ which is equivalent to (2.12). $\square$

We frequently use the following “transitivity” property for spherical subgroups, e.g., see [KR15, Lem. 2.3].

Lemma 2.13 (Weak Transitivity). *Let $H_1 \subseteq H_2 \subseteq G$ be connected reductive subgroups of G . If H_1 is spherical in G , then H_1 is spherical in H_2 and H_2 is spherical in G .*

This criterion is just necessary. To actually show that H_1 is spherical in G we need more detailed information. To this end, consider for any quasi-affine G -variety X the group $\Xi(X)$ of characters of B -semi-invariant rational functions on X . We define the rank of X as the $\mathbb{Z}$ -rank of $\Xi(X)$. Note that $\Xi(X)$ is generated by the set of highest weights occurring in the representation $k[X]$ of regular functions on X .

Let S_0 be the set of simple roots α of G (with respect to B) such that the coroot $\alpha^\vee$ is orthogonal to $\Xi(X)$. Then attached to S_0 , there is a parabolic subgroup $P = P(X)$ of G such that $\Xi(X) \subseteq \Xi(P)$, where $\Xi(P)$ is the character group of P . We define the subgroup P_0 of P by $P_0 = \{y \in P \mid \chi(y) = 1 \text{ for all } \chi \in \Xi(X)\}$.

Theorem 2.14 ([KR15, Thm. 4.1]). *Let X be a quasi-affine G -variety. Let $P = P(X)$ be as above. Then there is a P -invariant dense open subset X_0 of X such that $C_P(x)R_u(P) = P_0$ and $C_P(x) \cap R_u(P)$ is finite for all $x \in X_0$. In particular, $C_P(x)$ is a reductive group which is isogenous to a Levi subgroup of P_0 .*

Definition 2.15. In the setting of Theorem 2.14, let $L(X) := C_P(x)^0$ for any $x \in X_0$. It is isogenous to a Levi factor of P_0^0 .

Observe that if we put $H := C_G(x)$, then $L(X) = (P \cap H)^0$ can also be considered as a subgroup of H . The significance of $L(X)$ is a converse to Lemma 2.13:

Lemma 2.16 (Strong Transitivity). *Let G be a connected, reductive group and let $H_1 \subseteq H_2 \subseteq G$ be connected, reductive subgroups. Put $L := L(G/H_2)$. Then G/H_1 is spherical if and only if G/H_2 is spherical and H_2/H_1 is spherical as an L -variety.*

Proof. Consider the canonical morphism $\varphi : X_1 := G/H_1 \rightarrow X_2 := G/H_2$. Let x be a point in the open B -orbit X_2^0 of X_2 . Since X_2^0 is also P -invariant, we get a bijective morphism $P/C_P(x) \rightarrow X_2^0$, hence a bijective morphism $P \times^{C_P(x)} F \rightarrow \varphi^{-1}(X_2^0) \subseteq X_1$ with $F := \varphi^{-1}(x) \cong H_2/H_1$. Thus, B has an open orbit in X_1 if and only if $C_P(x) = C_B(x)$ has an open orbit in F . Now $C_B(x)$ is a Borel subgroup of $C_P(x)$. Thus X_1 is spherical for G if and only if F is spherical as an $L(= C_P(x)^0)$ -variety. $\square$

To compute $L(X)$ we use the following comparison result. For this put $\Xi_{\mathbb{Q}}(X) := \Xi(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 2.17 ([Kn14, Lem. 3.7]). *Assume that $X_{\mathbb{C}}$ has a $\mathbb{Z}$ -form with geometric fiber X_k . Then $\Xi_{\mathbb{Q}}(X_k) = \Xi_{\mathbb{Q}}(X_{\mathbb{C}})$. In particular, the group $L(X_k)$ is a reductive group whose root datum is, up to isogeny, the same sub-root system of the root system of G as that of $L(X_{\mathbb{C}})$.*

Remark 2.18. Over $\mathbb{C}$, Bravi–Pezzini calculated in [BP16] the Luna diagrams of all affine, homogeneous, spherical varieties. The group $L(G/H)$ can easily be read off: First, the simple roots of $P(X)$ are those without “decoration”. Second, the rank of the character group $\Xi(X)$ equals the number of spherical roots. From this, $L(G/H)$ can be determined up to isogeny. It is indicated in the fourth column of Table 2.

3. SOUNDNESS OF TABLE 2

Lemma 3.1. *All varieties G/H from Table 2 are spherical.*

Proof. All items of Table 2, except for (S9), are defined over $\mathbb{Z}$. Hence their sphericity follows via Lemma 2.9 from the work of Mikityuk and Brion.

It remains to show that the case (S9) is spherical: Let $\bar{H}$ be the subgroup $(\mathbb{G}_2 \times \mathrm{Sp}(2)) \times (\mathrm{Sp}(2) \times \mathrm{Sp}(2n))$ of G such that H is contained in $\bar{H}$ diagonally. Note that by Lemma 2.16, G/H is spherical if and only if $G/\bar{H}$ is spherical and $\bar{H}/H$ is spherical as an $L(G/\bar{H})$ -variety. Now $G/\bar{H} = X_1 \times X_2$, where $X_1 = \mathrm{Sp}(8)/(\mathbb{G}_2 \times \mathrm{Sp}(2))$ and $X_2 = \mathrm{Sp}(2n+2)/(\mathrm{Sp}(2) \times \mathrm{Sp}(2n))$. Since both X_1 and X_2 are spherical by [KR15], $G/\bar{H}$ is also spherical.

Now $L(G/\bar{H})$ decomposes as $L_1 \times L_2$, where $L_1 \cong \mathbb{G}_m$, see [KR15, Prop. 4.5], and $L_2 \cong \mathrm{Sp}(2) \times \mathrm{Sp}(2n-2)$. Since $\bar{H}/H \cong \mathrm{Sp}(2)$, we see that G/H is spherical if and only if $\mathrm{Sp}(2)$ is spherical as an $(S \times \mathrm{Sp}(2))$ -variety, where S is the image of L_1 in $\mathrm{Sp}(2)$. It remains to show that $S \neq 1$.

We have $\Xi(X_1) = \langle \omega_1 + \omega_4, \omega_2, \omega_3 \rangle_{\mathbb{Z}}$, thanks to [KR15, Prop. 4.5], where ω_i denotes the i -th fundamental weight of G with respect to the labeling in [Bou68, Planche III]. Hence L_1 is the 1-parameter subgroup of the Borel subgroup of $\mathrm{Sp}(8)$ corresponding to the cocharacter $\lambda := \alpha_1^{\vee} - \alpha_4^{\vee} = \varepsilon_1 - \varepsilon_2 - \varepsilon_4$. On the other hand, the characters of the maximal torus of $\mathbb{G}_2$

inside $\mathrm{Sp}(6)$ are those which are orthogonal to $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ (see [Bou68, Planche IX]). Now L_1 as a subgroup of H is $W_{\mathrm{Sp}(8)}$ -conjugate to λ . Hence it is of the form $\pm\varepsilon_i \pm \varepsilon_j \pm \varepsilon_k$ for $1 \leq i < j < k \leq 4$. Being orthogonal to $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ implies $k = 4$. This entails that the projection S of L_1 to the $\mathrm{Sp}(2)$ -factor of $\mathbf{G}_2 \times \mathrm{Sp}(2)$ is non-trivial. $\square$

4. COMPLETENESS OF TABLE 2

Definition 4.1. We say that a pair (G, H) is *known spherical* if it is isogenous to a product of trivial pairs, pairs contained in [KR15, Tables 1 and 2], or pairs from Table 2.

Since Table 1 of [KR15] is used extensively, we restate it here in a condensed form for the convenience of the reader. More precisely, the table is complete only *up to isogeny*, i.e., up to conjugation, outer automorphisms (such as triality), central isogenies, and totally inseparable isogenies (such as Frobenius morphisms).

TABLE 1. Spherical pairs $H \subset G$ with G classical up to isogeny.

	H	G	
(C1)	$\mathrm{SO}(n)$	$\mathrm{SL}(n)$	$n \geq 2$
(C2)	$S(\mathrm{GL}(m) \times \mathrm{GL}(n))$	$\mathrm{SL}(m+n)$	$m \geq n \geq 1$
(C3)	$\mathrm{SL}(m) \times \mathrm{SL}(n)$	$\mathrm{SL}(m+n)$	$m > n \geq 1$
(C4)	$\mathrm{Sp}(2n)$	$\mathrm{SL}(2n)$	$n \geq 2$
(C5)	$\mathbb{G}_m \cdot \mathrm{Sp}(2n)$	$\mathrm{SL}(2n+1)$	$n \geq 1$
(C6)	$\mathrm{Sp}(2n)$	$\mathrm{SL}(2n+1)$	$n \geq 1$
(C7)	$\mathrm{GL}(n)$	$\mathrm{Sp}(2n)$	$n \geq 1$
(C8)	$\mathbb{G}_m \times \mathrm{Sp}(2n-2)$	$\mathrm{Sp}(2n)$	$n \geq 2$
(C9)	$\mathrm{Sp}(2m) \times \mathrm{Sp}(2n)$	$\mathrm{Sp}(2m+2n)$	$m, n \geq 1$
(C10)	$\mathrm{GL}(n)$	$\mathrm{SO}(2n)$	$n \geq 2$
(C11)	$\mathrm{SL}(n)$	$\mathrm{SO}(2n)$	$n \geq 3$ odd
(C12)	$\mathbf{G}_2$	$\mathrm{SO}(8)$	
(C13)	$\mathrm{SO}(2) \times \mathrm{Spin}(7)$	$\mathrm{SO}(10)$	
(C14)	$\mathrm{GL}(n)$	$\mathrm{SO}(2n+1)$	$n \geq 2$
(C15)	$\mathrm{SO}(m) \times \mathrm{SO}(n)$	$\mathrm{SO}(m+n)$	$m \geq n \geq 1$
(C16)	$\mathrm{Spin}(7)$	$\mathrm{SO}(9)$	
(C17)	$\mathbf{G}_2$	$\mathrm{SO}(7)$	
(C18)	$\mathbf{G}_2 \times \mathrm{Sp}(2)$	$\mathrm{Sp}(8)$	$p = 2$

The proof of Theorem 1.1 reduces easily to the following statement.

Lemma 4.2. *Suppose that (G, H) is known spherical and $H' \subset H$ is a maximal, connected reductive subgroup. Then (G, H') is known spherical or not spherical.*

Proof of Theorem 1.1. In view of Lemma 3.1, it suffices to show that every spherical pair (G, H) is known spherical. If $H = G$, then this is clear by definition. Otherwise, there is a connected reductive intermediate subgroup $H \subset \bar{H} \subseteq G$ such that H is a maximal connected

reductive subgroup of $\bar{H}$. Then $G/\bar{H}$ is spherical by Lemma 2.13. Thus, (G, H) is known spherical by Lemma 4.2. $\square$

The vast majority of cases to be inspected are handled by the following lemma. It utilizes the classification results of Krämer, Mikityuk and Brion to their full extent.

Lemma 4.3. *Assume that the triple (G, H, H') in Lemma 4.2 is defined over $\mathbb{Z}$. Then the assertion of Lemma 4.2 holds for (G, H, H') .*

Proof. Assume that the pair (G, H') is spherical. Then, by Lemma 2.9, its lift to characteristic zero is spherical as well. Hence, this lift is known spherical by the work of Krämer, Mikityuk, and Brion. Since the reductions modulo p of their tables are contained in our tables, (G, H') is also known spherical. $\square$

Remark 4.4. Observe that every indecomposable factor of a known spherical pair is isogenous to one that lifts to characteristic zero, except for the cases isogenous to case (C18) in Table 1 or case (S9) in Table 2.

Let H be a connected reductive group and let $H' \subset H$ be a maximal connected reductive subgroup. Extending the version of Dynkin's Lemma [Kn14, Lem. 3.3] slightly from semisimple to reductive groups, we obtain the following cases where H' is obtained from H

- (a) by replacing the connected center of H with a subtorus of codimension one, or
- (b) by replacing a simple factor of H with a maximal connected reductive subgroup, or
- (c) by replacing two simple factors H_1 and H_2 of H with a connected subgroup $\tilde{H} \subset H_1 \times H_2$ for which both projections $\tilde{H} \rightarrow H_i/(H_1 \cap H_2)$ are isogenies.

Proof of Lemma 4.2 in case (a). By Remark 4.4, we may assume that all factors of (G, H) lift to characteristic zero, except for those of type (C18) or (S9). Since these exceptions do not contribute to $Z(H)^0$, we may assume that they do not occur. Thus, (G, H) lifts to characteristic zero. The torus $Z(H)^0$ lifts to characteristic zero as well, along with all of its subgroups. Thus, (G, H, H') lifts to characteristic zero, and the assertion follows from Lemma 4.3. $\square$

Proof of Lemma 4.2 in case (b). Assume that (G, H') is spherical. We may further assume that (G, H) is indecomposable, as only one indecomposable factor of (G, H) is modified in case (b). If G is simple, then (G, H') appears in [KR15, Tables 1 and 2] and is thus known spherical. If G is not simple, then (G, H) is isogenous to one of the items in Table 2. We now analyze these cases individually.

(S0) is dealt with by the following lemma.

Lemma 4.5. *Let G be simple and let $H \subset G$ be a maximal connected reductive subgroup. Then $(G \times G, \Delta H)$ is not spherical.*

Proof. We have $\Delta H \subset \Delta G \subset G \times G$. Since $L(G \times G/\Delta G) = T$, where T is a maximal torus of G (see Table 2), it suffices to show that $\Delta G/\Delta H \cong G/H$ is not spherical as a T -variety (see Lemma 2.16). By Lemma 2.3, G contains a proper subgroup H' such that $\dim G/H > \dim G/H'$. Moreover, by Lemma 2.4, we have $\dim G/H' \geq \dim T$. Thus, G/H is not spherical as a T -variety, as claimed. $\square$

For the cases where (G, H) is of type (S1) or (S2), we have $L(G/H) = 1$ by Table 2. Thus, the pair (G, H') is not spherical by Lemma 2.16.

Let (G, H) be one of the pairs (S3) – (S8). Since H/H' is also spherical (Lemma 2.13), it occurs in Table 1. Hence, for $p \neq 2$, the pair (H, H') can be lifted to characteristic zero. Consequently, (G, H, H') can also be lifted, and the assertion follows from Lemma 4.3.

Now suppose $p = 2$. Then we require the following lemma.

Lemma 4.6. *Let $p = 2$ and $n \geq 2$. Then $\mathrm{Sp}(2n)/\mathrm{SO}(2n)$ is not spherical as a $(\mathbb{G}_m \times \mathrm{Sp}(2n-2))$ -variety. In particular, it is not spherical as an $\mathrm{Sp}(2n-4)$ -variety. Moreover, $\mathrm{Sp}(6)/\mathbb{G}_2$ is not spherical as an $\mathrm{Sp}(4)$ -variety.*

Proof. Let $(x_1, \dots, x_{2n})$ be the standard basis of k^{2n} . Then $S^2 k^{2n}$ is the space of quadratic forms in the variables x_i . Since $p = 2$, we have the short exact sequence:

$$0 \longrightarrow (k^{2n})^{(2)} \longrightarrow S^2 k^{2n} \xrightarrow{\pi} \bigwedge^2 k^{2n} \longrightarrow 0,$$

where we identify $(k^{2n})^{(2)}$ with the subspace spanned by x_i^2 for $1 \leq i \leq 2n$ and π is the polarization map (e.g., see [EKM08, Def. 7.1]). Let $q := x_1 x_{2n} + \dots + x_n x_{n+1}$. Then $\omega := \pi(q) = x_1 \wedge x_{2n} + \dots + x_n \wedge x_{n+1}$ defines the group $\mathrm{Sp}(2n)$. Let $X := \pi^{-1}(\omega) = q + \langle x_1^2, \dots, x_{2n}^2 \rangle \cong \mathbb{A}^{2n}$. Then X can be identified with $\mathrm{Sp}(2n)/\mathrm{O}(2n)$. On the other hand, the map $X \rightarrow S^2 k^2 = S^2 \langle x_1, x_{2n} \rangle$ is $\mathrm{Sp}(2n-2)$ -invariant and non-constant. This shows that the quotient $X//\mathrm{Sp}(2n-2)$ is at least 2-dimensional. Consequently, not even $\mathbb{G}_m \times \mathrm{Sp}(2n-2)$ has a dense orbit in X , and therefore neither does its Borel subgroup. The same holds for the double cover $\mathrm{Sp}(2n)/\mathrm{SO}(2n)$. The last statement follows from the dimension criterion in Lemma 2.11. $\square$

We now return to the case $H' \subset H \subset G$ for $p = 2$. Note that all connected subgroups of $\mathrm{Sp}(2)$ still lift to characteristic zero. Thus, we can argue as before. Therefore only the additional cases $H/H' \sim \mathrm{Sp}(2n)/\mathrm{SO}(2n) \sim \mathrm{SO}(n+1)/\mathrm{SO}(n)$ for $n \geq 2$ and $H/H' \sim \mathrm{Sp}(6)/\mathbb{G}_2$, which do not lift, need to be considered. Observe that, by Lemma 4.6, H/H' is not spherical as an $L(G/H)$ -variety in each instance with the exception of case (S6) when $H/H' \cong \mathrm{Sp}(6)/\mathbb{G}_2$. In this case replacing the $\mathrm{Sp}(6)$ -factor with $\mathbb{G}_2$ results in a spherical pair, namely (S9). Hence, in all cases, (G, H') is either not spherical or known spherical.

Finally, let (G, H) be of type (S9), and let H_0 be a maximal reductive subgroup of $\mathbb{G}_2$. Let $H' = H_0 \times \mathrm{Sp}(2) \times \mathrm{Sp}(2n)$. If G/H' were spherical, then the quotient $\mathrm{Sp}(8)/(H_0 \times \mathrm{Sp}(2))$ would also be spherical. However, this contradicts the dimension bound (2.12). The same argument works for the $\mathrm{Sp}(2)$ -factor.

If, on the other hand, the $\mathrm{Sp}(2n)$ -factor is replaced with a maximal reductive subgroup H_0 , then H' is contained in $\mathrm{Sp}(6) \times \mathrm{Sp}(2) \times \mathrm{Sp}(2n)$. The discussion of case (S7) above shows that $\mathrm{Sp}(6) \times \mathrm{Sp}(2) \times H_0$ is spherical if and only if $H_0 = \mathbf{G}_2$. However, $\mathbf{G}_2 \times \mathrm{Sp}(2) \times \mathbf{G}_2$ inside $\mathrm{Sp}(8) \times \mathrm{Sp}(8)$ is not spherical by the dimension bound (2.12). This concludes the proof of Lemma 4.2 in case (b).

Proof of Lemma 4.2 in case (c). Let H'_1 and H'_2 be two different simple factors of H , and for $i = 1, 2$, let $\varrho_i : \tilde{H} \rightarrow H'_i$ be isogenies. Then, H' is obtained by replacing $H'_1 H'_2 \subseteq H$ with the image of $\varrho_1 \cdot \varrho_2$. There are now two cases to consider, namely:

- (i) the subgroups H'_i lie in the same indecomposable component of (G, H) , and
- (ii) the subgroups H'_i lie in two different components of (G, H) .

Note also that the isogenies ϱ_i might not be central, which could prevent liftability to characteristic zero.

We claim that in case (i), the pair (G, H') is not spherical. If G is simple, the cases where H has two isogenous factors are: $(\mathrm{SL}(4), \mathrm{SO}(4))$, $(\mathrm{SL}(2n), \mathrm{SL}(n) \mathbb{G}_m \mathrm{SL}(n))$ for $n \geq 2$, $(\mathrm{Sp}(4n), \mathrm{Sp}(2n) \mathrm{Sp}(2n))$ for $n \geq 1$, $(\mathrm{SO}(2n), \mathrm{SO}(n) \mathrm{SO}(n))$ for $n \geq 3$, and $(\mathrm{SO}(n), \mathrm{SO}(4) \mathrm{SO}(n-4))$. None of these cases lead to (G, H') being spherical, as can be seen by inspecting Table 1.

If G is not simple, then (G, H) belongs to Table 2. If H'_1 and H'_2 are subgroups of the same simple factor of G , then the projection of (G, H') is not spherical by the case where G is simple. Thus, we must still check case (S4) with $(m, n) = (2, 1)$ (which also settles (S5)), case (S6) with $m = n$ (which also settles (S7) with the first and third factors of H identified and (S8)), and case (S7) with $m = n$. By Lemma 2.16, we need to show that $H_0 \sim H'_1 H'_2 / H_0$ is not spherical as an $L(G/H)$ -variety. This results in an action of $\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2n-2)$ on $\mathrm{Sp}(2n)$, which is not spherical by Lemma 2.11. This concludes the proof in case (i).

In case (ii), let $(G, H) = (G_1, H_1) \times (G_2, H_2)$ be such that a factor H_0 of H' is embedded diagonally in $H_1 \times H_2$. First, we treat the case where (G_2, H_2) is a trivial pair, i.e., $G_2 = H_2$. Then, the diagram¹ of (G, H') looks like

$$(4.7) \quad \begin{array}{ccc} & G_1 & G_2 \\ & \swarrow & \searrow \\ H'_0 & \varrho_1 & H_0 \\ & \nwarrow & \nearrow \end{array}$$

where H'_0 is a connected reductive group (possibly trivial), ϱ_1 has a finite kernel, and ϱ_2 is an isogeny. Let $\tilde{H}$ be the image of H_0 in $G_1 \times G_2$, and let $p_i : \tilde{H} \rightarrow G_i$ be the projection for $i = 1, 2$. Then, $(\mathrm{id}, p_2) : G_1 \times \tilde{H} \rightarrow G_1 \times G_2$ is an isogeny that maps $(p_1, \mathrm{id})(\tilde{H})$ to $(\varrho_1, \varrho_2)(H_0)$. On the other hand, the isogeny $(\mathrm{id}, p_1) : G_1 \times \tilde{H} \rightarrow G_1 \times \varrho_1(H_0)$ maps $(p_1, \mathrm{id})(\tilde{H})$ to $(\mathrm{id}, \mathrm{id})(\varrho_1(H_0))$. Thus, we have shown that (G, H') is isogenous to a pair where ϱ_1 is an inclusion and ϱ_2 is the identity. In particular, this implies that if $(G_1, H'_0 \times H_0)$ lifts to characteristic zero, then (G, H') also does. Therefore, the assertion of Lemma 4.2 is proved in all cases except when (G_1, H_1) is of type (C18), which cannot be lifted to characteristic zero. If $H_0 = \mathrm{Sp}(2)$, we obtain case (S9) with $n = 2$ which is therefore known spherical.

¹Here, and in Table 2 in particular, this graphical description of a pair (G, H) generalizes the notation from [Mik86]: nodes symbolize simple (or trivial) factors, and edges indicate isogenies onto their images.

If, on the other hand, $H_0 = \mathbf{G}_2$, we obtain the pair $(\mathrm{Sp}(8) \times \mathbf{G}_2, \mathrm{Sp}(2) \times \mathbf{G}_2)$, which is not spherical by the dimension bound (2.12). Thus, we are done with pairs of type (4.7).

The analysis of pairs of type (4.7) also shows which factors can occur as (G_1, H_1) . In fact, these are the ones fitting into the diagram (4.7) that occur in Table 2. Thus, we obtain the following cases:

Type	G	H		L'
(S0')	H	$\underline{H}$	H simple	$\mathbf{T}$
(S1')	$\mathrm{SL}(n+1)$	$\mathbb{G}_m \mathrm{SL}(\underline{n})$	$n \geq 3$	$\mathrm{GL}(n-1)$
(S2')	$\mathrm{SO}(n+1)$	$\mathrm{SO}(\underline{n})$	$n \geq 5$	$\mathrm{SO}(n-1)$
(S3')	$\mathrm{Sp}(2n+4)$	$\mathrm{Sp}(2n) \mathrm{Sp}(4)$	$n \geq 1$	$\mathrm{Sp}(2) \mathrm{Sp}(2)$
(S4')	$\mathrm{SL}(n+2)$	$\mathbb{G}_m \mathrm{SL}(\underline{n}) \mathrm{Sp}(2)$	$n \geq 1$	$\mathbb{G}_m$
(S5')	$\mathrm{SL}(n+2)$	$\mathrm{SL}(n) \mathrm{Sp}(2)$	$n \geq 3$	$\mathbb{G}_m$
(S6')	$\mathrm{Sp}(2n+2)$	$\mathrm{Sp}(2n) \mathrm{Sp}(2)$	$n \geq 0$	$\mathrm{Sp}(2)$
(S7')	$\mathrm{Sp}(2n+2) \mathrm{Sp}(4)$	$\mathrm{Sp}(2n) \mathrm{Sp}(2) \mathrm{Sp}(2)$	$n \geq 0$	$\mathbb{G}_m$
(S8')	$\mathrm{Sp}(2m+2) \mathrm{Sp}(2n+2)$	$\mathrm{Sp}(2m) \mathrm{Sp}(2) \mathrm{Sp}(2n)$	$m, n \geq e0$	$\mathbb{G}_m$
(S9')	$\mathrm{Sp}(8)$	$\mathbf{G}_2 \mathrm{Sp}(2)$	$p = 2$	$\mathbb{G}_m$

Here, the factor H_0 is underlined. Note that we can ignore (S0') since we may assume that both factors (G_i, H_i) are non-trivial.

Assume first that $H_0 \sim \mathrm{Sp}(2)$. Then, by Lemma 2.16, the pair (G, H') is spherical if and only if $\mathrm{Sp}(2)$ is a spherical $(L_1 \times L_2)$ -variety, where L_i is the image of $L(G_i, H_i)$ in the H_0 -factor of H_i . It is easily verified that $L_i \cong \mathbb{G}_m$ in cases (S4')–(S9') except for case (S6'), where $L_i \cong \mathrm{Sp}(2)$. This implies that the fusion (G, H') of two cases among (S4')–(S9') is spherical if and only if one of the factors is of type (S6'). To show that in this case (G, H') is indeed known spherical, i.e., that it appears in our tables, we need to show that (G, H') is independent of the gluing isogenies ϱ_1 and ϱ_2 . Note that all self-isogenies of $\mathrm{Sp}(2)$ are powers of the Frobenius morphism F_{H_0} . Thus, $\varrho_1 = F_{H_0}^{a_1}$ and $\varrho_2 = F_{H_0}^{a_2}$. Then, the isogeny $\tau = (F_{G_1}^{a_2}, F_{G_2}^{a_1})$ of $G_1 \times G_2$ yields an isogeny from (G, H') to a known spherical pair.

Assume now that $H_0 \sim \mathrm{Sp}(4) \sim \mathrm{SO}(5)$, which occurs for case (S3') and case (S2') with $n = 5$. In all these cases, the projection of $L(G_i, H_i)$ is isogenous to $\mathrm{Sp}(2) \times \mathrm{Sp}(2)$. Since the action of $(\mathrm{Sp}(2) \times \mathrm{Sp}(2))^2$ on $\mathrm{Sp}(4)$ is not spherical (see (2.12)), the pair (G, H') is also not spherical.

Next, let $H_0 \sim \mathrm{SL}(4) \sim \mathrm{SO}(6)$. This occurs for (S1') with $n = 4$ and (S2') with $n = 5$. Since $L_1 \sim T^1 \mathrm{SL}(3)$ or $\mathrm{SO}(5)$, the action of $L_1 \times L_2$ on H_0 is not spherical due to the dimension bound.

The same holds for all other cases of type (S1') or (S2'), which concludes the proof of (ii) and therefore Lemma 4.2.

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TABLE 2. Indecomposable spherical pairs $H \subset G$ for G semisimple.

(G, H)		$L(G/H)$	
(S0)		H simple	max. torus
(S1)		$n \geq 2$	1
(S2)		$n \geq 5$	1
(S3)		$n \geq 1$	$\mathrm{Sp}(2n-4)$
(S4)		$m \geq 1$ $n \geq 0$	$T^1 \cdot \mathrm{SL}(m-2) \times \mathrm{Sp}(2n-2)$
(S5)		$m \geq 3$ $n \geq 0$	$\mathrm{SL}(m-2) \times \mathrm{Sp}(2n-2)$
(S6)		$m \geq 0$ $n \geq 0$	$\mathrm{Sp}(2m-2) \cdot T^1 \cdot \mathrm{Sp}(2n-2)$
(S7)		$m \geq 0$ $n \geq 0$	$\mathrm{Sp}(2m-2) \times \mathrm{Sp}(2n-2)$
(S8)		$l \geq 0$ $m \geq 0$ $n \geq 0$	$\mathrm{Sp}(2l-2) \times \mathrm{Sp}(2m-2) \times \mathrm{Sp}(2n-2)$
(S9)		char $k=2$ $n \geq 0$	$\mathrm{Sp}(2n-2)$